Scuola Internazionale Superiore di Studi Avanzati - Trieste

Ph.D. Thesis

Symplectic invariants of integrable Hamiltonian systems with singularities

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Chapter 1

Introduction

It is known that a completely integrable Hamiltonian system on a symplectic manifold (M, ω), seen as a Lagrangian fibration, defines an integer affine structure $\mathcal{A}_\omega$ on its base space [20]. We want to answer the following question:

In which cases does the affine structure $\mathcal{A}_\omega$ determine ω uniquely modulo momentum-preserving real-analytic automorphisms of M?

We are interested in integrable systems with singularities. In some cases, where some non-degeneracy condition is assumed, this is known to hold (see [18, 16, 19, 55]). Notice that the converse statement is obviously true: if a symplectic form is mapped to another one by a momentum-preserving map then the action variables of the two forms coincide, and so do the affine structures on the base space.

In Part I of the thesis we discuss the case of one degree of freedom, and we describe sufficient conditions for a positive answer in presence of *degenerate* singularities. The applicability of these criteria depends on the type of singularities. In Part II we study a typical degenerate case occurring in two degrees of freedom: *parabolic orbits* and *cuspidal tori*.

1 Completely integrable systems with singularities

All real objects throughout the thesis will be assumed to be real-analytic. The space of real-analytic k-forms will be denoted by A^k, while $A^k(\mathbb{R}^n, 0)$ will denote the space of germs of real-analytic k-forms at $0 \in \mathbb{R}^n$. For holomorphic k-forms we use the notation Ω^k and $\Omega^k(\mathbb{C}^n, 0)$.

Let (M, ω) be a symplectic manifold. For any function $f : M \to \mathbb{R}$, denote by $X_f = \omega^{-1} df$ the vector field satisfying $i_{X_f}\omega = df$, and let $\{f, g\} = \omega(X_f, X_g)$ be the Poisson bracket defined by ω. If two functions Poisson-commute, then the corresponding vector fields commute.

Definition 1.1. A **completely integrable Hamiltonian system**[1] on a compact symplectic manifold (M, ω) of dimension $2n$ is given by n real-analytic Poisson-commuting functions $f_1, \dots, f_n : M \to \mathbb{R}$ which are independent almost everywhere, and by a **Hamiltonian func-**

[1]For short we will often write "integrable systems", "integrable Hamiltonian systems", etc.

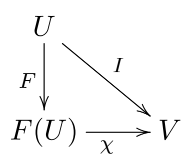
tion H such that $\{H, f_i\} = 0$ for all $i = 1, \dots, n$.

The map $F = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ is called the **momentum map**. A point $x \in M$ is **critical** for F if rank $dF(x) < n$. We denote by $K = \operatorname{Crit} F \subset M$ the set of critical points of the momentum map. The set of critical values $\Sigma := F(K) \subset \mathbb{R}^n$ is called the **bifurcation diagram** of F.

Since the vector fields $X_{f_1}, \dots, X_{f_n}$ commute, their joint flows define an action $\Phi : \mathbb{R}^n \times M \to M$, called the **Hamiltonian action** of the group $\mathbb{R}^n$ (defined by $f_1, \dots, f_n$). From the fact that this action preserves the momentum map it follows that the orbit through a point x of rank $k < n$, i.e., the **singular orbit** through x, is k-dimensional.

Let us recall the statement of Liouville-Arnol'd Theorem (see [20]). Assume M is compact. Let $M_r := \{x \in M : \text{rank } dF(x) = n\}$ be the set of regular points for F, then $F|_{M_r}$ is a submersion. For $x \in F^{-1}(c)$ denote by $F^{(x)}$ the connected component of $F^{-1}(c)$ containing x.

Theorem 1.1 (Liouville-Arnol'd)**.** *M_r is an open subset of M. For each $x \in M$ such that $F^{(x)} \subset M_r$ (i.e., is a regular fiber), there is an open neighborhood U of $F^{(x)}$ in M_r, invariant with respect to the flow of each X_{f_i}, and a diffeomorphism $(I, \varphi) : U \to V \times T^n$ with V open in $\mathbb{R}^n$ such that $I = \chi \circ F$ for some diffeomorphism $\chi : F(U) \to V$, and $\omega = dI_i \wedge d\varphi_i$ on U.*



The coordinates $(I_1, \dots, I_n, \varphi_1, \dots, \varphi_n)$ are called **action-angle variables**. Action variables, seen as a local coordinate system on the base space, are not unique: any two set of actions are related by an *integer affine transformation*, i.e., belonging to $\mathrm{GL}(n, \mathbb{Z}) \ltimes \mathbb{R}^n$. A completely integrable system can be understood as a Lagrangian fibration with singularities, i.e., a fibration $\pi : M \to B$ whose regular fibers are Lagrangian submanifolds of (M, ω). Here B denotes the quotient space obtained from M by identifying points on a same connected fiber. Then the collection of action variables defines an atlas of the regular part of B, whose transition functions are integer affine. In other words, the regular part of the base space is an **integer affine manifold**.

Liouville-Arnol'd theorem gives a complete description (i.e., a symplectic canonical form) of the fibration in a neighborhood of a regular fiber. It also shows that, if we are given two regular Lagrangian fibrations, then any integer affine local diffeomorphism between the bases can be lifted to a momentum-preserving symplectomorphism between the two systems (this amounts to choosing a Lagrangian section for the two fibrations). From the point of view of classification, this means that locally, regular Lagrangian fibrations have no topological, smooth nor symplectic invariants.

1.1 Singularities

The picture becomes more complicated when we consider the fibration in the neighborhood of a critical value. Consider a curve lying in the regular part of B, its pre-image consists of a family of Liouville-Arnol'd tori (not necessarily connected). If we let the curve cross the bifurcation diagram, these tori will typically undergo a nontrivial bifurcation, and the fibration near the singular fiber, i.e., at the point where this transition happens, can be quite complicated (see [7] for examples). In this sense, singular fibers contain the nontrivial topological information of the integrable system. For this reason, and because singularities are ubiquitous in integrable systems, the problem of their description and classification is very important in the theory of completely integrable Hamiltonian systems.

The typical classification problem can be formulated as follows: let $(M, \omega, f_1, \ldots, f_n)$ and $(\widetilde{M}, \widetilde{\omega}, \widetilde{f}_1, \ldots, \widetilde{f}_n)$ be two integrable systems. These two systems are **Liouville equivalent** if there exists a fiber-preserving homeomorphism $\Psi : M \to \widetilde{M}$. We can consider more rigid types of equivalence, requiring the map Ψ to be smooth or real-analytic. If there exists a fiber-preserving map Ψ as above which moreover is a symplectomorphism ($\Psi^* \widetilde{\omega} = \omega$), we say that the two systems are **symplectically equivalent**.

In describing the singularities of the fibration we can distinguish three kinds of problems, which are defined by restricting the system to some open domain $U \subset M$:

- the **local** problem, if U is a neighborhood of a singular point;
- the **semi-local** problem, if U is a neighborhood of a singular fiber;
- the **global** problem, if U is the whole manifold.

It is also common to consider U to be a neighborhood of a singular orbit (the local case correspond to zero-dimensional orbits).

A common type of singularity is that of *non-degenerate singularities* which are, essentially, the direct product of simpler singularities of the three kinds: elliptic, hyperbolic, or focus-focus (see [7] for a definition). Non-degenerate singularities admit a smooth symplectic normal form, as was shown by L. H. Eliasson [24] (see also [47]). This result was later generalized to the case of compact non-degenerate orbits of any dimension by E. Miranda and N. T. Zung [48]. The topological investigation of integrable systems with one and two degrees of freedom is due to A. T. Fomenko and H. Zieschang, and later developed by A. V. Bolsinov, V. S. Matveev, A. A. Oshemkov, N. T. Zung among others. For a detailed treatment and references see [7]. Other approaches to the study of topological phenomena of integrable systems have been developed by N. N. Nekhoroshev, D. A. Sadovskií and B. I. Zhilinskií [49], R. H. Cushman and L. M. Bates [17], K. Efstathiou [22], M. Audin [5], M. Symington [54].

From the point of view of global classification, other non-trivial invariants arise from the possibility that the fibration is globally non-trivial. The case of regular fibrations was completely clarified by J. J. Duistermaat [20], where he introduced the concept of *monodromy* of a Lagrangian fibration, together with *Chern* and *Lagrangian characteristic classes*. Duistermaat's results were later extended in different ways. N. T. Zung [62] has shown how to generalize these invariants to the case of Lagrangian fibrations with singularities. Under appropriate non-degeneracy assumptions, strong results were obtained showing that the affine base space classifies the system up to global symplectic equivalence. This was done by T. Delzant [18] under the assumption that the system admits a global torus action. A generalization to the case of *semi-toric* system is due to A. Pelayo and S. Vũ Ngọc [52].

In the thesis we will consider the problem of symplectic equivalence in the local and semi-local setting. Since a symplectomorphism between the fibrations will preserve action variables, the actions (or the affine structure on the base space) are a symplectic invariant. It was shown by J.-P. Dufour, P. Molino and A. Toulet [19], that in the case of semi-local symplectic classification in one degree of freedom for fibrations with non-degenerate hyperbolic singularities, action variables are (essentially) the only non-trivial invariants. In other words, if two Liouville equivalent systems have the same actions, then they are symplectically equivalent. The same is true for focus-focus singularities, as was shown by S. Vũ Ngọc [55].

The aim of this thesis is to extend this class to the case of *degenerate* singularities in one degree of freedom, and to that of *parabolic trajectories*, often occurring in systems with two degrees of freedom.

1.2 One degree of freedom

In order for two integrable systems to be symplectically equivalent, they must first of all be smoothly Liouville equivalent. For the case of one degree of freedom, we will assume that this latter condition is satisfied, i.e., that a fiber-preserving real-analytic diffeomorphism is given. In other words we reduce to the case of two different symplectic forms over the same foliated manifold.

Let M be a real-analytic 2-dimensional manifold, $H : M \to \mathbb{R}$ a real-analytic Hamiltonian function. Assume that $H^{-1}(0)$ is a singular fiber for H. We will consider the *semi-local* setting, i.e., we will assume that M is a neighborhood of the singular fiber of the form $M = H^{-1}[-\eta, \eta]$, where $\eta > 0$ is small enough so that $H^{-1}(0)$ is the only singular fiber in M.

Define the **Reeb graph** of (M, H, ω) to be the quotient $\Gamma := M/\sim$, where $x \sim y$ if and only if x and y belong to the same connected fiber of $H : M \to \mathbb{R}$. Denote by π the projection $M \to \Gamma$. The Reeb graph Γ is endowed with the quotient topology: a subset $U \subset \Gamma$ is open if and only if $\pi^{-1}(U)$ is open in M. We can decompose Γ as $\Gamma = \{0\} \cup_i \Gamma_i$ where each Γ_i represents a regular S^1-fibration in M. Consider now a symplectic form $\omega \in \Omega^2(M)$. For each i, the set of action variables defined over on $\pi^{-1}(\Gamma_i)$ defines an atlas over Γ_i having integer affine transition functions (i.e., an integer affine structure). The Reeb graph endowed with this integer affine structure on its regular part is called **affine Reeb graph**.

For each Γ_i we can define an action variable I_i through the formula:

$$I_i(t) = \oint_{\pi^{-1}(t)} \alpha$$

where α is the *action* 1-*form*, such that $d\alpha = \omega$, and $\pi^{-1}(t)$ is an oriented regular fiber. Notice that in this one-dimensional case, in a neighborhood of each regular $t \in \Gamma$, the action variables are defined up to a constant.

The problem of H-preserving symplectic equivalence in the case of non-degenerate singularities was studied in [16] and [19]. One way to approach degenerate singularities (e.g., cusps, etc.) is to translate the problem into a problem of *relative cohomology*, as we will explain better in a later chapter. This strategy was adopted in the local classification of volume forms by Y. Colin de Verdiere and J. Vey [16] for the non-degenerate case and by J.-P. Francoise [26, 27] for the degenerate case (see also [36, V.26]), and it is the starting point of this thesis. For this approach it is useful to reformulate affine equivalence in terms of *periods*. The period of the Hamiltonian vector field X_H at $t \in \Gamma_i$, i.e., the time it takes to the flow of X_H to complete a loop around a fiber $\pi^{-1}(t)$, is given by $\Pi_\omega(t) = dI_i(t)/dt$. Each period is a smooth function $\Gamma_i \to \mathbb{R}$ which may diverge at 0. We can collect all periods into a *period map* $\Pi_\omega : \Gamma \setminus \{0\} \to \mathbb{R}$. An equivalent expression for the period map is:

$$\Pi_\omega(h) = \oint_{\pi^{-1}(t)} \frac{\omega}{dH} \tag{1.1}$$

where ω/dH is the **Gelfand-Leray form** of ω, defined as follows (see [3]). Let $x \in M$ be a regular point for H, then we can find a neighborhood U of x and a real-analytic 1-form $\xi \in \Omega^1(U)$ such that $dH \wedge \xi = \omega$ on U. The form ξ is not uniquely defined, but its restriction to the level-sets $U \cap \pi^{-1}(t)$ is unique. To define a Gelfand-Leray form on the whole fiber $\pi^{-1}(t)$, we need to combine the local solutions ξ via a partition of unity covering $\pi^{-1}(t)$. The resulting form is not necessarily real-analytic, but its restriction on $\pi^{-1}(t)$ is real-analytic, because of unicity.

The final form of the problem for the case of one degree of freedom is the following:

Problem. *Suppose $\Pi_{\omega_0} = \Pi_{\omega_1}$ on Γ. Is it true that there exists a H-preserving real-analytic automorphism ψ of M such that $\psi^*\omega_1 = \omega_0$?*

If such a map ψ exists locally (resp. semi-locally, globally), we say that the local (resp. semi-local, global) problem has a solution. We do not restrict to non-degenerate saddle singularities, but consider more degenerate singularities. As it turns out, one way to address the problem is to start form the local case, complexify the singularities and to frame the problem into the theory of complex singularities. The equivalence of semi-local periods will impose some condition on the local singularity, namely that the periods over some open curves of the local fibration are real-analytic (see the end of Subsection 1.2 of Chapter 2). These conditions can be sufficient for the existence of a local solution (i.e., to find a local ψ in a neighborhood of the singularity). In this case the local singularity will be called a *good singularity.* Under some additional condition, local solutions can then be extended to a semi-local, or global one (see Section 3 of Chapter 3).

1.3 Two degrees of freedom

Let (M, Ω) be a 4-dimensional symplectic manifold, and let H, F be a pair of Poisson commuting real-analytic functions defining a completely integrable Hamiltonian system on M. The bifurcation diagram $\Sigma \subset \mathbb{R}^2$ in this case will generally consist of piecewise-regular curves with singular points plus, possibly, isolated points (see for example [7]). A fiber $\mathcal{L}_{h,f} = \{H = h, F = f\}$ is singular if it contains a point P such that $dH(P)$ and $dF(P)$ are linearly dependent. If these differentials are not both zero, the $\mathbb{R}^2$-orbit of P is a 1-dimensional *singular orbit* γ.

The most frequent examples of singular orbits are given by elliptic and hyperbolic orbits. These non-degenerate orbits were shown to have a symplectic canonical form (in a neighborhood of the singular orbit), in other words there is no symplectic invariant in the non-degenerate case [48]. It typically happens that a one-parameter family of non-degenerate singular orbits becomes degenerate at a given value of the parameter, or in other words, that a degenerate singularity splits into several non-degenerate ones (like for Morsification of singularities). In this thesis we consider a particular, but typical, type of degenerate singular orbit:

parabolic orbits (or *parabolic trajectories*). These can be considered as the simplest example of degenerate singular orbits. Because these singularities are stable under small integrable perturbation, they occur in many examples of integrable Hamiltonian systems. Unlike non-degenerate singularities, however, in the literature on topology and singularities of integrable systems there are only few papers devoted to degenerate singularites including parabolic ones. We refer, first of all, to the following six — L. Lerman, Ya. Umanskii [44], V. Kalashnikov [37], N. T. Zung [61], H. Dullin, A. Ivanov [21] and K. Efstathiou, A. Giacobbe [23], Y. Colin de Verdiere [15].

From the point of view of smooth classification, i.e., without taking into account the symplectic structure, a parabolic orbit can be reduced to a canonical model (see [23] or Proposition 5.3 from Chapter 5): there exist coordinates $(x, y, \lambda, \varphi) \in \mathbb{R}^3 \times S^1$ in a neighborhood of the singular orbit γ such that:

$$H = x^2 + y^3 + \lambda y, \quad F = \lambda.$$

The parabolic orbit is given by the curve $\gamma = (0, 0, 0, t)$. As we will show however, from the point of view of symplectic equivalence, these singularities admit non-trivial symplectic invariants.

If instead of a neighborhood of γ we consider a saturated neighborhood of a compact singular fiber containing γ, i.e., a *cuspidal torus*, then these invariants can be expressed in terms of the action variables (i.e., the affine structure on the base). The main result (Theorem 5.6) is that the only symplectic semi-local invariant of a cuspidal torus is the integer affine structure on the base of the fibration.

2 Content of the thesis

Part I. One degree of freedom.

- In Chapter 2 we introduce the notions of complex singularity theory which are needed in the local case. We define *analytically good singularities* (Definition 2.5), a class of local singularities for which the local problem in one degree of freedom admits a solution[2] (as explained in Theorem 2.4 and Corollary 2.2). Finally we describe toric resolutions (for the case of Γ-*non-degenerate* singularities, see Definition 2.6), which are an effective tool to simplify the study of singularities, both real and complex. We use them in Subsection 2.4 to give a description of the real local fibration.
- In Chapter 3 we define *topologically good singularities* (Definition 3.3), and we show that a topologically good singularity is also analytically good (Theorem 3.1). In Section 2, we describe an algorithm allowing to determine if a singularity is topologically, and hence analytically, good. We give a list of good singularities in Table 3.1. Finally, we move to the semi-local and global problems (Sections 3 and 4). We show that under some topological *rigidity* hypothesis, the solution of the local problem yields a solution to the semi-local one (Theorems 3.3 and 3.4).

  This gives sufficient conditions for a solution of the local, semi-local and (smooth) global problem (Theorem 3.5). As a particular example, we find that: if an integrable systems

[2] This notion is similar to the "condition $(*)$" of [27]. See also [36].

has only singularities which are non-degenerate or of the form $y^p - x^q$, with p, q different primes, and each singular fiber has only one critical point, then a symplectic form is uniquely determined (modulo smooth fiber-preserving isotopy) by its action variables (Corollary 3.3).

- In Chapter 4 we use toric resolutions to study the asymptotic expansion of period integrals, using the approach of Varchenko [56] and we show that such asymptotic expansions allow to recognize non-degenerate singularities: a singularity is non-degenerate if and only if it has *logarithmic* periods (Propositions 4.5, 4.6 and 4.7). Moreover, asymptotic expansions can sometimes be used to show that a singularity is analytically good (Proposition 4.2).

Part II. Two degress of freedom. Parabolic orbits and cuspidal tori.

- In Chapter 5 we discuss in detail the case of parabolic orbits in two degrees of freedom. First we discuss a normal form for the fibration in the neighborhood of a parabolic orbit (Proposition 5.3). We show that, given two systems with a parabolic orbit, a local diffeomorphism between the bases which respects the bifurcation diagrams can be lifted to a fiber-preserving diffeomorphism between neighborhoods of the parabolic trajectories (Proposition 5.2).

  Next we give necessary and sufficient conditions for the existence of a symplectic equivalence between two systems in a neighborhood of a parabolic orbit (Theorems 5.2 and 5.4). Finally we treat this problem from the semi-local point of view. We show that two cuspidal tori are semi-locally symplectically equivalent if and only if there exists a local integer affine equivalence between the corresponding bases (Theorem 5.6).

  This part is self-contained and consists of a joint work with A. V. Bolsinov and E. A. Kudryavtseva.

Part I

One degree of freedom

Chapter 2

Local study: singularity theory

1 Review of complex singularity theory

We briefly review the basic constructions of singularity theory (see [4, 2, 60] for a more detailed explanation), giving a proof of those statements which are most important or require to be adapted for our purposes. A germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, such that $df(0) = 0$ is called an **isolated singularity** if in a neighborhood of $0 \in \mathbb{C}^n$ the function f has no critical points other than zero. This is equivalent to requiring that the **local algebra** of the singularity

$$Q_f := \frac{\mathbb{C}\{z_1, \dots, z_n\}}{(\partial_{z_1} f, \cdots, \partial_{z_n} f)}$$

satisfies $\dim_{\mathbb{C}} Q_f < \infty$. Here $\mathbb{C}\{z_1, \dots, z_n\}$ denotes the ring of convergent complex power series of n-variables, and $(\partial_{z_1} f, \cdots, \partial_{z_n} f)$ denotes the Jacobian ideal of f.

Theorem 2.1 (Milnor, see [2]). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. Denote $B = B(0, \epsilon) \subset \mathbb{C}^n$ the open ball and $D = B(0, \eta) \subset \mathbb{C}$ the open disc. We put*

$$\begin{aligned} X &:= B \cap f^{-1}(D) \\ X_t &:= B \cap f^{-1}(t) \\ D^* &:= D \setminus \{0\} \\ X^* &:= X \setminus X_0 \end{aligned}$$

Then for $\epsilon > 0$ small enough (so that the fiber X_0 intersects transversally ∂B) and for $0 < \eta = \eta(\epsilon)$ small with respect to ϵ, we have

*i) $f : X^* \to D^*$ is a smooth locally trivial fibration, called the **Milnor fibration** of f (fibers have complex dimension $n-1$).*

*ii) For $t \in D^*$, X_t is homotopically equivalent to a bouquet of μ real $(n-1)$-dimensional spheres S^{n-1}, where $\mu = \dim_{\mathbb{C}} Q_f < \infty$ is the **Milnor number** at 0.*

iii) $X \sim X_0 \sim C(\partial X_0)$ is contractible (homotopic to the cone of X_0).

iv) $f|_{\partial X} : \partial X \to D$ is also a locally trivial fibration, but since D is contractible it is trivial.

v) The Milnor fibration does not depend on (ϵ, η) (two allowed pairs give diffeomorphic fibrations).

Notice that we drop ϵ and η from the notation, but keeping in mind that the construction is local, and the neighborhoods B and D can be shrinked if necessary.

Consider a closed curve $\gamma = \gamma(s) : [0,1] \to D^*$ in the base space with $\gamma(0) = t_0$. Let $\Gamma_s : X_{t_0} \to X_{\gamma(s)}$ be the continuous family of diffeomorphisms identifying the fibers along the loop, defined (up to isotopy) by the local triviality of the Milnor bundle. The diffeomorphism $h := \Gamma_1 : X_{t_0} \to X_{t_0}$ is called **Picard-Lefschetz monodromy** of the singularity. It induces the monodromy automorphism in the integral homology $M_{\mathbb{Z}} := h_* : H_{n-1}(X_{t_0}, \mathbb{Z}) \to H_{n-1}(X_{t_0}, \mathbb{Z})$. Using complex coefficients we get a map

$$M := h_* : H_{n-1}(X_{t_0}, \mathbb{C}) \to H_{n-1}(X_{t_0}, \mathbb{C})$$

called the **monodromy operator**. In this way we have defined an action of $\pi_1(D^*, t_0) \simeq \mathbb{Z}$ on $H_{n-1}(X_{t_0}, \mathbb{C})$. Similarly, identifying $H^{n-1}(X_{t_0}, \mathbb{C}) = \mathrm{Hom}(H_{n-1}(X_{t_0}, \mathbb{C}), \mathbb{C})$, we can define a "pull-back" operator in cohomology:

$$M_c := (h^*)^{-1} : H^{n-1}(X_{t_0}, \mathbb{C}) \to H^{n-1}(X_{t_0}, \mathbb{C}).$$

The collection of complex vector spaces $H^{n-1}(X_t, \mathbb{C})$ for $t \in D^*$ defines a holomorphic vector bundle, called the **cohomology bundle**, with total space

$$H^{n-1} := \bigcup_{t \in D^*} H^{n-1}(X_t, \mathbb{C}).$$

We denote by $\mathcal{H}^{n-1}$ the sheaf ot its holomorphic sections, which by Theorem 2.1 (*ii*), is a locally free $\mathcal{O}_{D^*}$-module of rank μ.

Definition 2.1. Let $\omega \in \Omega^n(\mathbb{C}^n, 0)$, and let $t \in D^*$. There exists a $(n-1)$-form ψ, defined in a neighborhood of X_t, such that $\omega = df \wedge \psi$. The restriction of ψ to X_t is a uniquely defined holomorphic $(n-1)$-form, called the **Gelfand-Leray form** of ω, and denoted by

$$\psi|_{X_t} = \frac{\omega}{df}\bigg|_{X_t}.$$

Definition 2.2. Let $\omega \in \Omega^n(\mathbb{C}^n, 0)$. Its Gelfand-Leray form on each fiber X_t induces a cohomology class

$$\left[\frac{\omega}{df}\bigg|_{X_t}\right] \in H^{n-1}(X_t, \mathbb{C})$$

and the collection of $[\omega/df|_{X_t}]$ for $t \in D^*$ defines a section $[\omega/df]$ of the cohomology bundle, called the **geometric section** of ω.

In the following we will consider the case of curve singularities ($n = 2$). In this case a regular Milnor fiber X_t is a non-compact Riemann surface. The first cohomology group of non-compact Riemann surfaces can be computed as the *holomorphic de Rham cohomology* group $H^1_{\mathrm{dR,hol}}(X) := \Omega^1(X)/d\mathcal{O}(X)$:

Theorem 2.2. *For a non-compact Riemann surface X there is an isomorphism:*

$$H^1_{\mathrm{dR,hol}}(X) \simeq H^1(X, \mathbb{C}). \tag{2.1}$$

Proof. For non-compact Riemann surfaces we have $H^1(X, \mathcal{O}) = 0$ [25, Theorem 26.1]. Now we argue like in [25, Theorem 15.13]: consider the exact sequence $0 \to \mathbb{C} \to \mathcal{O} \xrightarrow{d} \Omega^1 \to 0$, it induces a long exact sequence in cohomology, containing $\mathcal{O}(X) \xrightarrow{d} \Omega^1(X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}) = 0$, and so $\Omega^1(X)/d\mathcal{O}(X) \simeq H^1(X, \mathbb{C})$. □

Remark 2.1. These results can be generalized to the case of a singularity of n variables. The key fact is that the Milnor fiber for a n-variables singularity is a *Stein manifold* (of which non-compact Riemann surfaces are an example [25, Corollary 26.8]). For any Stein manifold we have $H^p(X, \mathcal{O}) = 0$ for $p > 0$, as a particular case of Cartan's Theorem B. The statement and proof of Theorem 2.2 can be generalized with little modification (see [53]).

