S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

October 5, 2006

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Let (f_n) be a sequence of functions in $L^\infty(0,1)$ such that

$$\liminf_{n\to\infty} f_n(x) = 0 \quad \text{for a.e. } x \in (0,1).$$

Suppose that f_n converges to 0 weakly* in $L^\infty(0,1)$ (considered as the dual of $L^1(0,1)$).

(a) Prove that $f_n \wedge 0$ converges to 0 strongly in $L^1(0,1)$.

(b) Prove that f_n converges to 0 strongly in $L^1(0,1)$.

2. Let X be a Banach space and let $(B_i)_i \subseteq X$ be a sequence of closed balls such that $B_{i+1} \subseteq B_i$ for every i. Prove that $\cap_i B_i$ is either a point of X or a closed ball of X.

3. Consider the 2×2 system of ordinary differential equations

$$\begin{cases} x'(t) = \alpha\, y(t)^2, \\ y'(t) = x(t)^2. \end{cases}$$

In the cases $\alpha = 0$ and $\alpha = 1$ say if it is true or false that for every initial condition the system has a solution $(x(\cdot), y(\cdot))$ defined in all $\mathbb{R}$ and justify the answer.

4. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function and let T be the operator defined by $Tf = \varphi f$. Prove that T is continuous from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ if and only if $\varphi \in L^\infty(\mathbb{R})$.

5. Let $u \in L^\infty([0,1])$. Prove that

$$\lim_{t\to+\infty} \frac{1}{t} \log \int_0^1 e^{tu(x)}\, dx = \text{ess sup } u.$$

6. Let ℓ^2 be the Hilbert space of all sequences of real numbers $x = (x_n)$ such that

$$\|x\|_2^2 := \sum_{n=1}^{\infty} |x_n|^2 < +\infty.$$

Given a sequence of positive real numbers $a = (a_n)$ such that $a_n \to +\infty$, prove that the set

$$E := \Big\{ x = (x_n) \in \ell^2 : \sum_{n=1}^{\infty} a_n x_n^2 \le 1 \Big\}$$

is compact in ℓ^2.

7. Let $f : (-1,1) \to \mathbb{R}$ be a differentiable function with the following property:

$$f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)} \tag{1}$$

for every $x, y \in (-1,1)$ such that $|x+y| < 1$.

(a) Determine the value $f(0)$.

(b) Find all differentiable functions $f : (-1,1) \to \mathbb{R}$ satisfying property (1). (Hint: first deduce a differential condition).

8. Prove that, if u_n converges to u strongly in $L^1(\mathbb{R})$, then

$$\frac{u_n}{1+u_n^2} \longrightarrow \frac{u}{1+u^2} \quad \text{strongly in } L^1(\mathbb{R}).$$

9. Consider the operator $T : C([0,1]) \to C([0,1])$ defined by

$$Tf(x) = \int_0^{1-x} f(t)\,dt.$$

Prove that T is compact and compute its spectrum.

10. Study the local and global existence and uniqueness for the solution of the Cauchy problem

$$\begin{cases} x'(t) = f(x(t)), \\ x(0) = 0, \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 - \sqrt{x} & \text{for } x \ge 0, \\ 1 - x^2 & \text{for } x < 0. \end{cases}$$