Sector of Functional Analysis

Entrance Examination, October 1993

Solve at most 5 of the following problems.

Pb. 1) Let $x_1, \dots x_n$ be real numbers. Prove that
1a) $\sum_{i=1}^n x_i^2 = 1 \implies (x_1^2 x_2^2 \cdots x_n^2)^{1/n} \le 1/n$;
and then deduce that
1b) $x_i > 0$ for every $i \implies (x_1 x_2 \cdots x_n)^{1/n} \le \frac{x_1 + \cdots x_n}{n}$.

Pb. 2) Let $f, g : \mathbb{C} \to \mathbb{C}$ be holomorphic. Prove that the condition

$$|f(z)| \le |g(z)| \qquad (z \in \mathbb{C}), \qquad\qquad (*)$$

implies the existence of a constant c such that, $f = cg$.

Pb. 3) Let H a Hilbert space, and for any $n \ge 1$ let $A_n : H \to H$ be a bounded linear operator. Prove that $\|A_n\| \to 0$ if the following condition holds:
$A_n x_n \to 0$ strongly for every weakly convergent sequence $(x_n)_{n \ge 1}$ in H.

Pb. 4) Let $(f_n)_{n \ge 1}$ be a sequence of real continuous functions defined on $[0, 1]$. Prove that the following conditions are equivalent:
a) $(f_n)_{n \ge 1}$ is equibounded and $f_n \to 0$ pointwise;
b) $\lim_n \int_{[0,1]} f_n d\mu = 0$ for every Borel measure μ which is positive and bounded on $[0, 1]$.

Pb. 5) Let X be a complete metric space and, for each $0 \le \lambda \le 1$, let $T_\lambda : X \to X$ satisfy

$$d(T_\lambda(x), T_\lambda(y)) \le \frac{1}{2} d(x, y) \qquad (\forall\ x, y, \lambda).$$

Prove that if D is a dense subset of X and for every $x \in D$

$$\lim_{\lambda \to 0} T_\lambda(x) = T_0(x),$$

then,

$$\lim_{\lambda \to 0} x_\lambda = x_0,$$

where $x_\lambda = T_\lambda(x_\lambda)$.

Pb. 6) In $L^\infty = L^\infty([0, 1])$, let V be the set of all characteristic function χ_E, with E running in the set of measurable subsets of $[0, 1]$. Is V compact in L^∞? Justify your answer.

Pb. 7) Let $T : \ell_2 \to \ell_2$ be the linear operator defined by

$$T(x) := (\frac{x_n}{n})_{n\geq 1},$$

for every $x = (x_n)_{n\geq 1} \in \ell_2$. Prove that $T(\ell_2)$ is dense in ℓ_2 and differs from ℓ_2 (that is, $\overline{T(\ell_2)} = \ell_2$ and $T(\ell_2) \neq \ell_2$).

Pb. 8) Let $f, f_n : [0,1] \to I\!R$ be continuous functions. Assume that for every sequence $(x_n)_{n\geq 1}$ converging to a point $x \in [0,1]$, we have

$$\lim_n f_n(x_n) = f(x).$$

Prove that f_n converges uniformly to f in $[0,1]$.

Pb. 9) Let $f : I\!R \to]0,+\infty[$ be continuous. Prove the uniqueness of the solution to the Cauchy problem

$$x' = f(x), \quad x(t_0) = x_0,$$

for every initial data t_0, x_0.

Pb. 10) Let $f : [0,1] \times I\!R^n \to I\!R^n$ be of class C^1. Let x be the solution to the Cauchy problem

$$x'(t) = f(t, x(t)), \quad x(0) = 0,$$

on $[0,\beta[\subset [0,1]$. Prove that if x cannot be extended at β as a solution, then for every compact subset $K \subset I\!R^n$ there exists $t_K < \beta$ such that $x(t) \notin K$ for $t_K < t < \beta$.