Sector of Functional Analysis

Entrance Examination, 1990

Solve at most five of the following problems.

Pb. 1) Let $\Omega = (0,\pi) \times (-1,1)$. Find the explicit solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of the problem

$$u_{xx} + u_{yy} = 0 \qquad \text{for all } (x,y) \in \Omega$$

$$u(0,y) = u(\pi,y) = 0 \qquad \text{for all } y \in [-1,1]$$

$$u(x,-1) = u(x,1) = \sin(2x) \qquad \text{for all } x \in [0,\pi].$$

Pb. 2) Prove that the family

$$\frac{1}{\sqrt{2\pi}}, \{\frac{\cos nx}{\sqrt{\pi}} | n \geq 1\}, \{\frac{\sin nx}{\sqrt{\pi}} | n \geq 1\}$$

is an orthonormal basis of $L^2([-\pi,\pi])$.

Pb. 3) Let $I = [a,b]$ be a compact interval, let $h : I \to I\!\!R$ be a continuous function, and let X be the set of all the continuous functions $f : I \to I\!\!R$ such that

$$||f||_h = \sup_{x \in I} e^{\int_a^x h(t)dt} |f(x)| < +\infty.$$

Prove that $|| \cdot ||_h$ on X is a norm, which is equivalent to the usual "sup"-norm on X.

Pb. 4) Let $\ell^2 = \{(u_n)_n | u_n \in I\!\!R, \sum_n u_n^2 < +\infty\}$. Given a sequence $(x_n)_n$ of real numbers such that $x_n \to 0$, let $T : \ell^2 \to \ell^2$ be the linear operator defined by $(Tu)_n = x_n u_n$ for every $u = (u_n)_n \in \ell^2$.
Prove that:
- T is continuous;
- T is compact and find the spectrum of T.

Pb. 5) Let $(A_n)_n$ be a sequence of linear and continuous operators between the two Banach spaces X and Y. We suppose that, for every $x \in X$, the sequence $(A_n x)_n$ converges weakly to Ax in Y. Is it true that $A : X \to Y$ is linear and continuous?

Pb. 6) Prove that, if E is a Banach space and $H \subset E$ is a hyperplane, then H is closed or dense in E. (*Hint: Suppose that* $0 \in H$. *If* H *is not closed, then it is a proper subspace of its closure.*)

Pb. 7) Let $\Omega \subset I\!R^{n+1}$ be an open set, $f : \Omega \to I\!R^n$ a continuous function, and $K \subset \Omega$ a compact set. Prove the existence of a $\delta = \delta(K) > 0$ such that for all $(t_0, x_0) \in K$ the Cauchy problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0$$

has a (unique) solution defined (at least) on $[t_0 - \delta, t_0 + \delta]$.

Pb. 8) Let X be a metric space and let $(K_n)_n$ be a sequence of closed subset of X such that

$$\ldots \subset K_{n+1} \subset K_n \subset K_{n-1} \subset \ldots$$

Prove the following statements:

a) If X is complete and $\lim_n diam K_n = 0$, then $\bigcap_n K_n$ is not empty and it consists of exactly one point.

b) If a K_n is compact for some n then $\bigcap_n K_n$ is not empty.

Pb. 9) Let $I = [0, 1]$, let C be a closed subset of $I\!R$, and let

$$X_C = \{f \in L^2(I) | f(x) \in C \text{ for a.e. } x \in I\}.$$

a) Prove that X_C is closed in the strong topology of $L^2(I)$.

b) Prove that, if C is a closed interval, then X_C is closed in the weak topology of $L^2(I)$.

c) Show, with an example, that in general X_C is not closed in the weak topology of $L^2(I)$.

Pb. 10) Let $f : [0, 1] \to I\!R$ be a measurable function. Prove that $f \in L^\infty([0, 1]; I\!R)$ if and only if $f \in L^p([0, 1]; I\!R)$ for all $p \geq 1$ and $\sup_{p\geq 1} ||f||_p$ is finite.