S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination - 13 October 2004

Solve at most five of the following problems.

1) Let (f_n) be a sequence bounded in $L^2([0,1])$ and let $f \in L^2([0,1])$. Consider the following functions defined in $[0,1]$:

$$F_n(x) = \int_0^x f_n(t)\,dt, \qquad F(x) = \int_0^x f(t)\,dt.$$

Prove that:

a) for every n the function F_n is Hölder continuous with exponent $1/2$;

b) if $f_n \rightharpoonup f$ weakly in $L^2([0,1])$, then $F_n \to F$ uniformly;

c) if $F_n \to F$ pointwise, then $f_n \rightharpoonup f$ weakly in $L^2([0,1])$.

2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^∞, periodic with period 2π. Prove that for every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that

$$\left| \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx \right| \leq \frac{C_k}{n^k} \quad \text{for every } n \in \mathbb{N}.$$

3) Let U be an open subset of $\mathbb{R}^N$. Let (f_n) be a sequence of functions in $C^\infty(U)$. Assume that for every $k \in \mathbb{N} \cup \{0\}$ there exists a constant $C_k > 0$ such that

$$\sup_{x \in U} |D^k f_n(x)| \leq C_k \quad \text{for every } n.$$

Prove that there exist $f \in C^\infty(U)$ and a subsequence (f_{n_j}) such that for every compact $K \subset U$ and for every $k \in \mathbb{N} \cup \{0\}$ there holds

$$D^k f_{n_j} \to D^k f \quad \text{uniformly in } K.$$

4) Prove that there are no analytic functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2} \quad \text{for every } n \in \mathbb{N}.$$

5) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of class C^1 with the following properties:

- $x \mapsto f(t,x)$ is strictly increasing for every t;
- $f(t+1,x) = f(t,x)$ for every t, x.

Prove that the equation
$$\dot{x}(t) = f(t, x(t))$$
has at most one periodic solution with period 1.

6) Prove that every monotone sequence bounded in $L^2([0,1])$ strongly converges.

7) For every $\alpha \in [0,1]$ consider the following 2×2 first order differential system in the unknown $(x(t), y(t))$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (1-\alpha) \left(\begin{pmatrix} -\frac{1}{10} & -1 \\ 1 & -\frac{1}{10} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ x^2 \end{pmatrix} \right).$$

a) For every $\alpha \in [0,1]$ determine whether the origin is stable, asymptotically stable, or unstable.

b) Prove that for every $\alpha \in [0,1]$ the origin is not globally asymptotically stable.

8) Let $f \in L^2([-1,1])$. For each one of the following properties determine whether or not it implies that $f = 0$ almost everywhere, justifying the answer:

a) $\displaystyle\int_{-1}^{1} x^n f(x)\, dx = 0$ for every $n \in \mathbb{N} \cup \{0\}$;

b) $\displaystyle\int_{-1}^{1} x^{2n} f(x)\, dx = 0$ for every $n \in \mathbb{N} \cup \{0\}$.

9) Prove that, for $\alpha \in [0,1]$, all the solutions of the differential equation
$$\ddot{x}(t) + x(t) - \sin(\alpha x(t)) = 0,$$
are periodic.

10) Find all the solutions of the following system and their domain of existence:
$$\begin{cases} \dot{x}(t) = 2\sqrt{x(t)}, & t \geq 0, \\ \dot{y}(t) = x(t)(y(t)+1)^2, & t \geq 0, \\ x(0) = 0, \ y(0) = 0. \end{cases}$$