The isomorphism of cohomology groups with holomorphic de Rham cohomology can be made locally fiberwise by the use of *relative de Rham cohomology* (see [40, Chapter I] for more details). As shown by Brieskorn [11] (see also [45, 40]), this description allows to define extensions of the sheaf $\mathcal{H}^{n-1}$ to the whole of D. More precisely, consider the following $\mathcal{O}_D$-module:

$$\mathcal{H}'' := \frac{f_*\Omega^n}{df \wedge d(f_*\Omega^{n-2})}$$

Theorem 2.3 (Brieskorn-Sebastiani)**.** *$\mathcal{H}''$ is a locally-free $\mathcal{O}_D$-module of rank μ (the Milnor number of the singularity). Moreover, the sheaf $\mathcal{H}''$ provides an extension of the cohomology bundle $\mathcal{H}^{n-1}$ from D^* to D, indeed:*

$$[\omega] \mapsto [\omega/df]$$

defines an isomorphism $\mathcal{H}''|_{D^*} \simeq \mathcal{H}^{n-1}$. The stalk of $\mathcal{H}''$ at $0 \in D$,

$$\mathcal{H}''_0 = \frac{\Omega^n(X,0)}{df \wedge d\Omega^{n-2}(X,0)},$$

is then a free $\mathbb{C}\{t\}$-module of rank μ.

Corollary 2.1. *Let $\omega \in \Omega^n(X,0)$. There exist unique germs $\psi_i \in \mathbb{C}\{t\}$, $i = 1, \dots, \mu$ and a germ $\varrho \in \Omega^{n-2}(X,0)$ such that:*

$$\omega = \sum_{i=1}^{\mu} \psi_i(f)\Omega_i + df \wedge d\varrho,$$

where the germs $\Omega_1, \dots, \Omega_\mu \in \Omega^n(X,0)$ induce a basis of $\mathcal{H}''_0$.

Definition 2.3. A set of forms $\{\Omega_1, \dots, \Omega_\mu\}$ as in the above corollary is called a **trivialization** for f.

If $f = f(x,y)$ is a quasi-homogeneous polynomial of weights $(w_1, w_2) \in \mathbb{Q}^2_+$, i.e., it is such that $f(t^{w_1}x, t^{w_2}y) = tf$, then a monomial basis of the local algebra of the singularity provides a trivialization (see [40, I.5.5]):

Proposition 2.1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a quasi-homogeneous singularity. Let $\{x^I = x^{i_1} \cdots x^{i_n} \in Q_f\}$ be a monomial basis of the local algebra of f. Then the germs $\Omega_I = x^I dx_1 \wedge \cdots \wedge dx_n \in \Omega^n(X,0)$ represent a basis of the $\mathbb{C}\{t\}$-module $\mathcal{H}''_0$.*

Now consider the case $n = 2$ of curve singularities. Consider a 1-cycle $\gamma(t)$ in X_t for $t \in D^*$. Letting t vary in D^* produces a family of cycles $\gamma(t)$, on the neighboring Milnor fibers, depending continuously on t. Let $\alpha \in \Omega^1(\mathbb{C}^2, 0)$ and $\omega \in \Omega^2(\mathbb{C}^2, 0)$. It can be shown (see [3] or Remark 3.1), that the functions

$$t \mapsto \oint_{\gamma(t)} \alpha$$
$$t \mapsto \oint_{\gamma(t)} \frac{\omega}{df}$$

are holomorphic functions of $t \in D^*$, and moreover, the following formula holds:

$$\frac{d}{dt} \oint_{\gamma(t)} \alpha = \oint_{\gamma(t)} \frac{d\alpha}{df}.$$

An important consequence of Brieskorn-Sebastiani theorem is that complex periods determine a class in $\mathcal{H}_0''$. This will be important for the problem of local symplectic equivalence. Indeed, as we will explain shortly (see Theorem 2.4), if two real-analytic symplectic forms define the same class in $\mathcal{H}_0''$, then there exists a local symplectic equivalence between them. It will be useful therefore to have a sufficient condition for a 2-form to induce the trivial class in $\mathcal{H}_0''$.

Proposition 2.2. *Let $f : X \to D$ be an isolated singularity.*

i) Let $\alpha \in \Omega^1(\mathbb{C}^2, 0)$. If for every $t \in D^*$ we have $\oint_\gamma \alpha = 0$ for all $\gamma \in H_1(X_t, \mathbb{C})$, then $\alpha = \varrho df + dg$ for some $\varrho, g \in \Omega^0(\mathbb{C}^2, 0)$.

ii) Let $\omega \in \Omega^2(\mathbb{C}^2, 0)$. If for every $t \in D^*$ we have $\oint_\gamma \omega/df = 0$ for all $\gamma \in H_1(X_t, \mathbb{C})$, then $\omega = df \wedge d\varrho$ for some $\varrho \in \Omega^0(\mathbb{C}^2, 0)$.

Point (*ii*) is a direct consequence of Brieskorn-Sebastiani theorem, indeed the hypothesis $[\omega/df] = 0$ means, after the isomorphism of Theorem 2.3, that $[\omega] = 0$ as a section of $\mathcal{H}''$ over D^*. But since the sheaf $\mathcal{H}''$ is locally free, this implies that $[\omega] = 0$ over the whole D. This implies the existence of a holomorphic germ $\varrho : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ such that $\omega = df \wedge d\varrho$.

However the proposition above admits an analytic proof, based on Riemann's First and Second Extension Theorems [32, Chapter 7]. We need one more lemma:

Lemma 2.1. *Let $\omega \in \Omega^2(\mathbb{C}^2, 0)$, and $\omega = d\alpha$ with $\alpha \in \Omega^1(\mathbb{C}^2, 0)$. If $\oint_{\gamma(t)} \omega/df \equiv 0$ for $t \in D^*$ then $\oint_{\gamma(t)} \alpha \equiv 0$ for $t \in D^*$.*

Proof. By assumption, for $t \in D^*$,

$$\frac{d}{dt} \oint_{\gamma(t)} \alpha = \oint_{\gamma(t)} \frac{d\alpha}{df} = \oint_{\gamma(t)} \frac{\omega}{df} \equiv 0,$$

therefore the function $t \mapsto \oint_{\gamma(t)} \alpha$ is constant on $t \in D^*$. But by Malgrange Lemma [45]

$$\lim_{\substack{t \to 0 \\ \arg t = 0}} \oint_{\gamma(t)} \alpha = 0. \qquad \square$$

Proof of Proposition 2.2. For point (i) we adapt the proof of [36, Theorem 26.13]. Fix a holomorphic section to the fibration $N : D \hookrightarrow X$, $N = N(t)$, intersecting every fiber transversally and not crossing the critical point $0 \in X$.

By hypothesis all periods of α vanish, this means that over each regular fiber X_t, $t \neq 0$, the restriction $\alpha|_{X_t}$ is an exact form, and admits a primitive. We construct it explicitly by integration. Let t be a regular value and $x \in X_t$. Consider a curve $\gamma(x)$ joining $N(t)$ to x; it exists because the fibers are connected. Define the following function g on $X \setminus X_0$:

$$g(x) = \int_{\gamma(x)} \alpha.$$

Since the periods vanish, the function is well-defined. It is a holomorphic function on $X \setminus X_0$ and it is bounded. Therefore it extends to a holomorphic function g on the whole X (by the First Riemann Extension Theorem).

Consider the form $\alpha - dg$. It vanishes on all vectors tangent to the fibers. Since df does the same, the two must be proportional, i.e. $\alpha - dg = \varrho df$ for some holomorphic function ϱ on $X \setminus 0$. Since ϱ is defined everywhere except for a single point then it extends to a holomorphic function on the whole X (now by the Second Riemann Extension Theorem). In conclusion, we have $\alpha = \varrho df + dg$ on X.

Finally we show that $(i) \Rightarrow (ii)$. Put $\omega = d\alpha$, then we know by Lemma 2.1 that $\int_{\gamma(t)} \alpha = 0$ for all $\gamma(t) \in H_1(X_t, \mathbb{Z})$. But then, by (i), we have $\alpha = -\varrho df + dg$ for some $\varrho, g \in \Omega^0(\mathbb{C}^2, 0)$. Taking the exterior differential we find $\omega = d\alpha = df \wedge d\varrho$. $\square$

1.1 Real-analytic singularities and their complexification

In the following, $\mathbb{R}\{x_1, \ldots, x_n\}$ (resp., $\mathbb{C}\{z_1, \ldots, z_n\}$) will denote real-analytic (resp., holomorphic) germs at 0, i.e., convergent power series in the respective fields.

Consider a real-analytic germ $f(x) = \sum_I a_I x^I \in \mathbb{R}\{x_1, \ldots, x_n\}$ (using multi-index notation $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$). Define the complexification of f as the holomorphic germ $f^{\mathbb{C}}(z) := \sum_I a_I z^I \in \mathbb{C}\{z_1, \ldots, z_n\}$, this complex power series is indeed convergent. For a germ of real-analytic differential form $\omega \in A^k(\mathbb{R}^n, 0)$, described as $\omega = \sum_I g_I(x) dx^I$ with $I = (i_1, \ldots, i_k) \in \mathbb{N}^k$ and $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we put $\omega^{\mathbb{C}} := \sum_I g_I^{\mathbb{C}}(z) dz^I$. This is a germ of holomorphic k-form on $\mathbb{C}^n$.

Now let $f \in \mathbb{C}\{z_1, \ldots, z_n\}$ be a holomorphic function germ, let $\mathrm{Re} f = (f + \overline{f})/2$ and $\mathrm{Im}\, f = (f - \overline{f})/2i$ denote its real and complex part respectively. We will consider also the real and complex "traces" on $\mathbb{R}^n$ defined by:

$$f^{\mathrm{Re}}(x) := \mathrm{Re} f|_{z_i = x_i + i0}, \qquad f^{\mathrm{Im}}(x) := \mathrm{Im}\, f|_{z_i = x_i + i0}.$$

If $f(z) = \sum_I a_I z^I$, then

$$f^{\mathrm{Re}}(x) = \sum_I \mathrm{Re}(a_I) x^I, \qquad f^{\mathrm{Im}}(x) = \sum_I \mathrm{Im}\,(a_I) x^I.$$

More generally, let $\alpha \in \Omega^k(\mathbb{C}^n, 0)$, given by $\alpha = \sum_I a_I(z) dz^I$, we define

$$\begin{aligned}
\overline{\alpha} &:= \textstyle\sum_I \overline{a}_I(z) d\overline{z}^I && \text{(conjugation)}\\
\mathrm{Re}(\alpha) &:= (\alpha + \overline{\alpha})/2 && \text{(real part)}\\
\mathrm{Im}\,(\alpha) &:= (\alpha - \overline{\alpha})/2i && \text{(imaginary part)}\\
\alpha^{\mathrm{Re}} &:= \mathrm{Re}(\alpha)|_{\mathbb{R}^n} = \textstyle\sum_I a_I^{\mathrm{Re}}(x) dx^I && \text{(real trace)}\\
\alpha^{\mathrm{Im}} &:= \mathrm{Im}\,(\alpha)|_{\mathbb{R}^n} = \textstyle\sum_I a_I^{\mathrm{Im}}(x) dx^I && \text{(imaginary trace)}
\end{aligned}$$

A real-analytic germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is said to have an **algebraically isolated** singularity at 0 if $\mu := \dim_{\mathbb{R}} \mathbb{R}\{x_1, \ldots, x_n\}/(\partial_1 f, \ldots, \partial_n f) < \infty$. This implies that the singular point is also topologically isolated in a neighborhood of $0 \in \mathbb{R}^n$, and even more: the complexified germ $f^{\mathbb{C}} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an isolated complex singularity with Milnor number μ. In the following, when talking about real singularities, we will always mean *algebraically isolated plane* germs $f \in \mathbb{R}\{x, y\}$. We will also assume that $f^{-1}(0) \subset \mathbb{R}^2$ is not an isolated point[1].

Definition 2.4. Define the real intervals $I_\eta^+ := (0, \eta]$ and $I_\eta^- := [-\eta, 0)$. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a real singularity, a **family of real relative cycles** is a connected family of level lines $f^{-1}(t)$, for either $t \in I_\eta^+$ or $t \in I_\eta^-$.

1.2 Analytically good singularities

Consider an algebraically isolated real singularity $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ and $\omega \in A^2(\mathbb{R}^2, 0)$. Consider a set of *real* relative cycles $\boldsymbol{\delta} = \boldsymbol{\delta}^+ \cup \boldsymbol{\delta}^-$ where $\boldsymbol{\delta}^\pm := \{\delta_i^\pm : i = 1, \ldots, r^\pm\}$ are positive (resp. negative) families of oriented real relative cycles. We allow relative cycles to be disconnected[2]. Now we want to bound these relative cycles by real-analytic sections: for each connected component of $f^{-1}(0) \smallsetminus \{0\}$ consider a point, and take a real-analytic cross-section N_i transversal to the real fibration and passing through this point (see Figure 2.1b). We will assume that the relative cycles $\delta \in \boldsymbol{\delta}^\pm$ have their end-points lying on the cross-sections $\{N_i\}$. Let $I_\eta^+ = (0, \eta]$ and $I_\eta^- = [-\eta, 0)$ as above. For each $\delta \in \boldsymbol{\delta}^\pm$, consider its (real relative) period map $J_\delta^\omega : I_\eta^\pm \to \mathbb{R}$:

$$J_\delta^\omega(t) := \int_{\delta(t)} \frac{\omega}{df}.$$

Since $\delta(t)$ is bounded by real-analytic sections, $J_\delta^\omega(t)$ is a real-analytic function on $I_\eta^\pm$. Notice that the above integrals are not proper "periods", meaning integrals over closed loops, but should be understood as *partial periods*.

Definition 2.5. We say that the singularity f is **analytically good** (with respect to $\boldsymbol{\delta}$) if for any $\omega \in A^2(\mathbb{R}^2, 0)$ the following implication is true:

$$J_\delta^\omega(t) \in \mathbb{R}\{t\}, \quad \forall \delta \in \boldsymbol{\delta} \quad \Longrightarrow \quad \omega = df \wedge d\varrho \quad \text{for some } \varrho \in A^0(\mathbb{R}^2, 0).$$

[1] This simpler case requires a slightly different discussion, as we explain in Remark 3.2 of the next chapter.

[2] The reason for this, and for the following discussion, comes from the problem of semi-local symplectic equivalence, as will be explained at the end of this section.

The first condition means that J_δ^ω can be extended to a real-analytic function in a neighborhood of zero. If it is satisfied we say that ω *has real-analytic periods over* $\boldsymbol{\delta}$. Similarly we say that it is **analytically bad** (with respect to $\boldsymbol{\delta}$) if it is not analytically good, i.e., if there exists a form $\omega \in A^2(\mathbb{R}^2, 0)$ having real-analytic periods over $\boldsymbol{\delta}$ but which cannot be put in the form $\omega = df \wedge d\varrho$.

When $\omega = df \wedge d\varrho$ for some $\varrho \in A^0(\mathbb{R}^2, 0)$, we say that ω is **relatively exact**.

The choice of the term "good" is motivated by the following proposition, which gives its connection with the problem of local symplectic equivalence (see also the corollary below):

Theorem 2.4. *Let $\omega_0, \omega_1 \in A^2(\mathbb{R}^2, 0)$ be two real-analytic symplectic forms. Suppose that $\omega_0 - \omega_1 = df \wedge d\varrho$ for some real-analytic function germ $\varrho(x, y)$ at $0 \in \mathbb{R}^2$. Then there is a local diffeomorphism ψ at $0 \in \mathbb{R}^2$ such that $\psi^* f = f$ and $\psi^* \omega_1 = \omega_0$.*

Proof. We adapt the proof of [26, Theorem 2.1]. The proof is based on the "Moser path method": put $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. In a neighborhood of zero, the forms ω_t, for $t \in [0, 1]$, are also non-degenerate, indeed the equation $\omega_0 - \omega_1 = df \wedge d\varrho$ implies that the two forms have the same sign/orientation at zero and near zero. Since ω_t is a convex combination of functions with the same sign, it will also be non-zero in a neighborhood of zero. Define a real-analytic time-dependent vector field X_t by

$$i_{X_t}\omega_t = -\varrho df.$$

Let φ_t be the flow generated by such X_t, we can integrate it for $t \in [0, 1]$. Notice that

$$L_{X_t}\omega_t = i_{X_t} d\omega_t + d i_{X_t}\omega_t = df \wedge d\varrho = \omega_0 - \omega_1,$$

therefore

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(L_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) = 0,$$

so that $\varphi_1^*\omega_1 = \omega_0$. Moreover $L_{X_t} f = 0$, because of the equality:

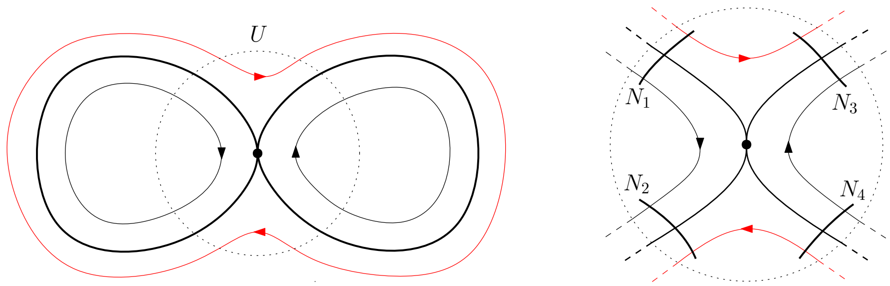
$$0 = i_{X_t}(df \wedge \omega_t) = (i_{X_t} df)\omega_t + df \wedge i_{X_t}\omega_t = (L_{X_t} f)\omega_t + df \wedge i_{X_t}\omega_t = (L_{X_t} f)\omega_t, \qquad (2.2)$$

and using $\omega_t \neq 0$. This means that $f \circ \varphi_t = f$. Finally take $\psi = \varphi_1$. □

In other words, we have:

Corollary 2.2. *Let f be an analytically good singularity with respect to a set of relative cycles $\boldsymbol{\delta}$. Let ω_0, ω_1 be two real-analytic symplectic forms. If the periods of ω_0 and ω_1 coincide (up to real-analytic functions) over the relative cycles $\delta \in \boldsymbol{\delta}$, then there is a local diffeomorphism ψ at $0 \in \mathbb{R}^2$ such that $\psi^* f = f$ and $\psi^* \omega_1 = \omega_0$.*

Remark 2.2. Notice that the converse is true as well. If ψ is a local symplectomorphism between ω_0 and ω_1, isotopic to the identity, and sending each oriented relative cycle to itself, then the periods of ω_0 and ω_1 coincide up to real-analytic functions.



(a) From semi-local to local (b) Local sections

Figure 2.1

It is necessary, at this point, to explain how the above definitions and results are related to the problem of symplectic equivalence. Consider a semi-local singularity $f = 0$ as in Figure 2.1a, and let ω_0, ω_1 be two symplectic forms. In this case the Reeb graph Γ has three edges, and correspondingly the period maps $\Pi_{\omega_0}, \Pi_{\omega_1}$ are given by the integrals over three regular families of circles (one red and two black in the figure).

Now assume that $\Pi_{\omega_0} = \Pi_{\omega_1}$ identically on $\Gamma \setminus \{0\}$, or equivalently that $\Pi_{\omega_1 - \omega_0} = 0$. Using cross-sections inside a small neighborhood U of the critical point (as in Figure 2.1b) we can split $\Pi_{\omega_1 - \omega_0}$ into a regular part (given by the integral over strips away from the critical point) plus a singular part (between the sections and inside U). Since the regular part of $\Pi_{\omega_1 - \omega_0}$ on each edge of Γ extends to a real-analytic function at $t = 0$, the condition $\Pi_{\omega_1 - \omega_0} = 0$ implies that the singular part of the integrals, over the relative cycles bounded by the sections $\{N_i\}$, extend to a real-analytic function at $t = 0$. Notice that in this example we have two connected relative cycles (in black) plus one disconnected relative cycle (in red): this is why we allow disconnected relative cycles in the definition of analytically good singularity. If the singularity is analytically good with respect to these relative cycles (cut from the semi-local periods), then by the above corollary we can find a fiber-preserving isotopy sending ω_0 to ω_1 inside U. We will show in the next chapter how, under some topological hypothesis on the semi-local fibration, it is possible to extend local solutions to semi-local ones (Theorems 3.3 and 3.4).

1.3 Symplectic equivalence in the complex case

We know that if two integrable systems are fiberwise symplectomorphic then their actions (or periods) coincide. Let's give the complex version of this statement.

Proposition 2.3. *Let $\omega_0, \omega_1 \in \Omega^2(\mathbb{C}^2, 0)$ be two holomorphic symplectic forms. Suppose there is a local holomorphic diffeomorphism ψ such that $\psi^* f = f$ and $\psi^*\omega_1 = \omega_0$. Moreover assume that the map $\psi_* : H_1(X_t) \to H_1(X_t)$ is the identity for $t \in D^*$. Then all complex periods of ω_0 and ω_1 on X_t coincide.*

Proof.

$$\oint_\gamma \frac{\omega_1}{df} = \oint_{\psi_*(\gamma)} \frac{\omega_1}{df} = \oint_\gamma \psi^* \left(\frac{\omega_1}{df}\right) = \oint_\gamma \frac{\psi^*\omega_1}{d\psi^* f} = \oint_\gamma \frac{\omega_0}{df}.$$ □

The last condition $\psi_* = \mathrm{id}$ is satisfied if ψ is isotopic to the identity map. In the complex case, the converse statement is also true:

Proposition 2.4. *Let $\omega_0, \omega_1 \in \Omega^2(\mathbb{C}^2, 0)$ be two symplectic forms. Suppose that all complex periods coincide:*

$$\oint_\gamma \frac{\omega_0}{df} = \oint_\gamma \frac{\omega_1}{df}$$

then there exists a local holomorphic diffeomorphism ψ such that $\psi^* f = f$ and $\psi^*\omega_1 = \omega_0$.

Proof. By Proposition 2.2 we have $\omega_1 - \omega_0 = df \wedge d\varrho$, for some $\varrho \in \Omega^0(\mathbb{C}^2, 0)$. Now apply [26, Theorem 2.1]. □

We can give an alternative proof of the above statement based on the natural idea of extending a local symplectomorphism through Hamiltonian flows. For $i = 0, 1$, let $v_i := \omega_i^{-1} df$ and let $\alpha_i \in \Omega^1(\mathbb{C}^2, 0)$ be holomorphic 1-form such that $\omega_i = d\alpha_i$.

Proposition 2.5. *There exists an holomorphic diffeomorphism ψ in a neighborhood of zero such that $\psi^* f = f$ and $\psi^*\omega_1 = \omega_0$ if and only if for any homology cycle γ the* complex actions *$I_i(t) = \oint_\gamma \alpha_i$, $i = 0, 1$ of the two symplectic forms coincide, i.e., $I_0(t) \equiv I_1(t)$, for $t \in D^*$.*

Proof. Fix a section $N : D \hookrightarrow X$ as in the proof of Proposition 2.2. Let Φ_{v_i} denote the flow of the complex vector field v_i. Define the "time" function $\tau_i(x)$ by $\Phi_{v_i}^{\tau_i(x)}(N(f(x))) = x$. It measures the complex time it takes to the Hamiltonian flow of v_i to go from the section N to x. On $X \smallsetminus X_0$, the function $\tau_i(x)$ can be computed as:

$$\tau_i(x) := \int_{\gamma(x)} \frac{\omega_i}{df}, \quad i = 0, 1.$$

Were $\gamma(x)$ is a path joining $N(t)$ to x on the fiber X_t. The path $\gamma(x)$ (and hence the functions τ_0 and τ_1) is not well-defined, more exactly it is defined modulo homology cycles. Nevertheless the difference $\tau := \tau_1 - \tau_0$ is well defined because, for any cycle γ, we have by hypothesis:

$$\oint_\gamma \frac{\omega_1 - \omega_0}{df} = \frac{d}{dt} \oint_\gamma (\alpha_1 - \alpha_0) = 0.$$

Now define the map $\psi = \psi(x)$ for $x \in X \smallsetminus X_0$ as

$$\psi(x) := \Phi_{v_1}^{\tau_0(x)} \circ \Phi_{v_0}^{-\tau_0(x)}(x) = \Phi_{v_1}^{\tau_0(x) - \tau_1(x)}(x). \tag{2.3}$$

Being the composition of Hamiltonian flows, ψ is a fiber-preserving symplectomorphism on $X \smallsetminus X_0$. Notice that any such map must satisfy (2.3). Now we need to extend the map ψ to the singular fiber X_0. Put $\alpha = \alpha_1 - \alpha_0$ and $\omega = \omega_1 - \omega_0$, and define the function:

$$g(x) = \int_{\gamma(x)} \alpha.$$

Consider a point $x \in X_0 \smallsetminus \{0\}$, and let (t, u) be a local complex coordinate system at x, such that $f = t$. Let $\alpha = \alpha_1(u, t)du + \alpha_2(u, t)dt$ in these coordinates. In this neighborhood we can split $\tau(u,t) = \tau(0,t) + \int_0^u \omega/df$ and $g(u,t) = g(0,t) + \int_0^u \alpha$. It follows from Proposition 3.2 (taking $N' = \{u = 0\}$ and working in coordinates (u,t)), that $\tau(u,t) = \partial g(u,t)/\partial t + \alpha_2(u,t)$. As was shown in Proposition 2.2, $g(x)$ can be extended to a holomorphic function on X. This implies that τ can be extended to a holomorphic function on $X \setminus \{0\}$, and therefore (by the Second Riemann Extension Theorem) to the whole X. Now (2.3) defines a holomorphic fiber-preserving local symplectomorphism on $(X,0)$. □

The above results mean that two symplectic forms with the same *complex periods* are holomorphically equivalent, i.e., that complex periods contain all required information for symplectic equivalence. In the real case we do not see all complex vanishing cycles, but only the real relative cycles. Nevertheless periods over real relative cycles can be enough to determine all complex periods (in which case we speak of *good* singularities). In Chapter 3 we give sufficient conditions for this to happen.

2 Toric resolutions

We briefly recall the definition of toric resolutions, following [50, 4]. Toric resolutions provide a way to "unfold" (or blow-up) the singularity, and make it simpler. We need them to give a description of the real fibration of the singularity and to understand its relationship with the complex Milnor fiber of its complexification. This will be used in the next chapter, when we will discuss sufficient conditions for a singularity to be analytically good.

Consider a matrix $\sigma \in \mathrm{SL}(2,\mathbb{Z})$,

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Associate to σ the following birational morphism

$$\begin{aligned} \pi_\sigma : (\mathbb{C}^*)^2 &\to (\mathbb{C}^*)^2 \\ (x,y) &\mapsto (x^\alpha y^\beta, x^\gamma y^\delta) \end{aligned}$$

called the **monomial map** (associated to σ). If $\alpha, \gamma \geq 0$ (resp. $\beta, \delta \geq 0$) the map can be extended to $x = 0$ (resp. $y = 0$). Notice that for $\sigma, \tau \in \mathrm{SL}(n,\mathbb{Z})$ we have:

$$\begin{aligned} \pi_{\sigma\tau} &= \pi_\sigma \circ \pi_\tau \\ \pi_\sigma^{-1} &= \pi_{\sigma^{-1}} \end{aligned}$$

Now consider in $\mathbb{R}^2_+$ a set of $m+2$ integral vectors $\{P_0, \dots, P_{m+1}\}$, $P_i = (a_i, b_i)^t \in \mathbb{Z}^2_+$, satisfying:

- $P_0 = (1,0)^t$, $P_{m+1} = (0,1)^t$
- $\gcd(a_i, b_i) = 1$ for all $i = 0, \dots, m$
- $\det(P_i, P_{i+1}) > 0$ for all $i = 0, \dots, m$.

