Scuola Internazionale Superiore di Studi Avanzati - Trieste

SISSA

Area of Mathematics

Ph.D. in Mathematical Analysis

Thesis

Sharp Inequalities and Blow-up Analysis for Singular Moser-Trudinger Embeddings

Candidate
Gabriele Mancini

Supervisor
Andrea Malchiodi

Academic Year 2014-2015

SISSA - Via Bonomea 265 - 34136 TRIESTE - ITALY

Il presente lavoro costituisce la tesi presentata da Gabriele Mancini, sotto la direzione del Prof. Andrea Malchiodi, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiæ presso la SISSA, Curriculum in Analisi Matematica, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della Sissa pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di Dottore di Ricerca in Matematica. Trieste, Anno Accademico 2014-2015.

Acknowledgements

I wish to thank my supervisor Andrea Malchiodi for introducing me to the world of research and for his guidance in the preparation of this work. Beside him, I am indebted to Luca Martinazzi and Daniele Bartolucci for many useful discussions and for their precious advises.

I am extremely grateful to Simona Dimase, Giovanna Catavitello and Maria Eleuteri, for continuously encouraging me during the last four years.

Finally, special thanks go to my colleagues Aleks, Elisa, Giancarlo, Luca, Lucia, Marks, Riccardo and Stefano. Their friendship accompanied and supported me since my first day at SISSA.

Contents

Chapter 1

Introduction

The present thesis deals with sharp Moser-Trudinger type inequalities and blow-up analysis for elliptic problems involving critical exponential nonlinearities in dimension two. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain, from the well known Sobolev's inequality

$$\|u\|_{L^{\frac{2p}{2-p}}(\Omega)} \leq S_p \|\nabla u\|_{L^p(\Omega)} \qquad p \in (1,2),\ u \in W_0^{1,p}(\Omega), \tag{1.1}$$

one can deduce that the Sobolev space $H_0^1(\Omega) := W^{1,2}(\Omega)$ is embedded into $L^q(\Omega)\ \forall\ q \geq 1$. A much more precise result was proved in 1967 by Trudinger [84]: on bounded subsets of $H_0^1(\Omega)$ one has uniform exponential-type integrability. Specifically, there exists $\beta > 0$ such that

$$\sup_{u \in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx \leq 1} \int_\Omega e^{\beta u^2} dx < +\infty. \tag{1.2}$$

This inequality was later improved by Moser in [68], who proved that the sharp exponent in (1.2) is $\beta = 4\pi$, that is

$$\sup_{u \in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx \leq 1} \int_\Omega e^{4\pi u^2} dx < +\infty, \tag{1.3}$$

and

$$\sup_{u \in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 dx \leq 1} \int_\Omega e^{\beta u^2} dx = +\infty \tag{1.4}$$

for $\beta > 4\pi$. The same inequality holds if $(\Omega, |dx|^2)$ is replaced by a smooth closed surface and the boundary condition by a zero mean value condition. More precisely, if (Σ, g) is a smooth, closed Riemannian surface and

$$\mathcal{H} := \left\{ u \in H^1(\Sigma)\ :\ \int_\Sigma |\nabla u|^2 dv_g \leq 1,\ \int_\Sigma u\, dv_g = 0 \right\}, \tag{1.5}$$

in [42] Fontana proved

$$\sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi u^2} dv_g < +\infty \tag{1.6}$$

and

$$\sup_{u\in\mathcal{H}}\int_\Sigma e^{\beta u^2}dv_g=+\infty \tag{1.7}$$

$\forall\ \beta>4\pi$. Moser's interest in finding sharp forms of (1.2) was motivated by the strict connection between these kind of inequalities and Nirenberg's problem of prescribing the curvature of S^2. More generally, given a smooth closed surface Σ and a function $K\in C^\infty(\Sigma)$, a classical problem consists in determining whether K can be realized as the Gaussian curvature of a smooth metric g on Σ. The Gauss-Bonnet condition

$$\int_\Sigma K dv_g = 4\pi\chi(\Sigma),$$

clearly gives the following necessary conditions on the sign of K:

$$\begin{aligned}
\chi(\Sigma)<0 \qquad &\Longrightarrow \qquad \min_\Sigma K<0;\\
\chi(\Sigma)=0 \qquad &\Longrightarrow \qquad K\equiv 0 \text{ or } K \text{ changes sign;}\\
\chi(\Sigma)>0 \qquad &\Longrightarrow \qquad \max_\Sigma K>0.
\end{aligned} \tag{1.8}$$

In [47] (see also [48]) Kazdan and Warner proved that if $\chi(\Sigma)\le 0$ the conditions in (1.8) are indeed necessary and sufficient. However they also proved that this is not true if $\Sigma=S^2$. A possible way of studying the Gaussian curvature problem consists in looking for solutions among the class of metrics on Σ which are pointwise conformally equivalent to a pre-assigned metric g. Indeed a metric of the form $e^u g$ has Gaussian curvature K if and only if u is a solution of

$$-\frac{1}{2}\Delta_g u = Ke^u - K_g \tag{1.9}$$

where K_g, Δ_g denote the Gaussian curvature and the Laplace-Beltrami operator of (Σ,g). It is not difficult to see that, if $\chi(\Sigma)\neq 0$ and K_g is constant, (1.9) is equivalent to

$$-\Delta_g u=\rho\left(\frac{Ke^u}{\int_\Sigma Ke^u dv_g}-\frac{1}{|\Sigma|}\right) \tag{1.10}$$

with $\rho=4\pi\chi(\Sigma)$, which is known as the Liouville equation on Σ. Solutions of (1.10) can be obtained as critical points of the functional

$$J_\rho^K(u):=\frac{1}{2}\int_\Sigma|\nabla u|^2dv_g+\frac{\rho}{|\Sigma|}\int_\Sigma u\,dv_g-\rho\log\left(\frac{1}{|\Sigma|}\int_\Sigma Ke^u dv_g\right). \tag{1.11}$$

As a consequence of inequality (1.3), Moser proved that $J_{8\pi}^K$ is bounded from below and J_ρ^K is coercive on the space

$$H_0:=\left\{u\in H^1(\Sigma)\ :\ \int_\Sigma u\,dv_g=0\right\} \tag{1.12}$$

for $\rho<8\pi$. In particular, using direct minimization, he was able to prove existence of solutions of (1.9) on the projective plane or, equivalently, on S^2 under the assumption $K(x)=K(-x)$ $\forall\ x\in S^2$. Without symmetry, minimization techniques are not sufficient to study equation

(1.9). We refer the reader to [24], [25] and [79], where existence of solutions is proved under nondegeneracy assumptions on the critical points of K, through min-max schemes or a curvature flow approach. Existence results for (1.10) with $\rho > 8\pi$ were obtained in [38], [80], [39], [62].

A more general problem consists in studying curvature functions for compact surfaces with conical singularities. We recall that, given a finite number of points $p_1, \dots, p_m \in \Sigma$, a metric with conical singularities of order $\alpha_1, \dots, \alpha_m > -1$ in $p_1, \dots, p_m$, is a metric of the form $e^u g$ where g is a smooth metric on Σ, and $u \in C^\infty(\Sigma \backslash \{p_1, \dots, p_m\})$ satisfies

$$|u(x) + 2\alpha_i \log d(x, p_i)| \leq C \qquad \text{near } p_i,\ i = 1, \dots, m.$$

It is possible to prove (see for example Proposition 2.1 in [6]) that a metric of this form has Gaussian curvature K if and only if u is a distributional solution of the singular Gaussian curvature equation

$$-\Delta_g u = 2Ke^u - 2K_g - 4\pi \sum_{i=1}^m \alpha_i \delta_{p_i}. \tag{1.13}$$

If $\chi(\Sigma) + \sum_{i=1}^m \alpha_i \neq 0$ and K_g is constant, (1.13) is equivalent to the singular Liouville equation

$$-\Delta_g u = \rho \left(\frac{Ke^u}{\int_\Sigma Ke^u dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{i=1}^m \alpha_i \left(\delta_{p_l} - \frac{1}{|\Sigma|} \right) \tag{1.14}$$

for

$$\rho = \rho_{geom} := 4\pi \left(\chi(\Sigma) + \sum_{i=1}^m \alpha_i \right). \tag{1.15}$$

Although we introduced equations (1.10) and (1.14) starting from the Gaussian curvature problem, they have also been widely studied in mathematical physics. For example, they appear in the description of Abelian vortices in Chern-Simmons-Higgs theory, and have applications in fluid dynamics ([67], [85]), Superconductivity and Electroweak theory ([81], [43]). Denoting by G the Green's function of (Σ, g), i.e. the solution of

$$\begin{cases} -\Delta_g G(x, \cdot) = \delta_x \text{ on } \Sigma \\ \int_\Sigma G(x,y) dv_g(y) = 0, \end{cases} \tag{1.16}$$

the change of variable $u \longleftrightarrow u + 4\pi \sum_{i=1}^m \alpha_i G(x, p_i)$ reduces (1.14) to

$$-\Delta_g u = \rho \left(\frac{he^u}{\int_\Sigma he^u dv_g} - \frac{1}{|\Sigma|} \right) \tag{1.17}$$

that is (1.10) with K replaced by the singular weight

$$h(x) = Ke^{-4\pi \sum_{i=1}^m \alpha_i G_{p_i}}. \tag{1.18}$$

Thus, as in absence of singularities, finding solutions of (1.14) is equivalent to proving existence of critical points for the singular Moser-Trudinger functional J^h_ρ. We stress that h satisfies

$$h \in C^\infty(\Sigma \backslash \{p_1, \dots, p_m\}) \qquad \text{and} \qquad h(x) \approx d(x, p_i)^{2\alpha_i} \text{ with } \alpha_i > -1 \text{ near } p_i, \tag{1.19}$$

$i = 1, \dots, m$. In the same spirit of Moser's work, in [83] Troyanov tried to minimize J_ρ^h by finding a sharp version of the Moser-Trudinger inequality for metrics with conical singularities. In particular he proved (see also [30]) that if $h \in C^0(\Sigma \setminus \{p_1, \dots, p_m\})$ satisfies (1.19), then

$$\sup_{u \in \mathcal{H}} \int_\Sigma h\, e^{\beta u^2} dx < +\infty \qquad \Longleftrightarrow \qquad \beta \leq 4\pi(1+\overline{\alpha}) \tag{1.20}$$

where

$$\overline{\alpha} = \min \left\{ 0, \min_{1 \leq i \leq m} \alpha_i \right\}. \tag{1.21}$$

As a consequence one has

$$\log \left(\frac{1}{|\Sigma|} \int_\Sigma h e^{u - \overline{u}} dv_g \right) \leq \frac{1}{16\pi(1+\overline{\alpha})} \int_\Sigma |\nabla u|^2 dv_g + C(\Sigma, g, h) \tag{1.22}$$

where the coefficient $\frac{1}{16\pi(1+\overline{\alpha})}$ is sharp. In particular

$$\begin{aligned} \rho < 8\pi(1+\overline{\alpha}) \quad &\Longrightarrow \quad J_\rho^h \text{ is bounded from below on } H^1(\Sigma) \text{ and coercive on } H_0; \\ \rho = 8\pi(1+\overline{\alpha}) \quad &\Longrightarrow \quad J_\rho^h \text{ is bounded from below on } H^1(\Sigma); \\ \rho > 8\pi(1+\overline{\alpha}) \quad &\Longrightarrow \quad \inf_{H^1(\Sigma)} J_\rho^h = -\infty. \end{aligned} \tag{1.23}$$

For $\rho < 8\pi(1+\overline{\alpha})$, the coercivity of J_ρ^h yields existence of minimum points. The case $\rho > 8\pi(1+\overline{\alpha})$ has been studied mainly with two different approaches: topological methods and the Leray-Schauder degree theory. In both methods, a fundamental role is played by blow-up analysis for sequences of solutions of (1.17) and, in particular, by the the following concentration-compactness alternative:

Theorem 1.1. *Let h be a positive function satisfying (1.18) with $K \in C^1(\Sigma)$, $K > 0$ and let $u_n \in H_0$ be a sequence of solutions of (1.17) with $\rho = \rho_n > 0$ and $\rho_n \longrightarrow \overline{\rho}$. Then, up to subsequences, one of the following holds:*

(i) $|u_n(x)| \leq C$ with C depending only on ρ, K, and $\alpha_1, \dots, \alpha_m$.

(ii) There exists a finite set $S := \{q_1, \dots, q_k\} \subseteq \Sigma$ such that

- *For any $j = 1, \dots, k$ there exists a sequence $\{q_n^j\}_{n \in \mathbb{N}}$ such that $q_n^j \longrightarrow q_j$ and $u_n(q_n^j) \longrightarrow +\infty$.*
- *$u_n \longrightarrow -\infty$ uniformly on any compact subset of $\Sigma \setminus S$.*
- *$\rho_n \frac{h e^{u_n}}{\int_\Sigma h e^{u_n} dv_g} \rightharpoonup \sum_{j=1}^k \beta_j \delta_{q_j}$ weakly as measures, where $\beta_j = 8\pi$ if $q_j \in \Sigma \setminus S$ and $\beta_j = 8\pi(1+\alpha_l)$ if $q_j = p_l$ for some $1 \leq l \leq m$.*

This statement is the combination of the work of several authors. Blow-up analysis for Liouville-type equations was first studied by Brezis and Merle in [18]. Their work was later completed by Li and Shafrir in [51] and [50] in the regular case $m = 0$, while the singular case was considered in [5] and [8] by Bartolucci, Montefusco and Tarantello. Clearly alternative *(ii)* in Theorem 1.1 is possible only if the limit parameter $\overline{\rho}$ belongs to the set

$$\Gamma(\alpha_1, \dots, \alpha_n) := \left\{ 8\pi k_0 + 8\pi \sum_{i=1}^{m} k_i(1+\alpha_i) \ : \ k_0 \in \mathbb{N}, k_i \in \{0,1\}, \sum_{i=0}^{m} k_i > 0 \right\}. \tag{1.24}$$

More precisely, combining Theorem 1.1 with standard elliptic estimates, one can prove that if Λ is a compact subset of $[0,+\infty)\setminus\Gamma(\alpha_1,\dots,\alpha_n)$, then the set of all the solutions in H_0 of (1.17) with $\rho \in \Lambda$ is a compact subset of $H^1(\Sigma)$. This compactness condition can be used to prove a deformation Lemma (see [60]) for the functional J_ρ^h: given $\rho \notin \Gamma(\alpha_1,\dots,\alpha_m)$ and $a, b \in \mathbb{R}$ with $a < b$, if there is no critical point of J_ρ^h in $\{a \le J_\rho^h \le b\}$, then the sublevel $\{J_\rho^h \le a\}$ is a deformation retract of $\{J_\rho^h \le b\}$. The boundedness of the set of solutions implies that high sublevels of J_ρ^h are contractible, thus one can prove existence of solutions by showing that low sublevels of J_ρ^h have nontrivial topology. In the regular case $m = 0$ this was done by Djadli and Malchiodi in [62] and [39]. They used an improved version of (1.22) to prove that, for $\rho \in (8\pi k, 8\pi(k+1))$, functions belonging to sufficiently low sublevels of J_ρ^h must be concentrated around at most k points on Σ. This concentration property shows that low sublevels are homotopy equivalent to the set of formal baricenters

$$\Sigma_k := \left\{ \sum_{i=1}^{k} t_i \delta_{x_i} \ : \ x_i \in \Sigma, \ t_i \in [0,1], \ \sum_{i=1}^{k} t_i = 1 \right\},$$

which in noncontractible. Therefore they prove existence of solutions of (1.10) for any positive function K and $\rho \in [0,+\infty)\setminus 8\pi\mathbb{N}$. In the presence of singularities, describing the topology of sublevels of J_ρ^h becomes more complicated. In [6], the authors considered the case of positive order singularities (i.e. $\alpha_i > 0 \ \forall \ i$). If Σ is orientable and $g(\Sigma)$ denotes the genus of Σ, they proved that is possible to embed a bouquet of $g(\Sigma)$ circles into sufficiently low sublevels. Hence, if $g(\Sigma) \ge 1$, one has existence of solutions of (1.17) whenever $\rho \notin \Gamma(\alpha_1,\dots,\alpha_m)$. The condition $g(\Sigma) \ge 1$ cannot be removed, indeed we will see that on S^2 it is possible to have nonexistence of solutions also for noncritical values of ρ. However in [7] it is proved that solutions exist provided $\rho \in \left(0, 8\pi\left(1 + \min_{1\le i\le m} \alpha_i\right)\right)$. The case $\alpha_i \in (-1,0)$ is treated in [21] and [22], where the authors prove existence of solutions if there exist $k \in \mathbb{N}$ and $I \subseteq \{1,\dots,m\}$ such that $k + |I| > 0$ and

$$8\pi\left(k + \sum_{i\in I} \alpha_i\right) < \rho < 8\pi\left(k + \sum_{i \in I\cup\{i_0\}} \alpha_i\right)$$

where $i_0 \in \{1,\dots,m\}$ is chosen so that $\alpha_{i_0} = \overline{\alpha}$. This condition is indeed necessary and sufficient for the noncontractibility of a generalized set of formal baricenters that can be embedded into low sublevels of J_ρ^h.

A different approach to equation (1.17) relies on the Leray-Schauder degree theory. For any $\rho > 0$ one can consider the operator $T_\rho : H_0 \longrightarrow H_0$ defined by

$$T_\rho(u) = \rho\, \Delta_g^{-1}\left(\frac{he^u}{\int_\Sigma he^u dv_{g_0}} - \frac{1}{|\Sigma|} \right), \tag{1.25}$$

and find solutions of (1.17) by proving that the Leray-Schauder degree

$$d_\rho := \deg_{LS}(Id + T_\rho, 0, B_R(0)) \tag{1.26}$$

is different from 0. Here $B_R := \{x \in H_0 \ : \ \|u\|_{H^1(\Sigma)} < R\}$. For $\rho \neq \Gamma(\alpha_1, \ldots, \alpha_m)$, the boundedness of the set of solutions of (1.17) implies that d_ρ is well defined, i.e. it does not depend on R if R is sufficiently large. Using Theorem 1.1 and the homotopy invariance of the Leray-Schauder degree, one can prove that d_ρ does not depend on the function h and is constant in ρ on the connected components of $[0, +\infty)\backslash\Gamma(\alpha_1, \ldots, \alpha_m)$. In a series of papers ([26], [27], [28], [29]) Chen and Lin were able to find and explicit formula for d_ρ by computing its jumps at each value of $\rho \in \Gamma(\alpha_1, \ldots, \alpha_m)$ due to the existence of blowing up families of solutions. They introduced the generating function

$$g(x) := (1 + x + x^2 + x^3 \ldots)^{m-\chi(\Sigma)} \prod_{i=1}^{m} (1 - x^{1+\alpha_i}) \tag{1.27}$$

and observed that

$$g(x) = 1 + \sum_{j=1}^{\infty} b_j x^{n_j} \tag{1.28}$$

where $n_1 < n_2 < n_3 < \ldots$ are such that

$$\Gamma(\alpha_1, \ldots, \alpha_m) = \{8\pi n_j \ : \ j \geq 1\}.$$

Moreover for $\rho \in (8\pi n_k, 8\pi n_{k+1})$ one has

$$d_\rho = \sum_{j=0}^{k} b_j \tag{1.29}$$

where $b_0 = 1$ and b_j are the coefficients in (1.28). While this formula holds only for $\rho \notin \Gamma(\alpha_1, \ldots, \alpha_m)$, the sharp blow-up analysis carried out in Chen and Lin's work can be exploited, under nondegeneracy assumptions on h, to prove existence of solutions also for the critical values of the parameter ρ.

1.1 Onofri-Type Inequalities for the First Critical Parameter

In Chapter 2 we will study sharp versions of (1.22). We are interested in determining the optimal value of the constant $C(\Sigma, g, h)$. Clearly one has

$$C(\Sigma, g, h) = -\frac{1}{8\pi(1 + \overline{\alpha})} \inf_{H^1(\Sigma)} J^h_{8\pi(1+\overline{\alpha})}, \tag{1.30}$$

thus this problem is strictly connected with the existence of minimum points of $J_{8\pi(1+\overline{\alpha})}$. Note that $\overline{\rho} := 8\pi(1 + \overline{\alpha}) = \min \Gamma(\alpha_1, \ldots, \alpha_m)$ is the first critical parameter for equation (1.17). For the standard Euclidean Sphere (S^2, g_0), the special case $m = 0$ and $K \equiv 1$ was studied by Onofri in [69]. He proved that $C(S^2, g_0, 1) = 0$ and gave a complete classification of the minima of $J^1_{8\pi}$, which turn out to be all the solutions of (1.10).

Theorem A (Onofri's inequality [69]). *$\forall\, u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} e^{u-\overline{u}}dv_{g_0}\right) \le \frac{1}{16\pi}\int_{S^2}|\nabla u|^2 dv_{g_0},$$

with equality holding if and only if $e^u g_0$ is a metric on S^2 with positive constant Gaussian curvature, or, equivalently, $u = \log|\det d\varphi| + c$ with $c \in \mathbb{R}$ and $\varphi : S^2 \longrightarrow S^2$ a conformal diffeomorphism of S^2.

Beside its geometric interest, this result has important applications in spectral analysis due to Polyakov's formula (see [72], [73], [71], [70]). Motivated by Theorem A, in [65] and [66] we studied Onofri-type inequalities and existence of energy-minimizing solutions on S^2 for the singular potential

$$h(x) = e^{-4\pi\sum_{i=1}^m \alpha_i G(x,p_i)}$$

(i.e. (1.18) with $K \equiv 1$). We determined the sharp constant $C(S^2, g_0, h)$ if $m \le 2$ or $\overline{\alpha} = 0$.

More generally we are able to give an estimate of $C(\Sigma, g, h)$ for an arbitrary surface Σ. Our key observation is that if $J^h_{\overline{\rho}}$ has no minimum point, then one can use blow-up analysis to describe the behavior of a suitable minimizing sequence and compute explicitly $\inf\limits_{H^1(\Sigma)} J^h_{\overline{\rho}}$. The same technique was used by Ding, Jost, Li and Wang [37] to give an existence result for (1.17) in the regular case. From their proof it follows that if $m = 0$ and there is no minimum point for $J^h_{8\pi}$, then

$$\inf_{H^1(\Sigma)} J^h_{8\pi} = -8\pi\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{p\in\Sigma}\{4\pi A(p) + \log h(p)\}\right)$$

where $A(p)$ is the value in p of the regular part of $G(\cdot, p)$. Here we extend this result to the general case proving:

Theorem 1.2. *Let h be a function satisfying (1.18) with $K \in C^\infty(\Sigma)$, $K > 0$, $\alpha_1, \ldots, \alpha_m \in (-1, +\infty)\setminus\{0\}$ and assume that $J_{\overline{\rho}}$ has no minimum point. If $\overline{\alpha} < 0$, then*

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} = -\overline{\rho}\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{1\le i\le m, \alpha_i = \overline{\alpha}}\left\{4\pi A(p_i) + \log\left(\frac{K(p_i)}{1+\overline{\alpha}}\prod_{j\ne i} e^{-4\pi\alpha_j G_{p_j}(p_i)}\right)\right\}\right)$$

while if $\overline{\alpha} > 0$

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} = -8\pi\left(1 + \log\left(\frac{\pi}{|\Sigma|}\right) + \max_{p\in\Sigma\setminus\{p_1,\ldots,p_m\}}\{4\pi A(p) + \log h(p)\}\right).$$

If $\Sigma = S^2$ and $K \equiv 1$, we will give a generalized version of the Kazdan-Warner identity and prove nonexistence of solutions of (1.17) provided $m = 1$ or $m = 2$, $p_1 = -p_2$, $\min\{\alpha_1, \alpha_2\} = \alpha_1 < 0$ and $\alpha_1 \ne \alpha_2$. In particular we obtain the following sharp inequalities:

Theorem 1.3. *If $h = e^{-4\pi\alpha G_{p_1}}$ with $\alpha \ne 0$, then $\forall\, u \in H^1(S^2)$*

$$\log\left(\frac{1}{4\pi}\int_{S^2} h e^{u-\overline{u}}dv_{g_0}\right) < \frac{1}{16\pi\min\{1, 1+\alpha\}}\int_{S^2}|\nabla u|^2 dv_{g_0} + \max\{\alpha, -\log(1+\alpha)\}.$$

Moreover, the Liouville equation (1.17) has no solution for $\rho = \overline{\rho} = 8\pi(1 + \min\{0, \alpha\})$.

Theorem 1.4. *Assume* $h = e^{-4\pi\alpha_1 G_{p_1} - 4\pi\alpha_2 G_{p_2}}$ *with* $p_2 = -p_1$, $\alpha_1 = \min\{\alpha_1, \alpha_2\} < 0$ *and* $\alpha_1 \neq \alpha_2$*; then* $\forall\, u \in H^1(S^2)$

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) < \frac{1}{16\pi(1+\alpha_1)}\int_{S^2}|\nabla u|^2 dv_{g_0} + \alpha_2 - \log(1+\alpha_1).$$

Moreover, the Liouville equation (1.17) *has no solution for* $\rho = \overline{\rho} = 8\pi(1+\alpha_1)$.

Note that the constant in Theorem 1.3 coincides with the one in Theorem 1.4 if we set $\alpha_1 = \min\{\alpha, 0\}$ and $\alpha_2 = \max\{\alpha, 0\}$.

The case $\alpha_1 = \alpha_2 < 0$ is particularly interesting because the critical parameter $\overline{\rho} = 8\pi(1+\overline{\alpha})$ coincides with the geometric value ρ_{geom} (see (1.15)) for which equation (1.17) is equivalent to the Gaussian curvature problem. This means that the functional acquires a natural conformal invariance that allows to use a stereographic projection and reduce (1.17) to the Liouville equation

$$-\Delta u = |x|^{2\alpha}e^u$$

on $\mathbb{R}^2$, whose solutions were completely classified in [74]. In particular combining Theorem 1.2 with a direct computation we will show that all solutions are minimum points of $J_{\overline{\rho}}$ and we will find the value of $\min_{H^1(S^2)} J_{\overline{\rho}}$.

Theorem 1.5. *Assume* $h = e^{-4\pi\alpha(G_{p_1}+G_{p_2})}$ *with* $\alpha \leq 0$ *and* $p_1 = -p_2$*; then* $\forall\, u \in H^1(S^2)$ *we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) \leq \frac{1}{16\pi(1+\alpha)}\int_{S^2}|\nabla u|^2 dv_{g_0} + \alpha - \log(1+\alpha).$$

Moreover the following conditions are equivalent:

- *u realizes equality.*
- *u is a solution of* (1.17) *for* $\rho = 8\pi(1+\alpha)$.
- *$he^u g_0$ is a metric with constant positive Gaussian curvature and conical singularities of order α_i in p_i, $i = 1, 2$.*
- *If π denotes the stereographic projection from p_1 then*

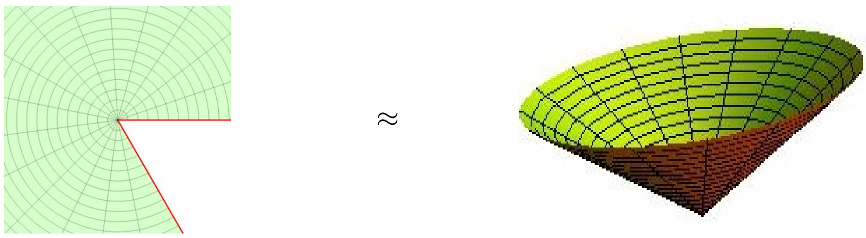
$$u \circ \pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha}}{1+e^{\lambda}|y|^{2(1+\alpha)}}\right) + c \tag{1.31}$$

  for some $\lambda, c \in \mathbb{R}$.

As in the original Onofri inequality, the family of solutions (1.31) can be interpreted in terms of of determinants of conformal transformations. Given $\alpha \leq 0$, let us consider the quotient space

$$C_\alpha := \frac{\{(r\cos t, r\sin t) \in \mathbb{R}^2 \ : \ r \geq 0,\ t \in [0, 2\pi(1+\alpha)\}}{\sim}$$

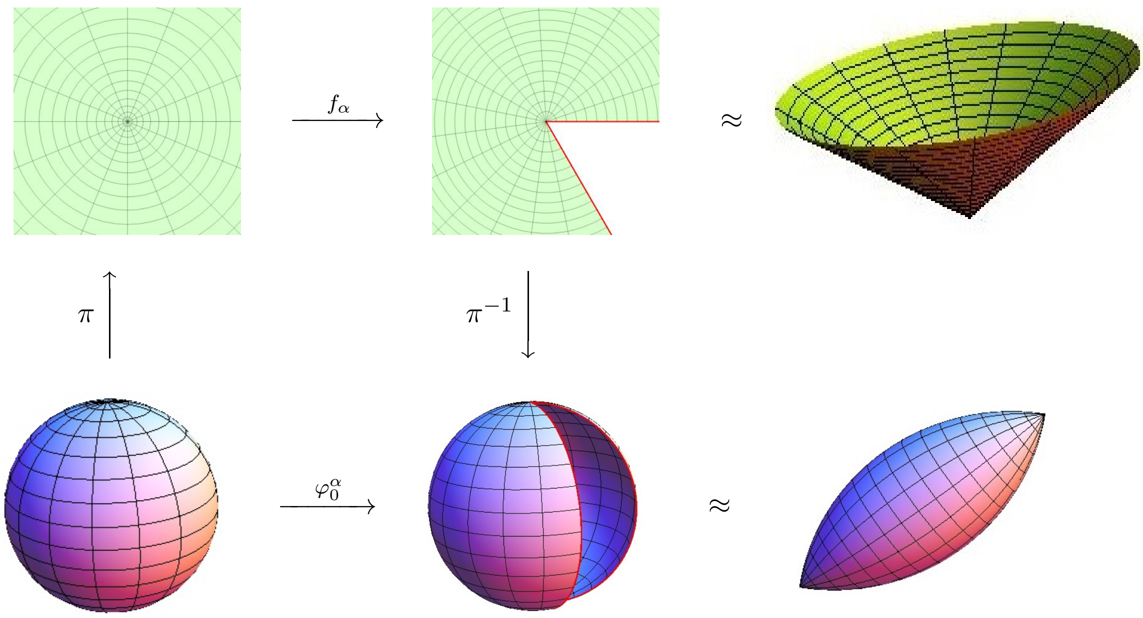
where $\sim$ is the identification of the boundary points $(r, 0) \sim (r\cos(2\pi(1+\alpha)), r\sin(2\pi(1+\alpha)))$. C_α can be identified with a cone of total interior angle equal to $2\pi(1+\alpha)$.



It is well known that the function $f_\alpha : \mathbb{R}^2 \longrightarrow C_\alpha,\ f_\alpha(z) = \frac{z^{1+\alpha}}{1+\alpha}$ is a well defined conformal diffeomorphism and $f_\alpha^*|dz|^2 = |z|^{2\alpha}|dz|^2$. Let π be the stereographic projection from the point p_1, then the surface $S_\alpha := \pi^{-1}(C_\alpha)$ is well defined and can be identified with an American football of interior angles $2\pi(1+\alpha)$. The map $\varphi_0^\alpha := \pi^{-1} \circ f_\alpha \circ \pi$ is a conformal diffeomorphism between S^2 and S_α and it is simple to verify that

$$|\det d\varphi_0^\alpha| = \frac{(1+|y|^2)^{1+\alpha}}{1+|y|^{2(1+\alpha)}}$$

so that $\log|\det d\varphi_0^\alpha|$ is a solution of (1.17).



The other solutions are obtained by taking the composition of φ_0^α with conformal diffeomorphisms of S^2 fixing the poles p_1, p_2.

In the last part of Chapter 2 we will consider the case of positive order singularities. We will assume (1.18), $\alpha_i \geq 0$ for $1 \leq i \leq m$ and

$$K \in C_+^\infty(S^2) := \left\{ f \in C^\infty(S^2) \ : \quad f(x) > 0 \quad \forall\, x \in S^2 \right\}.$$

Completing the results of Theorems 1.3, 1.4, 1.5, we give a further extension of Onofri's inequality.

Theorem 1.6. *Assume that h satisfies (1.18) with $K \in C^\infty_+(S^2)$ and $\alpha_1, \dots, \alpha_m \geq 0$, then*

$$\inf_{H^1(S^2)} J^h_{8\pi} = -8\pi \log \max_{S^2} h.$$

Moreover J^h_ρ has no minimum point, unless $\alpha_1 = \ldots = \alpha_m = 0$ (or, equivalently, $m = 0$) and K is constant.

Clearly, by (1.30), Theorem 1.6 yields the following sharp inequality:

Corollary 1.1. *If h satisfies (1.18) with $K \in C^\infty_+(S^2)$ and $\alpha_1, \dots, \alpha_m \geq 0$, then $\forall\, u \in H^1(S^2)$ we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) \leq \frac{1}{16\pi}\int_{S^2} |\nabla u|^2 dv_{g_0} + \log \max_{S^2} h$$

with equality holding if and only if $m = 0$, K is constant and u realizes equality in Theorem A.

Theorem 1.6 states that $J^h_{8\pi}$ has no minimum point, but does not exclude the existence of different kinds on critical points. In contrast to Theorem 1.4, if $\alpha_i > 0$ for $1 \leq i \leq m$, we will show that in many cases it is possible to find saddle points of $J^h_{8\pi}$. A simple example is given by the case in which h is axially symmetric. In this case an improved Moser-Trudinger inequality allows to minimize $J^h_{8\pi}$ in the class of axially symmetric functions and find a solution of (1.17).

Theorem 1.7. *Assume that h satisfies (1.18) with $m = 2$, $p_1 = -p_2$, $\min\{\alpha_1, \alpha_2\} = \alpha_1 > 0$ and $K \in C^\infty_+(S^2)$ axially symmetric with respect to the direction identified by p_1 and p_2. Then the Liouville equation (1.17) has an axially symmetric solution $\forall \rho \in (0, 8\pi(1 + \alpha_1))$.*

Further general existence results can be obtained using the sharp estimates proved in [26], [27], [28], [29], and the formula (1.26) for the Leray-Schauder degree. If $m \geq 2$ one has $d_\rho \neq 0$ for any $\rho \in (0, 8\pi(1 + \alpha_1))\backslash 8\pi\mathbb{N}$. While Theorem 1.6 implies blow-up of solutions as $\rho \nearrow 8\pi$, we can find solutions for $\rho = 8\pi$ by taking $\rho \searrow 8\pi$, provided the Laplacian of K is not too large at the critical points of h.

Theorem 1.8. *If h satisfies (1.18) with $K \in C^\infty_+(S^2)$, $m \geq 2$, $\alpha_1, \dots, \alpha_m > 0$ and*

$$\Delta_{g_0} \log K(x) < \sum_{i=1}^m \alpha_i \tag{1.32}$$

$\forall\, x \in \Sigma$ such that $\nabla h(x) = 0$, then equation (1.17) has a solution for $\rho = 8\pi$.

We stress that the same strategy can be used to find solutions of (1.17) for $\rho = 8k\pi$, with $k < 1 + \alpha_1$.

Theorem 1.9. *If h satisfies (1.18) with $K \in C^\infty_+(S^2)$, $m \geq 2$, $0 < \alpha_1 \leq \ldots \leq \alpha_m$ and*

$$\Delta_{g_0} \log K(x) < \sum_{i=1}^{m} \alpha_i + 2(1-k) \tag{1.33}$$

$\forall\, x \in S^2$, then equation (1.17) has a solution for $\rho = 8k\pi$, $k < 1 + \alpha_1$.

Note that Theorems 1.8 and 1.9 can be applied in the case $K \equiv 1$. If the sign condition (1.32) is not satisfied, then it is not possible to exclude blow-up of solutions as $\rho \longrightarrow 8\pi$. However, as it is pointed out in the introduction of [27], under some non-degeneracy assumptions on h, the Leray Schauder degree $d_{8\pi}$ is well defined and can be explicitly computed by taking into account the contributions of all the blowing-up families of solutions. In particular one can prove that $d_{8\pi} \neq 0$ under one of the following conditions.

Theorem 1.10. *Let h be a Morse function on $S^2 \backslash \{p_1, \ldots, p_m\}$ satisfying (1.18) with $K \in C^\infty_+(S^2)$, $m \geq 0$, $\alpha_1, \ldots, \alpha_m > 0$ and assume $\Delta_{g_0} h \neq 0$ at all the critical points of h. If h has r local maxima and s saddle points in which $\Delta_{g_0} h < 0$, then equation (1.17) has a solution for $\rho = 8\pi$ provided $r \neq s + 1$.*

Theorem 1.11. *Let h be a Morse function on $S^2 \backslash \{p_1, \ldots, p_m\}$ satisfying (1.18) with $K \in C^\infty_+(S^2)$, $m \geq 0$, $\alpha_1, \ldots, \alpha_m > 0$ and assume $\Delta_{g_0} h \neq 0$ at all the critical points of h. If h has r' local minima in $S^2 \backslash \{p_1, \ldots, p_m\}$ and s' saddle points in which $\Delta_{g_0} h > 0$, then equation (1.17) has a solution for $\rho = 8\pi$ provided $s' \neq r' + \overline{d}$, where*

$$\overline{d} := d_{8\pi+\varepsilon} = \begin{cases} 2 & m \geq 2, \\ 0 & m = 1, \\ -1 & m = 0. \end{cases}$$

In the regular case $m = 0$, Theorem 1.10 was first proved by Chang and Yang in [24] using a min-max scheme. A different proof was later given by Struwe [79] through a geometric flow approach.

1.2 Extremal Functions and Improved Inequalities.

Another interesting problem connected to Moser-Trudinger embeddings consists in studying the existence of extremal functions for (1.3). Indeed, while there is no function realizing equality in (1.1), one can show that the supremum in (1.3) is always attained. This was proved in [20] by Carleson and Chang for the unit disk $D \subseteq \mathbb{R}^2$, and by Flucher ([41]) for arbitrary bounded domains (see also [78] and [57]). The proof of these results is based on a concentration-compactness alternative stated by P. L. Lions ([58]): for a sequence $u_n \in H^1_0(\Omega)$ such that $\|\nabla u_n\|_{L^2(\Omega)} = 1$ one has, up to subsequences, either

$$\int_\Omega e^{4\pi u_n^2} dx \to \int_\Omega e^{4\pi u^2} dx$$

where u is the weak limit of u_n, or u_n concentrates in a point $x \in \overline{\Omega}$, that is

$$|\nabla u|^2 dx \rightharpoonup \delta_x \qquad \text{and} \qquad u_n \rightharpoonup 0. \tag{1.34}$$

The key step in [20] consists in proving that if a sequence of radially symmetric functions $u_n \in H_0^1(D)$ concentrates at 0, then

$$\limsup_{n\to\infty} \int_D e^{4\pi u_n^2} dx \le \pi(1+e). \tag{1.35}$$

Since for the unit disk the supremum in (1.3) is strictly grater than $\pi(1+e)$, (1.35) excludes concentration for maximizing sequences and yields existence of extremal functions for (1.3). In [41] Flucher observed that concentration at arbitrary points of a general domains Ω can always be reduced, through properly defined rearrangements, to concentration of radially symmetric functions on the unit disk. In particular he proved that if $u_n \in H_0^1(\Omega)$ satisfies $\|\nabla u_n\|_2 = 1$ and (1.34), then

$$\limsup_{n\to\infty} \int_\Omega e^{4\pi u_n^2} dx \le \pi e^{1+4\pi A_\Omega(x)} + |\Omega|. \tag{1.36}$$

where $A_\Omega(x)$ is the Robin function of Ω. He also proved

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega e^{4\pi u^2} dx > \pi e^{1+4\pi \max_{\overline{\Omega}} A} + |\Omega|, \tag{1.37}$$

which implies the existence of extremals for (1.3) on Ω. With similar techniques Li [53] proved existence of externals for (1.6) on compact surfaces (see also [54], [52]).

In Chapter 3 we will study Moser-Trudinger type inequalities in the presence of singular potentials. The simplest case is given by the singular metric $|x|^{2\alpha}|dx|^2$ on a bounded domain $\Omega \subset \mathbb{R}^2$ containing 0. In [2] Adimurthi and Sandeep proved that $\forall\, \alpha \in (-1, 0]$,

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx < +\infty \tag{1.38}$$

and

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega |x|^{2\alpha} e^{\beta u^2} dx = +\infty \tag{1.39}$$

if $\beta > 4\pi(1+\alpha)$. Existence of extremals for (1.38) has been proved in [35] and [34]. As for the case $\alpha = 0$, one can exclude concentration of maximizing sequences using the following estimate, which can be obtained from (1.35) using a clever change of variables (see [2], [35]).

Theorem 1.12. *Let $u_n \in H_0^1(D)$ be such that $\int_D |\nabla u_n|^2 \le 1$ and $u_n \rightharpoonup 0$ in $H_0^1(D)$, then $\forall\, \alpha \in (-1, 0]$ we have*

$$\limsup_{n\to\infty} \int_D |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \le \frac{\pi(1+e)}{1+\alpha}. \tag{1.40}$$

We will show that that (1.35) and (1.40) can be obtained from the singular Onofri-type inequalities proved in Chapter 2. More precisely we will deduce Theorem 1.12 from the following sharp inequality for the unit disk, that is a consequence of Theorem 1.5.

Theorem 1.13. *$\forall\, \alpha \in (-1, 0]$, $u \in H_0^1(D)$ we have*

$$\log\left(\frac{1+\alpha}{\pi}\int_D |x|^{2\alpha} e^u dx\right) \le \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx + 1.$$

We stress that our proof of Theorem 1.12 will not require (1.35), but will rather give a simplified version of its original proof in [20].

Theorem 1.12 can be used to prove existences of extremals for several generalized versions of (1.3). In (1.41) Adimurthi and Druet proved that

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega e^{4\pi u^2(1+\lambda\|u\|^2_{L^2(\Omega)})} dx < +\infty \tag{1.41}$$

for any $\lambda < \lambda(\Omega)$, where $\lambda(\Omega)$ is the first eigenvalue of $-\Delta$ with respect to Dirichlet boundary conditions. This bound on λ is sharp, that is

$$\sup_{u\in H_0^1(\Omega), \int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega e^{4\pi u^2(1+\lambda(\Omega)\|u\|^2_{L^2(\Omega)})} dx = \infty. \tag{1.42}$$

Similar inequalities have been proved for compact surfaces on the space $\mathcal{H}$ in [88] and [59], where the authors also prove existence of an extremal function for sufficiently small λ, again by excluding concentration for maximizing sequences. We refer the reader to [82], [89], [13] and references therein for further improved inequalities.

Using Theorem 1.12 as a local model in the analysis of concentration phenomena, we will combine (1.38) with (1.41) and the results, in [88], [59] proving an Adimurthi-Druet type inequality in the presence of singular weights. Given a smooth, closed Riemannian surface (Σ, g), and a finite number of points $p_1, \dots, p_m \in \Sigma$ we will consider functionals of the form

$$E^{\beta,\lambda,q}_{\Sigma,h}(u) := \int_\Sigma h e^{\beta u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g \tag{1.43}$$

where $\lambda, \beta \ge 0$, $q > 1$ and $h \in C^0(\Sigma\backslash\{p_1, \dots, p_m\})$ is a positive function satisfying (1.19). If $\lambda = 0$ we know by (1.20) that

$$\sup_{u\in\mathcal{H}} E^{\beta,0,q}_{\Sigma,h} < +\infty \qquad \Longleftrightarrow \qquad \beta \le 4\pi(1+\overline{\alpha}) \tag{1.44}$$

where $\overline{\alpha} = \min\left\{0, \min_{1\le i\le m} \alpha_i\right\}$. For $m = 0$ and $K \equiv 1$, $E^{\beta,\lambda,q}_{\Sigma,h}$ corresponds to the functional studied in [59]. In particular, one has

$$\sup_{u\in\mathcal{H}} E^{4\pi,\lambda,q}_{\Sigma,1} < +\infty \quad \Longleftrightarrow \quad \lambda < \lambda_q(\Sigma, g), \tag{1.45}$$

where

$$\lambda_q(\Sigma, g) := \inf_{u\in\mathcal{H}} \frac{\int_\Sigma |\nabla u|^2 dv_g}{\|u\|^2_{L^q(\Sigma,g)}}.$$

We will generalize the techniques used in [1], [59] and [88] to the singular case, proving the following singular version of (1.45):

Theorem 1.14. *Let (Σ, g) be a smooth, closed, surface. If $h \in C^0(\Sigma \backslash \{p_1, \ldots, p_m\})$ is a positive function satisfying (1.19), then $\forall\, \beta \in [0, 4\pi(1+\overline{\alpha})]$ and $\lambda \in [0, \lambda_q(\Sigma, g))$ we have*

$$\sup_{u \in \mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h}(u) < +\infty,$$

and supremum is attained if $\beta < 4\pi(1+\overline{\alpha})$ or if $\beta = 4\pi(1+\overline{\alpha})$ and λ is sufficiently small. Moreover

$$\sup_{u \in \mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h}(u) = +\infty$$

for $\beta > 4\pi(1+\overline{\alpha})$, or $\beta = 4\pi(1+\overline{\alpha})$ and $\lambda > \lambda_q(\Sigma, g)$.

In particular, for $\lambda = 0$ we always obtain existence of extremals for the singular functional $E^{\beta,0,q}_{\Sigma,h}$. In Theorem 1.14, it is possible to replace $\mathcal{H}$, $\|\cdot\|_{L^2(\Sigma,g)}$ and $\lambda_q(\Sigma, g)$ with H_{g_h}, $\|\cdot\|_{L^q(\Sigma,g_h)}$ and $\lambda_q(\Sigma, g_h)$, where $g_h := hg$. Thus we obtain an Adimurthi-Druet type inequality on compact surfaces with conical singularities.

Theorem 1.15. *Let (Σ, g) be a closed surface with conical singularities of order $\alpha_1, \ldots, \alpha_m > -1$ in $p_1, \ldots, p_m \in \Sigma$. Then for any $0 \le \lambda < \lambda_q(\Sigma, g)$ we have*

$$\sup_{u \in \mathcal{H}} \int_\Sigma e^{4\pi(1+\overline{\alpha})u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g < +\infty,$$

and the supremum is attained for $\beta < 4\pi(1+\overline{\alpha})$ or for $\beta = 4\pi(1+\overline{\alpha})$ and sufficiently small λ. Moreover

$$\sup_{u \in \mathcal{H}} \int_\Sigma e^{\beta u^2(1+\lambda\|u\|^2_{L^q(\Sigma,g)})} dv_g = +\infty,$$

if $\beta > 4\pi(1+\overline{\alpha})$ or $\beta = 4\pi(1+\overline{\alpha})$ and $\lambda > \lambda_q(\Sigma, g)$.

As in [53], [88] and [59], our technique can be adapted to treat the case of compact surfaces with boundary.

