S.I.S.S.A.

Sector of Functional Analysis and Applications

Entrance Examination – October 7, 1998

Solve at most five of the following problems.

1. Let $f\colon \mathbf{R} \to [0,1]$ be a nondecreasing continuous function and let

$$A := \{x \in \mathbf{R} : \text{ there exists } y > x \text{ such that } f(y) - f(x) > y - x\}.$$

(a) Prove that, if $]a,b[$ is a bounded open interval contained in A and $a, b \notin A$, then $f(b) - f(a) = b - a$.
(b) Prove that A contains no half line.
(c) Prove that the Lebesgue measure of A is less than or equal to one.

2. Let $a, b \in \mathbf{R}$ with $a < b$. For any $x \in]a,b]$ let $P(x)$ the set of all finite families $(x_i) = (x_0, x_1, \ldots, x_k)$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = x$ (the integer k depends on the family (x_i)). Let $u : [a,b] \to \mathbf{R}$ be a function such that

$$g(x) := \sup_{(x_i) \in P(x)} \sum_{i=1}^{k} |u(x_i) - u(x_{i-1})|^2 < +\infty$$

for every $x \in]a,b]$.
(a) Prove that, for any $x \in]a,b]$, the limit

$$\lim_{y \to x-} u(y)$$

exists and is finite.
(b) Prove that the set of points of $[a,b]$ at which u is discontinuous is at most countable.
(c) Prove that

$$\limsup_{y \to x} \frac{|u(y) - u(x)|^2}{|y - x|} < +\infty$$

for almost every $x \in [a,b]$.

3. Let $r : [0,1] \to \mathbf{R}$ be a continuous function and let u_λ be the unique solution of the following Cauchy problem:

$$\begin{cases} u''(t) + \lambda r(t) u(t) = 0, \quad \text{for } t \in [0,1], \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

It is well known that (u_{λ_n}) converges uniformly to u_λ on $[0,1]$ whenever $\lambda_n \to \lambda$. Define the map $\tau : \mathbf{R} \to \mathbf{R}$ by setting

$$\tau(\lambda) := \inf\{t \in]0,1] : u_\lambda(t) = 0\},$$

with the convention $\tau(\lambda) = 1$ if $u_\lambda(t) \neq 0$ for every $t \in]0,1]$. Prove that τ is continuous.

4. Let Ω be an open set of $\mathbf{R}^N$, let $(u_n)_{n\geq 1}$ be a sequence in $L^1(\Omega)$, and let $u \in L^1(\Omega)$. Suppose that (u_n) converges to u weakly in $L^1(\Omega)$ and that $|u_n| \leq |u|$ almost everywhere in Ω. Prove that (u_n) converges strongly to u in $L^1(\Omega)$.

5. Let X be an infinite dimensional separable Hilbert space with norm $\|\cdot\|$, and let $(e_n)_{n\geq 1}$ be a complete orthonormal system in X. For any $n \geq 1$, let X_n be the linear subspace generated by $\{e_1, \ldots, e_n\}$ and let $Y_n = X_n^\perp$ be orthogonal complement of X_n. Let $A : X \to X$ be a bounded linear operator. Prove that A is compact if and only if

$$\lim_{n\to\infty} \sup_{\substack{x\in Y_n \\ \|x\|=1}} \|Ax\| = 0.$$

6. Let l^2 be the Banach space of the real sequences $x = (x_n)_{n\geq 1}$ such that

$$\|x\|_{l^2}^2 := \sum_{n=1}^{\infty} |x_n|^2 < +\infty.$$

Let $A : l^2 \to l^2$ the bounded linear operator defined, for every $x \in l^2$, by $Ax = y$, where $y = (y_n)_{n\geq 1}$ with

$$\begin{cases} y_1 = 0, \\ y_n = \dfrac{x_{n-1}}{n-1}, \qquad n > 1. \end{cases}$$

Prove that A is compact and find the spectrum of A.

7. Let $\mathcal{E}$ be the set of all functions $u \in C^1([0,2])$ such that $u(x) \geq 0$ for every $x \in [0,2]$ and $|u'(x) + u(x)^2| < 1$ for every $x \in [0,2]$. Prove that the set $\mathcal{F} := \left\{u|_{[1,2]} : u \in \mathcal{E}\right\}$ is an equicontinuous subset of $C^0([1,2])$.

8. Let l^1 be the Banach space of the real sequences $x = (x_n)_{n\geq 1}$ such that

$$\|x\|_{l^1} := \sum_{n=1}^{\infty} |x_n| < +\infty.$$

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers with $a_n > 0$ for any n. Let X be the normed linear space of all sequences $x \in l^1$ such that

$$\|x\|_X := \sup_{n\geq 1} \frac{|x_n|}{a_n} < +\infty.$$

(a) Prove that the immersion of $(X, \|\cdot\|_X)$ into $(l^1, \|\cdot\|_{l^1})$ is continuous if and only if

$$(*) \qquad \sum_{n=1}^{\infty} a_n < +\infty.$$

(b) Prove that, if $(*)$ holds, then the immersion of $(X, \|\cdot\|_X)$ into $(l^1, \|\cdot\|_{l^1})$ is compact.

9. Let Ω be a bounded open subset of $\mathbf{R}^N$, let $p > 1$, and let $(u_n)_{n\geq 1}$ be a sequence in $L^p(\Omega)$. Suppose that (u_n) converges to zero pointwise almost everywhere in Ω and that there exists a constant $M \in \mathbf{R}$ such that $\|u_n\|_{L^p(\Omega)} \leq M$ for every $n \geq 1$. Prove that (u_n) converges to zero in $L^r(\Omega)$ for every $r \in [1, p[$.

10. Let $u_n : [0, 1] \to \mathbf{R}$ be the sequence of functions defined by

$$u_n(x) := \operatorname{sign}(\sin(2^n \pi x)), \qquad n = 0, 1, 2, \ldots$$

(a) Prove that $(u_n)_{n\geq 0}$ is an orthonormal system in $L^2([0, 1])$.
(b) Is $(u_n)_{n\geq 0}$ a complete orthonormal system in $L^2([0, 1])$? Justify the answer.