S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

October 1, 2009

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Prove that $f \in L^2(0,1)$ if and only if $f \in L^1(0,1)$ and there exists an increasing function $g\colon [0,1] \to \mathbb{R}$ such that

$$\Big|\int_a^b f(x)\,dx\Big|^2 \le (g(b)-g(a))(b-a)$$

for every $0 \le a \le b \le 1$.

2. Consider the differential equation in the plane

$$\begin{cases} \dot{x} = \cos(xy)\,x^3, \\ \dot{y} = \cos(xy)\,y^3. \end{cases}$$

For every initial condition $(x(0), y(0)) \in \mathbb{R}^2$ determine whether or not the corresponding solution is defined in the whole of $\mathbb{R}$.

3. Let f be a polynomial of one variable with simple roots and let f' be the derivative of f. Determine the quantity

$$\sum_{i,j} \frac{1}{t_i - s_j},$$

where $\{t_i\}$ is the set of all roots of f and $\{s_j\}$ is the set of all roots of f'.

4. Let X be the set of continuous functions on $\mathbb{R}$, which are periodic of period 2π. Prove that there exists a unique $u \in X$ satisfying the following identity:

$$u(x) = 1 + \tfrac{1}{2}\sin x\,\sin\big(u(x+\tfrac{\pi}{3})\big) \qquad \text{for every } x \in \mathbb{R}.$$

5. For every $r > 0$ consider the operator $T_r\colon L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ defined by

$$T_r f(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy \qquad \text{for every } x \in \mathbb{R}^N,\ f \in L^1(\mathbb{R}^N),$$

where $B(x,r)$ is the ball centered at x with radius r.

a) Compute the norm of the operator T_r.

b) Prove that $T_r f \to f$ in $L^1(\mathbb{R}^N)$ as $r \to 0^+$ for all $f \in L^1(\mathbb{R}^N)$.

6. Find all solutions defined on $[0, +\infty)$ to the differential equation

$$\begin{cases} \dot{x}^2 = 1 - x^2, \\ x(0) = 0, \\ \dot{x}(0) = 1, \end{cases}$$

and study their regularity.

7. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function such that

$$f(x+y) + f(x-y) = 2(f(x) + f(y)) \qquad \text{for every } x, y \in \mathbb{R}.$$

Show that there exists $a \in \mathbb{R}$ such that $f(x) = ax^2$ for every $x \in \mathbb{R}$.
(Hint: assume first that f is of class C^∞)

8. Consider the space $C([0,1])$ endowed with the $\|\cdot\|_\infty$ norm. Let $A\colon C([0,1]) \to C([0,1])$ be the operator defined by

$$Au(t) = \int_0^t e^{t-s} u(s)\, ds \qquad \text{for every } u \in C([0,1]),\ t \in [0,1].$$

a) Prove that A is continuous and compute its norm.

b) Prove that A is injective.

c) Determine the range of A.

9. Let $f_n \in L^1(0,1)$ and $C > 0$ be such that $f_n \geq 0$, $f_n \to 0$ a.e. in $(0,1)$, and

$$\int_0^1 \max\{f_1(x), \dots, f_n(x)\}\, dx \leq C \qquad \text{for every } n.$$

Prove that $f_n \to 0$ in $L^1(0,1)$.

10. Let $f_n \in C([0,+\infty))$ be defined by $f_n(t) = \sin\sqrt{t + 4n^2\pi^2}$ for $n \in \mathbb{N}$, $t \geq 0$.

a) Prove that f_n converges pointwise to $f \in C([0,+\infty))$ and determine f.

b) Study the uniform convergence of the sequence on bounded intervals and on $[0,+\infty)$.

c) Prove that the set $\{f_n : n \geq 1\}$ is equi-continuous. Is it true that this set is also compact in the space $C_b([0,+\infty))$ of bounded continuous functions, endowed with the $\|\cdot\|_\infty$ norm?