**S.I.S.S.A. - Sector of Functional Analysis and Application**

*Entrance Examination - October 11, 2000*

Solve at most five of the following problems.

1. Prove that the sequences of functions

$$f_n(x) = \sin(n^2 x^2) \qquad \text{and} \qquad g_n(x) = \sin(\sqrt{n}x)$$

converge weakly to zero in $L^2([0,1])$.

2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a convex function of class $C^1$. Let us suppose that there exists the limit

$$L := \lim_{x\to+\infty} \frac{f(x)}{x^2},$$

and that $0 < L < +\infty$.

(a) Prove that $0 < \liminf_{x\to+\infty} \frac{f'(x)}{x} \le \limsup_{x\to+\infty} \frac{f'(x)}{x} < +\infty$.

(b) Prove that there exists the limit $\lim_{x\to+\infty} \frac{f'(x)}{x}$, and compute it.

(Hint: use the inequality $f(y) \ge f(x) + f'(x)(y-x)$).

3. Let $(f_n)$ be a bounded sequence in $L^\infty([0,2\pi],\mathbf{C})$. For every $n \in \mathbf{N}$ and for every $k \in \mathbf{Z}$ let $a_k(f_n)$ be the Fourier coefficient defined by

$$a_k(f_n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(x)\, e^{-ikx}\, dx\,.$$

Let us suppose that $\lim_{n\to+\infty} a_k(f_n) = 0$ for every $k \in \mathbf{Z}$.

(a) Prove that $(f_n)$ converges to zero weakly in $L^p([0,2\pi],\mathbf{C})$ for all $p \in [1,+\infty[$.

(b) Find an example of a sequence $(f_n)$ satisfying all the hypotheses but not converging to zero strongly in $L^1([0,2\pi],\mathbf{C})$.

4. Let $K \in C([0,2])$ be positive, decreasing and such that $K(0) = 1$. Prove that for every $h \in C([0,1])$ there exists a unique solution $u \in C([0,1])$ to the equation

$$u(x) = h(x) + \int_0^1 K(x+y)\, u(y)\, dy \qquad \forall x \in [0,1]\,.$$

5. Let $\Omega$ be an open subset of $\mathbf{R}^n$ and let $u \in C^2(\Omega)$ be a function such that $|Du(x)| = 1$ for all $x \in \Omega$, where $Du$ is the gradient of $u$ and $|\cdot|$ is the Euclidean norm. Prove that every solution $x$ of the system $x'(t) = Du(x(t))$ is an affine function. (Hint: write the equation for $x''(t)$).

6. Let $Q\colon C^\infty(\mathbf{R}) \to \mathbf{R}$ be a linear function. Let us suppose that $Q(f) \geq 0$ for every $f \in C^\infty(\mathbf{R})$, with $f(0) = 0$, such that the set $\{x \in \mathbf{R} : f(x) \geq 0\}$ is a neighborhood of 0. Prove that there exist three constants $a$, $b$, $c$ in $\mathbf{R}$ such that

$$Q(f) = a\, f''(0) + b\, f'(0) + c\, f(0) \qquad \forall f \in C^\infty(\mathbf{R})\,.$$

(Hint: consider that, if the formula holds, then $a = Q(g)$, $b = Q(h)$ and $c = Q(1)$, where $g(x) = x^2/2$ and $h(x) = x$ for every $x \in \mathbf{R}$).

7. Given $b > 0$, find the constant $a \in \mathbf{R}$ such that the solution of the Cauchy problem

$$\begin{cases} x''(t) + a\, x'(t) + b\, x(t) = 0 \quad \text{for } t \in \mathbf{R}, \\ x(0) = 1\,, \\ x'(0) = 0\,, \end{cases}$$

tends to zero as faster as possible for $t \to +\infty$.

8. Let $\alpha$ be the supremum of the set of numbers $T > 0$ such that the Cauchy problem

$$\begin{cases} x'(t) = [x(t)]^2 + t^2\,, \\ x(0) = 1\,, \end{cases}$$

has a solution defined in $[0,T]$. Prove that $\frac{\pi}{4} \leq \alpha \leq 1$.

9. Let $X$ be a separable Banach space with dual $X^*$, let $B = \{x \in X : \|x\| \leq 1\}$ and let $(x_n)_{n\in\mathbf{N}}$ be a sequence dense in $B$. Setting $B^* = \{T \in X^* : \|T\|_{X^*} \leq 1\}$, let $d\colon B^* \times B^* \to \mathbf{R}$ be the distance defined by:

$$d(S,T) = \sum_{n=1}^{+\infty} 2^{-n}|S(x_n) - T(x_n)| \ , \quad S,T \in B^* \ .$$

Prove that:

(a) $d(T_n, T) \to 0$ if and only if $T_n \to T$ pointwise in $B$;

(b) $(B^*, d)$ is a metric and compact space.

10. Let $K$ be a compact subset of $\mathbf{R}^N$ and let $f_n\colon K \to \mathbf{R}$ be a sequence of continuous functions converging pointwise in $K$ to a continuous function $f\colon K \to \mathbf{R}$. Let us suppose that there exists a constant $k \geq 0$ such that

$$|f_m(x) - f(x)| \leq k|f_n(x) - f(x)| + \frac{1}{n}$$

for every $x \in K$ and for every $m, n \in \mathbf{N}$ with $m \geq n$. Prove that $f_n \to f$ uniformly in $K$.