S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

September 7, 2011

Solve five of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Consider the space $L^p = L^p(0,1)$, with $p \in [1,\infty]$. Show that the canonical inclusion $L^\infty \to L^1$ is continuons but not compact.
Is it possible to find $p, q \in [1,\infty]$ with $p < q$ such that the inclusion $L^q \to L^p$ is compact?

2. For $t \geq 0$, find all solutions of the Cauchy problem (defined for $x \geq 0$ and $|y| \leq 1$)

$$\begin{cases} \dot{x} = \sqrt{x}\, y \\ \dot{y} = \sqrt{1-y^2} \end{cases} \qquad x(0) = 0, \quad y(0) = 0.$$

3. Let $f \in L^1(0,\infty)$ be monotone. Prove that

$$\lim_{x\to\infty} x f(x) = 0.$$

4. Let $f \in L^2(0,1)$. Prove that $f(t) = t$ a.e. in $(0,1)$ if and only if

$$\int_0^1 t^n f(t)\, dt = \frac{1}{n+2} \qquad \text{for every } n \in \mathbb{N} \cup \{0\}.$$

5. Let $A \subset \mathbb{R}$ be a non-empty open set such that

$$\int_A \phi'(x) dx \leq 0$$

for all $\phi \in C^1_c(\mathbb{R})$ with $\phi \geq 0$. Prove that A is unbounded.
Hint: consider first the case $A = (a,b)$.

6. Let $f \in C(\mathbb{R})$ such that

$$f(0) \neq -2 \quad \text{and} \quad \int_0^1 f(t) = 0.$$

Show that there exists $\epsilon > 0$ such that the equation

$$\int_x^1 f(t)dt = 2x$$

has a unique solution for $|x| < \epsilon$.

7. Let (f_n) be a sequence in $L^2(\mathbb{R})$ and let $f \in L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Suppose that

$$f_n \rightharpoonup f \text{ weakly in } L^2(\mathbb{R}),$$
$$f_n^2 \rightharpoonup g \text{ weakly in } L^1(\mathbb{R}).$$

Prove that $g \geq f^2$ a.e. in $\mathbb{R}$.

8. Let y be a solution of the equation

$$y''(t) = y(t) - y(t)^3.$$

Suppose that $y \in L^2(\mathbb{R})$ e $y' \in L^2(\mathbb{R})$.

(a) Prove that $|y(t)| \leq \sqrt{2}$ for every $t \in \mathbb{R}$.

(b) Prove that either $y(t) = 0$ for every $t \in \mathbb{R}$ or $y(t)$ has constant sign.

9. Let $f \in C([0,1])$. Compute

$$\lim_{t \to +\infty} \frac{1}{t} \log \int_0^1 \cosh(tf(x))dx.$$

10. Let $(a_n)_{n \geq 1}$ be a non decreasing sequence of non negative numbers satisfying

$$a_{m \cdot n} \leq a_m + a_n \qquad \text{for every } n, m \geq 1.$$

Prove that there exists $C > 0$ such that

$$a_n \leq C \log n \qquad \text{for every } n \geq 2.$$

Hint: consider first the case $n = 2^k$.