S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination - October 6 2005

The candidate should solve at most five of the following exercises

(1) Consider the ordinary differential equation in polar coordinates

$$\dot{r} = \begin{cases} 0 & r = 0 \\ \sqrt{r}\sin(1/r) & r \neq 0 \end{cases} \quad (1)$$

$$\dot{\theta} = f(r), \quad (2)$$

with $f \in C^1$ and $0 < \alpha \leq f \leq \beta$.

(a) Study the local and global Lipschitzianity of the right hand side of (1), (2).

(b) For initial data $\theta_0 \in [0, 2\pi]$, $r_0 \in [0, +\infty)$, study the local and global existence and uniqueness of the solution.

(2) Consider the differential equation in $\mathbb{R}$

$$\dot{x} = x - e^{-t^2}.$$

Say if there are solutions $x(t)$ such that $x(t) \to 0$ for $t \to \pm\infty$. If the answer is positive, find how many solutions have this property.

(3) Consider the Cauchy problem in $\mathbb{R}$

$$\dot{x} = x^2(\alpha + \sin(x)), \quad x(0) = 1.$$

For every $\alpha \in \mathbb{R}$ give an estimate of the maximal interval of definition of the solution.

(4) Consider the linear system of partial differential equations on the torus $\mathbb{T}^2$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \sin(y) - \cos(x) \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

- Solve the system explicitly, assuming that

$$\int_0^{2\pi}\int_0^{2\pi} u(x,y)dxdy = \int_0^{2\pi}\int_0^{2\pi} v(x,y)dxdy = 0.$$

- Prove that if f is smooth, periodic and with zero average, the solution to the following system on $\mathbb{T}^2$
$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} = 0 \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = f(x,y)$$
satisfies
$$\int_0^{2\pi}\int_0^{2\pi}\Big(u(x,y)u'(x,y) + v(x,y)v'(x,y)\Big)dxdy = 0.$$

(5) In the space $L^\infty((0,1),\mathbb{R})$ consider the set

$$B = \{u \in L^\infty((0,1),\mathbb{R}) : 0 \le u(x) \le 1 \text{ almost everywhere}\}.$$

Let E be the set made of characteristic functions of unions of open intervals of $(0,1)$ with rational end points. Prove that the closure of the convex envelope of E in the weak* topology coincides with B.

(6) Consider the map $f : S^1 \to S^1$ defined as

$$f(x) = (3x + 2\sin x)\,/_{2\pi\mathbb{Z}}, \qquad x \in S^1.$$

Prove that for any open (non-empty) set $\mathcal{A} \subseteq S^1$ there exists $k \in \mathbb{N}$ such that $\underbrace{f \circ f \circ \ldots \circ f}_{k \text{ times}}(\mathcal{A}) = S^1$.

(7) Let $\mathcal{S} = \{(v_1,v_2,v_3) \in (0,+\infty)^3 \ : \ v_1v_2v_3 = 1\}$. Given a positive parameter τ, consider the function $f_\tau : \mathcal{S} \to \mathbb{R}$ defined by

$$f_\tau(v_1,v_2,v_3) = \left(v_1^2 + v_2^2 + v_3^2 - 3\right) - \tau\left(v_1 + v_2 + v_3\right).$$

Find the number of critical points of f_τ for τ varying in $(0,+\infty)$. They represent equilibrium configurations of a cube of incompressible neo-hookean rubber under hydrostatic traction

(8) Consider the linear operator $T : L^2([0,1]) \to L^2([0,1])$ given by the formula

$$(Tf)(x) = \int_0^x f(y)\,dy \quad \forall f \in L^2([0,1]).$$

(a) Prove that the adjoint operator T^* is given by

$$(T^* f)(x) = \int_x^1 f(y)\, dy \quad \forall f \in L^2([0,1]).$$

(b) Prove that

$$(T\,T^* f)(x) = \int_0^1 \min\{x, y\} f(y)\, dy \quad \forall f \in L^2([0,1]).$$

(c) Compute the spectral radius of $T\,T^*$ and the norm of T.

(9) (a) Let X be a Banach space and let $A \subset X$ be bounded. Prove that A is pre-compact if and only if for every $\epsilon > 0$ there exists a subspace F_ϵ of X of finite dimension such that

$$\text{dist}\,(x, F_\epsilon) \leq \epsilon \quad \forall x \in A.$$

(b) Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ and consider the linear operator $T : \ell^2 \to \ell^2$ given by

$$(T\,u)_n \doteq \lambda_n u_n \quad \forall u = (u_n)_{n\in\mathbb{N}} \in \ell^2.$$

Using if necessary point (a) prove that T is compact if and only if

$$\lim_{n\to\infty} \lambda_n = 0.$$

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that $f'(x) \geq 0$ for every $x \in \mathbb{R}$ and $f(0) = 0$. Prove that the solution $x(\cdot)$ of the Cauchy problem

$$\begin{cases} \dot{x} = \dfrac{1}{1 + t f(x)} \\ x(0) = 0, \end{cases}$$

is defined on the whole $\mathbb{R}$ and that

$$\lim_{t\to-\infty} x(t) = -\infty, \qquad \lim_{t\to+\infty} x(t) = +\infty.$$