For each $i = 0, \dots, m$ consider the cone $\mathrm{Cone}(P_i, P_{i+1}) \subseteq \mathbb{R}^2_+$ spanned by P_i and P_{i+1}. The vectors $\{P_0, \dots, P_{m+1}\}$ define a cone subdivision of $\mathbb{R}^2_+$. If $\det(P_i, P_{i+1}) = 1$ for all $i = 0, \dots, m$ we say that the cone subdivision is **regular**.

Suppose $\{P_0, \ldots, P_{m+1}\}$ define a regular cone subdivision. Identify each cone $\mathrm{Cone}(P_i, P_{i+1})$ with the matrix $\sigma_i \in \mathrm{SL}(2, \mathbb{Z})$

$$\sigma_i = \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix}.$$

For each cone σ_i consider a copy $\mathbb{C}^2_{\sigma_i} \approx \mathbb{C}^2$ with coordinates (x_i, y_i) and the monomial map $\pi_{\sigma_i} : \mathbb{C}^2_{\sigma_i} \to \mathbb{C}^2$ associated to σ_i. Now consider the complex manifold given by

$$X = \bigcup_{i=0}^{m+1} \mathbb{C}^2_{\sigma_i} / \sim$$

where two points $(x_i, y_i) \in \mathbb{C}^2_{\sigma_i}$ and $(x_j, y_j) \in \mathbb{C}^2_{\sigma_j}$ are identified if and only if the birational map $\pi_{\sigma_j^{-1}\sigma_i}$ is defined at (x_i, y_i) and $\pi_{\sigma_j^{-1}\sigma_i}(x_i, y_i) = (x_j, y_j)$. The projections π_{σ_i}, $i = 0, \ldots, m+1$, glue together to a well-defined projection $\pi : X \to \mathbb{C}^2$, called the (complex) **toric blow-up** of $(\mathbb{C}^2, 0)$ associated with $\{P_0, \ldots, P_{m+1}\}$. We will denote by (x, y) the coordinates of the base $\mathbb{C}^2$. The toric blow-up satisfies the following properties:

- $\{\mathbb{C}^2_{\sigma_i}, (x_i, y_i)\}$, $i = 0, \ldots, m$, are coordinate charts for X.
- For each $i = 1, \ldots, m$, the subsets $\{y_{i-1} = 0\} \subset \mathbb{C}^2_{\sigma_{i-1}}$ and $\{x_i = 0\} \subset \mathbb{C}^2_{\sigma_i}$ glue together to a projective line $E_i \approx \mathbb{P}^1(\mathbb{C})$ called an **exceptional divisor**, and $\pi^{-1}(0) = \cup_i E_i$.
- $\pi : X \setminus \pi^{-1}(0) \to \mathbb{C}^2 \setminus \{0\}$ is an isomorphism.
- $E_i \cap E_j \neq \emptyset$ if and only if $j = i \pm 1$ and in this case they intersect transversely at a point. Moreover $E_i \cap \mathbb{C}_{\sigma_j} \neq \emptyset$ if and only if $j = i \pm 1$.

By considering the monomial maps π_{σ_i} as real birational maps defined on the real subspace $\mathbb{R}^2_{\sigma_i} \subset \mathbb{C}^2_{\sigma_i} \to \mathbb{R}^2$ we obtain the **real part** $X^{\mathbb{R}} \subset X$ of the toric blow-up, again endowed with a projection $\pi_{\mathbb{R}} : X^{\mathbb{R}} \to \mathbb{R}^2$. Equivalently, the real part is the subset defined in each chart $\mathbb{C}^2_{\sigma_i}$ by requiring the coordinates (x_i, y_i) to be real-valued. Now $\pi_{\mathbb{R}}^{-1}(0)$ consists of real exceptional divisors which are real projective lines $E_i^{\mathbb{R}} \approx \mathbb{P}^1(\mathbb{R})$, the real part of the E_i's.

2.1 Algorithm to complete a fan to a regular fan

Suppose we are given a cone subdivision $\{Q_0, \ldots, Q_{k+1}\}$, not necessarily regular. We will need to complete this subdivision to a regular one, by adding primitive integer vectors. This can be done step-by-step as follows.

Consider two integer covectors $a, b \in \mathbb{Z}^2_+$, $a = (a_1, a_2)^t$ and $b = (b_1, b_2)^t$, with $\gcd(a_1, a_2) = \gcd(b_1, b_2) = 1$ and $\det(a, b) > 0$. We look for a new integer covector x such that:

$$\begin{cases} \det(a, x) = 1 \\ \det(x, b) \text{ is positive and minimal.} \end{cases}$$

To find x, apply the following procedure:

1. If $\det(a, b) = 1$, then put $x = b$.
2. Using the Euclid algorithm find $s = (s_1, s_2) \in \mathbb{Z}^2$ such that $a_1 s_2 - a_2 s_1 = 1$, in other words $\det(a, s) = 1$.

3. The integer vectors x such that $\det(a, x) = 1$ are the ones which belong to the family $k \mapsto s + ka$. We have $\det(a, s + ka) = 1$, for all k.

   There are two possibilities.

   – Case 1: there exists $k \in \mathbb{Z}$ such that $\det(s + ka, b) = 0$, then $s + ka$ is parallel to b. Put $s + ka = \lambda b$. But then the condition $1 = \det(a, s + ka) = \lambda \det(a, b)$ implies that $\lambda = 1$ and $\det(a, b) = 1$. Put $x = b$.
   – Case 2: such a k does not exist. Therefore the number $\ell := -\det(s, b)\det(a, b)^{-1}$ is not an integer. Let $\overline{k}$ be the (unique) integer in the open interval $(\ell, \ell + 1)$. Put $x = s + \overline{k}a$, then

$$\begin{aligned}\det(x, b) &= \det(s, b) + \overline{k}\det(a, b) > \det(s, b) + \ell\det(a, b) = 0\\ \det(x, b) &< \det(s, b) + (\ell + 1)\det(a, b) = \det(a, b).\end{aligned} \tag{2.4}$$

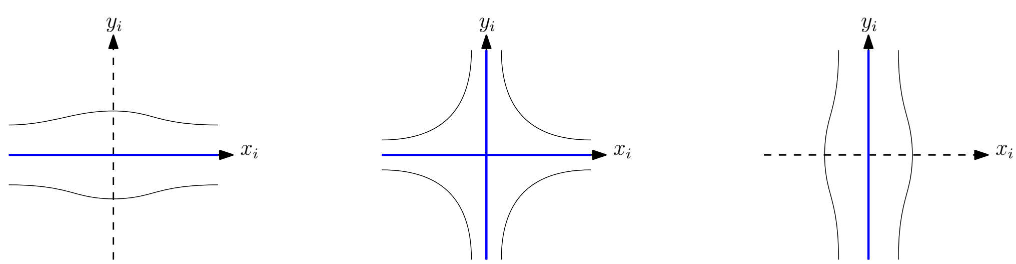
After repeating this procedure several times to the fan, we will eventually obtain a regular fan. Equation (2.4) shows that at each step the determinant is strictly decreasing. Since the determinants are all integer-valued, this implies that the algorithm will eventually stop.

2.2 Topology of the real part

We want to understand what a small neighborhood of $0 \in \mathbb{R}^2$ looks like in $\mathbb{R}^2_{\sigma_i}$. Consider a rectangular neighborhood of $0 \in \mathbb{R}^2$ of the form $U = [-\delta, \delta]^2$, then $\pi_{\sigma_i}(x_i, y_i) \in U$ if and only if

$$\begin{cases}|x|^{a_i}|y|^{a_{i+1}} \leq \delta\\ |x|^{b_i}|y|^{b_{i+1}} \leq \delta\end{cases}$$

This means we have the three possible cases represented in Figure 2.2.



(a) $P_i = (1, 0)$, P_{i+1} positive (b) P_i, P_{i+1} positive (c) P_i positive, $P_{i+1} = (0, 1)$

Figure 2.2

It remains to explain how these local pictures are glued together. To reconstruct the topology of the real part of the resolution, we need to understand how different charts $\{\mathbb{R}^2_{\sigma_i}\}$ are glued together, i.e., transition functions:

Lemma 2.2.

$$\pi^{-1}_{\sigma_j} \circ \pi_{\sigma_i}\begin{pmatrix}x_i\\ y_i\end{pmatrix} = \begin{pmatrix}x_i^{\det(P_i, P_{j+1})} y_i^{\det(P_{i+1}, P_{j+1})}\\ x_i^{-\det(P_i, P_j)} y_i^{-\det(P_{i+1}, P_j)}\end{pmatrix}$$

Proof.

$$\sigma_j^{-1}\sigma_i = \begin{pmatrix} a_j & a_{j+1} \\ b_j & b_{j+1} \end{pmatrix}^{-1} \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix} = \begin{pmatrix} b_{j+1} & -a_{j+1} \\ -b_j & a_j \end{pmatrix} \begin{pmatrix} a_i & a_{i+1} \\ b_i & b_{i+1} \end{pmatrix}$$
$$= \begin{pmatrix} \det(P_i, P_{j+1}) & \det(P_{i+1}, P_{j+1}) \\ -\det(P_i, P_j) & -\det(P_{i+1}, P_j) \end{pmatrix}. \qquad \square$$

If we consider just a neighborhood of the form $Y_U = \pi^{-1}(U)$, then the topology of the gluing is simpler. It follows from the properties of the toric blow-up that two charts corresponding to non-consecutive cones are identified away from the axes. For the case $j = i + 1$, of two consecutive cones, the monomial transition map is

$$\pi_{\sigma_{i+1}}^{-1} \circ \pi_{\sigma_i} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} x_i^{\det(P_i, P_{i+2})} y_i \\ 1/x_i \end{pmatrix} \tag{2.5}$$

Consider the oriented segments $(1, t)$ and $(-1, t)$, for $|t|$ small, in $\mathbb{R}^2_{\sigma_i}$. They are mapped to $(t, 1)$ and $((-1)^{\det(P_i, P_{i+2})}t, -1)$ respectively in $\mathbb{R}^2_{\sigma_{i+1}}$. Notice that $\det(P_i, P_{i+2}) > 0$, and so the topological behavior of the gluing map (2.5) depends only on one thing: if $\det(P_i, P_{i+2})$ is even or odd. The resulting picture, in the case of $\det(P_i, P_{i+2})$ odd, is shown in Figure 2.3.

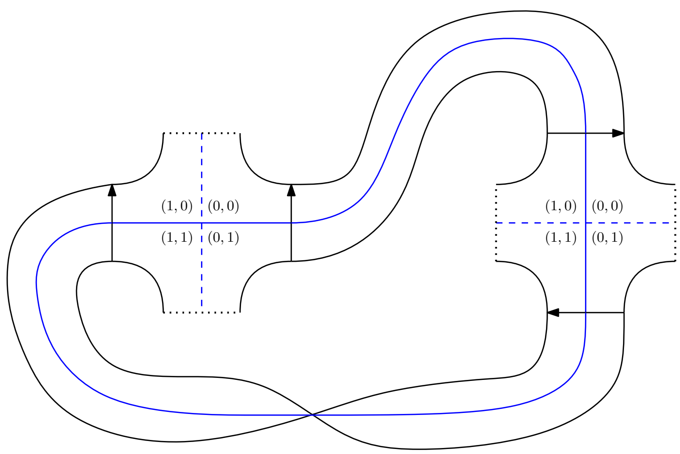


Figure 2.3: Gluing of consecutive charts

The final manifold, obtained gluing together several crosses, is a 2-dimensional surface Y_U called a **Möbius necklace** (borrowing the expression from [30]). See Figure 2.5 below for an example.

Proposition 2.6. *The real surface Y_U is not orientable.*

Lemma 2.3. *For any triple of plane vectors (a, b, c) the following formula holds:*

$$\det(a, b)c + \det(b, c)a = \det(a, c)b.$$

For each quadruple of plane vectors (a, b, c, d) we have the ***Plücker relations****:*

$$\det(a, b)\det(c, d) + \det(b, c)\det(a, d) = \det(a, c)\det(b, d).$$

Proof. Easy computation. The second equation follows by applying $\det(\cdot, d)$ to the first one. □

Proof of Proposition 2.6. Consider a sequence $\{P_0, P_1, \ldots, P_{m+1}\}$ of covectors in a regular fan. Recall that this means

- $\det(P_i, P_{i+1}) = 1$ for $i = 0, \ldots, m$,
- if $i < j$ then $\det(P_i, P_j) > 0$, since covectors are in counter-clockwise order.

The manifold will be orientable if and only if all transition maps for successive charts have no Möbius twist. In other words, if and only if $\det(P_i, P_{i+2})$ is always even. Assume by absurd that it is true. Then we show that the sequence $i \mapsto \det(P_0, P_i)$ is non-decreasing, i.e.,

$$\det(P_0, P_i) \leq \det(P_0, P_{i+1}). \tag{2.6}$$

First induction cases: for $i = 0$ we have $\det(P_0, P_0) = 0 \leq 1 = \det(P_0, P_1)$, and for $i = 1$ we have $\det(P_0, P_1) = 1 \leq 2 \leq \det(P_0, P_2)$ because $\det(P_0, P_2)$ is even and non-zero by hypothesis.

Now suppose equation (2.6) holds up to some $i \geq 1$, then the Plücker relation with $(a, b, c, d) = (P_0, P_i, P_{i+1}, P_{i+2})$, together with the induction step, gives

$$\begin{aligned}
\det(P_0, P_{i+2}) &= \det(P_0, P_{i+1}) \det(P_i, P_{i+2}) - \det(P_0, P_i) \\
&\geq 2 \det(P_0, P_{i+1}) - \det(P_0, P_{i+1}) \\
&= \det(P_0, P_{i+1}).
\end{aligned}$$

Finally, equation (2.6) gives the following contradiction

$$2 \leq \det(P_0, P_2) \leq \cdots \leq \det(P_0, P_m) \leq \det(P_0, P_{m+1}) = 1. \quad \square$$

2.3 Toric resolution of a singularity

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}, 0)$ be an analytic function germ with Taylor series $f(x, y) = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha y^\beta$. Its **Newton polygon** is the subset of $\mathbb{R}^2_+$ given by

$$\Gamma_+(f) = \text{the convex hull of the set } \bigcup_{a_{\alpha\beta} \neq 0} \{(\alpha, \beta) + \mathbb{R}^2_+\}.$$

The **Newton diagram** $\Gamma(f)$ is the union of the compact (0 and 1-dimensional) faces belonging to the boundary of $\Gamma_+(f)$. For each compact face $\gamma \subset \Gamma(f)$ denote by f_γ the quasi-homogeneous polynomial $f_\gamma(x, y) = \sum_{(\alpha,\beta)\in\gamma} a_{\alpha\beta} x^\alpha y^\beta$.

Definition 2.6. The germ f is **Γ-non-degenerate** over $\mathbb{K}$ if for every face $\gamma \subset \Gamma(f)$, the restriction $f_\gamma : (\mathbb{K}^*)^2 \to \mathbb{K}$ has no critical points.

The **supporting function** of the Newton polygon is defined, for $Q \in \mathbb{Z}^2_+$, by

$$\ell(Q) := \min_{\mathbf{x}\in\Gamma(f)} \langle Q, \mathbf{x} \rangle$$

and the **tangent line** to the Newton polygon associated to $Q \in \mathbb{Z}_+^2$ is the line of points achieving this minimum $t(Q) := \{\mathbf{x} \in \mathbb{R}^2 : \langle Q, \mathbf{x}\rangle = \ell(Q)\}$. The **trace** of $Q \in \mathbb{Z}_+^2$ is defined as $t_\Gamma(Q) = t(Q) \cap \Gamma(f)$.

Let $\{\gamma_1, \ldots, \gamma_k\}$ be the compact 1-dimensional faces of $\Gamma(f)$. For each γ_i let Q_i be the unique primitive integral vector such that $t_\Gamma(Q_i) = \gamma_i$. We assume $\gamma_1, \ldots, \gamma_k$ are ordered in such a way that $\det(Q_i, Q_{i+1}) > 0$, i.e., the vectors $Q_1, \ldots, Q_k$ are in counter-clockwise order. Consider the cone subdivision defined by the vectors $\{Q_0, \ldots, Q_{k+1}\}$ where $Q_0 = (1,0)^t$ and $Q_{k+1} = (0,1)^t$. This can be completed to a regular cone subdivision by adding in each cone $\mathrm{Cone}(Q_i, Q_{i+1})$ new primitive positive integer vectors $T_{i,1}, \ldots, T_{i,r_i}$ between Q_i and Q_{i+1} in counter-clockwise order (as explained in Subsection 2.1). Denote by $\{P_0, \ldots, P_{m+1}\}$ the resulting regular cone subdivision, and consider its associated toric blow-up $\pi : (X, X_\mathbb{R}, 0) \to (\mathbb{C}^2, \mathbb{R}^2, 0)$.

The next lemma shows that this blow-up provides a *resolution* of the singularity f, meaning that the pulled back function only has normal crossing singularities. The properties of the resolved functions, both in the real and complex blow-up, can be deduced from the following:

Lemma 2.4. *Consider the cone* $\sigma_i = \mathrm{Cone}(P_i, P_{i+1})$

i) $f \circ \pi_{\sigma_i}(x_i, y_i) = x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} \widetilde{f}_i(x_i, y_i)$ *with* $\widetilde{f}_i$ *not divisible by* x_i *nor* y_i.

ii) If $t_\Gamma(P_i) = t_\Gamma(P_{i+1}) = \{(\alpha, \beta)\}$ *then* $\widetilde{f}_i(x_i, y_i) = a_{\alpha\beta} + (x_i y_i)$.

iii) If $t_\Gamma(P_i) = \gamma_i$ *is a face and* $t_\Gamma(P_{i+1}) = \{(\alpha, \beta)\} \in \gamma_i$ *then*

$$\widetilde{f}_i(x_i, y_i) = \sum_{\gamma_i} a_\mathbf{k} y_i^{\langle P_{i+1}, \mathbf{k}\rangle - \ell(P_{i+1})} + (x_i y_i)$$

iv) If $t_\Gamma(P_{i+1}) = \gamma_{i+1}$ *is a face and* $t_\Gamma(P_i) = \{(\alpha, \beta)\} \in \gamma_{i+1}$ *then*

$$\widetilde{f}_i(x_i, y_i) = \sum_{\gamma_{i+1}} a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle - \ell(P_i)} + (x_i y_i)$$

v) If $t_\Gamma(P_i) = \gamma_i$ *and* $t_\Gamma(P_{i+1}) = \gamma_{i+1}$ *are both faces then*

$$\widetilde{f}_i(x_i, y_i) = \sum_{\gamma_i} a_\mathbf{k} y_i^{\langle P_{i+1}, \mathbf{k}\rangle - \ell(P_{i+1})} + \sum_{\gamma_{i+1}} a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle - \ell(P_i)} - \sum_{\gamma_i \cap \gamma_{i+1}} a_\mathbf{k} + (x_i y_i).$$

Proof. If $f(x, y) = \sum_{\alpha,\beta} a_{\alpha\beta} x^\alpha y^\beta$, then

$$f \circ \pi_{\sigma_i}(x_i, y_i) = \sum_{\alpha,\beta} a_{\alpha\beta} x_i^{a_i\alpha} y_i^{a_{i+1}\alpha} x_i^{b_i\beta} y_i^{b_{i+1}\beta} = \sum_{\alpha,\beta} a_{\alpha\beta} x_i^{\langle P_i, (\alpha,\beta)\rangle} y_i^{\langle P_{i+1}, (\alpha,\beta)\rangle}.$$

Let $\mathbf{k} = (\alpha, \beta)$ the summation index, then we can split the sum as follows

$$\sum_\Gamma a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle} y_i^{\langle P_{i+1}, \mathbf{k}\rangle} = \sum_{t_\Gamma(P_i) \cup t_\Gamma(P_{i+1})} a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle} y_i^{\langle P_{i+1}, \mathbf{k}\rangle} + x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} (x_i y_i)$$

$$= x_i^{\ell(P_i)} \sum_{t_\Gamma(P_i)} a_\mathbf{k} y_i^{\langle P_{i+1}, \mathbf{k}\rangle} + y_i^{\ell(P_{i+1})} \sum_{t_\Gamma(P_{i+1})} a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle} - x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} \sum_{t_\Gamma(P_i) \cap t_\Gamma(P_{i+1})} a_\mathbf{k} + x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} (x_i y_i)$$

$$= x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} \left(\sum_{t_\Gamma(P_i)} a_\mathbf{k} y_i^{\langle P_{i+1}, \mathbf{k}\rangle - \ell(P_{i+1})} + \sum_{t_\Gamma(P_{i+1})} a_\mathbf{k} x_i^{\langle P_i, \mathbf{k}\rangle - \ell(P_i)} - \sum_{t_\Gamma(P_i) \cap t_\Gamma(P_{i+1})} a_\mathbf{k} + (x_i y_i) \right)$$

All the assertions follow from this decomposition. □

Definition 2.7. Consider the divisor E_i for $i = 1, \ldots, m$ defined by $\{y_{i-1} = 0\}$ on $\mathbb{C}^2_{\sigma_{i-1}}$ and by $\{x_i = 0\}$ on $\mathbb{C}^2_{\sigma_i}$. Then we define the **weight** of the divisor E_i as $\ell_i := \ell(P_i)$.

Next we need to describe the proper preimage of $f^{-1}(0)$.

Lemma 2.5. *If f is Γ-non-degenerate over $\mathbb{C}$, then the proper preimage of $f^{-1}(0)$ in X intersects the exceptional divisors transversely (idem for the real exceptional divisors in $X^{\mathbb{R}}$, if f is real and Γ-non-degenerate over $\mathbb{R}$).*

Proof. Indeed consider the affine part $\{y_i = 0\} \subset E_{i+1}$ of the exceptional divisor, and let $(x_i, 0) \in E_{i+1}$ be such that $\widetilde{f}_i(x_i, 0) = 0$, then $\{\widetilde{f}_i = 0\}$ intersects E_{i+1} transversely at $(x_i, 0)$ if and only if $\partial \widetilde{f}_i / \partial x_i(x_i, 0) \neq 0$. It follows from point (v) above that for $\gamma = t_\Gamma(P_{i+1})$,

$$f_\gamma(x_i^{a_i} y_i^{a_{i+1}}, x_i^{b_i} y_i^{b_{i+1}}) = x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} \widetilde{f}_i(x_i, 0).$$

Suppose $\widetilde{f}_i(\overline{x}_i, 0) = \partial \widetilde{f}_i / \partial x_i(\overline{x}_i, 0) = 0$. Taking $x_i \partial / \partial x_i$ and $y_i \partial / \partial y_i$ in the equation above, we find $y_i \neq 0$ such that:

$$\begin{cases} a_i(\overline{x}_i^{a_i} y_i^{a_{i+1}}) \cdot \partial f_\gamma / \partial x(\pi_{\sigma_i}(\overline{x}_i, y_i)) + b_i(\overline{x}_i^{b_i} y_i^{b_{i+1}}) \cdot \partial f_\gamma / \partial x(\pi_{\sigma_i}(\overline{x}_i, y_i)) = 0 \\ a_{i+1}(\overline{x}_i^{a_i} y_i^{a_{i+1}}) \cdot \partial f_\gamma / \partial x(\pi_{\sigma_i}(\overline{x}_i, y_i)) + b_{i+1}(\overline{x}_i^{b_i} y_i^{b_{i+1}}) \cdot \partial f_\gamma / \partial x(\pi_{\sigma_i}(\overline{x}_i, y_i)) = 0 \end{cases}$$

Composition with $\pi_\sigma^{-1} : (\mathbb{R}^*)^2 \to (\mathbb{R}^*)^2$ gives

$$\begin{cases} a_i x \cdot \partial f_\gamma / \partial x(x, y) + b_i y \cdot \partial f_\gamma / \partial x(x, y) = 0 \\ a_{i+1} x \cdot \partial f_\gamma / \partial x(x, y) + b_{i+1} y \cdot \partial f_\gamma / \partial x(x, y) = 0 \end{cases}$$

where $(x, y) = \pi_{\sigma_i}^{-1}(\overline{x}_i, y_i) \in (\mathbb{R}^*)^2$. Since $a_i b_{i+1} - a_{i+1} b_i \neq 0$ this implies $\nabla f_\gamma = 0$ at (x, y), contradicting the Γ-non-degeneracy. □

This means we are reduced to studying the restriction of $\widetilde{f}_i$ to E_{i+1}. We have

$$\widetilde{f}_i(x_i, 0) = \sum_{\mathbf{k} \in t_\Gamma(P_{i+1})} a_{\mathbf{k}} x^{\langle P_i, \mathbf{k} \rangle - \ell(P_i)}. \tag{2.7}$$

For $i = 0, \ldots, m$ let $\mathbf{k}_i = t_\Gamma(P_i) \cap t_\Gamma(P_{i+1})$. Then we can rewrite

$$\widetilde{f}_i(x_i, 0) = \sum_{\mathbf{k} \in t_\Gamma(P_{i+1})} a_{\mathbf{k}} x^{\langle P_i, \mathbf{k} - \mathbf{k}_i \rangle}.$$

For $\mathbf{k} = (\alpha, \beta)^t$ denote $\overline{\mathbf{k}} = (\beta, -\alpha)^t$, so that $\langle y, \overline{\mathbf{k}} \rangle = \det(y, \mathbf{k})$. All the integer points $\mathbf{k} \in t_\Gamma(P_{i+1})$ are of the form $\mathbf{k} = \mathbf{k}_i + j\overline{P}_{i+1}$, for $j = 0, 1, 2, \ldots$. Notice that $\overline{P}_{i+1}$ is the smallest integer vector in its direction, otherwise its coordinates would not be coprime. Since $\langle P_i, \mathbf{k} - \mathbf{k}_i \rangle = \langle P_i, j\overline{P}_{i+1} \rangle = j \det(P_i, P_{i+1}) = j$, we can write

$$\widetilde{f}_i(x_i, 0) = \sum_{j=0}^{\lambda(P_{i+1})} \widetilde{a}_j(P_{i+1}) x^j, \quad \text{with } \widetilde{a}_j(P_{i+1}) := a_{\mathbf{k}_i + j\overline{P}_{i+1}},$$

where $\lambda(P_{i+1})$ is the **integer length** of $t_\Gamma(P_{i+1})$, i.e. the number of integer points in $t_\Gamma(P_{i+1})$:

$$\begin{aligned}\lambda(P_{i+1}) &= \max\{j \in \mathbb{N} : \mathbf{k}_i + j\overline{P}_{i+1} \in t_\Gamma(P_{i+1})\}\\ &= \max\{j \in \mathbb{N} : \widetilde{a}_j(P_{i+1}) \neq 0\}\\ &= \langle P_i, \mathbf{k}_{i+1} - \mathbf{k}_i\rangle\\ &= \langle P_{i+2}, \mathbf{k}_i - \mathbf{k}_{i+1}\rangle.\end{aligned}$$

Lemma 2.6. *We have*

i) $\partial\widetilde{f}_i(x_i, 0) = \det(P_i, P_{i+2})\ell(P_{i+1}) - (\ell(P_{i+2}) + \ell(P_i))$

ii) $\gcd(\ell(P_i) + \partial\widetilde{f}_i(x_i, 0), \ell(P_{i+1})) = \gcd(\ell(P_{i+1}), \ell(P_{i+2}))$.

