SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

September 5, 2017

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1 Mathematical Analysis

1.

(a) Let V be a closed subspace of a Hilbert space H and $\ell : V \to \mathbb{R}$ be a continuous linear functional. Prove that there exists a unique norm-preserving extension of ℓ to H.

(b) Give an example of a closed subspace W of $L^1([0,1])$ and a continuous linear functional $\ell : W \to \mathbb{R}$ with at least two norm-preserving extensions to $L^1([0,1])$.

2.

(a) Let $1 \le p_1 \le p \le p_2 \le \infty$ and

$$\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}, \quad 0 \le \alpha \le 1 .$$

Prove that if $f \in L^{p_1} \cap L^{p_2}$, then $f \in L^p$ and the following inequality holds:

$$\|f\|_{L^p} \le \|f\|_{L^{p_1}}^{\alpha} \|f\|_{L^{p_2}}^{1-\alpha} .$$

(b) Let f_n be a bounded sequence in $L^\infty([0,1])$ and $f_n \to f$ in $L^2([0,1])$. Deduce that f is in $L^\infty([0,1])$ and that f_n converges to f in $L^p([0,1])$ for any $2 \le p < +\infty$.

3. Let E be a closed subspace of $L^p([0,1])$ with $1 \le p < +\infty$ such that $E \subset L^\infty([0,1])$.

(a) Prove that E is complete for both the $\| \, \|_{L^p}$ and $\| \, \|_{L^\infty}$ norm.

(b) Prove that there is a constant $C > 0$ such that

$$\|u\|_{L^\infty} \le C \|u\|_{L^p} , \quad \forall u \in E.$$

4. Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of nonzero self-adjoint operators, everywhere defined on the Hilbert space H and such that for every $n \in \mathbb{N}$:

$$T_n^2 = \left(1 + \frac{1}{n}\right) T_n, \quad \mathrm{Im}(T_n) \subset \mathrm{Im}(T_{n+1}) \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} \mathrm{Im}(T_n) = H.$$

(a) Prove that each T_n is a bounded operator with operator norm $\|T_n\| = 1 + \frac{1}{n}$.

(b) Prove that the sequence $\{T_n\}_n$ converges strongly to the identity on H, i.e. that for every $x \in H$ we have $T_n x \to x$.

5. Let $T : H \to H$ be a *compact* self-adjoint operator on the Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$ and let $p : \mathbb{R} \to \mathbb{R}$ be a real polynomial with real zeroes such that:

$$p(T) = 0.$$

(a) Prove that if H is infinite dimensional, then 0 is an eigenvalue of T.

(b) Assume that $p(s) > 0$ for all $s < 0$. Prove that for every $x \in H$ we have $\langle x, Tx \rangle$.

6. Consider the Cauchy problem

$$\begin{cases} u'(t) = (\sin t)^2 - u(t)^2 \\ u(0) = -1. \end{cases} \tag{1}$$

Let $u :]\alpha, \beta[\to \mathbb{R}$ the unique maximal solution to (1). Prove that $\beta < +\infty$.

7. Let X, Y be two Banach spaces with X reflexive. Let $X \subseteq Y$ and suppose that the immersion $X \hookrightarrow Y$ is continuous. Consider $y \in Y$ and $\{x_n\}_n$ a bounded sequence in X. Prove that if $x_n \rightharpoonup y$ weakly in Y then $y \in X$ and $x_n \rightharpoonup y$ weakly in X.

8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be C^1 functions, bounded, and such that $f(0) = 0$. Prove that for $\epsilon > 0$ small enough the following system has a solution:

$$\begin{cases} u''(t) = f(\epsilon u(t)) + \epsilon g(t) \\ u(0) = 0 = u(1). \end{cases}$$

(Hint: use the Implicit Function Theorem.)

9.

(a) Prove that if all the zeroes of a complex polynomial p lie in the upper half plane $\{z \in \mathbb{C} \,|\, \mathrm{im}(z) > 0\}$, then the derivative of p cannot vanish at the origin.

