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Sector of Functional Analysis and Applications

Entrance Examination – October 8, 1997

Solve at most five of the following problems.

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous function such that

$$\int_0^1 f(u(x))\,dx = 0 \tag{1}$$

for every $u \in C^0([0,1])$ satisfying

$$\int_0^1 u(x)\,dx = 0\,. \tag{2}$$

(a) Prove that (1) holds for every $u \in L^\infty([0,1])$ satisfying (2).
(b) Prove that f is linear.

2. Let X be a separable Hilbert space with infinite dimension, let $(\cdot,\cdot)$ and $\|\cdot\|$ be the scalar product and the norm of X, and $\{e_n\}_{n\geq 1}$ be an orthonormal basis of X.
(a) Prove that the function $\|\cdot\|_0$ defined by

$$\|x\|_0 = \left(\sum_{n=1}^{\infty} \frac{|(x,e_n)|^2}{n^2}\right)^{\frac{1}{2}}$$

is a norm in X.
(b) Prove that $\{x \in X \ :\ \|x\| \leq 1\}$ is compact in $(X, \|\cdot\|_0)$.
(c) Prove that the normed space $(X, \|\cdot\|_0)$ is not complete.

3. Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be a convex function of class C^2 and let $x\colon I \to \mathbf{R}^n$ be a solution of the equation

$$\dot{x}(t) = -\nabla f(x(t))$$

defined on a connected open set $I \subseteq \mathbf{R}$.
(a) Prove that the function $t \mapsto f(x(t))$ is non-increasing.
(b) Prove that the function $t \mapsto |\dot{x}(t)|^2$ is non-increasing.
(c) Prove that, if $I = \mathbf{R}$ and x is periodic, then there exists an absolute minimum point x_0 of f such that $x(t) = x_0$ for every $t \in \mathbf{R}$.

4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function with compact support. Prove that the two following conditions are equivalent:

(i) f is the uniform limit of a sequence of step functions, continuous from the right, namely of functions of the kind $\sum_{i=1}^{k} c_i \chi_{[a_i,b_i)}$, where χ_E is the characteristic function of E;

(ii) f is continuous from the right and admits a finite limit from the left at each point.

5. Let $a > 0$ and let $g \in C^0([-a,a])$. Prove that there exists a unique function $u \in C^0([-a,a])$ such that

$$u(x) = \frac{x}{2} u\Big(\frac{x}{2}\Big) + g(x)$$

for every $x \in [-a,a]$.

6. Let (u_n) be a sequence of functions of class $C^1([0,1])$ pointwise converging in $[0,1]$ to a function $u\colon [0,1] \to \mathbf{R}$. Suppose that

$$\sup_n \int_0^1 |u_n'(x)|\, dx < +\infty\,.$$

(a) Prove that u has bounded variation on $[0,1]$.

(b) Prove that

$$\int_0^1 |u'(x)|\, dx \le \liminf_{n\to\infty} \int_0^1 |u_n'(x)|\, dx\,,$$

where u' denotes the derivative of u, that is defined almost everywhere in $[0,1]$.

7. Let $1 \le p < +\infty$ and let $L^p = L^p(0,1)$. For every integer $n \ge 1$ let $T_n\colon L^p \to L^p$ be the linear operator defined by

$$(T_n f)(x) = n \int_{\frac{i-1}{n}}^{\frac{i}{n}} f(t)\, dt \quad \text{for } \frac{i-1}{n} \le x < \frac{i}{n}, \qquad i = 1, \dots, n\,.$$

(a) Prove that for every $n \ge 1$ we have $\|T_n\|_{\mathcal{L}(L^p,L^p)} = 1$, where $\mathcal{L}(L^p,L^p)$ is the normed space of all bounded linear operators from L^p into L^p.

(b) Prove that $T_n f \to f$ strongly in L^p for every $f \in L^p$.

(c) Prove that for every $n \ge 1$ we have $\|T_n - I\|_{\mathcal{L}(L^p,L^p)} \ge 1$, where $I\colon L^p \to L^p$ is the identity operator. (Hint: determine the kernel of T_n).

8. Let $u\colon [0,1] \to \mathbf{R}$ be a bounded function such that for every $t < \sup\limits_{x\in[0,1]} u(x)$ the set

$$\{x \in [0,1] \ :\ u(x) \ge t\}$$

is a closed interval. Prove that u has bounded variation and that the total variation $V(u)$ of u satisfies the inequality

$$V(u) \le 2\Big(\sup_{x\in[0,1]} u(x) - \inf_{x\in[0,1]} u(x)\Big)\,.$$

9. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function of class C^1 with $f(1) = 0$. Prove that for $0 < \alpha < 1$ the Cauchy problems

$$\begin{cases} x'(t) = x(t) f(x(t)^2 + y(t)^2) - y(t)\,, & x(0) = \alpha\,, \\ y'(t) = y(t) f(x(t)^2 + y(t)^2) + x(t)\,, & y(0) = 0\,, \end{cases}$$

have a bounded solution defined on $\mathbf{R}$. (*)

10. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a function of class C^1 such that for every $\xi \in \mathbf{R}$

$$\begin{aligned} & f(-\xi) = -f(\xi)\,, \\ & f'(\xi) > 0\,, \\ & \lim_{\xi \to +\infty} f(\xi) = l < +\infty\,. \end{aligned}$$

For every $\alpha > 0$ let x be the maximal solution of the Cauchy problem

$$x'(t) = f(tx(t))\,, \quad x(0) = \alpha\,.$$

(a) Prove that x is defined on $\mathbf{R}$. (*)
(b) Prove that $x'(t) > 0$ for $t > 0$.
(c) Prove that $x(t) = x(-t)$ for every $t \in \mathbf{R}$.
(d) Prove that $x(t) \geq \alpha$ for every $t \in \mathbf{R}$.
(e) Prove that $\lim_{t \to +\infty} x'(t) = l$.

(*) To answer this question one can use general theorems on global existence of solutions, but in this case one must write explicitly the statements of the theorems that are used.