S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

October 9, 2007

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Let X be a separable reflexive real Banach space and let (φ_n) be a dense sequence in

$$\{\varphi \in X' : \ \|\varphi\| \leq 1\} \ .$$

Consider in X the scalar product $(\ |\ )_0$ defined by

$$(x|y)_0 = \sum_{n=1}^{\infty} 2^{-n} \langle \varphi_n, x\rangle \langle \varphi_n, y\rangle \ .$$

Prove that:

(a) if $\|\ \|_0$ is the norm induced by such scalar product, then $\|x\|_0 \leq \|x\|$ for all $x \in X$;

(b) every bounded sequence in X admits a Cauchy subsequence with respect to the norm $\|\ \|_0$.

2. Consider the differential system

$$\begin{cases} \dot{x} = -xy \\ \dot{y} = x^2 \end{cases} , \quad (x,y) \in \mathbb{R}^2 \, .$$

Show that every solution is defined in $\mathbb{R}$ and admits a limit as $t \to +\infty$ and $t \to -\infty$.

3. Let (f_n) be a sequence of functions converging to f in $L^1([0,1])$. Assume, in addition, that the sequence is equibounded; that is, there exists a constant $M > 0$ such that $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and almost every $x \in [0,1]$. Prove that $f_n g \to fg$ in $L^1([0,1])$ for every $g \in L^1([0,1])$. Show with an example that the above conclusion may fail if the sequence (f_n) is not equibounded, even if each function f_n is assumed to be bounded.

4. Consider the Cauchy problem

$$\begin{cases} \dot{x} = x(t)^2 + t \\ x(0) = 0 \end{cases}$$

Show that its solution is not defined in $[0,3]$.

5. Let $(a_{i,j})$ be a $n \times n$ matrix such that for each $j \in \{1, ..., n\}$ the following holds:

- $a_{j,j} > 1$
- there exists $k \in \{1, ..., n\}$, $k \neq j$ such that $a_{k,j} = 1$ and $a_{i,j} = 0$ for every $i \notin \{k, j\}$.

Show that $(a_{i,j})$ is nonsingular.

6. Consider the Cauchy problem

$$\begin{cases} x'' + x + (x^2 + x'^2 - 1)x' = 0 \\ x(0) = a \\ x'(0) = b \end{cases} \tag{1}$$

Prove that:

(a) the solutions of (1) are defined for all $t \geq 0$;

(b) if $a^2 + b^2 > 0$, then $\lim_{t \to +\infty} (x^2(t) + x'^2(t)) = 1$.

Hint: consider $\rho = x^2 + x'^2$.

7. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally integrable function such that

$$f(x + y) = f(x) + f(y) \quad \text{for all } x \text{ e } y \in \mathbb{R}^N.$$

Prove that:

(a) f is of class C^∞;

(b) f is linear.

Hint: for (a) consider the convolution of f with a nonnegative C^∞ function with compact support.

8. Let (f_n) be a sequence of functions converging in $L^1(\mathbb{R})$ and let (E_n) be a sequence of Lebesgue-measurable sets such that $|E_n| \to 0$. Prove that

$$\int_{E_n} f_n \, dx \to 0 \, .$$

9. Let $f : [0, +\infty) \to [0, +\infty]$ be a measurable function such that

$$\int_0^{+\infty} f(x) x \, dx < +\infty \, .$$

Prove that

$$\sum_{n=1}^{\infty} \int_0^{+\infty} f(x+n)\, dx < +\infty\,.$$

10. Let $u \in C^\infty(\mathbb{R}^2)$ be a solution of

$$u_t = u_{xx} - \alpha u\,, \qquad \alpha \in \mathbb{R}\,,$$

such that $x \mapsto u(x,0)$ is periodic, with period 2π, and

$$\int_0^{2\pi} u(x,0)\, dx = 0\,.$$

Determine the values of $\alpha \in \mathbb{R}$ for which the function

$$t \mapsto \int_0^{2\pi} |u(x,t)|^2\, dx$$

is bounded in $[0,+\infty)$ for all choices of the initial condition $u(\cdot,0)$.

Hint: expand u in Fourier series with respect to the x variable.