Proof. Applying the Plücker relations to the quadruple $(P_i, P_{i+1}, P_{i+2}, \overline{\mathbf{k}}_i)$ we get

$$\begin{aligned}\langle P_i, \mathbf{k}_i\rangle + \langle P_{i+2}, \mathbf{k}_i\rangle &= -\big(\det(P_i, \overline{\mathbf{k}}_i) + \det(P_{i+2}, \overline{\mathbf{k}}_i)\big)\\ &= -\det(P_i, P_{i+2})\det(P_{i+1}, \overline{\mathbf{k}}_i)\\ &= \det(P_i, P_{i+2})\langle P_{i+1}, \mathbf{k}_i\rangle\\ &= \det(P_i, P_{i+2})\ell(P_{i+1})\end{aligned}$$

We have $\partial\widetilde{f}_i(x_i, 0) = \lambda(P_{i+1}) = \langle P_{i+2}, \mathbf{k}_i - \mathbf{k}_{i+1}\rangle = \langle P_{i+2}, \mathbf{k}_i\rangle - \ell(P_{i+2})$, therefore

$$\begin{aligned}\ell(P_i) + \partial\widetilde{f}_i(x_i, 0) &= \langle P_i, \mathbf{k}_i\rangle + \langle P_{i+2}, \mathbf{k}_i\rangle - \ell(P_{i+2})\\ &= \det(P_i, P_{i+2})\ell(P_{i+1}) - \ell(P_{i+2}).\end{aligned}$$

This implies (i), for (ii) take $\gcd(\cdot, \ell(P_{i+1}))$ on both sides. □

2.4 Sign layouts and real relative cycles

For the rest of the chapter we will only work with the real part of the toric resolution, which we denote by $\pi : X \to \mathbb{R}^2$, hence dropping the suffix. For the same reason, we will denote the real part of the exceptional divisors by E_i. We want to give a description of the real relative cycles inside the toric resolution.

Consider the real toric resolution $\pi : Y \subset X \to \mathbb{R}^2$, where Y denotes the Möbius necklace associated to f. On the surface Y we can distinguish the connected regions of the set $Y \smallsetminus (f \circ \pi)^{-1}(0)$ and label them according to the sign of $f \circ \pi$. In the cone chart $\mathbb{R}^2_{\sigma_i}$, consider a neighborhood V_i of $\{y_i = 0\} \subset E_{i+1}$ in Y. The sign-layout of $f \circ \pi$ in this neighborhood, i.e., the decomposition of Y into positive and negative regions, is determined by:

- the sign of $\widetilde{f}_i(0, 0)$;
- the weights $\ell(P_i)$, $\ell(P_{i+1})$ (for the sign-layout at $(x_i, y_i) = (0, 0)$);
- the number of positive and negative zeros of $\widetilde{f}_i(x_i, 0)$.

This is sufficient because, due to the assumption of Γ-non-degeneracy, the function $\widetilde{f}_i$ changes sign whenever crossing a zero-level line $\subset \{\widetilde{f}_i = 0\} \cap V_i$ (Lemma 2.5).

Remark 2.3. If f is polynomial, the number of positive and negative zeros of $\widetilde{f}(x_i, 0)$ can be computed by the Sturm theorem.

Knowing the sign-layout of $f \circ \pi$ and the structure of the Möbius necklace it is possible to describe the "chord diagram" of the original singularity $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ (i.e., the order of the intersections of $S^1_\delta = \partial\{(x,y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq \delta^2\}$ with the branches of $\{f = 0\}$) and to identify the connected components of the positive/negative level-sets of f (i.e., the real relative cycles). To do this we need to "follow" a connected level line along the different cone charts. The preimage of S^1_δ is clearly connected on Y, and the strict preimage of $f^{-1}(0)$ will intersect it transversely as well. If f is divisible by x (resp. y) then we will find the preimage of $x = 0$ (resp. $y = 0$) only in the first (resp. last) cone chart.

For $i = 0, \dots, m-1$, consider the real divisor E_{i+1} defined by the equations $\{y_i = 0\}$ on $\mathbb{R}^2_{\sigma_i}$ and by $\{x_{i+1} = 0\}$ on $\mathbb{R}^2_{\sigma_{i+1}}$. Let $r^\pm_{i+1}$ denote the number of positive (resp. negative) roots of $\widetilde{f}_i(x_i, 0)$, and $r_i = r^+_i + r^-_i$. Let $Z^\pm_i = (z^\pm_{i,1}, \dots, z^\pm_{i,r^\pm_i})$ denote the oriented sequences of positive (resp. negative) zeros of $\widetilde{f}_i(x_i, 0)$, i.e., the intersections of the proper preimage of $f^{-1}(0)$ with the divisor E_{i+1}, oriented from $x_i = -\infty$ to $x_i = \infty$.

Consider the following graph D: for each real divisor E_{i+1} consider the points $x_i = 0$ and $x_i = \infty$ as vertices. Orient the divisor E_{i+1} from $x_i = -\infty$ to $x_i = \infty$, and consider the segments $e^-_{i+1} := (-\infty, 0]$ and $e^+_{i+1} := [0, \infty)$ as oriented edges connecting the vertices $x_i = 0$ and $x_i = \infty$ of the graph D.

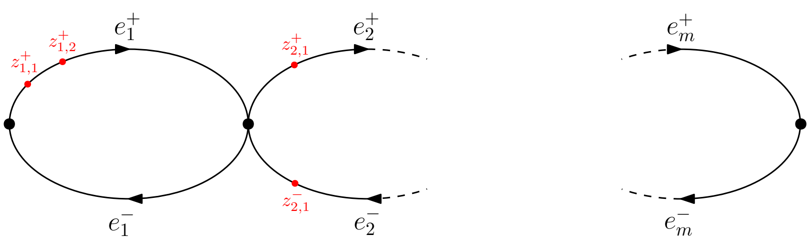


Figure 2.4: The graph D

The preimage $\pi^{-1}(S^1_\delta)$ is still a connected boundary circle. Consider the closed path going along this circle, starting from the positive quadrant of the first chart $\mathbb{R}^2_{\sigma_0}$. After contracting the Möbius necklace to $\pi^{-1}(0) = \cup_{i=1}^m E_i$, this path becomes a loop on the graph D (not necessarily following the orientation of the edges). It is clear that the chord diagram of f is determined by this path, more precisely: by the order of the zeros $z^\pm_{i,j}$ encountered along this path.

On $\mathbb{R}^2_{\sigma_i}$, $i = 0, \dots, m$, label the quadrants in counter-clockwise order, starting from the positive one, with $(0,0), (1,0), (1,1), (0,1) \in \mathbb{Z}^2_2$, as in Figure 2.3. Denote by $Q_i \simeq \mathbb{Z}^2_2$ the set of quadrants of $\mathbb{R}^2_{\sigma_i}$. An element $q \in Q_i$ will denote, depending on the context, both an element of $\mathbb{Z}^2_2$ and the corresponding quadrant of $\mathbb{R}^2_{\sigma_i}$.

The gluing rules of Section 2.2, can be described as follows: the quadrant $y_i \in \mathbb{Z}^2_2$ of $\mathbb{R}^2_{\sigma_i}$ is identified with a quadrant $\varphi(y_i)$ of $\mathbb{R}^2_{\sigma_{i+1}}$ according to the linear bijection $\varphi : \mathbb{Z}^2_2 \to \mathbb{Z}^2_2$ defined by:

$$y_i \mapsto \varphi(y_i) = \begin{pmatrix} s_i & 1 \\ 1 & 0 \end{pmatrix} y_i, \quad s_i := \det(P_i, P_{i+2}) \mod 2.$$

In the opposite direction

$$\varphi^{-1}(y_{i+1}) = \begin{pmatrix} 0 & 1 \\ 1 & s_i \end{pmatrix} y_{i+1}.$$

To understand the path along the boundary, we must apply φ and its inverse repeatedly. We assume for simplicity that the Newton polygon of f intersects both axes, so that $\ell(P_0) = \ell(P_{m+1}) = 0$. The description of real relative cycles, and of the chord diagram of the singularity, is contained in the following proposition.

Proposition 2.7. *Suppose $\Gamma(f)$ intersects both the x-axis and the y-axis. The curve $\{f = 0\} \subset \mathbb{R}^2$ is composed of $r = \sum_1^m r_i$ branches, whose chord diagram is determined by the following path along the graph D:*

$$e_1^+ \cdots e_m^+ \, e_m^{\alpha_m''} \cdots e_1^{\alpha_1''} \, e_1^{\beta_0'} \cdots e_m^{\beta_{m-1}'} \, e_m^{\gamma_m''} \cdots e_1^{\gamma_1''},$$

where

$$\alpha_j = \varphi^j(b), \quad \beta_j = \varphi^j(b + (1,0)), \quad \gamma_j = \varphi^j(1,0),$$

$b = \varphi^{-m}(0,1)$, and $z' = (-1)^{z_1}$, $z'' = (-1)^{z_2}$ for $z = (z_1, z_2) \in \mathbb{Z}_2^2$.

Proof. Consider the path starting from the $(0,0)$-quadrant of $\mathbb{R}^2_{\sigma_0}$ and going around the boundary. This path goes through all the $(0,0)$-quadrants from the first to the last chart. When the last chart $\mathbb{R}^2_{\sigma_m}$ is reached, the path continues to $\alpha_m = (0,1)$ of $\mathbb{R}^2_{\sigma_m}$, and then moves back through the quadrants $\alpha_m, \alpha_{m-1}, \ldots$ up to $\alpha_0 = b = \varphi^{-m}(0,1)$ in $\mathbb{R}^2_{\sigma_0}$, which is either $b = (1,1)$ or $b = (0,1)$, otherwise the loop will become closed.

In either case the path then continues to $\beta_0 = b + (1,0) \in Q_0$ in $\mathbb{R}^2_{\sigma_0}$. Denote by $\beta_0, \ldots, \beta_m$ the second sequence of quadrants in the positive direction, and by $\gamma_m, \ldots, \gamma_0$ the final path going back to $\mathbb{R}^2_{\sigma_0}$. This means that $\alpha_0 = b$, $\beta_0 = b + (1,0)$ and $\gamma_0 = (1,0)$, and the subsequent terms can be obtained applying φ multiple times. It is then easy to see that the sign of the edge $e_i^{\pm}$ is determined by the first coordinate of $\mathbb{Z}_2^2$ when going in the positive direction (i.e., for β_i), and by the second coordinate when going in the negative direction (α_i and γ_i). □

Example 2.1. We consider a simple example: $f(x,y) = y^2 - x^4$. A regular fan is given by: $\{(1,0), (1,1), (1,2), (0,1)\}$, consequently we have:

$$\begin{cases} f \circ \pi_{\sigma_0}(x_0, y_0) & = f(x_0 y_0, y_0) & = y_0^2(1 - x_0^4 y_0^2) \\ f \circ \pi_{\sigma_1}(x_1, y_1) & = f(x_1 y_1, x_1 y_1^2) & = x_1^2 y_1^4 (1 - x_1^2) \\ f \circ \pi_{\sigma_2}(x_2, y_2) & = f(x_2, x_2^2 y_2) & = x_2^4 (y_2^2 - 1) \end{cases}$$

There are two exceptional divisors E_1, E_2 with weight $\ell_1 = \ell(1,1) = 2$ and $\ell_2 = \ell(1,2) = 4$, respectively. The real neighborhood of E_1 is orientable, and for E_2 is not. The topology of the Möbius necklace associated to f is represented in Figure 2.5. The exceptional divisors are represented by blue circles.

The proper preimage of $f^{-1}(0)$ intersects only the second divisor, in two real points ($x_1 = \pm 1$). By applying φ and its inverse several times (as in the above proposition) we find the

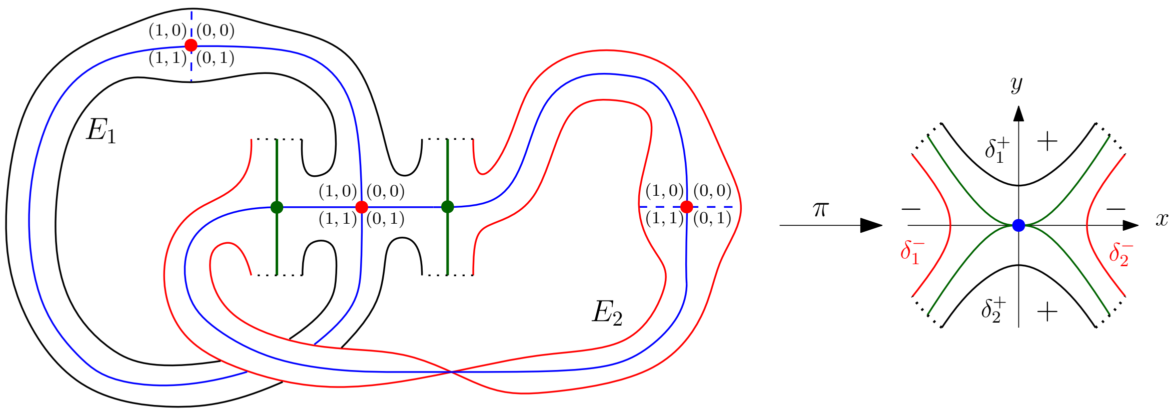


Figure 2.5: Möbius necklace for $f = y^2 - x^4$.

following sequence of quadrants:

$$
\begin{array}{ccccc}
(0,0) & \longrightarrow & (0,0) & \longrightarrow & (0,0) \\
 & & & & \downarrow \\
(1,1) & \longleftarrow & (1,1) & \longleftarrow & (0,1) \\
\downarrow & & & & \\
(0,1) & \longrightarrow & (1,0) & \longrightarrow & (1,1) \\
 & & & & \downarrow \\
(1,0) & \longleftarrow & (0,1) & \longleftarrow & (1,0)
\end{array}
$$

This corresponds to the path $e_1^+ \, e_2^+ \, e_2^- \, e_1^- \, e_1^+ \, e_2^- \, e_2^+ \, e_1^-$ on D. There are 4 real relative cycles visible from the figure, two positive approaching only E_2 (in red) and two negative approaching both E_1 and E_2 (in black).

Chapter 3

Good singularities

1 Topologically good singularities

Now we want to characterize good singularities in a topological way. First we need to introduce a few preliminary notions. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be an algebraically isolated real singularity. We still denote by f its complexification $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. Let $f : X \to D$ the associated Milnor fibration, and $M : H_1(X_\eta, \mathbb{C}) \to H_1(X_\eta, \mathbb{C})$ the monodromy operator acting on the fiber $f = \eta > 0$.

For each $t \in D$ consider the collar $C_t = X_t \cap (B_\epsilon \setminus B_{\epsilon'})$ with $0 < \epsilon' < \epsilon$. This defines a smooth trivial fibration $C := \cup_t C_t \to D$, let $\varphi : C \xrightarrow{\sim} C_0 \times D$ be a trivialization. Consider a loop $\gamma = \gamma(s) : [0, 1] \to D^*$, with $\gamma(0) = \gamma(1) = \eta$ making a counter-clockwise loop around 0, and let $\Gamma_s : X_{\gamma(0)} \to X_{\gamma(s)}$ be the continuous family of maps identifying the fibers along the loop defined by the local triviality of the Milnor bundle. These maps can be chosen to be compatible with the trivialization of C, meaning that

$$\varphi \circ \Gamma_s \circ \varphi^{-1}(x, t_0) = (x, \gamma(s)), \quad x \in C_0,$$

(see for example [59]). This implies that if we consider a cycle $\delta \in H_1(X_\eta, X_\eta \cap C_\eta)$, then the composition $\Gamma_1 \delta - \delta$ defines a class $\mathrm{Var}\ \delta \in H_1(X_\eta)$. In this way we define the classical **variation** operator:

$$\mathrm{Var}\ : H_1(X_\eta, X_\eta \cap C_\eta) \longrightarrow H_1(X_\eta).$$

In the following, when working with real singularities, we will need to relate the Milnor fibers corresponding to negative and positive values. Consider the path γ^- in D^* which connects $-\eta$ to η below zero (see Figure 3.1), and let $T^- : H_1(X_{-\eta}, \mathbb{C}) \to H_1(X_\eta, \mathbb{C})$ be the map induced by identifying the fibers of the Milnor fibration along γ^-.

The last preliminary notion is that of cyclic subspaces of a linear map:

Definition 3.1. Let V be a complex vector space, $L : V \to V$ a linear operator and let v be a vector of V. The **L-cyclic subspace generated by** v is defined as

$$\begin{aligned} Z(v, L) &:= \{g(L)v : g \in \mathbb{C}[x]\} \\ &= \mathrm{Span}_{\mathbb{C}}\{v, Lv, \cdots, L^k v, \cdots\} \end{aligned}$$

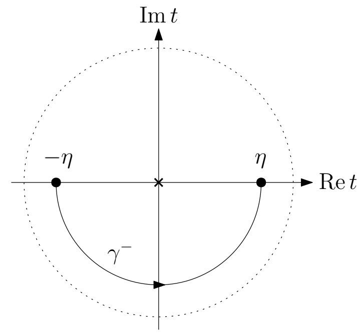


Figure 3.1: Definition of T^-

Definition 3.2. Given a set of real relative cycles $\boldsymbol{\delta} = \boldsymbol{\delta}^+ \cup \boldsymbol{\delta}^-$, the subspace of **visible vanishing cycles** (generated by $\boldsymbol{\delta}$) of $H_1(X_\eta, \mathbb{C})$ is by definition the subspace

$$Z(\boldsymbol{\delta}(\eta)) := \sum_{\delta \in \boldsymbol{\delta}^+} Z(\mathrm{Var}\ \delta(\eta), M) + \sum_{\delta \in \boldsymbol{\delta}^-} Z(T^-(\mathrm{Var}\ \delta(-\eta)), M) \quad \subseteq H_1(X_\eta, \mathbb{C}).$$

Definition 3.3. An algebraically isolated real singularity is **topologically good** (with respect to $\boldsymbol{\delta}$) if $Z(\boldsymbol{\delta}(\eta)) = H_1(X_\eta, \mathbb{C})$, in which case $\boldsymbol{\delta}$ is said to be a *good set of relative cycles* for f. The singularity is **topologically bad** if it is not topologically good, i.e., if $Z(\boldsymbol{\delta}(\eta)) \subsetneq H_1(X_\eta, \mathbb{C})$.

In other words, a singularity is topologically good with respect to a set of relative cycles if applying repeatedly the monodromy map to their variation we obtain a set of cycles which generates the whole homology group.

1.1 Topologically good implies analytically good

We use the same notation of the previous section. Consider two real points $P, P' \in (f^{-1}(0) \smallsetminus 0) \cap (B_\epsilon \smallsetminus B_{\epsilon'})$ and two holomorphic sections $N, N' : D \to X$ of the Milnor fibration with $N(0) = P$ and $N'(0) = P'$. By continuity, if D is small enough, the sections N and N' are entirely contained in $B_\epsilon \smallsetminus B_{\epsilon'}$, and therefore in the collar C.

Definition 3.4. A (N, N')**-cycle** on X_t is a path δ lying in X_t and connecting $N(t)$ to $N'(t)$.

Consider a curve $\gamma = \gamma(s) : [0, 1] \to D^*$ with $\gamma(0) = \eta$. Starting from a (N, N')-cycle $v \mapsto \delta(v)$ on $X_{\gamma(0)}$ consider its lift $\delta_N^{N'}(v, s)$ over $\gamma(s)$, with end-points prescribed by $N(\gamma(s))$ and $N'(\gamma(s))$, in other words a lift for the pair $(I, \partial I)$.

If γ makes a counter-clockwise loop around 0, we obtain a curve on $X_{\gamma(1)} = X_{\gamma(0)}$ which differs from the starting one, because of monodromy, but the end-points for both curves will be again $N(\gamma(0))$ and $N'(\gamma(0))$. Therefore we get a variation operator

$$\mathrm{Var}\ _N^{N'} : H_1(X_\eta, X_\eta \cap C_\eta) \longrightarrow H_1(X_\eta).$$

Without loss of generality we can assume that the sections N and N' are constant in the trivialization of C, which implies that this variation operator coincides with the classical one. A more detailed proof is as follows.

Proposition 3.1. *Let δ be a (N, N')-cycle, then* $\operatorname{Var}{}_N^{N'}\delta = \operatorname{Var}\delta$.

Proof. It is sufficient to show that the final curves on $X_{\gamma(1)}$ are homotopic with the same end-points. Consider a neighborhood of $(P, 0)$ in $C_0 \times T$ of the form $U \times T$, where $U \subset C_0$ is contractible. Because of continuity we can restrict T in order to have $\varphi \circ N(T) \subset U \times T$.

For each $s \in [0, 1]$ on the surface $X_{\gamma(s)}$ we have two points: $\Gamma_s(N(t_0))$ and $N(\gamma(s))$, both belonging to the contractible "slice" $\varphi^{-1}(U \times \{\gamma(s)\})$. Therefore we can find a homotopy $\mathbf{N} = \mathbf{N}(u, s) : I \times I \to X$ over $\gamma(s)$ with $\mathbf{N}(0, s) = \Gamma_s(N(\gamma(0)))$ and $\mathbf{N}(1, s) = N(\gamma(s))$. In the same way we find $\mathbf{N}' = \mathbf{N}'(u, s)$ for the second section. Let $\delta = \delta(v)$ be the starting (N, N')-cycle, $\delta(v, s) = \Gamma_s(\delta(v))$ be its ordinary deformation, and $\delta_N^{N'}(v, s)$ be family of (N, N')-cycles defined above.

Now consider the pair $(I^2, \partial I^2)$, and the homotopy $G : I^2 \times I \to T^*$ given by $G(v, u, s) = \gamma(s)$. Let $\widehat{G} : I^2 \times I \to X^*$ be a lift of G, $f \circ \widehat{G}(v, u, s) = G(v, u, s) = \gamma(s)$, with initial condition $G(v, u, 0) = \delta(v)$, and prescribed on ∂I^2 as follows:

$$\begin{cases} \widehat{G}(v, 0, s) = \delta(u, s) \\ \widehat{G}(v, 1, s) = \delta_N^{N'}(u, s) \\ \widehat{G}(0, u, s) = \mathbf{N}(u, s) \\ \widehat{G}(1, u, s) = \mathbf{N}'(u, s) \end{cases}$$

The restriction $\widehat{G}(v, u, 1)$ provides the required homotopy. $\square$

Consider the function

$$J(t) := \int_{\delta_N^{N'}(t)} \frac{\omega}{df}, \quad t \in D^*. \tag{3.1}$$

It follows from its definition that

$$J \equiv J_\delta^\omega, \quad \text{on } (0, \eta].$$

Moreover, as we proceed to show, J is holomorphic on D^*. First, we have:

Proposition 3.2. *Let $\alpha \in \Omega^1(\mathbb{C}^2, 0)$, then the function*

$$I(t) = \int_{\delta_N^{N'}(t)} \alpha$$

is holomorphic on D^* and

$$dI(t) = \left(\int_{\delta_N^{N'}(t)} \frac{d\alpha}{df} \right) dt + (N \circ f)^* \alpha - (N' \circ f)^* \alpha. \tag{3.2}$$

For the proof we adapt the usual technique of Leray coboundaries, used to prove the analyticity of periods over vanishing cycles, to the case of relative curves. Let $t \in D^*$ and $D_\epsilon \subset D$ a small disc around t. Consider a smooth family of (N, N')-curves $\delta(s)$ on D_ϵ and define a real "tube" surface around $\delta(t)$:

$$\Gamma_\epsilon := \bigcup_{s \in \partial D_\epsilon} \delta(s)$$

Define the following 2-form

$$\Omega_\alpha(t) := \frac{df}{f-t} \wedge \alpha$$

Lemma 3.1.

$$\lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} \Omega_\alpha(t) = 2\pi i \int_{\delta(t)} \alpha$$

Proof.

$$\begin{aligned}\int_{\Gamma_\epsilon} \frac{df}{f-t} \wedge \alpha &= \int_{\partial D_\epsilon} \left(\int_{\delta(s)} \alpha \right) \frac{ds}{s-t} \\ &= \int_{\partial D_\epsilon} \left(\int_{\delta(t)} \alpha \right) \frac{ds}{s-t} + \int_{\partial D_\epsilon} \left(\int_{\delta(s)} \alpha - \int_{\delta(t)} \alpha \right) \frac{ds}{s-t}.\end{aligned}$$

The first integral in the last line is $2\pi i \int_{\delta(t)} \alpha$, the second one tends to 0 as $\epsilon \to 0$. Therefore taking the limit $\epsilon \to 0$ in the first equation we obtain the formula. □

Let $\Gamma := \Gamma_{\epsilon_0}$ be a fixed tube around $\delta(t)$.

Lemma 3.2.

$$2\pi i \int_{\delta(t)} \alpha = \int_\Gamma \Omega_\alpha(t)$$

Proof. The difference $\Gamma - \Gamma_\epsilon$ is given by two annuli, A_ϵ on N and A'_ϵ on N'. Consider the boundary surface $\Sigma_\epsilon := \partial\left(\cup_{\eta\in[\epsilon,1]}\Gamma_\eta\right) = \Gamma - \Gamma_\epsilon + A_\epsilon - A'_\epsilon$. The integral of the closed form Ω_α over Σ_ϵ is zero by Stokes' formula, and:

$$\int_{\Gamma_\epsilon} \Omega_\alpha = \int_\Gamma \Omega_\alpha + \int_{A_\epsilon} \Omega_\alpha - \int_{A'_\epsilon} \Omega_\alpha.$$

Since we choose N and N' to be holomorphic (complex submanifolds of X of complex dimension 1), then $\Omega_\alpha|_N = \sigma^*\Omega_\alpha = 0$ (holomorphic 2-form on a 1-dimensional complex manifold). Similarly $\Omega_\alpha|_{N'} = 0$. This means that the last two integrals vanish, and we have:

$$\int_{\Gamma_\epsilon} \Omega_\alpha = \int_\Gamma \Omega_\alpha$$

Taking $\epsilon \to 0$ and using Lemma 3.1, we get the formula. □

The above lemma is used to prove that $t \mapsto \int_{\delta(t)} \alpha$ is a holomorphic function, being the integral over a fixed surface of a holomorphic 2-form.