(b) Prove that every critical point of a complex polynomial p (i.e. a point $z_0 \in \mathbb{C}$ such that $p'(z_0) = 0$) lies in the convex hull of the zeroes of p.

10. Construct a sequence of positive real numbers $\{a_n\}_{n\in\mathbb{N}}$ with $a_n \neq o\left(\frac{1}{n}\right)$ such that:

$$\sum_{n=0}^{\infty} a_n < \infty.$$

2 Numerical Analysis

11. Consider the steady Navier-Stokes problem with non-homogeneous Dirichlet boundary conditions.

$$\begin{cases} \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega \subset \mathbb{R}^2, \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{2}$$

where $\mathbf{f}$ and $\mathbf{g}$ are given vector functions, while $\mathbf{u}$ and p are the velocity and pressure unknowns, respectively.

(a) Write the weak formulation of the problem under proper assumptions.

(b) Show that $\int_\Gamma \mathbf{g} \cdot \mathbf{n} = 0$ is a necessary condition for the existence of a weak solution, in order to fulfil a compatibility condition.

(c) Introduce and discuss stable finite element approximations of (2).

12. Consider the diffusion-advection-reaction problem

$$\begin{cases} -\varepsilon \Delta u + \mathbf{b} \cdot \nabla u + \gamma u = f, & \text{in } \Omega \subset \mathbb{R}^2, \\ u = \varphi, \text{ on } \Gamma_D, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \text{ on } \Gamma_N, \end{cases} \tag{3}$$

where $\Gamma_D \subset \partial\Omega$ and $\Gamma_N \subset \partial\Omega$ are two non-empty disjoint sets such that $\Gamma_D \cup \Gamma_N = \partial\Omega$.

(a) Discuss the assumptions on $\varepsilon = \varepsilon(x, y)$, $\mathbf{b} = (b_1(x, y), b_2(x, y))^T$, $\gamma = \gamma(x, y)$, $f = f(x, y)$ and $\varphi = \varphi(x, y)$ that guarantee existence and uniqueness of the solution $u = u(x, y)$.

(b) Introduce a Galerkin finite element approximation of (3).

(c) Suppose that $|b| \gg \varepsilon$. Introduce suitable stabilization methods (e.g. SUPG), discussing advantages and disadvantages with respect to the classic Galerkin finite element method.

13. Let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$, and the following optimal control problem

$$\min_{(y,u)\in Y\times U} \left\{ J(y,u) = \frac{1}{2}\int_\Omega (y-y_d)^2 + \frac{\alpha}{2}\int_\Omega u^2 \right\} \tag{4}$$

subject to

$$\begin{cases} -\Delta y = f + u, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where y_d and f are given functions, while $y \in Y$ and $u \in U$ are the unknown state solution and control, respectively.

(a) Introduce a weak formulation of (5), of the following form:

$$\text{given } u \in U, \quad \text{find } y \in Y \quad \text{such that} \quad a(y,q) = c(u,q) + F(q), \quad \forall q \in Y,$$

for some suitable bilinear forms $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$, and a suitable linear form $F(\cdot)$. Specify a functional space for f, as well as the state space Y and control space U. Discuss the well posedeness of the resulting weak problem.

(b) Rewrite the cost functional $J(y,u)$ as

$$J(y,u) = \frac{1}{2} m(y-y_d, y-y_d) + \frac{\alpha}{2} n(u,u),$$

for some suitable bilinear forms $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$. Specify any additional assumption on y_d.

