S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance Examination for the curricula of Mathematical Analysis and Applied Mathematics

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Solve at most five of the following problems.

1. Let $f \in C(\mathbf{R}^2, \mathbf{R})$ be such that $|f(t,y)| < 1$ for any $(t,y) \in \mathbf{R}^2$. Consider the solution y of the Cauchy problem

$$\begin{cases} y''(t) + 3y'(t) + 2y(t) = f(t, y(t)), \\ y(0) = y'(0) = 0. \end{cases}$$

Prove that $|y(t)| < 1$ for any $t \geq 0$.
(*Hint:* Use the methods of variations of constants.)

2. Let $f \in C^1(\mathbf{R}, \mathbf{R})$ be such that $f'(t) > 0$ for any $t \in \mathbf{R}$.
(i) Prove that the image $f(\mathbf{R})$ is an open interval (possibly unbounded).
(ii) Assuming that

$$f'(t) \geq \frac{1}{1 + f(t)^2}, \qquad \text{for any} \quad t \in \mathbf{R}$$

prove that $f(\mathbf{R}) = \mathbf{R}$.

3. Given $E \subseteq [0,1]$, let χ_E be the indicatrix function of E, defined by

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases}$$

Let $(E_n)_{n \geq 0}$ be a sequence of measurable subsets of $[0,1]$ and E a measurable subset of $[0,1]$ such that $\chi_{E_n} \rightharpoonup \chi_E$ weakly in $L^2([0,1])$.
Prove that $\chi_{E_n} \to \chi_E$ strongly in $L^2([0,1])$.

4. For any $f \in C([0,1], \mathbf{R})$ set

$$Tf(x) = \int_0^1 [\min\{x, y\} \cdot f(y)] dy.$$

Prove that T is a compact operator from $C([0,1], \mathbf{R})$ into itself and find its spectrum.

5. Let (X, Σ, μ) be a σ-finite measure space and let $f : X \to \mathbf{R}^+$ be a measurable function such that

$$\mu(\{x \in X : f(x) > t\}) > \frac{1}{1+t}.$$

Prove that f is not integrable.

6. Let $a = (a_n)_{n\in\mathbf{N}}$ be a sequence of real positive numbers such that $a_n \to +\infty$ as $n \to \infty$. Consider the Hilbert space

$$\ell^2_a = \left\{ u = (u_n)_{n\in\mathbf{N}} \quad \text{sequence in } \mathbf{C} \text{ such that } \quad \sum_{n=1}^{\infty} a_n |u_n|^2 < +\infty \right\},$$

endowed with the scalar product

$$(u, v) = \sum_{n=1}^{\infty} a_n u_n \overline{v}_n.$$

Prove that the closed unit ball in the Hilbert space

$$\ell^2 = \left\{ u = (u_n)_{n\in\mathbf{N}} \quad \text{sequence in } \mathbf{C} \text{ such that } \quad \sum_{n=1}^{\infty} |u_n|^2 < +\infty \right\},$$

endowed with the usual scalar product, is a compact subset of ℓ^2_a.

7. Let $K = \{x : [0, T] \to \mathbf{R} \text{ such that } x'(t) = x^2(t) \text{ and } 0 \le x(T) \le 1\}$.
Prove that K is a compact subset of $C([0, T], \mathbf{R})$.

8. Let $f \in C^2(\mathbf{R}^2, \mathbf{R})$ be a function with first and second derivatives globally bounded and such that $f(x, 0) = f(0, y) = 0$ for all $(x, y) \in \mathbf{R}^2$.
Prove that there exists $C > 0$ such that $|f(x, y)| \le C|xy|$ for any $(x, y) \in \mathbf{R}^2$.

9. Let $f \in C^1(\mathbf{R}, \mathbf{R})$. Consider a not identically zero solution y of the equation

$$y''(t) = f(y(t)), \qquad t \in [0, 1].$$

Prove that y has at most a finite number of zeroes.

10. Consider the system of ordinary differential equations

$$\begin{cases} x' = -x^2 - xy + x \\ y' = -y^2 - xy + y, \end{cases}$$

with initial conditions $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$.
Prove that the solution $(x(t), y(t))$ is defined for any $t \ge 0$ and that $x(t) > 0$ and $y(t) > 0$ per for any $t \ge 0$.