Lemma 3.3. *Let $\Sigma \subset X$ be any real surface such that $\Sigma \subset X \smallsetminus X_t$, then*

$$\frac{d}{dt} \int_\Sigma \frac{df}{f-t} \wedge \alpha = \int_\Sigma \frac{df}{f-t} \wedge \frac{d\alpha}{df} - \int_{\partial\Sigma} \frac{\alpha}{f-t}.$$

Proof.

$$\begin{aligned}\frac{d}{dt}\int_\Sigma \frac{df\wedge\alpha}{f-t} &= \int_\Sigma \frac{df\wedge\alpha}{(f-t)^2}\\ &= \int_\Sigma d\left(\frac{-1}{f-t}\right)\wedge\alpha\\ &= -\int_\Sigma d\left(\frac{\alpha}{f-t}\right)+\int_\Sigma\frac{d\alpha}{f-t}\\ &= -\int_{\partial\Sigma}\frac{\alpha}{f-t}+\int_\Sigma\frac{df}{f-t}\wedge\frac{d\alpha}{df}.\end{aligned}$$ □

Proof of Proposition 3.2. By Lemma 3.2

$$2\pi i\int_{\delta(t)}\alpha=\int_\Gamma\Omega_\alpha.$$

Applying Lemma 3.3 to Γ we get

$$\begin{aligned}2\pi i\frac{d}{dt}\int_{\delta(t)}\alpha &= \frac{d}{dt}\int_\Gamma\Omega_\alpha\\ &= \int_\Gamma\Omega_{d\alpha/df}-\oint_{\partial\Gamma}\frac{\alpha}{f-t}\\ &= 2\pi i\left(\int_{\delta(t)}\frac{d\alpha}{df}-\mathrm{Res}_{f=t}\frac{N^*\alpha}{f-t}+\mathrm{Res}_{f=t}\frac{(N')^*\alpha}{f-t}\right).\end{aligned}$$

For the first term in the last line we applied again Lemma 3.2. □

This implies that

Corollary 3.1. *Let $\omega\in\Omega^2(X)$. Then $t\mapsto\int_{\delta(t)}\omega/df$ is holomorphic on D^*.*

Proof. Let α be such that $d\alpha=\omega$, and apply Proposition 3.2. □

Remark 3.1. The same arguments prove that the integrals over families of *vanishing* cycles are holomorphic. For this it is sufficient to take $N=N'$ in the above proof.

Start from a real value $t=\eta>0$, and consider the counter-clockwise loop around the origin. This gives multi-valuedness:

$$\begin{aligned}J(e^{2\pi i}\cdot t) &= J(t)+\oint_{\mathrm{Var}\,{}_N^{N'}\delta_N^{N'}(t)}\frac{\omega}{df}\\ &= J(t)+\oint_{\mathrm{Var}\;\delta(t)}\frac{\omega}{df}\end{aligned}$$

Lemma 3.4. *Let δ be a positive (resp., negative) relative cycle. Suppose J_δ^ω admits a real-analytic extension to a neighborhood of $t=0$, then*

$$\oint_{\mathrm{Var}\,\delta(t)}\frac{\omega}{df}=0\quad \textit{for } t\in I_\eta^+ \textit{ (resp., } I_\eta^-\textit{)}.$$

Proof. Let $J_\delta^\omega(t) = \Theta(t)$ for $t > 0$, where $\Theta \in \mathbb{R}\{t\}$ is a real convergent power series at $t = 0$. The complexification $\Theta^{\mathbb{C}}$ is a holomorphic function around $t = 0$ and therefore single-valued. Moreover $\Theta^{\mathbb{C}} \equiv J$ on $I_\eta^\pm$, and hence everywhere. So J is single-valued, in other words $\oint_{\mathrm{Var}\,\delta(t)} \omega/df = 0$. □

Now we can prove:

Theorem 3.1. *If a singularity is topologically good then it is analytically good (with respect to the same set of relative cycles).*

Proof. Assume $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a topologically good singularity, we want to prove that it is analytically good. Starting with positive relative cycles, assume that all periods $J_\delta^\omega(t)$, for $t \in (0, \eta]$ and $\delta \in \boldsymbol{\delta}^+$, are real-analytic at zero. Consider the holomorphic extension J of J_δ^ω defined by (3.1). It follows from the lemma above that $\langle \omega/df, \mathrm{Var}\ \delta(t)\rangle = 0$ for all $t \in (0, \eta]$.

Now consider the real-analytic, complex-valued function $I(t) := \langle \omega/df, \mathrm{Var}\ \delta(t)\rangle$ on $t \in (0, \eta]$. By continuously deforming $\mathrm{Var}\ \delta(t)$ to complex values of t, this function extends to a holomorphic function $\widehat{I}$ on D^*. But since $\widehat{I} = 0$ on $(0, \eta]$, then $\widehat{I} \equiv 0$ on D^*. This implies, after taking a loop around the origin, that $\langle \omega/df, M(\mathrm{Var}\ \delta(t))\rangle = 0$. By iterating the same argument we deduce that $\langle \omega/df, M^k(\mathrm{Var}\ \delta(t))\rangle = 0$ for all $k \in \mathbb{N}$. This means that $[\omega/df] = 0$ on $Z(\mathrm{Var}\ \delta(t), M)$.

For a negative relative cycle $\delta \in \boldsymbol{\delta}^-$ we get $\langle \omega/df, \mathrm{Var}\ \delta(t)\rangle = 0$ for $t \in [-\eta, 0)$ and by analytic continuation, $\langle \omega/df, T^-(\mathrm{Var}\ \delta(t))\rangle = 0$ for $t \in (0, \eta]$. Doing this for every $\delta \in \boldsymbol{\delta}$ we deduce that $[\omega/df] = 0$ on $Z(\boldsymbol{\delta}(t))$. Since by hypothesis, the singularity is topologically good we have $Z(\boldsymbol{\delta}(t)) = H_1(X_t, \mathbb{C})$. Using Proposition 2.2 we find a holomorphic function germ ϱ such that $\omega^{\mathbb{C}} = df^{\mathbb{C}} \wedge d\varrho$, and taking the real trace of this equation

$$\omega = (\omega^{\mathbb{C}})^{\mathrm{Re}} = (df^{\mathbb{C}} \wedge d\varrho)^{\mathrm{Re}} = (df^{\mathbb{C}})^{\mathrm{Re}} \wedge (d\varrho)^{\mathrm{Re}} - \cancel{(df^{\mathbb{C}})^{\mathrm{Im}}} \wedge (d\varrho)^{\mathrm{Im}} = df \wedge d(\varrho^{\mathrm{Re}})$$

we find $\omega = df \wedge d\varrho^{\mathrm{Re}}$. We have shown that any germ of 2-form which has analytic periods on $\boldsymbol{\delta}$ is relatively exact. In other words, that the singularity f is analytically good with respect to the set $\boldsymbol{\delta}$. □

Remark 3.2. The case of $f^{-1}(0) = \{0\}$ can be treated similarly. Depending on the sign of f outside zero, $\delta = f^{-1}(\pm\eta) \subset X_{\pm\eta}$ will be a closed curve representing an element of $H_1(X_{\pm\eta})$. The singularity f is then *topologically good* if $Z(\delta, M) = H_1(X_{\pm\eta})$. The simplest example is $f = x^2 + y^2$, in which case the cycle $f^{-1}(\eta)$ already gives a homology basis for the complex Milnor fiber $X_\eta = \{f^{\mathbb{C}} = \eta\}$.

1.2 Counterexamples

It is easy to find examples of singularities which are not analytically good. The following counterexample is known in the literature [36]. Consider the singularity $f(x, y) = y^4 - x^4 + x^6$. It is an algebraically isolated singularity with Milnor number $\mu = 9$. The manifold $U = f^{-1}[-\eta, \eta] \subset \mathbb{R}^2$ is already globally embedded in $\mathbb{R}^2$, and it is eight-shaped with symmetry with respect to both axes. Consider on U the symplectic forms: $\omega_0 = dx \wedge dy$ and $\omega_1 =$

$(1+\lambda y)\,dx\wedge dy$ (which is non-degenerate on the whole U if $\lambda > 0$ is small enough), so that $\omega_1 - \omega_0 = \lambda y\,dx\wedge dy$.

From the symmetry, and being $\omega_1 - \omega_0$ odd in y, it follows that all its areas on the three families of ovals vanish, for each value of f, therefore all global periods are zero. The difference $\omega_1 - \omega_0$ however, is not relatively exact, since $df\wedge d\varrho = y\,dx\wedge dy$ can't be solved for ϱ (as can be seen by comparing the order of vanishing of the two hand sides).

1.3 Naive general criteria

Let $M \in \mathrm{GL}(\mu,\mathbb{Z})$ denote the monodromy matrix in some basis of vanishing cycles. There are naive criteria for a singularity to be good, which can be readily verified. For instance we may use the following sufficient condition:

If for any integer vector $v \in \mathbb{Z}^n$, the cyclic subspace $Z(v,M)$ is the whole $\mathbb{C}^\mu \simeq H_1(X_\eta,\mathbb{C})$, i.e., if there is no proper cyclic subspace generated by an integer vector, then the singularity is topologically good (with respect to any relative cycle having non-zero variation).

This is guaranteed if, e.g., the characteristic polynomial $p_M(t)$ of M is irreducible over $\mathbb{Z}$:

Proposition 3.3.

i) If there exists $v \in \mathbb{Z}^n \setminus 0$ such that $Z(v,M) \neq \mathbb{C}^\mu \Rightarrow p_M(t)$ is reducible over $\mathbb{Z}$.

ii) If $p_M(t) = q(t)r(t)$ for some $q,r \in \mathbb{Z}[t]$ with $\deg q, \deg r > 0$, and $q(M)$ and $r(M)$ are not both zero $\Rightarrow \exists v \in \mathbb{Z}^n \setminus 0$ such that $Z(v,M) \neq \mathbb{C}^\mu$.

Lemma 3.5. *If M is an integer matrix, then $\operatorname{Ker} M$ admits a $\mathbb{C}$-basis consisting of integer vectors.*

Proof. Let rank $M = k$. We can assume that M is of the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is a $k \times k$ invertible integer matrix. Since the lower rows are dependent from the upper ones, $\operatorname{Ker} M = \operatorname{Ker}(A|B)$ and therefore the equation $Mv = 0$ for $v = (\boldsymbol{x},\boldsymbol{y}) = (x_1,\dots,x_k,y_{k+1},\dots,y_n)$, is reduced to the equation:

$$\boldsymbol{x} = -A^{-1}B\boldsymbol{y} = -\frac{1}{\det A}A^*B\boldsymbol{y}$$

where A^* denotes the adjugate matrix of A. Consider the solutions of the form

$$(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{\det A}(A^*B\boldsymbol{y},\boldsymbol{y})$$

for $\boldsymbol{y} = (1,0,\dots),(0,1,\dots)$, and so on. To get the integer solutions it suffices to multiply these solutions by $\det A$. If we write all solutions as a matrix, it is clear that the rows are independent, because we have a $(n-k)\times(n-k)$ block of the form $(\det A)\mathrm{Id}$ on the right. □

Proof of Proposition 3.3. (i) Let $v \in \mathbb{Z}^\mu \setminus 0$, and $e_i = M^{i-1}v$. Assume $e_1, \dots, e_k$ are linearly independent over $\mathbb{C}$, and $Z(v, M) = \mathrm{Span}_{\mathbb{C}}\{e_1, \dots, e_k\}$. Complete the set $\{e_1, \dots, e_k\}$ to a basis $\{e_1, \dots, e_\mu\}$ of integer vectors. With respect to this basis, the monodromy matrix is block upper triangular with rational coefficients. So its characteristic polynomial $p_M(t)$ splits in $\mathbb{Q}[t]$. Since, in the original basis, M is an integer matrix, the polynomial $p_M(t)$ has integer coefficients. Therefore, by Gauss's Lemma, it also splits in $\mathbb{Z}[t]$.

(ii) Assume $q(M) \neq 0$. The integer matrix $q(M)$ is not zero, but also not invertible, indeed if $q(t) = \prod_i (t - \lambda_i)^{k_i}$, where λ_i are eigenvalues of M and k_i are not all zero, then $\det q(M) = \prod_i \det(M - \lambda_i \mathrm{Id})^{k_1} = 0$. Let v be an integer vector in $\mathrm{Ker}\, q(M)$ (Lemma 3.5). Then all $M^i v \in \mathrm{Ker}\, q(M)$, therefore $Z(v, M) \subset \mathrm{Ker}\, q(M) \neq \mathbb{C}^\mu$. □

However the irreducibility of $p_M(t)$ is a rare event, since the characteristic polynomial is generally a product of polynomials, according to the following theorem, holding for Γ-non-degenerate[1] singularities.

Theorem 3.2 (Varchenko [57]). *Let $f(x, y)$ be Γ-non-degenerate isolated singularity, with $f(x, 0) = x^a$ and $f(0, y) = y^b$. Let γ_j, $j = 1, \dots, k$ be the faces of $\Gamma(f)$, and Q_j the primitive integer vector corresponding to γ_j. Then*

$$p_M(t) = (t-1)\frac{\prod_{j=1}^{k}(1 - t^{\ell(Q_j)})^{\lambda(Q_j)-1}}{(1 - t^a)(1 - t^b)}.$$

Proof. See [51]. □

Notice that if the Newton polygon has more than one face, the characteristic polynomial will be always divisible by $t - 1$. Assume now that $\Gamma(f)$ has just one face $\gamma = t_\Gamma(Q)$ where Q is a primitive integer vector. In this case $\lambda(Q) = \gcd(a, b) + 1$ and $\ell(Q) = \mathrm{lcm}(a, b)$, therefore

$$p_M(t) = \frac{(t-1)(1 - t^{\mathrm{lcm}(a,b)})^{\gcd(a,b)}}{(1 - t^a)(1 - t^b)}. \tag{3.3}$$

Suppose a, b are different prime numbers, then using the relation $t^n - 1 = \prod_{d|n} \Phi_d(t)$, where $\Phi_n(t)$ denotes the n-th cyclotomic polynomial, we find $p_M(t) = -\Phi_{ab}(t)$, which is irreducible.

Corollary 3.2. *The singularities of type $f(x, y) = y^p - x^q$ with p, q different primes are topologically good. Any relative cycle constitutes a good set for the singularity.*

Example 3.1 (Cusps). Let $f(x, y) = y^2 - x^r$ with $r > 2$ prime. Then $p_M(t) = (t-1)(1 - t^{2r})/(1 - t^2)(1 - t^r) = -\Phi_{2r}(t)$, therefore it is irreducible. Notice that this type of singularities can have arbitrarily large Milnor number, while they only have two real relative cycles. Still they are (topologically and analytically) good with respect to each relative cycle.

Remark 3.3. One possible way to improve the above criterion is to consider an M-invariant decomposition of the Milnor fiber, and compute the homology group starting from this decomposition (see [51]). Moreover, one could improve the criterion by taking into account the position of relative cycles.

[1] See Definition 2.6.

2 Algorithm using toric resolutions

In this section we describe an algorithm allowing to determine if a given Γ-non-degenerate singularity is topologically good.

2.1 Local description at a point of normal crossing

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be an algebraically isolated real singularity. In order to determine if the singularity is topologically good with respect to a given set of relative cycles, we need to compute the cyclic subspaces generated by the variation of such cycles. This essentially amounts to computing the monodromy and variation operators (for real relative cycles). In order to do so, it is convenient to consider a resolution of the singularity. After the toric resolution (discussed in the previous chapter) we are left with a normal crossing fibration, equivalent to the original one, and whose monodromy is easier to study. The monodromy of normal crossing fibration was studied by Clemens in [14] (see also [12, Section 8.5], [46, Chapter 7], [60, Chapter 9,10]).

The first step in order to understand the topology and the action of the monodromy is to look at what happens locally, so let us consider the function $f : U \subset \mathbb{C}^2 \to \mathbb{C}$ given by $f(x, y) = x^m y^n$, for some positive natural numbers $m, n \in \mathbb{N}$, where U is a neighborhood of 0. We can assume that U is the polydisc $U = \{|x|, |y| \leq 1\}$.

Let $\epsilon > 0$ be a small real positive number, and consider the level set $X_\eta = \{x^m y^n = \epsilon\} \subset \mathbb{C}^2$. Consider the projection $p_x(x, y) = x$ to the x-coordinate. Its restriction to X_η defines a covering map $p_x : X_\eta \to \{|x| \leq 1\}$ of degree n. The fiber above $x = e^{i\theta}$ consists of the y's which are solution of $e^{im\theta} y^n = \epsilon$, that is $y = \epsilon^{1/n} e^{i\psi}$ with $m\theta + n\psi = 0 \mod 2\pi$. We denote the solutions as

$$y_k(\theta) = \epsilon^{1/n} \exp\left(-\frac{m}{n}\theta - \frac{2\pi k}{n}\right), \quad k = 0, \ldots, n-1. \tag{3.4}$$

We see that

$$y_k(\theta + 2\pi) = y_{k+m}(\theta).$$

Consider the path $x = \gamma(\theta) = e^{i\theta}$ with $\theta \geq 0$, and consider the lifted path $\widehat{\gamma}(\theta)$ on the covering space defined by $\widehat{\gamma}(0) = y_a(0)$. The equation above shows that the curve $\widehat{\gamma}(\theta)$ passes through the preimages $y_a(0), y_{a+m}(0), y_{a+2m}(0), \ldots, y_{a+\ell m}(0)$ as $x = \gamma(\theta)$ loops ℓ times around 0. Here the indices $a + km$ should be considered as congruence classes modulo n. The lifted curve becomes closed at $\ell =$ the smallest positive number satisfying $a + \ell m \equiv a \mod n$, i.e. for $\ell = n/\gcd(m, n)$.

More precisely, we have

Lemma 3.6.

i) $y_a(0)$ *and* $y_b(0)$ *belong to the same connected circle in the preimage of* $\{|x| = 1\}$ *if and only if* $a \equiv b \mod \gcd(m, n)$.

ii) The preimage of the unit circle $|x| = 1$ *consists of* $\gcd(m, n)$ *circles.*

Before proving this we recall an elementary fact about congruence equations:

Lemma 3.7. *The equation* $x \equiv \ell m \mod n$ *admits a solution* $\ell \in \mathbb{Z}$ *if and only if* $x \equiv 0 \mod \gcd(m, n)$.

Proof. By Bezout theorem we can find integers r, s such that $\gcd(m, n) = rm+sn$. Now assume $x = q\gcd(m, n)$, then $\ell = qr$ is a solution of the first congruence, indeed $x = q\gcd(m, n) = (qr)m + (qs)n = \ell m + n(qs)$. The other implication is obvious. □

Proof of Lemma 3.6. Let $a, b \in \{0, \dots, n-1\}$. Now $y_b(0)$ belongs to the same component of $y_a(0)$ if and only if $b \equiv a + \ell m \mod n$ for some $\ell \in \mathbb{N}$. By the above Lemma the congruence $b - a \equiv \ell m \mod n$ admits a solution ℓ if and only if $\gcd(m, n)|b - a$. This proves (i), and (ii) is an immediate consequence. □

2.2 Cell decomposition of the Milnor fiber

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a Γ-non-degenerate singularity, and consider a toric resolution of f, as defined in Section 2 of the previous chapter (we will use the same notations here). Recall that the pullback of f on each chart $\mathbb{R}^2_{\sigma_i}$ takes the form:

$$f \circ \pi_{\sigma_i}(x_i, y_i) = x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})} \widetilde{f}_i(x_i, y_i),$$

where $\{P_0, \dots, P_{N+1}\}$ is the regular fan defining the toric resolution. Then we will have N exceptional divisors E_i, $i = 1, \dots, N$ of weight $\ell_i = \ell(P_i)$.

Let η be a small positive number[2]. In order to describe the Milnor fiber X_η we will describe it as a cell complex of dimension 2. Later we will describe relative cycles and the monodromy operator in terms of this cell decomposition.

Consider one of the divisors $E_i \simeq \mathbb{P}^1(\mathbb{C})$, which is homeomorphic to a 2-sphere. On this sphere we consider the points 0_i and ∞_i and also the roots $z_{i,1}, \dots, z_{i,q_i}$ of $\widetilde{f}_i(x_i, 0) = 0$ as **marked points**. These roots are simple because of the hypothesis of Γ-non-degeneracy, so that $q_i = \deg \widetilde{f}_i(x_i, 0)$.

At each point $p \in E_i$ we can find local coordinates (X, Y) such that $f \circ \pi_{\sigma_i}(X, Y) = X^m Y^{\ell_i}$, for some $m \in \mathbb{N}$. The number $m_i(p) := m$ is called the **multiplicity of** p **on** E_i. Clearly

$$\begin{cases} m_i(z_{i,j}) = 1 & j = 1, \dots, q_i \\ m_i(0_i) = \ell_{i-1} & \\ m_i(\infty_i) = \ell_{i+1} & \\ m_i(p) = 0 & \text{for any other } p \end{cases}$$

Let N_i be a tubular neighborhood of E_i. For each marked point $p \in E_i$ choose a small polydisc of the form $\Delta(p) = \{(X, Y) : |X|, |Y| \le \delta\}$. Let Δ_i denote the union of all such polydiscs.

Let $D(p) = \Delta(p) \cap E_i$, which is a small ball around p and put $E_i^* := E_i \setminus \cup_p D(p)$. The part of the Milnor fiber contained in N_i, i.e., $X_i := X_\eta \cap N_i$ is then decomposed into two pieces:

[2] For simplicity we restrict to positive η. It is not difficult to include the case of negative η, and to describe the action of T^- between the corresponding homology groups.

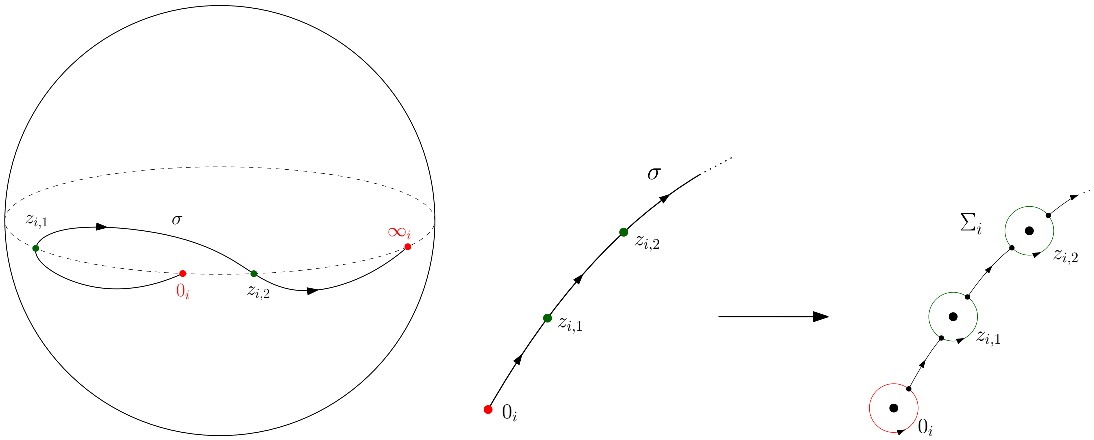


Figure 3.2

$X_\eta \cap \Delta$, which is a collection of annuli, and $X_i \cap (N_i \smallsetminus \Delta)$ which is a cyclic ℓ_i-fold covering of E_i^*. Let $p_i : X_i \cap (N_i \smallsetminus \Delta) \to E_i^*$ denote the corresponding projection. We can assume that p preserves the real part of the toric resolution. The previous section shows that if we denote the preimages of a point $x_0 \in E_i^*$ by an element of $\mathbb{Z}_{\ell_i}$, then the monodromy of the covering after a loop around p is given by $x \mapsto x + m_i(p)$ in $\mathbb{Z}_{\ell_i}$.

Consider a tree σ lying on E_i and connecting all the marked points of the divisor. For each marked point p, let $\lambda(p)$ denote the oriented small loop $\partial D(p)$ around p. In the graph σ, replace each marked point p with the small loop $\lambda(p)$, let Σ_i be the resulting graph on E_i (see Figure 3.2). The small loops $\lambda(0_i)$, $\lambda(\infty_i)$ will be called **red loops**, and the remaining ones, $\lambda(z_{i,j})$, **green loops**.

The Milnor fiber X_η is obtained by gluing together the preimages $X_i := p^{-1}(E_i^*)$ of the divisors along their common boundaries, which are given by the preimages of the red loops. The graph Σ_i provides a cell decompostion of the surface E_i^*, which can be lifted to a cell decomposition of the Milnor fiber, as we now explain. The preimage $\widehat{\Sigma}_i := p^{-1}(\Sigma_i)$ in the Milnor fiber is easy to describe according to the local description of the previous section. Let $d_i(p) := \gcd(m_i(p), \ell_i)$, then:

- Each point $p \in \Sigma_i$ has ℓ_i preimages;
- $p^{-1}(\lambda(z_{i,j}))$ is a connected circle;
- $p^{-1}(\lambda(0_i))$ consists of $\gcd(\ell_{i-1}, \ell_i) = d_i(0_i)$ connected circles;
- $p^{-1}(\lambda(\infty_i))$ consists of $\gcd(\ell_i, \ell_{i+1}) = d_i(\infty_i)$ connected circles.

In short, for each marked point p, the preimage of $\lambda(p)$ consists of $d_i(p)$ connected circles, each containing $\ell_i/d_i(p)$ vertices. The connected circles are parametrized as in Equation (3.4), so that their orientation is induced from the corresponding loop on E_i. The circles belonging to the preimage of a red (resp. green) loop will still be called red (resp. green) loops (of $\widehat{\Sigma}_i$). The preimages of red loops coming from consecutive divisors must be identified respecting the cyclic order, while green loops are the boundary components of the Milnor fiber X_η. The graph resulting from gluing the graphs $\widehat{\Sigma}_i$ along red loops will be denoted by $\widehat{\Sigma}$.

In order to reconstruct the whole preimage X_i of E_i^* we need to attach to the 1-skeleton $\widehat{\Sigma}_i$ the boundary components given by $p^{-1}(E_i \smallsetminus \Sigma_i)$. Since $E_i \smallsetminus \Sigma_i$ is a 2-disk its preimage will consist of ℓ_i disconnected disks, so that we have ℓ_i boundary components. In this way we obtain a cell decomposition of X_i. Let $X^{(i)}$ denote the union of cells of dimension $\leq i$.

We need to describe these sets more explicitly, together with the attaching maps. The preimage of each loop $\lambda(p)$ contains $\deg(p)\ell_i$ vertices, where $\deg(p)$ is the degree of p in the graph σ. We label these vertices according to the corresponding class modulo ℓ_i. The ℓ_i preimages of each edge of Σ_i will connect pairs of vertices with the same label. We label each edge of $\widehat{\Sigma}_i$ with the same label of its end-points.

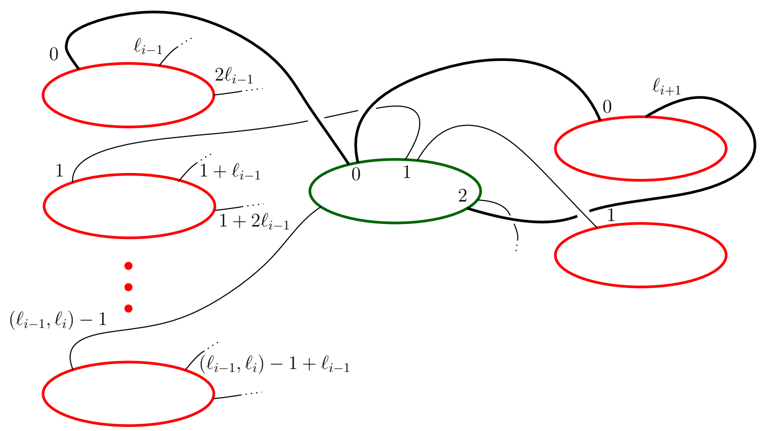


Figure 3.3: $\widehat{\Sigma}_i = p^{-1}(\Sigma_i)$

The attaching map of the boundary 2-cells are given by the following paths on $\widehat{\Sigma}_i$: for each $j \in \mathbb{Z}_{\ell_i}$, start from a red loop in $p^{-1}(\lambda(0_i))$ at the vertex labeled by j, and follow the edges labeled by j until a red loop at infinity is reached. Each time the path reaches a loop, it follows the orientation of that loop. In the same way we go back to the red loop at zero. Each time the path completes a loop $\lambda(p)$ we have an identification of the preimages in the covering map, which is given by $x \mapsto x + m_i(p) \mod \ell_i$. The list of edges belonging to the loop now described, is:

$$j,\ j,\ \ldots,\ j,\ \ j+\ell_{i+1},\ j+\ell_{i+1}+1,\ \ldots,\ j+\ell_{i+1}+q_i,\ \ j+\ell_{i+1}+\ell_{i-1}+q_i.$$

We denote this loop by $r_{i,j}$. Notice that this path is indeed a closed curve, because $\ell_{i-1} + \ell_{i+1} + q_i \equiv 0 \mod \ell_i$ due to Lemma 2.6.

