Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Course in Geometry and Mathematical Physics

Written test, September 12, 2017

Each candidate is required to solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. CLASSICAL MECHANICS

Let (q_1, q_2, p_2, p_2) be canonical coordinates. Consider the following Hamiltonian

$$H := p_1 q_1 - p_2 q_2 - a q_1^2 + b q_2^2, \qquad a, b \in \mathbb{R},$$

and a one-parameter $s \in \mathbb{R}$ group of diffeomorphisms

$$\Phi_s(q_1, q_2, p_1, p_2) := \left(q_1 + \frac{s}{q_2},\ q_2,\ p_1 + \frac{as}{q_2},\ p_2 + \frac{(p_1 - aq_1)\ s}{q_2^2} \right), \qquad q_2 \neq 0.$$

1. Verify that Φ_s is a symmetry of H and find an associated conserved quantity $H_1(q_1, q_2, p_1, p_2)$.
2. Find another conserved quantity $H_2(q_1, q_2, p_1, p_2)$ and the associated one-parameter group of symmetries (*you may wish to calculate the explicit solutions of the Hamilton equations*).
3. Consider the Hamilton-Jacobi equation

   $$H(q_i, \frac{\partial S}{\partial q_i}) + \frac{\partial S}{\partial t} = 0 \quad .$$

   Find a general integral $S(q_1, q_2)$ of the Hamilton-Jacobi equation. Using $S(q_1, q_2)$, explicitly compute the solution of the equations of motion for H.

B. QUANTUM MECHANICS

Consider a free quantum particle on a circle $q \equiv q+1$, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}\dot{q}^2$$

1. discuss the spectrum of the associated Hamiltonian H and its eigenfunctions
2. add to $\mathcal{L}$ the term $\frac{\theta}{2\pi}\dot{q}$ so that the new Langrangian is
$$\mathcal{L}_\theta = \frac{1}{2}\dot{q}^2 + \frac{\theta}{2\pi}\dot{q}$$
and compute the new Hamiltonian H_θ. Do the eigenfunctions change ? Does the spectrum change ?
3. Are there degeneracies in the spectrum of H_θ - namely different eigenfunctions with the same energy eigenvalue - for some special values of θ ? Motivate the answer in terms of the symmetries of H_θ.
4. Discuss the symmetry group acting on the Hilbert space of the system for generic θ. Is there an enhancement of symmetry for some special values of θ ?

C. COMPLEX ANALYSIS

1. Compute the integral
$$\int_0^\infty \frac{(\ln z)^2 dz}{z^2+1} \tag{1}$$
2. Prove also that all integrals
$$\int_0^\infty \frac{(\ln z)^{2k+1} dz}{z^2+1}\ , \quad k \in \mathbb{N} \tag{2}$$
are zero.
3. Provide a suitable generating function for the numbers
$$I_n = \int_0^\infty \frac{(\ln z)^n dz}{z^2+1}, \tag{3}$$
that is, a closed (simple!) expression for a series of the type
$$\sum_{\ell=0}^{\infty} \xi^\ell c_\ell I_\ell \tag{4}$$
with c_ℓ a convenient sequence of your choice. The computation can use formal manipulations if convenient.

D. RIEMANNIAN GEOMETRY

Let $S \subset \mathbb{R}^n$ be a compact surface, regularly embedded in the n-dimensional Euclidean space. Endow S with the induced Riemannian metric g. Write κ_g for the Gaussian curvature of g.

(a) Prove that when $n = 3$ there exists a point $p \in S$ such that $\kappa_g(p) > 0$.

(b) Provide a counterexample to the claim in (a) when $n > 3$.

(c) Prove that when $n = 3$ if S is also connected, orientable and $\kappa_g \neq 0$ everywhere then S is diffeomorphic to the sphere S^2.

Let $X \subset \mathbb{R}^{n+1}$ be a compact n-dimensional regular submanifold of the $(n+1)$-dimensional Euclidean space. Endow X with the induced Riemannian metric g. Write s_g for the scalar curvature of g.

(d) Prove that there exists a point $x \in X$ such that $s_g(p) > 0$.

(e) Prove that if X is also connected, orientable with strictly positive sectional curvatures then X is diffeomorphic to the sphere S^n.

E. ALGEBRAIC OR COMPLEX GEOMETRY

This exercise comes in two versions, one for classical algebraic geometry and the other (in square brackets) for complex manifolds. You should declare at the beginning which version you are choosing.

Consider on $\mathbb{C}^4$ the bilinear skew-symmetric form

$$Q(v, w) := v_1 w_3 + v_2 w_4 - v_3 w_1 - v_4 w_2.$$

Let B be the set of pairs $(v, w) \in \mathbb{C}^4 \times \mathbb{C}^4$ which are linearly independent.

1. Let Y be the set of pairs $(v, w) \in B$ which satisfy $Q(v, w) = 0$. For $(v, w) \in Y$, show that the form Q restricted to the vector space V generated by v and w is zero (i.e., for all $x, y \in V$, one has $Q(x, y) = 0$).

2. Show that Y is a smooth closed subvariety [closed complex submanifold] of B of dimension 7.

3. Let $G(2, 4)$ be the Grassmann variety parametrizing 2-dimensional subspaces of $\mathbb{C}^4$. Let LG (the Lagrangian Grassmannian) be the subset of $G(2, 4)$ defined as

$$LG := \{V \in G(2, 4) \,|\, \forall x, y \in V \times V,\ Q(x, y) = 0\};$$

in other words, a subspace is in LG if and only if it is Lagrangian. Let $p : Y \to LG$ map (v, w) to the subspace V they generate. Show that p is a surjective morphism [holomorphic map].

4. Construct an action of $GL(2, \mathbb{C})$ on Y with no fixed points and such that two points in Y are in the same orbit if and only if they have the same image in LG.

5. Show that LG is a smooth projective variety [compact complex manifold] of dimension three.

6. (Optional) Show that $Y \to LG$ is a principal $GL(2, \mathbb{C})$-bundle. If you do not know what this means, show that there is a finite open cover $\{U_i\}_{i \in I}$ of LG such that (for all $i \in I$) there are sections s_i : of p on U_i, that is morphisms [holomorphic maps] $s_i : U_i \to Y$ such that $p \circ s_i$ is the inclusion of U_i in LG.