1.3 Systems of Liouville-type Equations.

Let (Σ, g) be a smooth, closed Riemannian surface. We consider Systems of Liouville-type equations of the form

$$-\Delta_g u_i = \sum_{j=1}^N a_{ij}\rho_j \left(\frac{K_j e^{u_j}}{\int_\Sigma K_j e^{u_j} dv_g} - \frac{1}{|\Sigma|} \right) - 4\pi \sum_{j=1}^m \alpha_{ij} \left(\delta_{p_j} - \frac{1}{|\Sigma|} \right) \qquad i = 1, \ldots, N, \quad (1.46)$$

where A is a $N \times N$ symmetric positive definite matrix, $\rho_i > 0$, $0 < K_i \in C^\infty(\Sigma)$, $\alpha_{ij} > -1$, $p_j \in \Sigma$. One of the most important cases is

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix} \tag{1.47}$$

when (1.46) is known as the $SU(N+1)$ Toda system. This system is widely studied in both geometry (description of holomorphic curves in $\mathbb{CP}^N$, see e.g. [16], [19], [32]) and mathematical physics (non-abelian Chern-Simons vortices theory, see [40], [81], [87]. Note that for $N = 1$ (1.46) coincides with (1.14).

As in the scalar case, it is convenient to write the system (1.46) in an equivalent form through the change of variables

$$u_i \to u_i + 4\pi \sum_{j=1}^{m} \alpha_{ij} G(\cdot, p_j). \tag{1.48}$$

The new u_i's solve

$$-\Delta_g u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} dv_g} - \frac{1}{|\Sigma|} \right) \qquad i = 1, \dots, N. \tag{1.49}$$

with

$$h_i = K_i \prod_{j=1}^{m} e^{-4\pi\alpha_{ij} G_{p_j}} \qquad \Rightarrow \qquad h_i \approx d(\cdot, p_j)^{2\alpha_{ij}} \quad \text{near } p_j.$$

We can associate to (1.49) the functional

$$J_{\underline{\rho}}(\underline{u}) := \frac{1}{2} \int_\Sigma \sum_{i,j=1}^{N} a^{ij} \nabla u_i \cdot \nabla u_j \, dv_g - \sum_{i=1}^{N} \rho_i \log \left(\int_\Sigma h_i e^{u_i - \overline{u}_i} dv_g \right)$$

where a^{ij} are the coefficients of A^{-1}. In Chapter 4 we will address two main problems. The first one consists in finding lower bounds for $J_{\underline{\rho}}$. In the regular case $\alpha_{ij} = 0$ Jost and Wang [45] proved that, for the special case of the matrix (1.47), one has

$$\inf_{H^1(\Sigma)^N} J_{\underline{\rho}} > -\infty \qquad \Longleftrightarrow \qquad \rho_i \leq 4\pi \quad \text{for} \quad i = 1, \dots, N.$$

General systems were considered in [77] and [76], using a dual approach first introduced in [86] and [33] for the equivalent problem on bounded domains of $\mathbb{R}^2$. Specifically, in [76] a necessary and sufficient condition for the boundedness of $J_{\overline{\rho}}$ is proved for matrices A satisfying the following condition: there exists $I_1, \dots I_k \subseteq \{1, \dots, N\}$ such that $\{1, \dots, N\} = I_1 \sqcup \cdots \sqcup I_k$ and

$$a_{ij} \geq 0 \quad \text{for } i, j \in I_l,\ l = 1, \dots, k \qquad \text{and} \qquad a_{ij} \leq 0 \quad \text{if } i \in I_l,\ j \in I_s \text{ with } l \neq s. \tag{1.50}$$

Note that (1.50) is satisfied by the matrix (1.47) and by any positive definite 2×2 matrix. For any $I \subseteq \{1, \ldots, N\}$ we consider the polynomial

$$\Lambda_I(y_1, \ldots, y_N) = 8\pi \sum_{i \in I} y_i - \sum_{i,j \in I} a_{ij} y_i y_j. \tag{1.51}$$

If A is positive definite and satisfies (1.50), then (see [76])

$$\inf_{H^1(\Sigma)^N} J_{\underline{\rho}} > -\infty \qquad \Longleftrightarrow \qquad \Lambda_I(\underline{\rho}) \geq 0 \qquad \forall\, I \subseteq \{1, \ldots, N\}.$$

In the singular case, sharp Moser-Trudinger type inequalities for the $SU(3)$ Toda System were proved in [12].

Here we consider the class of positive definite matrices satisfying (1.50) with $k = N$, that is

$$a_{ij} \leq 0 \qquad \text{for } i \neq j. \tag{1.52}$$

Generalizing the dual approach to the singular case we will give a simple proof of the following Moser-Trudinger inequality:

Theorem 1.16. *Let A be a symmetric positive definite matrix satisfying (1.52), then*

$$\inf_{H^1(\Sigma)^N} J_{\underline{\rho}} > -\infty \qquad \Longleftrightarrow \qquad \rho_i \leq 8\pi \left(1 + \min\left\{0, \min_{1 \leq j \leq m} \alpha_{ij}\right\}\right) \qquad i = 1, \ldots N. \tag{1.53}$$

Moreover $J_{\underline{\rho}}$ has a minimum point if

$$\rho_i < 8\pi \left(1 + \min\left\{0, \min_{1 \leq j \leq m} \alpha_{ij}\right\}\right) \qquad i = 1, \ldots N.$$

We stress that a different proof of Theorem 1.16 has been recently given by Luca Battaglia in [9]. In the same paper he also treated arbitrary positive definite matrices introducing the polynomials

$$\Lambda_{I,x}(y_1, \ldots, y_N) := 8\pi \sum_{i \in I} (1 + \alpha_i(x)) y_i - \sum_{i,j \in I} a_{ij} y_i y_j \tag{1.54}$$

where $x \in \Sigma$ and $\alpha_i(x) = 0$ if $x \in \Sigma \backslash \{p_1, \ldots, p_m\}$ and $\alpha_i(p_j) = \alpha_{ij}$, $j = 1, \ldots, m$. He proved

$$\inf_{x \in \Sigma, I \subseteq \{1, \ldots, N\}} \Lambda_I(\underline{\rho}) > 0 \qquad \Longrightarrow \qquad \inf_{H^1(\Sigma)^N} J_{\underline{\rho}} > -\infty, \tag{1.55}$$

and

$$\inf_{x \in \Sigma, I \subseteq \{1, \ldots, N\}} \Lambda_I(\underline{\rho}) < 0 \qquad \Longrightarrow \qquad \inf_{H^1(\Sigma)^N} J_{\underline{\rho}} = -\infty. \tag{1.56}$$

Observe that if (1.52) holds, then

$$\inf_{x \in \Sigma, I \subseteq \{1, \ldots, N\}} \Lambda_I(\underline{\rho}) \geq 0 \qquad \Longleftrightarrow \qquad \rho_i \leq 8\pi \left(1 + \min\left\{0, \min_{1 \leq j \leq m} \alpha_{ij}\right\}\right) \qquad i = 1, \ldots N,$$

and

$$\inf_{x\in\Sigma, I\subseteq\{1,\ldots,N\}} \Lambda_I(\underline{\rho}) > 0 \qquad \Longleftrightarrow \qquad \rho_i < 8\pi\left(1+\min\left\{0, \min_{1\leq j\leq m}\alpha_{ij}\right\}\right) \qquad i=1,\ldots N,$$

therefore (1.55), (1.56) generalize Theorem 1.16.

The second problem we will address, is the analysis of concentration and blow-up phenomena for (1.49). In the same spirit of Theorem 1.2, we will prove, still assuming (1.52), a concentration-compactness alternative for sequences of solutions of (1.49). Our analysis is particularly relevant in the case $N=2$ and

$$A = (a_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{1.57}$$

because it can be combined with mass-quantization results. For the regular case, Jost, Lin and Wang [44] proved:

Theorem B. *Assume* (1.57) *and $\alpha_{ij}=0$ for any i,j. Let $u_n=(u_{1,n},u_{2,n})\in H_0\times H_0$ be a sequence of solutions of* (1.49) *with $\rho_i=\rho_{i,n}\longrightarrow\overline{\rho}_i$ and define, for $x\in\Sigma$, $\sigma_i(x)$ as*

$$\sigma_i(x) := \lim_{r\to 0}\lim_{n\to+\infty}\rho_{i,n}\frac{\int_{B_r(x)} h_i e^{u_{i,n}}dv_g}{\int_\Sigma h_i e^{u_{i,n}}dv_g} \qquad i=1,\ldots,N. \tag{1.58}$$

Then,

$$(\sigma_1(x),\sigma_2(x)) \in \{(0,0),(0,4\pi),(4\pi,0),(4\pi,8\pi),(8\pi,4\pi),(8\pi,8\pi)\}. \tag{1.59}$$

In the same paper, the authors stated that Theorem *B* immediately implies the following compactness result.

Theorem 1.17. *Suppose $\alpha_{ij}=0$ for any i,j and let Λ_1,Λ_2 be compact subsets of $\mathbb{R}^+\backslash 4\pi\mathbb{N}$. Then, the space of solutions in H_0 of* (1.49) *with $\rho_i\in K_i$ is compact in $H^1(\Sigma)$.*

Theorem 1.17 is a necessary step to find solutions of (1.46) by variational methods, as was done in [11], [63], [64]. Although Theorem 1.17 has been widely used during the last years, it was not explicitly proved how it follows from Theorem *B*. Recently, in [55], a proof was given in the case $\rho_1<8\pi$. The purpose of the last part of Chapter 4 is to give a complete proof of Theorem 1.17. Actually, the proof follows quite directly from [23].

Our arguments, which were presented in [14], also work in the presence of singularities. In this case, an analogue of Theorem C was proved in [56].

Theorem C. *Assume* (1.57) *and let $u_n=(u_{1,n},u_{2,n})\in H_0\times H_0$ be a sequence of solutions of* (1.49) *with $\rho_i=\rho_{i,n}$. If $\sigma_1(x),\sigma_2(x)$ are defined as in* (1.58) *we have $(\sigma_1(x),\sigma_2(x))\in\Gamma$ where*

$$\Gamma = \Gamma_0 \cup \bigcup_{k=1}^{\infty}\Gamma_k^1\cup\Gamma_k^2 \tag{1.60}$$

with

$$\Gamma_0 = \Gamma_0^i = \{(0,0),\ (4\pi(1+\alpha_1(x)),0),\ (0,4\pi(1+\alpha_2(x)),\ (4\pi(1+\alpha_1(x)),4\pi(2+\alpha_1(x)+\alpha_2(x))),$$

$$(4\pi(2+\alpha_1(x)+\alpha_2(x)), 4\pi(1+\alpha_2(x))),\ (4\pi(2+\alpha_1(x)+\alpha_2(x)), 4\pi(2+\alpha_1(x)+\alpha_2(x)))\},$$

$$\Gamma_k^1 = \{(y_1, y_2) \in E : y_1 = x_1 + 4n\pi, y_2 \geq x_2,\ (x_1, x_2) \in \Gamma_{k-1}^1 \cup \Gamma_{k-1}^2,\ n \in \mathbb{N}\}$$

$$\Gamma_k^2 = \{(y_1, y_2) \in E : y_2 = x_2 + 4n\pi, y_1 \geq x_1,\ (x_1, x_2) \in \Gamma_{k-1}^1 \cup \Gamma_{k-1}^2,\ n \in \mathbb{N}\}$$

and

$$E = \{(y_1, y_2)\ :\ \Lambda_{\{1,2\},x}(y_1, y_2) = 0\}.$$

Theorem C gives a finite number of possible values for the local blow-up masses $(\sigma_1(x), \sigma_2(x))$. We will show that this quantization result implies compactness of solutions outside a closed, zero-measure set of $\mathbb{R}^{+2}$.

Theorem 1.18. *There exist two discrete subsets $\Gamma_1, \Gamma_2 \subset \mathbb{R}^+$, depending only on the α_{ij}'s, such that for any $\Lambda_i \subset\subset \mathbb{R}^+ \backslash \Gamma_i$, the space of solutions in H_0 of (1.49) with $\rho_i \in \Lambda_i$ is compact in $H^1(\Sigma)$.*

As in the regular case, Theorem 1.18 has important applications in the variational analysis of (1.46), see for instance [11], [10].

Chapter 2

Onofri Type Inequalities for Singular Liouville Equations

In this Chapter we study singular Onofri-Type Inequalities on S^2. Onofri's original proof of Theorem A was based on the conformal invariance of the Moser-Trudinger functional and on an improved inequality proved by Aubin [4]. Another proof was later given by Beckner [15] using a duality principle similar to the one presented in section 4.1. Similar arguments might work also in the presence of singularities when J_ρ is conformal invariant, that is when $\overline{\rho} = \rho_{geom}$ (see (1.15)). Here, however we present a different approach based on blow-up analysis for sequences of solutions of the Liouville equation (1.17) which can be applied also if J_ρ does not have good geometric properties.

In the first part of the Chapter we will work on an arbitrary smooth compact, connected, Riemannian surface (Σ, g). We will fix $p_1, \ldots, p_m \in \Sigma$ and consider a function h satisfying (1.18) with $K \in C^\infty(\Sigma)$, $K > 0$ and $\alpha_i \in (-1, +\infty)\setminus\{0\}$. In order to distinguish the singular points of h from the regular ones, we introduce a singularity index function

$$\alpha(p) := \begin{cases} \alpha_i & \text{if } p = p_i \\ 0 & \text{if } p \notin S \end{cases} . \tag{2.1}$$

We will denote $\overline{\alpha} := \min_{p\in\Sigma} \alpha(p) = \min\left\{0, \min_{1\le i\le m} \alpha_i\right\}$ the minimum singularity order. We shall consider the functional

$$J_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla_g u|^2 dv_g + \frac{\rho}{|\Sigma|}\int_\Sigma u\, dv_g - \rho \log\left(\frac{1}{|\Sigma|}\int_\Sigma h e^u dv_g\right). \tag{2.2}$$

Our goal is to give a sharp version of (1.22) finding the explicit value of

$$C(\Sigma, g, h) = -\frac{1}{8\pi(1+\overline{\alpha})} \inf_{u\in H^1(\Sigma)} J_{8\pi(1+\overline{\alpha})}(u). \tag{2.3}$$

To simplify the notation we will set $\overline{\rho} := 8\pi(1+\overline{\alpha})$, $\rho_\varepsilon = \overline{\rho} - \varepsilon$, $J_\varepsilon := J_{\rho_\varepsilon}$ and $J := J_{\overline{\rho}}$. From (1.23) it follows that $\forall\, \varepsilon > 0$ there exists a function $u_\varepsilon \in H^1(\Sigma)$ satisfying

$$J_\varepsilon(u_\varepsilon) = \inf_{u\in H^1(\Sigma)} J_\varepsilon(u) \tag{2.4}$$

and

$$-\Delta_g u_\varepsilon = \rho_\varepsilon \left(\frac{h e^{u_\varepsilon}}{\int_\Sigma h e^{u_\varepsilon} dv_g} - \frac{1}{|\Sigma|} \right). \tag{2.5}$$

Since J_ε is invariant under addition of constants $\forall\, \varepsilon > 0$, we may also assume

$$\int_\Sigma h\, e^{u_\varepsilon} dv_g = 1. \tag{2.6}$$

In the first section of this Chapter we will state some preliminary Lemmas and, assuming nonexistence of minima of $J_{\overline{\rho}}$, we will describe the blow-up behavior of u_ε. These results will be used in Section 2.2 to give in an estimate from below of

$$\liminf_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon).$$

In Section 2.3 we will prove the sharpness of this estimate and complete the proof of Theorem 1.2. In the remaining two sections we will discuss the case of the sphere. In section 2.4 we will prove a generalized Kazdan-Warner identity and give some nonexistence results for (1.17). As a consequence we will then prove Theorems 1.3 and 1.4. Theorem 1.5 will also be proved in section 2.4 using the conformal invariance of the functional $J_{\overline{\rho}}$. The case of positive order singularities will be treated separately in Section 2.5, where we give the proof of Theorems 1.6-1.11. Due to the lack conformal invariance and the ineffectiveness of the Kazdan-Warner identity, this case will require different techniques. Theorem 1.6 will be deduced form the standard Onofri's inequality (Theorem A). Theorem 1.7 will follow from an improved inequality for radially symmetric functions (Lemma 2.9). As a consequence we will obtain a multiplicity result for equation (1.17) with $\rho \in (8\pi - \varepsilon_0, 8\pi)$. This is particularly interesting since in this range the Leray-Schauder degree is equal to 1. Theorems 1.8, 1.9, 1.10, 1.11 will be proved using the estimates in [26], [28] and the formula (1.29).

2.1 Preliminaries and Blow-up Analysis

In this section we consider a family $u_\varepsilon \in H^1(\Sigma)$ satisfying (2.4), (2.5), (2.6).

Lemma 2.1. *$u_\varepsilon \in C^{0,\gamma}(\Sigma) \cap W^{1,s}(\Sigma)$ for some $\gamma \in (0,1)$ and $s > 2$.*

Proof. It is easy to see that $h \in L^q(\Sigma)$ for some $q > 1$ ($q = +\infty$ if $\alpha = 0$ and $q < -\frac{1}{\alpha}$ for $\alpha < 0$). Applying locally Remarks 2 and 5 in [18] one can show that $u_\varepsilon \in L^\infty(\Sigma)$ so $-\Delta u_\varepsilon \in L^q(\Sigma)$ and by standard elliptic estimates $u_\varepsilon \in W^{2,q}(\Sigma)$. Since $q > 1$ the conclusion follows by Sobolev's embedding theorems. □

The behaviour of u_ε is described by Theorem 1.2. More precisely we will use the following more general concentration-compactness alternative:

Proposition 2.1. *Let u_n be a sequence satisfying*

$$-\Delta_g u_n = V_n e^{u_n} - \psi_n$$

and

$$\int_\Sigma V_n e^{u_n} dv_g \leq C,$$

where $\|\psi_n\|_{L^s(\Sigma)} \leq C$ for some $s > 1$, and

$$V_n = K_n \prod_{1\leq i\leq m} e^{-4\pi\alpha_i G_{p_i}}$$

with $K_n \in C^\infty(\Sigma)$, $0 < a \leq K_n \leq b$ and $\alpha_i > -1$, $i = 1, \dots, m$. Then there exists a subsequence u_{n_k} of u_n such that one of the following holds:

i. u_{n_k} is uniformly bounded in $L^\infty(\Sigma)$;

ii. $u_{n_k} \longrightarrow -\infty$ uniformly on Σ;

iii. there exist a finite blow-up set $B = \{q_1, \dots, q_l\} \subseteq \Sigma$ and a corresponding family of sequences $\{q_k^j\}_{k\in\mathbb{N}}$, $j = 1, \dots l$ such that $q_k^j \overset{k\to\infty}{\longrightarrow} q_j$ and $u_{n_k}(q_k^j) \overset{k\to\infty}{\longrightarrow} +\infty$ $j = 1, \dots, l$. Moreover $u_{n_k} \overset{k\to\infty}{\longrightarrow} -\infty$ uniformly on compact subsets of $\Sigma\backslash B$ and $V_{n_k} e^{u_{n_k}} \rightharpoonup \sum_{j=1}^l \sigma_j \delta_{q_j}$ weakly in the sense of measures where $\sigma_j = 8\pi(1 + \alpha(q_j))$ for $j = 1, \dots, l$.

A proof of Proposition 2.1 in the regular case can be found in [50] while the general case is a consequence of the results in [5] and [8]. A unified proof can be given following the arguments presented in Sections 4.2, 4.3. In our analysis we will also need the following local version of Proposition 2.1 proved by Li and Shafrir ([51]):

Proposition 2.2. *Let Ω be an open domain in $\mathbb{R}^2$ and v_n be a sequence satisfying $\|e^{v_n}\|_{L^1(\Omega)} \leq C$ and*

$$-\Delta v_n = V_n e^{v_n}$$

where $0 \leq V_n \in C_0(\overline{\Omega})$ and $V_n \longrightarrow V$ uniformly in $\overline{\Omega}$. If v_n is not uniformly bounded from above on compact subsets of Ω, then $V_n e^{v_n} \rightharpoonup 8\pi \sum_{i=1}^{l} m_j \delta_{q_j}$ as measures, with $q_j \in \Omega$ and $m_j \in \mathbb{N}^+$, $j = 1, \dots, l$.

Applying Proposition 2.1 to u_ε under the additional condition (2.6) we obtain that either u_ε is uniformly bounded in $L^\infty(\Sigma)$ or its blows-up set contains a single point p such that $\alpha(p) = \overline{\alpha}$. In the first case, one can use elliptic estimates to find uniform bounds on u_ε in $W^{2,q}(\Sigma)$, for some $q > 1$; consequently, a subsequence of u_ε converges in $H^1(\Sigma)$ to a function $u \in H^1(\Sigma)$ that is a minimum point of J and a solution of (1.17) for $\rho = \overline{\rho}$. We now focus on the second case, that is

$$\lambda_\varepsilon := \max_{\Sigma} u_\varepsilon = u_\varepsilon(p_\varepsilon) \longrightarrow +\infty \quad \text{and} \quad p_\varepsilon \longrightarrow p \quad \text{with} \quad \alpha(p) = \overline{\alpha}. \tag{2.7}$$

In the following $G(x,y)$ will denote the Green's function defined in (1.16). It will also be convenient to set $G_x(y) := G(x,y)$. By Proposition 2.1 we also get:

Lemma 2.2. *If u_ε satisfies (2.5), (2.6) and (2.7), then, up to subsequences,*

1. $\rho_\varepsilon h e^{u_\varepsilon} \rightharpoonup \overline{\rho}\, \delta_p$;
2. $u_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} -\infty$ *uniformly in* Ω, $\forall\, \Omega \subset\subset \Sigma \backslash \{p\}$;
3. $\overline{u}_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} -\infty$;
4. *There exist* $\gamma \in (0,1)$, $s > 2$ *such that* $u_\varepsilon - \overline{u_\varepsilon} \overset{\varepsilon \to 0}{\longrightarrow} \overline{\rho}\, G_p$ *in* $C^{0,\gamma}(\overline{\Omega}) \cap W^{1,s}(\Omega)$ $\forall\, \Omega \subset\subset \Sigma \backslash \{p\}$;
5. ∇u_ε *is bounded in* $L^q(\Sigma)$ $\forall\, q \in (1,2)$.

Proof. 1., 2. and 3. are direct consequences of Proposition 2.1. To prove 4. we consider Green's representation formula

$$u_\varepsilon(x) - \overline{u}_\varepsilon = \rho_\varepsilon \int_\Sigma G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y).$$

We stress that Green's function has the following properties:

- $|G(x,y)| \le C_1(1 + |\log d(x,y)|)$ $\forall\, x,y \in \Sigma$, $x \neq y$.
- $|\nabla_g^x G(x,y)| \le \dfrac{C_2}{d(x,y)}$ $\forall\, x,y \in \Sigma$, $x \neq y$.
- $G(x,y) = G(y,x)$ $\forall\, x,y \in \Sigma$, $x \neq y$.

Take $q > 1$ such that $h \in L^q(\Sigma)$. The first property also yields

$$\sup_{x \in \Sigma} \|G_x\|_{L^{q'}(\Sigma)} \le C_3. \tag{2.8}$$

Let us fix $\delta > 0$ such that $B_{3\delta}(p) \subset \Sigma \backslash \Omega$ and take a cut-off function φ such that $\varphi \equiv 1$ in $B_\delta(p)$ and $\varphi \equiv 0$ in $\Sigma \backslash B_{2\delta}(p)$.

$$u_\varepsilon(x) - \overline{u}_\varepsilon = \rho_\varepsilon \int_\Sigma \varphi(y) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) + \rho_\varepsilon \int_\Sigma (1 - \varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y).$$

By (2.8) and 2. we have

$$\left| \int_\Sigma (1 - \varphi(y)) G_x(y) h(y) e^{u_\varepsilon(y)} dv_g(y) \right| \le \int_{\Sigma \backslash B_\delta(p)} |G_x(y)|\, h(y) e^{u_\varepsilon(y)} dv_g(y) \le$$

$$\leq C_3\|h\|_{L^q(\Sigma)}\|e^{u_\varepsilon}\|_{L^\infty(\Sigma\setminus B_\delta(p))} \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

By 1. and the smoothness of φG_x for $x \in \overline{\Omega}$ and $y \in \Sigma$ we get

$$\int_\Sigma \varphi(y)G_x(y)h(y)e^{u_\varepsilon(y)}dv_g(y) \overset{\varepsilon\to 0}{\longrightarrow} \varphi(p)G_x(p) = G_p(x)$$

uniformly for $x \in \Omega$. Similarly we have

$$\nabla_g u_\varepsilon(x) = \rho_\varepsilon \int_\Sigma \varphi(y)\nabla_g^x G_x(y)h(y)e^{u_\varepsilon(y)}dv_g(y) + \rho_\varepsilon \int_\Sigma (1-\varphi(y))\nabla_g^x G_x(y)h(y)e^{u_\varepsilon(y)}dv_g(y)$$

with

$$\int_\Sigma \varphi(y)\nabla_g^x G_x(y)h(y)e^{u_\varepsilon(y)}dv_g(y) \overset{k\to\infty}{\longrightarrow} \nabla_g^x G_p(x)$$

uniformly in Ω and, assuming $q \in (1,2)$, by the Hardy-Littlewood-Sobolev inequality

$$\int_\Sigma \left(\int_\Sigma (1-\varphi(y))\nabla_g^x G_x(y)h(y)e^{u_\varepsilon(y)}dv_g(y)\right)^s dv_g(x) \leq$$

$$\leq C_2^s \int_\Sigma \left(\int_{\Sigma\setminus B_\delta(p)} \frac{h(y)e^{u_\varepsilon(y)}}{d(x,y)}dv_g(y)\right)^s dv_g(x) \leq C\|h\|^s_{L^q(\Sigma)}\|e^{u_n}\|^s_{L^\infty(\Sigma\setminus B_\delta(p))} \overset{\varepsilon\to 0}{\longrightarrow} 0$$

where

$$\frac{1}{s} = \frac{1}{q} - \frac{1}{2}.$$

Note that $q > 1$ implies $s > 2$. Finally, to prove 5., we shall observe that for any $1 < q < 2$ there exists a positive constant C_q such that

$$\int_\Sigma \varphi\, dv_g = 0 \quad \text{and} \quad \int_\Sigma |\nabla_g\varphi|^{q'}dv_g \leq 1 \quad \Longrightarrow \quad \|\varphi\|_\infty \leq C_q.$$

Hence $\forall\, \varphi \in W^{1,q'}(\Sigma)$

$$\int_\Sigma \nabla_g u_\varepsilon \cdot \nabla_g \varphi\, dv_g = -\int_\Sigma \Delta u_\varepsilon \varphi\, dv_g \leq C_q\|\Delta u_\varepsilon\|_{L^1(\Sigma)} \leq \tilde{C}_q$$

so that

$$\|\nabla u_\varepsilon\|_{L^q} \leq \sup\left\{\int_\Sigma \nabla_g u_\varepsilon\cdot\nabla_g\varphi\, dv_g \;:\; \varphi \in W^{1,q'}(\Sigma), \|\nabla\varphi\|_{L^{q'}} \leq 1\right\} \leq \tilde{C}_q.$$

□

We now focus on the behaviour of u_ε near the blow-up point. First we consider the case $\overline{\alpha} < 0$. Let us fix a system of normal coordinates in a small ball $B_\delta(p)$, with p corresponding to 0 and p_ε corresponding to x_ε. We define

$$\varphi_\varepsilon(x) := u_\varepsilon(t_\varepsilon x) - \lambda_\varepsilon, \quad t_\varepsilon := e^{-\frac{\lambda_\varepsilon}{2(1+\overline{\alpha})}}. \tag{2.9}$$

Lemma 2.3. *If* $\overline{\alpha} < 0$, $\dfrac{|x_\varepsilon|}{t_\varepsilon}$ *is bounded.*

Proof. We define

$$\psi_\varepsilon(x) = u_\varepsilon(|x_\varepsilon|x) + 2(1+\overline{\alpha})\log|x_\varepsilon| + s_\varepsilon(|x_\varepsilon|x)$$

where $s_\varepsilon(x)$ is the solution of

$$\begin{cases} -\Delta s_\varepsilon = \frac{\rho_\varepsilon}{|\Sigma|} & \text{in } B_\delta(0) \\ s_\varepsilon = 0 & \text{if } |x| = \delta \end{cases}.$$

The function ψ_ε satisfies

$$-\Delta\psi_\varepsilon = |x_\varepsilon|^{-2\overline{\alpha}}\rho_\varepsilon h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)}e^{\psi_\varepsilon} = V_\varepsilon e^{\psi_\varepsilon}$$

in $B_{\frac{\delta}{|x_\varepsilon|}}(0)$. We stress that, by standard elliptic estimates, s_ε is uniformly bounded in $C^1(\overline{B_\delta})$ and that G_p has the expansion

$$G_p(x) = -\frac{1}{2\pi}\log|x| + A(p) + O(|x|) \tag{2.10}$$

in $B_\delta(0)$. Thus

$$|x_\varepsilon|^{-2\overline{\alpha}}h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)} =$$

$$= |x_\varepsilon|^{-2\overline{\alpha}}e^{2\overline{\alpha}\log(|x_\varepsilon||x|)-4\pi\overline{\alpha}A(p)+O(|x_\varepsilon||x|)}e^{-s_\varepsilon(|x_\varepsilon|x)}K(|x_\varepsilon|x)\prod_{1\le i\le m, p_i\neq p} e^{-4\pi\overline{\alpha}_i G_{p_i}(|x_\varepsilon|x)} =$$

$$= |x|^{2\overline{\alpha}}e^{-4\pi\overline{\alpha}A(p)}e^{O(|x_\varepsilon||x|)}e^{-s_\varepsilon(|x_\varepsilon|x)}K(|x_\varepsilon|x)\prod_{1\le i\le m, p_i\neq p} e^{-4\pi\overline{\alpha}_i G_{p_i}(|x_\varepsilon|x)} = |x|^{2\overline{\alpha}}\tilde{h}(|x_\varepsilon|x)$$

where $\tilde{h} \in C^1(\overline{B_\delta})$. In particular V_ε is uniformly bounded in $C^1_{loc}(\mathbb{R}^2\setminus\{0\})$. If there existed a subsequence such that $\dfrac{|x_\varepsilon|}{t_\varepsilon} \longrightarrow +\infty$ then

$$\psi_\varepsilon\left(\frac{x_\varepsilon}{|x_\varepsilon|}\right) = 2(1+\overline{\alpha})\log\left(\frac{|x_\varepsilon|}{t_\varepsilon}\right) + s_\varepsilon(x_\varepsilon) \longrightarrow +\infty,$$

so $y_0 := \lim_{\varepsilon\to 0}\dfrac{x_\varepsilon}{|x_\varepsilon|}$ would be a blow-up point for ψ_ε. Since $y_0 \neq 0$, applying Proposition 2.2 to ψ_ε in a small ball $B_r(y_0)$ we would get

$$\liminf_{\varepsilon\to 0}\int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon}dx \ge 8\pi.$$

But this would be in contradiction to (2.6) since

$$\int_{B_r(y_0)} V_\varepsilon e^{\psi_\varepsilon}dx = \int_{B_{r(y_0)}} \rho_\varepsilon\, |x_\varepsilon|^{-2\overline{\alpha}}h(|x_\varepsilon|x)e^{-s_\varepsilon(|x_\varepsilon|x)}e^{\psi_\varepsilon}dx \le \rho_\varepsilon\int_{B_\delta(p)} he^{u_\varepsilon}dv_g \le 8\pi(1+\overline{\alpha}) < 8\pi.$$

□

Lemma 2.4. *Assume $\overline{\alpha} < 0$. Then, possibly passing to a subsequence, φ_ε converges uniformly on compact subsets of $\mathbb{R}^2$ and in $H^1_{loc}(\mathbb{R}^2)$ to*

$$\varphi_0(x) := -2\log\left(1 + \frac{\pi c(p)}{1+\overline{\alpha}}|x|^{2(1+\overline{\alpha})}\right)$$

where $c(p) = K(p)e^{-4\pi\overline{\alpha}A(p)} \prod_{1\le i\le m, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(p)}$.

Proof. The function φ_ε is defined in $B_\varepsilon = B_{\frac{\delta}{t_\varepsilon}}(0)$ and satisfies

$$-\Delta\varphi_\varepsilon = t_\varepsilon^2\rho_\varepsilon\left(h(t_\varepsilon x)e^{\varphi_\varepsilon}e^{\lambda_\varepsilon} - \frac{1}{|\Sigma|}\right) = t_\varepsilon^{-2\overline{\alpha}}\rho_\varepsilon h(t_\varepsilon x)e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2\rho_\varepsilon}{|\Sigma|}$$

and

$$t_\varepsilon^{-2\overline{\alpha}}\int_{B_{\frac{\delta}{t_\varepsilon}}} h(t_\varepsilon x)e^{\varphi_\varepsilon} \le 1.$$

As in the previous proof we have

$$t_\varepsilon^{-2\overline{\alpha}}h(t_\varepsilon x) = t_\varepsilon^{-2\overline{\alpha}}e^{2\overline{\alpha}\log(t_\varepsilon|x|) - 4\pi\overline{\alpha}A(p) + O(t_\varepsilon|x|)}K(t_\varepsilon x)\prod_{1\le i\le m, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)} =$$

$$= |x|^{2\overline{\alpha}}e^{-4\pi\overline{\alpha}A(p)}e^{O(t_\varepsilon|x|)}K(t_\varepsilon x)\prod_{1\le i\le m, p_i\neq p} e^{-4\pi\alpha_i G_{p_i}(t_\varepsilon x)} \stackrel{\varepsilon\to 0}{\longrightarrow} c(p)|x|^{2\overline{\alpha}}$$

in $L^q_{loc}(\mathbb{R}^2)$ for some $q > 1$. Fix $R > 0$ and let ψ_ε be the solution of

$$\begin{cases} -\Delta\psi_\varepsilon = t_\varepsilon^{-2\overline{\alpha}}\rho_\varepsilon h(t_\varepsilon x)e^{\varphi_\varepsilon} - \frac{t_\varepsilon^2\rho_\varepsilon}{|\Sigma|} & \text{in } B_R(0) \\ \psi_\varepsilon = 0 & \text{su } \partial B_R(0) \end{cases}.$$

Since $\Delta\psi_\varepsilon$ is bounded in $L^q(B_R(0))$ with $q > 1$, elliptic regularity shows that ψ_ε is bounded in $W^{2,q}(B_R(0))$ and by Sobolev's embeddings we may extract a subsequence such that ψ_ε converges in $H^1(B_R(0)) \cap C^{0,\lambda}(B_R(0))$. The function $\xi_\varepsilon = \varphi_\varepsilon - \psi_\varepsilon$ is harmonic in B_R and bounded from above. Furthermore $\xi_\varepsilon\left(\frac{x_\varepsilon}{t_\varepsilon}\right) = -\psi_\varepsilon\left(\frac{x_\varepsilon}{t_\varepsilon}\right)$ is bounded from below, hence by Harnack inequality ξ_ε is uniformly bounded in $C^2(\overline{B_{\frac{R}{2}}}(0))$. Thus φ_ε is bounded in $W^{2,q}(B_{\frac{R}{2}})$ and we can extract a subsequence converging in $H^1(B_{\frac{R}{2}}) \cap C^{0,\lambda}(B_{\frac{R}{2}})$. Using a diagonal argument we find a subsequence for which φ_ε converges in $H^1_{loc}(\mathbb{R}^2) \cap C^{0,\lambda}_{loc}(\mathbb{R}^2)$ to a function φ_0 solving

$$-\Delta\varphi_0 = 8\pi(1+\overline{\alpha})c(p)|x|^{2\overline{\alpha}}e^{\varphi_0}$$

on $\mathbb{R}^2$ with

$$\int_{\mathbb{R}^2}|x|^{2\overline{\alpha}}e^{\varphi_0(x)}dx < \infty.$$

The classification result in [74] yields

$$\varphi_0(x) = -2\log\left(1 + \frac{\pi e^{\lambda} c(p)}{1+\overline{\alpha}}|x|^{2(1+\overline{\alpha})}\right) + \lambda$$

for some $\lambda \in \mathbb{R}$. To conclude the proof it remains to note that, since 0 is the unique maximum point of φ_0, the uniform convergence of φ_ε implies $\frac{x_\varepsilon}{t_\varepsilon} \longrightarrow 0$ and $\lambda = 0$. □

As in [37], to give a lower bound on $J_\varepsilon(u_\varepsilon)$ we need the following estimate from below for u_ε:

Lemma 2.5. *Fix $R > 0$ and define $r_\varepsilon = t_\varepsilon R$. If $\overline{\alpha} < 0$ and u_ε satisfies (2.5), (2.6), (2.7), then*

$$u_\varepsilon \geq \overline{\rho}\, G_p - \lambda_\varepsilon - \overline{\rho}\, A(p) + 2\log\left(\frac{R^{2(1+\overline{\alpha})}}{1 + \frac{\pi c(p)}{1+\overline{\alpha}} R^{2(1+\overline{\alpha})}}\right) + o_\varepsilon(1)$$

in $\Sigma \backslash B_{r_\varepsilon}(p)$.

Proof. $\forall\, C > 0$ we have

$$-\Delta_g(u_\varepsilon - \overline{\rho}\, G_p - C) = \rho_\varepsilon\left(he^{u_\varepsilon} - \frac{1}{|\Sigma|}\right) + \frac{\overline{\rho}}{|\Sigma|} = \rho_\varepsilon h e^{u_\varepsilon} + \frac{\varepsilon}{|\Sigma|} \geq 0.$$

Let us consider normal coordinates near p. We know that

$$G_p(x) = -\frac{1}{2\pi}\log|x| + A(p) + O(|x|),$$

so by Lemma 2.4 if $x = t_\varepsilon y$ with $|y| = R$ we have

$$u_\varepsilon(x) - \overline{\rho}\, G_p = \varphi_\varepsilon(y) + \lambda_\varepsilon + 4(1+\overline{\alpha})\log(t_\varepsilon R) - \overline{\rho} A(p) + o_\varepsilon(1) =$$

$$= -2\log\left(1 + \frac{\pi c(p)}{1+\overline{\alpha}} R^{2(1+\overline{\alpha})}\right) - \lambda_\varepsilon + \log R^{4(1+\overline{\alpha})} - \overline{\rho}\, A(p) + o_\varepsilon(1).$$

Thus, taking

$$C_\varepsilon = -\lambda_\varepsilon - \overline{\rho}\, A(p) + 2\log\left(\frac{R^{2(1+\overline{\alpha})}}{1 + \frac{\pi c(p)}{1+\overline{\alpha}} R^{2(1+\overline{\alpha})}}\right) + o_\varepsilon(1)$$

we have $u_\varepsilon - \overline{\rho} G_p - C_\varepsilon \geq 0$ on $\partial B_{r_\varepsilon}(p)$ and the conclusion follows from the maximum principle. □

As a consequence we also have

Lemma 2.6. $t_\varepsilon^2 \overline{u}_\varepsilon \longrightarrow 0$.

Proof. By Lemma 2.4

$$\int_{B_{t_\varepsilon}(p)} u_\varepsilon \, dv_g = t_\varepsilon^2 \int_{B_1(0)} \varphi_\varepsilon(y)dy + \lambda_\varepsilon |B_{t_\varepsilon}| = o_\varepsilon(1).$$

and by the previous Lemma

$$\lambda_\varepsilon |\Sigma| \geq \int_{\Sigma \setminus B_{t_\varepsilon}(p)} u_\varepsilon \geq \overline{\rho} \int_{\Sigma \setminus B_{t_\varepsilon}(p)} G_p \, dv_g - \lambda_\varepsilon |\Sigma \setminus B_{t_\varepsilon}(p)| + O(1).$$

Thus $\dfrac{|\overline{u}_\varepsilon|}{\lambda_\varepsilon}$ is bounded and, since $\lambda_\varepsilon t_\varepsilon^2 = o_\varepsilon(1)$, we get the conclusion. □

The case $\overline{\alpha} = 0$ can be studied in a similar way. The main difference is that, since we do not know whether $\frac{|x_\varepsilon|}{t_\varepsilon}$ is bounded, we have to center the scaling in p_ε and not in p. Note that $\alpha(p) = 0$ means that $p \in \Sigma \setminus S$ is a regular point of h.