2.3 Homology and Milnor number

Lemma 3.8. *Let S be an orientable surface with boundary consisting of r connected circles. Let $\widetilde{S}$ the surface obtained by attaching a 2-cell on each connected component of the boundary. Then the genus of $\widetilde{S}$ is*

$$g(\widetilde{S}) = \frac{2 - r - \chi(S)}{2}.$$

where $\chi(S)$ is the Euler number of S.

Proof. We have $2 - 2g(\widetilde{S}) = \chi(\widetilde{S}) = \chi(S) + r$, since we are adding r faces. $\square$

As known the Euler characteristic of a surface S can be described as the alternating sum of Betti numbers $b_i = \text{rank } H_i(S, \mathbb{Z})$. In the case of a connected surface with boundary, we have $b_1 = 1$ and $b_2 = 0$, therefore we find $\chi(S) = \text{rank } H_1(S, \mathbb{Z}) + 1$, or equivalently $\text{rank } H_1(S, \mathbb{Z}) = \chi(S) - 1$. Applying this to the Milnor fiber we get: $\mu = 1 - \chi(X_\eta)$. The Milnor number can be easily computed in terms of the toric resolution:

Proposition 3.4. $\mu = \sum_i \ell_i q_i - (\ell_1 + \ell_N) + 1$.

Remark 3.4. This result follows from Theorem 3.2, here we present an elementary proof.

Proof. Let N be the number of divisors. The Euler characteristic is additive with respect to the operation of gluing along boundary circles, therefore:

$$\chi(X_\eta) = \chi\Big(p^{-1}(D(0_1)) \cup p^{-1}(D(\infty_N)) \cup \bigcup_i X_i\Big) = \ell_1 + \ell_N + \sum_i \chi(X_i).$$

Next, by removing from X_i the ℓ_i boundary faces given by $p^{-1}(E_i^* \setminus \Sigma_i)$, we find

$$\chi(X_i) = \chi(\widehat{\Sigma}_i) + \ell_i.$$

Now we can simplify $\widehat{\Sigma}_i$ with operations which leave the Euler characteristic unchanged. On each circle (corresponding to a marked point), we concentrate all vertices in a single point by contracting small arcs on the circle. Each circle becomes a loop in the new graph. The last operation is to identify all these loops to a single one. The resulting graph has the same Euler characteristic as the original one, and is composed by one loop (counting as 1 vertex and 1 edge) plus $\ell_i \cdot |E(\sigma)| = \ell_i(|V(\sigma)| - 1) = \ell_i(q_i + 1)$ edges. In conclusion $\chi(\widehat{\Sigma}_i) = -\ell_i(q_i + 1)$, therefore $\chi(X_i) = -\ell_i q_i$. Putting everything together,

$$\mu = 1 - \chi(X_\eta) = 1 - (\ell_1 + \ell_N) - \sum_i (-\ell_i q_i). \quad \square$$

2.4 Monodromy

Given the cell decomposition above, we can describe the homology $H_1(X_\eta)$ by giving generators and relations (i.e., as a finitely presented $\mathbb{Z}$-module). Consider again the cell decomposition $\{X^{(i)}\}$ of X_η. In the long exact homology sequence of the pair $(X^{(2)}, X^{(1)})$ we find:

$$H_2(X^{(2)}) \longrightarrow H_2(X^{(2)}, X^{(1)}) \xrightarrow{\partial} H_1(X^{(1)}) \longrightarrow H_1(X^{(2)}) \longrightarrow H_1(X^{(2)}, X^{(1)})$$

where ∂ is the connecting boundary homomorphism. Since $X^{(2)} = X_\eta$ is a manifold with boundary, and $H_1(X^{(2)}, X^{(1)}) = 0$, the sequence becomes

$$0 \longrightarrow H_2(X_\eta, X^{(1)}) \xrightarrow{\partial} H_1(X^{(1)}) \longrightarrow H_1(X_\eta) \longrightarrow 0 \tag{3.5}$$

so that we have

$$H_1(X_\eta) \simeq \frac{H_1(X^{(1)})}{\operatorname{Im} \partial}.$$

The first homology group of the graph $X^{(1)}$ can be computed by finding a spanning tree $T \subset X^{(1)}$ for $X^{(1)}$. The group $H_1(X^{(1)})$ is then a free abelian group with one generator for every edge in $X^{(1)} \setminus T$. To find a basis of $H_1(X_\eta)$, we first express $\operatorname{Im}\partial$ in terms of the basis of $H_1(X^{(1)})$. A basis for $H_1(X_\eta)$ is then given by a set of vectors completing $\operatorname{Im}\partial$ to a basis of $H_1(X^{(1)})$.

The monodromy homeomorphism acts nicely on the cell decomposition, and induces a map $\widehat{M} : \widehat{\Sigma} \to \widehat{\Sigma}$, given by a rotation on $\widehat{\Sigma}$ according to the cyclic order. On the edges it acts as the multiplication by $\exp(2\pi i/\ell_i)$, i.e., sending each edge $x \in \mathbb{Z}_{\ell_i}$ to $x+1 \in \mathbb{Z}_{\ell_i}$. Moreover it preserves the boundary components $\operatorname{Im}\partial$, thus inducing a homomorphism on the quotient $M_{\mathbb{Z}} : H_1(X_\eta) \to H_1(X_\eta)$. We can complexify the above sequence (3.5) and get a morphism $M : H_1(X_\eta, \mathbb{C}) \to H_1(X_\eta, \mathbb{C})$.

2.5 Real relative cycles and variation operator

Finally we need to compute the subspace $Z(\delta, M)$, for a given real relative cycle δ. A relative cycle of X_η is given by path on $\widehat{\Sigma}$ connecting two green loops. *Real* relative cycles are those relative cycles which project to $E_i^{\mathbb{R}}$. In order to simplify the description of real relative cycles in $\widehat{\Sigma}$ we choose the tree σ on each divisor E_i to be symmetric with respect to complex conjugation in the corresponding cone chart $\mathbb{C}^2_{\sigma_i}$. Since $\widetilde{f}_i(x_i, 0)$ has real coefficients its complex roots are conjugated, and we can label the roots accordingly. Let $u_1, u_2, \ldots$ be the real roots and $w_1, \overline{w}_1, w_2, \overline{w}_2, \ldots$ the conjugate pairs of complex roots. We order the real roots $u_1, u_2, \ldots$ from $x_i = -\infty$ to $x_i = +\infty$, while for the complex roots we assume $\operatorname{Im} w_i > 0$ and order the w_i first by their real part, and then by their imaginary part. The tree σ which connects all marked points is described as follows: all real roots, 0_i and ∞_i are connected by a real segment. Then connect the point 0_i to the complex roots $w_1, w_2, \ldots$ (if present) with a new curve, and finally connect 0_i to the conjugated roots $\overline{w}_1, \overline{w}_2, \ldots$ using the curve obtained by complex-conjugating the first one.

Next we need to understand the respective positions of the real relative cycles on $\widehat{\Sigma}$. This can be understood by locally studying each real marked point. By performing a half-loop around the marked point (like in (3.4)) we can determine which preimages of the vertex at the end of the half-loop will be real (if any). This depends on the weights $m_i(p)$ and ℓ_i. Finally we consider the relative cycle represented by the real interval $[-\infty, x]$, where $x \in \mathbb{R}$ is the minimum real marked point. We can deform this path in E_i^* to a path connecting $\lambda(x)$ to $\lambda(\infty_i)$ and following the orientations along the loops. The lift of this new path in $\widehat{\Sigma}$ (with the same end-points) will represent the original relative cycle in X_η.

Finally we describe the variation operator. Let $\delta : [0,1] \to \widehat{\Sigma}$ be a real relative cycle in $\widehat{\Sigma}$. Its image $\widehat{M}\delta$ is again a relative cycle starting and ending at the same green loops of δ. The end-points of the two relative cycles δ and $\widehat{M}\delta$ will be consecutive vertices on the respective green loops. Let β_0 and β_1 denote the directed paths, over the green loops, connecting the end-points of δ with the end-points of $\widehat{M}\delta$. Then the variation of δ is represented by the cycle

$$\operatorname{Var}\delta = \widehat{M}\delta - \beta_1 - \delta + \beta_0$$

over the lifted graph $\widehat{\Sigma}$.

In this way we obtain, for each real relative cycle, an explicit expression for its variation and for the cyclic subspace it generates. It is now possible to determine if a set of relative cycles is good for a given Γ-non-degenerate singularity.

2.6 Examples

Example 3.2. $f(x, y) = y^2 - x^2$ (non-degenerate saddle), i.e., the simplest example. A regular fan is: $\{(1,0),(1,1),(0,1)\}$. There is just one exceptional divisor E_1, of degree 2, with one positive and one negative root. The graph $\widehat{\Sigma}$ is represented in Figure 3.4.

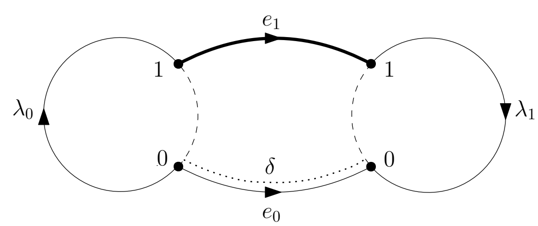


Figure 3.4: $\widehat{\Sigma}$

In order to simplify the description of generators, we retract the dashed part of each loop to a point (we use the same notation for the edges before and after the contraction). We shall keep track of which edges were retracted for later, when we will express boundary components and the action of monodromy in terms of the retracted graph. A spanning tree (in the retracted graph) is $T = e_1$. A basis of $H_1(X^{(1)})$ is given by the loops: $\{b_0 = e_1 - e_0, \lambda_0, \lambda_1\}$ while $\operatorname{Im}\partial$ is generated by the two cycles:

$$\begin{aligned} r_0 &= -b_0 && = e_0 - e_1 \\ r_1 &= b_0 + \lambda_0 + \lambda_1 && = e_1 + \lambda_1 - e_0 + \lambda_0 \end{aligned}$$

The Milnor number is then, of course $\mu = 3 - 2 = 1$. The element λ_0 completes $\{r_0, r_1\}$ to a basis of $H_1(X^{(1)})$, so that its class is a basis of $H_1(X_\eta)$.

The edge $\delta = e_0$ represents a real relative cycle, with $\operatorname{Var}\delta = b_0 + \lambda_0$, whose class modulo $\operatorname{Im}\partial$ obviously generates $H_1(X_\eta)$. The same is true for every real relative cycle, so that $f(x, y) = y^2 - x^2$ is topologically good with respect to any of its real relative cycles (in this trivial case we don't even need to compute the cyclic subspaces).

Example 3.3. $f(x, y) = y^2 - x^4$. We continue from Example 2.1 of Chapter 2, using the same toric resolution. Recall that there are two exceptional divisors E_1, E_2 with weight $\ell_1 = \ell(1,1) = 2$ and $\ell_2 = \ell(1,2) = 4$, respectively. Since $\gcd(\ell_1, \ell_2) = 2$, the red loop $\lambda(0_2)$ has 2 connected preimages. Since E_1 contains no marked points other than 0_1 and ∞_1, the corresponding surface X_1 is composed of two disjoint discs. Analogously, $p^{-1}(D(\infty_2))$ is made of 4 disjoint discs. After contracting the corresponding red loops of $\widehat{\Sigma}$ to a point, $\widehat{\Sigma}$ becomes as in Figure 3.5.

The thick line in the figure represents a spanning tree T of $\widehat{\Sigma}$. As in the previous example, we retract the dashed part of each loop to a point and keep using the same notation for the

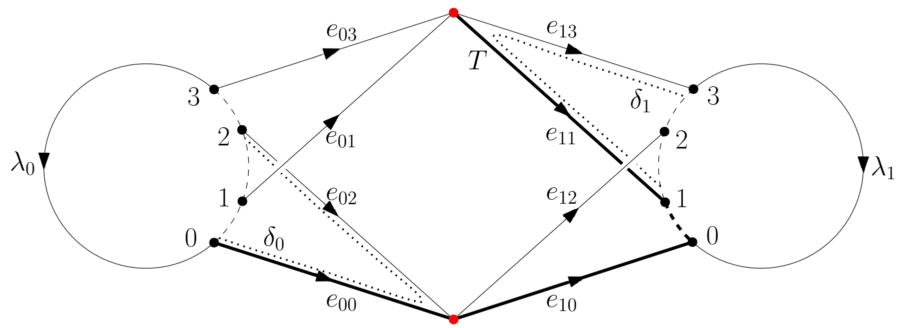


Figure 3.5: $\widehat{\Sigma}$

edges. After this, by looking at the edges which do not belong to T, we find the following basis $\{b_0, b_1, b_2, b_3, b_4, \lambda_0, \lambda_1\}$ for $H_1(X^{(1)})$:

$$\begin{aligned} b_0 &= e_{02} - e_{00} \\ b_1 &= e_{03} - e_{01} \\ b_2 &= e_{12} - e_{10} \\ b_3 &= e_{13} - e_{11} \\ b_4 &= e_{03} + e_{11} - e_{10} - e_{00} \end{aligned}$$

The monodromy matrix in this basis is given by:

$$M = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

The subgroup of relations $\operatorname{Im}\partial$ is generated by the four cycles:

$$\begin{aligned} r_0 &= e_{00} + e_{10} - e_{11} - e_{03} + \lambda_0 &&= -b_4 + \lambda_0 \\ r_1 &= e_{01} + e_{11} - e_{12} - e_{00} &&= -b_1 - b_2 + b_4 \\ r_2 &= e_{02} + e_{12} - e_{13} - e_{01} &&= b_0 + b_1 + b_2 - b_3 - b_4 \\ r_3 &= e_{03} + e_{13} - e_{10} - e_{02} + \lambda_1 &&= -b_0 + b_3 + b_4 + \lambda_1 \end{aligned}$$

The Milnor number is therefore $\mu = 7 - 4 = 3$. The elements $\{b_0, b_1, b_2\}$ complete $\operatorname{Im}\partial$ to a basis of $H_1(X^{(1)})$, and therefore provide a basis $\{\bar{b}_0, \bar{b}_1, \bar{b}_2\}$ of the quotient $H_1(X_\eta)$. The monodromy matrix in the new basis $\{b_0, b_1, b_2, r_0, r_1, r_2, r_3\}$ is given by:

$$M' = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Finally, the matrix of the induced monodromy morphism on $H_1(X_\eta)$ in the basis $\{\bar{b}_0, \bar{b}_1, \bar{b}_2\}$ is:

$$\overline{M} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Its characteristic polynomial is $p_M(t) = (1+t^2)(t-1)$ (in agreement with Theorem 3.2), which is not irreducible. The two positive relative cycles identified in Example 2.1 of Chapter 2 are given by:

$$\begin{aligned} \delta_0 &= e_{00} - e_{02} \\ \delta_1 &= -e_{13} + e_{11}. \end{aligned}$$

They are represented by dotted curves in Figure 3.5. Their variation is:

$$\begin{aligned} \text{Var } \delta_0 &= e_{01} - e_{03} + e_{02} - e_{00} &&= b_0 - b_1 \\ \text{Var } \delta_1 &= -e_{10} + e_{12} - e_{11} + e_{13} + \lambda_1 &&= b_2 + b_3 + \lambda_1 \end{aligned}$$

We have Var $\delta_1 =$ Var $\delta_0 + \operatorname{Im}\partial$, so that the two cycles in fact induce the same element $\gamma = \bar{b}_0 - \bar{b}_1 \in H_1(X_\eta)$. Since $\dim Z(\gamma, \overline{M}) = 2$, we conclude that no combination of the relative cycles δ_0 and δ_1 gives a good set of relative cycles for $f(x, y)$. However, with a little reflection one can see that the paths

$$\begin{aligned} \delta_2 &= e_{01} + e_{11} \\ \delta_3 &= -e_{13} - e_{03} \end{aligned}$$

represent the two negative relative cycles (after the identification T^-). In this case

$$\begin{aligned} \text{Var } \delta_2 &= e_{02} + e_{12} - e_{01} - e_{11} &&= b_0 + b_1 + b_2 - b_4 &&= b_1 + \operatorname{Im}\partial \\ \text{Var } \delta_3 &= e_{13} + e_{03} - e_{10} - e_{00} + \lambda_1 - \lambda_0 &&= b_3 + b_4 + \lambda_1 - \lambda_0 &&= b_0 - b_1 - b_2 + \operatorname{Im}\partial \end{aligned}$$

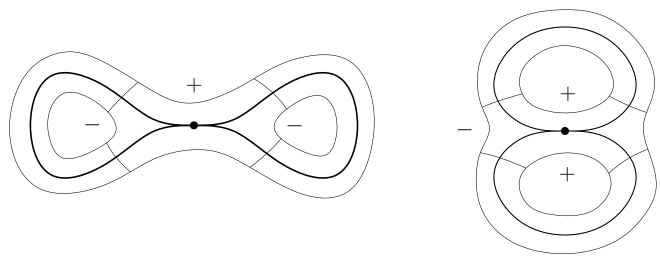
and $\dim Z(\bar{b}_1, \overline{M}) = \dim Z(\bar{b}_0 - \bar{b}_1 - \bar{b}_2, \overline{M}) = 3$. In conclusion, $f(x, y)$ is topologically good. Despite the characteristic polynomial of the monodromy is not irreducible, each negative real relative cycle alone is a good set for of relative cycles for the singularity.

Finally, we consider those sets of relative cycles which are given by the restriction of the fibers of a semi-local singularity (see Figure 3.6). There are two possibilities arising from a semi-local picture: either the positive cycles appear together (Case 1), or the negative cycles appear together (Case 2). The corresponding subspaces of visible vanishing cycles are given, respectively, by:

$$\begin{aligned} Z_1 &= Z(\text{Var } \delta_0 + \text{Var } \delta_1, \overline{M}) + Z(\text{Var } \delta_2, \overline{M}) + Z(\text{Var } \delta_3, \overline{M}) \\ Z_2 &= Z(\text{Var } \delta_0, \overline{M}) + Z(\text{Var } \delta_1, \overline{M}) + Z(\text{Var } \delta_2 + \text{Var } \delta_3, \overline{M}) \end{aligned}$$

It is interesting to notice that $\dim Z_1 = 3$ but $\dim Z_2 = 2$. In other words, the two cases are different, and only Case 1 yields a good set of relative cycles.

Remark 3.5. Pictures and computations become much more complicated for singularities with high Milnor number. However, the above algorithm only requires to know the Newton



(a) Case 1 (b) Case 2

Figure 3.6: Semi-local realizations of $f(x, y) = y^2 - x^4$.

polygon and the number of positive, negative and complex roots of the functions $\widetilde{f}_i(x_i, 0)$. This allows in principle to implement the algorithm on a computer and to produce a list of Γ-non-degenerate good singularities. Notice that similar algorithms can be obtained also for singularities which are not Γ-non-degenerate. A resolution in this case can be obtained by ordinary blow-ups or towers of toric blowing-ups (see [50] for definitions).

Remark 3.6. We mention an alternative way to study particular examples based on the methods of A'Campo [1] and Gusein-Zade [35]. In these methods, the intersection numbers among vanishing cycles is computed starting from a "real Morsification" of the singularity, and the monodromy matrix then follows from Picard-Lefschetz formulas. It is not difficult to see that the formulas for intersection numbers can be used to compute the intersection index of real relative cycles as well.

We collect in Table 3.1 a list of examples of good singularities, together with corresponding good sets of relative cycles. For some of the examples, we also indicate a "bad" set of relative cycles $\boldsymbol{\delta}$, i.e., such that $Z(\boldsymbol{\delta})$ is not the whole homology.

3 Semi-local case

Let $M = f^{-1}[-\eta, \eta]$ be a semi-local singularity, and $\omega_0, \omega_1 \in \Omega^2(M)$ be two symplectic forms on M inducing the same orientation. The following proposition gives the connection between our problem and relative cohomology:

Proposition 3.5. *If $\omega_1 - \omega_0 = df \wedge d\widehat{\varrho}$ for some real-analytic function $\widehat{\varrho} : M \to \mathbb{R}$, then there exists an isotopy φ_t on M, such that $\varphi_0 = \mathrm{id}$, $\varphi_t^*(f) = f$ and $\varphi_1^*\omega_1 = \omega_0$*

Proof. As for Theorem 2.4, the proof is based on Moser path method. Let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. Let's write $\omega_1 = \rho\omega_0$ with $\rho \in C^\infty(M)$. By compactness $\nu := \min_M \rho > 0$. Now consider $\omega_t = (1 - t + t\rho)\omega_0$. It is symplectic if $1 - t + t\rho > 0$ on M, but $1 - t + t\rho \geq (1 - t) + t\nu \geq \min\{1, \nu\} > 0$.

The rest of the proof is identical to the proof of Theorem 2.4. Define a real-analytic time-dependent vector field X_t by $i_{X_t}\omega_t = -\widehat{\varrho} df$. Let φ_t be the flow generated by such X_t, we can integrate over all times since M is compact. Notice that $L_{X_t}\omega_t = df \wedge d\varrho$ therefore $\frac{d}{dt}\varphi_t^*\omega_t = 0$, so that $\varphi_1^*\omega_1 = \omega_0$. Moreover $L_{X_t} f = 0$, which means that $f \circ \varphi_t = f$. □

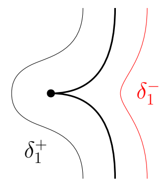
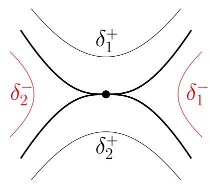
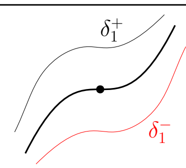
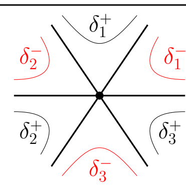
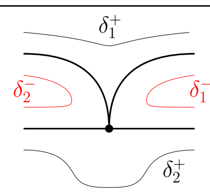
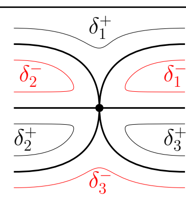
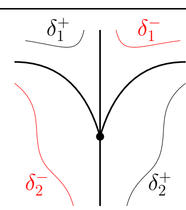
Singularity		Good $\boldsymbol{\delta}$	Bad $\boldsymbol{\delta}$
$y^2 - x^p$ p prime $p > 2$		$\{\delta_1^+\}$, $\{\delta_1^-\}$	
$y^2 - x^{2k}$ $k = 2, 3$		$\{\delta_1^+ + \delta_2^+,\ \delta_1^-, \delta_2^-\}$	$\{\delta_1^+, \delta_2^+,\ \delta_1^- + \delta_2^-\}$
$y^p - x^q$ p, q prime $p, q > 2$, $p \neq q$		$\{\delta_1^+\}$, $\{\delta_1^-\}$	
$y^3 - x^2y$		$\{\delta_1^+, \delta_2^+, \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$, same with $+ \leftrightarrow -$	$\{\delta_1^+ + \delta_2^+ + \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$
$y^{2k} - x^2y$ $k = 2, 3$		$\{\delta_1^+ + \delta_2^+,\ \delta_1^-, \delta_2^-\}$	$\{\delta_1^+, \delta_2^+,\ \delta_1^- + \delta_2^-\}$
$y^5 - x^2y$		$\{\delta_1^+, \delta_2^+, \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$, same with $+ \leftrightarrow -$	$\{\delta_1^+ + \delta_2^+ + \delta_3^+,$ $\delta_1^- + \delta_2^- + \delta_3^-\}$
$x^3 - xy^3$		$\{\delta_1^+, \delta_2^+, \delta_1^- + \delta_2^-\}$, same with $+ \leftrightarrow -$	

Table 3.1: Topologically good singularities

3.1 Extension of relative exactness from local to semi-local, one critical point

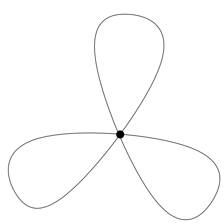
Consider a semi-local singularity $f : M \to \mathbb{R}$, where $M = f^{-1}[-\eta, \eta]$, and let ω_0 and ω_1 be two symplectic forms on M defining the same orientation. Suppose the critical leaf $\{f = 0\}$ contains only one algebraically isolated critical point x_0 (see below for the general case). Let $K := f^{-1}(0)$, it consists of a finite number of edges $\{K_1, K_2, \ldots\}$ starting and ending at x_0 and oriented by the Hamiltonian flow of $\omega_0^{-1} df$. Consider the Reeb graph Γ of the singularity. We can distinguish positive and negative edges of Γ according to the value of f. Let e_K be the number of edges of K, and e_Γ^+ and e_Γ^- the number of positive and negative edges of Γ respectively; we have of course $e_\Gamma^\pm \leq e_K$. We define an $e_\Gamma \times e_K$ matrix R with values in $\{0, 1\}$ as follows

$$R_{ij} = \begin{cases} 1 & \text{if for } f \to 0 \text{ the circles } \Gamma_i(f) \text{ get arbitrarily close to the edge } K_j \\ 0 & \text{otherwise} \end{cases}$$

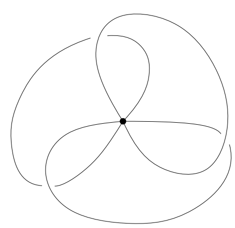
We say that the semi-local singularity is **rigid** if $\operatorname{Ker} R = 0$.

Example 3.4.

- The "annulus" shape is rigid. This corresponds, for example, to the semi-local neighborhood of regular fibers or of singularities of given locally by $f(x, y) = y^2 - x^k$ with $k > 2$ odd (like the cusp).
- The classical "eight shape", occurring e.g. for non-degenerate saddles, or for $f(x, y) = y^2 - x^k$, $k > 1$ even, etc. (see e.g. Figure 2.1a) is rigid.
- Consider now the singularity $f(x, y) = x^2 y - y^3$ (three intersecting lines). It has two semi-local realizations (shown in Figure 3.7). In Case (a) the Reeb graph Γ has 4 edges, in Case (b) it has only 2 edges. Case (a) is rigid, while (b) is not.



(a) Rigid



(b) Not rigid

Figure 3.7: Semi-local realizations of $f(x, y) = x^2 y - y^3$.

