S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance Examination - October 11, 2001

Solve at most five of the following problems.

1. Consider the differential equation

$$y'' + q(t)y = 0,$$

where q is a continuous function and $q(t) \leq 0$ for every $t \in \mathbf{R}$. Prove that any nonconstant solution y satisfying $y(0) = 0$ is strictly monotone.

2. Let $F : \mathbf{R}^2 \to \mathbf{R}$ be a strictly convex nonnegative function of class C^2 such that $F(0,0) = 0$.

(a) Prove that the equation $F(x,y) = F(0,1)$ defines implicitly a function $y = \phi(x)$ of class C^2 for x belonging to a neighborhood of zero and satisfying $\phi(0) = 1$;

(b) Prove that $\phi''(0) \leq 0$.

Note: F strictly convex means that

$$F(\lambda P + (1-\lambda)Q) < \lambda F(P) + (1-\lambda)F(Q)$$

for every $\lambda \in]0,1[$ and for every $P, Q \in \mathbf{R}^2$ $P \neq Q$.

3. Let $f, g, h : [0,1] \to \mathbf{R}$ be nonnegative integrable functions. Prove that the following conditions are equivalent:

(a) $(f(x))^2 \leq g(x)h(x)$ for a.e. $x \in [0,1]$;

(b) $(\int_E f(x)dx)^2 \leq \int_E g(x)dx \int_E h(x)dx$ for every measurable set $E \subseteq [0,1]$.

4. Given $f : [0,1] \to \mathbf{R}$, define, for $n = 1, 2, 3, \ldots$

$$\sigma_n(f) := \frac{1}{n} \sum_{k=1}^{n} f(\frac{k}{n}).$$

Prove that for every $f \in C^1([0,1])$ there exist two constants $a_0 = a_0(f)$ and $a_1(f)$ such that

$$\sigma_n(f) = a_0 + \frac{\sigma_n(f')}{2n} + o\left(\frac{1}{n}\right) = a_0 + \frac{a_1}{n} + o\left(\frac{1}{n}\right)$$

Find an expression for a_0 and a_1 in terms of f.

5. Let $a, b \in \mathbf{R}$ with $a < b$. Suppose that there exists a solution y of the differential equation

$$y'' + e^t y = 0$$

such that $y(a) = y(b) = 0$ and $y(t) < 0$ for every $t \in]a, b[$. Prove that

$$\inf_{a<t<b} \frac{y(t)}{(t-a)(t-b)} > 0.$$

6. Let $f, g \in C(\mathbf{R}, \mathbf{R})$ and suppose that $g(x+1) = g(x)$ for every $x \in \mathbf{R}$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x)g(nx)dx = \left(\int_0^1 f(x)dx\right)\left(\int_0^1 g(x)dx\right).$$

7. Let $1 \leq p < \infty$.

(a) Let $g : [0,1] \to \mathbf{R}$ be a measurable function such that fg belongs to $L^p([0,1])$ for every $f \in L^p([0,1])$. Prove that g belongs to $L^\infty([0,1])$.

(b) Let $1 \leq p < \infty$. For any $a \in L^\infty([0,1])$ consider the linear continuous operator $T_a : L^p([0,1]) \to L^p([0,1])$ given by $T_a(f) = af$. Let $T : L^p([0,1]) \to L^p([0,1])$ be a linear and continuous operator such that $TT_a = T_aT$ for every $a \in L^\infty([0,1])$. Prove that there exists $h \in L^\infty([0,1])$ such that $T = T_h$.

8. Let $f \in C^1(\mathbf{R}^2, \mathbf{R})$ satisfying

$$\frac{\partial f}{\partial x}(t,x) \leq 0 \quad \forall (t,x) \in \mathbf{R}^2, \quad f(t,0) = 0 \quad \forall t \in \mathbf{R}.$$

Consider the Cauchy problem

$$\begin{cases} y' = f(t,y) \\ y(0) = x. \end{cases}$$

(a) Prove that, for any $x \in \mathbf{R}$, the solution y_x of the Cauchy problem is well defined on the whole half line $[0, +\infty[$.

(b) Prove that the set of maps $\{y_x(\cdot), x \in [0,1]\}$ is compact in $C([0, +\infty[)$.

9. Consider the differential equation

$$y'' = \frac{y'}{2\sqrt{y}} \quad (y > 0).$$

(a) Prove that any nonconstant solution is strictly monotone.

(b) Consider the Cauchy problem

$$\begin{cases} y'' = \frac{y'}{2\sqrt{y}} \\ y(0) = u \\ y'(0) = v. \end{cases}$$

Find the set of points $(u, v) \in \mathbf{R}^+ \times \mathbf{R}$ such that the maximal solution of the Cauchy problem is nonconstant and defined on the whole real line $\mathbf{R}$. For such points (u, v) compute the limits of the solution for $t \to \pm\infty$.

10. For every $r > 0$ consider the function $f^r : \mathbf{R} \to \mathbf{R}$ defined by

$$f^r(x) = \begin{cases} \sqrt{r^2 - x^2}, & \text{for} |x| \le r, \\ 0, & \text{for} |x| > r. \end{cases}$$

(a) Find all $p \ge 1$ such that the map $r \to f^r$ from $]0, +\infty[$ to $L^p(\mathbf{R})$ is continuous.

(b) Find all $p \ge 1$ such that the map $r \to f^r$ from $]0, +\infty[$ to $L^p(\mathbf{R})$ is differentiable.