Denote by $\mathcal{J}(u)$ the reduced cost functional obtained as $\mathcal{J}(u) = J(y(u),u)$, being $y(u)$ the unique weak solution of (5) under the assumptions in a), and denote by $\mathcal{J}'(u) \in U'$ its derivative, being U' the dual space of U. It can be shown that

$$\langle \mathcal{J}'(u), v\rangle = \alpha\ n(u,v) - c(v, p(u))$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between U' and U, and $p(u)$ is the solution of the so-called *adjoint problem*

$$\text{given } u \in U, \quad \text{find } p \in Y \quad \text{such that} \quad a(z,p) = -m(y-y_d, z), \quad \forall z \in Y.$$

(c) Discuss a numerical method based on a finite element approximation to solve the optimal control problem (4)-(5), resorting either to an iterative strategy (e.g. gradient type) or to a one shot approach (e.g. saddle point).

14. Let $\mathbf{x} \in \mathbb{R}^n$ be the solution to the linear system $A\mathbf{x} = \mathbf{b}$, for $A \in \mathbb{R}^{n\times n}$ and $\mathbf{b} \in \mathbb{R}^n$. Consider the following stationary Richardson method

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha P^{-1}\mathbf{r}^{(k)}$$

where α is a relaxation parameter, P is a preconditioner of A and $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$.

(a) Under which conditions on A and P does $\mathbf{x}^{(k)}$ converge to $\mathbf{x}$?

(b) Describe how to choose α in order to have the fastest convergence of $\mathbf{x}^{(k)}$ to $\mathbf{x}$.

15. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function and denote by α one of its roots (i.e., $f(\alpha) = 0$).

(a) Introduce the Newton method to numerically solve $f(x) = 0$.

(b) Recast the Newton method as a fixed point method $x^{(n+1)} = g(x^{(n)})$.

(c) Define the notion of order of convergence of the method in b). Under what assumptions on g (and, possibly, $x^{(0)}$) does the fixed point method converge, and with what order? Take into consideration also special cases, such as $g'(\alpha) = 0$.

(d) Apply the result in c) to the Newton method. How does the multiplicity of α affect the order of convergence?

3 Continuum Mechanics

16. Consider the two-dimensional, unsteady flow of a fluid whose velocity field is given by $\mathbf{v}(x, y, t) = xy\,\mathbf{e}_1 + \exp(-5x^3t)\mathbf{e}_2$, where $\mathbf{e}_1$-$\mathbf{e}_2$ is an orthonormal basis in $\mathbb{R}^2$, $t > 0$ is the time, and x, y are the space variables. Given the temperature field $T(x, y, t) = 2x + y^2 + A\sin(t)$, find the constant A such that the instantaneous rate of change of the temperature of the fluid particles crossing the point $x = 0$, $y = 1$ vanishes for $t = 2k\pi$, with k a positive integer. For the value of A just found, compute the time rate of the temperature field in the same point and for the same times.

17. A cylindrical body with length L and circular cross-section with radius R is subject to applied tractions $\mathbf{t} = \sigma\mathbf{n}$, $\sigma > 0$, where $\mathbf{n} = \pm\mathbf{e}_1$ is the outward unit normal to the bases of the cylinder and $\mathbf{e}_1$ is the unit vector along the axis of the cylinder. Within the framework of linear elasticity and assuming a homogeneous, isotropic and incompressible constitutive response for the body, compute the stress, the pressure and the infinitesimal strain fields. In addition, compute the displacement field, up to a rigid body motion.

18. Consider an elastic, inextensible and straight (in its unloaded configuration) rod of length L with solid circular cross-section of constant radius R. Let the rod be in equilibrium with two bending moments of magnitude M applied at its extremities. Compute the value of M such that the rod deforms into a closed loop (for a solid circular cross-section of radius R the moment of inertia reads $\pi R^4/4$).

19. A sphere with initial radius R made of a linearly elastic, isotropic material is immersed in a fluid at pressure p. Using linear elasticity theory, compute the value of the bulk modulus of the material such that the relative change in radius of the sphere (with respect to the initial radius) equals 1%.

20. Let $\mathbf{A}$ be an invertible second-order tensor. Compute the derivative of its determinant $\det(\mathbf{A})$ with respect to $\mathbf{A}$.