Theorem 3.3. *Let f be a semi-local singularity with one critical point x_0. Assume f is rigid. Suppose there exists a real-analytic germ $\varrho : U \to \mathbb{R}$ defined in a small neighborhood U of x_0 and such that $\omega_1 - \omega_0 = df \wedge d\varrho$ on U. If the period maps of the two symplectic forms coincide, i.e., $\Pi_{\omega_0} = \Pi_{\omega_1}$ on $\Gamma \setminus \{0\}$, then ϱ can be extended to a real-analytic function $\widehat{\varrho}$ on M such that $\omega_1 - \omega_0 = df \wedge d\widehat{\varrho}$ on the whole M.*

Proof. For each oriented edge of the singular fiber $f^{-1}(0)$ we fix a *source* section $f \mapsto \sigma_i(f)$ and a *target* section $f \mapsto \tau_i(f)$ which are both real-analytic and contained in U. We extend ϱ

by integration:

$$\widehat{\varrho}(f,y) := \varrho(\sigma_i(f)) + \int_{\sigma_i(f)}^{y} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)}.$$

We need to show that: (a) $\widehat{\varrho}$ is well-defined, (b) it is real-analytic, (c) it satisfies $\widehat{\varrho}|_U = \varrho$ and (d) $\omega_1 - \omega_0 = df \wedge d\widehat{\varrho}$ on M. Point (b) is immediate, (c) follows from the fact that $\frac{\omega_1-\omega_0}{df}\big|_{\Gamma_i(f)} = d\varrho\big|_{\Gamma_i(f)}$ inside U, and (d) is easy to see in coordinates (f,t), where t is the time variable defined by the flow of $\omega_0^{-1}df$ (these are analytic coordinates due to Cauchy-Kovalevskaya theorem). The non-trivial part is to prove (a). To this end, let's introduce the *obstruction functions*:

$$\begin{aligned}\delta_i(f) &:= \widehat{\varrho}(\tau_i(f)) - \varrho(\tau_i(f)) \\ &= \int_{\sigma_i(f)}^{\tau_i(f)} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} + \varrho(\sigma_i(f)) - \varrho(\tau_i(f)).\end{aligned} \tag{3.5}$$

Let's show that if $\delta_i = 0$ for all i then $\widehat{\varrho}$ is well-defined. Indeed let $y \in U$ be the end-point of a path starting from $\sigma_i(f)$ and ending inside U just once (let's call it a *simple path*). Then y and $\tau_i(f)$ can be joined by a path lying entirely inside U, and we have:

$$\begin{aligned}\widehat{\varrho}(f,y) &= \varrho(\sigma_i(f)) + \int_{\sigma_i(f)}^{\tau_i(f)} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} + \int_{\tau_i(f)}^{y} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} \\ &= \widehat{\varrho}(f,\tau_i(f)) + \int_{\tau_i(f)}^{y} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} \\ &= \widehat{\varrho}(f,\tau_i(f)) + \varrho(f,y) - \varrho(f,\tau_i(f)) \\ &= \varrho(f,y) + \delta_i(f).\end{aligned}$$

Therefore if $\delta_i = 0$, then the two functions agree at y. For points y which are end-points of paths entering U multiple times it suffices now to split the paths into simple ones, and apply the above argument to each piece.

It follows by the same argument that the obstruction functions are independent of the choice of source and target sections. Moreover they are related to the periods in the following way. Let's consider the period maps of Γ_i. For $f > 0$ (or $f < 0$, depending on the edge Γ_i) we have, by choosing a proper reordering of source and target sections, that

$$\begin{aligned}\Pi^i_{\omega_1}(f) - \Pi^i_{\omega_0}(f) &= \int_{\Gamma_i(f)} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} \\ &= \sum_{j\,:\,R_{ij}\neq 0} \left(\int_{\sigma_j(f)}^{\tau_j(f)} \frac{\omega_1-\omega_0}{df}\Big|_{\Gamma_i(f)} + \varrho(\sigma_j(f)) - \varrho(\tau_j(f)) \right) \\ &= \sum_j R_{ij}\delta_j(f)\end{aligned}$$

If $\Pi^i_{\omega_1}(f) = \Pi^i_{\omega_0}(f)$ for, let's say, $f > 0$ then $\sum_j R_{ij}\delta_j(f) = 0$ for $f > 0$. But the obstruction functions δ_j are analytic, therefore we get $\sum_j R_{ij}\delta_j(f) = 0$ for all f. The assumption that all periods are the same becomes: $R\delta(f) = 0$, where $\delta(f) = (\delta_1(f), \ldots, \delta_{e_K}(f))^t$. But now our rigidity assumption implies $\delta = 0$. □

In order to generalize to the case of several critical points we need to introduce the notion of *ribbon graph.*

3.2 Ribbon graphs

A ribbon graph, in short, is a graph endowed with a cyclic orientation on the half-edges incident to each vertex. To give a rigorous definition, it is convenient to consider a graph as a directed graph with pairs of (opposite) directed edges. We follow the exposition of [41].

A **directed graph** is a triple $G = (V, E, \varphi)$ where V and E are sets whose elements are called **edges** and **vertices** respectively and $\varphi : E \to V \times V$, $e \mapsto (e_-, e_+)$ is a map. The vertices e_- and e_+ are called, respectively, the **tail** and the **head** of e.

A **graph** is a pair (G, ι) where $G = (V, E, \varphi)$ is a directed graph, and $\iota : E \to E$, $e \mapsto \overline{e}$ is a fixed point free involution on E satisfying $\overline{e}_+ = e_-$, $\overline{e}_- = e_+$. A pair $(e, \overline{e})$ in E^2 is called a **geometric edge** of (G, ι).

By **cyclic ordering** on a finite set S we mean a bijection $s : S \to S$ such that for any $x \in S$ the orbit $\{x, s(x), s^2(x), \dots\}$ is the whole S. For a given $x \in S$, the element $s(x)$ is called the **successor** of x and $s^{-1}(x)$ the **predecessor** of x.

Definition 3.5. Let (G, I) be a graph. For $v \in V$ the **star** of v is the set of edges starting from v: $E_v := \{e \in E : e_- = v\}$. A **ribbon graph** is a graph equipped with a cyclic order on the star of every vertex. A **face** of the ribbon graph is an equivalence class (up to cyclic permutation) of n-tuples $(e_1, \dots, e_n)$ of edges such that $e_p^+ = e_{p+1}^-$ and $s_{e_p^+}(\overline{e}_p) = e_{p+1}$ for all $1 \le p \le n$, where we put $e_{n+1} = e_1$.

Ribbon graphs represent triangulations of oriented surfaces. Consider a graph G as modeled by a 1-dimensional cell complex. The graph G embedded in a surface S is **filling** if each connected component of $S \setminus G$ is homeomorphic to the disc. Then we have:

Proposition 3.6 (See [41])**.** *Every compact oriented surface admits a filling ribbon graph. Conversely, for any ribbon graph G there exists a unique compact oriented surface S_G (up to homeomorphism) such that G can be embedded into S_G as a filling ribbon graph.*

We are going to use the cohomology of a ribbon graph. Let V, E, F denote the sets of vertices, edges and faces of a ribbon graph G. Define the following cochain groups:

$$
\begin{aligned}
C^0(G) &= \{f : V \to \mathbb{R}\} \\
C^1(G) &= \{f : E \to \mathbb{R} : f(\overline{e}) = -f(e)\} \\
C^2(G) &= \{f : F \to \mathbb{R}\}
\end{aligned}
$$

Define the two differentials:

$$
\begin{aligned}
\delta^0 &: C^0(G) \to C^1(G), \quad (\delta^0 f)(e) = f(e_+) - f(e_-) \\
\delta^1 &: C^1(G) \to C^2(G), \quad (\delta^0 \alpha)(f) = \sum_{e \in f} \alpha(e).
\end{aligned}
$$

It is easily verified that $\delta^1 \circ \delta^0 = 0$, therefore we can define the first cohomology group of the ribbon graph as:

$$H^1(G) := \frac{\operatorname{Ker} \delta^1}{\operatorname{Im} \delta^0}.$$

Remark 3.7. As usual, one can define cohomology $H^1(G, A)$ with coefficients in a ring A.

3.3 Several critical points

In order to define *rigid singularities* in the case of several critical points we need to interpret M as the surface associated to a ribbon graph R_M. Each edge $\Gamma_i \subset \Gamma$ of the Reeb graph represents one face of the ribbon graph R_M, the oriented edges are the ones of K, and the vertices are the critical points of f lying inside K.

Definition 3.6. The semi-local singularity defined by f is **rigid** if $H^1(R_M, \mathbb{R}\{t\}) = 0$.

Remark 3.8. One can check that, in the case of a single critical point, this definition agrees with the previously given one.

Now we can generalize Proposition 3.3 to the case of general semi-local singularities:

Theorem 3.4. *Let $f : M \to \mathbb{R}$ be a rigid semi-local singularity (possibly with several critical points). Suppose for each critical point $x_i \in \{f = 0\}$ there exists a real-analytic germ $\varrho_i : U_i \to \mathbb{R}$ defined in a small neighborhood U_i of x_i and such that $\omega_1 - \omega_0 = df \wedge d\varrho_i$ on U_i. If the period maps of the two symplectic forms coincide, i.e., $\Pi_{\omega_0} = \Pi_{\omega_1}$ on $\Gamma \setminus \{0\}$, then there exists a real-analytic function $\widehat{\varrho}$ on M such that $\omega_1 - \omega_0 = df \wedge d\widehat{\varrho}$ on the whole M.*

Proof. We show that if $H^1(R_M, \mathbb{R}\{t\}) = 0$ and $\Pi_{\omega_0} = \Pi_{\omega_1}$ then the obstruction functions can be made vanish by proper adjustments of the local solutions ϱ_i. Each critical point x_i corresponds to a vertex $v \in V(R_M)$. We denote U_v and ϱ_v the corresponding neighborhood and function germ (given by the hypothesis). As in Proposition 3.3, we fix, for each oriented edge $e \in E(R_M)$, a source section $\sigma_e \subset U_{e^-}$ and a target section $\tau_e \subset U_{e^+}$, and we define an obstruction function δ_e as in Equation (3.5). The collection of obstruction functions defines an element $\Delta \in C^1(R_M, \mathbb{R}\{t\})$.

The condition that one period $\Pi : \Gamma_i \to \mathbb{R}$ is zero means that the *sum* of the obstruction functions of the edges belonging to the face of R_M represented by Γ_i are zero. In other words, we can write $\Pi = \delta\Delta$ and we are assuming $0 = \Pi = \delta\Delta$ as an element of $C^2(R_M, \mathbb{R}\{t\})$. Since $\delta\Delta = 0$, and because f is rigid, then $\Delta = \delta g$ for some $g \in C^0(R_M, \mathbb{R}\{t\})$. For each $v \in V(R_M)$, replace ϱ_v with $\widetilde{\varrho}_v = \varrho_v + g_v(f)$, then the obstruction Δ transforms, by definition, to $\widetilde{\Delta} = \Delta - \delta g = 0$. □

4 Smooth global case

Consider a compact surface M foliated by a real-analytic function H having algebraically isolated singularities and let Γ be the Reeb graph of (M, H). Let ω_0, ω_1 be two symplectic forms on M. We consider the original problem for one degree of freedom systems mentioned in the Introduction: assume the period maps of the two symplectic forms coincide, i.e., $\Pi_{\omega_0} = \Pi_{\omega_1}$ on $\Gamma \setminus \{0\}$. Is it true that there exists a H-preserving automorphism ψ of M such that $\psi^*\omega_1 = \omega_0$?

If we require ψ to be only smooth, and not real-analytic, then a solution to the semi-local problem extends to a global solution without obstructions. To each vertex $v \in V(\Gamma)$ there corresponds a critical point $x_v \in M$ of H, and an associated semi-local fibration. Consider two symplectic forms ω_0 and ω_1. Suppose that the semi-local equivalence problem is solved around every singular fiber, i.e., for each $v \in V(\Gamma)$ the two forms are (semi-locally) relatively exact: there exists a real-analytic function ϱ_v in a semi-local neighborhood of the critical point x_v, solving $\omega_1 - \omega_0 = dH \wedge d\varrho_v$.

The functions ϱ_v can be smoothly extended to semi-local neighborhoods U_v that cover the whole M: take smooth cross-sections to the fibration, extend ϱ_v to a smooth function on the section, and then integrate $(\omega_1 - \omega_0)/dH$ along the fibers to extend ϱ_v (like in Proposition 3.3). Again the identity of the periods guarantees that the result is well-defined.

Now choose partition of unity $\{\rho_v : v \in V(\Gamma)\}$ on the graph Γ, such that ρ_v is identically equal to 1 in a neighborhood of v, and $\operatorname{Supp} \rho_v \subset U_v$. Then we see that $\omega_1 - \omega_0 = dH \wedge d\widetilde{\varrho}$ where $\widetilde{\varrho} = \sum_{v \in V(\Gamma)} \rho_v(H)\varrho_v$, and the Moser path method can be applied as in the semi-local case (Proposition 3.5).

Putting together the results of the above sections, we obtain sufficient conditions for the affine Reeb graph to determine uniquely the symplectic form modulo H-preserving smooth isotopy. We assume that every singular fiber contains just one critical point. Remember that the equality of semi-local periods imply, after restricting to a local neighborhood of the critical point, the real-analyticity of partial periods over some set of local relative cycles (see Figure 2.1b). We have:

Theorem 3.5. *Let M be a real-analytic surface and $H : M \to \mathbb{R}$ be a real-analytic Hamiltonian function. Let Γ be the corresponding Reeb graph. Let $\omega_0, \omega_1 \in A^2(M)$ be two symplectic forms. Assume that:*

i) Each singular fiber of H contains one algebraically-isolated critical point,

ii) Each singularity of H is good with respect to the relative cycles obtained by restricting semi-local cycles to a neighborhood of the critical point,

iii) Each singular fiber is rigid.

If the period maps of the two symplectic forms coincide, i.e., $\Pi_{\omega_0} = \Pi_{\omega_1}$ on $\Gamma \setminus \{0\}$, then there exists an H-preserving smooth diffeomorphism $\psi : M \to M$ such that $\psi^*\omega_1 = \omega_0$.

Proof. Let $H^{-1}(c)$ be a critical fiber with critical point x_0. Choose real-analytic source and target sections in a neighborhood U of x_0 like in Proposition 3.3. Each period is then split into an integral inside U (i.e., over some real relative cycle $\delta_i(t)$), plus an integral which exits U, which is a real-analytic function of t. There is one relative cycle for each edge of the Reeb graph incident to the vertex $c \in \Gamma$ representing the critical fiber $H^{-1}(c)$, i.e., there are $\deg(c)$ relative cycles. The condition $\Pi_{\omega_0}(t) = \Pi_{\omega_1}(t)$ for t in a neighborhood of $c \in \Gamma$, then implies that $\omega_1 - \omega_0$ has real-analytic periods over the relative cycles $\delta_i(t), i = 1, \ldots, \deg(c)$, which by assumption form a good set of relative cycles. This means that $\omega_1 - \omega_0$ is relatively exact locally at each critical point. Since each semi-local singularity is rigid then, by the results of Section 3, we can extend local solutions to semi-local solutions, and finally, as explained above, to a smooth global solution. □

As a particular example, we find:

Corollary 3.3. *Consider an integrable system whose singularities are either non-degenerate or of the form $y^p - x^q$, with p, q different primes. Assume each singular fiber only has one critical point. Then a symplectic form is uniquely determined (modulo smooth fiber-preserving isotopy) by its period map.*

Proof. These singularities are good for any choice of relative cycles (see Table 3.1), and their semi-local singularity is an annulus, and thus rigid. So in this case the conditions of the theorem are satisfied. □

Chapter 4

Asymptotics of period integrals

In this chapter we study the asymptotic behavior of actions and periods, and we show how this asymptotic behavior can give information about the singularity.

1 Real quasi-homogeneous case

Consider a real quasi-homogeneous polynomial singularity $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ of weights $(w_1, w_2) \in \mathbb{Q}_+^2$, i.e., such that $f(t^{w_1}x, t^{w_2}y) = tf$. Let $\delta(t)$ denote a family of connected fibers of $\{f = t\}$ with $t > 0$, having its end-points in two real-analytic transversal sections $\nu(t)$ and $\nu'(t)$. We are interested in the behavior for $t \to 0^+$ of the partial periods

$$J_\delta^\omega(t) = \int_{\delta(t)} \frac{\omega}{df}, \qquad t > 0,$$

(with ω not necessarily symplectic).

Proposition 4.1. *Let $\omega = x^i y^j \, dx \wedge dy$, then*

$$J_\delta^\omega(t) = \begin{cases} Ct^{r(i,j)} + \mathbb{R}\{t\} & \text{if } r(i,j) \notin \mathbb{N} \\ Ct^{r(i,j)} \log t + \mathbb{R}\{t\} & \text{if } r(i,j) \in \mathbb{N} \end{cases}$$

where $r(i,j) := \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} i+1 \\ j+1 \end{smallmatrix}\right) - 1$ and $C \in \mathbb{R}$.

Recall that $\mathbb{R}\{t\}$ denotes real-analytic germs at zero. Before giving the proof we need two lemmas. The first one is the real version of Lemma 3.2 of Chapter 2:

Lemma 4.1. *Let $\alpha \in \Omega^1(\mathbb{R}^2, 0)$, then $\frac{d}{dt} \int_{\delta(t)} \alpha = \int_{\delta(t)} \frac{d\alpha}{df} + \mathbb{R}\{t\}$.*

Proof. Let $t_0 > 0$ be fixed. Let $A(t_0, t)$ be a connected strip $\{t_0 \le f \le t\}$ between ν and ν'. For $|t - t_0|$ small enough we can divide the strip into a number of small rectangles $A_i(t_0, t)$, delimited by small transversal sections $\nu_i(t)$ and $\nu_i'(t)$, and in which $f_x \neq 0$ or $f_y \neq 0$.

Consider one such rectangle $A_i(t_0, t)$ and suppose $f_y \neq 0$ there, then we can use coordinates (x, f). In these coordinates we have:

$$\begin{aligned} \alpha &= p(x,f)dx + q(x,f)df \\ d\alpha &= (p_f(x,f) - q_x(x,f))df \wedge dx \\ \frac{d\alpha}{df} &= (p_f(x,f) - q_x(x,f))dx \end{aligned}$$

Denote $\delta_i(t) := \delta(t) \cap A_i(t_0, t)$ and apply Stokes and Fubini theorems:

$$\begin{aligned} \int_{\delta_i(t)} \alpha - \int_{\delta_i(t_0)} \alpha + \int_{\nu_i} \alpha - \int_{\nu_i'} \alpha &= \int_{A_i(t_0,t)} d\alpha \\ &= \int_{t_0}^{t} \left(\int_{\nu_i(f)}^{\nu_i'(f)} (p_f(x,f) - q_x(x,f))dx \right) df \\ &= \int_{t_0}^{t} \left(\int_{\nu_i(f)}^{\nu_i'(f)} \frac{d\alpha}{df} \right) df \end{aligned}$$

Taking the sum over all rectangles we obtain:

$$\int_{\delta(t)} \alpha - \int_{\delta(t_0)} \alpha + \beta(t_0, t) = \int_{t_0}^{t} \left(\int_{\nu(f)}^{\nu'(f)} \frac{d\alpha}{df} \right) df.$$

Where $\beta(t_0, t) = \int_{t_0}^{t} \nu^* \alpha - \int_{t_0}^{t} (\nu')^* \alpha$ is a real-analytic function of (t_0, t). Taking $\partial/\partial t|_{t_0}$ we obtain

$$\frac{d}{dt}\Big|_{t_0} \int_{\delta(t)} \alpha = \int_{\delta(t_0)} \frac{d\alpha}{df} + b(t_0)$$

where $b(t)dt = -\partial\beta(t_0, t)/\partial t|_{t_0} = (\nu')^*\alpha - \nu^*\alpha$. □

Lemma 4.2. *Let* $\omega_{ij} = x^i y^j \, dx \wedge dy$,

$$\frac{d}{dt} \int_{\delta(t)} \frac{\omega_{ij}}{df} = \frac{r(i,j)}{t} \int_{\delta(t)} \frac{\omega_{ij}}{df} + \frac{1}{t} b(t)$$

where b is real-analytic.

Proof. Define the following 1-forms inside U:

$$\alpha_{ij} = x^i y^j (-w_2 y \, dx + w_1 x \, dy)$$

They satisfy the following equation:

$$\begin{aligned} df \wedge \alpha_{ij} &= x^i y^j (f_x \, dx + f_y \, dy) \wedge (-w_2 y \, dx + w_1 x \, dy) \\ &= x^i y^j f \, dx \wedge dy \\ &= f \omega_{ij} \end{aligned}$$

telling us that:

$$\frac{\omega_{ij}}{df} = \frac{\alpha_{ij}}{f}.$$

Moreover, the following (differential) equation is satisfied:

$$\begin{aligned} d\alpha_{ij} &= (j+1)w_2 x^i y^j \, dx \wedge dy + (i+1)w_1 x^i y^j \, dx \wedge dy \\ &= (r(i,j)+1)\omega_{ij}. \end{aligned}$$

Therefore, using Lemma 1

$$\begin{aligned} \frac{d}{dt}\int_{\delta(t)} \frac{\omega}{df} &= \frac{d}{dt}\left(\frac{1}{t}\int_{\delta(t)} \alpha\right) \\ &= -\frac{1}{t^2}\int_{\delta(t)} \alpha + \frac{1}{t}\left(\int_{\delta(t)} \frac{d\alpha}{df} + b(t)\right) \\ &= \cancel{-\frac{1}{t^2}\int_{\delta(t)}} \alpha + \frac{r(i,j)+\cancel{1}}{t}\int_{\delta(t)} \frac{\alpha}{f} + \frac{1}{t}b(t) \\ &= \frac{r(i,j)}{t}\int_{\delta(t)} \frac{\omega}{df} + \frac{1}{t}b(t). \end{aligned}$$

□

Proof of Proposition 4.1. Put $J(t) := J_\delta^\omega(t)$ and $r := r(i,j)$. By Lemma 2 we have the following differential equation

$$tJ'(t) = rJ(t) + b(t). \tag{4.1}$$

After multiplying by t^{-r-1} this becomes $(t^{-r}J(t))' = t^{-r-1}b(t)$, so that for any r we have

$$J(t) = Ct^r + t^r \int s^{-r-1} b(s) ds$$

for some constant C. Let k be any natural number $> r$. Put $b(s) = \sum_{i=0}^k b_i s^i + s^{k+1}\widetilde{b}(s)$ with $\widetilde{b}(s) = \sum_{i=k+1}^\infty b_i s^{i-(k+1)}$ still convergent in the same interval of b. We can suppose that t lies in the interval of convergence of b and $\widetilde{b}$. We have

$$\begin{aligned} J(t) &= Ct^r + t^r\left(\sum_{i=0}^k b_i \int s^{i-r-1} ds + \int s^{k-r}\widetilde{b}(s) ds\right) \\ &= Ct^r + t^r\left(\sum_{\substack{i=0 \\ i\neq r}}^k b_i \frac{t^{i-r}}{i-r} + b_r \log t + \int \sum_{i=k+1}^\infty b_i s^{i-r-1} ds\right) \end{aligned}$$

Here we mean $b_r = 0$ for $r \notin \mathbb{N}$. Notice that the power series defining $\widetilde{b}$ converges uniformly on the compact sets inside its interval of convergence, and since $k - r > 0$ then also $s^{k-r}\sum_{i=k+1}^\infty b_i s^{i-(k+1)}$ is uniformly convergent. This means that we can interchange sum and integral signs:

$$
\begin{aligned}
J(t) &= Ct^r + t^r \left(\sum_{\substack{i=0 \\ i\neq r}}^{k} b_i \frac{t^{i-r}}{i-r} + b_r \log t + \sum_{i=k+1}^{\infty} b_i \int s^{i-r-1} ds \right) \\
&= Ct^r + \left(\sum_{\substack{i=0 \\ i\neq r}}^{\infty} b_i \frac{t^{i-r}}{i-r} + b_r \log t \right) t^r \\
&= Ct^r + b_r t^r \log t + \sum_{\substack{i=0 \\ i\neq r}}^{\infty} \frac{b_i}{i-r} t^i \qquad (b_r = 0 \text{ if } r \notin \mathbb{N}).
\end{aligned}
$$

Summarizing: if $r \in \mathbb{N}$, then the term Ct^r is analytic, and we find $J(t) = b_r t^r \log t + \mathbb{R}\{t\}$. Otherwise $b_r = 0$ and $J(t) = Ct^r + \mathbb{R}\{t\}$. □

1.1 Criterion for analytically good singularities

The above formulas suggest a possible criterion to check if a singularity is analytically good, involving the independence over $\mathbb{R}[\![t]\!]$ (formal power series) of the asymptotic expansions of basic period integrals. To illustrate this, we consider the example of the cusp $f = y^3 - x^2$ (which will be treated in more detail in Chapter 5). The cusp is quasi-homogeneous with weight $(1/2, 1/3)$. It follows from Corollary 2.1 that any holomorphic 2-form ω can be decomposed, in a sufficiently small neighborhood U of $0 \in \mathbb{C}^2$, as follows

$$
\omega = \alpha(f) dx \wedge dy + \beta(f) y dx \wedge dy + df \wedge d\varrho \tag{4.2}
$$

for some holomorphic germ $\varrho(x, y)$, and unique $\alpha, \beta \in \mathbb{C}\{t\}$. It follows from Proposition 4.1 that a partial period $J_\delta^\omega(t)$ of a real symplectic form ω can be written for $t > 0$ as

$$
\Pi(t) = a(t) t^{-1/6} + b(t) t^{1/6} + c(t), \quad t > 0, \tag{4.3}
$$

where $a, b, c \in \mathbb{R}\{t\}$ and $a(t) = C_0 \alpha(t)$ and $b(t) = C_1 \beta(t)$ for some constants $C_0, C_1 \in \mathbb{R}$. We will show in Chapter 5, Section 4 that the coefficients C_0, C_1 are non-zero. Assume now that ω has real-analytic periods over the family $\delta(t)$: $\Pi(t) \in \mathbb{R}\{t\}$. It is easy to check, by expanding $a(t), b(t), c(t)$ in Equation (4.3) into a power series and working modulo 6, that this implies $a(t) = b(t) = 0$ (i.e., the periods of the trivialization are *independent over* $\mathbb{R}[\![t]\!]$), which in turn implies $\alpha(t) = \beta(t) = 0$, so that $\omega = df \wedge d\varrho$. Therefore f is analytically good.

In general, the partial periods J_δ^ω of a 2-form ω have an asymptotic expansion of the form

$$
J_\delta^\omega(t) \approx \sum_{k=1}^{\infty} \sum_{m=1}^{2} a_{k,m} t^{\lambda_k - 1} (\log t)^{m-1}, \quad a_{k,m} \in \mathbb{R},
$$

where the $\lambda_k' s$ belong to a finite number of rational arithmetic sequences [4] (see also the next section). The above example suggests the following criterion:

Proposition 4.2. *Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a singularity and let $\boldsymbol{\delta} = \{\delta_1, \dots, \delta_r\}$ be a set of relative cycles. Let $\{\Omega_1, \dots, \Omega_\mu\}$ be a trivialization for $f^{\mathbb{C}}$ (Definition 2.3). Consider the vectors $\boldsymbol{J}^i = (J_{\delta_1}^{\Omega_i}, \dots J_{\delta_r}^{\Omega_i})^t$ for $i = 1, \dots, \mu$, where each element is considered as a formal asymptotic expansion. If the vectors $\boldsymbol{J}^i$ (modulo terms in $\mathbb{R}[\![t]\!]^r$) are independent over $\mathbb{R}[\![t]\!]$, then f is analytically good with respect to $\boldsymbol{\delta}$.*

Notice that in the example above the non-vanishing of the coefficients C_0, C_1 is essential. Let's consider another example.