Lemma 2.7. *Assume that $\overline{\alpha} = 0$ and that u_ε satisfies (2.5), (2.6) and (2.7). In normal coordinates near p define*

$$\psi_\varepsilon(x) = u_\varepsilon(x_\varepsilon + t_\varepsilon x) - \lambda_\varepsilon \qquad \textit{where} \qquad t_\varepsilon = e^{-\frac{\lambda_\varepsilon}{2}}.$$

Then

1. *ψ_ε converges in $C^1_{loc}(\mathbb{R}^2)$ to*

$$\psi_0(x) = -2\log(1 + \pi h(p)|x|^2)$$

2. *$\forall\, R > 0$ one has*

$$u_\varepsilon \geq 8\pi G_{p_\varepsilon} - \lambda_\varepsilon - 8\pi A(p) + 2\log\left(\frac{R^2}{1 + \pi h(p) R^2}\right) + o_\varepsilon(1)$$

   in $\Sigma \setminus B_{R t_\varepsilon}(p_\varepsilon)$;

3. *$t_\varepsilon^2 \overline{u}_\varepsilon \to 0$.*

2.2 A Lower Bound

In this section and in the next one we present the proof of Theorem 1.2. We begin by giving an estimate from below of $\inf\limits_{H^1(\Sigma)} J$. As before we consider u_ε satisfying (2.4), (2.5), (2.6), and (2.7). Again we will focus on the case $\overline{\alpha} < 0$ since the computation for $\overline{\alpha} = 0$ is equivalent to the one in [37]. We consider normal coordinates in a small ball $B_\delta(p)$ and assume that G_p has the expansion (2.10) in $B_\delta(p)$. Let t_ε be defined as in (2.9), then $\forall\, R > 0$ we shall consider the decomposition

$$\int_{\Sigma}|\nabla_g u_\varepsilon|^2 dv_g = \int_{\Sigma\backslash B_\delta(p)}|\nabla_g u_\varepsilon|^2 dv_g + \int_{B_\delta\backslash B_{r_\varepsilon}(p)}|\nabla_g u_\varepsilon|^2 dv_g + \int_{B_{r_\varepsilon}(p)}|\nabla_g u_\varepsilon|^2 dv_g.$$

On $\Sigma\backslash B_\delta(p)$ we can use Lemma 2.2 and an integration by parts to obtain:

$$\begin{aligned}\int_{\Sigma\backslash B_\delta}|\nabla_g u_\varepsilon|^2 dv_g &= \overline{\rho}^2\int_{\Sigma\backslash B_\delta}|\nabla_g G_p|^2 dv_g + o_\varepsilon(1) = \\ &= -\frac{\overline{\rho}^2}{|\Sigma|}\int_{\Sigma\backslash B_\delta} G_p\, dv_g - \overline{\rho}^2\int_{\partial B_\delta} G_p\frac{\partial G_p}{\partial n}\, d\sigma_g + o_\varepsilon(1) = \\ &= -\overline{\rho}^2\int_{\partial B_\delta} G_p\frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) + o_\delta(1). \qquad (2.11)\end{aligned}$$

On $B_{r_\varepsilon}(p)$ the convergence result for the scaling (2.9) stated in Lemma 2.4 yields

$$\begin{aligned}\int_{B_{r_\varepsilon}}|\nabla_g u_\varepsilon|^2 dv_g &= \int_{B_R(0)}|\nabla\varphi_0|^2 dx + o_\varepsilon(1) = 2\overline{\rho}\left(\log\left(1+\frac{\pi\, c(p)}{1+\overline{\alpha}}R^{2(1+\overline{\alpha})}\right)-1\right) + \\ &\quad + o_\varepsilon(1) + o_R(1). \qquad (2.12)\end{aligned}$$

For the remaining term we can use (2.5) and Lemma 2.2 to obtain

$$\begin{aligned}\int_{B_\delta\backslash B_{r_\varepsilon}}|\nabla_g u_\varepsilon|^2 dv_g &= \rho_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon}u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|}\int_{B_\delta\backslash B_{r_\varepsilon}} u_\varepsilon dv_g + \\ &\quad + \int_{\partial B_\delta} u_\varepsilon\frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon\frac{\partial u_\varepsilon}{\partial n} d\sigma_g = \\ &= \rho_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon}u_\varepsilon dv_g - \frac{\rho_\varepsilon}{|\Sigma|}\int_{B_\delta\backslash B_{r_\varepsilon}} u_\varepsilon dv_g + \overline{u}_\varepsilon\int_{\partial B_\delta}\frac{\partial u_\varepsilon}{\partial n} d\sigma_g \\ &\quad - \int_{\partial B_{r_\varepsilon}} u_\varepsilon\frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \overline{\rho}^2\int_{\partial B_\delta} G_p\frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1). \qquad (2.13)\end{aligned}$$

By Lemma 2.5 and (2.6) we get

$$\begin{aligned}\rho_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon}u_\varepsilon dv_g &\geq \rho_\varepsilon\overline{\rho}\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon}G_p dv_g - \rho_\varepsilon\lambda_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g \\ &\quad + O_R(1)\rho_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g = \\ &= \rho_\varepsilon\overline{\rho}\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon}G_p dv_g - \rho_\varepsilon\lambda_\varepsilon\int_{B_\delta\backslash B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g + o_\varepsilon(1). \quad (2.14)\end{aligned}$$

Again by (2.5) and Lemma 2.2

$$
\begin{aligned}
\rho_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} G_p dv_g &= \int_{B_\delta \setminus B_{r_\varepsilon}} G_p \left(-\Delta u_\varepsilon + \frac{\rho_\varepsilon}{|\Sigma|} \right) dv_g = \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \int_{\partial B_\delta} u_\varepsilon \frac{\partial G_p}{\partial n} - G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \qquad (2.15) \\
&\quad + \int_{\partial B_{r_\varepsilon}} G_p \frac{\partial u_\varepsilon}{\partial n} - u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g + o_\delta(1) = \\
&= -\frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g + \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g + \\
&\quad + \int_{\partial B_{r_\varepsilon}} G_p \frac{\partial u_\varepsilon}{\partial n} d\sigma_g - \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g + \ o_\varepsilon(1) + o_\delta(1), \qquad (2.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho_\varepsilon \lambda_\varepsilon \int_{B_\delta \setminus B_{r_\varepsilon}} h e^{u_\varepsilon} dv_g &= -\lambda_\varepsilon \int_{\partial B_\delta \setminus B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} \left(Vol(B_\delta) - Vol(B_{r_\varepsilon}) \right) = \qquad (2.17) \\
&= -\lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_{r_\varepsilon}} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) + o_\varepsilon(1).
\end{aligned}
$$

Using (2.13), (2.14), (2.15) and (2.17) we get

$$
\begin{aligned}
\int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g \ \geq\ & -(16\pi(1+\overline{\alpha}) - \varepsilon) \frac{1}{|\Sigma|} \int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon \, dv_g - \frac{\rho_\varepsilon \lambda_\varepsilon}{|\Sigma|} Vol(B_\delta) + \\
& + \overline{\rho}\, \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g + \lambda_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g + \\
& + \overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g - \overline{\rho} \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g + \qquad (2.18) \\
& - \int_{\partial B_{r_\varepsilon}} \left(u_\varepsilon - \overline{\rho}\, G_p + \lambda_\varepsilon \right) \frac{\partial u_\varepsilon}{\partial n} + o_\varepsilon(1) + o_\delta(1).
\end{aligned}
$$

By Lemmas 2.2 and 2.6 we can say that

$$
\int_{B_\delta \setminus B_{r_\varepsilon}} u_\varepsilon dv_g = \int_{B_\delta \setminus B_{r_\varepsilon}} (u_\varepsilon - \overline{u}_\varepsilon) dv_g + \overline{u}_\varepsilon (Vol(B_\delta) - Vol(B_{r_\varepsilon})) = \overline{u}_\varepsilon Vol(B_\delta) + o_\delta(1) + o_\varepsilon(1).
$$

Using Green's formula

$$
\overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial G_p}{\partial n} d\sigma_g = -\overline{u}_\varepsilon \int_{\Sigma \setminus B_\delta} \Delta_g G_p \, dv_g = -\overline{u}_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|} \right).
$$

Similarly

$$\int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g = -\int_{\Sigma\setminus B_\delta} \Delta u_\varepsilon \, dv_g = \int_{\Sigma\setminus B_\delta} \rho_\varepsilon \left(he^{u_\varepsilon} - \frac{1}{|\Sigma|}\right) dv_g \geq -\rho_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|}\right)$$

and

$$\begin{aligned} \overline{u}_\varepsilon \int_{\partial B_\delta} \frac{\partial u_\varepsilon}{\partial n} d\sigma_g &= \overline{u}_\varepsilon \rho_\varepsilon e^{\overline{u}_\varepsilon} \int_{\Sigma\setminus B_\delta(p)} h \, e^{u_\varepsilon - \overline{u}_\varepsilon} dv_g - \overline{u}_\varepsilon \rho_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|}\right) = \\ &= -\overline{u}_\varepsilon \rho_\varepsilon \left(1 - \frac{Vol(B_\delta)}{|\Sigma|}\right) + o_\varepsilon(1). \end{aligned}$$

Lemma 2.4 yields

$$\begin{aligned} \int_{\partial B_{r_\varepsilon}} u_\varepsilon \frac{\partial G_p}{\partial n} d\sigma_g &= \lambda_\varepsilon \int_{\partial B_{r_\varepsilon}} \frac{\partial G_p}{\partial n} d\sigma_g + t_\varepsilon \int_{\partial B_R(0)} \varphi_\varepsilon \frac{\partial G_p}{\partial n}(t_\varepsilon x)(1 + o_\varepsilon(1)) d\sigma = \\ &= -\lambda_\varepsilon \left(1 - \frac{Vol(B_{r_\varepsilon})}{|\Sigma|}\right) + t_\varepsilon \int_{\partial B_R(0)} \varphi_0 \left(-\frac{1}{2\pi t_\varepsilon R} + O(1)\right) d\sigma = \\ &= -\lambda_\varepsilon + 2\log\left(1 + \frac{\pi \, c(p)}{1+\overline{\alpha}} R^{2(1+\overline{\alpha})}\right) + o_\varepsilon(1) \end{aligned}$$

and the estimate in Lemma 2.5 gives

$$\begin{aligned} &-\int_{\partial B_{r_\varepsilon}} \left(u_\varepsilon - \overline{\rho}\, G_p + \lambda_\varepsilon\right) \frac{\partial u_\varepsilon}{\partial n} d\sigma_g \geq \\ &\geq \left(2\log\left(\frac{R^{2(1+\overline{\alpha})}}{1 + \frac{\pi c(p)}{(1+\overline{\alpha})} R^{2(1+\overline{\alpha})}}\right) - \overline{\rho} A(p)\right) \frac{8\pi^2 c(p) R^{2(1+\overline{\alpha})}}{\left(1 + \frac{\pi c(p) R^{2(1+\overline{\alpha})}}{1+\overline{\alpha}}\right)} + o_\varepsilon(1) = \\ &= -\overline{\rho}^2 A(p) - 2\,\overline{\rho}\, \log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) + o_\varepsilon(1) + o_R(1). \end{aligned}$$

Hence

$$\begin{aligned} \int_{B_\delta \setminus B_{r_\varepsilon}} |\nabla_g u_\varepsilon|^2 dv_g &\geq -(16\pi(1+\overline{\alpha}) - \varepsilon)\overline{u}_\varepsilon + \varepsilon\lambda_\varepsilon + \overline{\rho}^2 \int_{\partial B_\delta} G_p \frac{\partial G_p}{\partial n} d\sigma_g + \\ &- 2\overline{\rho}\log\left(1 + \frac{\pi c(p)}{1+\overline{\alpha}} R^{2(1+\overline{\alpha})}\right) - \overline{\rho}^2 A(p) - 2\overline{\rho}\log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) + \\ &+ o_\varepsilon(1) + o_\delta(1) + o_R(1). \end{aligned} \tag{2.19}$$

By (2.11), (2.12) and (2.19) we can therefore conclude

$$\int_\Sigma |\nabla_g u_\varepsilon|^2 dv_g \geq -(16\pi(1+\overline{\alpha})-\varepsilon)\overline{u}_\varepsilon + \varepsilon\lambda_\varepsilon - \overline{\rho}^2 A(p) - 2\overline{\rho}\log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) - 2\overline{\rho} + + o_\varepsilon(1) + o_\delta(1) + o_R(1),$$

so that

$$\begin{aligned} J_\varepsilon(u_\varepsilon) &\geq \frac{\varepsilon}{2}(\lambda_\varepsilon - \overline{u}_\varepsilon) - \frac{\overline{\rho}^2}{2}A(p) - \overline{\rho}\log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) - \overline{\rho} + \rho_\varepsilon \log|\Sigma| + o_\varepsilon(1) + o_\delta(1) + o_R(1) \\ &\geq -\overline{\rho}\left(4\pi(1+\overline{\alpha})A(p) + 1 + \log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) - \log|\Sigma|\right) + o_\varepsilon(1) + o_\delta(1) + o_R(1). \end{aligned}$$

As $\varepsilon, \delta \to 0$ and $R \to \infty$ we obtain

$$\begin{aligned} \inf_{H^1(\Sigma)} J &\geq -\overline{\rho}\left(4\pi(1+\overline{\alpha})A(p) + 1 + \log\left(\frac{\pi c(p)}{1+\overline{\alpha}}\right) - \log|\Sigma|\right) = \\ &= -\overline{\rho}\left(1 + \log\frac{\pi}{|\Sigma|} + 4\pi A(p) + \log\left(\frac{K(p)}{1+\overline{\alpha}}\prod_{q\in S, q\neq p} e^{-4\pi\alpha(q)G_q(p)}\right)\right). \end{aligned} \tag{2.20}$$

Using Lemma 2.7 it is possible to prove that (2.20) holds even for $\overline{\alpha} = 0$. About the blow-up point p we only know that $\alpha(p) = \overline{\alpha}$, so we have proved

Proposition 2.3. *If J has no minimum point, then*

$$\inf_{H^1(\Sigma)} J \geq -\overline{\rho}\left(1 + \log\frac{\pi}{|\Sigma|} + \max_{p\in\Sigma, \alpha(p)=\overline{\alpha}}\left\{4\pi A(p) + \log\left(\frac{K(p)}{1+\overline{\alpha}}\prod_{q\in S, q\neq p} e^{-4\pi\alpha(q)G_q(p)}\right)\right\}\right).$$

Notice that, if $\overline{\alpha} < 0$, the set

$$\{p \in \Sigma \ : \ \alpha(p) = \overline{\alpha}\} = \{p_i \ : \ i \in \{1, \dots, m\}, \ \overline{\alpha}_i = \overline{\alpha}\}$$

is finite, while if $\overline{\alpha} = 0$

$$\{p \in \Sigma \ : \ \alpha(p) = \overline{\alpha}\} = \Sigma \backslash S.$$

Although this set is not finite, the maximum in the above expression is still well defined since the function

$$p \longmapsto 4\pi A(p) + \log\left(K(p)\prod_{q\in S} e^{-4\pi\alpha(q)G_q(p)}\right) = 4\pi A(p) + \log h(p)$$

is continuous on $\Sigma\backslash S$ and approaches $-\infty$ near S.

2.3 An Estimate From Above

In order to complete the proof of Theorem 1.2 we need to exhibit a sequence $\varphi_\varepsilon \in H^1(\Sigma)$ such that

$$J(\varphi_\varepsilon) \longrightarrow -\overline{\rho}\left(1+\log\frac{\pi}{|\Sigma|}+\max_{p\in\Sigma,\alpha(p)=\overline{\alpha}}\left\{4\pi A(p)+\log\left(\frac{K(p)}{1+\overline{\alpha}}\prod_{q\in S,q\neq p}e^{-4\pi\alpha(q)G_q(p)}\right)\right\}\right)$$

Let us define $r_\varepsilon := \gamma_\varepsilon \varepsilon^{\frac{1}{2(1+\overline{\alpha})}}$ where γ_ε is chosen so that

$$\gamma_\varepsilon \to +\infty, \quad r_\varepsilon^2 \log\varepsilon \longrightarrow 0, \quad r_\varepsilon^2\log(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}) \longrightarrow 0. \tag{2.21}$$

Let $p \in \Sigma$ be such that $\alpha(p) = \overline{\alpha}$ and

$$4\pi A(p)+\log\left(\frac{K(p)}{1+\overline{\alpha}}\prod_{q\in S,q\neq p}e^{-4\pi\alpha(q)G_q(p)}\right) =$$

$$= \max_{\xi\in\Sigma,\alpha(\xi)=\overline{\alpha}}\left\{4\pi A(\xi)+\log\left(\frac{K(\xi)}{1+\overline{\alpha}}\prod_{q\in S,q\neq \xi}e^{-4\pi\alpha(q)G_q(\xi)}\right)\right\}$$

and consider a cut-off function η_ε such that $\eta_\varepsilon \equiv 1$ in $B_{r_\varepsilon}(p)$, $\eta_\varepsilon \equiv 0$ in $\Sigma\backslash B_{2r_\varepsilon}(p)$ and $|\nabla_g \eta_\varepsilon| = O(r_\varepsilon^{-1})$. Define

$$\varphi_\varepsilon(x) = \begin{cases} -2\log(\varepsilon + r^{2(1+\overline{\alpha})}) + \log\varepsilon & r \le r_\varepsilon \\ \overline{\rho}\,(G_p - \eta_\varepsilon\sigma) + C_\varepsilon + \log\varepsilon & r \ge r_\varepsilon \end{cases}$$

where $r = d(x,p)$, $\sigma(x) = O(r)$ is defined by

$$G_p(x) = -\frac{1}{2\pi}\log r + A(p) + \sigma(x), \tag{2.22}$$

and

$$C_\varepsilon = -2\log\left(\frac{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}{\gamma_\varepsilon^{2(1+\overline{\alpha})}}\right) - \overline{\rho}\,A(p).$$

In the case $\overline{\alpha}_i = 0\ \forall\, i$, a similar family of functions was used in [37] to give an existence result for (1.17) by proving, under some strict assumptions on h, that

$$\inf_{H^1(\Sigma)} J_{\overline{\rho}} < -8\pi\left(1+\log\left(\frac{\pi}{|\Sigma|}\right)+\max_{p\in\Sigma}\{4\pi A(p)+\log h(p)\}\right).$$

Here we only prove a weak inequality but we have no extra assumptions on h. Taking normal coordinates in a neighborhood of p it is simple to verify that

$$\begin{aligned}\int_{B_{r_\varepsilon}} |\nabla_g\varphi_\varepsilon|^2 dv_g &= 16\pi(1+\overline{\alpha})\left(\log\left(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}\right)+\frac{1}{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}-1\right)+o_\varepsilon(1) = \\ &= 16\pi(1+\overline{\alpha})\left(\log\left(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}\right)-1\right)+o_\varepsilon(1).\end{aligned}$$

By our definition of φ_ε

$$\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g \varphi_\varepsilon|^2 dv_g = \overline{\rho}^2 \left(\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g G_p|^2 dv_g + \int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon \sigma)|^2 dv_g - 2\int_{\Sigma\setminus B_{r_\varepsilon}} \nabla_g G_p \cdot \nabla_g(\eta_\varepsilon\sigma)\, dv_g \right)$$

and by the properties of η_ε

$$\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g(\eta_\varepsilon\sigma)|^2 dv_g = \int_{B_{2r_\varepsilon}\setminus B_{r_\varepsilon}} |\nabla_g \eta_\varepsilon|^2\sigma^2 + 2\eta_\varepsilon\sigma\, \nabla_g\eta_\varepsilon\cdot\nabla_g\sigma + \eta_\varepsilon^2|\nabla_g\sigma|^2\, dv_g = O(r_\varepsilon^2).$$

Hence, integrating by parts and using (2.22), one has

$$\begin{aligned}
\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla_g \varphi_\varepsilon|^2 dv_g &= \overline{\rho}^2 \left(\int_{\Sigma\setminus B_{r_\varepsilon}} |\nabla G_p|^2 dv_g - 2\int_{\Sigma\setminus B_{r_\varepsilon}} \nabla_g G_p\cdot\nabla_g(\eta_\varepsilon\sigma)\, dv_g + \right) + o_\varepsilon(1) = \\
&= -\overline{\rho}^2 \left(\frac{1}{|\Sigma|}\int_{\Sigma\setminus B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon\sigma)\, dv_g + \int_{\partial B_{r_\varepsilon}} (G_p - 2\eta_\varepsilon\sigma)\frac{\partial G_p}{\partial n} d\sigma_g \right) + o_\varepsilon(1) = \\
&= -\overline{\rho}^2 \int_{\partial B_{r_\varepsilon}} (G_p - 2\sigma)\frac{\partial G_p}{\partial n} d\sigma_g + o_\varepsilon(1) = \\
&= -\overline{\rho}^2 \int_{\partial B_{r_\varepsilon}} \left(-\frac{1}{2\pi}\log(r_\varepsilon) + A(p) - \sigma \right)\left(-\frac{1}{2\pi r_\varepsilon} + \nabla\sigma \right)(1 + O(r_\varepsilon^2))d\sigma \\
&\quad + o_\varepsilon(1) = \\
&= -\overline{\rho}^2 \int_{\partial B_{r_\varepsilon}} \left(\frac{\log r_\varepsilon}{4\pi^2 r_\varepsilon} - \frac{1}{2\pi r_\varepsilon}A(p) + O(\log r_\varepsilon) + O(1) \right) d\sigma + o_\varepsilon(1) = \\
&= -\frac{\overline{\rho}^2}{2\pi}\log(\gamma_\varepsilon \varepsilon^{\frac{1}{2(1+\overline{\alpha})}}) + \overline{\rho}^2 A(p) + o_\varepsilon(1) = \\
&= -2\overline{\rho}\left(\log\gamma_\varepsilon^{2(1+\overline{\alpha})} + \log\varepsilon - 4\pi(1+\overline{\alpha})A(p) \right) + o_\varepsilon(1).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_\Sigma |\nabla_g\varphi_\varepsilon|^2 dv_g &= 2\overline{\rho}\left(\log\left(\frac{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}{\gamma_\varepsilon^{2(1+\overline{\alpha})}} \right) - 1 + 4\pi(1+\overline{\alpha})A(p) - \log\varepsilon \right) + o_\varepsilon(1) = \\
&= -2\overline{\rho}\,(1 - 4\pi(1+\overline{\alpha})A(p) + \log\varepsilon) + o_\varepsilon(1). \qquad (2.23)
\end{aligned}$$

Similarly one has

$$\begin{aligned}
\int_{B_{r_\varepsilon}} \varphi_\varepsilon\, dv_g &= |B_{r_\varepsilon}|\log\varepsilon - 4\pi\int_0^{r_\varepsilon} r\log\left(\varepsilon + r^{2(1+\overline{\alpha})}\right)(1+o_\varepsilon(1))dr = \\
&= |B_{r_\varepsilon}|\log\varepsilon - 2\pi r_\varepsilon^2\log\varepsilon - 4\pi\int_0^{r_\varepsilon} r\log\left(1 + \frac{r^{2(1+\overline{\alpha})}}{\varepsilon}\right)(1+o_\varepsilon(1))dr = \\
&= O(r_\varepsilon^2\log\varepsilon) - 4\pi\int_0^1 r_\varepsilon^2 s\log\left(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}s^{2(1+\overline{\alpha})}\right)(1+o_\varepsilon(1))dr =
\end{aligned}$$

$$= O(r_\varepsilon^2 \log\varepsilon) + O(r_\varepsilon^2 \log(1+\gamma_\varepsilon^{2(1+\overline{\alpha})})) = o_\varepsilon(1)$$

and

$$\begin{aligned}\int_{\Sigma\setminus B_{r_\varepsilon}} \varphi_\varepsilon \, dv_g &= \overline{\rho}\int_{\Sigma\setminus B_{r_\varepsilon}} (G_p - \eta_\varepsilon\sigma)dv_g + (C_\varepsilon + \log\varepsilon)|\Sigma\setminus B_{r_\varepsilon}(p)| = \\ &= |\Sigma|\log\varepsilon - \overline{\rho}|\Sigma|A(p) + o_\varepsilon(1)\end{aligned}$$

so that

$$\frac{1}{|\Sigma|}\int_\Sigma \varphi_\varepsilon dv_g = \log\varepsilon - \overline{\rho}\, A(p) + o_\varepsilon(1). \tag{2.24}$$

To compute the integral of the exponential term we fix a small $\delta > 0$ and observe that

$$\begin{aligned}\int_\Sigma h e^{\varphi_\varepsilon} dv_g &= \tilde{h}(p)\int_{B_{r_\varepsilon}} e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g + \int_{B_{r_\varepsilon}} \left(\tilde{h} - \tilde{h}(p)\right) e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g + \\ &+ \int_{B_\delta\setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g + \int_{\Sigma\setminus B_\delta} h e^{\varphi_\varepsilon} dv_g\end{aligned}$$

where $\tilde{h} = h\, e^{4\pi\overline{\alpha}G_p} = K \prod_{q\in S, q\neq p} e^{-4\pi\alpha(q)G_q}$. For the first term we have

$$\begin{aligned}\int_{B_{r_\varepsilon}} e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g &= \varepsilon\int_{B_{r_\varepsilon}} e^{2\overline{\alpha}\log r - 4\pi\overline{\alpha}A(p) - 4\pi\overline{\alpha}\sigma} e^{-2\log(\varepsilon + r^{2(1+\overline{\alpha})})} dv_g = \\ &= \varepsilon e^{-4\pi\overline{\alpha}A(p)}\int_{B_{r_\varepsilon}} \frac{r^{2\overline{\alpha}}}{(\varepsilon + r^{2(1+\overline{\alpha})})^2}(1 + o_\varepsilon(1))dv_g = \\ &= \frac{\pi e^{-4\pi\overline{\alpha}A(p)}}{1+\overline{\alpha}} \frac{\gamma_\varepsilon^{2(1+\overline{\alpha})}}{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}(1 + o_\varepsilon(1)) = \\ &= \frac{\pi e^{-4\pi\overline{\alpha}A(p)}}{1+\overline{\alpha}} + o_\varepsilon(1).\end{aligned} \tag{2.25}$$

Since $\tilde{h}$ is smooth in a neighbourhood of p we obtain

$$\int_{B_{r_\varepsilon}} \left(\tilde{h} - \tilde{h}(p)\right) e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1)\int_{B_{r_\varepsilon}} e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g = o_\varepsilon(1) \tag{2.26}$$

and

$$\begin{aligned}\left|\int_{B_\delta\setminus B_{r_\varepsilon}} h e^{\varphi_\varepsilon} dv_g\right| &= \left|\int_{B_\delta\setminus B_{r_\varepsilon}} \tilde{h} e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g\right| \le \sup_{B_\delta}|\tilde{h}|\int_{B_\delta\setminus B_{r_\varepsilon}} e^{-4\pi\overline{\alpha}G_p} e^{\varphi_\varepsilon} dv_g = \\ &= \varepsilon e^{C_\varepsilon}\sup_{B_\delta}|\tilde{h}|\int_{B_\delta\setminus B_{r_\varepsilon}} e^{4\pi(2+\overline{\alpha})G_p} e^{-\overline{\rho}\eta_\varepsilon\sigma} dv_g = \\ &= O(\varepsilon)\int_{B_\delta\setminus B_{r_\varepsilon}} e^{4\pi(2+\overline{\alpha})G_p} dx = O(\varepsilon)\int_{B_\delta\setminus B_{r_\varepsilon}} \frac{1}{|x|^{2(2+\overline{\alpha})}} dx = \\ &= O(\varepsilon)\left(\frac{1}{r_\varepsilon^{2(1+\overline{\alpha})}} - \frac{1}{\delta^{2(1+\overline{\alpha})}}\right) = O\left(\frac{1}{\gamma_\varepsilon^{2(1+\overline{\alpha})}}\right) + O(\varepsilon) = o_\varepsilon(1).\end{aligned} \tag{2.27}$$

Finally

$$\int_{\Sigma\setminus B_\delta} he^{\varphi_\varepsilon}dv_g = \varepsilon e^{C_\varepsilon}\int_{\Sigma\setminus B_\delta} he^{\overline{\rho}G_p}dv_g = O(\varepsilon) \tag{2.28}$$

so by (2.25), (2.26), (2.27) and (2.28) we have

$$\int_\Sigma he^{\varphi_\varepsilon}dv_g = \frac{\pi\tilde{h}(p)e^{-4\pi\overline{\alpha}A(p)}}{1+\overline{\alpha}} + o_\varepsilon(1). \tag{2.29}$$

Using (2.23), (2.24) and (2.29) we get

$$\lim_{\varepsilon\to 0} J(\varphi_\varepsilon) = -\overline{\rho}\left(1+4\pi A(p)+\log\left(\frac{1}{|\Sigma|}\frac{\pi\tilde{h}(p)}{1+\overline{\alpha}}\right)\right) =$$

$$= -\overline{\rho}\left(1+\log\frac{\pi}{|\Sigma|}+\max_{\xi\in\Sigma,\alpha(\xi)=\overline{\alpha}}\left\{4\pi A(\xi)+\log\left(\frac{K(\xi)}{1+\overline{\alpha}}\prod_{q\in S,q\neq\xi}e^{-4\pi\alpha(q)G_q(\xi)}\right)\right\}\right).$$

This, together with Proposition 2.3, completes the proof of Theorem 1.2.

2.4 Onofri's Inequalities on S^2

In this section we will consider the special case of the standard sphere (S^2, g_0) with $m \le 2$ and $K \equiv 1$. We fix $\alpha_1, \alpha_2 \in \mathbb{R}$ with $-1 < \alpha_1 \le \alpha_2$ and as before we consider the singular weight

$$h = e^{-4\pi\alpha_1 G_{p_1} - 4\pi\alpha_2 G_{p_2}}. \tag{2.30}$$

In order to apply Theorem 1.2 and obtain sharp versions of (1.22), we need to study the existence of minimum points for the functional J_ρ^h. Let us fix a system of coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$ such that $p_1 = (0,0,1)$. When $h \in C^1(S^2)$ the Kazdan-Warner identity (see [47]) states that any solution of (1.17) has to satisfy

$$\int_{S^2} \nabla h\cdot\nabla x_i\, e^u\, dv_{g_0} = \left(2-\frac{\rho}{4\pi}\right)\int_{S^2} he^u x_i\, dv_{g_0} \quad i=1,2,3.$$

We claim that if $p_2 = -p_1$ the same identity holds, at least in the x_3-direction, even when h is singular.

Lemma 2.8. *Let u be a solution of (1.17) on S^2, then there exist $C, \delta_0 > 0$ such that*

- $|\nabla u(x)| \le C d(x,p_i)^{2\alpha_i+1}$ *if* $\alpha_i < -\frac{1}{2}$;
- $|\nabla u(x)| \le C\,(-\log d(x,p_i))$ *if* $\alpha_i = -\frac{1}{2}$;
- $|\nabla u(x)| \le C$ *if* $\alpha_i > -\frac{1}{2}$;

for $0 < d(x,p_i) < \delta_0$, $i = 1,2$.

Proof. Let us fix $0 < r_0 < \frac{1}{2}\min\{\frac{\pi}{2}, d(p_1,p_2)\}$ and $i \in \{1,2\}$. If $\alpha_i > -\frac{1}{2}$ then, by standard elliptic regularity, $u \in C^1(\overline{B_{r_0}(p_i)})$ and the conclusion holds for $\delta_0 = r_0$ and $C = \|\nabla u\|_{L^\infty(B_{r_0}(p_i))}$. Let us now assume $\alpha_i \le -\frac{1}{2}$. We know that $h(y) \le C_1 d(y,p_i)^{2\alpha_i}$ for $y \in B_{2r_0}(p_i)$ so, if $\delta_0 < r_0$, by Green's representation formula we have

$$|\nabla u|(x) \le \rho e^{\|u\|_\infty} \int_{S^2} \frac{h(y)}{d(x,y)} dv_{g_0}(y) \le \frac{\rho e^{\|u\|_\infty} \|h\|_{L^1(S^2)}}{r_0} + \rho e^{\|u\|_\infty} C_1 \int_{B_{r_0}(x)} \frac{d(y,p_i)^{2\alpha_i}}{d(x,y)} dv_{g_0}(y).$$

Let π be the stereographic projection from the point $-p_i$. It is easy to check that there exist $C_2, C_3 > 0$ such that

$$C_2\, d(q,q') \le |\pi(q) - \pi(q')| \le C_3\, d(q,q')$$

$\forall\, q, q' \in B_{\frac{\pi}{2}}(p_i)$. Thus we have

$$\int_{B_{r_0}(x)} \frac{d(y,p_i)^{2\alpha_i}}{d(x,y)} dv_{g_0}(y) \le \int_{B_{\frac{\pi}{2}}(p_i)} \frac{d(y,p_i)^{2\alpha_i}}{d(x,y)} dv_{g_0}(y) \le C_4 \int_{\{|z|\le 1\}} \frac{|z|^{2\alpha_i}}{|\pi(x) - z|} dz =$$

$$= C_4 |\pi(x)|^{2\alpha_i+1} \int_{\left\{|z| \le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz \le C_5 d(x,p_i)^{2\alpha_i+1} \int_{\left\{|z| \le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz.$$

Notice that

$$\int_{\left\{|z| \le \frac{1}{|\pi(x)|}\right\}} \frac{|z|^{2\alpha_i}}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz \le$$

$$\le \frac{1}{2^{2\alpha_i}} \int_{\left\{\left|\frac{\pi(x)}{|\pi(x)|} - z\right| \le \frac{1}{2}\right\}} \frac{1}{\left|\frac{\pi(x)}{|\pi(x)|} - z\right|} dz + 2 \int_{\{|z|\le 2\}} |z|^{2\alpha_i} dz + 2 \int_{\left\{2 \le |z| \le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz \le$$

$$\le C_6 + 2 \int_{\left\{2 \le |z| \le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz.$$

If $\alpha_i < -\frac{1}{2}$

$$\int_{\left\{2 \le |z| \le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz \le C_7,$$

while if $\alpha_i = -\frac{1}{2}$

$$\int_{\left\{2 \le |z| \le \frac{1}{|\pi(x)|}\right\}} |z|^{2\alpha_i - 1} dz = 2\pi \log\left(\frac{1}{2|\pi(x)|}\right) \le C_8\left(-\log d(x,p_i)\right).$$

Thus we get the conclusion for δ_0 sufficiently small. □

In any case there exists $s \in [0,1)$ such that

$$|\nabla u(x)| \le C d(x,p_i)^{-s}\left(-\log d(x,p_i)\right) \tag{2.31}$$

for $0 < d(x,p_i) < \delta_0$, $i = 1,2$.

Proposition 2.4. *If* $p_2 = -p_1$ *then any solution of* (1.17) *satisfies*

$$\int_{S^2} \nabla h \cdot \nabla x_3 \, e^u \, dv_{g_0} = \left(2 - \frac{\rho}{4\pi}\right) \int_{S^2} h e^u x_3 \, dv_{g_0}.$$

Proof. Without loss of generality we may assume

$$\int_{S^2} h e^u dv_{g_0} = 1. \tag{2.32}$$

Let us denote $S_\delta = S^2 \backslash B_\delta(p_1) \cup B_\delta(p_2)$. Since u is smooth in S_δ, multiplying (1.17) by $\nabla u \cdot \nabla x_3$ and integrating on S_δ we have

$$-\int_{S_\delta} \Delta u \, \nabla u \cdot \nabla x_3 \, dv_{g_0} = \rho \int_{S_\delta} \left(h \, e^u - \frac{1}{4\pi}\right) \nabla u \cdot \nabla x_3 \, dv_{g_0} \tag{2.33}$$

Integrating by parts we obtain

$$-\int_{S_\delta} \Delta u \, \nabla u \cdot \nabla x_3 \, dv_{g_0} = \int_{S_\delta} \nabla u \cdot \nabla(\nabla u \cdot \nabla x_3) dv_{g_0} + \sum_{i=1}^{2} \int_{\partial B_\delta(p_i)} \nabla u \cdot \nabla x_3 \frac{\partial u}{\partial n} d\sigma_{g_0}$$

and by (2.31)

$$\left| \int_{\partial B_\delta(p_i)} \nabla u \cdot \nabla x_3 \, \frac{\partial u}{\partial n} \, d\sigma_{g_0} \right| \leq \int_{\partial B_\delta(p_i)} |\nabla u|^2 |\nabla x_3| d\sigma_{g_0} = O(\delta^{2(1-s)} \log^2 \delta) = o_\delta(1).$$

Using the identities

$$\nabla u \cdot \nabla(\nabla u \cdot \nabla x_3) = \frac{1}{2} \nabla |\nabla u|^2 \cdot \nabla x_3 - x_3 |\nabla u|^2$$

and

$$-\Delta x_3 = 2 x_3,$$

and applying again (2.31) to estimate the boundary term, we get

$$-\int_{S_\delta} \Delta u \, \nabla u \cdot \nabla x_3 \, dv_{g_0} = \int_{S_\delta} \frac{1}{2} \nabla |\nabla u|^2 \cdot \nabla x_3 \, dv_{g_0} - \int_{S_\delta} x_3 |\nabla u|^2 dv_{g_0} + o_\delta(1) =$$

$$= -\frac{1}{2} \int_{S_\delta} \Delta x_3 \, |\nabla u|^2 dv_{g_0} - \sum_{i=1}^{2} \int_{\partial B_\delta(p_i)} |\nabla u|^2 \frac{\partial x_3}{\partial n} d\sigma_{g_0} - \int_{S_\delta} x_3 |\nabla u|^2 dv_{g_0} = o_\delta(1).$$

Thus (2.33) becomes

$$\int_{S_\delta} h e^u \nabla u \cdot \nabla x_3 \, dv_{g_0} - \frac{1}{4\pi} \int_{S_\delta} \nabla u \cdot \nabla x_3 \, dv_{g_0} = o_\delta(1). \tag{2.34}$$

Moreover

$$\begin{aligned}\int_{S_\delta} \nabla u \cdot \nabla x_3 \, dv_{g_0} &= -\int_{S_\delta} \Delta u \, x_3 \, dv_{g_0} - \sum_{i=1}^{2} \int_{\partial B_\delta(p_i)} x_3 \frac{\partial u}{\partial n} \, d\sigma_{g_0} = \\ &= \rho \int_{S_\delta} \left(h e^u - \frac{1}{4\pi} \right) x_3 \, dv_{g_0} + O(\delta^{1-s}(-\log \delta)) \\ &= \rho \int_{S_\delta} h e^u x_3 \, dv_{g_0} + o_\delta(1)\end{aligned}$$

and

$$\begin{aligned}\int_{S_\delta} h e^u \, \nabla u \cdot \nabla x_3 \, dv_{g_0} &= \int_{S_\delta} \nabla e^u \cdot h \nabla x_3 \, dv_{g_0} = \\ &= -\int_{S_\delta} e^u \, \mathrm{div}(\, h \nabla x_3) dv_{g_0} - \sum_{i=1}^{2} \int_{\partial B_\delta(p_i)} h e^u \frac{\partial x_3}{\partial n} \, d\sigma_{g_0} = \\ &= -\int_{S_\delta} \nabla h \cdot \nabla x_3 \, e^u \, dv_{g_0} + 2 \int_{S_\delta} h e^u x_3 dv_{g_0} + O(\delta^{2(1+\alpha)}).\end{aligned}$$

Thus by (2.34) we have

$$\int_{S_\delta} \nabla h \cdot \nabla x_3 \, e^u \, dv_{g_0} = \left(2 - \frac{\rho}{4\pi} \right) \int_{S_\delta} h e^u x_3 \, dv_{g_0} + o_\delta(1).$$

Since u is continuous on S^2 and $h, \nabla h \cdot \nabla x_3 \in L^1(S^2)$ as $\delta \to 0$ we get the conclusion. □

Remark 2.1. *In the above proof there is no need to assume $K \equiv 1$.*

Assuming $p_1 = (0,0,1)$ and $p_2 = (0,0,-1)$, one may easily verify that

$$G_{p_1}(x) = -\frac{1}{4\pi} \log(1 - x_3) - \frac{1}{4\pi} \log \left(\frac{e}{2} \right)$$

and

$$G_{p_2}(x) = -\frac{1}{4\pi} \log(1 + x_3) - \frac{1}{4\pi} \log \left(\frac{e}{2} \right),$$

so that

$$\nabla h \cdot \nabla x_3 = -4\pi h(\alpha_1 \nabla G_1 + \alpha_2 \nabla G_2) \cdot \nabla x_3 = (\alpha_2 - \alpha_1) h - (\alpha_1 + \alpha_2) h x_3.$$

Thus we can rewrite the identity in Proposition 2.4 as

$$\alpha_2 - \alpha_1 = \left(2 - \frac{\rho}{4\pi} + \alpha_1 + \alpha_2 \right) \int_{S^2} h e^u x_3 \, dv_{g_0}. \tag{2.35}$$

Proof of Theorem 1.3. Assume $m = 1$ (i.e. $\alpha_2 = 0$). We claim that equation (1.17) has no solutions for $\rho = \overline{\rho} = 8\pi(1 + \min\{0, \alpha_1\})$, unless $\alpha_1 = 0$. Indeed if u were a solution of (1.17) satisfying (2.32), then applying (2.35) with $\rho = \overline{\rho}$ we would get

$$-\alpha_1 = (\alpha_1 - 2\min\{0, \alpha_1\}) \int_{S^2} he^u x_3 \, dv_{g_0}$$

so that, if $\alpha_1 \neq 0$,

$$\left| \int_{S^2} he^u x_3 \, dv_{g_0} \right| = 1.$$

This contradicts (2.32). In particular we proved non-existence of minimum points for $J_{\overline{\rho}}$ so we can exploit Theorem 1.2 and (2.3) to prove that (1.22) holds with

$$C = \max_{p \in S^2, \alpha(p) = \alpha} \left\{ \log \left(\frac{1}{1+\alpha} \prod_{q \in S, q \neq p} e^{-4\pi\alpha(q) G_q(p)} \right) \right\}.$$

If $\alpha_1 < 0$ one has

$$C = -\log(1 + \alpha_1).$$

If $\alpha_1 > 0$,

$$C = \max_{p \in S^2 \setminus \{p_1\}} \{-4\pi\alpha_1 G_{p_1}(p)\} = -4\pi\alpha_1 G_{p_1}(p_2) = \alpha_1.$$

□

Remark 2.2. *More generally (2.35) implies that, for $m = 1$, $K \equiv 1$ and $\alpha_1 \neq 0$, equation (1.17) has no solutions for $\rho \in [8\pi(1 + \min\{0, \alpha_1\}), 8\pi(1 + \max\{0, \alpha_1\})]$.*

Proof of Theorem 1.4. As in the previous proof, applying (2.35) with $\rho = \overline{\rho} = 8\pi(1 + \alpha_1)$, we obtain that any critical point of $J_{\overline{\rho}}$ for which (2.32) holds has to satisfy

$$\alpha_2 - \alpha_1 = (\alpha_2 - \alpha_1) \int_{S^2} he^u x_3 dv_{g_0}.$$

Since $\alpha_1 \neq \alpha_2$ one has

$$\int_{S^2} he^u x_3 dv_{g_0} = 1$$

which is impossible. Thus $J_{\overline{\rho}}$ has no critical points and by Theorem 1.2 one has

$$C = \log \left(\frac{1}{1 + \alpha_1} e^{-4\pi\alpha_2 G_{p_2}(p_1)} \right) = \alpha_2 - \log(1 + \alpha_1).$$

□

Remark 2.3. *More generally (2.35) implies that, for $m = 2$, $K \equiv 1$ and $\alpha_1 < \alpha_2$, equation (1.17) has no solutions for $\rho \in [8\pi(1 + \alpha_1), 8\pi(1 + \alpha_2)]$.*

Now we assume $\alpha_1 = \alpha_2 < 0$. In this case identity (2.35) gives no useful condition. Let us denote by π the stereographic projection from the point p_1. It is easy to verify that u satisfies (1.17) and (2.32) if and only if

$$v := u \circ \pi^{-1} + (1+\alpha)\log\left(\frac{4}{(1+|y|^2)^2}\right) + 2\alpha\log\left(\frac{e}{2}\right)$$

solves

$$-\Delta_{\mathbb{R}^2} v = 8\pi(1+\alpha)|y|^{2\alpha}e^v \tag{2.36}$$

in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2}|y|^{2\alpha}e^v dy = 1.$$

As we pointed out in the proof of Lemma 2.4, equation (2.36) has a one-parameter family of solutions:

$$v_\lambda(y) = -2\log\left(1 + \frac{\pi}{1+\alpha}e^l|y|^{2(1+\alpha)}\right)$$

$l \in \mathbb{R}$. Thus we have a corresponding family $\{u_{\lambda,c}\}$ of critical points of $J_{\overline{\rho}}$ given by the expression

$$u_{\lambda,c}\circ\pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha}}{1+\lambda|y|^{2(1+\alpha)}}\right) + c, \tag{2.37}$$

$c \in \mathbb{R}, \lambda > 0$. A priori we do not know whether these critical points are minima for $J_{\overline{\rho}}$ (as it happens for $\alpha = 0$), so a direct application of 1.2 is not possible. However, we can still get the conclusion by comparing $J_{\overline{\rho}}(u_{\lambda,c})$ with the blow-up value provided by Theorem 1.2.

Proof of Theorem 1.5. Let us first compute $J(u_{\lambda,c})$. Let $\varphi_t : S^2 \longrightarrow S^2$ be the conformal transformation defined by $\pi(\varphi_t(\pi^{-1}(y))) = ty$. It is not difficult to prove that $\forall\, t > 0$

$$J_{\overline{\rho}}(u) = J_{\overline{\rho}}(u\circ\varphi_t + (1+\alpha)\log|\det d\varphi_t|);$$

in particular, since

$$u_{\lambda,c} = u_{1,0}\circ\varphi_{\lambda^{\frac{1}{2(1+\alpha)}}} + (1+\alpha)\log|\det\varphi_{\lambda^{\frac{1}{2(1+\alpha)}}}| + c - \log\lambda,$$

we have that $J(u_{\lambda,c})$ does not depend on λ and c. Thus we may assume $\lambda = 1$ and $c = 0$. A simple computation shows that

$$\int_{S^2} h\, e^{u_{1,0}}dv_{g_0} = 4e^{2\alpha}\int_{\mathbb{R}^2}\frac{|y|^{2\alpha}}{\left(1+|y|^{2(1+\alpha)}\right)^2}dy = \frac{4e^{2\alpha}\pi}{1+\alpha}. \tag{2.38}$$

Since $u_{1,0}(p_1) = 0$ and $u_{1,0}$ solves

$$-\Delta u_{1,0} = \omega\, h\, e^{u_{1,0}} - 2(1+\alpha) \quad \text{with} \quad \omega := 2(1+\alpha)^2e^{-2\alpha}$$

one has

$$\int_{S^2} u_{1,0}\, dv_{g_0} = 4\pi\int_{S^2}\Delta u_{1,0}\, G_{p_1}dv_{g_0} = -4\pi\omega\int_{S^2} he^{u_{1,0}}G_{p_1}dv_{g_0}$$

and

$$\frac{1}{2}\int_{S^2}|\nabla u_{1,0}|^2 dv_{g_0} + 2(1+\alpha)\int_{S^2} u_{1,0}\, dv_{g_0} = \frac{1}{2}\omega\int_{S^2} he^{u_{1,0}}u_{1,0}\, dv_{g_0} + (1+\alpha)\int_{S^2} u_{1,0}\, dv_{g_0} =$$

$$= \frac{\omega}{2}\int_{S^2} he^{u_{1,0}}(u_{1,0} - \overline{\rho}G_{p_1})dv_{g_0}. \tag{2.39}$$

Since

$$G_{p_1}(\pi^{-1}(y)) := \frac{1}{4\pi}\log(1+|y|^2) - \frac{1}{4\pi}$$

we get

$$\int_{S^2} he^{u_{1,0}}(u_{1,0} - \overline{\rho}G_{p_1}) = 2(1+\alpha)\int_{S^2} he^{u_{1,0}}dv_{g_0} - 8e^{2\alpha}\int_{\mathbb{R}^2}\frac{|y|^{2\alpha}\log\left(1+|y|^{2(1+\alpha)}\right)}{\left(1+|y|^{2(1+\alpha)}\right)^2}dy =$$

$$= 8\pi e^{2\alpha} - \frac{8\pi e^{2\alpha}}{1+\alpha}\int_0^{+\infty}\frac{\log(1+s)}{(1+s)^2}ds = \frac{8\pi\alpha e^{2\alpha}}{1+\alpha}. \tag{2.40}$$

Using (2.38), (2.39) and (2.40) we obtain

$$J(u_{\lambda,c}) = J(u_{1,0}) = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right) \qquad \forall\, \lambda > 0, c \in \mathbb{R}.$$

To conclude the proof it is sufficient to observe that $u_{\lambda,c}$ have to be minimum points for $J_{\overline{\rho}}$ that is

$$\inf_{H^1(S^2)} J_{\overline{\rho}} = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right).$$

Indeed if this were false then $J_{\overline{\rho}}$ would have no minimum points but, by Theorem 1.2, we would get

$$\inf_{H^1(S^2)} J_{\overline{\rho}} = 8\pi(1+\alpha)\left(\log(1+\alpha) - \alpha\right) = J(u_{\lambda,c}).$$

This is clearly a contradiction. □

Remark 2.4. *There is no need to assume $p_1 = -p_2$.*

Indeed given two arbitrary points $p_1, p_2 \in S^2$ with $p_1 \neq p_2$ it is always possible to find a conformal diffeomorphism $\varphi : S^2 \longrightarrow S^2$ such that $\varphi^{-1}(p_1) = -\varphi^{-1}(p_2)$. Moreover one has

$$J_{\overline{\rho}}(u) = \widetilde{J}_{\overline{\rho}}(u\circ\varphi + (1+\alpha)\log|\det d\varphi|) + c_{\alpha,p_1,p_2}$$

$\forall\, u \in H^1(S^2)$, where $\widetilde{J}$ is the Moser-Trudinger functional associated to

$$\widetilde{h} = e^{-4\pi\alpha G_{\varphi^{-1}(p_1)} - 4\pi\alpha G_{\varphi^{-1}(p_2)}}.$$

and c_{α,p_1,p_2} is an explicitly known constant depending only on α, p_1 and p_2. In particular one can still compute $\min\limits_{H^1(S^2)} J_{\overline{\rho}}$ and describe the minimum points of $J_{\overline{\rho}}$ in terms of φ and the family (2.37).

To complete the discussion of Onofri-Type inequalities with $m \leq 2$, it remains to consider the case $\alpha_1, \alpha_2 > 0$. This will be done in the next section.

2.5 Spheres with Positive Order Singularities

In this section we will assume (1.18) with $K \in C^\infty(\Sigma)$, $K > 0$ and $\alpha_1, \dots \alpha_m \geq 0$. The proof of Theorem 1.6 is a rather simple consequence of Theorem A.

Proof of Theorem 1.6. By the results of section 2.3 we have

$$\inf_{H^1(S^2)} J_{8\pi} \leq -8\pi \log \max_{S^2} h. \tag{2.41}$$

Remember that on S^2 $A(p) = \frac{1-2\log(2)}{4\pi}$. Let us consider

$$J^1_{8\pi}(u) := \frac{1}{2}\int_{S^2} |\nabla u|^2 dv_{g_0} + 2\int_{S^2} u\, dv_{g_0} - 8\pi \log\left(\frac{1}{4\pi}\int_{S^2} e^u dv_{g_0}\right).$$

By Theorem A we have $J^1_{8\pi}(u) \geq 0 \ \forall\ u \in H^1(S^2)$. The condition $\alpha_1, \dots, \alpha_m > 0$ guarantees $h \in C^0(S^2)$. Thus we have

$$J^h_{8\pi}(u) \geq \frac{1}{2}\int_{S^2} |\nabla u|^2 dv_{g_0} + 2\int_{S^2} u\, dv_{g_0} - 8\pi \log\left(\frac{1}{4\pi}\max_{\Sigma} h \int_{S^2} e^u dv_{g_0}\right) = \tag{2.42}$$

$$= J^1_{8\pi}(u) - 8\pi \log \max_{S^2} h \geq -8\pi \log \max_{S^2} h.$$

Since $e^u > 0$ on S^2, equality can hold only if

$$h \equiv \max_{S^2} h$$

which, by (1.18), is possible only if $\alpha_1 = \dots = \alpha_m = 0$ and K is constant. From (2.41), the lower bound in (2.42) is sharp and the proof is concluded. □

We will now discuss existence of solutions of (1.17) for $\rho = 8\pi$. Theorem 1.6 proves nonexistence of energy-minimizing solutions. However, in contrast to Theorems 1.3 and 1.4 we will prove that (1.17) (and thus (1.10)) has always a solution for $K \equiv 1$, and in many other cases.

