Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Course in Geometry and Mathematical Physics

Written test, September 12, 2018

Each candidate is required to solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. CLASSICAL MECHANICS

Consider a point particle of mass m moving in 3-space under the action of a central force F_G and a constant force directed along the z axis, and let $U_G = -g/r$ and $\phi = -\kappa\, z$ be the associated potential energies ($r = \sqrt{x^2 + y^2 + z^2}$, where (x, y, z) are Cartesian coordinates in $\mathbb{R}^3$).

1. Write the Lagrangian $\mathcal{L}$ and the Hamiltonian $\mathcal{H}$ of the system in cylindrical coordinates
$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \qquad\qquad z \in \mathbb{R}, \rho > 0, \varphi \in (0, 2\pi) \end{cases}$$

2. Consider *parabolic* coordinates defined by
$$\begin{cases} \rho = \sqrt{\xi\eta} \\ z = \dfrac{1}{2}(\xi - \eta) \\ \varphi = \varphi \qquad\qquad \xi, \eta > 0 \end{cases}$$
and write $\mathcal{L}$ and $\mathcal{H}$ in these coordinates.

3. Show that in the parabolic coordinates of point 2 the Hamilton-Jacobi equations associated with $\mathcal{H}$ admit a complete separated integral.
(*Hint:* observing that φ is cyclic, start by separating this variable).

4. Explicitly compute the third integral of the motion obtained via the separation procedure.

5. Discuss these integrals of the motion in the light of Noether's theorem in the Lagrangian and the Hamiltonian setting.

B. QUANTUM MECHANICS

Consider the following Hamiltonian for a particle on $\mathbb{R}^2$

$$H = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}\left(x^2 + y^2\right) - \lambda xy$$

where (x, y) are the Cartesian coordinates on $\mathbb{R}^2$ and $\lambda \geq 0$ a real number.

1. compute the exact spectrum of H for $\lambda = 0$.
2. for $\lambda = 0$, compute the eigenfunctions of the lowest energy level and the first excited state. Discuss possible degeneracies of the energy spectrum, namely whether there exist different eigenfunctions with the same energy eigenvalue.
3. compute the exact spectrum for $\lambda \neq 0$.
4. calculate by using perturbation theory the energies of the ground state and the first excited states up to the first order in λ and compare with the exact result of the previous item. Discuss if there are degeneracies of the spectrum for $\lambda \neq 0$.

Recall that the Gaussian integral formula reads

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

C. COMPLEX ANALYSIS

Let $\mathcal{D}$ be a domain containing the unit disk $\{|z| \leq 1\}$ and let f be a meromorphic function on the domain $\mathcal{D}$ with finitely many simple poles at the points $z_1, \ldots, z_k$ on the unit circle, $|z_j| = 1$. Let $A_1, \ldots, A_k$ be the residues at those points.

Let $f(z) = \sum_{k\geq 0} a_k z^k$ be the Taylor series at $z = 0$. Prove that there is a positive constant M such that $|a_\ell| \leq M$ for all $\ell = 0, 1, \ldots$. Provide an estimate for M.

D. DIFFERENTIAL GEOMETRY

(a) Compute the curvature of the parabola $y = x^2$ at the point $(x, y) = (0, 0)$.

(b) Compute the curvature and torsion of the following parametrised space curves at the point $t = 0$:

$$\begin{aligned} \alpha(t) &= (t, t^2, t^3); \\ \beta(t) &= (t, t^2, t^5); \\ \gamma(t) &= (t, t^4, t^5). \end{aligned}$$

(c) Prove that a space curve with everywhere zero torsion lies on a plane.

(d) Prove that there exists a *closed* space curve with nonzero constant curvature and whose torsion does not vanish identically.

E. ALGEBRAIC GEOMETRY

Let $\pi\colon X \to \mathbb{A}^2$ be the blowup of $\mathbb{A}^2$ at $O = (0,0)$, and let C be the curve $y^2 = x^3$ in $\mathbb{A}^2$. Let C' be the proper transform of C, i.e.,

$$C' = \overline{\pi^{-1}(C - O)}.$$

a) Prove that C' is nonsingular and it is isomorphic to $\mathbb{A}^1$.

b) Let π' be the restriction of π to C'. Prove that π' is bijective, regular and birational, but is not an isomorphism.

Let $\pi\colon X \to \mathbb{A}^3$ be the blowup of $\mathbb{A}^3$ at $O = (0,0,0)$, and let Y be the quadric cone $xy = z^2$ in $\mathbb{A}^3$. Let Y' be the proper transform of Y, i.e.,

$$Y' = \overline{\pi^{-1}(Y - O)}.$$

c) Prove that Y' is nonsingular.

d) Let π' be the restriction of π to Y'. Describe the inverse image $\pi'^{-1}(O)$.