Sector of Functional Analysis

Entrance Examination, 1992

Solve at most 5 of the following problems.

Pb. 1) Let $f : I\!R \to I\!R$. Define

$$\omega(x) := \inf_{\epsilon>0} \sup_{\substack{|y-x|<\epsilon \\ |z-x|<\epsilon}} |f(y) - f(z)|.$$

Prove that

1a) ω is upper semicontinuous;

1b) The set of points of continuity of f is a countable intersection of open sets (that is, a G_δ).

Pb. 2) Let (z_n) a sequence of complex numbers, with $\mathrm{Re}(z_n) \geq 0$ for each n. Show that the convergence of the series $\sum z_n$ and $\sum z_n^2$, implies the convergence of the series $\sum |z_n|^2$.

Pb. 3) Let $(f_n)_n$ be a sequence of measurable functions from $I\!R$ into $I\!R$. Prove that the set

$$\{x \in I\!R \mid (f_n(x))_n \ \text{ converges}\}$$

is measurable.

Pb. 4) Set L_1 be the linear subspace of ℓ_2 spanned by the set of vectors $(1, 2, 0, 0, \ldots)$, $(0, 1, 2, 0, \ldots)$, $(0, 0, 1, 2, 0, \ldots), \ldots$, and let L_2 be the linear subspace of ℓ_2 spanned by the vector $(1, 0, 0, \ldots)$. Prove that $L_1 + L_2$ is dense in ℓ_2.

Pb. 5) Find the maximum domain of existence of the solution of each of the following Cauchy problems:

$$y' = \sin y^2, \quad y(0) = 1,$$

$$y' = y + \sin y^2, \quad y(0) = 1,$$

$$y' = y^2, \quad y(0) = 1.$$

Pb. 6) Let $u, v \in C^1([0, 1])$ and $\omega \in C^0([0, 1] \times I\!R)$. Show that the conditions

$$v(0) < u(0) \qquad \text{and} \qquad v'(t) < \omega(t, v(t)), \qquad u'(t) = \omega(t, u(t)) \quad \forall\, t \in [0, 1]$$

imply $v(t) < u(t) \quad \forall t \in [0, 1]$.

Pb. 7) Find a solution u of

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

on $\Omega = [0, \pi] \times \mathbb{R}$ satisfying

$$u(x, 0) = \sin x \quad \text{for } 0 \leq x \leq \pi,$$

$$\frac{\partial}{\partial t} u(x, 0) = \sin 2x \quad \text{for } 0 \leq x \leq \pi,$$

$$u(0, t) = u(\pi, t) = 0 \quad \text{for } t \in \mathbb{R}.$$

Pb. 8) Let $(f_n)_n$ be a sequence in L^1 with $f_n \geq 0$ for every n. Assume that $(f_n)_n$ converges pointwise to a function $f \in L^1$. Prove that $\int f_n \to \int f$ implies $f_n \to f$ in L^1.

(Hint: consider the functions $\min\{f_n, f\}$ and $\max\{f_n - f, 0\}$.)

Pb. 9) Let $a_0, \ldots, a_n \in C^1(\mathbb{R})$. For every $x \in \mathbb{R}$ consider the polynomial

$$P_x(z) = \sum_{k=0}^{n} a_k(x) z^k$$

in the real variable z. Assume that for each x the equation $P_x(z) = 0$ has n different real roots $z_1(x) < z_2(x) < \ldots < z_n(x)$. Show that $z_1, z_2, \ldots, z_n$ are C^1 function from $\mathbb{R}$ into $\mathbb{R}$.

Pb. 10) Let H be a separable Hilbert space with basis $(e_n)_n$. Let $T : H \to H$ be a bounded linear operator such that

$$\sum_n ||Te_n||^2 < \infty.$$

Prove that T is a compact operator.

Pb. 11) Let X be a Banach space and A, B closed vector subspaces of X with $A \cap B = \{0\}$ and $A + B = X$. It is well known that every $x \in X$ can be written in a unique way as $x = P_A x + P_B x$, with $P_A x \in A$ and $P_B x \in B$. Prove that the above defined projections $P_A : X \to A$ and $P_B : X \to B$ are continuous.