Let us first focus on the case of two antipodal singular points $p_1 = -p_2$. Given any point $p \in S^2 \subset \mathbb{R}^3$ we consider the space

$$H_{rad,p} := \left\{u \in H^1(S^2) \ :\ \exists\, \varphi : [-1,1] \longrightarrow \mathbb{R} \ \text{ measurable s.t. } \ u(x) = v(x \cdot p) \text{ for a.e. } x \in S^2\right\}.$$

Lemma 2.9. *Suppose $m = 2$, $\min\{\alpha_1, \alpha_2\} = \alpha_1 > 0$ and $p_2 = -p_1$. If h is a positive function satisfying (1.19), then the Moser-Trudinger functional J^h_ρ is bounded from below on H_{rad,p_1} for any $\rho \in (0, 8\pi(1+\alpha_1))$.*

Proof. Let us consider

$$\widetilde{h}(x) := e^{-4\pi\alpha_1(G(x,p_1)+G(x,p_2))}.$$

Since $h = Ke^{-4\pi\alpha_1 G(x,p_1)-4\pi\alpha_2 G(x,p_2)} \leq \widetilde{h}\max_{x\in S^2} K(x)e^{4\pi(\alpha_1-\alpha_2)G(x,p_2)}$ it is sufficient to prove that the functional

$$\widetilde{J}_\rho(u) := J_\rho^{\widetilde{h}}(u) = \frac{1}{2}\int_{S^2}|\nabla u|^2 dv_{g_0} + \frac{\rho}{4\pi}\int_{S^2} u\, dv_{g_0} - \rho\log\left(\frac{1}{4\pi}\int_{S^2}\widetilde{h}e^u dv_{g_0}\right)$$

is bounded from below for any $\rho < 8\pi(1+\alpha_1)$. Let us consider Euclidean coordinates (x_1,x_2,x_3) on S^2 such that $p_1 = (0,0,-1)$, $p_2 = (0,0,1)$, and let π be the stereographic projection from the point p_2. Given a function $u \in H^1(S^2)$ we define $v(|y|) := (u(\pi^{-1}(y)))$, $v_{\alpha_1}(y) := v(|y|^{\frac{1}{1+\alpha_1}})$ and $u_{\alpha_1}(x) := v_{\alpha_1}(|\pi(x)|)$. Then we have

$$\int_{S^2}|\nabla u|^2 dv_{g_0} = 2\pi\int_0^\infty t|v'(t)|^2 dt = (1+\alpha_1)\int_0^{+\infty} s|v_{\alpha_1}'(s)|^2 ds = (1+\alpha_1)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0}, \tag{2.43}$$

and, using that $\sup_{t>0}\dfrac{1+t^{2(1+\alpha_1)}}{(1+t^2)^{1+\alpha_1}} < +\infty$,

$$\int_{S^2}\widetilde{h}e^u dv_{g_0} = 8\pi\int_0^{+\infty} e^{2\alpha_1}\frac{t^{2\alpha_1+1}e^{v(t)}}{(1+t^2)^{2(1+\alpha_1)}}dt \leq c_{\alpha_1}\int_0^{+\infty}\frac{t^{2\alpha_1+1}e^{v_{\alpha_1}(t^{1+\alpha_1})}}{(1+t^{2(1+\alpha_1)})^2}dt =$$

$$= 4\widetilde{c}_{\alpha_1}\int_0^{+\infty}\frac{se^{v_{\alpha_1}(s)}}{(1+s^2)^2} = \widetilde{c}_{\alpha_1}\int_{S^2} e^{v_{\alpha_1}}dv_{g_0}. \tag{2.44}$$

Finally, $\forall\, \varepsilon > 0$, $t \in \mathbb{R}^+$

$$|v(t) - v_{\alpha_1}(t)| \leq \left|\int_t^{t^{\frac{1}{1+\alpha_1}}}|v'(s)|ds\right| \leq \left|\int_t^{t^{\frac{1}{1+\alpha_1}}} s|v'(s)|^2 ds\right|^{\frac{1}{2}}\left|\frac{\alpha_1}{1+\alpha_1}\log t\right| \leq$$

$$\leq \frac{\varepsilon}{4\pi}\|\nabla u\|_2^2 + c_{\varepsilon,\alpha_1}|\log t|$$

from which

$$\left|\int_{S^2} u\, dv_{g_0} - \int_\Sigma u_{\alpha_1}\, dv_{g_0}\right| \leq 8\pi\int_0^{+\infty}\frac{|v(t)-v_{\alpha_1}(t)|}{(1+t^2)^2} \leq \varepsilon\|\nabla u\|_2^2 + C_{\varepsilon,\alpha_1}. \tag{2.45}$$

(2.43), (2.44), (2.45) and the Moser-Trudinger inequality (1.22) imply

$$\widetilde{J}_\rho(u) \geq (1+\alpha_1)\left(\frac{1}{2}-\rho\,\varepsilon\right)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0} + \rho\int_{S^2} u_{\alpha_1}dv_{g_0} - \rho\log\left(\frac{1}{4\pi}\int_{S^2}e^{u_{\alpha_1}}dv_{g_0}\right) - C_{\epsilon,\alpha_1,\rho} =$$

$$= (1+\alpha_1)\left(\left(\frac{1}{2}-\rho\,\varepsilon\right)\int_{S^2}|\nabla u_{\alpha_1}|^2 dv_{g_0} - \frac{\rho}{1+\alpha_1}\log\left(\frac{1}{4\pi}\int_{S^2}e^{u_{\alpha_1}-\overline{u}_{\alpha_1}}dv_{g_0}\right)\right) - C_{\epsilon,\alpha_1,\rho} \geq -\widetilde{C}_{\epsilon,\alpha_1,\rho}$$

if $\rho < 8\pi(1+\alpha_1)$ and ε is sufficiently small. □

Remark 2.5. *Arguing as in sections 2.2, 2.3, 2.4, it is possible to describe the behavior of sequences of minimum points of* J_ρ^h *in* $H^1_{rad,p_1}(S^2)$ *as* $\rho \nearrow 8\pi(1+\alpha_1)$ *to prove that also* $J^h_{8\pi(1+\alpha_1)}$ *is bounded from below. Moreover if* $K \equiv 1$ *and* $\alpha_1 = \alpha_2 = \alpha$ *then we have*

$$\log\left(\frac{1}{4\pi}\int_{S^2} he^{u-\overline{u}}dv_{g_0}\right) \le \frac{1}{16\pi(1+\alpha)}\int_{S^2}|\nabla u|^2 dv_{g_0} + \alpha - \log(1+\alpha) \quad \forall\, u \in H_{rad,p_1}(S^2),$$

with equality holding for

$$u\circ\pi^{-1}(y) = 2\log\left(\frac{(1+|y|^2)^{1+\alpha}}{1+e^{\lambda}|y|^{2(1+\alpha)}}\right) + c,$$

where $\lambda, c \in \mathbb{R}$ *and* π *is the stereographic projection from* p_1*.*

Proof of Theorem 1.7. By Lemma 2.9, $\forall\, \rho < 8\pi(1+\alpha_1)$ $\exists\, \delta_\rho, C_\rho > 0$ such that

$$J^h_\rho(u) \ge \delta\int_{S^2}|\nabla u|^2 dv_{g_0} - C_\rho$$

$\forall\, u \in H_{rad,p_1}$. Thus J^h_ρ is coercive on the space

$$\left\{u \in H_{rad,p_1}, \int_\Sigma u\, dv_{g_0} = 0\right\},$$

and by direct methods we can find a minimum point of J^h_ρ in $H^1_{rad,p}$. Since $h \in H^1_{rad,p_1}$, by Palais' criticality principle (see Remark 11.4 in [3]), this minimum point is a solution of (1.17). $\square$

As a consequence of Theorems 1.6 and 1.7 we obtain a multiplicity result for equation (1.17). Indeed we can observe that if $\rho < 8\pi$ is sufficiently close to 8π, one has

$$\min_{u\in H^1(S^2)} J^h_\rho < \min_{u\in H_{rad,p_1}} J^h_\rho.$$

Corollary 2.1. *Suppose* h *satisfies the hypotheses of Theorem 1.7. There exists* $\varepsilon_0 > 0$ *such that* $\forall\, \rho \in (8\pi - \varepsilon_0, 8\pi)$*, equation* (1.17) *has at least two solutions* u*,* v *such that* $u \in H_{rad,p_1}$ *and* $v \in H^1(S^2)\backslash H_{rad,p_1}$*.*

Proof. For any $\rho < 8\pi$ let us take two functions $u_\rho \in H^1(S^2), v_\rho \in H_{rad,p_1}$, such that

$$J^h_\rho(u_\rho) = \min_{H^1(S^2)} J^h_\rho, \qquad J^h_\rho(v_\rho) = \min_{H_{rad,p_1}(S^2)} J^h_\rho(u) \quad \text{and} \quad \int_\Sigma u_\rho dv_{g_0} = \int_\Sigma v_\rho dv_{g_0} = 0.$$

We claim that, for ε sufficiently small and $\rho \in (8\pi-\varepsilon, 8\pi)$, $u_\rho \notin H_{rad,p_1}$ and in particular $u_\rho \ne v_\rho$. Assume by contradiction that there exists a sequence $\rho_n \nearrow 8\pi$ for which $u_{\rho_n} =\in H_{rad,p_1}$. Then, applying Lemma 2.9 as in the proof Theorem 1.7, we would have

$$J^h_{\rho_m}(u_{\rho_m}) \ge \delta\int_{S^2}|\nabla u_{\rho_n}|^2 dv_{g_0} - C$$

for some $\delta, C > 0$. Therefore $\|\nabla u_{\rho_n}\|_2$ would be uniformly bounded and, up to subsequences, $u_{\rho_n} \rightharpoonup u$ in $H^1(S^2)$ with $J^h_{8\pi}(u) = \inf_{H^1(S^2)} J^h_{8\pi}$. This is not possible because we know by Theorem 1.6 that $J^h_{8\pi}$ has no minimum point. $\square$

Now we will discuss some sufficient conditions for the existence of solutions of (1.17), without symmetry assumptions on h. Let H_0, $\Gamma(\alpha_1,\dots,\alpha_m)$, T_ρ and d_ρ be defined as in (1.12), (1.24), (1.25) and (1.26). By Theorem 1.2, if $u_n \in H_0$ is a sequence of solutions of (1.17) with $\rho = \rho_n$ uniformly bounded we have, up to subsequences, either

(i) $|u_n| \le C$ with C depending only on $\alpha_1,\dots,\alpha_m$, $\max_\Sigma K$, $\min_\Sigma K$ and $\overline{\rho}$.

or

(ii) u_n blows-up in a finite number of points, that is

$$\frac{\rho_n h e^{u_n}}{\int_\Sigma h e^{u_n} dv_g} \rightharpoonup 8\pi \sum_{i=1}^{k} (1+\alpha(q_i))\delta_{q_i}$$

with $q_1,\dots,q_k \in \Sigma$.

Case *(ii)* is possible only if $\overline{\rho} \in \Gamma(\alpha_1,\dots,\alpha_m)$. As we pointed out in the Introduction, a direct consequence is that the Leray Schauder degree d_ρ is well defined and is constant on every connected component of $(0,+\infty)\backslash\Gamma(\alpha_1,\dots,\alpha_m)$. From Chen and Lin's formula (1.29) for d_ρ we deduce existence of solutions for any $\rho \in (0, 8\pi(1+\alpha_1))\backslash 8\pi\mathbb{N}$.

Lemma 2.10. *Suppose that h satisfies (1.18) with $K \in C^\infty_+(S^2)$, $m \ge 2$ and $0 < \alpha_1 \le \ldots \le \alpha_m$. Then equation (1.17) has a solution $\forall\ \rho \in (0, 8\pi(1+\alpha_1))\backslash 8\pi\mathbb{N}$.*

Proof. Let $g(x)$ be the generating function in (1.27). If $m \ge 2$, then the first negative coefficient appearing in the expansion

$$g(x) = (1 + x + x^2 + x^3 \ldots)^{m-2} \prod_{i=1}^{m} (1 - x^{1+\alpha_i}) = 1 + \sum_{j=1}^{\infty} b_j x^{n_j}$$

is the coefficient of $x^{1+\alpha_1}$, i.e.

$$g(x) = \sum_{j=0}^{\infty} b_j x^{n_j}$$

with $b_0 = 1$ and $b_j \ge 0$ for any $j \ge 1$ such that $n_j < 1 + \alpha_1$. From (1.29) it follows that $d_\rho \ge 1$ for $\rho \in (0, 8\pi(1+\alpha_1))\backslash 8\pi\mathbb{N}$. □

Remark 2.6. *Lemma 2.10 only holds for $m \ge 2$. Indeed for $m = 1$ and $K \equiv 1$, Remark 2.2 states that (1.17) has no solutions for $\rho \in [8\pi, 8\pi(1+\alpha_1)]$. Also, for $m = 2$ the bound $8\pi(1+\alpha_1)$ is sharp by Remark 2.3.*

Remark 2.7. *A different proof of Lemma 2.10 was given in [7] by Bartolucci and Malchiodi using topological methods.*

By Theorem 1.2, if $\rho_n \longrightarrow 8k\pi$ with $k < 1 + \alpha_1$, then any blowing-up sequence of solutions of (1.17) must concentrate around exactly k points $q_1,\dots,q_k \in \Sigma\backslash\{p_1,\dots,p_m\}$. A more precise description of the blow-up set is given in [26] (see also [28], [29]):

Proposition 2.5 ([26], [28]). *Let u_n be a sequence of solutions of (1.17) with $\rho = \rho_n \longrightarrow 8\pi k$ and $k < 1 + \alpha_1$. If alternative (ii) of Theorem 1.2 holds, then u_n has exactly k blow-up points $q_1, \ldots, q_k \in \Sigma \backslash \{p_1, \ldots, p_m\}$ and $(q_1, \ldots, q_k)$ is a critical point of the function*

$$f_h(x_1, \ldots, x_k) := \sum_{j=1}^{k} \left(\log h(x_j) + \sum_{l \neq j} G(x_l, x_j) \right)$$

on the set

$$\{(x_1, \ldots, x_k) \in (S^2)^k \ : \ x_i \neq x_j \ \ \textit{for } i \neq j\}.$$

Moreover we have

$$\rho_n - 8k\pi = \sum_{j=1}^{k} h(q_{j,n})^{-1} \left(\Delta_{g_0} \log h(q_{j,n}) + 2(k-1)\right) \frac{\lambda_{j,n}}{e^{\lambda_{j,n}}} + O(e^{-\lambda_{j,n}})$$

where $q_{j,n}$ are the local maxima of u_n near q_j and $\lambda_{j,n} = u_n(q_{j,n})$.

Proof of Theorems 1.8 and 1.9. Take a sequence $\rho_n \searrow 8k\pi$ and a solution $u_n \in H_0$ of (1.17) for $\rho = \rho_n$. By Theorem 1.2, Proposition 2.5 and standard elliptic estimates, either u_n is uniformly bounded in $W^{2,q}(S^2)$ for any $q \geq 1$ or u_n blows-up at $(q_1, \ldots, q_k) \in \Sigma \backslash \{p_1, \ldots, p_m\}$. In the former case we have $u_n \longrightarrow u$ in $H^1(S^2)$ and u satisfies (1.17) with $\rho = 8\pi k$. The latter case can be excluded using (1.32), (1.33). Indeed we have

$$\Delta_{g_0} \log h(q_j) + 2(k-1) = \Delta_{g_0} \log K - \sum_{i=1}^{m} \alpha_i + 2(k-1) < 0$$

for any j. Denoting $q_{n,j}$ the maximum point of u_n near q_j and $\lambda_{j,n} = u_n(q_{j,n})$, by Proposition 2.5 we get

$$\rho_n - 8\pi k = \sum_{j=1}^{k} h(q_{j,n})^{-1} \left(\Delta_{g_0} \log h(q_{j,n}) + 2(k-1)\right) \frac{\lambda_{j,n}}{e^{\lambda_{j,n}}} + O(e^{-\lambda_{j,n}}) =$$

$$= \sum_{j=1}^{k} h(q_j)^{-1} \left(\Delta_{g_0} \log h(q_j) + 2(k-1)\right) \lambda_{j,n} e^{-\lambda_{j,n}} + o(\lambda_{j,n} e^{-\lambda_{j,n}}) < 0$$

which contradicts $\rho_n \searrow 8k\pi$. □

In order to prove Theorems 1.10, 1.11 we need to compute the Leray-Schauder degree for $\rho = 8\pi$.

Lemma 2.11. *Let h be a function satisfying (1.18) with $K \in C^\infty_+(\Sigma)$ and $\alpha_1, \ldots, \alpha_m > 0$. If $\Delta_{g_0} h(q) \neq 0$ for any $q \in \Sigma \backslash \{p_1, \ldots, p_m\}$ critical point of h, then $d_{8\pi}$ is well defined.*

Proof. It is sufficient to prove that the set of solutions of (1.17) in H_0 with $\rho = 8\pi$ is a bounded subset of H_0. Assume by contradiction that there exists $u_n \in H_0$ solution of (1.17) for $\rho = 8\pi$ such that $\|u_n\|_{H_0} \longrightarrow +\infty$. By Theorem 1.2 and Proposition 2.5, there exists $q \in \Sigma\backslash\{p_1, \dots, p_m\}$ such that $u_n \rightharpoonup 8\pi\delta_q$, $\nabla h(q) = 0$ and

$$0 = h(q_n)^{-1}\Delta_{g_0} \log h(q_n)\lambda_n e^{-\lambda_n} + O(e^{-\lambda_n}) = h(q)^{-2}\Delta_{g_0} h(q)\lambda_n e^{-\lambda_n} + o(\lambda_n e^{-\lambda_n})$$

where $\lambda_n := \max_\Sigma u_n$ and $u_n(q_n) = \lambda_n$. Since $\Delta_{g_0} h(q) \neq 0$ this is not possible. □

Under nondegeneracy assumptions, Chen and Lin proved that for any critical q point of h there exists a blowing-up sequence of solutions which concentrates at q. Moreover they were able to compute the total contribution to the Leray-Schauder degree of all the solutions concentrating at q.

Proposition 2.6 (see [27], [29])**.** *Assume that h is a Morse function on $\Sigma\backslash\{p_1, \dots, p_m\}$. Given a critical point $q \in \Sigma\backslash\{p_1, \dots, p_m\}$ of h, the total contribution to $d_{8\pi}$ of all the solutions of (1.17) concentrating at q is equal to $\mathrm{sgn}(\rho - 8\pi)(-1)^{\mathrm{ind}_p}$, where ind_p is the Morse index of p as critical point of h.*

Proof of Theorems 1.10, 1.11. Let us denote

$$\Lambda_- = \{q \in \Sigma\backslash\{p_1, \dots, p_m\} \ : \ \nabla h(q) = 0, \ \Delta_{g_0} h(q) < 0\},$$
$$\Lambda_+ = \{q \in \Sigma\backslash\{p_1, \dots, p_m\} \ : \ \nabla h(q) = 0, \ \Delta_{g_0} h(q) > 0\}.$$

By Proposition 2.6 we have

$$d_{8\pi} = 1 - \sum_{q\in\Lambda_-} (-1)^{\mathrm{ind}_q} = \overline{d} + \sum_{q\in\Lambda_+} (-1)^{\mathrm{ind}_q},$$

where $\overline{d}$ is the Leray-Schauder degree for $\rho \in (8\pi, 8\pi + \varepsilon)$. Clearly Λ_- contains only the local maxima of h and the saddle points of h in which $\Delta_{g_0} h < 0$, thus

$$d_{8\pi} = 1 - r + s.$$

Therefore we get existence of solutions if $r \neq s + 1$. Similarly we have

$$d_{8\pi} = \overline{d} - s' + r'$$

and we get solutions if $s' \neq r' + \overline{d}$. $\overline{d}$ can be computed using 1.29. If $m \geq 2$,

$$g(x) = 1 + x + \cdot... \quad \Longrightarrow \quad \overline{d} = 2.$$

If $m = 1$ we have

$$g(x) := 1 - x - x^{1+\alpha} + x^{2(1+\alpha)} \quad \Longrightarrow \quad \overline{d} = 0.$$

If $m = 0$, then

$$g(x) = 1 - 2x + x^2 \quad \Longrightarrow \quad \overline{d} = -1.$$

This concludes the proof. □

Chapter 3

Extremal Functions for Singular Moser Trudinger Embeddings

Most of the results in literature concerning existence of extremal functions for the Moser-Trudinger inequalities (1.3), (1.6), (1.41) rely deeply on the original estimates proved by Carleson and Chang in [20] for the unit disk. The main ingredient in the proof of these estimates (and of (1.35)) is the following inequality (cfr. Lemma 1 in [20]):

Proposition 3.1. *$\forall \ \delta, \tau > 0$ $c \in \mathbb{R}$ and $\alpha \in (-1, 0]$ we have*

$$\int_{D_\delta} e^{cu} dx \le \pi e^{1+\frac{c^2\tau}{16\pi}} \delta^2$$

$\forall \ u \in H_0^1(D_\delta)$ radially symmetric and such that $\int_{D_\delta} |\nabla u|^2 dx \le \tau$.

Here, and in the rest of the Chapter, $D_\delta := \left\{x \in \mathbb{R}^2 \ : \ |x| \le \delta\right\}$ and $D := D_1$. Moreover $\forall \ x_0 \in \mathbb{R}^2$, $D_\delta(x_0) := \{x \in \mathbb{R}^2 \ : \ x - x_0 \in D_\delta\}$ will denote the disk of radius δ centered at x.

Proposition 3.1 is a different way of writing the Onofri inequality for the unit disk:

$$\log\left(\frac{1}{\pi}\int_D e^u dx\right) \le \frac{1}{16\pi}\int_\Sigma |\nabla u|^2 dx + 1. \tag{3.1}$$

Using ODE techniques, Carleson and Chang gave a direct proof of (3.1), but it can also be deduced from Theorem A.

Onofri-type inequalities can thus be used to control blow-up phenomena for the nonlinearity $e^{4\pi u^2}$. In this Chapter we will use this technique in the presence of singularities. Starting form Theorem 1.5, in Section 3.1 we will prove Theorem 1.13 which is a singular version of (3.1). Then, in section 3.2, we will be able to reproduce, in a simplified version, the argument in [20] and prove Theorem 1.12. As a consequence we obtain existence of extremal functions for (1.38).

The rest of the Chapter is devoted to the proof of Theorem 1.14. We will take a smooth compact surface (Σ, g) and study uniform bounds and existence of extremals for the functional

(1.43) on the space (1.5). Differently from the previous section, where the change of variable (1.48) suggested to consider singular weight satisfying (1.18), here we will just assume (1.19). More precisely we will assume that any point $p \in \Sigma$ has a neighborhood $\Omega_p \subseteq \Sigma$ such that

$$\frac{h}{d(\,\cdot\,,p_i)^{2\alpha_i}} \in C^0_+(\Omega_p) := \left\{ f \in C^0(\Omega_p) \;:\; f > 0 \right\} \qquad \text{for } i = 1, \dots, m. \tag{3.2}$$

In section 3.3 we will introduce some notations and prove the subcritical case of Theorem 1.14. The critical functional will be studied in sections 3.4, 3.5. Similarly to what we have seen for Liouville equations a sequence of subcritical extremals for (1.43) on the space $\mathcal{H}$ can either be compact or concentrate at a point $p \in \Sigma$. We stress that this concentration-compactness alternative is strictly related to the condition $\|\nabla u\|_2 \le 1$. Indeed if we only assume $\|\nabla u\|_2 \le C$, a general concentration-compactness theory for critical points of (1.43) has not yet been developed. In section 3.4 we will prove an upper bound for concentrating maximizing sequences similar to (1.36). Lower bounds on $\sup_{\mathbb{H}} E^{\lambda,\beta,q}_{\Sigma,h}$ will be studied in section 3.5, where we complete the proof of Theorem 1.14.

3.1 Onofri-type Inequalities for Disks.

Let us fix Euclidean coordinates (x_1, x_2, x_3) on $S^2 \subseteq \mathbb{R}^3$ and denote $N := (0, 0, 1)$ and $S = (0, 0, -1)$ the north and the south pole. Let us consider the stereographic projection $\pi : S^2 \backslash \{N\} \longrightarrow \mathbb{R}^2$

$$\pi(x) := \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right),$$

and the Green's functions

$$G_N(x) = -\frac{1}{4\pi} \log(1 - x_3) - \frac{1}{4\pi} \log \frac{e}{2}$$

$$G_S(x) = -\frac{1}{4\pi} \log(1 + x_3) - \frac{1}{4\pi} \log \frac{e}{2}$$

It is well known that π is a conformal diffeomorphism and

$$\left(\pi^{-1}\right)^* g_0 = e^{u_0} |dx|^2 \tag{3.3}$$

where

$$u_0 = \log \left(\frac{4}{(1 + |x|^2)^2} \right) \tag{3.4}$$

satisfies

$$-\Delta u_0 = 2e^{u_0} \qquad \text{on } \mathbb{R}^2. \tag{3.5}$$

Proof of Theorem 1.13. We want to apply Theorem 1.5 with $p_1 = N$, $p_2 = S$. Given $r > 0$, we consider the set $S^2_r = \pi^{-1}(D_r)$ and the map $T_r : H^1_0(D_r) \longrightarrow H^1(S^2)$ defined by

$$T_r u(x) := \begin{cases} u(\pi(x)) - (1 + \alpha) u_0(\pi(x)) & \text{on } S^2_r \\ 2(1 + \alpha) \log(\frac{1 + r^2}{2}) & \text{on } S^2 \backslash S^2_r. \end{cases}$$

Using (3.3) and $h(\pi^{-1}(y)) = \left(\frac{e}{2}\right)^{2\alpha} |y|^{2\alpha} e^{\alpha u_0}$ we find

$$\int_{S^2} he^{T_r u} dv_{g_0} \geq \int_{S^2_r} he^{T_r u} dv_{g_0} = \int_{D_r} h(\pi^{-1}(y)) e^{T_r u(\pi^{-1}(y))} e^{u_0} dy = \\ = \left(\frac{e}{2}\right)^{2\alpha} \int_{D_r} |y|^{2\alpha} e^{u(y)} dy. \tag{3.6}$$

Moreover, by (3.5),

$$\int_{S^2_r} |\nabla T_r u|^2 dv_{g_0} = \int_{D_r} |\nabla u|^2 dx - 2(1+\alpha) \int_{D_r} \nabla u_0 \cdot \nabla u \, dy + (1+\alpha)^2 \int_{D_r} |\nabla u_0|^2 dy =$$

$$= \int_{D_r} |\nabla u|^2 dy - 4(1+\alpha) \int_{D_r} u e^{u_0} dy + (1+\alpha)^2 \int_{D_r} |\nabla u_0|^2 dy =$$

$$= \int_{D_r} |\nabla u|^2 dy - 4(1+\alpha) \int_{S^2_r} T_r u \, dv_{g_0} + (1+\alpha)^2 \left(\int_{D_r} |\nabla u_0|^2 dy - 4 \int_{D_r} u_0 e^{u_0} dy \right).$$

A direct computation shows

$$\int_{D_r} |\nabla u_0|^2 dy = 16\pi \left(\log(1+r^2) - \frac{r^2}{1+r^2} \right)$$

and

$$\int_{D_r} u_0 e^{u_0} dy = 8\pi \log 2 - 8\pi + o_r(1),$$

where $o_r(1) \longrightarrow 0$ as $r \to +\infty$. Moreover

$$\int_{S^2 \setminus S_r} T_r u \, dv_{g_0} = o(1),$$

thus we get

$$\int_{S^2} |\nabla T_r u|^2 dv_{g_0} + 4(1+\alpha) \int_{S^2} T_r u \, dv_{g_0} = \\ = \int_{D_r} |\nabla u|^2 dy + 16\pi(1+\alpha)^2 \left(\log(1+r^2) + 1 - 2\log 2 + o_r(1) \right). \tag{3.7}$$

Using (3.6), (3.7) and Theorem 1.5 we can so conclude

$$\log\left(\frac{1}{\pi} \int_{D_r} |y|^{2\alpha} e^u dy \right) \leq \log\left(\frac{1}{\pi} \int_{S^2} he^{T_r u} dv_{g_0} \right) + 2\alpha \log 2 - 2\alpha \leq$$

$$\leq \frac{1}{16\pi(1+\alpha)} \left(\int_{S^2} |\nabla T_r u|^2 dv_{g_0} + 2(1+\alpha) \int_{S^2} T_r u \, dv_{g_0} \right) + 2(1+\alpha)\log 2 - \alpha - \log(1+\alpha) \leq$$

$$\leq \frac{1}{16\pi(1+\alpha)} \int_{D_r} |\nabla u|^2 dy + (1+\alpha)\log(1+r^2) + 1 - \log(1+\alpha) + o_r(1). \tag{3.8}$$

Now, if $u \in H_0^1(D)$, we can apply (3.8) to $u_r(y) = u(\frac{y}{r})$. Since

$$\int_D |x|^{2\alpha} e^u dx = \frac{1}{r^{2(1+\alpha)}} \int_{D_r} |y|^{2\alpha} e^{u_r(y)} dy \qquad \text{and} \qquad \int_D |\nabla u|^2 dx = \int_{D_r} |\nabla u_r|^2 dy,$$

we find

$$\log\left(\frac{1}{\pi}\int_D |x|^{2\alpha} e^u dx\right) \le \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx + 1 - \log(1+\alpha) + o_r(1).$$

As $r \to \infty$ we get the conclusion. □

Since

$$\int_D |x|^{2\alpha} dx = \frac{\pi}{1+\alpha},$$

Theorem 1.13 can be written in a simpler form in terms of the singular metric $g_\alpha = |x|^{2\alpha}|dx|^2$.

Corollary 3.1. *For any $u \in H_0^1(D)$ and $\alpha \le 0$, we have*

$$\log\left(\frac{1}{|D|_\alpha}\int_D e^u dv_{g_\alpha}\right) \le \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dv_{g_\alpha} + 1$$

where $|D|_\alpha = \frac{\pi}{(1+\alpha)}$ is the measure of D with respect to g_α.

We stress that the constant 1 appearing in Theorem 1.13 is sharp.

Proposition 3.2. $\forall \;\; \alpha \in (-1, 0]$

$$\inf_{u \in H_0^1(D)} \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx - \log\left(\frac{1}{|D|_\alpha}\int_D |x|^{2\alpha} e^u dx\right) = -1.$$

Proof. Let us denote $J_\alpha(u) := \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dx - \log\left(\frac{1}{|D|_\alpha}\int_D |x|^{2\alpha} e^u dv_g\right)$. It is sufficient to exhibit a family of functions $u_\varepsilon \in H_0^1(D)$ such that $J_\alpha(u_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} -1$. Take $\gamma_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} +\infty$ such that $\varepsilon\gamma_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0$, and define

$$u_\varepsilon(x) = \begin{cases} -2\log\left(1 + \left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon & \text{for } |x| \le \gamma_\varepsilon\varepsilon \\ -4(1+\alpha)\log|x| & \text{for } \gamma_\varepsilon\varepsilon \le |x| \le 1 \end{cases}$$

where $L_\varepsilon := 2\log\left(\frac{1+\gamma_\varepsilon^{1+\alpha}}{\gamma_\varepsilon^{1+\alpha}}\right) - 4(1+\alpha)\log\varepsilon$ is chosen so that $u_\varepsilon \in H_0^1(D)$. Simple computations show that

$$\frac{1}{16\pi(1+\alpha)}\int_D |\nabla u_\varepsilon|^2 dx = \log\left(\frac{1+\gamma_\varepsilon^{2(1+a)}}{\gamma_\varepsilon^{2(1+a)}}\right) - 1 + \frac{1}{1+\gamma_\varepsilon^{2(1+\alpha)}} - 2(1+\alpha)\log\varepsilon$$

$$= -1 - 2(1+\alpha)\log\varepsilon + o_\varepsilon(1)$$

and

$$\int_D |x|^{2\alpha} e^{u_\varepsilon} dx = \frac{\varepsilon^{2(1+\alpha)}\gamma_\varepsilon^{2(1+\alpha)} e^{L_\varepsilon}\pi}{(1+\alpha)(1+\gamma_\varepsilon^{2(1+\alpha)})} + \frac{\pi}{1+\alpha}\left(\frac{1}{(\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}} - 1\right) = \frac{\pi\varepsilon^{-2(1+\alpha)}}{1+\alpha}(1+o_\varepsilon(1)).$$

as $\varepsilon \to 0$. Thus

$$J_\alpha(u_\varepsilon) \longrightarrow -1.$$

□

In order prove to Theorem 1.12, in the next section we will need to apply Theorem 1.13 on arbitrarily small disks to functions with a precise Dirichlet energy. Thus it will be convenient to use the following formulation of Theorem 1.13 (cfr Proposition 3.1).

Corollary 3.2. *$\forall\ \delta, \tau > 0$ $c \in \mathbb{R}$ and $\alpha \in (-1, 0]$ we have*

$$\int_{D_\delta} |x|^{2\alpha} e^u dx \le \frac{\pi}{1+\alpha} e^{1+\frac{c^2\tau}{16\pi(1+\alpha)}} \delta^{2(1+\alpha)}$$

$\forall\ u \in H_0^1(D_\delta)$ such that $\int_{D_\delta} |\nabla u|^2 dv_g \le \tau$.

We conclude this section with a Remark concerning the case $\alpha > 0$. If $h = e^{-4\pi\alpha(G_N+G_S)}$, with $\alpha > 0$ then by Theorem 1.6 one has

$$\log\left(\int_{S^2} h e^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi}\int_{S^2} |\nabla u|^2 dv_{g_0} + 2\alpha \log\left(\frac{e}{2}\right) \tag{3.9}$$

where the constants $\frac{1}{16\pi}$ and $2\alpha\log\left(\frac{e}{2}\right)$ are sharp. This inequality is not conformally invariant, thus it does not give a sharp inequality for the unit disk. However, by Lemma 2.9 and Remark 2.5, if we only consider functions that are axially symmetric with respect to the direction identified by p_1, p_2, (3.9) can be improved to

$$\log\left(\int_{S^2} h e^{u-\overline{u}} dv_{g_0}\right) \le \frac{1}{16\pi(1+\alpha)}\int_{S^2} |\nabla u|^2 dv_{g_0} + \alpha - \log(1+\alpha).$$

Therefore, arguing as before, we recover Theorem 1.12 in the class of radially symmetric functions on D:

Proposition 3.3. *If $\alpha > 0$, then we have*

$$\log\left(\frac{1+\alpha}{\pi}\int_D |x|^{2\alpha} e^u dx\right) \le \frac{1}{16\pi(1+\alpha)}\int_D |\nabla u|^2 dv_g + 1$$

for any radially symmetric function $u \in H_0^1(D)$.

3.2 A Carleson-Chang Type Estimate.

In this section we will use Corollary 3.2 to prove Theorem 1.12. We will consider the space

$$H := \left\{ u \in H_0^1(D) \ : \ \int_D |\nabla u|^2 dx \le 1 \right\}$$

and, $\forall\, \alpha \in (-1, 0]$, the functional

$$E_\alpha(u) := \int_D |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx.$$

By (1.38) we have $\sup_H E_\alpha < +\infty$. As in the previous section, for any $\delta > 0$, D_δ will denote the disk with radius δ. With a trivial change of variables, one immediately gets:

Lemma 3.1. *If $\delta > 0$ and $u \in H_0^1(D_\delta)$ are such that $\int_{D_\delta} |\nabla u_n|^2 dx \le 1$, then*

$$\int_{D_\delta} |x|^{2\alpha} e^{4\pi(1+\alpha)u^2} dx \le \delta^{2(1+\alpha)} \sup_H E_\alpha.$$

As in the original proof in [20], we will start by proving Theorem 1.12 for radially symmetric functions. For this reason we introduce the space

$$H_{rad} := \{ u \in H \ : \ u \text{ is radially symmetric and decreasing} \}.$$

Functions in H_{rad} satisfy the following useful decay estimate.

Lemma 3.2. *If $u \in H_{rad}$, then*

$$u(x)^2 \le -\frac{1}{2\pi} \left(1 - \int_{D_{|x|}} |\nabla u|^2 dy \right) \log |x| \qquad \forall\, x \in D \backslash \{0\}.$$

Proof.

$$|u(x)| \le \int_{|x|}^1 |u'(t)| dt \le \left(\int_{|x|}^1 t u'(t)^2 dt \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}} \le \frac{1}{\sqrt{2\pi}} \left(\int_{D \backslash D_{|x|}} |\nabla u|^2 dy \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}} \le$$

$$\le \frac{1}{\sqrt{2\pi}} \left(1 - \int_{D_{|x|}} |\nabla u|^2 dy \right)^{\frac{1}{2}} (-\log |x|)^{\frac{1}{2}}.$$

□

On a sufficiently small scale, it is possible to control E_α using only Corollary 3.2 and Lemmas 3.1, 3.2.

Lemma 3.3. *Assume $\alpha \in (-1, 0]$. If $u_n \in H_{rad}$ and $\delta_n \longrightarrow 0$ satisfy*

$$\int_{D_{\delta_n}} |\nabla u_n|^2 dx \longrightarrow 0, \tag{3.10}$$

then

$$\limsup_{n\to\infty} \int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \le \frac{\pi e}{1+\alpha}.$$

Proof. Take $v_n := u_n - u_n(\delta_n) \in H_0^1(D_{\delta_n})$ and set $\tau_n := \int_{D_{\delta_n}} |\nabla u_n|^2 dx$.
If $\tau_n = 0$, then $u_n \equiv u_n(\delta_n)$ in D_{δ_n} and, using Lemma 3.2, we find

$$\int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx = \frac{\pi}{1+\alpha} \delta_n^{2(1+\alpha)} e^{4\pi(1+\alpha)u_n(\delta_n)^2} \le \frac{\pi}{1+\alpha} \le \frac{\pi e}{1+\alpha}.$$

Thus we can assume $\tau_n > 0$. By Holder's inequality and Lemma 3.1 we have

$$\int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx = e^{4\pi(1+\alpha)u_n(\delta_n)^2} \int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)v_n^2 + 8\pi(1+\alpha)u_n(\delta_n)v_n} dx \le$$

$$\le e^{4\pi(1+\alpha)u_n(\delta_n)^2} \left(\int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)\frac{v_n^2}{\tau_n}} dx \right)^{\tau_n} \left(\int_{D_{\delta_n}} |x|^{2\alpha} e^{\frac{8\pi(1+\alpha)u_n(\delta_n)v_n}{1-\tau_n}} dx \right)^{1-\tau_n} \le$$

$$\le e^{4\pi(1+\alpha)u_n(\delta_n)^2} \left(\delta_n^{2(1+\alpha)} \sup_H E_\alpha \right)^{\tau_n} \left(\int_{D_{\delta_n}} |x|^{2\alpha} e^{\frac{8\pi(1+\alpha)u_n(\delta_n)v_n}{1-\tau_n}} dx \right)^{1-\tau_n}. \tag{3.11}$$

Applying Corollary 3.2 with $\tau = \tau_n, \delta = \delta_n$ and $c = \frac{8\pi(1+\alpha)u_n(\delta_n)}{1-\tau_n}$ we find

$$\int_{D_{\delta_n}} |x|^{2\alpha} e^{\frac{4\pi(1+\alpha)u_n(\delta_n)^2 v_n}{1-\tau_n}} dx \le \delta_n^{2(1+\alpha)} \frac{\pi e^{1+\frac{4\pi(1+\alpha)u_n(\delta_n)^2}{(1-\tau_n)^2}\tau_n}}{1+\alpha}$$

thus from (3.11)

$$\int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \le \delta_n^{2(1+\alpha)} \left(\sup_H E \right)^{\tau_n} \left(\frac{\pi e}{1+\alpha} \right)^{1-\tau_n} e^{4\pi(1+\alpha)u_n^2(\delta_n) + \frac{4\pi(1+\alpha)u_n(\delta_n)^2 \tau_n}{(1-\tau_n)}} =$$

$$= \delta_n^{2(1+\alpha)} \left(\sup_H E_\alpha \right)^{\tau_n} \left(\frac{\pi e}{1+\alpha} \right)^{1-\tau_n} e^{\frac{4\pi(1+\alpha)u_n(\delta)^2}{1-\tau_n}}.$$

Lemma 3.2 yields

$$\delta_n^{2(1+\alpha)} e^{4\pi(1+\alpha)\frac{u_n(\delta_n)^2}{1-\tau_n}} \le 1,$$

therefore

$$\int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \le \left(\sup_H E_\alpha \right)^{\tau_n} \left(\frac{\pi e}{1+\alpha} \right)^{1-\tau_n}.$$

Since $\tau_n \longrightarrow 0$, we obtain the conclusion by taking the lim sup as $n \to \infty$ on both sides. □

In order to prove Theorem 1.12 for H_{rad} it is sufficient to show that, if $u_n \rightharpoonup 0$, there exists a sequence δ_n satisfying the hypotheses of Lemma 3.3 and such that

$$\int_{D_{\delta_n}} |x|^{2\alpha}\left(e^{4\pi(1+\alpha)u_n^2} - 1\right) dx \longrightarrow 0. \tag{3.12}$$

Note that, by the dominated convergence Theorem, (3.12) holds if there exists $f \in L^1(D)$ such that

$$|x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} \le f \tag{3.13}$$

in $D\backslash D_{\delta_n}$. In the next Lemma we will chose a function $f \in L^1(D)$ with critical growth near 0 (i.e. $f(x) \approx \frac{1}{|x|^2 \log^2 |x|}$) and define δ_n so that (3.13) is satisfied.

Lemma 3.4. *Assume $\alpha \in (-1, 0]$. Take $u_n \in H_{rad}$ such that*

$$\sup_{D\backslash D_r} u_n \longrightarrow 0 \qquad \forall\, r \in (0,1). \tag{3.14}$$

Then there exists a sequence $\delta_n \in (0,1)$ such that

1. $\delta_n \longrightarrow 0.$
2. $\tau_n := \int_{D_{\delta_n}} |\nabla u_n|^2 dx \longrightarrow 0.$
3. $\int_{D\backslash D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \longrightarrow \frac{\pi}{1+\alpha}.$

Proof. Let r_0 be the smallest value in $(0,1)$ such that $\frac{1}{r_0^{2(1+\alpha)} \log^2 r_0} = e^2$. Observe that r_0 exists since $\min_{t\in(0,1)} \dfrac{1}{t^{2(1+\alpha)} \log^2 t} = e^2(1+\alpha)^2 \le e^2$ and $\lim_{t\to 0} \dfrac{1}{t^{2(1+\alpha)} \log^2 t} = +\infty$. We consider the function

$$f(x) := \begin{cases} \frac{1}{|x|^2 \log^2 |x|} & |x| \le r_0 \\ e^2 |x|^{2\alpha} & |x| \in (r_0, 1]. \end{cases} \tag{3.15}$$

Note that $f \in L^1(D)$ and

$$\inf_{x\in D} |x|^{-2\alpha} f(x) = e^2. \tag{3.16}$$

Let us fix $\gamma_n \in (0, \frac{1}{n})$ such that $\int_{D_{\gamma_n}} |\nabla u_n|^2 dx \le \frac{1}{n}$. We define

$$\widetilde{\delta}_n := \inf\left\{ r \in (0,1) \;:\; |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2(x)} \le f(x) \;\text{ for } r \le |x| \le 1 \right\} \in [0,1),$$

and

$$\delta_n := \begin{cases} \widetilde{\delta}_n & \text{if } \widetilde{\delta}_n > 0 \\ \gamma_n & \text{if } \widetilde{\delta}_n = 0. \end{cases}$$

By definition we have

$$|x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} \le f(x) \qquad \text{in} \quad D\backslash D_{\delta_n},$$

thus *3* follows by the dominated convergence Theorem. To conclude the proof it suffices to show that if $n_k \nearrow +\infty$ is chosen so that $\delta_{n_k} = \widetilde{\delta}_{n_k}$ $\forall\, k$, then

$$\lim_{k\to\infty} \delta_{n_k} = \lim_{k\to\infty} \tau_{n_k} = 0. \tag{3.17}$$

For such n_k one has

$$\delta_{n_k}^{2\alpha} e^{4\pi(1+\alpha)u_{n_k}(\delta_{n_k})^2} = f(\delta_{n_k}), \tag{3.18}$$

in particular using (3.16) we obtain

$$e^{4\pi(1+\alpha)u_{n_k}(\delta_{n_k})^2} = \delta_{n_k}^{-2\alpha} f(\delta_{n_k}) \geq e^2 > 1$$

which, by (3.14), yields $\delta_{n_k} \overset{k\to\infty}{\longrightarrow} 0$. Finally, Lemma 3.2 and (3.18) imply

$$1 \geq \delta_{n_k}^{2(1+\alpha)(1-\tau_{n_k})} e^{4\pi(1+\alpha)u_{n_k}(\delta_{n_k})^2} = \frac{\delta_{n_k}^{-2(1+\alpha)\tau_{n_k}}}{\log^2 \delta_{n_k}}$$

so that $\tau_{n_k} \overset{k\to\infty}{\longrightarrow} 0$ (otherwise the limit of the RHS would be $+\infty$). □

Combining Lemma 3.3 with Lemma 3.4 we immediately get Theorem 1.12 for radially symmetric functions:

Proposition 3.4. *If* $u \in H_{rad}$ *and*

$$\sup_{D\setminus D_r} u_n \longrightarrow 0 \qquad \forall\, r \in (0,1),$$

then

$$\limsup_{n\to\infty} E_\alpha(u_n) \leq \frac{\pi(1+e)}{1+\alpha}.$$

Proof. Let $\delta_n \in (0,1)$ be as in Lemma 3.4. Then,

$$\int_{D\setminus D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \longrightarrow \frac{\pi}{1+\alpha}$$

and by Lemma 3.3

$$\limsup_{n\to\infty} \int_{D_{\delta_n}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx \leq \frac{\pi e}{1+\alpha}.$$

□

To pass from Proposition 3.4 to Theorem 1.12 we will use rearrangements. We recall that given a measurable function $u : \mathbb{R}^2 \longrightarrow [0,+\infty)$, the symmetric decreasing rearrangement of u is the unique right-continuous radially symmetric and decreasing function $u^* : \mathbb{R}^2 \longrightarrow [0,+\infty)$ such that

$$|\{u > t\}| = |\{u^* > t\}| \qquad \forall\, t > 0.$$

Among the properties of u^* we recall that

1. If $u \in L^p(\mathbb{R}^2)$, then $u^* \in L^p(\mathbb{R}^2)$ and $\|u^*\|_p = \|u\|_p$.
2. If $u \in H_0^1(D)$, then $u^* \in H_0^1(D)$ and $\int_D |\nabla u^*|^2 dx \le \int_D |\nabla u|^2 dx$. In particular if $u \in H$, then $u^* \in H_{rad}$.
3. If $u, v : \mathbb{R}^2 \longrightarrow [0, +\infty)$, then

$$\int_{\mathbb{R}^2} u^*(x)v^*(x)dx \ge \int_{\mathbb{R}^2} u(x)v(x)dx.$$

In particular if $u \in H$ and $\alpha \le 0$,

$$E_\alpha(u^*) \ge E_\alpha(u). \tag{3.19}$$

Note that the last property does not hold if $\alpha > 0$. We refer the reader to [49] for a more detailed introduction to symmetric rearrangements.

