SISSA, Area of Mathematics

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics

September 6, 2012

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

1. Consider in $C([0,1],\mathbb{R})$ the family of functions

$$f_r(x) := \begin{cases} 0 & 0 \le x \le r, \\ \frac{x-r}{1-r} & r < x \le 1, \end{cases}$$

parameterized by $r \in [0,1]$. Show that the closure of the convex envelope of $\{f_r, r \in [0,1]\}$ in $C([0,1],\mathbb{R})$ is the set of functions which are convex, non decreasing and such that $f(0) = 0$ e $f(1) = 1$.

2. Consider the partial differential equation

$$u_t + u_x = u_{xx}, \quad t \in (0,+\infty),\ x \in (0, 2\pi),\ u(t,x) \in \mathbb{R}.$$

(a) Write explicitly the unique solution $\bar{u}(x)$ of class $C^2([0,2\pi],\mathbb{R})$ which is time independent and such that

$$\bar{u}(0) = 1, \qquad \bar{u}(2\pi) = 0.$$

(b) Show that all the other solutions $u(t,x)$ of class $C^2([0,+\infty) \times [0,2\pi],\mathbb{R})$ with the same boundary values

$$u(t,0) = 1, \qquad u(t,2\pi) = 0 \qquad \text{per ogni } t \in [0,+\infty)$$

converge to $\bar{u}(x)$ w.r.t. the norm $C^0([0,2\pi])$ as $t \to +\infty$.

3. Let f be a continuous positive strictly monotone function on the segment $[0,1]$. For all $p > 0$ consider the point x_p such that

$$[f(x_p)]^p = \int_0^1 [f(x)]^p \, dx.$$

Find $\lim\limits_{p\to+\infty} x_p$.

4. Find a map $f : [0,1] \to (0,1)$ satisfying the following two properties:

(a) f establishes a one-to one correspondence between the points of the segment $[0,1]$ and the points of the open interval $(0,1)$,

(b) $f(x)=x$ for almost all $x\in[0,1]$ (with respect to the Lebesgue measure).

5. Consider the system of ordinary differential equations

$$\begin{cases} \dot{x} = 15x + x^2 + y - 3x^3; \\ \dot{y} = 4y^2 - x - 5y^3. \end{cases}$$

Prove that every solution of the system stays uniformly bounded for positive times.

6. Let $f:[0,1]\to\mathbb{R}$ be continuous. Let $f_n:[0,1]\to\mathbb{R}$ be defined by

$$f_n(t) = f(t^n); \qquad n\in\mathbb{N}.$$

Prove that if there exists $g:[0,1]\to\mathbb{R}$ continuous such that $f_n\to g$ uniformly on $[0,1]$ then f is constant.

7. Let X be a separable Hilbert space, let (e_n) be a complete orthonormal system in X, and let (λ_n) be a bounded sequence of complex numbers.

(a) Prove that there exists a unique continuous linear operator $A\colon X\to X$ such that $Ae_n = \lambda_n e_n$ for every $n\in\mathbb{N}$.

(b) Prove that A is compact if and only if $\lambda_n\to 0$ for $n\to\infty$.

8. Let $f\in L^1(\mathbb{R})$ and let F, $G\colon\mathbb{R}\to\mathbb{R}$ be the functions defined by

$$F(x) := \int_x^{x+1} f(t)\,dt \qquad \text{and} \qquad G(x) := \left| \int_x^{x+1} f(t)\,dt \right|.$$

(a) Prove that G has a maximum point on $\mathbb{R}$.

(b) Give an example of $f\in L^1(\mathbb{R})$ such that F has no maximum point on $\mathbb{R}$.

9. Consider the linear system $Ax=b$ in $\mathbb{R}^2$, where

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ -8 \end{bmatrix}.$$

(a) Write the sequences generated by the Steepest Descent method, and by the Conjugate Gradient method, starting with initial guess

$$x_0 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}.$$

(b) In how many steps do the Conjugate Gradient method converge? In how many steps do the Steepest Descent method converge?

10. Let Ω be an open set of $\mathbb{R}^n$ and let $H^1(\Omega)$ be the corresponding Sobolev space of first order. Let $a\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be a bilinear and coercive form, and let $f\colon H^1(\Omega) \to \mathbb{R}$ be continuous linear functional. Consider the following variational problem: find $u \in H^1(\Omega)$ such that

$$a(u, v) = f(v) \qquad \forall v \in H^1(\Omega),$$

and assume that the solution u belongs to the Sobolev space of second order $H^2(\Omega)$.

For each $h > 0$ let $V_h \subset H^1(\Omega)$ be a finite dimensional subspace and let $u_h \in V_h$ be the solution to

$$a(u_h, v_h) = f(v_h) \qquad \forall v_h \in V_h.$$

Assume that for each $h > 0$ there exists a projection operator $\mathcal{P}_h : H^1(\Omega) \mapsto V_h$ such that

$$\|v - \mathcal{P}_h v\|_{H^1(\Omega)} \leq Ch|v|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega),$$

where $C > 0$ does not depend on h and $|v|_{H^2(\Omega)} := \sum_{i,j=1}^{n} \|D_{ij}v\|_{L^2(\Omega)}$.

Prove that there exists a constant $C_1 > 0$ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq C_1 h|u|_{H^2(\Omega)} \qquad \forall h > 0.$$

11. Let $I_p\colon C^0([0,1]) \to \mathbb{R}$ be the linear functional

$$I_p(f) := \sum_{i=0}^{p} f(q_i)\alpha_i,$$

where q_i are $p+1$ distinct points of the interval $[0,1]$ and α_i are $p+1$ real numbers. Denote with P_n the space of polynomials of degree n on the interval $[0,1]$. The functional I_p is a *quadrature formula* of order n, if

$$I_p(f) = \int_0^1 f(x)dx \qquad \forall f \in P_n.$$

(a) Write the necessary and sufficient conditions on q_i e α_i such that the functional I_p is a quadrature formula of order n.

(b) Given the points q_i, write an explicit formula to compute the values of the weights α_i such that I_p is a quadrature formula of order p.