S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

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Solve at most five of the following problems.

1. Let $f : \mathbf{R}^3 \to \mathbf{R}$ be continuos. Consider the system of differential equations

$$x''(t) = f(x(t))x(t),$$

where $x(\cdot)$ takes values in $\mathbf{R}^3$. Prove that the trajectory of any solution $x(\cdot)$ is contained in the subspace of $\mathbf{R}^3$ generated by the vectors $x(0)$ and $x'(0)$.

2. Given $a \in \mathbf{R}$, consider the 2×2 system of ordinary differential equations

$$\begin{cases} x'(t) = x(t) + ay(t) \\ y'(t) = ay(t). \end{cases}$$

Say for which values of a there exists a solution $(x(\cdot), y(\cdot))$ not identically zero such that $(x(t), y(t)) \to (0,0)$ as $t \to +\infty$.

3. In this problem consider known the completeness in $L^2((-\pi,\pi))$ of the system of functions $\{\cos(kx) : k = 0,1,2,\ldots\} \cup \{\sin(kx) : k = 1,2,\ldots\}$.

(a) Prove that the closed subspace of $L^2((0,\pi))$ generated by $\{\sin(kx) : k = 1,2,\ldots\}$ coincides with $L^2(0,\pi))$.

(b) Prove that the closed subspace of $L^2((-\pi,\pi))$ generated by $\{1\} \cup \{\sin(kx) : k = 1,2,\ldots\}$ does not coincide with $L^2((-\pi,\pi))$.

4. Let $f \in C^1(\mathbf{R},\mathbf{R})$ be a function such that $|f(x) - \cos x| \leq 1$ for every $x \in \mathbf{R}$. Prove that all the solutions of the equation $x'(t) = f(x(t))$ are bounded.

5. Say if the Cauchy problem

$$\begin{cases} x'(t) = \sqrt[3]{(x(t)-1)^2} \\ x(0) = 0. \end{cases}$$

has a unique solution defined on the whole $\mathbf{R}$, and justify the answer.

6. Set

$$B^0 = \{u \in C^0([0,1]) : \|u\|_{C^0} \leq 1\};$$

$$B^1 = \{u \in C^1([0,1]) : \|u\|_{C^1} \leq 1\}.$$

Answer to the following questions and justify the answers.
(a) Is B^0 a closed subset of $L^1([0,1])$?
(b) Does B^0 have empty interior in $L^1([0,1])$?
(c) Is B^1 a relatively compact subset of $L^1([0,1])$?

7. Let $f,g:[0,1]\to\mathbf{R}$ be measurable functions and define the sets $E_k=\{x\in[0,1]:|f(x)|\le k\}$ for $k=1,2,\dots$ Assuming that $fg\in L^1([0,1])$ prove that

$$\lim_{k\to\infty}\int_{E_k} f(x)g(x)dx=\int_0^1 f(x)g(x)dx.$$

8. Prove that, for every $\varphi\in C_c^1(\mathbf{R})$,

$$\lim_{n\to\infty}\int_{\mathbf{R}}\frac{1}{2n}|x|^{\frac{1}{n}-1}\varphi(x)dx=\varphi(0).$$

9. Set

$$(Tf)(x)=\int_0^1\frac{f(tx)}{\sqrt{1-t^2}}dt,\qquad x\in[0,1].$$

Prove that T is a linear and continuous operator from the space $C([0,1])$ into itself and compute its norm.

10. Prove that there exists no function $f:\mathbf{R}^2\to\mathbf{R}$ of class C^1 injective.