Proof of Theorem 1.12. Take $u_n \in H$ such that $u_n \rightharpoonup 0$ and let u_n^* be the symmetric decreasing rearrangement of u_n. Then $u_n^* \in H_{rad}$ and, since $\|u_n^*\|_2 = \|u_n\|_2 \longrightarrow 0$, we have $\sup_{D\setminus D_r} u_n^* \longrightarrow 0$ $\forall\, r > 0$. Thus from (3.19) and Proposition 3.4 we get

$$\limsup_{n\to\infty} E_\alpha(u_n) \le \limsup_{n\to\infty} E_\alpha(u_n^*) \le \frac{\pi(1+e)}{1+\alpha}.$$

□

In the next section we will need the following local version of Theorem 1.12.

Corollary 3.3. *Fix $\delta > 0$, and take $u_n \in H_0^1(D_\delta)$ such that $\int_{D_\delta} |\nabla u_n|^2 dx \le 1$ and $u_n \rightharpoonup 0$ in $H_0^1(D_\delta)$. For any choice of sequences $\delta_n \to 0$, $x_n \in \Omega$ such that $D_{\delta_n}(x_n) \subset D_\delta$ we have*

$$\limsup_{n\to\infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dv_g \le \frac{\pi e}{1+\alpha}\delta^{2(1+\alpha)}.$$

Proof. Let us consider $\tilde{u}_n(x) := u_n(\delta x)$. Note that $\tilde{u}_n \in H$ and satisfies the hypotheses of Theorem 1.12, hence

$$\limsup_{n\to\infty} \int_{D_\delta} |x|^{2\alpha}(e^{4\pi u_n^2} - 1)dx = \delta^{2(1+\alpha)} \limsup_{n\to\infty} \int_D |x|^{2\alpha}(e^{4\pi \tilde{u}_n^2} - 1)dx \le \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}.$$

Thus we get

$$\limsup_{n\to\infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha} e^{4\pi(1+\alpha)u_n^2} dx = \limsup_{n\to\infty} \int_{D_{\delta_n}(x_n)} |x|^{2\alpha}\left(e^{4\pi(1+\alpha)u_n^2} - 1\right) dx \le$$

$$\le \int_{D_\delta} |x|^{2\alpha}(e^{4\pi u_n^2} - 1)dx \le \delta^{2(1+\alpha)} \frac{\pi e}{1+\alpha}.$$

□

We conclude this section with a proof of the existence of extremals for E_α, $\alpha \in (-1, 0]$.

Proposition 3.5.

$$\sup_H E_\alpha > \frac{\pi(1+e)}{1+\alpha}.$$

Proof. Let us consider the family of functions

$$u_\varepsilon(x) = \begin{cases} c_\varepsilon - \dfrac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right)+L_\varepsilon}{4\pi(1+\alpha)c_\varepsilon} & |x| \le \gamma_\varepsilon \varepsilon \\ -\dfrac{1}{2\pi c_\varepsilon}\log|x| & \gamma_\varepsilon\varepsilon \le |x| \le 1. \end{cases}$$

where $\gamma_\varepsilon = |\log\varepsilon|^{\frac{1}{1+\alpha}}$ and c_ε, L_ε will be chosen later. In order to have $u_\varepsilon \in H_0^1(D)$ we require

$$4\pi(1+\alpha)c_\varepsilon^2 - L_\varepsilon = \log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - 2(1+\alpha)\log\varepsilon \tag{3.20}$$

By direct computations

$$\int_{D_{\gamma_\varepsilon \varepsilon}} |\nabla u_\varepsilon|^2 dx = \frac{1}{4\pi(1+\alpha)c_\varepsilon^2}\left(\log(1+\gamma_\varepsilon^{2(1+\alpha)}) - \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{21+\alpha}}\right)$$

and

$$\int_{D\setminus D_{\gamma_\varepsilon\varepsilon}} |\nabla u_\varepsilon|^2 dx = -\frac{1}{2\pi c_\varepsilon^2}\log(\varepsilon\gamma_\varepsilon),$$

so that

$$\int_D |\nabla u_\varepsilon|^2 dx = \frac{1}{4\pi(1+\alpha)c_\varepsilon^2}\left(\log\left(\frac{1+\gamma^{2(1+\alpha)}}{\gamma^{2(1+\alpha)}}\right) - \frac{\gamma^{2(1+\alpha)}}{1+\gamma^{2(1+\alpha)}} - 2(1+\alpha)\log\varepsilon\right).$$

In particular $u_\varepsilon \in H$ if we choose c_ε so that

$$4\pi(1+\alpha)c_\varepsilon^2 = \log\left(\frac{1+\gamma_\varepsilon^{2(1+\alpha)}}{\gamma_\varepsilon^{2(1+\alpha)}}\right) - \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} - 2(1+\alpha)\log\varepsilon. \tag{3.21}$$

From (3.20) and (3.21) we have

$$L_\varepsilon = -\frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} = -1 + O(\gamma_\varepsilon^{-2(1+\alpha)}) \tag{3.22}$$

and

$$2\pi c_\varepsilon^2 = |\log\varepsilon|(1+o_\varepsilon(1)). \tag{3.23}$$

To estimate $E_\alpha(u_\varepsilon)$ we observe first that in $D_{\gamma_\varepsilon \varepsilon}$

$$u_\varepsilon^2 = c_\varepsilon^2 \left(1 - \frac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon}{4\pi(1+\alpha)c_\varepsilon^2}\right)^2 \geq c_\varepsilon^2 \left(1 - \frac{\log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) + L_\varepsilon}{2\pi(1+\alpha)c_\varepsilon^2}\right) =$$

$$= c_\varepsilon^2 - \frac{1}{2\pi(1+\alpha)} \log\left(1+\left(\frac{|x|}{\varepsilon}\right)^{2(1+\alpha)}\right) - \frac{L_\varepsilon}{2\pi(1+\alpha)}.$$

Thus, using also (3.20) and (3.22),

$$\int_{D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_\varepsilon^2} dx \geq \frac{\pi \varepsilon^{2(1+\alpha)}}{1+\alpha} \frac{\gamma_\varepsilon^{2(1+\alpha)}}{1+\gamma_\varepsilon^{2(1+\alpha)}} e^{4\pi(1+\alpha)c_\varepsilon^2 - 2L_\varepsilon} = \frac{\pi e^{-L_\varepsilon}}{1+\alpha} =$$

$$= \frac{\pi e}{1+\alpha} + O(\gamma_\varepsilon^{-2(1+\alpha)}).$$

Finally, since $e^{4\pi(1+\alpha)u_\varepsilon^2} \geq 1 + 4\pi(1+\alpha)u_\varepsilon^2$ and

$$(1+\alpha)\int_{D \setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} \log^2 |x| dx \geq \delta > 0,$$

using (3.23) we get

$$\begin{aligned}
\int_{D \setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} e^{4\pi(1+\alpha)u_\varepsilon^2} dx \quad &\geq \quad \int_{D \setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} dx + \frac{(1+\alpha)}{\pi c_\varepsilon^2} \int_{D \setminus D_{\gamma_\varepsilon \varepsilon}} |x|^{2\alpha} \log^2 |x| dx \geq \\
&\geq \quad \frac{\pi}{1+\alpha} + O((\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}) + \frac{\delta}{\pi c_\varepsilon^2} = \\
&= \quad \frac{\pi}{1+\alpha} + \frac{2\delta}{|\log \varepsilon|}(1 + o_\varepsilon(1)) + O((\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}).
\end{aligned}$$

Therefore

$$E(u_\varepsilon) \geq \frac{\pi(1+e)}{1+\alpha} + \frac{2\delta}{|\log \varepsilon|}(1 + o_\varepsilon(1)) + O((\gamma_\varepsilon \varepsilon)^{2(1+\alpha)}) + O(\gamma_\varepsilon^{-2(1+\alpha)}).$$

Since $\gamma_\varepsilon = |\log \varepsilon|^{\frac{1}{1+\alpha}}$ one has

$$|\log \varepsilon|(\gamma_\varepsilon \varepsilon)^{2(1+\alpha)} = |\log \varepsilon|^3 \varepsilon^{2(1+\alpha)} = o_\varepsilon(1)$$

and

$$|\log \varepsilon| \gamma_\varepsilon^{-2(1+\alpha)} = |\log \varepsilon|^{-1} = o_\varepsilon(1)$$

so that, for sufficiently small ε,

$$E(u_\varepsilon) \geq \frac{\pi(1+e)}{1+\alpha} + \frac{2\delta}{|\log \varepsilon|}(1 + o_\varepsilon(1)) > \frac{\pi(1+e)}{1+\alpha}.$$

□

Corollary 3.4. $\forall\, \alpha \in (-1, 0]$ *there exists a function* $u_\alpha \in H$ *such that*

$$E_\alpha(u_\alpha) = \sup_H E_\alpha.$$

Proof. Let $u_n \in H$ be a maximizing sequence for E_α. Up to subsequences, we may assume $u_n \rightharpoonup u$. If $u = 0$, then by Theorem 1.12 we would have

$$\sup_H E_\alpha = \lim_{n\to\infty} E_\alpha(u_n) \le \frac{\pi(1+e)}{1+\alpha},$$

which contradicts Proposition 3.5. Thus $u \neq 0$. Since

$$\limsup_{n\to\infty} \|\nabla(u_n - u)\|_2^2 = \limsup_{n\to\infty} \left(\|\nabla u_n\|_2^2 + \|\nabla u\|_2^2 - 2\int_D \nabla u_n \cdot \nabla u \; dx \right) = 1 - \|\nabla u\|_2 < \gamma < 1,$$

by (1.38) we find

$$\int_D |x|^{2\alpha} e^{\frac{4\pi s(1+\alpha)}{\gamma}(u_n - u)^2} dx \le C$$

for some $s > 1$. If we take $1 < p < \frac{1}{\gamma}$, then

$$p u_n^2 = p(u_n - u)^2 + pu^2 + 2pu(u_n - u) \le \frac{1}{\gamma}(u_n - u)^2 + C_{\gamma,p} u^2$$

so that

$$\int_D |x|^{2\alpha} e^{4\pi p(1+\alpha)u_n^2} dx \le \int_D |x|^{2\alpha} e^{\frac{4\pi(1+\alpha)}{\gamma}(u_n-u)^2} e^{C_{\gamma,p}u^2} dx \le$$

$$\le \left(\int_D |x|^{2\alpha} e^{\frac{4\pi s(1+\alpha)}{\gamma}(u_n-u)^2} dx \right)^{\frac{1}{s}} \left(\int_D |x|^{2\alpha} e^{s' C_{\gamma,\varepsilon} u^2} dx \right)^{\frac{1}{s'}} \le C.$$

Here we used $e^{u^2} \in L^q(D)\ \forall\, q \ge 1$ which was proved by Moser in [68] (see also Lemma 3.5). Applying Vitali's convergence Theorem to the measure $|x|^{2\alpha}dx$ we find

$$E_\alpha(u_n) \longrightarrow E_\alpha(u),$$

which concludes the proof. □

3.3 Subcritical Problems, Notations and Prelimiaries

Let (Σ, g) be a smooth, closed Riemannian surface. In this section, and in the rest of the Chapter, we will fix $p_1, \dots, p_m \in \Sigma$ and consider a positive function $h \in C^0(\Sigma \backslash \{p_1, \dots, p_m\})$ satisfying (3.2). Clearly condition (3.2) implies that the limit

$$K(p) := \lim_{q\to p} \frac{h(q)}{d(q,p)^{2\alpha(p)}} \tag{3.24}$$

exists and is strictly positive for any $p \in \Sigma$. Here $\alpha(p)$ is the singularity index (2.1) and d is the Riemannian distance on Σ. We will study the functionals (1.43) on the space (1.5). Let us consider the critical exponent

$$\overline{\beta} := 4\pi(1+\overline{\alpha})$$

where

$$\overline{\alpha} := \min\left\{0, \min_{1\le i\le m} \alpha_i\right\}.$$

Given $s \geq 1$, the symbols $\|\cdot\|_s$, $L^s(\Sigma)$ will denote the standard L^s-norm and L^s-space on Σ with respect to the metric g. Since in many computations we will deal with the singular metric $g_h = hg$, we will also consider

$$\|u\|_{s,h} := \int_\Sigma |u|^s dv_{g_h} = \int_\Sigma h\,|u|^s dv_g$$

and

$$L^s(\Sigma, g_h) := \{u : \Sigma \longrightarrow \mathbb{R} \text{ Borel-measurable}, \; \|u\|_{s,h} < +\infty\}.$$

In this section we will prove the existence of an extremal function for $E^{\beta,\lambda,q}_{\Sigma,h}$ for the subcritical case $\beta < \overline{\beta}$. We begin by stating some well known but useful Lemmas:

Lemma 3.5. *If $u \in H^1(\Sigma)$ then $e^{u^2} \in L^s(\Sigma) \cap L^s(\Sigma, g_h)$, $\forall\, s \geq 1$.*

Proof. Clearly since $h \in L^r(\Sigma)$ for some $r > 1$, it is sufficient to prove that $e^{u^2} \in L^s(\Sigma)$, $\forall\, s \geq 1$. Moreover, since

$$e^{su^2} = e^{s(u-\overline{u})^2+2s(u-\overline{u})\overline{u}+\overline{u}^2} \leq e^{2s(u-\overline{u})^2} e^{2s\overline{u}^2},$$

without loss of generality we can assume $\overline{u} = 0$. Take $\varepsilon > 0$ such that $2s\varepsilon \leq 4\pi$ and a function $v \in C^1(\Sigma)$ satisfying $\|\nabla_g(v-u)\|_2^2 \leq \varepsilon$ and $\overline{v} = 0$. By (1.6), we have

$$\|e^{2s(u-v)^2}\|_1 + \|e^{2s\varepsilon\frac{u^2}{\|\nabla u\|_2}}\|_1 < +\infty. \tag{3.25}$$

Note that

$$e^{su^2} \leq e^{s(u-v)^2} e^{2suv}. \tag{3.26}$$

By (3.25), we have $e^{s(u-v)^2} \in L^2(\Sigma)$ and, since $v \in L^\infty(\Sigma)$,

$$e^{2suv} \leq e^{s\varepsilon\frac{u^2}{\|\nabla u\|_2^2}} e^{C(\varepsilon,s,\|\nabla u\|_2)v^2} \in L^2(\Sigma),$$

Hence using Holder's inequality we get $e^{su^2} \in L^1(\Sigma)$. □

Lemma 3.6. *If $u_n \in \mathcal{H}$ and $u_n \rightharpoonup u \neq 0$ weakly in $H^1(\Sigma)$, then*

$$\sup_n \int_\Sigma h e^{p\overline{\beta}u_n^2} dv_g < +\infty$$

$\forall\, 1 \leq p < \frac{1}{1-\|\nabla u\|_2^2}$.

Proof. Observe that

$$e^{p\overline{\beta}u_n^2} \le e^{p\overline{\beta}(u_n-u)^2} e^{2p\overline{\beta}u_n u}. \tag{3.27}$$

Since

$$\frac{1}{p} > 1 - \|\nabla u\|_2^2 \ge \|\nabla u_n\|_2^2 - \|\nabla u\|_2^2 = \|\nabla(u_n-u)\|_2^2 + o(1) \quad \Longrightarrow \quad \limsup_{n\to\infty} \|\nabla(u_n-u)\|_2^2 < \frac{1}{p},$$

by (1.20) we get $\|e^{p\overline{\beta}(u_n-u)^2}\|_{s,h} \le C$ for some $s > 1$. Taking $\frac{1}{s} + \frac{1}{s'} = 1$, since by Lemma 3.5 $e^{u^2} \in L^q(\Sigma, g_h)$ $\forall\, q \ge 1$, we have

$$e^{2ps'\overline{\beta}u_n u} \le e^{\frac{\overline{\beta}}{2}u_n^2}\, e^{C_{s,\alpha,p}u^2} \in L^1(\Sigma, g_h) \qquad \Longrightarrow \qquad \|e^{2p\overline{\beta}u_n u}\|_{s',h} \le C.$$

Thus from (3.27) we get $\|e^{p\overline{\beta}u_n^2}\|_{1,h} \le C$. □

Existence of extremals for $\beta < \overline{\beta}$ is a simple consequence of Lemma 3.6 and Vitali's convergence Theorem.

Lemma 3.7. *$\forall\, \beta \in (0, \overline{\beta})$, $\lambda \in [0, \lambda_q(\Sigma, g))$, $q > 1$ we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} < +\infty$$

and the supremum is attained.

Proof. Let $u_n \in \mathcal{H}$ be a maximizing sequence for $E_{\Sigma,h}^{\beta,\lambda,q}$, and assume $u_n \rightharpoonup u$ weakly in $H^1(\Sigma)$. We claim that $e^{\beta u_n^2(1+\lambda\|u_n\|_q^2)}$ is uniformly bounded in $L^p(\Sigma, g_h)$ for some $p > 1$. In particular by Vitali's convergence Theorem we get $E_{\Sigma,h}^{\beta,\lambda,q}(u_n) \longrightarrow E_{\Sigma,h}^{\beta,\lambda,q}(u)$ and $E_{\Sigma,h}^{\beta,\lambda,q}(u) = \sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q}$. Since by Lemma 3.5 $E_{\Sigma,h}^{\beta,\lambda,q}(u) < +\infty$, we obtain the conclusion.
If $u = 0$, then

$$\beta(1 + \lambda\|u_n\|_q^2) \longrightarrow \beta < \overline{\beta},$$

and the claim is proved taking $1 < p < \frac{\overline{\beta}}{\beta}$ and using (1.6). If $u \neq 0$, since

$$(1 - \|\nabla u\|_2)(1 + \lambda\|u_n\|_q^2) \le 1 - \|\nabla u\| + \lambda\|u\|_q^2 + o(1) \le 1 - (\lambda_q(\Sigma) - \lambda)\|u\|_q^2 + o(1) < 1$$

we can find $p > 1$ such that $\limsup_{n\to\infty} p(1 + \lambda\|u_n\|_q^2) < \dfrac{1}{1 - \|\nabla u\|_2^2}$, and the claim follows from Lemma 3.6. □

Lemma 3.8. *As $\beta \nearrow \overline{\beta}$ we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} \longrightarrow \sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q}.$$

Proof. Clearly, since $\beta < \overline{\beta}$, we have

$$\limsup_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h} \leq \sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h}.$$

On the other hand, by monotone convergence Theorem we have

$$\liminf_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h} \geq \liminf_{\beta\nearrow+\infty} E^{\beta,\lambda,q}_{\Sigma,h}(v) = E^{\overline{\beta},\lambda,q}_{\Sigma,h}(v) \quad \forall\, v \in \mathcal{H},$$

which gives

$$\liminf_{\beta\nearrow\overline{\beta}} \sup_{\mathcal{H}} E^{\beta,\lambda,q}_{\Sigma,h} \geq \sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h}.$$

□

We conclude this section with some Remarks concerning isothermal coordinates and Green's functions. We recall that, given any point $p \in \Sigma$, we can always find a small neighborhood Ω of p and a local chart

$$\psi : \Omega \longrightarrow D_{\delta_0} \subseteq \mathbb{R}^2 \tag{3.28}$$

such that

$$\psi(p) = 0 \tag{3.29}$$

and

$$(\psi^{-1})^* g = e^{\varphi}|dx|^2 \tag{3.30}$$

with

$$\varphi \in C^{\infty}(\overline{D_\delta}) \qquad \text{and} \qquad \varphi(0) = 0. \tag{3.31}$$

For any $\delta < \delta_0$ we will denote $\Omega_\delta := \psi^{-1}(D_\delta)$. More generally if $D_r(x) \subseteq D_{\delta_0}$ we define $\Omega_r(\psi^{-1}(x)) := \psi^{-1}(D_r(x))$. We stress that (3.30) also implies

$$(\varphi^{-1})^* g_h = |x|^{2\alpha(p)} V(x) e^{\varphi}|dx|^2. \tag{3.32}$$

with

$$0 < V \in C^0(\overline{D_{\delta_0}}) \qquad \text{and} \qquad V(0) = K(p) \tag{3.33}$$

(see (3.24)).

For any $p \in \Sigma$ we denote as G^λ_p the solution of

$$\begin{cases} -\Delta_g G^\lambda_p = \delta_p + \lambda \|G^\lambda_p\|_q^{2-q} |G^\lambda_p|^{q-2} G^\lambda_p - \dfrac{1}{|\Sigma|}\left(1 + \lambda\|G^\lambda_p\|_q^{2-q} \displaystyle\int_\Sigma |G^\lambda_p|^{q-2} G^\lambda_p dv_g\right) \\ \displaystyle\int_\Sigma G^\lambda_p dv_g = 0. \end{cases} \tag{3.34}$$

In local coordinates satisfying (3.28)-(3.33) we have

$$G^\lambda_p(\psi^{-1}(x)) = -\frac{1}{2\pi}\log|x| + A^\lambda_p + \xi(x) \tag{3.35}$$

with $\xi \in C^1(\overline{D_{\delta_0}})$ and $\xi(x) = O(|x|)$. Observe that G^0_p is the standard Green's function for $-\Delta_g$.

Lemma 3.9. *As $\lambda \to 0$ we have $G_p^\lambda \longrightarrow G_p^0$ in $L^s(\Sigma)$ $\forall\, s \geq 1$ and $A_p^\lambda \longrightarrow A_p^0$.*

Proof. Let us denote $c_\lambda := \frac{\lambda}{|\Sigma|}\|G_p^\lambda\|_q^{2-q}\int_\Sigma |G_p^\lambda|^{q-2}G_p^\lambda dv_g$. Observe that

$$-\Delta_g(G_p^\lambda - G_p^0) := \lambda\|G_p^\lambda\|_q^{2-q}|G_p^\lambda|^{q-2}G_p^\lambda - c_\lambda.$$

Since

$$\left\|\|G_p^\lambda\|_q^{2-q}|G_p^\lambda|^{q-2}G_p^\lambda\right\|_{\frac{q}{q-1}} = \|G_p^\lambda\|_q$$

and

$$|c_\lambda| \leq \lambda\|G_p^\lambda\|_q|\Sigma|^{\frac{1-q}{q}},$$

by elliptic estimates we find

$$\|G_p^\lambda - G_p^0\|_{L^\infty(\Sigma)} \leq \|G_p^\lambda - G_p^0\|_{W^{2,\frac{q}{q-1}}(\Sigma)} \leq C\lambda\|G_p^\lambda\|_q. \tag{3.36}$$

In particular

$$\|G_p^\lambda\|_q \leq \|G_p^0\|_q + \|G_p^\lambda - G_p^0\|_q \leq \|G_p^0\|_q + C\|G_p^\lambda - G_p^0\|_\infty \leq \|G_p^0\|_q + C\lambda\|G_p^\lambda\|_q,$$

thus for sufficiently small λ we have

$$\|G_p^\lambda\|_q \leq C\|G_p^0\|_q.$$

Thus by (3.36), as $\lambda \to 0$ we find

$$\|G_p^\lambda - G_p^0\|_{L^\infty(\Sigma)} \longrightarrow 0.$$

In particular $G_p^\lambda \longrightarrow G_p^0$ in L^s for any $s > 1$. Since $A_p^\lambda - A_p^0 = (G_p^\lambda - G_p^0)(p)$ we also get the convergence of A_λ^p. □

Lemma 3.10. *Let (Ω, ψ) be a local chart satisfying (3.28)-(3.33). As $\delta \to 0$ we have*

$$\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\frac{1}{2\pi}\log\delta + A_p^\lambda + \lambda\|G_p^\lambda\|_q^2 + O(\delta|\log\delta|)$$

where $\Omega_\delta = \psi^{-1}(D_\delta)$.

Proof. Integrating by parts we have

$$\int_{\Sigma\setminus\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\int_{\Omega_\delta} \Delta_g G_p^\lambda\, G_p^\lambda dv_g - \int_{\partial\Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial\nu} d\sigma_g. \tag{3.37}$$

For the first term, using the definition of G_p^λ we get

$$\begin{aligned} -\int_{\Omega_\delta} \Delta_g G_p^\lambda\, G_p^\lambda dv_g &= \lambda\|G_p^\lambda\|_q^{2-q}\int_{\Sigma\setminus\Omega_\delta} |G_p^\lambda|^q dv_g - \left(\frac{1}{|\Sigma|} + c_\lambda\right)\int_{\Sigma\setminus\Omega_\delta} G_p^\lambda\, dv_g = \\ &= \lambda\|G_p^\lambda\|_q^2 + O(\delta^2|\log\delta|^q). \end{aligned} \tag{3.38}$$

For the second term we use (3.35) to find

$$-\int_{\partial\Omega_\delta} G_p^\lambda \frac{\partial G_p^\lambda}{\partial\nu} d\sigma_g = \int_{\partial D_\delta} \left(\frac{1}{2\pi}\log\delta - A_p^\lambda + O(\delta)\right)\left(-\frac{1}{2\pi\delta} + O(1)\right) d\sigma = $$
$$= -\frac{1}{2\pi}\log\delta + A_p^\lambda + O(\delta|\log\delta|). \tag{3.39}$$

□

3.4 Blow-up Analysis for the Critical Exponent.

In this section we will study the critical case $\beta = \overline{\beta}$ and prove

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} < +\infty \tag{3.40}$$

Let us fix $q > 1$, $\lambda \in [0, \lambda_q(\Sigma, g))$ and take a sequence $\beta_n \nearrow \overline{\beta}$, $\beta_n < \overline{\beta}$. To simplify the notation we will set $E_n := E_{\Sigma,h}^{\beta_n,\lambda,q}$. By Lemma 3.7, for any n we can take a function $u_n \in \mathcal{H}$ such that

$$E_n(u_n) = \sup_{\mathcal{H}} E_n. \tag{3.41}$$

Up to subsequences, we can always assume that

$$u_n \rightharpoonup u_0 \qquad \text{in } H^1(\Sigma) \tag{3.42}$$

and

$$u_n \longrightarrow u_0 \quad \text{in } L^s(\Sigma) \quad \forall\, s \geq 1. \tag{3.43}$$

Lemma 3.11. *If $u_0 \neq 0$, then*

$$E_n(u_n) \longrightarrow E_{\Sigma,h}^{\overline{\beta},\lambda,q}(u_0). \tag{3.44}$$

In particular we get (3.40) and u_0 is an extremal function.

Proof. If $u_0 \neq 0$ we can argue as in Lemma 3.7 to find $p > 1$ such that $e^{\beta_n u_n^2(1+\lambda\|u_n\|_q^2)}$ is uniformly bounded in $L^p(\Sigma, g_h)$. Vitali's convergence Theorem yields (3.44). Since by Lemma 3.5 we have $E_{\Sigma,h}^{\overline{\beta},\lambda,q}(u_0) < +\infty$, (3.44) and Lemma 3.8 imply

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} = E_{\Sigma,h}^{\overline{\beta},\lambda,q}(u_0) < +\infty.$$

□

Thus it is sufficient to study the case $u_0 = 0$. In the same spirit of Theorem 1.12 and (1.36) we will prove that if $u_0 = 0$, then

$$\limsup_{n\to\infty} E_n(u_n) \leq \frac{\pi e}{1+\overline{\alpha}} \max_{p\in\Sigma,\ \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta} A_p^\lambda} + |\Sigma|_{g_h}, \tag{3.45}$$

where A_p^λ is defined as in (3.35) and $|\Sigma|_{g_h} := \int_\Sigma h\, dv_g$.

Lemma 3.12. *There exists $s > 1$ such that $u_n \in \mathcal{H} \cap W^{2,s}(\Sigma)$ $\forall\, n$. Moreover $\|\nabla u_n\|_2 = 1$ and, if $u_n \rightharpoonup 0$, we have*

$$-\Delta_g u_n = \gamma_n h(x) u_n e^{b_n u_n^2} + s_n(x) \tag{3.46}$$

where

$$b_n := \beta_n (1 + \lambda \|u_n\|_q^2) \longrightarrow \overline{\beta}, \tag{3.47}$$

$$\limsup_n \gamma_n < +\infty \quad \text{and} \quad \gamma_n \int_\Sigma h\, u_n^2 e^{b_n u_n^2} dv_g \longrightarrow 1, \tag{3.48}$$

and

$$s_n := \lambda_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n \tag{3.49}$$

with

$$\lambda_n \longrightarrow \lambda, \tag{3.50}$$

and

$$c_n := \frac{1}{|\Sigma|} \left(\gamma_n \int_\Sigma u_n e^{b_n u_n^2} dv_{g_h} + \lambda_n \|u_n\|_q^{2-q} \int_\Sigma |u_n|^{q-2} u_n dv_g \right) \longrightarrow 0. \tag{3.51}$$

In particular we have

$$\|s_n\|_{\frac{q}{q-1}} \longrightarrow 0. \tag{3.52}$$

Proof. The maximality of u_n clearly implies $\|\nabla u_n\|_2 = 1$. Using Lagrange's multipliers Theorem, it is simple to verify that u_n satisfies

$$-\Delta_g u_n = 2\nu_n b_n h(x) u_n e^{b_n u_n^2} + 2\lambda \nu_n \beta_n \mu_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n. \tag{3.53}$$

where b_n is defined as in (3.47), $\mu_n := \int_\Sigma h\, u_n^2 e^{b_n u_n^2} dv_g$,

$$c_n := \frac{1}{|\Sigma|} \left(2\nu_n b_n \gamma_n \int_\Sigma h u_n e^{b_n u_n^2} dv_g + 2\lambda \nu_n \beta_n \mu_n \|u_n\|_q^{2-q} \int_\Sigma |u_n|^{q-2} u_n dv_g \right), \tag{3.54}$$

and $\nu_n \in \mathbb{R}$. We define $\gamma_n := 2\nu_n b_n$, $\lambda_n := 2\lambda \nu_n \beta_n \mu_n$ and $s_n(x) := \lambda_n \|u_n\|_q^{2-q} |u_n|^{q-2} u_n - c_n$ so that (3.46), (3.49) and (3.51) are satisfied. Observe also that

$$\left\| \|u_n\|_q^{2-q} |u_n|^{q-2} u_n \right\|_{\frac{q}{q-1}} = \|u_n\|_q \longrightarrow 0. \tag{3.55}$$

and

$$\|u_n\|_q^{2-q} \left| \int_\Sigma |u_n|^{q-2} u_n dv_g \right| \leq \|u_n\|_q |\Sigma|^{\frac{1}{q}} \longrightarrow 0 \tag{3.56}$$

If $s_0 > 1$ is such that $h \in L^{s_0}(\Sigma)$, using Lemma 3.5 and standard Elliptic regularity, we find $u_n \in W^{2,s}(\Sigma)$ $\forall\, 1 < s < s_0$. Multiplying (3.53) by u_n and integrating on Σ we get

$$1 = 2\nu_n b_n \mu_n + 2\lambda\nu_n\beta_n\mu_n\|u_n\|_q^2 = 2\nu_n b_n\mu_n\left(1 + \frac{\lambda\beta_n\|u_n\|_q^2}{b_n}\right) = \gamma_n\mu_n(1+o(1))$$

from which we get the second part of (3.48). As a consequence we also have

$$\lambda_n = 2\lambda\nu_n\beta_n\mu_n = \lambda\gamma_n\mu_n\frac{\beta_n}{b_n} \longrightarrow \lambda. \tag{3.57}$$

Now we prove $\limsup_{n\to\infty}\gamma_n < +\infty$ or, equivalently, $\liminf_{n\to\infty}\mu_n > 0$. For any $t > 0$, we have

$$E_n(u_n) \le \frac{1}{t^2}\int_{\{|u_n|>t\}} h\, u_n^2 e^{b_n u_n^2} dv_g + \int_{\{|u_n|\le t\}} h e^{b_n u_n^2} dv_g \le \frac{1}{t^2}\int_\Sigma h u_n^2 e^{b_n u_n^2} dv_g + |\Sigma|_{g_h} + o(1)$$

from which

$$\liminf_{n\to\infty}\mu_n = \liminf_{n\to\infty}\int_\Sigma h\, u_n^2 e^{b_n u_n^2} dv_g \ge t^2\left(\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} - |\Sigma|_{g_h}\right) > 0.$$

It remains to prove that $c_n \longrightarrow 0$ which, by (3.50) and (3.55), completes the proof of (3.52). For any $t > 0$

$$\gamma_n\int_\Sigma h|u_n|e^{b_n u_n^2} dv_g \le \frac{\gamma_n}{t}\int_{\{|u_n|>t\}} h u_n^2 e^{b_n u_n^2} dv_g + \gamma_n\int_{\{|u_n|\le t\}} h|u_n|e^{b_n u_n^2} dv_g = \frac{1+o(1)}{t} + o(1).$$

Since t can be taken arbitrarily large we find

$$\gamma_n\int_\Sigma h|u_n|e^{b_n u_n^2} dv_g \longrightarrow 0.$$

Combined with (3.51) and (3.56), this yields $c_n \to 0$. □

By Lemma 3.12 we know that $u_n \in C^0(\Sigma)$, thus we can take a sequence p_n such that

$$m_n := \max_\Sigma u_n = u_n(p_n). \tag{3.58}$$

Clearly if $\sup_n m_n < +\infty$, then we would have $E_n(u_n) \longrightarrow |\Sigma|_{g_h}$ which contradicts Lemma 3.8. Thus, up to subsequences, we will assume

$$m_n \longrightarrow +\infty \qquad \text{and} \qquad p_n \longrightarrow p. \tag{3.59}$$

For our maximizing sequence u_n it is natural to expect concentration in the regions in which h is larger. In the next Lemma we will indeed show that p must be a minimum point of the singularity index α defined in (2.1). This clarifies the difference between the cases $\overline{\alpha} < 0$ and $\overline{\alpha} = 0$: in the former, the blow-up point p is one of the singular points $p_1, \dots, p_m$, while in the latter $p \in \Sigma\setminus\{p_1, \dots, p_m\}$.

Lemma 3.13. *If $u_n \rightharpoonup 0$, then we have (3.59) with $\alpha(p) = \overline{\alpha}$. Moreover $|\nabla u_n|^2 \rightharpoonup \delta_p$ weakly as measures.*

Proof. Assume by contradiction that $\alpha(p) > \overline{\alpha}$. Let (Ω, ψ) be a local chart in p satisfying (3.28)-(3.33). If $v \in H_0^1(\Omega)$ is such that $\int_\Omega |\nabla v|^2 dv_g \le 1$, then by (1.38) we have

$$\int_\Omega h e^{4\pi(1+\alpha(p))v^2} dv_g \le \sup_{D_{\delta_0}} V e^{\varphi} \int_{D_{\delta_0}} |x|^{2\alpha(p)} e^{4\pi(1+\alpha(p))v(\psi^{-1}(x))^2} dy \le C. \tag{3.60}$$

Take a cut-off function $\xi \in C_0^\infty(\Omega)$ such that $0 \le \xi \le 1$ and $\xi \equiv 1$ in $\Omega_{\frac{\delta_0}{2}}$. Since

$$\int_\Omega |\nabla(u_n\xi)|^2 dv_g = \int_\Sigma |\nabla u_n|^2 \xi^2 dv_g + 2\int_\Omega u_n \xi \nabla u_n \cdot \nabla \xi \, dv_g + \int_\Sigma |\nabla \xi|^2 u_n^2 dv_g \le$$

$$\le (1+\varepsilon)\int_\Sigma |\nabla u_n|^2 \xi^2 dv_g + C_\varepsilon \int_\Sigma |\nabla \xi|^2 u_n^2 dv_g \le (1+\varepsilon) + o(1)$$

and ε can be taken arbitrarily small, we find

$$\limsup_{n\to\infty} \|\nabla(u_n\xi)\|^2_{L^2(\Omega)} \le 1.$$

Thus, applying (3.60) to $v_n := u_n\xi$ and using $\|u_n\|_q \longrightarrow 0$, we find

$$\int_\Omega h e^{\beta(u_n\xi)^2(1+\lambda\|u_n\|_q^2)} \le C$$

for any $\beta < 4\pi(1+\alpha(p))$. In particular, since we are assuming $\overline{\beta} < 4\pi(1+\alpha(p))$,

$$\left\| e^{\overline{\beta} u_n^2 (1+\lambda\|u_n\|_q^2)} \right\|_{L^{s_0}\left(\Omega_{\frac{\delta_0}{2}}, g_h\right)} \le C \tag{3.61}$$

for some $s_0 > 1$. From (3.52), (3.61) and Lemma 3.5, $-\Delta_g u_n$ is uniformly bounded in $L^s(\Omega)$ $\forall\ s < \min\{s_0, \frac{q}{q-1}\}$. If we take another cut-off function $\widetilde{\xi} \in C_0^\infty\left(\Omega_{\frac{\delta_0}{2}}\right)$ such that $\widetilde{\xi} \equiv 1$ in $\Omega_{\frac{\delta_0}{4}}$, applying elliptic estimates to $\widetilde{\xi} u_n$ we find $\sup_{\Omega_{\frac{\delta_0}{4}}} u_n \le C$. This contradicts (3.58)-(3.59).

Therefore we proved $\alpha(p) = \overline{\alpha}$. To prove $|\nabla u_n|^2 \rightharpoonup \delta_p$ we can argue in a similar way. If there existed $r_0 > 0$ such that $\int_{B_{r_0}(p)} |\nabla u_n|^2 dv_g < 1$, then we could find a uniform bound for $-\Delta_g u_n$ in $L^s(B_{r_0}(p))$ for some $s > 1$. Then elliptic estimates would yield $\sup_{\Omega_{\frac{\delta_0}{2}}} u_n \le C$ which, again, contradicts (3.58)-(3.59). □

The next step consists in studying the behavior of u_n near p. Arguing as in [53] and Lemma 2.4, we will prove that a suitable scaling of u_n converges to a solution of the singular Liouville equation

$$-\Delta u = |x|^{2\overline{\alpha}} e^u$$

on $\mathbb{R}^2$. Again we consider a local chart (Ω,ψ) satisfying (3.28)-(3.33). From now on we will denote $x_n := \psi(p_n)$ and $v_n = u_n \circ \psi$. Let us take $r_n > 0$ such that

$$r_n^{2(1+\overline{\alpha})}\gamma_n m_n^2 e^{b_n m_n^2} = 1 \tag{3.62}$$

and consider the scaling

$$\eta_n(x) := m_n(v_n(x_n + r_n x) - m_n).$$

Lemma 3.14. *$m_n^2 r_n^{2(1+\overline{\alpha})} e^{\beta m_n^2} \longrightarrow 0 \ \forall\, \beta < \overline{\beta}$. In particular $r_n m_n^s \longrightarrow 0 \ \forall\, s > 0$.*

Proof. By (3.47), (3.48) and (3.62)

$$\begin{aligned} e^{\beta m_n^2} r_n^{2(1+\overline{\alpha})} m_n^2 &= \frac{e^{(\beta-b_n)m_n^2}}{\gamma_n} = e^{(\beta-b_n)m_n^2}\int_\Sigma h u_n^2 e^{b_n u_n^2} dv_g (1+o(1)) = \\ &\le (1+o(1))\int_\Sigma h u_n^2 e^{\beta u_n^2} dv_g. \end{aligned}$$

Take $s = \frac{\overline{\beta}}{\beta}'$ $\left(\text{i.e. } \frac{1}{s} + \frac{\beta}{\overline{\beta}} = 1\right)$ and $s_0 > 1$ such that $h \in L^{s_0}(\Sigma)$. Then

$$\int_\Sigma h u_n^2 e^{\beta u_n^2} dv_g \le \|u_n^2\|_{s,h} \|e^{\overline{\beta} u_n^2}\|_{1,h}^{\frac{\beta}{\overline{\beta}}} \le C\|h\|_{s_0}^{\frac{1}{s}} \|u_n^2\|_{s s_0'} \longrightarrow 0.$$

□

As in Lemma 2.3, in order to prove the convergence of η_n it is important to verify that, if $\overline{\alpha} < 0$, $\frac{|x_n|}{r_n}$ is bounded. Indeed if $\frac{|x_n|}{r_n} \longrightarrow +\infty$ the disk $D_{r_n}(x_n)$ would not contain the origin and we would not see any singularity in the limit equation for η_n, even if p is a singular point of h. This is excluded by the following Lemma.

Lemma 3.15. *If $\overline{\alpha} = \alpha(p) < 0$, then*

$$\limsup_{n\to\infty} \frac{|x_n|}{r_n} < +\infty$$

.

Proof. Assume by contradiction that $\frac{|x_n|}{r_n} \longrightarrow +\infty$ for a subsequence. Then we take $t_n > 0$ such that

$$|x_n|^{2\overline{\alpha}} t_n^2 \gamma_n m_n^2 e^{b_n m_n^2} = 1.$$

Observe that

$$|x_n|^{2\overline{\alpha}} r_n^2 \gamma_n m_n^2 e^{b_n m_n^2} = \frac{|x_n|^{2\overline{\alpha}}}{r_n^{2\overline{\alpha}}} r_n^{2(1+\overline{\alpha})}\gamma_n m_n^2 e^{b_m m_n^2} = \left(\frac{|x_n|}{r_n}\right)^{2\overline{\alpha}} \longrightarrow 0 \quad \Longrightarrow \quad \frac{t_n}{r_n} \longrightarrow +\infty.$$

and

$$\frac{|x_n|^{2\overline{\alpha}}}{t_n^{2\overline{\alpha}}} = \frac{1}{t_n^{2(1+\overline{\alpha})}\gamma_n m_n^2 e^{b_m m_n^2}} = \left(\frac{r_n}{t_n}\right)^{2(1+\overline{\alpha})} \longrightarrow 0 \quad \Longrightarrow \quad \frac{|x_n|}{t_n} \longrightarrow +\infty.$$

Furthermore, arguing as in Lemma 3.14 we have

$$t_n|x_n|^{2\overline{\alpha}} m_n^2 e^{\beta m_n^2} \longrightarrow 0 \qquad \forall\, \beta < \overline{\beta}$$

and in particular

$$t_n m_n^s \longrightarrow 0 \qquad \forall\ \ s > 0. \tag{3.63}$$

Let us define $\widetilde{\eta}_n(x) = m_n\left(v_n(x_n + t_n x) - m_n\right)$. Then

$$-\Delta\widetilde{\eta}_n = m_n t_n^2 e^{\varphi(x_n+t_n x)}\left(\gamma_n |x_n + t_n x|^{2\overline{\alpha}} V(x_n + t_n x) e^{b_n v_n^2} v_n(x_n + r_n x) + s_n(x_n + t_n x)\right) =$$

$$= e^{\varphi(x_n+t_n x)}\left(\left|\frac{x_n}{|x_n|} + \frac{t_n}{|x_n|}x\right|^{2\overline{\alpha}} V(x_n + t_n x)\left(1 + \frac{\widetilde{\eta}_n}{m_n^2}\right) e^{b_n\left(2\widetilde{\eta}_n + \frac{\widetilde{\eta}_n^2}{m_n^2}\right)} + m_n t_n^2 s_n(x_n + r_n x)\right).$$

Using (3.63) and (3.49), $\forall\, L > 0$ we have

$$\begin{aligned} \int_{D_L} (m_n t_n^2 s_n(x_n + t_n x))^{\frac{q}{q-1}} &= m_n^{\frac{q}{q-1}} t_n^{\frac{2}{q-1}} \int_{D_{Lr_n}(x_n)} |s_n(x)|^{\frac{q}{q-1}} dv_g \\ &\le C m_n^{\frac{q}{q-1}} t_n^{\frac{2}{q-1}} \|s_n\|_{\frac{q}{q-1}} \to 0. \end{aligned} \tag{3.64}$$

Since $\widetilde{\eta}_n \le 0$ and $|\widetilde{\eta}_n| \le m_n$, for any $L > 0$, using (3.64), we find $\|-\Delta\widetilde{\eta}_n\|_{L^\infty(D_L)} \le C$. Moreover $\widetilde{\eta}_n(0) = 0$ thus we can exploit Harnack's inequality to find a uniform bound for $\widetilde{\eta}$ in $W^{2,s}(D_L)$ $\forall\, s > 1$. Using Sobolev's embedding Theorems and a diagonal argument, we find a subsequence such that $\widetilde{\eta}_n \longrightarrow \eta_0$ in $C^1_{loc}(\mathbb{R}^2)$, where η_0 is a solution of

$$-\Delta\eta_0 = V(0)e^{2\overline{\beta}\eta_0}$$

with

$$\eta_0(0) = 0 = \sup_{\mathbb{R}^2} \eta_0,$$

and

$$\int_{\mathbb{R}^2} e^{2\overline{\beta}\eta_0} dv_{g_0} < +\infty.$$

A classification result contained in [31] yields

$$\eta_0 := -\frac{1}{\overline{\beta}}\log\left(1 + \frac{\overline{\beta}V(0)}{4}|x|^2\right).$$

From (3.46) and (3.49) we get

$$1 = -\int_\Sigma \Delta_g u_n u_n dv_g = \gamma_n \int_\Sigma h\, u_n^2 e^{b_n u_n^2} dv_g + \lambda_n \|u_n\|_q^2 \ge \gamma_n \int_{\Omega_{Lt_n}} h\, u_n^2 e^{b_n u_n^2} dv_g + o(1) =$$

$$= V(0)\int_{D_L} e^{2\overline{\beta}\eta_0}dx + o(1) = \frac{V(0)L^2\pi}{1+\frac{\overline{\beta}V(0)}{4}L^2} + o(1). \tag{3.65}$$

Note that

$$\lim_{L\to\infty}\frac{V(0)L^2\pi}{1+\frac{\overline{\beta}V(0)}{4}L^2} = \frac{1}{1+\overline{\alpha}} > 1$$

hence, for sufficiently large L, we get a contradiction in (3.65). □

Lemma 3.16. $\eta_n \longrightarrow \eta_0 := -\frac{1}{\beta}\log(1+\frac{\overline{\beta}V(0)}{4(1+\overline{\alpha})^2}|y|^{2(1+\overline{\alpha})})$ *in* $C^0_{loc}(\mathbb{R}^2)\cap H^1_{loc}(\mathbb{R}^2)$. *Moreover,* $\frac{|x_n|}{r_n}\longrightarrow 0$.

