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Sector of Functional Analysis and Applications

Entrance examination for the curricula for Mathematical Analysis and Applied Mathematics
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Solve at most five of the following problems.

1. Let $E \subseteq \mathbb{R}^N$ be a measurable set with $\mu(E) < +\infty$. Given $p \in [1, +\infty[$ let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^p(E)$ such that $f_n(x) \to f(x)$ for almost every $x \in E$.

(i) Prove that $f \in L^p(E)$.

(ii) If $1 < p < +\infty$, prove that $f_n \to f$ in $L^q(E)$ for any $q \in [1, p[$.

(*Hint*: For *(ii)* recall that $f_n \to f$ in measure.)

2. Let $\mathcal{H}$ be a Hilbert space and D a fundamental system in $\mathcal{H}$. Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ such that

$$(u_n, e) \to (u, e), \quad \forall e \in D.$$

Prove that $u_n \rightharpoonup u$ weakly.

3. Prove that there exist a constant $c \in \mathbb{R}$ and a periodic function $U : \mathbb{R} \to \mathbb{R}$ (non identically zero) such that $u(x,t) = U(x - ct)$ solves the following differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + u^3 = 0.$$

4. Let $A(t) = (a_{ij}(t))$ be a real symmetric matrix $N \times N$ with continuous coefficients a_{ij}. Suppose that, for any $t \in \mathbb{R}$, all the eigenvalues of $A(t)$ are less than -1. Defining u the solution of the Cauchy problem

$$\frac{du(t)}{dt} = A(t)u(t), \quad u(0) = u_0 \in \mathbb{R}^N,$$

prove that $\lim_{t\to+\infty} |u(t)|^2 = 0$.

5. Let $f \in L^2(\mathbb{R}^N)$ with $f > 0$ almost everywhere. Let define $F : \mathbb{R}^N \to [0, +\infty)$ by setting

$$F(x) := \int_{B(x,1)} f(y)dy.$$

Prove that:

(i) F is continuous;

(ii) $F(x) \to 0$ as $|x| \to +\infty$;

(iii) F admits at least one absolute maximum point but does not attains absolute minimum on $\mathbb{R}^N$.

6. Let $f \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ be such that $|f(y) - f(x)| \geqslant |y - x|$, $\forall x, y \in \mathbb{R}^N$.

(i) Prove that $f(\mathbb{R}^N)$ is closed.

(ii) Prove that the Jacobian matrix $Df(x)$ is invertible $\forall x \in \mathbb{R}^N$ and deduce that $f(\mathbb{R}^N)$ is open.

(iii) Prove that $f : \mathbb{R}^N \to \mathbb{R}^N$ is injective and surjective.

7. Let $A := \{(x, y) \in \mathbb{R}^2 \,|\, x > 0, y > 0\}$. Prove that any solution $u \in C^1(A, \mathbb{R})$ of the differential equation

$$x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$$

has the form $u(x, y) = \varphi(xy)$. Say if this is true even for $A = \mathbb{R}^2$.

8. Let $\varphi \in C^1_c(\mathbb{R})$. Prove that

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \frac{\sin(nx)}{x}\varphi(x)dx = \pi\varphi(0).$$

9. Find all the solutions of the Cauchy problem:

$$\frac{dx}{dt} = \sin(t)x^{1/3}, \quad x(0) = 0.$$

10. Prove that the solution of the Cauchy problem

$$\frac{dx}{dt} = t - x^2, \quad x(1) = 1$$

is defined in $[1, +\infty)$, $\lim_{t\to+\infty} x(t) = +\infty$ and moreover $\lim_{t\to+\infty} x(t) - t^{1/2} = 0$.