Sector of Functional Analysis

Entrance Examination, 1991

Solve at most five of the following problems.

Pb. 1) Let $f : (a, b) \to I\!R$ be a convex function. Prove that it is continuous. Is the result true in the case of a closed interval $[a, b]$?
(Hint: Remember that a function f is convex if and only if the set $\{(x, y) \mid y \geq f(x)\}$ is convex.)

Pb. 2) Let $f : I\!R \to I\!R$ be a continuous function such that
i) $f(0) = 0$; $f(x) > 0 \;\; \forall x \neq 0$;
ii) there exists $x > 0$ such that $\lim_{\epsilon \to 0} \int_\epsilon^x \frac{du}{f(u)} = +\infty$.
Prove that $x(t) \equiv 0$ is the only solution to

$$x' = f(x), \qquad x(0) = 0.$$

Pb. 3) Let H be a separable Hilbert space of infinite dimension. Prove that $S = \{x \in H \mid ||x|| = 1\}$ is not compact.

Pb. 4) Let $I = [0, 1]$ and let $f : I \times I\!R \to I\!R$ be a function such that
i) $f(\cdot, x)$ is measurable for all $x \in I\!R$;
ii) $f(t, \cdot)$ is continuous for a.e. $t \in I$.
Prove that, for every continuous function $x : I \to I\!R$, the function $g_x : I \to I\!R$, $g_x(t) = f(t, x(t))$ is measurable.

Pb. 5) Let $I = [0, 1]$ and let X be the set of all $f : I \times I\!R \to I\!R$ such that

$$\sup_{(t,x) \in I \times I\!R} \frac{|f(t, x)|}{1 + |x|} < +\infty.$$

Prove that X with the norm

$$||f|| = \sup_{(t,x) \in I \times I\!R} \frac{|f(t, x)|}{1 + |x|},$$

is a Banach space.

Pb. 6) Let $(X_1, ||\cdot||_1)$ and $(X_2, ||\cdot||_2)$ be Banach spaces with X_1 compactly embedded in X_2. Let T be an operator in X_2 with domain $D(T)$ contained in X_1 and such that there exists $a > 0$ so that

$$a||x||_1 \leq ||Tx||_2, \qquad \forall x \in D(T).$$

Prove that:

i) $T^{-1} : R(T) \to X_2$ exists and is a compact operator;

ii) there exist constants $A, B > 0$, such that for all $y \in R(T)$:

$$||T^{-1}y||_1 \leq A||y||_2, \qquad ||T^{-1}y||_2 \leq B||y||_2;$$

iii) if moreover $R(T)$ is dense in X_2, then T^{-1} has a unique continuous extension in the whole X_2.

Pb. 7) Let

$$L_o = \{\phi \in L^2[-a,a] \mid \phi(t) = -\phi(-t) \text{ a.e.}\},$$

$$L_e = \{\phi \in L^2[-a,a] \mid \phi(t) = \phi(-t) \text{ a.e.}\}.$$

Find the distance (in $L^2[-a,a]$) of $f(t) = t^2 + t$ from L_o and from L_e. Find also the distance of an arbitrary $f \in L^2[-a,a]$ from L_o and from L_e.

Pb. 8) Let H be a Hilbert space and let M be a closed subset of H. Find the eigenvalues and the eigenvectors of the orthogonal projection on M.

Pb. 9) Let $(f_n)_n$ be a sequence of function in $L^1[0,1]$ with $f_n \to f$ a.e. and

$$\lim_{n\to\infty} ||f_n||_{L^1} = ||f||_{L^1}. \tag{1}$$

Prove that $\lim_n ||f_n - f||_{L^1} = 0$. Prove with an example that, if the condition (1) is changed in

$$||f_n||_{L^1} \text{ converges,}$$

the thesis is not true.

Pb. 10) Let $I = [0,1]$. Given $g \in C^1(I \times I)$, consider the operator

$$(Tu)(s) = \int_0^1 g(t,s)u(t)dt$$

defined for all $u \in C^0(I)$. Discuss the spectrum of T.

Pb. 11) Let $I = [0,r]$. Consider the differential equation

$$u_{xy}(x,y) = f(x,y,u(x,y)), \qquad (x,y) \in I \times I,$$

$$u(0,\cdot) = 0 \text{ on } I,$$

$$u(\cdot,0) = 0 \text{ on } I.$$

Give conditions on r and f assuring the existence of a solution to the problem.