Proof. The function η_n is defined in $D_{\frac{\delta_0}{r_n}}$ and satisfies

$$-\Delta\eta_n = m_n r_n^2 e^{\varphi(x_n+r_ny)}\left(\gamma_n\,|x_n+r_nx|^{2\overline{\alpha}}\,V(x_n+r_nx)e^{b_nv_n^2}v_n(x_n+r_nx)+s_n(x_n+r_nx)\right) =$$

$$= e^{\varphi(x_n+r_ny)}\left(\left|\frac{x_n}{r_n}+x\right|^{2\overline{\alpha}}V(x_n+r_nx)\left(1+\frac{\eta_n}{m_n^2}\right)e^{2b_n\eta_n+b_n\frac{\eta_n^2}{m_n^2}}+r_m^2m_ns_n(x_n+r_nx^2)\right).$$

By Lemma 3.15 if $\overline{\alpha}<0$ we can assume, up to subsequences, that $\frac{x_n}{r_n}\longrightarrow\overline{x}\in\mathbb{R}^2$, so that

$$\left|\frac{x_n}{r_n}+x\right|^{2\overline{\alpha}}\longrightarrow|\overline{x}+x|^{2\overline{\alpha}} \tag{3.66}$$

in $L^s_{loc}(\mathbb{R}^2)$ for some $s>1$. Clearly (3.66) holds also for $\overline{\alpha}=0$. Arguing as in the previous Lemma we can find a subsequence such that $\eta_n\longrightarrow\eta_0$ in $C^0_{loc}(\mathbb{R}^2)\cap H^1_{loc}(\mathbb{R}^2)$, where η_0 is a solution of

$$-\Delta\eta_0 = V(0)|\overline{x}+x|^{2\overline{\alpha}}e^{2\overline{\beta}\eta_0} \tag{3.67}$$

with

$$\eta_0(0) = 0 = \max_{\mathbb{R}^2}\eta_0 \tag{3.68}$$

and

$$\int_{\mathbb{R}^2}|\overline{x}+x|^{2\overline{\alpha}}e^{2\overline{\beta}\eta_0}dv_g < +\infty. \tag{3.69}$$

In [74] is is proved that solutions of (3.67), (3.69) have the form

$$\eta_0 = -\frac{1}{\overline{\beta}}\log\left(1+\frac{\overline{\beta}V(0)e^l}{4(1+\overline{\alpha})^2}|x+\overline{x}|^{2(1+\overline{\alpha})}\right)+\frac{l}{2\overline{\beta}}.$$

for some $l\in\mathbb{R}$. Note that all these functions are radially symmetric and decreasing with respect to $-\overline{x}$. Thus (3.68) is satisfied only if $\overline{x}=0$ and $l=0$. □

The next Lemmas follow the standard arguments in [53], [2].

Lemma 3.17. *For any $A > 1$ we define $u_n^A := \min\{u_n, \frac{m_n}{A}\}$. Then we have*

$$\limsup_{n\to\infty} \int_\Sigma |\nabla u_n^A|^2 dv_g = \frac{1}{A}.$$

Proof. Fix $L > 0$. By Lemma 3.16, for sufficiently large n, $\Omega_{Lr_n} \subseteq \{u_n > \frac{m_n}{A}\}$, hence using (3.46) and (3.49) we find

$$-\int_\Sigma \Delta_g u_n \, u_n^A dv_g = \gamma_n \int_\Sigma h u_n e^{b_n u_n^2} u_n^A dv_g + o(1) \geq \frac{\gamma_n m_n}{A} \int_{\Omega_{Lr_n}} h \, u_n e^{b_n u_n^2} dv_g + o(1) =$$

$$= \frac{m_n \gamma_n}{A} \int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} V(x) v_n e^{b_n v_n^2} e^{\varphi(x)} dx + o(1) =$$

$$= \frac{\gamma_n r_n^{2(1+\overline{\alpha})} m_n^2 e^{b_n m_n^2}}{A} \int_{D_L} \left| \frac{x_n}{r_n} + x \right|^{2\overline{\alpha}} V(x_n + r_n x) e^{\varphi(x_n + r_n x)} \left(1 + \frac{\eta_n}{m_n}\right) e^{2b_n \eta_n + \frac{b_n \eta_n^2}{m_n^2}} dx + o(1) =$$

$$= \frac{V(0)}{A} \int_{D_L} |x|^{2\overline{\alpha}} e^{2\overline{\beta}\eta_0} dx + o(1) = \frac{1}{A} \frac{\pi V(0) L^{2(1+\overline{\alpha})}}{1 + \pi V(0) L^{2(1+\overline{\alpha})}} + o(1).$$

Passing to limit as $n, L \to \infty$ we obtain

$$\liminf_{n\to\infty} \int_\Sigma |\nabla u_n^A|^2 dv_g = \liminf_{n\to\infty} \int_\Sigma \nabla u_n^A \cdot \nabla u_n dv_g = -\int_\Sigma \Delta_g u_n u_n^A \geq \frac{1}{A}. \tag{3.70}$$

Similarly, since

$$-\int_\Sigma \Delta_g u_n \left(u_n - \frac{m_n}{A}\right)^+ dv_g \geq \gamma_n \int_{\Omega_{Lr_n}} h \, u_n e^{b_n u_n^2} \left(u_n - \frac{m_n}{A}\right) dv_g + o(1) =$$

$$= \frac{A-1}{A} V(0) \int_{D_L} |x|^{2\overline{\alpha}} e^{2\overline{\beta}\eta_0} + o(1),$$

we get

$$\liminf_{n\to\infty} \int_\Sigma |\nabla \left(u_n - \frac{m_n}{A}\right)^+ |^2 dv_g \geq \frac{A-1}{A}. \tag{3.71}$$

Clearly $u_n = u_n^A + (u_n - \frac{m_n}{A})^+$ and $\int_\Sigma \nabla u_n^A \cdot \nabla (u_n - \frac{m_n}{A})^+ dv_g = 0$ thus

$$1 = \int_\Sigma |\nabla u_n|^2 dv_g = \int_\Sigma |\nabla u_n^A|^2 dv_g + \int_\Sigma |\nabla \left(u_n - \frac{m_n}{A}\right)^+ |^2 dv_g$$

and from (3.70) and (3.71) we find

$$\lim_{n\to\infty} \int_\Sigma |\nabla u_n^A|^2 dv_g = \frac{1}{A} \quad \text{and} \quad \lim_{n\to\infty} \int_\Sigma |\nabla \left(u_n - \frac{m_n}{A}\right)^+ |^2 dv_g = \frac{A-1}{A}.$$

□

Lemma 3.18.
$$\limsup_{n\to\infty} E_n(u_n) \le \limsup_{n\to\infty} \frac{1}{\gamma_n m_n^2} + |\Sigma|_{g_h}.$$

Proof. For any $A > 1$ we have
$$E_n(u_n) = \int_{\{u_n \ge \frac{m_n}{A}\}} h e^{b_n u_n^2} dv_g + \int_{\{u_n \le \frac{m_n}{A}\}} h e^{b_n (u_n^A)^2} dv_g.$$
By (3.48),
$$\int_{\{u_n \ge \frac{m_n}{A}\}} h e^{b_n u_n^2} dv_g \le \frac{A^2}{m_n^2}\int_\Sigma h u_n^2 e^{b_n u_n^2} dv_g = \frac{A^2}{\gamma_n m_n^2}(1+o(1)).$$
For the last integral we apply Lemma 3.17. Since $\limsup_{n\to\infty} \|\nabla u_n^A\|_2^2 \le \frac{1}{A} < 1$, (1.44) implies that $e^{b_n (u_n^A)^2}$ is uniformly bounded in $L^s(\Sigma, g_h)$ for some $s > 1$. Thus by Vitali's Theorem
$$\int_{\{u_n \le \frac{m_n}{A}\}} h e^{b_n (u_n^A)^2} dv_g \le \int_\Sigma h e^{b_n (u_n^A)^2} dv_g \longrightarrow |\Sigma|_{g_h}.$$
Therefore we proved
$$\limsup_{n\to\infty} E_n(u_n) \le \limsup_{n\to\infty} \frac{A^2}{\gamma_n m_n^2} + |\Sigma|_{g_h}.$$
As $A \to 1$ we get the conclusion. □

Using a similar strategy we prove:

Lemma 3.19. *$\gamma_n m_n h u_n e^{b_n u_n^2} \rightharpoonup \delta_p$ weakly as measures.*

Proof. Take $\xi \in C^0(\Sigma)$. For $L > 0$, $A > 1$ we have
$$\gamma_n m_n \int_\Sigma h\, u_n e^{b_n u_n^2} \xi dv_g =$$
$$= \gamma_n m_n \int_{\Omega_{Lr_n}} h u_n e^{b_n u_n^2} \xi dv_g + \gamma_n m_n \int_{\{u_n > \frac{m_n}{A}\}\setminus\Omega_{Lr_n}} u_n h e^{b_n u_n^2} \xi dv_g + \gamma_n m_n \int_{\{u_n \le \frac{m_n}{A}\}} h u_n e^{b_n u_n^2} \xi dv_g =$$
$$=: I_n^1 + I_n^2 + I_n^3.$$
By Lemma 3.16 we find
$$I_n^1 = \int_{D_L(0)} \left|\frac{x_n}{r_n} + x\right|^{2\overline{\alpha}} V(x_n + r_n x)\left(1 + \frac{\eta_n}{m_n^2}\right) e^{2b_n\eta_n + \frac{b_n\eta_n^2}{m_n^2}} \xi(x_n + r_n x) e^{\varphi(x_n + r_n x)} dx =$$
$$= \xi(p) V(0) \int_{D_L(0)} |x|^{2\overline{\alpha}} e^{2\overline{\beta}\eta_0} dx + o(1) = \xi(p) \frac{\pi V(0) L^{2(1+\overline{\alpha})}}{1 + \pi V(0) L^{2(1+\overline{\alpha})}} + o(1).$$

Similarly, using also (3.48),

$$I_n^2 = m_n \int_{\{u_n > \frac{m_n}{A}\}\setminus \Omega_{Lr_n}} \gamma_n h u_n e^{b_n u_n^2} \xi dv_g \le A \int_{\{u_n > \frac{m_n}{A}\}\setminus \Omega_{Lr_n}} \gamma_n h u_n^2 e^{b_n u_n^2} \xi dv_g =$$

$$= A \max_{\Sigma} \xi \left(\int_\Sigma \gamma_n h u_n^2 e^{b_n u_n^2} dv_g - \int_{\Omega_{Lr_n}} \gamma_n h u_n^2 e^{b_n u_n^2} dv_g \right) =$$

$$= A \left(1 - V(0) \int_{D_L} |x|^{2\overline{\alpha}} e^{2\overline{\beta}\eta_0} dx + o(1) \right) = \frac{A}{1 + \pi V(0) L^{2(1+\overline{\alpha})}}.$$

Therefore

$$\lim_{L\to\infty} \lim_{n\to\infty} I_n^1 = \xi(p) \qquad \text{and} \qquad \lim_{L\to\infty} \lim_{n\to\infty} I_n^2 = 0.$$

For the last integral we apply Lemma 3.17. Since $\limsup_{n\to\infty} \|\nabla u_n^A\|_2^2 \le \dfrac{1}{A} < 1$, (1.44) implies the existence of $s > 1, C > 0$ such that

$$\int_\Sigma h e^{s\overline{\beta}(u_n^A)^2} dv_g \le C$$

thus

$$|I_n^3| \le \gamma_n m_n \|\xi\|_\infty \int_\Sigma h u_n^A e^{b_n (u_n^A)^2} dv_g \le \gamma_n m_n \|\xi\|_\infty \|u_n\|_{q',h} \|e^{\overline{\beta}(u_n^A)^2}\|_{q,h} = \gamma_n m_n o(1).$$

Since by Lemma 3.18 $\gamma_n m_n \longrightarrow 0$, we find $|I_n^3| \longrightarrow 0$ which gives the conclusion. □

Let now G_p^λ be the Green's function defined in (3.34). Using Lemma 3.19 we obtain:

Lemma 3.20. *$m_n u_n \longrightarrow G_p^\lambda$ in $C^0_{loc}(\Sigma\setminus\{p\}) \cap H^1_{loc}(\Sigma\setminus\{p\}) \cap L^r(\Sigma)$ $\forall\, r > 1$.*

Proof. First we observe that $\|m_n u_n\|_q$ is uniformly bounded. If not we could consider the sequence $w_n := \frac{u_n}{\|u_n\|_q}$ which satisfies

$$-\Delta_g w_n = \gamma_n h \frac{m_n u_n}{\|m_n u_n\|_q} e^{b_n u_n^2} + \frac{s_n}{\|u_n\|_q}$$

Being $\|\gamma_n h m_n u_n e^{b_n u_n^2}\|_1 \le C$ and $|s_n| \le C\|u_n\|_q$, we have a uniform bound for $-\Delta_g w_n$ in $L^1(\Sigma)$ and, arguing as the proof of Lemma (2.2), u_n is uniformly bounded in $W^{1,s}(\Sigma)$ for any $1 < s < 2$. The weak limit w of w_n will satisfy

$$\int_\Sigma \nabla w \cdot \nabla \varphi \, dv_g = \lambda \int_\Sigma |w|^{q-2} w \varphi dv_g.$$

for any $\varphi \in C^1(\Sigma)$ such that $\int_\Sigma \varphi dv_g = 0$. But, since $\lambda < \lambda_q(\Sigma, g)$, this implies $w = 0$ which contradicts $\|w_n\|_q = 1$.
Hence $\|m_n u_n\|_q \le C$. This implies that $-\Delta_g(m_n u_n)$ is uniformly bounded in $L^1(\Sigma)$ and, as before, $m_n u_n$ is uniformly bounded in $W^{1,s}(\Sigma)$ for any $s \in (1, 2)$. By Lemma 3.19 we have

$m_n u_n \rightharpoonup G_p^\lambda$ weakly in $W^{1,s}(\Sigma)$, $s \in (1,2)$ and strongly in L^r for any $r \geq 1$. Since $|\nabla u_n|^2 \rightharpoonup \delta_p$, arguing as in Lemma 3.13 one can show that u_n is uniformly bounded in $L^\infty_{loc}(\Sigma\backslash\{p\})$. This implies the boundedness of $-\Delta_g(m_n u_n)$ in $L^s_{loc}(\Sigma\backslash\{p\})$ for some $s > 1$ which gives a uniform bound for $m_n u_n$ in $W^{2,s}_{loc}(\Sigma\backslash\{p\})$. Then, by elliptic estimates, we get $m_n u_n \longrightarrow G_p^\lambda$ in $H^1_{loc}(\Sigma\backslash\{p\}) \cap C^0_{loc}(\Sigma\backslash\{p\})$. □

Using Lemma 3.20 and Corollary 3.3 we can now start the proof of (3.45).

Proposition 3.6. *For any $L > 0$, we have*

$$\limsup_{n\to\infty} \int_{\Omega_{Lr_n}} h e^{b_n u_n^2} dv_g \leq \frac{\pi K(p) e^{1+\overline{\beta} A_p^\lambda}}{1+\overline{\alpha}}.$$

Proof. Fix $\delta > 0$ and set $\tau_n = \int_{\Omega_\delta} |\nabla u_n|^2 dv_g = \int_{D_\delta} |\nabla v_n|^2 dy$. Observe that, by Lemma 3.20,

$$m_n^2(1-\tau_n) = \int_{\Sigma\backslash\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g + o(1), \tag{3.72}$$

and

$$m_n^2 \|u_n\|_q^2 = \|G_p^\lambda\|_q^2 + o(1). \tag{3.73}$$

Since by Lemma 3.10 we have

$$\int_{\Sigma\backslash\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g = -\frac{1}{2\pi}\log\delta + O(1) \overset{\delta\to 0}{\longrightarrow} +\infty \tag{3.74}$$

if δ is sufficiently small, we have

$$\tau_n(1+\lambda\|u_n\|_q^2) = \left(1 - \frac{1}{m^2}\int_{\Sigma\backslash\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g + o\left(\frac{1}{m_n^2}\right)\right)\left(1 + \frac{\lambda}{m_n^2}\|G_p^\lambda\|_q^2 + o\left(\frac{1}{m_n^2}\right)\right) =$$
$$= 1 - \left(\int_{\Omega_\delta} |\nabla G_p^\lambda|^2 dv_g - \lambda\|G_p^\lambda\|_q^2\right)\frac{1}{m_n^2} + o\left(\frac{1}{m_n^2}\right) < 1. \tag{3.75}$$

We denote $d_n := \sup_{\partial D_\delta} v_n$ and $w_n := (v_n - d_n)^+ \in H^1_0(D_\delta)$. Observe that $\frac{w_n}{\tau_n} \longrightarrow 0$ uniformly on $D_\delta\backslash D_{\delta'}$ for any $0 < \delta' < \delta$. Thus applying Corollary 3.3 with $\delta_n = Lr_n$, we find

$$\limsup_{n\to\infty} \int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{\overline{\beta}\frac{w_n^2}{\tau_n}} dx \leq \frac{\pi e}{1+\overline{\alpha}}\delta^{2(1+\overline{\alpha})}. \tag{3.76}$$

Applying Holder's inequality we have

$$\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{b_n v_n^2} dx = e^{b_n d_n^2} \int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{b_n w_n^2 + 2b_n d_n w_n} dx \leq$$

$$\leq e^{b_n d_n^2}\left(\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{\beta_n \frac{w_n^2}{\tau_n}} dx\right)^{\tau_n(1+\lambda\|u_n\|_q^2)} \left(\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}}\right)^{1-\tau_n(1+\lambda\|u_n\|_q^2)}. \tag{3.77}$$

Using Corollary 3.2 we find

$$\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}} \leq \int_{D_\delta} |x|^{2\overline{\alpha}} e^{\frac{2b_n w_n d_n}{1-\tau_n(1+\lambda\|u_n\|_q^2)}} \leq \frac{\pi e^{1+\frac{4b_n^2 d_n^2 \tau_n}{16\pi(1+\overline{\alpha})(1-\tau_n(1+\lambda\|u_n\|_q^2)^2}}}{1+\overline{\alpha}}\delta^{2(1+\overline{\alpha})} \leq$$

$$\leq \frac{\pi e^{1+\frac{b_n d_n^2 \tau_n(1+\lambda\|u_n\|_q^2)}{(1-\tau_n(1+\lambda\|u_n\|_q^2)^2}}}{1+\overline{\alpha}}\delta^{2(1+\overline{\alpha})}.$$

Combining this with (3.76) and (3.77), we find

$$\limsup_{n\to\infty}\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{b_n v_n^2} dx \leq \frac{\pi e \delta^{2(1+\overline{\alpha})}}{1+\overline{\alpha}} \limsup_{n\to\infty} e^{\frac{b_n d_n^2}{1-\tau_n(1+\lambda\|u_n\|_q^2)}}. \tag{3.78}$$

Using (3.75) and Lemma 3.20,

$$\lim_{n\to\infty}\frac{b_n d_n^2}{1-\tau_n(1+\lambda\|u_n\|_q^2)} = \frac{\overline{\beta}(\sup_{\partial\Omega_\delta} G_p^\lambda)^2}{\left(\int_{\Omega_\delta}|\nabla G_p^\lambda|^2 dv_g - \lambda\|G_p^\lambda\|_q^2\right)} =: H(\delta). \tag{3.79}$$

By Lemma 3.10 and (3.35) we find

$$H(\delta) = -2(1+\overline{\alpha})\log\delta + \overline{\beta}A_p^\lambda + o_\delta(1),$$

and from (3.78), (3.79) we obtain

$$\limsup_{n\to\infty}\int_{D_{Lr_n}(x_n)} |x|^{2\overline{\alpha}} e^{b_n v_n^2} dx \leq \frac{\pi e\delta^{2(1+\overline{\alpha})}}{1+\overline{\alpha}} e^{H(\delta)} = \frac{\pi e^{1+\overline{\beta}A_p^\lambda + o_\delta(1)}}{1+\overline{\alpha}}. \tag{3.80}$$

□

Proposition 3.7.

$$\limsup_{n\to\infty} E_n(u_n) \leq \frac{\pi K(p) e^{1+\overline{\beta}A_p^\lambda}}{1+\overline{\alpha}} + |\Sigma|_{g_h}.$$

Proof. $\forall\, L>0$, by Lemma 3.16, we have

$$\gamma_n m_n^2 \int_{\Omega_{Lr_n}} h e^{b_n u_n^2} dv_g = V(0)\int_{D_L} |x|^{2\overline{\alpha}} e^{2\overline{\beta}\eta_0} dx = \frac{\pi V(0) L^{2(1+\overline{\alpha})}}{1+\pi V(0) L^{2(1+\alpha)}} = 1 + o_L(1)$$

where $o_L(1) \longrightarrow 0$ as $L\to\infty$. Thus, using Proposition 3.6,

$$\limsup_{n\to\infty}\frac{1}{\gamma_n m_n^2} = (1+o_L(1))\limsup_{n\to\infty}\int_{\Omega_{Lr_n}} h e^{b_n u_n^2} dv_g \leq (1+o_L(1))\frac{\pi K(p) e^{1+\overline{\beta}A_p^\lambda}}{1+\overline{\alpha}}.$$

The conclusion follows by Lemma 3.18. □

We can summarize the results of this section in the following Proposition.

Proposition 3.8. *$\forall\lambda \in [0, \lambda_q(\Sigma, g))$, $q > 1$ we have*

$$\sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h} < +\infty.$$

Moreover if the supremum is not attained we have

$$\sup_{\mathcal{H}} E^{\overline{\beta},\lambda,q}_{\Sigma,h} \le \frac{\pi e}{1+\overline{\alpha}} \max_{p\in\Sigma,\ \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta}A^{\lambda}_p} + |\Sigma|_{g_h}.$$

3.5 Test Functions and Existence of Extremals.

By Proposition 3.8, in order to prove existence of extremals for $E^{\overline{\beta},\lambda,q}_{\Sigma,h}$ it suffices to show that the value

$$\frac{\pi e}{1+\overline{\alpha}} \max_{p\in\Sigma\ \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta}A^{\lambda}_p} + |\Sigma|_{g_h}.$$

is exceeded.

Proposition 3.9. *There exists $\lambda_0 > 0$ such that $\forall\ 0 \le \lambda \le \lambda_0$ one has*

$$\sup_{u\in H} E^{\overline{\beta},\lambda,q}_{\Sigma,h} > \frac{\pi e}{1+\overline{\alpha}} \max_{p\in\Sigma,\ \alpha(p)=\overline{\alpha}} K(p) e^{\overline{\beta}A^{\lambda}_p} + |\Sigma|_{g_h}.$$

Proof. In local coordinates (Ω, ψ) satisfying (3.28)-(3.33) we define

$$w_\varepsilon(x) := \begin{cases} c_\varepsilon - \frac{\log\left(1+\left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right)+L_\varepsilon}{\overline{\beta}c_\varepsilon} & x\in\Omega_{\gamma_\varepsilon\varepsilon} \\ \frac{G^{\lambda}_p-\eta_\varepsilon\xi}{c_\varepsilon} & x\in\Omega_{2\gamma_\varepsilon\varepsilon}\backslash\Omega_{\gamma_\varepsilon\varepsilon} \\ \frac{G^{\lambda}_p}{c_\varepsilon} & x\in\Sigma\backslash\Omega_{2\gamma_\varepsilon\varepsilon} \end{cases}$$

and

$$u_\varepsilon := \frac{w_\varepsilon}{\sqrt{1+\frac{\lambda}{c_\varepsilon^2}\|G^{\lambda}_p\|_q^2}}$$

where $c_\varepsilon, L_\varepsilon$ will be chosen later, $\gamma_\varepsilon = |\log\varepsilon|^{\frac{1}{1+\overline{\alpha}}}$, ξ is defined as in (3.35) and η_ε is a cut-off function such that $\eta_\varepsilon \equiv 1$ in $\Omega_{\gamma_\varepsilon\varepsilon}$, $\eta_\varepsilon \in C^\infty_c(\Omega_{2\gamma_\varepsilon,\varepsilon})$ and $\|\nabla\eta_\varepsilon\| = O(\frac{1}{\gamma_\varepsilon\varepsilon})$. In order to have $u_\varepsilon \in H^1(\Sigma)$ we have to require

$$\overline{\beta}c_\varepsilon - L_\varepsilon = \log\left(\frac{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}{\gamma_\varepsilon^{2(1+\overline{\alpha})}}\right) + \overline{\beta}A^{\lambda}_p - 2(1+\overline{\alpha})\log\varepsilon. \tag{3.81}$$

Observe that

$$\int_{B_{\gamma_\varepsilon\varepsilon}} |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\overline{\beta}c_\varepsilon^2}\left(\log(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}) - \frac{\gamma_\varepsilon^{2(1+\overline{\alpha})}}{1+\gamma_\varepsilon^{2(1+\overline{\alpha})}}\right) =$$

$$= \frac{1}{\overline{\beta} c_\varepsilon^2} \left(\log(1+\gamma_\varepsilon^{2(1+\overline{\alpha})}) - 1 + O(|\log\varepsilon|^{-2}) \right).$$

Since $\xi \in C^1(\Omega)$ and $\xi(x) = O(|x|)$ we have

$$\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla(\eta_\varepsilon\xi)|^2 = \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla\eta|^2\xi^2 + \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla\xi|^2\eta_\varepsilon^2 + 2\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \eta_\varepsilon\xi\nabla\eta_\varepsilon\cdot\nabla\xi = O((\gamma_\varepsilon\varepsilon)^2),$$

and similarly

$$\int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \nabla G_p^\lambda \cdot \nabla(\eta_\varepsilon\xi) dv_g = O(\gamma_\varepsilon\varepsilon),$$

by Lemma 3.10 we have

$$c_\varepsilon^2 \int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla w_\varepsilon|^2 dv_g = \int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} |\nabla G_p^\lambda|^2 + O(\gamma_\varepsilon\varepsilon) =$$

$$= -\frac{1}{2\pi}\log\gamma_\varepsilon\varepsilon + A_p^\lambda + \lambda\|G_p^\lambda\|_q^2 + O(\gamma_\varepsilon\varepsilon|\log(\gamma_\varepsilon\varepsilon)|).$$

Observe that $\gamma_\varepsilon\varepsilon\log(\gamma_\varepsilon\varepsilon) = o(|\log\varepsilon|^{-2})$, therefore we get

$$\int_\Sigma |\nabla w_\varepsilon|^2 dv_g = \frac{1}{\overline{\beta} c_\varepsilon^2}\left(-1 - 2(1+\overline{\alpha})\log\varepsilon + \overline{\beta}A_p^\lambda + \overline{\beta}\lambda\|G_p^\lambda\|_q^2 + O(|\log\varepsilon|^{-2}) \right).$$

If we chose c_ε so that

$$\overline{\beta} c_\varepsilon^2 = -1 - 2(1+\overline{\alpha})\log\varepsilon + \overline{\beta}A_p^\lambda + O(|\log\varepsilon|^{-2}), \tag{3.82}$$

then $u_\varepsilon - \overline{u}_\varepsilon \in \mathcal{H}$. Observe also that (3.81), (3.82) yield

$$L_\varepsilon = -1 + O(|\log\varepsilon|^{-2}), \tag{3.83}$$

and

$$2\pi c_\varepsilon^2 = |\log\varepsilon| + O(1). \tag{3.84}$$

Since $0 \le w_\varepsilon \le c_\varepsilon$ in $\Omega_{\gamma_\varepsilon\varepsilon}$ we get

$$\int_{\Omega_{\gamma_\varepsilon\varepsilon}} w_\varepsilon dv_g = O(c_\varepsilon(\gamma_\varepsilon\varepsilon)^2) = o(|\log\varepsilon|^{-2}).$$

Moreover

$$\begin{aligned}\int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} w_\varepsilon dv_g &= \int_{\Sigma\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \frac{G_p^\lambda}{c_\varepsilon} dv_g + \int_{\Omega_{2\gamma_\varepsilon\varepsilon}\setminus\Omega_{\gamma_\varepsilon\varepsilon}} \frac{\eta_\varepsilon\xi}{c_\varepsilon} dv_g = \\ &= O\left(\frac{\gamma_\varepsilon\varepsilon|\log(\gamma_\varepsilon\varepsilon)|}{c_\varepsilon}\right) + O\left(\frac{(\gamma_\varepsilon\varepsilon)^3}{c_\varepsilon}\right) = o(|\log\varepsilon|^{-2})\end{aligned}$$

therefore

$$\overline{w}_\varepsilon = o(|\log\varepsilon|^{-2}) = o(c_\varepsilon^{-4}).$$

From this, (3.82), (3.83) it follows that

$$\overline{\beta}(w_\varepsilon - \overline{w}_\varepsilon)^2 \geq \overline{\beta} c_\varepsilon^2 - 2L_\varepsilon - 2\log\left(1 + \left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right) + o(c_\varepsilon^{-2})$$

in $\Omega_{\gamma_\varepsilon \varepsilon}$. Since

$$c_\varepsilon^2 \|w_\varepsilon - \overline{w}_\varepsilon\|_q^2 \geq \left(\int_{\Omega \setminus \Omega_{\gamma_\varepsilon \varepsilon}} |G_p^\lambda|^q dv_g + o(c_\varepsilon^{-2})\right)^{\frac{2}{q}} \geq \|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-2})$$

we find

$$\frac{1}{1 + \frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\left(1 + \frac{\lambda \|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{1 + \frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\right) \geq \frac{1 + 2\frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2 + o(c_\varepsilon^{-4})}{\left(1 + \frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2\right)^2} = 1 - \frac{\lambda^2 \|G_p^\lambda\|_q^4}{c_\varepsilon^4} + o(c_\varepsilon^{-4}).$$

Therefore

$$\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2(1 + \lambda\|u_\varepsilon - \overline{u}_\varepsilon\|_q^2) = \frac{\overline{\beta}(w_\varepsilon - \overline{w}_\varepsilon)}{1 + \frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\left(1 + \frac{\lambda\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{1 + \frac{\lambda}{c_\varepsilon^2}\|G_p^\lambda\|_q^2}\right) \geq$$

$$\geq \overline{\beta} c_\varepsilon^2 - 2L_\varepsilon - 2\log\left(1 + \left(\frac{|\psi(x)|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right) - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2}).$$

It follows that

$$\int_{\Omega_{\gamma_\varepsilon \varepsilon}} h e^{\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon - \overline{u}_\varepsilon\|_q^2)} dv_g \geq \int_{D_{\gamma_\varepsilon \varepsilon}} |x|^{2\overline{\alpha}}(V(0) + O(\gamma_\varepsilon \varepsilon)) \frac{e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})}}{\left(1 + \left(\frac{|x|}{\varepsilon}\right)^{2(1+\overline{\alpha})}\right)^2} dx =$$

$$= \frac{\pi V(0)\varepsilon^{2(1+\overline{\alpha})}\gamma_\varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})(1 + \gamma_\varepsilon^{2(1+\overline{\alpha})})} e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})}(1 + O(\gamma_\varepsilon \varepsilon)) =$$

$$= \frac{\pi K(p)\varepsilon^{2(1+\overline{\alpha})}}{(1+\overline{\alpha})} e^{\overline{\beta}c_\varepsilon^2 - 2L_\varepsilon - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})}(1 + O(c_\varepsilon^{-4})).$$

Using (3.82) and (3.83) we find

$$\overline{\beta} c_\varepsilon^2 - 2L_\varepsilon = -2(1+\overline{\alpha})\log\varepsilon + 1 + \overline{\beta}A_p^\lambda + o(c_\varepsilon^{-2})$$

so that

$$\int_{\Omega_{\gamma_\varepsilon \varepsilon}} h e^{\overline{\beta}(u_\varepsilon - \overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon - \overline{u}_\varepsilon\|_q^2)} dv_g = \frac{\pi K(p) e^{1+\overline{\beta}A_p^\lambda}}{(1+\overline{\alpha})}\left(1 - \frac{\overline{\beta}\lambda^2\|G_p^\lambda\|_q^4}{c_\varepsilon^2} + o(c_\varepsilon^{-2})\right). \tag{3.85}$$

Finally we observe that

$$\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} he^{\overline{\beta}(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u_\varepsilon}\|_q^2)}dv_g \geq \int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} hdv_g+\overline{\beta}\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} h(u_\varepsilon-\overline{u}_\varepsilon)^2(1+\lambda\|u_\varepsilon-\overline{u_\varepsilon}\|_q^2)dv_g \geq$$

$$\geq |\Sigma|_{g_h} + O((\gamma_\varepsilon\varepsilon)^{2(1+\overline{\alpha})}) + \overline{\beta}\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} h(w_\varepsilon-\overline{w}_\varepsilon)^2(1+o(c_\varepsilon^{-4})) =$$

$$= |\Sigma|_{g_h} + \overline{\beta}\int_{\Sigma\setminus\Omega_{2\gamma_\varepsilon\varepsilon}} hw_\varepsilon^2 dvg + O(c_\varepsilon^{-4}) =$$

$$= |\Sigma|_{g_h} + \frac{\overline{\beta}\|G_p^\lambda\|_{L^2(\Sigma,g_h)}}{c_\varepsilon^2} + O(c_\varepsilon^{-4}). \tag{3.86}$$

From (3.85) and (3.86) we find

$$E_{\Sigma,h}^{\overline{\beta},\lambda,q}(u_\varepsilon-\overline{u}_\varepsilon) > \frac{\pi K(p)}{1+\alpha}e^{1+\overline{\beta}A_p^\lambda} + |\Sigma|_{g_h} + \frac{\overline{\beta}}{c_\varepsilon^2}\left(\|G_p^\lambda\|_{L^2(\Sigma,g_h)} - \frac{\pi K(p)e^{1+\overline{\beta}A_p^\lambda}\lambda^2\|G_p^\lambda\|_q^4}{1+\overline{\alpha}}\right) + o(c_\varepsilon^{-2}).$$

By Lemma 3.9, we know that

$$\|G_p^\lambda\|_{L^2(\Sigma,g_h)} - \frac{\pi K(p)e^{1+\overline{\beta}A_p^\lambda}\lambda^2\|G_p^\lambda\|_q^4}{1+\alpha} \longrightarrow \|G_p^0\|_{L^2(\Sigma,g_h)} > 0$$

as $\lambda \to 0$. Thus for sufficiently small λ we get the conclusion. □

We have so proved the existence of extremals for $E_{\Sigma,h}^{\overline{\beta},\lambda,q}$ for $\lambda \in [0,\lambda_0]$. To finish the proof of Theorem 1.14 we have to treat the case $\lambda > \lambda_q(\Sigma,g)$. We will use a family of test functions similar to the one used in [59].

Lemma 3.21. *If $\beta > \overline{\beta}$ or $\beta = \overline{\beta}$ and $\lambda > \lambda_q(\Sigma,g)$, we have*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\beta,\lambda,q} = +\infty.$$

Proof. Take $p \in \Sigma$ such that $\alpha(p) = \overline{\alpha}$ and a local chart (Ω,ψ) satisfying (3.28)-(3.33). Let us define $v_\varepsilon : D_{\delta_0} \longrightarrow [0,+\infty)$,

$$v_\varepsilon(x) := \frac{1}{\sqrt{2\pi}}\begin{cases} \sqrt{\log\frac{\delta_0}{\varepsilon}} & |x| \leq \varepsilon \\ \dfrac{\log\frac{\delta_0}{|x|}}{\sqrt{\log\frac{\delta_0}{\varepsilon}}} & \varepsilon \leq |x| \leq \delta_0 \end{cases}$$

and

$$u_\varepsilon(x) := \begin{cases} v_\varepsilon(\psi(x)) & x \in \Omega \\ 0 & x \in \Sigma\setminus\Omega. \end{cases}$$

It is simple to verify that

$$\int_\Sigma |\nabla u_\varepsilon|^2 dv_g = \int_{D_{\delta_0}} |\nabla v_\varepsilon|^2 dx = 1,$$

thus $u_\varepsilon - \overline{u}_\varepsilon \in \mathcal{H}$. Moreover one has $\overline{u}_\varepsilon = O\left(\left(\log \frac{1}{\varepsilon}\right)^{-\frac{1}{2}}\right)$, hence in Ω_ε we have

$$(u_\varepsilon - \overline{u}_\varepsilon)^2 = \frac{1}{2\pi}\log\left(\frac{\delta_0}{\varepsilon}\right) + O(1).$$

Thus if $\beta > \overline{\beta}$ we have

$$E^{\beta,\lambda,q}_{\Sigma,h}(u_\varepsilon - \overline{u}_\varepsilon) \geq E^{\beta,0,q}_{\Sigma,h}(u_\varepsilon - \overline{u}_\varepsilon) \geq \int_{\Omega_\varepsilon} h e^{\beta(u_\varepsilon - \overline{u}_\varepsilon)^2} dv_g \geq \frac{c}{\varepsilon^{\frac{\beta}{2\pi}}} \int_{D_\varepsilon} |x|^{2\overline{\alpha}} dx =$$

$$= \frac{c\pi}{1+\overline{\alpha}} \varepsilon^{2(1+\overline{\alpha}) - \frac{\beta}{2\pi}} = \tilde{c}\varepsilon^{\frac{\overline{\beta}-\beta}{2\pi}} \longrightarrow +\infty.$$

For the case $\beta = \overline{\beta}$ we take a function $u_0 \in H^1(\Sigma)$ such that

$$\begin{cases} \|\nabla u_0\|_2^2 = \lambda_q(\Sigma, g)\|u_0\|_q^2 \\ \int_\Sigma u_0 \, dv_g = 0 \\ \|u_0\|_q^2 = 1. \end{cases} \tag{3.87}$$

The function u_0 will also satisfy

$$-\Delta_g u_0 = \lambda_q \|u_0\|_q^{2-q} |u_0|^{q-2} u_0 - c \tag{3.88}$$

where

$$c = \frac{\lambda_q}{|\Sigma|}\|u_0\|_q^{2-q} \int_\Sigma |u_0|^{q-2} u_0 \, dv_g.$$

Let us take $t_\varepsilon, r_\varepsilon \longrightarrow 0$ such that

$$t_\varepsilon^2 |\log \varepsilon| \longrightarrow +\infty, \qquad \frac{r_\varepsilon}{\varepsilon}, \longrightarrow +\infty \qquad \text{and} \qquad \frac{\log^2 r_\varepsilon}{t_\varepsilon^2 |\log \varepsilon|} \longrightarrow 0. \tag{3.89}$$

We define

$$w_\varepsilon := u_\varepsilon \eta_\varepsilon + t_\varepsilon u_0$$

where $\eta_\varepsilon \in C^\infty(\Omega_{2r_\varepsilon})$ is a cut-off function such that $\eta_\varepsilon \equiv 1$ in Ω_{r_ε}, $0 \leq \eta_\varepsilon \leq 1$ and $|\nabla \eta_\varepsilon| = O(r_\varepsilon^{-1})$. Observe that

$$\|\nabla w_\varepsilon\|_2^2 = \int_\Sigma |\nabla(u_\varepsilon \eta_\varepsilon)|^2 dv_g + t_\varepsilon^2 \|\nabla u_0\|_2^2 + 2t_\varepsilon \int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g.$$

Using the definition of $u_\varepsilon, \eta_\varepsilon$ and (3.89) we find

$$\int_\Sigma |\nabla \eta_\varepsilon|^2 u_\varepsilon^2 dv_g = O(r_\varepsilon^{-2}) \int_{\Omega_{2r_\varepsilon} \setminus \Omega_{r_\varepsilon}} u_\varepsilon^2 dv_g = O\left(|\log \varepsilon|^{-1} \log^2 r_\varepsilon\right) = o(t_\varepsilon^2)$$

and

$$\left|\int_\Sigma u_\varepsilon \eta_\varepsilon \nabla u_\varepsilon \cdot \nabla \eta_\varepsilon dv_g\right| \le O(r_\varepsilon^{-1}) \int_{\Omega_{2r_\varepsilon}\setminus \Omega_{r_\varepsilon}} |\nabla u_\varepsilon| u_\varepsilon dv_g = O(|\log r_\varepsilon||\log \varepsilon|^{-1}) = o(t_\varepsilon^2).$$

Thus

$$\|\nabla(u_\varepsilon \eta_\varepsilon)\|_2^2 = \int_\Sigma |\nabla u_\varepsilon|^2 \eta_\varepsilon^2 dv_g + o(t_\varepsilon^2) \le 1 + o(t_\varepsilon^2).$$

Moreover (3.87) gives $\|\nabla u_0\|_2^2 = \lambda_q$ and

$$\left|\int_\Sigma \nabla u_0 \cdot \nabla(u_\varepsilon \eta_\varepsilon) dv_g\right| = \lambda_q \|u_0\|_q^{2-q} \left|\int_\Sigma |u_0|^{q-2} u_0 \eta_\varepsilon u_\varepsilon dv_g\right| = O(1)\int_\Sigma u_\varepsilon dv_g = O(|\log\varepsilon|^{-\frac{1}{2}}) = o(t_\varepsilon).$$

Hence we have

$$\|\nabla w_\varepsilon\|_2^2 \le 1 + \lambda_q t_\varepsilon^2 + o(t_\varepsilon^2).$$

Furthermore,

$$\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2 \ge t_\varepsilon^2 \left(\int_{\Sigma\setminus\Omega_{2r_\varepsilon}} |u_0 - \overline{w}_\varepsilon|^q \, dv_g\right)^{\frac{2}{q}} = t_\varepsilon^2 \left(\int_{\Sigma\setminus\Omega_{2r_\varepsilon}} |u_0|^q dv_g\right)^{\frac{2}{q}} + o(t_\varepsilon^2) = t_\varepsilon^2 + o(t_\varepsilon^2)$$

thus

$$\frac{1}{\|\nabla w_\varepsilon\|_2^2}\left(1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2}\right) \ge 1 + (\lambda - \lambda_q) t_\varepsilon^2 + o(t_\varepsilon^2).$$

Finally, since $\overline{w}_\varepsilon = O(|\log\varepsilon|^{-\frac{1}{2}})$, on Ω_ε we find

$$\frac{4\pi(1+\overline{\alpha})(w_\varepsilon - \overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2}\left(1 + \lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2}\right) = (2(1+\overline{\alpha})|\log\varepsilon| + O(1))\left(1 + (\lambda-\lambda_q)t_\varepsilon^2 + o(t_\varepsilon^2)\right) =$$

$$= -2(1+\overline{\alpha})\log\varepsilon + (\lambda - \lambda_q) t_\varepsilon^2 |\log\varepsilon| + o(t_\varepsilon^2|\log\varepsilon|) + O(1),$$

so that

$$E_{\Sigma,h}^{\lambda,\overline{\beta},q}\left(\frac{w_\varepsilon - \overline{w}_\varepsilon}{\|\nabla w_\varepsilon\|_2}\right) \ge \int_{\Omega_\varepsilon} h\, e^{\frac{4\pi(1+\overline{\alpha})(w_\varepsilon - \overline{w}_\varepsilon)^2}{\|\nabla w_\varepsilon\|_2^2}\left(1+\lambda \frac{\|w_\varepsilon - \overline{w}_\varepsilon\|_q^2}{\|\nabla w_\varepsilon\|_2^2}\right)} dv_g \ge$$

$$\ge c\varepsilon^{-2(1+\overline{\alpha})} e^{(\lambda-\lambda_q)t_\varepsilon^2|\log\varepsilon| + o(t_\varepsilon^2|\log\varepsilon|)} \int_{D_\varepsilon} |y|^{2\overline{\alpha}} dy = \tilde{c} e^{(\lambda-\lambda_q)t_\varepsilon^2|\log\varepsilon| + o(t_\varepsilon^2|\log\varepsilon|)} \longrightarrow +\infty$$

as $\varepsilon \to 0$. □

Remark 3.1. *If there exists a point $p \in \Sigma$ such that $\alpha(p) = \overline{\alpha}$ and $u_0(p) > 0$, then one can argue as in [59] to prove that,*

$$\sup_{\mathcal{H}} E_{\Sigma,h}^{\overline{\beta},\lambda,q} = +\infty$$

also for $\lambda = \lambda_q(\Sigma, g_0)$. This is always true if $\overline{\alpha} = 0$.

Chapter 4

Sharp Inequalities and Mass-Quantization for Singular Liouville Systems

Let (Σ, g) be a smooth, closed, connected Riemannian surface. We consider singular Liouville Systems of the form

$$-\Delta_g u_i = \sum_{j=1}^{N} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j} dv_g} - \frac{1}{|\Sigma|} \right) \qquad i = 1, \ldots, N \tag{4.1}$$

where $\rho_i > 0$, $A = (a_{ij})$ is symmetric positive definite matrix and $h_i \in C^\infty(\Sigma \setminus \{p_1, \ldots, p_m\})$ are positive singular weights satisfying (1.19). More precisely, motivated by the equivalence between (4.1) and (1.46) and by the change of variables (1.48) we will assume

$$h_i = K_i e^{-4\pi \sum_{j=1}^{m} \alpha_{ij} G_{p_j}} \tag{4.2}$$

with $K_i \in C^\infty(\Sigma)$, $K_i > 0$ and some coefficients $\alpha_{ij} > -1$. Throughout this Chapter, α_i will denote the singularity index associated to h_i, that is

$$\alpha_i(x) = \begin{cases} \alpha_{ij} & x = p_j \\ 0 & x \in \Sigma \setminus \{p_1, \ldots, p_m\}. \end{cases}$$

System (1.46) is the Euler-Langrange equation for the functional

$$J_{\underline{\rho}}(\underline{u}) = \frac{1}{2} \sum_{i,j=1}^{N} a^{ij} \nabla u_i \cdot \nabla u_j \, dv_g - \sum_{i=1}^{N} \rho_i \log \left(\int_\Sigma h_i e^{u_i - \overline{u}_i} dv_g \right).$$

Here, and in the rest of the chapter $\underline{u} = (u_1, \ldots, u_N)$ and $\underline{\rho} = (\rho_1, \ldots, \rho_N)$. The simplest way of finding solutions of (4.1) is trying to minimize $J_{\underline{\rho}}$ on $H^1(\Sigma)^N$. In the first section will give the proof of Theorem 1.16. which gives necessary and sufficient conditions for the boundedness

of $J_{\underline{\rho}}$ from below. The dual approach that we will present is a special case of a general duality principle.
Let X be a Banach space and let $F : X \longrightarrow (-\infty, +\infty]$ be a convex, lower semicountinuous map. We recall that the domain of F and the Legendre transform $F^* : X^* \longrightarrow \mathbb{R}$ of F are defined as

$$D(F) := \{x \in X \ : \ F(x) < +\infty\}$$

and

$$F^*(y) = \sup_{x\in X} < y,\ x > -F(x) = \sup_{x\in D(F)} < y,\ x > -F(x) \qquad \forall\, y \in X^*.$$

Here X^* denotes the dual space of X and $< \cdot, \cdot >$ the duality product. The Legendre transform is involutive, that is

$$F(x) = F^{**}(x) := \sup_{y\in X^*} < y, x > -F^*(y).$$

see [17]. Given two convex, lower semicontinuous functions $F,\ G : X \longrightarrow (-\infty, +\infty]$ one can consider the map $W : D(G) \times D(F^*) \longrightarrow \mathbb{R}$ defined by

$$W(x, y) = F^*(y) + G(x) - < y, x > .$$

Observe that

$$\inf_{y\in D(F^*)} W(x, y) = G(x) - F^{**}(x) = G(x) - F(x)$$

and

$$\inf_{x\in D(G)} W(x, y) = F^*(y) - G^*(y).$$

This proves that for the functionals

$$J(x) := \begin{cases} G(x) - F(x) & x \in D(G) \\ +\infty & x \notin D(G) \end{cases} \qquad \text{and} \qquad J^*(y) := \begin{cases} F^*(y) - G^*(y) & y \in D(F^*) \\ +\infty & y \notin D(F^*) \end{cases}$$

one has

$$\inf_{x\in X} J(x) = \inf_{y\in X^*} J^*(y). \tag{4.3}$$

If $X = H_0^N$, then we can write $J_{\underline{\rho}}$ in the form $J_{\underline{\rho}} = G(\underline{u}) - F(\underline{u})$ where

$$G(\underline{u}) := \sum_{i=1}^{N} a^{ij} \int_\Sigma \nabla u_i \cdot \nabla u_j dv_g$$

and

$$F(\underline{u}) := \sum_{i=1}^{N} \rho_i \log\left(\int_\Sigma h_i e^{u_i - \overline{u}_i} dv_g \right).$$

Therefore (4.3) shows that the minimization problem for $J_{\underline{\rho}}$ can be reduced to a minimization problem on H_0^* (more precisely on $D(F^*)$). The explicit expression of the dual functional and a more rigorous proof of the duality principle will be given in section 4.1.

