S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

October 6, 2010

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Let $f_n, f \in L^1(0,1)$ be such that $f_n \to f$ a.e. in $(0,1)$ and $\|f_n\|_{L^1} \to \|f\|_{L^1}$. Prove that f_n converges to f in $L^1(0,1)$.

2. Compute

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \max\{x_1, \ldots, x_n\}\, dx_1 \ldots dx_n.$$

3. Consider the equation

$$\ddot{x}(t) + a(t) f(x(t)) = 0,$$

where $a, f \in C^0(\mathbb{R})$, $a \geq 1$, $f \geq 0$, and

$$\int_0^{+\infty} f(y)\, dy = +\infty.$$

Let $x(t)$ be any solution and let (t_0, t_1) be its maximal interval of definition. Prove that $x(t)$ is bounded from above, as $t \to t_1^-$.

4. Let $f \in L^1(0,1)$ be such that $f \geq 0$ a.e. in $(0,1)$. Assume that there exists a constant $c \geq 0$ such that

$$\int_0^1 (f(x))^n\, dx = c \quad \text{for every } n \in \mathbb{N},\ n \neq 0.$$

a) Prove that f is the characteristic function of a measurable set.

b) Is the previous conclusion still true if the assumption $f \geq 0$ is removed?

5. Let B_k be a sequence of closed balls in $\mathbb{R}^n$ (with respect to the euclidean metric) such that $B_{k+1} \subset B_k$ for every k. Prove that $\cap_k B_k$ is either a point or a closed ball.

6. Prove that all the solutions to the system

$$\begin{cases} \dot{x} = e^{-y^2} \sin(x^n + y^n), \\ \dot{y} = x^n \sin(x^n + y^n), \end{cases}$$

where n is a fixed natural number, are defined on $[0, +\infty)$.

7. a) Find a function φ such that

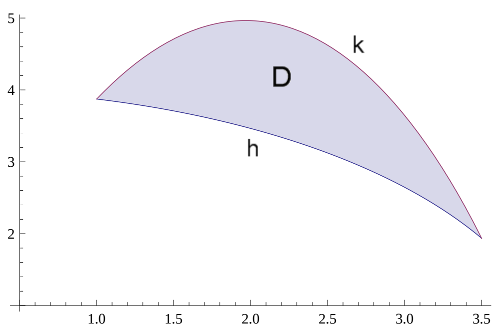
$$\sqrt{1+y^2} - \sqrt{1+x^2} \geq \varphi(x)(y-x)$$

for every $x, y \in \mathbb{R}$.

b) Let $h, k : [a,b] \to \mathbb{R}$ be two functions of class C^1 coinciding at the end-points of $[a,b]$ and such that $h \leq k$ on $[a,b]$. Assume also that the graph of h is an arc of circle of radius R. Prove that

$$L(k) - L(h) \geq \frac{1}{R}|D|,$$

where $L(k)$ and $L(h)$ denote the length of the graphs of k and h, respectively, and $|D|$ is the area of the region D enclosed by the graphs of h and k.



8. a) Let f be an integrable function on $(0,1)$. Prove that there exists $x \in (0,1)$ such that

$$f(x) \leq \int_0^1 f(y)\, dy. \tag{1}$$

b) Given $\varepsilon > 0$, exhibit an integrable function f on $(0,1)$ such that the measure of the set of points $x \in (0,1)$ that satisfy (1) is less than ε.

9. Let X be the set of real polynomials defined on $\mathbb{R}$. For every $p \in X$ set

$$\|p\| := \sup_{0 \le t \le 1} |p(t)|.$$

Show that:

a) $\|\cdot\|$ is a norm on X;

b) the functional $\varphi : X \to \mathbb{R}$ defined as $\varphi(p) = p(2)$ for every $p \in X$ is not continuous.

10. Consider the following initial problem for the partial differential equation

$$u_{yy} = u_x, \qquad u(x,0) = \sin(2x).$$

Exhibit a solution that is 2π-periodic in x and such that for all $x \in \mathbb{R}$

$$\lim_{y \to +\infty} u(x,y) = 0.$$