S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

October 2, 2008

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Let $f\colon (0,+\infty) \to \mathbb{R}$ be a bounded and differentiable function. Prove that

$$\liminf_{t\to+\infty} f'(t) \le 0 \le \limsup_{t\to+\infty} f'(t),$$

and deduce from that the existence of a sequence (t_n) such that $t_n \to +\infty$ and $f'(t_n) \to 0$.

2. Consider the function

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } x \in [0,1), \\ \sqrt{1-(x-2)^2} & \text{if } x \in [1,2), \\ (x-2)^2+1 & \text{if } x \ge 2. \end{cases}$$

For every $x_0 \in [0,+\infty)$ study existence and uniqueness of the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(x(t)), & t \ge 0, \\ x(0) = x_0, \end{cases}$$

and determine the values of x_0 for which there exists at least one solution defined on $[0,+\infty)$.

3. Let $f_n, f \in L^2(0,1)$ be such that f_n converges to f weakly in $L^2(0,1)$. Let $g_n, g\colon (0,1) \to \mathbb{R}$ be measurable functions such that g_n converges to g in measure and $\|g_n\|_{L^\infty} \le M$ for every n. Prove that

(a) $g \in L^\infty(0,1)$;

(b) the sequence $f_n g_n$ converges to fg weakly in $L^2(0,1)$.

(Recall that a sequence g_n converges to g in measure if for every $\varepsilon > 0$ the measure of the set $\{x \in (0,1) : |g_n(x) - g(x)| > \varepsilon\}$ tends to zero, as $n \to \infty$).

4. Let $f\colon [0,+\infty) \to \mathbb{R}$ be continuous. For every $n \geq 1$ let

$$f_n\colon [0,+\infty) \to \mathbb{R} \qquad f_n(x) = f(x^n) \quad \forall x \in [0,+\infty).$$

Prove that the family of continuous functions $\{f_n\colon n \geq 1\}$ is equi-continuous at the point 1 if and only if f is constant.

5. Make the qualitative study of the ODE on the plane

$$\begin{cases} \dot{x} = -y^2 \\ \dot{y} = x^2 \end{cases}$$

and determine how many solutions satisfy $y(0) = x(1) = 0$.

6. Let X be a Banach space. Let $\phi\colon [0,1] \to X$ be a sequentially weakly continuous function, that is,

$$\forall (t_n) \subset [0,1] \quad t_n \to t \quad \Rightarrow \quad \phi(t_n) \rightharpoonup \phi(t) \text{ weakly in } X.$$

Consider the function $f\colon [0,1] \to \mathbb{R}$ defined by $f(t) = \|\phi(t)\|$ for every $t \in [0,1]$. Prove that

(a) f is bounded;

(b) there exists the minimum of f on $[0,1]$.

7. Let $C_c(0,+\infty)$ be the space of continuous functions with compact support in $(0,+\infty)$ and let $p > 1$. For every $f \in C_c(0,+\infty)$ define

$$Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt.$$

(a) Prove that $Tf \in L^p(0,+\infty)$ for every $f \in C_c(0,+\infty)$.

(b) Prove that, if $f \in C_c(0,+\infty)$ and $f \geq 0$, then

$$\int_0^\infty (Tf(x))^p\,dx = \frac{p}{p-1}\int_0^\infty f(x)(Tf(x))^{p-1}\,dx.$$

(Hint: integrate by parts).

(c) Deduce that T is a bounded linear operator from $C_c(0,+\infty)$, endowed with the L^p norm, into $L^p(0,+\infty)$ and that $\|T\| \leq p/(p-1)$.

8. Let $f\colon \mathbb{R}\times(0,1) \to \mathbb{R}$ be a measurable function such that

(a) $f(t,\cdot) \in L^1(0,1)$ for a.e. $t \in \mathbb{R}$;

(b) there exists $C > 0$ such that for every interval $(a,b) \subset (0,1)$ and for every $s \le t$ there holds

$$\left| \int_a^b (f(t,x) - f(s,x))\, dx \right| \le C(b-a)(t-s).$$

Prove that, up to redefining f on a subset of $\mathbb{R}\times(0,1)$ of zero measure, there holds

$$|f(t,x) - f(s,x)| \le C(t-s)$$

for every $s \le t$ and every $x \in (0,1)$.

9. Determine for which values of the parameters $\lambda, \mu \in \mathbb{R}$ there exists a unique continuous function φ such that

$$\varphi(t) = \lambda + \mu^2 \int_0^1 s\,\varphi(s)\, ds + \mu^2 \int_t^1 (t-s)\varphi(s)\, ds$$

for $0 \le t \le 1$.

10. Let $f \in C^2([0,1])$. Prove that

$$\lim_{n\to\infty} n \left(\int_0^1 f(t)\, dt - \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \right) = \frac{f(1)-f(0)}{2}.$$