The last two sections of the thesis are devoted to blow-up analysis for Liouville systems. In section 4.2 we will prove the following concentration compactness Theorem, which is a generalized version of a result by Lucia and Nolasco [61].

Theorem 4.1. *Assume that A is a symmetric positive definite matrix satisfying* (1.52) *and h_i has the form* (4.2) *with $K_i \in C^\infty(\Sigma)$ and $K_i > 0$. Let $\underline{u}_n = (u_{1,n} \dots, u_{N,n}) \in H_0^N$ be a sequence of solutions of* (4.1) *with $\rho_i = \rho_{i,n} \longrightarrow \overline{\rho}_{i,n}$ for $i = 1, \dots, N$. Up to subsequences, one of the following alternatives holds:*

- *(Compactness) $u_{i,n}$ is bounded in $W^{2,q}(\Sigma)$ for $i = 1, \dots, N$, $q > 1$.*
- *(Blow-up) There exist N finite sets $S_1, \dots, S_N$ such that $u_{i,N}^+$ is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S_i)$, $i = 1, \dots, N$. If $S = S_1 \cup \cdots S_N$ then, $\forall\, i \in \{1, \dots, N\}$, either $u_{i,n}$ is bounded in $L^\infty_{loc}(\Sigma \backslash S)$ or $u_{i,n} \longrightarrow -\infty$ locally uniformly in $\Sigma \backslash S$.*

Moreover, denoting by μ_i the weak limit of the sequence of measures $V_i e^{u_{i,n}}$, one has

$$\mu_i = r_i + \sum_{x \in S_i} \sigma_i(x) \delta_x$$

with $r_i \in L^1(\Sigma) \cap L^q_{loc}(\Sigma \backslash S_i) \cap L^\infty_{loc}(\Sigma \backslash (S_i \cup \{p_1, \dots, p_m\}))$ for some $q > 1$, and $\sigma_i(x) \geq \frac{4\pi}{a_{ii}} \min\{1, 1 + \alpha_i(x)\}$ $\forall x \in S_i$, $i = 1, \dots, N$.

Theorem 4.1 is weaker that its scalar version Theorem 1.2 for two main reasons. The first is that it does not give a complete description of the local concentration values $\sigma_1(x), \dots, \sigma_N(x)$. The second is the presence of the residual terms r_i, $i = 1, \dots, N$. For the special case of the $SU(3)$ Toda System, that is for $N = 2$ and

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

the first issue was addressed in [44] and [56]. Theorem B gives a complete description of the values $\sigma_1(x), \sigma_2(x)$ in the regular case, while, for the singular case, Theorem C gives a partial characterization showing that σ_1, σ_2 can only assume a finite number of values. In order to prove Theorems 1.17 and 1.18 one has to deal with the presence of the residual terms. Observe that

$$r_i \equiv 0 \qquad \Longrightarrow \qquad \overline{\rho}_i = \sum_{x \in S_i} \sigma_i(x) \tag{4.4}$$

and, in particular, in this case the limit parameter ρ_i must be a sum of the finitely many possible values of σ_i. In general, one can not prove that both r_1 and r_2 vanish. Some examples were given in [36]. A local example is also given by the family of functions

$$u_1^\alpha(x) = \log\left(8 \frac{1 + \alpha^2(|x|^4 + 2|x|^2)}{(1 + 2|x|^2 + \alpha^2|x|^4)^2}\right) \qquad u_2^\alpha(x) = \log\left(8 \frac{\alpha^2\left(1 + 2|x|^2 + \alpha^2|x|^4\right)}{(1 + \alpha^2(|x|^4 + 2|x|^2))^2}\right)$$

on the unit disk of D. These functions solve the Toda System

$$\begin{cases} -\Delta u_1^\alpha = 2e^{u_1^\alpha} - e^{u_2^\alpha} \\ -\Delta u_2^\alpha = 2e^{u_2^\alpha} - e^{u_1^\alpha} \end{cases} \tag{4.5}$$

on $\mathbb{R}^2$ (actually a complete classification of the solutions of (4.5) on $\mathbb{R}^2$ was given in [46]). As $a \to +\infty$ both the components blow-up since $u_1^\alpha(1/\alpha) \longrightarrow +\infty$ and $u_2^\alpha(0) \longrightarrow +\infty$. Moreover one has $u_1^\alpha \longrightarrow -\infty$ in uniformly on compact subsets on $D\backslash\{0\}$ and $u_2^\alpha \longrightarrow \log\left(\frac{8}{(2+x^2)^2}\right)$ in $L^\infty_{loc}(D\backslash\{0\})$. Thus $r_2 \not\equiv 0$.

In section 4.3 we will prove that in Theorem 4.1 at least one of the r_i's must always vanish. Using this and (4.4), we will obtain Theorems 1.17, 1.18 for $SU(3)$ Toda Systems.

4.1 Lower Bounds: A Dual Approach.

Let us consider the convex function $\Phi(t) = (1+|t|)\log(1+|t|) - |t|$ and the space

$$X := \left\{ v : \Sigma \longrightarrow \mathbb{R} \ : \ \int_\Sigma \Phi(v) dv_g < +\infty \right\}$$

endowed with the norm

$$\|v\|_X := \inf\left\{ \lambda > 0 \ : \ \int_\Sigma \Phi\left(\frac{v}{\lambda}\right) \le 1 \right\}.$$

$(X, \|\cdot\|_X)$ is known as the Orlicz's space associated to Φ. In particular, for our choice of Φ, $(X, \|\cdot\|_X)$ is a reflexive Banach space. We refer the reader to [75] for a general introduction on the theory of Orlicz spaces.

Consider now the set

$$\Gamma(\underline{\rho}) = \left\{ \underline{v} = (v_1, \ldots, v_n) \in X^N \ : \ v_i \ge 0, \ \int_\Sigma v_i dv_g = \rho_i, \ i = 1, \ldots, N \right\} \tag{4.6}$$

and the functional

$$\Psi(\underline{v}) := \sum_{i=N}^{\infty} \int_\Sigma v_i \left(\log v_i - \log h_i\right) dv_g - \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \int_\Sigma \int_\Sigma G(x,y) v_i(x) v_j(y) dv_g(x) dv_g(y). \tag{4.7}$$

The main goal of this section is to prove that $J_{\underline{\rho}}$ is bounded from below on $H^1(\Sigma)^N$ if and only if Ψ is bounded from below on $\Gamma(\underline{\rho})$. We shall begin by proving that Ψ is well defined on X^N. A crucial role will be played by the following elementary inequality:

Lemma 4.1. $\forall\, a \in \mathbb{R}$, $b \in \mathbb{R}^+$ *one has*

$$ab \le e^a + b\log b - b. \tag{4.8}$$

Proof. It follows from the duality relation between the functions $f_1(x) = e^x$ and $f_2(x) = x\log x - x$. Specifically, $\forall\, b > 0$ one has

$$\sup_{a\in\mathbb{R}} \left(ab - e^a\right) = b\log b - b,$$

which implies the conclusion. □

Lemma 4.2. *Let $\xi : \Sigma \longrightarrow \mathbb{R}$ be such that $e^{\delta|\xi|} \in L^1(\Sigma)$ for some $\delta > 0$. For any $v \in X$ we have $v, v\log|v|, \xi v \in L^1(\Sigma)$. Moreover the functional $l_\xi(v) := \int_\Sigma v\xi dv_g$ is continuous on X.*

Proof. Since $\lim_{t\to\infty} \frac{\Phi(t)}{t\log t} = 1$, there exists $t_0 > 1$ such that $t\log t \leq 2\Phi(t)$ for $t \geq t_0$. It follows that

$$\int_\Sigma |v\log|v||dv_g \leq C + \int_{\{|v|\geq t_0|\}} |v|\log|v|dv_g \leq C + 2\int_\Sigma \Phi(v)dv_g < +\infty.$$

By definition of $\|\cdot\|_X$, if $v \not\equiv 0$, we have

$$\int_\Sigma \Phi\left(\frac{v}{\|v\|_X}\right)dv_g \leq 1,$$

therefore, using (4.8) we find

$$\left|\int_\Sigma \frac{v\xi}{\|v\|_X}dv_g\right| \leq \frac{1}{\delta}\int_\Sigma \frac{|v|}{\|v\|_X}\log\left(\frac{|v|}{\delta\|v\|_X}\right)dv_g + \int_\Sigma e^{\delta|\xi|} \leq C_{\delta,\xi} + \frac{2}{\delta}\int_\Sigma \Phi\left(\frac{v}{\|v\|_X}\right)dv_g \leq \widetilde{C}_{\delta,\xi}.$$

Hence

$$\left|\int_\Sigma v\;\xi\;dv_g\right| \leq \widetilde{C}_{\delta,\xi}\|v\|_X.$$

□

Lemma 4.3. *For any $v \in X$. we have*

$$\int_\Sigma\int_\Sigma |G(x,y)||v(x)||v(y)|dv_g(x)dv_g(y) < +\infty.$$

Proof. Without loss of generality we may assume $\|v\|_X > 0$. Let us denote

$$\xi(x) := \int_\Sigma |G(x,y)||v(y)|dv_g(y).$$

By the properties of the Green function it is possible to find $\delta > 0$ such that

$$\sup_{y\in\Sigma}\int_\Sigma e^{\delta\|v\|_{L^1(\Sigma)}|G(x,y)|}dv_g(x) < +\infty.$$

For such δ, applying Jensen's inequality we find

$$\int_\Sigma e^{\delta\,\xi}dv_g \leq \int_\Sigma\int_\Sigma e^{\delta\,\|v\|_{L^1(\Sigma)}|G(x,y)|}\frac{|v(y)|}{\|v\|_{L^1(\Sigma)}}dv_g(y)dv_g(x) \leq C\int_\Sigma \frac{|v(y)|}{\|v\|_{L^1(\Sigma)}}dv_g(y) \leq C.$$

Therefore $e^{\delta\xi} \in L^1(\Sigma)$ and the conclusion follows from Lemma 4.2. □

Lemmas 4.2 and 4.3 show that Ψ is well defined on $\Gamma(\underline{\rho})$.

Lemma 4.4. • *If $v_n \in X$ then*

$$\|v_n\|_X \longrightarrow +\infty \qquad \Longrightarrow \qquad \int_\Sigma v_n \log v_n dv_g \longrightarrow +\infty.$$

• *If $v_n \rightharpoonup v$ weakly in X, $v_n \geq 0$ then*

$$\int_\Sigma v \log v dv_g \leq \liminf_{n\to\infty} \int_\Sigma v_n \log v_n dv_g.$$

Proof. Assume that $\|v_n\|_X \longrightarrow +\infty$. Since $\forall\, \lambda > 1$ we have

$$\int_\Sigma \Phi\left(\frac{|v_n|}{\lambda}\right) dv_g \leq \frac{1}{\lambda}\int_\Sigma \Phi(|v_n|) dv_g,$$

we get $\int_\Sigma \Phi(|v_n|)dv_g \longrightarrow +\infty$. Let us now take t_0 such that $\Phi(t) \leq 2t\log t$ for $t \geq t_0$. Clearly

$$\int_{\{|v_n|\leq t_0\}} \Phi(|v_n|)dv_g \leq |\Sigma|\Phi(t_0) \qquad \Longrightarrow \qquad \int_{\{|v_n|\geq t_0\}} \Phi(|v_n|)dv_g \longrightarrow +\infty.$$

Since

$$\int_{\{|v_n|\geq t_0\}} \Phi(|v_n|)dv_g \leq 2\int_{\{|v_n|\geq t_0\}} |v_n|\log|v_n|dv_g \leq 2\int_\Sigma |v_n|\log|v_n|dv_g + C$$

we obtain

$$\int_\Sigma |v_n|\log|v_n|dv_g \longrightarrow +\infty.$$

Assume now that $v_n \rightharpoonup v$. Let us select a subsequence such that

$$\liminf_{n\to\infty} \int_\Sigma v_n \log v_n dv_g = \lim_{k\to\infty}\int_\Sigma v_{n_k}\log v_{n_k} dv_g.$$

By Lemma 4.2 we know that $\int_\Sigma v_{n_k} dv_g \longrightarrow \int_\Sigma v dv_g$, therefore extracting a further subsequence we may assume $v_{n_k} \longrightarrow v$ a.e. on Σ. Thus, using Fatou's Lemma we get

$$\int_\Sigma v\log v dv_g \leq \liminf_{k\to\infty}\int_\Sigma v_{n_k}\log v_{n_k} dv_g = \liminf_{n\to\infty}\int_\Sigma v_n \log v_n dv_g.$$

□

Let us consider the functional $W : H_0^N \times X^N \longrightarrow H_0^1(\Sigma)$ defined by

$$W(\underline{u}, \underline{v}) = \sum_{i=1}^N \int_\Sigma v_i \log v_i dv_g + \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \; dv_g - \sum_{i=1}^N \int_\Sigma (u_i + \log h_i)v_i dv_g.$$

Lemma 4.5. *For any $\underline{u} \in H_0^N$ we have*

$$\min_{\underline{v}\in\Gamma(\underline{\rho})} W(\underline{u},\underline{v}) = J_{\underline{\rho}}(\underline{u}) + \sum_{i=1}^{N} \rho_i \log\left(\frac{\rho_i}{|\Sigma|}\right).$$

Moreover the minimum is attained by the functions

$$v_{0,i} = \frac{\rho_i h_i e^{u_i}}{\int_\Sigma h_i e^{u_i} dv_g} \qquad i = 1,\dots,N.$$

Proof. By Lemmas 4.2, 4.4, $\Gamma(\underline{\rho})$ is a weakly closed subset of X and the functional $\underline{v} \longrightarrow W(\underline{u},\underline{v})$ is convex and weakly lower semicontinuous on $\Gamma(\underline{\rho})$. Take $p > 1$ such that $h_i \in L^p(\Omega)$, $i = 1,\dots,N$ and $\gamma,\varepsilon > 0$ such that $\gamma + \frac{1}{p} < 1 - \varepsilon$. By (4.8) we have

$$\int_\Sigma (u_i + \log h) v_i \, dv_g \le \int_\Sigma e^{\frac{u_i}{\gamma}} dv_g + \gamma \int_\Sigma v_i \log(\gamma v_i) dv_g + \int_\Sigma h^p dv_g + \frac{1}{p}\int_\Sigma v_i \log\left(\frac{v_i}{p}\right) dv_g \le$$
$$\le C_{p,h,\gamma,\rho_i} + \left(\gamma + \frac{1}{p}\right)\int_\Sigma v_i \log v_i dv_g.$$

Therefore we get

$$W(\underline{u},\underline{v}) \ge \varepsilon \sum_{i=1}^{N} \int_\Sigma v_i \log v_i dv_g - C_{p,h\gamma,\underline{\rho},\varepsilon,\underline{u}}. \tag{4.9}$$

By Lemma 4.4, this implies the coercivity condition

$$\|\underline{v}_n\|_{X^N} := \sum_{i=1}^{N} \|v_{i,n}\|_X \longrightarrow +\infty \qquad \Longrightarrow \qquad W(\underline{u},\underline{v}_n) \longrightarrow +\infty.$$

Therefore, using standard minimization techniques we find $\underline{v}_0 \in \Gamma(\underline{\rho})$ such that

$$W(\underline{u},\underline{v}_0) = \min_{\underline{v}\in\Gamma(\underline{\rho})} W(\underline{u},\underline{v}).$$

Moreover $\underline{v}_0$ must satisfy

$$\log v_{0,i} - (u_i + \log h_i) = \lambda_i \qquad i = 1,\dots,N, \tag{4.10}$$

or, equivalently

$$v_{0,i} = e^{\lambda_i} h_i e^{u_i} \qquad i = 1,\dots,N, \tag{4.11}$$

for some $\lambda_i \in \mathbb{R}$. Integrating (4.11) over Σ we find

$$\lambda_i = \log\rho_i - \log\left(\int_\Sigma h_i e^{u_i} dv_g\right) = \log\frac{\rho_i}{|\Sigma|} - \log\left(\frac{1}{|\Sigma|}\int_\Sigma h_i e^{u_i} dv_g\right). \tag{4.12}$$

Replacing (4.10), (4.12) into the definition of $W(\underline{u},\underline{v}_0)$ we find

$$W(\underline{u},\underline{v}_0) = \frac{1}{2}\sum_{i,j=1}^{N} a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \, dv_g + \sum_{i=1}^{N} \lambda_i \int_\Sigma v_i dv_g = J_{\underline{\rho}}(\underline{u}) + \sum_{i=1}^{N} \rho_i \log\left(\frac{\rho_i}{|\Sigma|}\right).$$

which concludes the proof. □

Lemma 4.6. *For any $\underline{v} \in \Gamma(\underline{\rho})$ we have*

$$\min_{\underline{u} \in H_0^N} W(\underline{u}, \underline{v}) = \Psi(\underline{v}).$$

Moreover the minimum is attained by the functions $u_{0,i} \in H_0$ satisfying

$$-\Delta_{g_0} u_{0,i} = \sum_{j=1}^{N} a_{ij} \left(v_{0,j} - \frac{\rho_j}{|\Sigma|} \right).$$

Proof. By (4.8) and (1.22) we find that

$$\left| \int_\Sigma \frac{u_i}{\|\nabla u_i\|_2} v_i \, d_g \right| \leq \int_\Sigma e^{\frac{u_i}{\|\nabla u_i\|_2}} dv_g + \int_\Sigma v_i \log v_i dv_g$$
$$\leq C + \int_\Sigma v_i \log v_i dv_g \leq C_{\underline{v}}.$$

It follows that

$$J_{\underline{\rho}}(\underline{u}) \geq \frac{1}{\theta} \sum_{i=1}^{N} \|\nabla u_i\|_2^2 - C_v \sum_{i=1}^{N} \|\nabla u\|_2 - C_{\underline{v}, h_i, A}$$

so that $\underline{u} \longrightarrow W(\underline{u}, \underline{v})$ is a coercive and lower semicontinuous functional on H_0. Therefore it has a minimum point $\underline{u}_0 \in H_0^N$ which satisfies

$$\sum_{k=1}^{N} a^{jk} \Delta_g u_{0,k} + v_j = \lambda_j \qquad j = 1, \ldots, N.$$

Integrating over Σ one finds $\lambda_j = \frac{\rho_j}{|\Sigma|}$, $j = 1, \ldots N$. Multiplying the j^{th} equation for a_{ij} and taking the sum over j we get

$$-\Delta_g u_{0,i} = \sum_{j=1}^{N} a_{ij} \left(v_j - \frac{\rho_j}{|\Sigma|} \right).$$

Integrating by parts and applying Green's representation formula we have

$$\sum_{i,j=1}^{N} a^{ij} \int_\Sigma \nabla u_{0,i} \cdot \nabla u_{0,j} \, dv_g = \sum_{i,j=1}^{N} a^{ij} \int_\Sigma \int_\Sigma G(x,y) \Delta_g u_{0,i}(x) \Delta u_{0,j}(y) dv_g(x) dv_g(y) =$$
$$= \sum_{i,j=1}^{N} a_{ij} \int_\Sigma \int_\Sigma G(x,y) v_i(x) v_j(y) dv_g(x) dv_g(y).$$

Similarly

$$\int_\Sigma v_i \; u_{0,i} \, dv_g = \sum_{j=1}^{N} a_{ij} \int_\Sigma \int_\Sigma v_i(x) G(x,y) v_j(y) dv_g(y)$$

so that

$$W(\underline{u}_0, \underline{v}) = \Psi(\overline{v}).$$

□

We have so proved the following duality property:

Proposition 4.1.

$$\inf_{\underline{v}\in\Gamma(\underline{\rho})} \Psi(\underline{v}) = \inf_{\underline{u}\in H_0^N} J_\rho(\underline{u}) + \sum_{i=1}^N \rho_i \log\left(\frac{\rho_i}{|\Sigma|}\right).$$

Moreover existence of minimizers for the two problems is equivalent.

Proof. It follows from

$$\inf_{\underline{v}\in\Gamma(\underline{\rho})} \inf_{\underline{u}\in H_0^N} W(\underline{u},\underline{v}) = \inf_{\underline{u}\in H_0} \inf_{\underline{v}\in\Gamma(\underline{\rho})} W(\underline{u},\underline{v}).$$

By Lemmas 4.5, 4.6 the LHS is equal to $\inf_{\underline{v}\in\Gamma(\underline{\rho})} \Psi$ and the RHS to $\inf_{\underline{u}\in H_0} J_\rho(\underline{u}) + \sum_{i=1}^N \rho_i \log\left(\frac{\rho_i}{|\Sigma|}\right)$. □

We can now give a very simple proof of Theorem 1.16.

Proof of Theorem 1.16. Let $\Gamma(\underline{\rho})$, Ψ, be defined as in (4.6), (4.7). For any $i = 1, \dots, N$ let us denote

$$\Gamma^i := \left\{ v \in X \;:\; \int_\Sigma v dv_g = \rho_i \right\}$$

and consider the functionals $\Psi^i : \Gamma^i \longrightarrow \mathbb{R}$, $J^i : H_0 \longrightarrow \mathbb{R}$, defined by

$$\Psi^i(v) := \int_\Sigma v \log v dv_g - \frac{a_{ii}}{2} \int_\Sigma \int_\Sigma G(x,y) v(x) v(y) dv_g(x) dv_g(y),$$

$$J^i(u) := \frac{1}{2a_{ii}} \int_\Sigma |\nabla u|^2 dv_g - \rho_i \log\left(\frac{1}{|\Sigma|}\int_\Sigma h e^u dv_g\right)$$

Applying Proposition 4.1 to J^i and Ψ^i and using (1.23) we find

$$\begin{aligned} \Psi^i \text{ is bounded from below on } \Gamma^i \quad &\Longleftrightarrow \quad J^i \text{ is bounded from below on } H_0 \\ &\Longleftrightarrow \quad \rho_i \le \frac{8\pi\left(1+\min\left\{0, \min_{1\le j\le m} \alpha_{ij}\right\}\right)}{a_{ii}}. \end{aligned}$$

Clearly

$$\begin{aligned} \Psi \text{ is bounded from below on } \Gamma(\underline{\rho}) \quad &\Longrightarrow \quad \Psi^i \text{ is bounded from below on } \Gamma^i \quad i = 1, \dots, N \\ &\Longrightarrow \quad \rho_i \le \frac{8\pi\left(1+\min\left\{0, \min_{1\le j\le m} \alpha_{ij}\right\}\right)}{a_{ii}} \quad i = 1, \dots, N. \end{aligned}$$

On the other hand, since $G(x,y) \ge -C$, $\forall\, \underline{v} \in \Gamma(\underline{\rho})$ we have

$$\Psi(\underline{v}) = \sum_{i=1}^N \Psi^i(v_i) - \frac{1}{2}\sum_{i\ne j}^N a_{ij} \int_\Sigma\int_\Sigma G(x,y) v_i(x) v_j(y) dv_g(x) dv_g(y) \ge \tag{4.13}$$

$$\ge \sum_{i=1}^N \Psi^i(v_i) - \frac{C}{2}\sum_{i\ne j} a_{ij}\rho_i\rho_j. \tag{4.14}$$

Therefore

$$\Psi \text{ is bounded from below on } \Gamma(\rho) \quad\Longleftrightarrow\quad \Psi^i \text{ is bounded from below on } \Gamma^i \quad i=1,\dots,N$$
$$\Longleftrightarrow\quad \rho_i \le \frac{8\pi\left(1+\min\left\{0,\min_{1\le j\le m}\alpha_{ij}\right\}\right)}{a_{ii}} \quad i=1,\dots,N.$$

The conclusion follows from Proposition 4.1. ☐

We conclude this section with some remarks on the case of arbitrary positive definite matrices A. Let us consider the polynomials $\Lambda_{I,x}$ defined in (1.54).

Lemma 4.7. *If there exists $I \subseteq \{1,\dots,N\}$, $x_0 \in \Sigma$ such that*

$$\Lambda_{I,x_0}(\underline{\rho}) < 0 \qquad \textit{then} \qquad \inf_{\Gamma(\underline{\rho})} \Psi = -\infty \qquad \textit{and} \qquad \inf_{H_0^N} J_{\underline{\rho}}(\underline{u}) = -\infty.$$

Proof. Take $\varphi_\lambda(x) := \begin{cases} \frac{\lambda^2}{\pi} & \text{if } x \in B_{\frac{1}{\lambda}}(x_0) \\ 0 & \text{if } x \in \Sigma \setminus B_{\frac{1}{\lambda}}(x_0). \end{cases}$ Then we have

$$\begin{aligned}\int_\Sigma\int_\Sigma G(x,y)\varphi_\lambda(x)\varphi_\lambda(y)dv_g(x)dv_g(u) &= \frac{\lambda^4}{\pi^2}\int_{B_{\frac{1}{\lambda}}(x_0)}\int_{B_{\frac{1}{\lambda}}(x_0)} G(x,y)dv_g(x)dv_g(u) = \\ &= \frac{1}{2\pi}\log\lambda + O(1).\end{aligned}$$

Moreover

$$\int_\Sigma \varphi_\lambda \log\varphi_\lambda \, dv_g = 2\log\lambda + O(1),$$

$$\int_\Sigma \varphi_\lambda \, dv_g = 1 + O(\lambda^{-2}),$$

and

$$\begin{aligned}\int_\Sigma \varphi_\lambda \log h_i dv_g &= \frac{\lambda^2}{\pi}\int_{B_{\frac{1}{\lambda}}(x_0)} \log h_i \, dv_g = \\ &= -4\alpha_i(x)\lambda^2 \int_{B_{\frac{1}{\lambda}}(x_0)} G(x_0,y)dv_g + O(1) = \\ &= -2\alpha_i(x)\log\lambda + O(1).\end{aligned}$$

Let us consider $v \in \Gamma(\underline{\rho})$ defined by

$$v_i = \begin{cases} \dfrac{\rho_i\varphi_\lambda}{\int_\Sigma \varphi_\lambda \, dv_{g_0}} & \text{if } i \in I \\ \dfrac{\rho_i}{|\Sigma|} & \text{if } i \notin I. \end{cases}$$

Then we have

$$\sum_{i=1}^{N}\int_\Sigma v_i\left(\log v_i-\log h_i\right)dv_g \;=\; \sum_{i\in I}\frac{\rho_i}{\int_\Sigma \varphi_\lambda dv_g}\int_\Sigma \varphi_\lambda\left(\log\varphi_\lambda-\log h_i\right)dv_g+O(1)=$$
$$=\;2\sum_{i\in I}(1+\alpha_i(x))\rho_i\log\lambda+O(1)$$

and

$$\sum_{i,j=1}^{N}a_{ij}\int_\Sigma\int_\Sigma G(x,y)v_i(x)v_j(y)dv_g(x)dv_g(y) \;=\; \sum_{i,j\in I}^{N}\frac{a_{ij}\rho_i\rho_j}{\int_\Sigma\varphi_\lambda dv_g}\int_\Sigma\int_\Sigma G(x,y)\varphi_\lambda(x)\varphi_\lambda(y)dv_g(x)dv_g(y)$$
$$=\;-\frac{1}{2\pi}\sum_{i,j\in I}a_{ij}\rho_i\rho_j\log\lambda+O(1).$$

Therefore

$$\Psi(v)=\frac{1}{4\pi}\Lambda_{I,x_0}(\underline{\rho})\log\lambda+O(1)\longrightarrow-\infty\qquad\text{as}\qquad\lambda\to-\infty.$$

Finally, Proposition 4.1 yields also $\inf_{u\in H_0^N}J_{\underline{\rho}}=-\infty.$ □

Under the assumption (1.50) one can argue as in the proof of Theorem 1.16 to show that

$$\Psi\quad\text{is bounded from below}\qquad\Longleftrightarrow\qquad\Psi_{I_j}\quad\text{is bounded from below}\qquad j=1,\dots k,$$

where

$$\Psi_I(v)=\sum_{i\in I}\int_\Sigma v_i\log v_d v_g-\sum_{i,j\in I}\frac{a_{ij}}{2}\int_\Sigma\int_\Sigma G(x,y)v_i(x)v_j(y)dv_g(x)dv_g\quad\forall\, I\subseteq\{1,\dots,N\}.$$

This reduces the problem to the case of matrices with nonnegative coefficients. In the regular case Shafrir and Wolansky [76] proved that, for such matrices, the condition

$$\inf_{I,x}\Lambda_{I,x}\geq 0$$

is indeed necessary and sufficient for the boundedness of $J_{\underline{\rho}}$ and Ψ. It is conjectured that this should be true also for general matrices and in the presence of singularities.

4.2 A Concentration-Compactness Alternative for Liouville Systems.

In this section and in the next one, we study blow-up phenomena for sequences of solutions of (4.1), and give the proof of Theorem 4.1. We will actually work in a slightly more general setting. Given a matrix A satisfying (1.52), we will consider a sequence $\underline{u}_n = (u_{1,n}, \dots, u_{N,n})$ of solutions of a Liouville-type system of the form

$$-\Delta_g u_{i,n} = \sum_{j=1}^{N} a_{ij} V_{j,n} e^{u_{j,n}} - c_{i,n} \qquad i = 1, \dots, N. \tag{4.15}$$

where

$$V_{i,n} = K_{i,n} e^{-4\pi \sum_{j=1}^{m} \alpha_{i,j} G_{p_j}} \tag{4.16}$$

with

$$K_{i,n} \in C^\infty(\Sigma), \qquad 0 < a \le K_{i,n} \le b, \qquad \alpha_{i,n} > -1, \tag{4.17}$$

and

$$c_{i,n} = \frac{1}{|\Sigma|} \sum_{j=1}^{N} a_{ij} \int_\Sigma V_{j,n} e^{u_{j,n}} dv_g. \tag{4.18}$$

We will also assume the condition

$$\int_\Sigma V_{i,n} e^{u_{i,n}} dv_g \le C \qquad i = 1, \dots, N. \tag{4.19}$$

which implies the boundedness of $c_{i,n}$.

Remark 4.1. *More generally we could consider*

$$-\Delta_g u_{i,n} = \sum_{j=1}^{N} a_{ij} V_{j,n} e^{u_j} - \psi_{j,n} \tag{4.20}$$

with $\psi_{j,n}$ bounded in $L^s(\Sigma)$ for some $s > 1$ and

$$\int_\Sigma \psi_{j,n} dv_g = \sum_{j=1}^{N} a_{ij} \int_\Sigma V_{j,n} e^{u_j} dv_g$$

Adding to $u_{i,n}$ a solution of

$$\begin{cases} -\Delta_g v_{j,n} = \psi_{j,n} - \overline{\psi}_{j,n} \\ \int_\Sigma v_{j,n} dv_g = 0, \end{cases} \tag{4.21}$$

one reduces (4.20) to the case in which $\psi_{j,n}$ is constant, that is to (4.15).

For $i = 1, \dots, N$, let us denote

$$S_i := \left\{ x \in \Sigma \ : \ \exists \{x_n\}_{n\in\mathbb{N}} \subset \Sigma, \ u_{i,n}(x_n) \underset{n\to+\infty}{\longrightarrow} +\infty \right\}.$$

the blow-up set of $u_{i,n}$, and

$$\sigma_i(x) := \lim_{r\to 0} \limsup_{n\to\infty} \int_{B_r(x)} V_{i,n} e^{u_{i,n}} dv_g.$$

the local concentration value at x. We will prove the following concentration-compactness result:

Proposition 4.2. *Let A be a symmetric positive definite matrix satisfying (1.52) and assume $u_{i,n}$, $V_{i,n}$ satisfy (4.15)-(4.18). Up to subsequences, one of the following alternatives holds:*

(i) (Compactness/Vanishing) For $i = 1, \dots, N$, $u^+_{i,n}$ is uniformly bounded from above and either $u_{i,n}$ is bounded in $L^\infty(\Sigma)$ or $u_{i,n} \longrightarrow -\infty$ uniformly on Σ, $i = 1, \dots, N$.

(ii) (Blow-up) The blow-up set $S := S_1 \cup \cdots \cup S_N$ is non-empty and finite and $u^+_{i,n}$ is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S_i)$ $\forall\, i \in \{1, \dots, N\}$. Moreover, for any i we have either $u_{i,n}$ bounded in $L^\infty_{loc}(\Sigma\backslash S)$ or $u_{i,n} \longrightarrow -\infty$ locally uniformly in $\Sigma\backslash S$.

Furthermore, denoting by μ_i the weak limit of the sequence of measures $V_i e^{u_{i,n}}$, one has

$$\mu_i = r_i + \sum_{x \in S_i} \sigma_i(x) \delta_x \tag{4.22}$$

with $r_i \in L^1(\Sigma) \cap L^q_{loc}(\Sigma\backslash S_i) \cap L^\infty_{loc}(\Sigma\backslash(S_i \cup \{p_1, \dots, p_m\}))$ for some $q > 1$ and $\sigma_i(x) \geq \frac{4\pi}{a_{ii}} \min\{1, 1 + \alpha_i(x)\}$ $\forall x \in S_i$, $i = 1, \dots, N$.

The proof will be split into several simple steps. We begin with two general Lemmas. The first one was proved by Brezis and Merle in [18].

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain and let $u \in L^1_{loc}(\Omega)$ be a distributional solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$. Then $\forall\, \delta \in (0, 4\pi)$ we have

$$\int_\Omega e^{\frac{(4\pi - \delta)|u(x)|}{\|f\|_1}} dx \leq \frac{4\pi^2}{\delta} (\operatorname{diam} \Omega)^2.$$

Proof. Let $\tilde{f}(x) := \begin{cases} |f| & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$ be the 0 extension of $|f|$. We take $R = \frac{1}{2}\operatorname{diam}(\Omega)$ and consider the function

$$\tilde{u}(x) = \frac{1}{2\pi} \int_{B_R} \log\left(\frac{2R}{|x - y|}\right) \tilde{f}(y) dy.$$

Since $\tilde{u}$ solves $-\Delta \tilde{u} = \tilde{f}$ in $\mathbb{R}^2$ and $\tilde{u} \geq 0$ in B_R, by the maximum principle we have $|u| \leq \tilde{u}$ in Ω. Moreover by Jensen's inequality

$$\int_\Omega e^{\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}} dx \leq \int_{B_R} e^{\frac{(4\pi-\delta)\tilde{u}(x)}{\|f\|_1}} dx \leq \int_{B_R} dx \int_{B_R} dy \left(\frac{2R}{|x-y|}\right)^{2 - \frac{\delta}{2\pi}} \frac{\tilde{f}(y)}{\|f\|_1} \leq$$

$$\leq \int_{B_R} dy \frac{f(y)}{\|f\|_1} \int_{B_R} dx \left(\frac{2R}{|x-y|}\right)^{2 - \frac{\delta}{2\pi}}.$$

Since the function $\Phi(y) := \int_{B_R} \left(\frac{2R}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} dx$ is radially symmetric and decreasing we may deduce

$$\int_{\Omega} e^{\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}} dx \leq \Phi(0) = \frac{4\pi^2}{\delta} 2^{2-\frac{\delta}{2\pi}} R^2 \leq \frac{4\pi^2}{\delta} (\operatorname{diam}\Omega)^2 .$$

□

The following Lemma is a consequence of Harnack's inequality. It describes the behavior of $u_{i,n}$ on $\Sigma \backslash S$.

Lemma 4.9. *Let $\Omega \subseteq \Sigma$ be a connected open domain and let f_n be a bounded sequence in $L^1(\Omega) \cap L^q_{loc}(\Omega)$, $q > 1$. If u_n is sequence of solutions of $-\Delta_g u_n = f_n$ and u_n^+ is uniformly bounded in $L^\infty_{loc}(\Omega)$ then, up to subsequences, one of the following holds:*

(i) u_n is uniformly bounded in $L^\infty_{loc}(\Omega)$;

(ii) $u_n \longrightarrow -\infty$ uniformly on any compact subset of Ω.

Proof. Assume that the second alternative does not hold. Then we can find a point $x_0 \in \Omega$ such that, up to subsequences, $u_n(x_0) \geq -C$. Let $K \subset\subset \Omega$ be a compact subset of Ω. Since Ω is connected we can find $x_1, \ldots, x_L \in \Omega$ and $r_0, \ldots, r_L > 0$ such that

$$K \subset \bigcup_{i=0}^{L} B_{\frac{r_i}{2}}(x_i) \subset \bigcup_{i=0}^{L} B_{r_i}(x_i) \subset\subset \Omega \qquad \text{and} \qquad B_{\frac{r_i}{2}}(x_i) \cap B_{\frac{r_{i+1}}{2}}(x_{i+1}) \neq \emptyset \text{ for } i = 0, \ldots, L-1.$$

Without loss of generality, one can assume that it is possible to take isothermal coordinates in each of the balls $B_{r_i}(x_0)$. Let v_n be the solution of

$$\begin{cases} -\Delta_g v_n = f_n & \text{in } B_{r_0}(x_0) \\ v_n = 0 & \text{su } \partial B_{r_0}(x_0). \end{cases}$$

By elliptic estimates we find that v_n is uniformly bounded in $W^{2,q}(B_{r_0}(x_0))$ and, since $q > 1$, in $L^\infty(B_{r_0}(x_0))$. Being u_n bounded from above we can find $C' > 0$ such that $z_n := C' - u_n + v_n > 0$ in $B_{r_0}(x_0)$. Note that z_n is harmonic and $\inf_{B_{\frac{r_0}{2}}(x_0)} \leq z_n(x_0)$ is bounded from above, thus applying Harnack's inequality in local coordinates, we get that z_n and u_n are uniformly bounded in $L^\infty(B_{\frac{r_0}{2}})$. Since $B_{\frac{r_1}{2}}(x_1) \cap B_{\frac{r_0}{2}(x_0)} \neq \emptyset$, we have $\sup_{B_{\frac{r_1}{2}}(x_1)} u_n \geq -C$. We can so repeat the argument and find a uniform bound for u_n in $L^\infty(B_{\frac{r_1}{2}}(x_0))$. Iterating the procedure we find uniform bounds for u_n of each of the balls $B_{\frac{r_i}{2}}(x_i)$ and thus on K. □

Now we prove the lower bound for the concentration values at blow-up points.

Lemma 4.10. *For $i = 1, \ldots, N$, if $\sigma_i(x_0) < \frac{4\pi}{a_{ii}}(1 + \min\{0, \alpha_i(x)\})$ then $\exists\, r_0 > 0$ such that $u_{i,n}^+$ is uniformly bounded in $L^\infty(B_r(x_0))$.*

Proof. Without loss of generality we will consider the case $i = 1$. Let $r_0 > 0$ be such that

$$\int_{B_{r_0}(x_0)} V_{1,n} e^{u_{1,n}} dv_g < \frac{4\pi}{a_{11}}(1 + \min\{0, \alpha_1(x_0)\})$$

for sufficiently large n. Let us denote

$$f_n := a_{11} V_{1,n} e^{u_{1,n}}$$

and write $u_{1,n} = z_n + w_n - \xi_n$ where z_n and ξ_n are the solutions of

$$\begin{cases} -\Delta_g z_n = f_n & \text{in } B_{r_0}(x_0) \\ z_n = 0 & \text{on } \partial B_{r_0}(x_0) \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta_g \xi_n = c_{1,n} & \text{in } B_{r_0}(x_0) \\ \xi_n = 0 & \text{on } \partial B_{r_0}(x_0). \end{cases}$$

Since $c_{i,n}$ is bounded and f_n by elliptice estimates and the maximum principle we have

$$z_n \geq -C \qquad \text{and} \qquad |\xi_n| \leq C. \tag{4.23}$$

Applying Lemma 4.8 in local coordinates, we find $q > \frac{1}{1+\min\{0,\alpha_1(x_0)\}}$ such that $\|e^{qz_n}\|_{L^1(B_{r_0}(x))} \leq C$. We claim that $V_{1,n} \in L^s(B_{r_0}(x))$ for some $s > q'$. Indeed, $V_{1,n} \in L^\infty(B_{r_0}(x_0))$ if $\alpha_1(x_0) \geq 0$ and and $V_{1,n} \in L^s$ for $s < \frac{-1}{\alpha_1(x_0)}$ if $\alpha_1(x_0) < 0$. Since $q' = 1 + \frac{1}{q-1} < -\frac{1}{\alpha_1(x_0)}$ the claim is proved. In particular, by Holder's inequality we have $V_{1,n} e^{z_n} \in L^{1+\delta}(B_{r_0}(x))$ for some $\delta > 0$. Observe now that

$$-\Delta_g w_n = \sum_{j=2}^{N} a_{ij} V_{j,n} e^{u_{j,n}} \leq 0.$$

Applying the mean value Theorem for subharmonic functions we find

$$w_n(x) \leq C \int_{B_{\frac{r_0}{2}}(x)} w_n dv_g \leq \int_{B_{r_0}(x_0)} w_n^+ dv_g \leq \int_{B_{\frac{r_0}{2}}(x)} u_{1,n}^+ dv_g + C$$

$\forall\, x \in B_{\frac{r_0}{2}(x)}$. If we now take $\theta \in (0, 1]$ such that $V_{1,n}^{-\theta}$ is uniformly bounded in $L^1(B_{r_0}(x_0))$, then

$$\begin{aligned} \int_{B_{r_0}(x_0)} u_{1,n}^+ dv_g &\leq \frac{1+\theta}{\theta} \int_{B_{r_0}(x_0)} e^{\frac{\theta}{\theta+1} u} dv_g \leq \\ &\leq C \int_{B_{r_0}(x_0)} V_{1,n}^{-\frac{\theta}{\theta+1}} V_{1,n}^{\frac{\theta}{\theta+1}} e^{\frac{\theta}{\theta+1} u_{1,n}} dv_g \leq \\ &\leq \|V_{11,n}^{-\theta}\|_{L^1(B_{r_0}(x_0))}^{\frac{1}{1+\theta}} \|V_{1,n} e^{u_{1,n}}\|_{L^1(B_{r_0}(x_0))}^{\frac{\theta}{1+\theta}} \leq C. \end{aligned}$$

Thus w_n is uniformly bounded from above in $B_{\frac{r_0}{2}}(x_0)$. It follows that

$$f_n = a_{11} V_{1,n} e^{z_n} e^{w_n} e^{-\xi_n}$$

is uniformly bounded in $L^{1+\delta}(B_{\frac{r_0}{2}}(x_0))$. To conclude we consider the solution $\widetilde{v}_n$ of

$$\begin{cases} -\Delta_g \widetilde{v}_n = f_n & \text{in } B_{\frac{r_0}{2}}(x_0) \\ \widetilde{v}_n = 0 & \text{on } \partial B_{\frac{r_0}{2}}(x_0). \end{cases}$$

By elliptic estimates $\widetilde{v}_n$ is uniformly bounded in $B_{\frac{r_0}{2}}(x_0)$ and, arguing as before, one can prove that $(u_{1,n} - \widetilde{v}_n)$ is bounded from above in $B_{\frac{r_0}{4}}(x_0)$. It follows that $u_{1,n}$ is uniformly bounded from above in $B_{\frac{r_0}{4}}(x_0)$. □

Remark 4.2. *If $V_{i,n}e^{u_{i,n}} \rightharpoonup \mu_i$ as measures, then $\forall\, x \in \Sigma$ we have*

$$\sigma_i(x) = \mu_i(\{x\}) = \lim_{r\to 0} \liminf_{n\to\infty} \int_\Sigma V_{i,n}e^{u_{i,n}} dv_g.$$

In particular one can have $\sigma_i(x) \geq \frac{4\pi}{a_{ii}}(1 + \min\{0, \alpha_i(x)\})$ only for a finite number of points.

Proof. By the properties of the weak convergence of measures we have

$$\mu_i(B_r(x)) \leq \liminf_{n\to\infty} \int_{B_r(x)} V_n e^{u_n} dv_g \leq \limsup_{n\to\infty} \int_\Omega V_n e^{u_n} dv_g \leq \mu_i(\overline{B_r(x)}).$$

Since $\lim_{r\to 0} \mu_i(B_r(x)) = \lim_{r\to 0} \mu_i(\overline{B_r(x)}) = \mu_i(\{x\})$, the conclusion follows by taking the limit as $r \to 0$. □

We can thus characterize the blow-up set S_i as the set of points in which σ_i is positive.

Lemma 4.11. *Assume that $V_{i,n}e^{u_{i,n}} \rightharpoonup \mu_i$ as measures. The following conditions are equivalent:*

- $x_0 \in S_i$;
- $\sigma_i(x_0) \geq \frac{4\pi}{a_{ii}}(1 + \min\{0, \alpha_i(x_0)\})$;
- $\sigma_i(x_0) > 0$.

Moreover S_i is finite and $u^+_{i,n}$ is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S_i)$ for $i = 1, \ldots, N$.

Proof. By Lemma 4.10 the first condition implies the second and clearly the second implies the third one. Moreover, by Remark 4.2, $\sigma_i(x_0) > 0$ implies $\sup_{B_r(x_0)} u_n \longrightarrow +\infty\ \forall\ r > 0$. Let us choose $r_0 > 0$ such that $\overline{B_{r_0}(x_0)} \backslash \{x_0\}$ does not contain any point such that $\sigma_i(x) \geq \frac{4\pi}{a_{ii}}(1 + \min\{0, \alpha_i(x)\})$. Then using Lemma 4.10 we find $\sup_{\overline{B_{r_0}} \backslash B_r(x_0)} u_{i,n} \leq C$. Therefore, taking $x_n \in B_{r_0}(x_0)$ such that $u_n(x_n) = \sup_{\overline{B_{r_0}(x_0)}} u_n$, we have $u_n(x_n) \longrightarrow +\infty$ and $x_n \longrightarrow x_0$. This shows that $x_0 \in S_i$ and proves the equivalence of the three conditions. The finiteness of S_i and the bound on $u^+_{i,n}$ follow from Remark 4.2 and Lemma 4.10. □

The following Lemma describes the limit measures $\mu_1, \ldots, \mu_N$.

Lemma 4.12. *Let $q > 1$ be such that $V_{i,n} \in L^q(\Sigma)$ $i = 1, \dots, N$. Then $\exists\ r_i \in L^1(\Sigma) \cap L^q_{loc}(\Sigma \backslash S_i) \cap L^\infty_{loc}(\Sigma \backslash (S_i \cup \{p_1, \dots, p_m\}))$ such that*

$$\mu_i = \sum_{x \in S_i} \sigma_i(x)\delta_x + r_i. \tag{4.24}$$

Proof. First we observe that $\mu|_{\Sigma \backslash S_i}$ is absolutely continuous with respect to the Riemannian measure. Let $\Omega_k \subset\subset \Sigma \backslash S_i$ be an increasing sequence of open subsets of Σ such that $\Sigma \backslash S_i = \cup_{k=1}^\infty \Omega_k$. Let $E \subseteq \Sigma \backslash S_i$ be such that $|E| = 0$ and take $E_k = E \cap \Omega_k$. If $\{A^l_k\}$ is a sequence of opens sets such that $E_k \subseteq A^l_k \subseteq \Omega_k$ and $|A^l_k| \longrightarrow 0$ as $l \to 0$. Then $\forall\ l, k$, using the boundednes of $u^+_{i,n}$ on Ω_k, we get

$$\mu_i(E_k) \le \mu_i(A^l_k) \le \liminf_{n \to \infty} \int_{A^l_k} V_{i,n} e^{u_{i,n}} dx \le \|e^{u_{i,n}}\|_{L^\infty(\Omega_k)} \|V_{i,n}\|_{L^q(\Omega)} |A^l_k|^{\frac{1}{q'}} \le C(k) |A^l_k|^{\frac{1}{q'}}.$$

As $l \to 0$ we find $\mu_i(E_k) = 0\ \forall\ k$ and thus $\mu_i(E) = 0$. By the Radon-Nikodym Theorem we can find $r_i \in L^1(\Sigma)$ such that 4.24 holds. Moreover, since $V_{i,n} e^{u_{i,n}}$ is bounded in $L^q_{loc}(\Sigma \backslash S_i) \cap L^\infty_{loc}(\Sigma \backslash (S_i \cup \{p_1, \dots, p_m\}))$, $r_i \in L^q_{loc}(\Sigma \backslash S_i) \cap L^\infty_{loc}(\Sigma \backslash (S_i \cup \{p_1, \dots, p_m\}))$. □

We stress that Lemma 4.12 holds also if $S_i = \emptyset$.

Proof of Proposition 4.2. By Lemmas 4.11, $u^+_{i,n}$ is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S_i)$. If $S_1 = \dots = S_N = \emptyset$ then, by lemma 4.9, we have *(i)*. If instead $S = S_1 \cup \ldots \cup S_N \neq \emptyset$ then, for any $i \in \{1, \dots, N\}$, $-\Delta_g u_{i,n}$ is uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S)$ and, again by Lemma 4.9, we have either $u_{i,n} \longrightarrow -\infty$ locally uniformly or $u_{i,n}$ uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S)$. Finally (4.24) follows from Lemma 4.12. □

The following was also observed in [61].

Remark 4.3. *If there exists $x_0 \in S_i \backslash \cup_{j \neq i} S_j$ then $r_i \equiv 0$.*

Proof. In local isothermal coordinates around x_0 we have

$$-\Delta u_{i,n} = |x|^{2\alpha_i(x_0)} \widetilde{V}_{i,n} e^{2u_{i,n}} + \psi_{i,n}$$

in D_{r_0} with $0 < c_1 \le \widetilde{V}_{i,n} \le c_2$ and $\psi_{i,n} \in L^q(D_{r_0})$ for some $q > 1$. Thus one can exploit the results in [8] and [5] to prove that $u_{i,n} \longrightarrow -\infty$ uniformly in D_{r_0}. This proves that $u_{i,n}$ cannot be uniformly bounded in $L^\infty_{loc}(\Sigma \backslash S)$ and thus $u_{i,n} \longrightarrow -\infty$ locally uniformly in $\Sigma \backslash S$. In particular $r_i \equiv 0$. □

Proof of Theorem 4.1. We apply Proposition 4.2 to the functions

$$w_{i,n} := u_{i,n} - \log \int_\Sigma h_i e^{u_{i,n}} dv_g + \log \rho_{i,n} \tag{4.25}$$

which solve

$$-\Delta_g w_{i,n} = \sum_{j=1}^{N} a_{ij} \left(h_j e^{w_{j,n}} - \rho_j \right)$$

and

$$\int_\Sigma h_i e^{w_{i,n}} dv_g = \rho_{i,n} \qquad i = 1, \dots, N.$$

If $w_{i,n}^+$ is bounded in $L^\infty(\Sigma)$ $\forall\, i \in \{1, \dots, N\}$,then $-\Delta_g u_{i,n}$ is bounded in $L^q(\Sigma)$ for some $q > 1$ and by elliptic estimates we get a uniform bound for $\underline{u}_n$ in $W^{2,q}(\Sigma)$. Otherwise, since by Jensen's inequality we get

$$\int_\Sigma h_j e^{u_{i,n}} dv_g \geq |\Sigma| e^{\frac{1}{|\Sigma|} \int_\Sigma \log h_j dv_g} > 0,$$

we get *(ii)* with $S_1, \dots, S_N$ equal to the blow-up sets of $w_{i,n}$. □

4.3 Mass quantization for the $SU(3)$ Toda System

In order to prove Theorems 1.17 and 1.18 we need to prove the vanishing of at least one of the residual terms r_i in Theorem 4.1 and Proposition 4.2. As in the previous section, we will assume that $u_{i,n}$ and $V_{i,n}$ satisfy (4.15)-(4.18). In addition to (4.17) we will assume

$$K_{i,n} \longrightarrow K_{i,0} \quad \text{in} \quad C^1(\Sigma), \qquad i = 1, \dots, N. \tag{4.26}$$

We shall also denote

$$V_{i,0} = K_{i,0} e^{-4\pi \sum_{j=1}^m, \alpha_{ij} G_{p_j}}.$$

As a first thing, we can show that the profile of $u_{i,n} - \overline{u}_{i,n}$ near blow-up points resembles a combination of Green's functions:

Lemma 4.13. *$u_{i,n} - \overline{u}_{i,n} \longrightarrow \sum_{j=1}^N \sum_{x \in S_j} a_{ij} \sigma_j(x) G_x + s_i$ in $L^\infty_{loc}(\Sigma \backslash S)$ and weakly in $W^{1,q}(\Sigma)$ for any $q \in (1, 2)$ with $e^{s_i} \in L^p(\Sigma)$ $\forall p \geq 1$.*

Proof. If $q \in (1, 2)$

$$\int_\Sigma \nabla u_{i,n} \cdot \nabla \varphi dv_g \leq \|\Delta u_{i,n}\|_{L^1(\Sigma)} \|\varphi\|_\infty \leq C \|\varphi\|_{W^{1,q'}(\Sigma)}$$

$\forall\ \varphi \in W^{1,q'}(\Sigma)$ with $\int_\Sigma \varphi = 0$, hence one has $\|\nabla u_{i,n}\|_{L^q(\Sigma)} \leq C$. In particular $u_{i,n} - \overline{u}_{i,n}$ converges to a function $w_i \in W^{1,q}(\Sigma)$ weakly in $W^{1,q}(\Sigma)$ $\forall q \in (1, 2)$.
The limit functions w_i are distributional solutions of

$$-\Delta_g w_i = \sum_{j=1}^{N} a_{ij} \left(r_j + \sum_{x \in S_j} \sigma_j(x) \delta_x \right) - c_i,$$

where

$$c_i = \lim_{n\to\infty} c_{i,n} = \frac{1}{|\Sigma|}\sum_{j=1}^{N} a_{ij}\left(\int_\Sigma r_j dv_g + \sum_{x\in S_j}\sigma_j(x)\right).$$

In particular $s_i := w_i - \sum_{j=1}^N \sum_{x\in S_j} a_{ij}\sigma_j(x)G_x$ solves

$$-\Delta_g s_i = \sum_{j=1}^{N} a_{ij}\left(r_j + \frac{1}{|\Sigma|}\sum_{x\in S_j}\sigma_j(x)\right) - c_i = \sum_{j=1}^{N} a_{ij}\left(r_j - \overline{r}_j\right).$$

Since $-\Delta_g s_i \in L^1(\Sigma)$ we can exploit Remark 2 in [18] to prove that $e^{s_i} \in L^p(\Sigma)\ \forall\ p \geq 1$. The convergence in $L^\infty_{loc}(\Sigma\backslash S)$ follows by elliptic estimates and the boundedness of $-\Delta_g u_{i,n}$ in $L^q_{loc}(\Sigma\backslash S)$, $q > 1$. □

The following Lemma shows the main difference between the case of vanishing and non-vanishing residual.

Lemma 4.14.

- $r_i \equiv 0 \implies \overline{u}_{i,n} \longrightarrow -\infty$.
- $r_i \not\equiv 0 \implies \overline{u}_{i,n}$ *is bounded.*

Proof. First of all, $\overline{u}_{i,n}$ is bounded from above due to Jensen's inequality.
Now, take any non-empty open set $\Omega \subset\subset \Sigma\backslash S$.

$$\int_\Omega V_{i,n}e^{u_{i,n}}dv_g = e^{\overline{u}_{i,n}}\int_\Omega V_{i,n}e^{u_{i,n}-\overline{u}_{i,n}}dv_g$$

and by Lemma 4.13 and (4.26)

$$\int_\Omega V_{i,n}e^{u_{i,n}-\overline{u}_{i,n}}dv_g \underset{n\to+\infty}{\longrightarrow} \int_\Omega V_{i,0}e^{\sum_{j=1}^N\sum_{x\in S_j} a_{ij}\sigma_j(x)G_x + s_i}dv_g \in (0,+\infty).$$

On the other hand,

$$\int_\Omega V_{i,n}e^{u_{i,n}}dv_g \underset{n\to+\infty}{\longrightarrow} \mu_i(\Omega) = \int_\Omega r_i(x)dv_g(x).$$

If $r_i \equiv 0$ one has $\overline{u}_{i,n} \longrightarrow -\infty$. If instead $r_i \not\equiv 0$, choosing Ω such that $\int_\Omega r_i dv_g > 0$ we must have $\overline{u}_{i,n}$ necessarily bounded. □

Remark 4.4. *From the previous two lemmas, we can write* $r_i = \widehat{V}_i e^{s_i}$, *where*

$$\widehat{V}_i := V_{i,0}e^{\lim_{n\to+\infty}\overline{u}_{i,n}}e^{\sum_{j=1}^N\sum_{x\in S_j} a_{ij}\sigma_j(x)G_x}$$

satisfies $\widehat{V}_i \sim d(\cdot,x)^{2\alpha_i(x)-\frac{\sum_{j=1}^N a_{ij}\sigma_j(x)}{2\pi}}$ *around each* $x \in S_i$, *provided* $r_i \not\equiv 0$.

Now we state a technical Lemma that will be needed in the proof of Lemma 4.16.

Lemma 4.15. *Let A be a symmetric positive definite $L \times L$ matrix, then there exists $\gamma = (\gamma_1, \dots, \gamma_L) \in \mathbb{R}^L$ such that*

- $\gamma_i \geq 0, \qquad i = 1, \dots, L;$
- $\displaystyle\sum_{i=1}^{L} \gamma_i a_{ij} \geq 0 \qquad j = 1, \dots L$
- $\displaystyle\sum_{i=1}^{L} \gamma_i = 1.$

Proof. Let us consider the set $E := \{x \in \mathbb{R}^L \, : \, xA \geq 0, \; x \geq 0\}$ and the linear map $F : \mathbb{R}^L \longrightarrow \mathbb{R}$, $F(x) := x_1 + \ldots + x_L$. Clearly one has either $\sup_{x \in E} F = +\infty$ or $F(0) = \sup_{x \in E} F = 0$. In the former holds, then there exists $\overline{x} \in E$, $\overline{x} \neq 0$ and we can conclude by taking $\gamma = \frac{\overline{x}}{\sum_{i=1}^{L} x_i}$. In the latter case, by the Strong Duality Theorem in Linear Programming, there exists $y \in \mathbb{R}^L \backslash \{0\}$ such that $y \geq 0$ and $\sum_{j=1}^{L} a_{ij} y_j \leq -1$ for $j = 1, \dots, L$. But then we would have

$$y \cdot Ay = \sum_{i,j=1}^{L} y_i a_{ij} y_j \leq 0$$

which contradicts the assumptions on A. □

The key Lemma is an extension of Chae-Ohtsuka-Suzuki [23] to the singular case. Basically, it gives necessary conditions on the σ_i's to have non-vanishing residual.

Lemma 4.16. *For $i = 1, \dots, N$ we have $s_i \in W^{2,p}(\Sigma)$, $p > 1$. Moreover, if $\sum_{j=1}^{N} a_{ij} \sigma_j(x_0) \geq 4\pi(1 + \alpha_i(x_0))$ for some $x_0 \in S_i$, then $r_i \equiv 0$.*

Proof. If all the r_i's are identically zero, then also all the s_i's are identically zero and there is nothing to prove.
Assume that $r_i \neq 0$ for some $i \in \{1, \dots, N\}$. Up to reordering the indices, we can assume $r_1, \dots, r_{L_0} \not\equiv 0$ and $r_{L_0+1}, \dots, r_N \equiv 0$, for some $L_0 \in \{1, \dots, N\}$. Observe that

$$\begin{cases} -\Delta_g s_i = \sum_{i=1}^{L_0} a_{ij} \left(r_j - \overline{r}_j \right) & 1 \leq i \leq L_0 \\ s_i \equiv 0 & L_0 + 1 \leq i \leq N. \end{cases}$$

We have to prove that for $i = 1, \dots, L_0$ one has

$$x_0 \in S_i \qquad \Longrightarrow \qquad \sum_{j=1}^{N} a_{ij} \sigma_j(x_0) < 4\pi(1 + \alpha_i(x_0)) \quad \text{and} \quad s_i \in W^{2,q}(B_r(x_0)), \; q > 1, r > 0.$$

Take $x_0 \in S_1 \cup \cdots \cup S_{L_0}$. Up to relabeling the indices, we can assume $x_0 \in S_1 \cap \cdots \cap S_L$ and $x_0 \notin S_{L+1} \cup \cdots \cup S_{L_0}$, for some $1 \le L \le L_0$. Observe that this implies $r_i \in L^q(B_{r_0}(x_0))$ and $s_i \in W^{2,q}(B_{r_0}(x_0))$ for $L+1 \le i \le L_0$. Let us consider the $L \times L$ matrix $A_L := (a_{i,j})_{1\le i,j\le L}$. Since A_L is symmetric and positive definite, by Lemma 4.15 we can find $\gamma_1, \dots, \gamma_L \ge 0$ such that $\sum_{j=1}^L \gamma_i a_{ij} \ge 0$ and $\sum_{j=1}^L \gamma_j = 1$. Then, being $G(x,y) \ge -C$, we have for $x \in B_{\frac{r_0}{2}}(x_0)$

$$\begin{aligned}
\sum_{i=1}^L \gamma_i s_i &= \sum_{i=1}^L \sum_{j=1}^{L_0} \gamma_i a_{ij} \int_\Sigma G(x,y) r_j(y) dv_g(y) = \\
&= \sum_{i=1}^L \sum_{j=1}^{L} \gamma_i a_{ij} \int_\Sigma G(x,y) r_j(y) dv_g(y) + \sum_{i=1}^L \sum_{j=L+1}^{L_0} \gamma_i a_{ij} \int_\Sigma G(x,y) r_j(y) dv_g(y) \ge \\
&\ge -C \sum_{i,j=1}^L \gamma_i a_{ij} \int_\Sigma r_j dv_g + \sum_{i=1}^L \sum_{j=L+1}^{L_0} \gamma_i a_{ij} \int_{B_{r_0}(x_0)} G(x,y) r_j(y) dv_g(y) - C \ge \\
&\ge -C - \sum_{i=1}^L |a_{ij}| \gamma_i \sum_{j=L+1}^{L_0} \sup_{z\in\Sigma} \|G(\cdot,z)\|_{L^{q'}(\Sigma)} \|r_j\|_{L^q(B_r(x_0))} \ge -C'.
\end{aligned}$$

Therefore, using the convexity of $t \to e^t$ we get

$$\begin{aligned}
e^{-C'} \int_\Sigma \min\left\{\widehat{V}_1, \dots, \widehat{V}_L\right\} dv_g &\le \int_\Sigma \min\left\{\widehat{V}_1, \dots, \widehat{V}_L\right\} e^{\sum_{i=1}^L \gamma_i s_i} dv_g \le \\
&\le \sum_{i=1}^L \gamma_i \int_\Sigma \widehat{V}_i e^{s_i} dv_g = \sum_{i=1}^L \gamma_i \int_\Sigma r_i dv_g < +\infty. \qquad (4.27)
\end{aligned}$$

By Remark 4.4 we must have $\sum_{j=1}^N a_{ij}\sigma_j(x_0) < 4\pi(1+\alpha_i(x_0))$ and $r_i \in L^{\widetilde{q}}(B_{\frac{r_0}{2}}(x_0))$ for some $i \in \{1, \dots, L\}$. Suppose, without loss of generality, that this is true for $i = L$. Reducing eventually q, we have $r_i \in L^q(B_{\frac{r_0}{2}}(x_0))$ and $s_i \in W^{2,q}(B_{\frac{r_0}{2}}(x_0))$ for $i = L, \dots, L_0$. The procedure can be iterated to prove that $\sum_{j=1}^N a_{ij}\sigma_j(x_0) < 4\pi(1+\alpha_i(x_0))$ for $i = 1, \dots, L$ and $r_i \in L^p(B_r(x_0))$, $w \in W^{2,p}(B_r(x_0))$ for any i and for small r. Hence, being x_0 an arbitrary point in S, the proof is complete. □

Remark 4.5. *By Remark 4.4 and Lemma 4.16 one finds that if* $s_i \not\equiv 0$, *then* $-\Delta_g s_i \approx d(\cdot,x_0)^{2\beta(x_0)}$ *where* $\beta(x_0) = \alpha(x_0) - \frac{1}{2}\sum_{i=1}^N a_{ij}\sigma_j(x_0) > -1$ *near each point* $x_0 \in S$. *Then, one can argue as in the proof of Lemma 2.8 to prove that near* x_0

- $|\nabla s_i(x)| = O(d(x,x_0)^{2\beta(x_0)})$ *if* $\beta(x_0) < -\frac{1}{2}$;
- $|\nabla s_i(x)| = O(-\log d(x,x_0))$ *if* $\beta(x_0) = -\frac{1}{2}$;
- $|\nabla s_i(x)| \le C$ *if* $\beta(x_0) > -\frac{1}{2}$.

In any case one has

$$\lim_{r\to 0}\int_{\partial B_r(x_0)} r|\nabla s_i|^2 dv_g = 0 \qquad \forall\, i\in\{1,\dots,N\},\ x_0\in S.$$

From Lemmas 4.13 and 4.16 we can deduce, through a Pohozaev identity, the following information about the local blow-up values.

Lemma 4.17. *If $x_0\in S$ then*

$$\sum_{i,j=1}^N a_{ij}\sigma_i(x_0)\sigma_j(x_0) = 8\pi\sum_{i=1}^N (1+\alpha_i(x_0))\sigma_i(x_0). \tag{4.28}$$

Proof. Let us take local isothermal coordinates on D_{δ_0} in which x_0 corresponds to 0. In these coordinates $u_{i,n}$ satisfies

$$-\Delta u_{i,n} = \sum_{j=1}^N a_{ij}\widetilde{V}_{i,n}e^{u_{i,n}} + \psi_{i,n}$$

with $\psi_{i,n}\in C^1(D_{\delta_0})$ and $\widetilde{V}_{i,n} = |x|^{2\alpha_i(x_0)}\widetilde{K}_{i,n}$ where $\widetilde{K}_{i,n}\longrightarrow \widetilde{K}_{i,0}$ in $C^1(D_{\delta_0})$, $\widetilde{K}_{i,0}>0$. Moreover by Lemmas 4.13, 4.16 and Remark 4.5 we have

$$u_{i,n}-\overline{u}_{i,n}\longrightarrow \sum_{j=1}^N a_{ij}\sigma_j(x_0)G_{x_0}+\widetilde{s}_i \qquad \text{in } C^1_{loc}(D_{\delta_0}\backslash\{0\}) \tag{4.29}$$

with $\widetilde{s}_i\in W^{2,q}(D_{\delta_0})$ and

$$\lim_{r\to 0} r\int_{\partial D_r}|\nabla\widetilde{s}_i|^2 d\sigma = 0. \tag{4.30}$$

Integrating by parts on D_r for $r\in(0,\delta_0)$ we get

$$\sum_{i,j=1}^N a^{ij}\left(-\int_{D_r}\Delta u_{i,n}\nabla u_{j,n}\cdot x\,dx + r\int_{\partial D_r}\frac{\partial u_{i,n}}{\partial\nu}\frac{\partial u_{j,n}}{\partial\nu}d\sigma\right) = \sum_{i,j=1}^N a^{ij}\int_{D_r}\nabla u_{i,n}\cdot\nabla\left(\nabla u_{j,n}\cdot x\right)dx =$$

$$= \sum_{i,j=1}^N a^{ij}\int_{D_r}\left(\frac{1}{2}\nabla\left(\nabla u_{i,n}\cdot\nabla u_{j,n}\right)\cdot x + \nabla u_{i,n}\cdot\nabla u_{j,n}\right)dx =$$

$$= \frac{1}{2}\sum_{i,j=1}^N a^{ij} r\int_{\partial D_r}\nabla u_{i,n}\cdot\nabla u_{j,n}d\sigma$$

On the other hand we have

$$
\begin{aligned}
-\sum_{i,j=1}^{N} a^{ij}\int_{D_r}\Delta u_{i,n}\nabla u_{j,n}\cdot x\,dx &= \sum_{k=1}^{N}\int_{D_r}\widetilde{V}_{k,n}e^{u_{k,n}}\nabla u_{k,n}\cdot x d\sigma+\sum_{i,j=1}^{N}a^{ij}\int_{D_r}\psi_{i,n}\nabla u_{j,n}\cdot x\,dx\\
&= \sum_{k=1}^{N} r\int_{\partial D_r}\widetilde{V}_{k,n}e^{u_{k,n}}d\sigma-\sum_{k=1}^{N}\int_{D_r}\left(\widetilde{V}_{k,n}+\nabla\widetilde{V}_{k,n}\cdot x\right)e^{u_{i,n}}dx\\
&\quad+\sum_{i,j=1}^{N}a^{ij}\int_{D_r}\psi_{i,n}\nabla u_{j,n}\cdot x\,dx
\end{aligned}
$$

thus we obtain the Pohozaev-type identity

$$
\sum_{i,j=1}^{N} a^{ij}\int_{\partial D_r} r\left(\frac{\partial u_{i,n}}{\partial\nu}\frac{\partial u_{j,n}}{\partial\nu}-\frac{1}{2}\nabla u_{i,n}\cdot\nabla u_{j,n}\right)d\sigma+\sum_{k=1}^{N} r\int_{\partial D_r}\widetilde{V}_{k,n}e^{u_{k,n}}d\sigma=
$$

$$
=\sum_{k=1}^{N}\int_{D_r}\left(\widetilde{V}_{k,n}+\nabla\widetilde{V}_{k,n}\cdot x\right)e^{u_{i,n}}dx-\sum_{i,j=1}^{N}a^{ij}\int_{D_r}\psi_{i,n}\nabla u_{j,n}\cdot x\,dx. \tag{4.31}
$$

Using (4.29) we find

$$
\begin{aligned}
\lim_{n\to\infty} r\int_{\partial D_r}\nabla u_{i,n}\cdot\nabla u_{j,n}d\sigma &= r\sum_{k,l=1}^{N} a_{ik}a_{jl}\sigma_k(x_0)\sigma_l(x_0)\int_{\partial D_r}|\nabla G_{x_0}|^2 d\sigma+r\int_{\partial D_r}\nabla\widetilde{s}_i\cdot\nabla\widetilde{s}_j d\sigma+\\
&+ r\sum_{k=1}^{N}\sigma_k(x_0)\left(a_{ik}\int_{\partial D_r}\nabla G_{x_0}\cdot\nabla\widetilde{s}_j+a_{jk}\int_{\partial D_r}\nabla G_{x_0}\cdot\nabla\widetilde{s}_i\right)d\sigma.
\end{aligned}
$$

therefore, by (4.30),

$$
\lim_{r\to 0}\lim_{n\to\infty} r\sum_{i,j=1}^{N} a_{ij}\int_{\partial D_r}\nabla u_{i,n}\cdot\nabla u_{j,n}d\sigma=\sum_{i,j=1}^{N} a_{ij}\sigma_i(x_0)\sigma_j(x_0). \tag{4.32}
$$

Similarly

$$
\lim_{r\to 0}\lim_{n\to\infty} r\sum_{i,j=1}^{N} a^{ij}\int_{\partial D_r}\frac{\partial u_{i,n}}{\partial\nu}\frac{\partial u_{j,n}}{\partial\nu}\,d\sigma=\sum_{i,j=1}^{N} a_{ij}\sigma_i(x_0)\sigma_j(x_0). \tag{4.33}
$$

We also claim that

$$
\lim_{r\to 0}\lim_{n\to\infty} r\int_{\partial D_r}\widetilde{V}_{i,n}e^{u_{i,n}}dx=0 \qquad i=1,\dots,N. \tag{4.34}
$$

If $r_i\equiv 0$ this follows by Lemmas 4.14, 4.13 (actually the limit in n is 0 for any r sufficiently small). If instead $r_i\neq 0$ then by Lemma 4.16 we have $\sum_{j=1}^{N} a_{ij}\sigma_j<4\pi(1+\alpha_i(x_0))$ so that

$$
\begin{aligned}
\lim_{n\to\infty} r\int_{\partial D_r}\widetilde{V}_{i,n}e^{u_{i,n}}dx &= r\int_{\partial D_r}|x|^{2\alpha_i(x_0)}K_{i,0}e^{\lim_{n\to\infty}\overline{u}_{i,n}}e^{\sum_{j=1}^{N}a_{ij}\sigma_j G_{x_0}+\widetilde{s}_i}d\sigma=\\
&= O\left(r^{2(1+\alpha(x_0))-\sum_{j=1}^{N}a_{ij}\sigma_j}\right)\stackrel{r\to 0}{\longrightarrow}0.
\end{aligned}
$$

Since $\nabla\widetilde{V}_{i,n}\cdot x = 2\alpha(x_0)\widetilde{V}_{i,n} + |x|^{2\alpha(x_0)}\nabla\widetilde{K}_{i,n}\cdot x$, if r is sufficiently small we get

$$\begin{aligned}\lim_{n\to\infty}\int_{D_r}\left(2\widetilde{V}_{i,n}+\nabla\widetilde{V}_{i,n}\cdot x\right)e^{u_{i,n}}dv_g &= 2(1+\alpha_i(x_0))\sigma_i(x_0)+\\ &+\int_{D_r}\left(2(1+\alpha_i(x_0))\widetilde{K}_{i,0}+\nabla\widetilde{K}_{i,0}\cdot x\right)|x|^{2\alpha_i(x_0)}\widetilde{s}_i dx\end{aligned}$$

so that

$$\lim_{r\to 0}\lim_{n\to\infty}\int_{D_r}\left(2\widetilde{V}_{i,n}+\nabla\widetilde{V}_{i,n}\cdot x\right)e^{u_{i,n}}dv_g = 2(1+\alpha_i(x_0))\sigma_i(x_0) \quad i=1,\dots,N. \tag{4.35}$$

Finally we have

$$\lim_{n\to\infty}\int_{D_r}\psi_{i,n}\nabla u_{j,n}\cdot x\,dx = \int_{D_r}\psi_{i,n}\sum_{k=1}^{N}a_{jk}\sigma_k(x_0)\nabla G_{x_0}\cdot x\,dx + \int_{D_r}\psi_{i,n}\nabla\widetilde{s}_j\cdot x\,dx = O(r) \tag{4.36}$$

which implies

$$\lim_{r\to 0}\lim_{n\to\infty}\int_{D_r}\psi_{i,n}\nabla u_{i,n}\cdot x\,dx = 0. \tag{4.37}$$

Using (4.31) - (4.37) we find

$$\frac{1}{4\pi}\sum_{i,j=1}^{N}a_{ij}\sigma_i(x_0)\sigma_j(x_0) = 2\sum_{k=1}^{N}(1+\alpha_k(x_0))\sigma(x_0).$$

□

Lemma 4.18. *If $x_0\in S$, then there exists i such that $\sum_{j=1}^{N}a_{ij}\sigma_j(x_0)\geq 4\pi(1+\alpha_i(x_0))$.*

Proof. Suppose the statement is not true. Then

$$\sum_{j=1}^{N}a_{ij}\sigma_j(x_0) < 4\pi(1+\alpha_i(x_0)) \qquad i=1,\dots N. \tag{4.38}$$

Multiplying the i^{th} equation by $\sigma_i(x_0)$ and taking the sum over i one finds

$$\sum_{i,j=1}^{N}a_{ij}\sigma_i(x_0)\sigma_j(x_0) < 4\pi\sum_{j=1}^{N}(1+\alpha_j(x_0))\sigma_j(x_0)$$

which contradicts Lemma 4.17.
For $N=2$, the scenario is described by the picture.

□

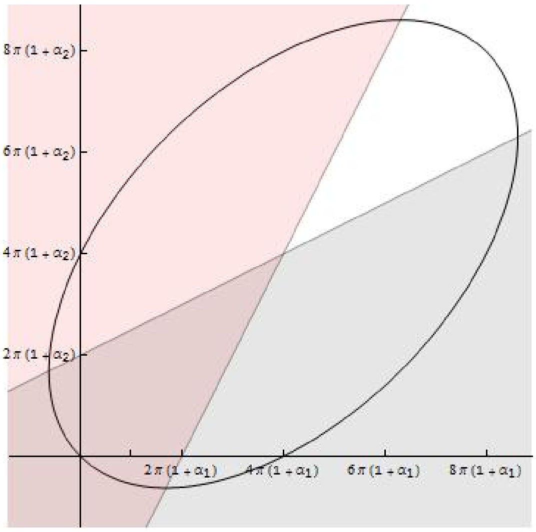


Figure 4.1: The algebraic conditions (4.38), (4.28) satisfied by $\sigma_1(x_0), \sigma_2(x_0)$

Corollary 4.1. *Suppose $\underline{u}_n$ satisfies (4.15)-(4.18) and that (4.26) holds. If $S \neq \emptyset$ then (4.22) holds with $r_i \equiv 0$ for some $i \in \{1, \ldots, N\}$. In particular there exists i such that*

$$\lim_{n \to \infty} \int_\Sigma V_{i,n} e^{u_{i,n}} dv_g = \sum_{x \in S_i} \sigma_i(x).$$

Similarly we get:

Corollary 4.2. *Let $\underline{u}_n$ be a sequence of solutions of (4.1) with $\rho_i = \rho_{i,n} \longrightarrow \overline{\rho}_i$, $i = 1, \ldots, N$. If alternative (ii) holds in Theorem 4.1, then $r_i \equiv 0$ for some i. In particular there exists $i \in \{1, \ldots, N\}$ such that $\overline{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$.*

Proof. As in the proof of Theorem 4.1, it is sufficient to apply Corollary 4.1 to the functions w_i defined in (4.25). □

We can so prove the compactness condition for the $SU(3)$ Toda System.

Proof of Theorems 1.17 and 1.18.
Assume $N = 2$ and $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Let u_n be a sequence of solutions of (4.1) with $\rho_i = \rho_{i,n} \underset{n \to +\infty}{\longrightarrow} \overline{\rho}_i$ and $\int_\Sigma u_{1,n} dv_g = \int_\Sigma u_{2,n} dv_g = 0$. If $u_{1,n}$, $u_{2,n}$ are both uniformly bounded in $W^{2,p}(\Sigma)$, then $\underline{u}_n$ is compact in $H^1(\Sigma)$.
Otherwise, from Corollary 4.2 we must have $\overline{\rho}_i = \sum_{x \in S_i} \sigma_i(x)$ for some $i \in \{1, 2\}$. In the regular case, from Theorem *B* follows that ρ_i must be an integer multiple of 4π, hence the proof of Theorem 1.17 is complete.

In the singular case, local blow-up values at regular points are still multiples of 4π, whereas for any $j = 1, \ldots, l$ there exists a finite Γ_j such that $(\sigma_1(p_j), \sigma_2(p_j)) \in \Gamma_j$. Therefore, it must hold

$$\rho_i \in \Lambda_i := \left\{ 4\pi k + \sum_{j=1}^{l} n_j \sigma_j, \ k \in \mathbb{N}, \ n_j \in \{0,1\}, \ \sigma_j \in \Pi_i(\Gamma_j) \right\},$$

where Π_i is the projection on the i^{th} component; being Λ_i discrete we can also conclude the proof of Theorem 1.18. □

Bibliography

[1] Adimurthi and O. Druet. Blow-up analysis in dimension 2 and a sharp form of Trudinger-Moser inequality. *Comm. Partial Differential Equations*, 29(1-2):295–322, 2004.

[2] Adimurthi and K. Sandeep. A singular Moser-Trudinger embedding and its applications. *NoDEA Nonlinear Differential Equations Appl.*, 13(5-6):585–603, 2007.

[3] Antonio Ambrosetti and Andrea Malchiodi. *Nonlinear analysis and semilinear elliptic problems*, volume 104 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[4] Thierry Aubin. Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire. *J. Funct. Anal.*, 32(2):148–174, 1979.

[5] D. Bartolucci and G. Tarantello. Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory. *Comm. Math. Phys.*, 229(1):3–47, 2002.

[6] Daniele Bartolucci, Francesca De Marchis, and Andrea Malchiodi. Supercritical conformal metrics on surfaces with conical singularities. *Int. Math. Res. Not. IMRN*, (24):5625–5643, 2011.

[7] Daniele Bartolucci and Andrea Malchiodi. An improved geometric inequality via vanishing moments, with applications to singular Liouville equations. *Comm. Math. Phys.*, 322(2):415–452, 2013.

[8] Daniele Bartolucci and Eugenio Montefusco. Blow-up analysis, existence and qualitative properties of solutions for the two-dimensional Emden-Fowler equation with singular potential. *Math. Methods Appl. Sci.*, 30(18):2309–2327, 2007.

[9] L. Battaglia. Moser-Trudinger inequalities for singular liouville systems. *preprint*, 2015.

[10] Luca Battaglia. Existence and multiplicity result for the singular Toda system. *J. Math. Anal. Appl.*, 424(1):49–85, 2015.

[11] Luca Battaglia, Aleks Jevnikar, Andrea Malchiodi, and David Ruiz. A general existence result for the Toda System on compact surfaces. *arXiv* 1306.5404, 2013.

[12] Luca Battaglia and Andrea Malchiodi. A Moser-Trudinger inequality for the singular Toda system. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 9(1):1–23, 2014.

[13] Luca Battaglia and Gabriele Mancini. Remarks on the Moser-Trudinger inequality. *Adv. Nonlinear Anal.*, 2(4):389–425, 2013.

[14] Luca Battaglia and Gabriele Mancini. A note on compactness properties of the singular Toda system. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 26(3):299–307, 2015.

[15] William Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[16] J. Bolton and L. M. Woodward. Some geometrical aspects of the 2-dimensional Toda equations. In *Geometry, topology and physics (Campinas, 1996)*, pages 69–81. de Gruyter, Berlin, 1997.

[17] Haïm Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[18] Haïm Brezis and Frank Merle. Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. *Comm. Partial Differential Equations*, 16(8-9):1223–1253, 1991.

[19] Eugenio Calabi. Isometric imbedding of complex manifolds. *Ann. of Math. (2)*, 58:1–23, 1953.

[20] Lennart Carleson and Sun-Yung A. Chang. On the existence of an extremal function for an inequality of J. Moser. *Bull. Sci. Math. (2)*, 110(2):113–127, 1986.

[21] Alessandro Carlotto. On the solvability of singular Liouville equations on compact surfaces of arbitrary genus. *Trans. Amer. Math. Soc.*, 366(3):1237–1256, 2014.

[22] Alessandro Carlotto and Andrea Malchiodi. Weighted barycentric sets and singular Liouville equations on compact surfaces. *J. Funct. Anal.*, 262(2):409–450, 2012.

[23] Dongho Chae, Hiroshi Ohtsuka, and Takashi Suzuki. Some existence results for solutions to SU(3) Toda system. *Calc. Var. Partial Differential Equations*, 24(4):403–429, 2005.

[24] Sun-Yung A. Chang and Paul C. Yang. Conformal deformation of metrics on S^2. *J. Differential Geom.*, 27(2):259–296, 1988.

[25] Sun-Yung Alice Chang and Paul C. Yang. Prescribing Gaussian curvature on S^2. *Acta Math.*, 159(3-4):215–259, 1987.

[26] Chiun-Chuan Chen and Chang-Shou Lin. Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. *Comm. Pure Appl. Math.*, 55(6):728–771, 2002.

[27] Chiun-Chuan Chen and Chang-Shou Lin. Topological degree for a mean field equation on Riemann surfaces. *Comm. Pure Appl. Math.*, 56(12):1667–1727, 2003.

[28] Chiun-Chuan Chen and Chang-Shou Lin. Mean field equations of Liouville type with singular data: sharper estimates. *Discrete Contin. Dyn. Syst.*, 28(3):1237–1272, 2010.

[29] Chiun-Chuan Chen and Chang-Shou Lin. Mean field equation of Liouville type with singular data: topological degree. *Comm. Pure Appl. Math.*, 68(6):887–947, 2015.

[30] Wen Xiong Chen. A Trüdinger inequality on surfaces with conical singularities. *Proc. Amer. Math. Soc.*, 108(3):821–832, 1990.

[31] Wen Xiong Chen and Congming Li. Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.*, 63(3):615–622, 1991.

[32] Shiing Shen Chern and Jon G. Wolfson. Harmonic maps of the two-sphere into a complex Grassmann manifold. II. *Ann. of Math. (2)*, 125(2):301–335, 1987.

[33] M. Chipot, I. Shafrir, and G. Wolansky. On the solutions of Liouville systems. *J. Differential Equations*, 140(1):59–105, 1997.

[34] G. Csatò and P. Roy. The singular moser-trudinger inequality on simply connected domains. *preprint*, http://arxiv.org/pdf/1507.00323v1.pdf, 2015.

[35] G. Csatò and P. Roy. Extremal functions for the singular moser-trudinger inequality in 2 dimensions. *Calc. Var. Partial Differential Equations*, 10.1007/s00526-015-0867-5, to appear.

[36] Teresa D'Aprile, Angela Pistoia, and David Ruiz. Asymmetric blow-up for the $SU(3)$ Toda System. *arXiv* 1411.3482, 2014.

[37] Weiyue Ding, Jürgen Jost, Jiayu Li, and Guofang Wang. The differential equation $\Delta u = 8\pi - 8\pi h e^u$ on a compact Riemann surface. *Asian J. Math.*, 1(2):230–248, 1997.

[38] Weiyue Ding, Jürgen Jost, Jiayu Li, and Guofang Wang. Existence results for mean field equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(5):653–666, 1999.

[39] Zindine Djadli. Existence result for the mean field problem on Riemann surfaces of all genuses. *Commun. Contemp. Math.*, 10(2):205–220, 2008.

[40] G. Dunne. *Self-dual Chern-Simons Theories*. Lecture notes in physics. New series m: Monographs. Springer, 1995.

[41] Martin Flucher. Extremal functions for the Trudinger-Moser inequality in 2 dimensions. *Comment. Math. Helv.*, 67(3):471–497, 1992.

[42] Luigi Fontana. Sharp borderline Sobolev inequalities on compact Riemannian manifolds. *Comment. Math. Helv.*, 68(3):415–454, 1993.

[43] Jooyoo Hong, Yoonbai Kim, and Pong Youl Pac. Multivortex solutions of the abelian Chern-Simons-Higgs theory. *Phys. Rev. Lett.*, 64(19):2230–2233, 1990.

[44] Jürgen Jost, Changshou Lin, and Guofang Wang. Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. *Comm. Pure Appl. Math.*, 59(4):526–558, 2006.

[45] Jürgen Jost and Guofang Wang. Analytic aspects of the Toda system. I. A Moser-Trudinger inequality. *Comm. Pure Appl. Math.*, 54(11):1289–1319, 2001.

[46] Jürgen Jost and Guofang Wang. Classification of solutions of a Toda system in $\mathbb{R}^2$. *Int. Math. Res. Not.*, (6):277–290, 2002.

[47] Jerry L. Kazdan and F. W. Warner. Curvature functions for compact 2-manifolds. *Ann. of Math. (2)*, 99:14–47, 1974.

[48] Jerry L. Kazdan and F. W. Warner. Curvature functions for open 2-manifolds. *Ann. of Math. (2)*, 99:203–219, 1974.

[49] S. Kesavan. *Symmetrization & applications*, volume 3 of *Series in Analysis*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[50] Yan Yan Li. Harnack type inequality: the method of moving planes. *Comm. Math. Phys.*, 200(2):421–444, 1999.

[51] Yan Yan Li and Itai Shafrir. Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two. *Indiana Univ. Math. J.*, 43(4):1255–1270, 1994.

[52] Yu Xiang Li. Remarks on the extremal functions for the Moser-Trudinger inequality. *Acta Math. Sin. (Engl. Ser.)*, 22(2):545–550, 2006.

[53] Yuxiang Li. Moser-Trudinger inequality on compact Riemannian manifolds of dimension two. *J. Partial Differential Equations*, 14(2):163–192, 2001.

[54] Yuxiang Li. Extremal functions for the Moser-Trudinger inequalities on compact Riemannian manifolds. *Sci. China Ser. A*, 48(5):618–648, 2005.

[55] Changshou Lin, Juncheng Wei, and Wen Yang. Degree counting and shadow system for $SU(3)$ Toda system: one bubbling. *arXiv* 1408.5802, 2014.

[56] Changshou Lin, Juncheng Wei, and Lei Zhang. Classification of blowup limits for $SU(3)$ singular Toda systems. *arXiv* 1303.4167, 2014.

[57] Kai-Ching Lin. Extremal functions for Moser's inequality. *Trans. Amer. Math. Soc.*, 348(7):2663–2671, 1996.

[58] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[59] Guozhen Lu and Yunyan Yang. Sharp constant and extremal function for the improved Moser-Trudinger inequality involving L^p norm in two dimension. *Discrete Contin. Dyn. Syst.*, 25(3):963–979, 2009.

[60] Marcello Lucia. A deformation lemma with an application to a mean field equation. *Topol. Methods Nonlinear Anal.*, 30(1):113–138, 2007.

[61] Marcello Lucia and Margherita Nolasco. SU(3) Chern-Simons vortex theory and Toda systems. *J. Differential Equations*, 184(2):443–474, 2002.

[62] Andrea Malchiodi. Topological methods for an elliptic equation with exponential nonlinearities. *Discrete Contin. Dyn. Syst.*, 21(1):277–294, 2008.

[63] Andrea Malchiodi and Cheikh Birahim Ndiaye. Some existence results for the Toda system on closed surfaces. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 18(4):391–412, 2007.

[64] Andrea Malchiodi and David Ruiz. A variational analysis of the Toda system on compact surfaces. *Comm. Pure Appl. Math.*, 66(3):332–371, 2013.

[65] G. Mancini. Onofri-type inequalities for singular liouville equations. *Journal of Geometric Analysis*, 2015.

[66] G. Mancini. Singular liouville equations on S^2: Sharp inequalities and existence results. *preprint*, 2015.

[67] David Montgomery and Glenn Joyce. Statistical mechanics of “negative temperature” states. *Phys. Fluids*, 17:1139–1145, 1974.

[68] J. Moser. A sharp form of an inequality by N. Trudinger. *Indiana Univ. Math. J.*, 20:1077–1092, 1970/71.

[69] E. Onofri. On the positivity of the effective action in a theory of random surfaces. *Comm. Math. Phys.*, 86(3):321–326, 1982.

[70] B. Osgood, R. Phillips, and P. Sarnak. Compact isospectral sets of surfaces. *J. Funct. Anal.*, 80(1):212–234, 1988.

[71] B. Osgood, R. Phillips, and P. Sarnak. Extremals of determinants of Laplacians. *J. Funct. Anal.*, 80(1):148–211, 1988.

[72] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103(3):207–210, 1981.

[73] A. M. Polyakov. Quantum geometry of fermionic strings. *Phys. Lett. B*, 103(3):211–213, 1981.

[74] J. Prajapat and G. Tarantello. On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(4):967–985, 2001.

[75] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.

[76] Itai Shafrir and Gershon Wolansky. The logarithmic HLS inequality for systems on compact manifolds. *J. Funct. Anal.*, 227(1):200–226, 2005.

[77] Itai Shafrir and Gershon Wolansky. Moser-Trudinger and logarithmic HLS inequalities for systems. *J. Eur. Math. Soc. (JEMS)*, 7(4):413–448, 2005.

[78] Michael Struwe. Critical points of embeddings of $H_0^{1,n}$ into Orlicz spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(5):425–464, 1988.

[79] Michael Struwe. A flow approach to Nirenberg's problem. *Duke Math. J.*, 128(1):19–64, 2005.

[80] Michael Struwe and Gabriella Tarantello. On multivortex solutions in Chern-Simons gauge theory. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(1):109–121, 1998.

[81] Gabriella Tarantello. *Selfdual gauge field vortices.* Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston, Inc., Boston, MA, 2008. An analytical approach.

[82] Cyril Tintarev. Trudinger-Moser inequality with remainder terms. *J. Funct. Anal.*, 266(1):55–66, 2014.

[83] Marc Troyanov. Prescribing curvature on compact surfaces with conical singularities. *Trans. Amer. Math. Soc.*, 324(2):793–821, 1991.

[84] Neil S. Trudinger. On imbeddings into Orlicz spaces and some applications. *J. Math. Mech.*, 17:473–483, 1967.

[85] Anatoly Tur and Vladimir Yanovsky. Point vortices with a rational necklace: new exact stationary solutions of the two-dimensional Euler equation. *Phys. Fluids*, 16(8):2877–2885, 2004.

[86] Guofang Wang. Moser-Trudinger inequalities and Liouville systems. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(10):895–900, 1999.

[87] Yisong Yang. *Solitons in field theory and nonlinear analysis.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.

[88] Yunyan Yang. A sharp form of the Moser-Trudinger inequality on a compact Riemannian surface. *Trans. Amer. Math. Soc.*, 359(12):5761–5776 (electronic), 2007.

[89] Yunyan Yang. Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two. *J. Differential Equations*, 258(9):3161–3193, 2015.