Example 4.1. Let $f(x, y) = y^2 - x^4$, which is quasi-homogeneous of weights $(w_1, w_2) = (1/4, 1/2)$. A trivialization for f is given by the forms $\Omega_1 = dx \wedge dy$, $\Omega_2 = xdx \wedge dy$ and $\Omega_3 = x^2 dx \wedge dy$. It follows from Proposition 4.1 that the corresponding periods have singular part, respectively: $C_1 t^{-1/4}$, $C_2 \log t$ and $C_3 t^{1/4}$. Consider the four relative cycles $\delta_i^{\pm}$ from Figure 2.5. Remember that two possible sets of relative cycles arise as restriction of semi-local periods: $\boldsymbol{\delta} = \{\delta_1^+ + \delta_2^+, \delta_1^-, \delta_2^-\}$ or $\boldsymbol{\delta'} = \{\delta_1^+, \delta_2^+, \delta_1^- + \delta_2^-\}$. We have already observed at the end of Example 3.3 that f is (topologically) good with respect to $\boldsymbol{\delta}$ and not for $\boldsymbol{\delta'}$. This difference is visible also from the analytic point of view: if we consider the period over a positive cycle we find $C_2 = 0$ because of the x-simmetry of Ω_2, while for negative cycles all constants C_i are non-zero. This implies that f is analytically good with respect to $\boldsymbol{\delta}$. However, the *sum* of the periods of Ω_2 over δ_1^- and δ_2^- is zero. Therefore the constant C_2 is zero for all the cycles in $\boldsymbol{\delta'}$.

The explicit computation of period integrals is often difficult. Although for this reason the above criterion is not always applicable, it is important to study the asymptotic expansions of period integrals. In particular, we are interested in the periods of monomial forms (especially for quasi-homogeneous singularities, in view of Proposition 2.1). One way to derive such properties (for the cusp, and in general) when explicit computation is not available, is by looking at the resolution of the singularity, as indicated in the next section.

2 General results using resolution of singularities

Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a real singularity. Let A denote a connected component of $U \setminus f^{-1}(0) \subset \mathbb{R}^2$, for a small neighborhood U of 0. We consider the following function:

$$J_A(t) = \int_{\delta_A(t)} \varphi(x, y) \frac{x^{p_1} y^{p_2} \, dx \wedge dy}{df}$$

where $\varphi \in C_0^\infty(\mathbb{R}^2, 0)$ is such that $\varphi \equiv 1$ near the origin and $\delta_A(t) = \{f = t\} \cap A$. We are interested in the asymptotic expansion of $J_A(t)$ as $t \to 0^+$. For this it is convenient to consider the Mellin transform of J_A: the so-called **zeta function**:

$$Z_A(\lambda) := \mathcal{M}[J_A](\lambda) = \int_0^\infty t^\lambda J_A(t) \, dt = \int_{\mathbb{R}^2} (f(x, y)\chi_A(x, y))^\lambda \varphi(x, y) \, dx \, dy$$

It is clear that $Z_A(\lambda)$ is holomorphic for $\operatorname{Re} \lambda > 0$. As we will show later, this function admits a meromorphic continuation to the whole complex plane, with poles belonging to a finite number of arithmetic sequences. The asymptotic expansion of $J_A(t)$ is determined by the Laurent expansion of (the analytic continuation of) $Z_A(\lambda)$ around its poles, indeed:

Proposition 4.3. *[29, Chapter III, Section 4.5] Let $Z_A(\lambda)$ have poles $-\lambda_1, -\lambda_2, \dots$ with $0 < \lambda_1 < \lambda_2 < \cdots$ and let m_k = multiplicity of the k-th pole, then*

$$J_A(t) \approx \sum_{k=1}^{\infty} \sum_{m=1}^{m_k} a_{k,m} t^{\lambda_k - 1} (\log t)^{m-1}$$

where

$$a_{k,m} = \frac{(-1)^{m-1}}{(m-1)!} \times (\text{coefficient of } (\lambda+\lambda_k)^{-m} \text{ in the Laurent expansion of } Z_A(\lambda) \text{ around } \lambda = -\lambda_k).$$

For this reason we are going to study the poles of the zeta function. A good way to do this is to consider a resolution of the singularity. As we will explain in the next subsection, the resolution allows to reduce the problem to the following basic case:

$$Z(\lambda) = \int_0^{\infty}\int_0^{\infty} X^{k_1\lambda+p_1} Y^{k_2\lambda+p_2} \varphi(X,Y,\mu)\, dX dY.$$

where $\varphi(X,Y,\mu)$ is a C_0^∞ function of (X,Y) and an holomorphic function of $\mu \in \mathbb{C}$, and $k_1, k_2, p_1, p_2 \in \mathbb{N}$. Define the two arithmetic sequences:

$$\mathcal{P}_i(j) := -\frac{p_i + 1 + j}{k_i}, \qquad i = 1,2, \quad j \in \mathbb{N}$$

and let $\mathcal{P}_i := \cup_j \mathcal{P}_i(j) \subset \mathbb{C}$. We have:

Lemma 4.3. *The integral $Z(\lambda)$ admits a meromorphic continuation to $\mathbb{C}$ with poles belonging to $\mathcal{P}_1 \cup \mathcal{P}_2$. A pole λ_0 has multiplicity ≤ 2 if it belongs to $\mathcal{P}_1 \cap \mathcal{P}_2$ and ≤ 1 otherwise. Moreover:*

- *If $\lambda_0 \in \mathcal{P}_1 \setminus \mathcal{P}_2$, with $\lambda_0 = \mathcal{P}_1(\ell_1)$, then*

$$\mathrm{Res}_{\lambda=\lambda_0} Z(\lambda) = K_{Y,\ell_1}(\lambda_0)$$

- *If $\lambda_0 \in \mathcal{P}_2 \setminus \mathcal{P}_1$, with $\lambda_0 = \mathcal{P}_2(\ell_2)$, then*

$$\mathrm{Res}_{\lambda=\lambda_0} Z(\lambda) = K_{X,\ell_2}(\lambda_0)$$

- *If $\lambda_0 \in \mathcal{P}_1 \cap \mathcal{P}_2$, with $\lambda_0 = \mathcal{P}_1(\ell_1) = \mathcal{P}_2(\ell_2)$, then*

$$\mathrm{Res}_{\lambda=\lambda_0} Z(\lambda) = \mathrm{Res}_{\lambda=\lambda_0} \left(K_{Y,\ell_1}(\lambda) + K_{X,\ell_2}(\lambda)\right)$$
$$\lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)^2 Z(\lambda) = \frac{1}{k_1 k_2} \frac{1}{\ell_1! \ell_2!} \frac{\partial^{\ell_1+\ell_2} \varphi(0,0,\lambda_0)}{\partial X^{\ell_1} \partial Y^{\ell_2}}.$$

where

$$K_{X,\ell_2}(\lambda) := \text{an.cont}\left(\frac{1}{k_2}\frac{1}{\ell_2!} \int_0^{\infty} X^{k_1\lambda+p_1} \frac{\partial^{\ell_2}\varphi(X,0,\lambda)}{\partial Y^{\ell_2}}\, dX \right)$$
$$K_{Y,\ell_1}(\lambda) := \text{an.cont}\left(\frac{1}{k_1}\frac{1}{\ell_1!} \int_0^{\infty} Y^{k_2\lambda+p_2} \frac{\partial^{\ell_1}\varphi(0,Y,\lambda)}{\partial X^{\ell_1}}\, dY \right)$$

Here an.cont $\left(\int_0^\infty x^{\lambda k+m}\varphi(x)\,dx\right)$ denotes, for given $k,m\in\mathbb{N}$ and $\varphi\in C_0^\infty(\mathbb{R},0)$, the analytic continuation of the function $\lambda\mapsto\int_0^\infty x^{\lambda k+m}\varphi(x)\,dx$, explicitly given in the half-plane $\mathrm{Re}(\lambda+m+N)>-1$ (for any positive integer N) by the formula:

$$\text{an.cont}\left(\int_0^\infty x^{\lambda k+m}\varphi(x)\,dx\right)=\int_0^1 x^{\lambda k+m}R_N(x)\,dx+\int_1^\infty x^{k\lambda+m}\varphi(x)\,dx+\sum_{j=0}^{N-1}\frac{\varphi^{(j)}(0)}{j!}\frac{1}{k\lambda+m+j+1},$$

where $R_N(x)=\varphi(x)-\sum_{j=0}^{N-1}\varphi^{(j)}(0)x^j/j!$ denotes the Taylor remainder. Notice that this formula coincides with $\int_0^\infty x^{\lambda k+m}\varphi(x)\,dx$ for $\mathrm{Re}(\lambda k+m)>-1$, and therefore gives its analytic continuation. In a similar way, on can define the analytic continuation of integrals over an interval for functions having algebraic singularities at the end points [29, Chapter I, Section 3.8].

Proof. The first two points follow from [4, Lemma 7.2 and Lemma 7.3]. The residue formula in the third case can be proved similarly. □

2.1 Asymptotics and Newton diagram

Let f be a Γ-non-degenerate singularity. Let A denote a connected component of $U\smallsetminus f^{-1}(0)\subset\mathbb{R}^2$, for a small neighborhood U of 0. We assume that $f>0$ on A.

Remark 4.1. The case $A=U\cap\{\pm f>0\}$ is typically considered in the study of *oscillating integrals*. See [56, Proposition 1.4, Part 3], for the non-vanishing of the leading coefficient in the case of ω symplectic, and [13, Section 5.1] for $\omega=x^py^q\,dx\wedge dy$ where p,q are even. In this section we adapt the same arguments to the case of general A.

Let $\{P_0,\dots,P_{m+1}\}$ be the regular cone subdivision defined from the Newton polygon of f, and $\pi:Y\to(\mathbb{R}^2,0)$ be (the real part of) the corresponding toric resolution. We assume for simplicity that the Newton polygon of f intersects both axes. This implies that $\ell(P_0)=\ell(P_{m+1})=0$. For $i=1,\dots,m$ define the arithmetic sequences:

$$\mathcal{P}_i(j):=-\frac{\langle P_i,\mathbf{p}+\mathbf{1}\rangle+j}{\ell(P_i)},\quad j\in\mathbb{N},$$

where $\mathbf{p}=(p_1,p_2)$, $\mathbf{1}=(1,1)$, and put $\mathcal{P}_i:=\cup_j\mathcal{P}_i(j)$.

Together with the pullback of f on the charts $\mathbb{R}^2_{\sigma_i}$, we need to understand the pullback of $\omega=\varphi(x,y)x^{p_1}y^{p_2}\,dx\wedge dy$. However, because Y is not orientable (see Chapter 2, Proposition 2.6) we shall consider ω as a density, $\omega=\varphi(x,y)x^{p_1}y^{p_2}\,dxdy$.

Lemma 4.4. *Let* $\pi_{\sigma_i}(x_i,y_i)=(x_i^{a_i}y_i^{a_{i+1}},x_i^{b_i}y_i^{b_{i+1}})$,

i) $\det J_{\pi_{\sigma_i}}(x_i,y_i)=x_i^{a_1+a_2-1}y_i^{b_1+b_2-1}$

ii) $\pi^*_{\sigma_i}(x^{p_1}y^{p_2}\,dxdy)=x_i^{\langle P_i,\mathbf{p}\rangle}y_i^{\langle P_{i+1},\mathbf{p}\rangle}\left|x_i^{a_1+a_2-1}y_i^{b_1+b_2-1}\right|\,dx_idy_i$, *where* $\mathbf{p}=(p_1,p_2)$.

Summarizing, on $\mathbb{R}^2_{\sigma_i}$ we have

$$\begin{cases}\pi^*_{\sigma_i}\omega &=x_i^{\langle P_i,\mathbf{p}\rangle}y_i^{\langle P_{i+1},\mathbf{p}\rangle}\left|x_i^{a_1+a_2-1}y_i^{b_1+b_2-1}\right|\varphi(\pi_{\sigma_i}(x_i,y_i))\,dx_idy_i\\ \pi^*_{\sigma_i}f(x_i,y_i) &=x_i^{\ell(P_i)}y_i^{\ell(P_{i+1})}\widetilde{f}_i(x_i,y_i)\end{cases}\tag{4.4}$$

and $\operatorname{sgn} \widetilde{f}_i(0,0) = \operatorname{sgn}\,(t_\Gamma(P_i) \cap t_\Gamma(P_{i+1}))$.

Consider a partition of unity $\{(U_i, \rho_i) : i = 0, \ldots, m+1\}$ on Y covering $\pi^{-1}(0)$ and such that $\operatorname{Supp}\rho_i \subset \mathbb{R}^2_{\sigma_i}$. We can assume that for $|x_i|$ small (resp. $|y_i|$ small) ρ_i depends only on y_i (resp. x_i). Moreover we can assume that $\operatorname{Supp}\rho_i$ contains all zeros of $\widetilde{f}_i(x_i, 0)$ but no zeros of $\widetilde{f}_i(0, y_i)$. Then we can rewrite $Z_A(\lambda)$ as:

$$\begin{aligned}
Z_A(\lambda) &= \int_{\mathbb{R}^2} (f(x,y)\,\chi_A(x,y))^\lambda\,\varphi(x,y)\,x^{p_1}y^{p_2}\,dxdy \\
&= \sum_i \int_{U_i} (f(\pi_{\sigma_i}(x_i,y_i))\,\chi_A(\pi_{\sigma_i}(x_i,y_i)))^\lambda\,\rho_i(x_i,y_i)\,\pi^*_{\sigma_i}\omega \\
&= \sum_i \int_{\mathbb{R}^2} \left(x_i^{\ell(P_i)} y_i^{\ell(P_{i+1})}\widetilde{f}_i(x_i,y_i)\right)_A^\lambda x_i^{\langle P_i,\mathbf{p}\rangle} y_i^{\langle P_{i+1},\mathbf{p}\rangle}\left|x_i^{a_1+a_2-1}y_i^{b_1+b_2-1}\right|\,\rho_i(x_i,y_i)\,\varphi(\pi_{\sigma_i}(x_i,y_i))\,dx_idy_i \\
&= \sum_i \sum_{q=(q_1,q_2)\in Q_i} (-1)^{q_1\langle P_i,\mathbf{p}\rangle+q_2\langle P_{i+1},\mathbf{p}\rangle}\times \\
&\qquad \times \int_{\pi^{-1}(A)\cap q} |x_i|^{\ell(P_i)\lambda+\langle P_i,\mathbf{p+1}\rangle-1}|y_i|^{\ell(P_{i+1})\lambda+\langle P_{i+1},\mathbf{p+1}\rangle-1}|\widetilde{f}_i(x_i,y_i)|^\lambda\,\rho_i(x_i,y_i)\,\varphi(\pi_{\sigma_i}(x_i,y_i))\,dx_idy_i
\end{aligned}$$

where $(F(x_i,y_i))_A := F(x_i,y_i)\cdot\chi_{\pi^{-1}(A)}$ and $Q_i \subset \mathbb{Z}_2^2$ is the set of quadrants of $\mathbb{R}^2_{\sigma_i}$ according to the notation of Proposition 2.7 of Chapter 2. Denote by $Q_i^A \subseteq Q_i$ the quadrants of $\mathbb{R}^2_{\sigma_i}$ having non-empty intersection with $\pi^{-1}(A)$.

Notice that the function $\chi_{\pi^{-1}(A)} = \chi_A \circ \pi$ is the characteristic function of the set $\pi^{-1}(A) \subset Y$ of points which are projected to A. This region depends on the configuration of the Möbius strips in the resolution, as described by Proposition 2.7 of Chapter 2.

Before applying Lemma 4.3 to the integral

$$\int_{\pi^{-1}(A)\cap q} |x_i|^{\ell(P_i)\lambda+\langle P_i,\mathbf{p+1}\rangle-1}|y_i|^{\ell(P_{i+1})\lambda+\langle P_{i+1},\mathbf{p+1}\rangle-1}|\widetilde{f}_i(x_i,y_i)|^\lambda\,\rho_i(x_i,y_i)\,\varphi(\pi_{\sigma_i}(x_i,y_i))dx_idy_i \tag{4.5}$$

we must take into account the presence of the term $|\widetilde{f}_i(x_i,y_i)|^\lambda$ and the zeros of $\widetilde{f}_i$. If $\widetilde{f}_i(x_i,0)$ has no zeros for $x_i \in \mathbb{R}$, then (after shrinking U if necessary) Lemma 4.3 applies and shows that the integral (4.5), as a function of $\lambda \in \mathbb{C}$, admits a meromorphic continuation to $\mathbb{C}$ with poles in $\mathcal{P}_i \cup \mathcal{P}_{i+1}$. The poles have order ≤ 2 if they belong to $\mathcal{P}_i \cap \mathcal{P}_{i+1}$ and ≤ 1 otherwise.

Consider in particular the contribution to the residue of Z_A at $\lambda_0 = \mathcal{P}_{i+1}(0)$ coming from this integral. According to Lemma 4.3 and Subsection 2.2, this residue is given by:

$$C_{q,i} = \frac{1}{\ell(P_{i+1})}\int_{\mathbb{R}^\pm} |x_i|^{\ell(P_i)\lambda_0+\langle P_i,\mathbf{p+1}\rangle-1}|\widetilde{f}_i(x_i,0)|^{\lambda_0}\rho_i(x_i,0)dx_i \tag{4.6}$$

where $\pm = (-1)^{q_1}$ and the integral is intended as a regularized integral. Notice that the integral (4.6) is convergent if and only if $\lambda_0 = \mathcal{P}_{i+1}(0) > \mathcal{P}_i(0)$.

If instead the closure of $\pi^{-1}(A)$ contains a zero z_i of $\widetilde{f}(x_i,0)$, then by a local change of variables we can assume that $\widetilde{f}(x_i,y_i) = x_i$ in a neighborhood of z_i. By the use of a second partition of unity we are reduced to an integral of the form

$$\int_q |x_i|^\lambda|y_i|^{\ell(P_{i+1})\lambda+\langle P_{i+1},\mathbf{p+1}\rangle-1}\,\widetilde{\varphi}_i(x_i,y_i)dx_idy_i \tag{4.7}$$

where q is a new quadrant, relative to the new local coordinates. By Lemma 4.3 this integral admits a meromorphic continuation with poles in $\mathcal{P}_{i+1} \cup (-\mathbb{N})$. In general, we see that chart $\mathbb{R}^2_{\sigma_i}$ contributes to the poles in $\mathcal{P}_i \cup \mathcal{P}_{i+1} \cup (-\mathbb{N})$.

The above formula for the residue can be used in this second case as well. Similarly to the previous case, the integral formula for the residue will be convergent if and only if $\lambda_0 = \mathcal{P}_{i+1}(0) > -1$. Let (α, β) denote the interval obtained as the intersection of the closure of $\pi^{-1}(A) \cap q$ with the half-line $\mathbb{R}_\pm$, without the end-points. The end-points α, β can be zeros of $\widetilde{f}_i(x_i, 0)$, but also $0, \pm\infty$. The residue of $\lambda_0 = \mathcal{P}_{i+1}(0)$ is given by:

$$C_{q,i} = \frac{1}{\ell(P_{i+1})} \int_{(\alpha,\beta)} |x_i|^{\ell(P_i)\lambda_0 + \langle P_i, \mathbf{p}+\mathbf{1}\rangle - 1} |\widetilde{f}_i(x_i, 0)|^{\lambda_0} \rho_i(x_i, 0) dx_i. \tag{4.8}$$

The above results, used in conjunction with Proposition 2.7 of Chapter 2, allow to study the properties of the coefficients in the asymptotic expansion of period integrals over real relative cycles. We limit ourselves to point out some results which follow from the previous discussion, summarizing them in a proposition.

For each divisor E_i, $i = 1, \dots, m$, defined by $\{y_{i-1} = 0\} \subset \mathbb{R}^2_{\sigma_{i-1}}$ and $\{x_i = 0\} \subset \mathbb{R}^2_{\sigma_i}$, put

$$\beta(E_i) := \mathcal{P}_i(0).$$

Proposition 4.4. *Let $A \subset U \cap \{f > 0\}$, let its pre-image $\pi^{-1}(A)$ be described by the path $\gamma_A \subset e_i^\pm e_{i+1}^\pm \dots e_k^\pm$ on the graph D with end-points $\alpha \in e_i^\pm$ and $\beta \in e_k^\pm$, and let I_A be the set of indices appearing in the path γ_A.*

- *The poles of Z_A belong to the set: $\cup_{i \in I_A} \mathcal{P}_i \cup (-\mathbb{N})$ and are of order ≤ 2.*
- *Let $\beta_A(\mathbf{p}) = \max\{\beta(E_i) : i \in I_A\}$. Assume that the divisors which achieve this maximum are non-consecutive and that $\beta_A(\mathbf{p}) > -1$, so that the integral formulas for the residue at $\beta_A(\mathbf{p})$ are convergent. If for all j satisfying $\beta(E_j) = \beta_A(\mathbf{p})$ or $\beta(E_{j+1}) = \beta_A(\mathbf{p})$ we have*
$$q_1 \langle P_j, \mathbf{p}\rangle + q_2 \langle P_{j+1}, \mathbf{p}\rangle \equiv 0 \mod 2, \quad \forall q \in Q_j^A,$$
then $\mathrm{Res}_{\lambda = \beta_A(\mathbf{p})} Z_A(\lambda) \neq 0$. (In particular, this happens if p_1, p_2 are even)
- *Let $\lambda_0 \notin (-\mathbb{N})$, then:*
$$\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^2 Z_A(\lambda) = \sum_{i \in J} \frac{1}{\ell(P_i)\ell(P_{i+1})} \sum_{q \in Q_i^A} \frac{(-1)^{q_1 \ell_i + q_2 \ell'_i} \delta_{i,q}}{\ell_i! \ell'_i!} \frac{\partial^{\ell_i + \ell'_i} |\widetilde{f}_i(x_i, y_i)|^{\lambda_0}}{\partial x_i^{\ell_i} \partial y_i^{\ell'_i}} \Big|_{(0,0)},$$
where $J = \{i : \lambda_0 \in \mathcal{P}_i \cap \mathcal{P}_{i+1}\}$, $\delta_{i,q} = (-1)^{q_1 \langle P_i, \mathbf{p}\rangle + q_2 \langle P_{i+1}, \mathbf{p}\rangle}$ and $\ell_i = \mathcal{P}_i^{-1}(\lambda_0)$, $\ell'_i = \mathcal{P}_{i+1}^{-1}(\lambda_0)$.

2.2 Connection with Gelfand-Leray derivatives

It is interesting to note that the formulas for residues of Lemma 4.3 and therefore also equations (4.6), (4.8), admit a more intrinsic interpretation in terms of Gelfand-Leray residues. Consider again the situation

$$\begin{cases} \omega &= X^{p_1} Y^{p_2} \varphi(X, Y) dX \wedge dY \\ f &= X^{k_1} Y^{k_2} \end{cases}$$

and denote $a_j(X) = \partial^j\varphi/\partial Y^j(X,0)$. In the region $X > 0$, introduce the function $h = f^{1/k_2} = X^{\frac{k_1}{k_2}}Y$ and the form

$$\begin{aligned}
\widetilde{\omega} &= h^{-p_2}\omega \\
&= X^{p_1 - p_2\frac{k_1}{k_2}}\left(a_0(X) + a_1(X)Y + \frac{1}{2}a_2(X)Y^2 + \dots\right) dX \wedge dY \\
&= X^{p_1 - \frac{k_1}{k_2}(p_2+1)}\left(a_0(X) + a_1(X)X^{-\frac{k_1}{k_2}}h + \frac{1}{2}a_2(X)X^{-2\frac{k_1}{k_2}}h^2 + \dots\right) dX \wedge dh \\
&= \sum_{j=0}^{\infty} X^{p_1 - \frac{k_1}{k_2}(p_2+1+j)}\frac{1}{j!}a_j(X)h^j dX \wedge dh
\end{aligned}$$

From this expression we derive the formula

$$X^{p_1 - \frac{k_1}{k_2}(p_2+j+1)}a_j(X)dX = \frac{d^j}{dh^j}\left(\frac{\widetilde{\omega}}{dh}\right)\Big|_{Y=0}$$

More precisely the form $\widetilde{\omega} = f^{-\frac{p_2}{k_2}}\omega$ extends to a real-analytic form on $Y = 0$, as well as its higher order Gelfand-Leray derivatives with respect to dh. Using $dh = \frac{1}{k_2}f^{\frac{1}{k_2}-1}df$, we can reformulate in terms of the original Gelfand-Leray forms:

$$X^{p_1 - \frac{k_1}{k_2}(p_2+k+1)}a_k(X)dX = D^k\left(g(f)\cdot f^{-\frac{p_2}{k_2}}\frac{\omega}{df}\right), \tag{4.9}$$

where

$$D : \{\text{1-forms}\} \to \{\text{1-forms}\}, \qquad D(\beta) = g(f)\frac{d\beta}{df},\ g(f) = k_2 f^{-\frac{1}{k_2}+1}.$$

Remark 4.2. The above definition in the case $j > 0$ requires some further justification. Indeed, higher order Gelfand-Leray residues are not uniquely defined, even on the $\{h = 0\}$ level set [29]. The formulas above correspond therefore to a particular choice of the Gelfand-Leray residue.

Remark 4.3. A less elementary but probably more straightforward way to prove the above results, at the expense of working with orbifolds, is to first apply Mumford semi-stable reduction (see e.g. [58, Section 4.2]).

3 Recognizing non-degenerate singularities

3.1 Quasi-homogeneous case

Let f be a real quasi-homogeneous singularity of weights (w_1, w_2). The expansion of the period of the basic form $\omega = x^i y^j\, dx \wedge dy$ is given by Proposition 4.1.

Proposition 4.5. *Let $f : (\mathbb{R}^2, 0) \to \mathbb{R}$ be a quasi-homogeneous singularity, and let $\omega \in \Omega^2(\mathbb{R}^2, 0)$ be symplectic. Let δ be a positive relative cycle. If $J_\delta^\omega(t) = C\log t + \mathbb{R}\{t\}$ with $C \neq 0$, then the singularity is a non-degenerate saddle.*

Remark 4.4. Since f is quasi-homogeneous its Newton polygon consists of a straight line in the plane (i, j) of equation $w_1 i + w_2 j = 1$. The distance $d(f)$ of the Newton polygon to the origin, i.e., the value s at which the diagonal line $s \mapsto (s, s)$ intersects the Newton polygon, is then given by $s = 1/(w_1 + w_2)$. Therefore we have $r(0, 0) = 1/d(f) - 1$.

Proof of Proposition 4.5. The proof consists of the following simple observations.

- If ω is a *symplectic* form, it must contain $dx \wedge dy$ in its decomposition into basic forms $\omega = \sum_i \psi_i(f)\Omega_i + df \wedge d\varrho$, otherwise it will vanish at 0.
- The value $r = r(0, 0)$ is achieved only for $i = j = 0$, that is for the form $dx \wedge dy$, because it is the strict minimum value taken by $r(i, j)$. Therefore in the asymptotic expansion of J^ω_δ, the term, $Ct^{r(0,0)}$ or $C \log t$, coming from $dx \wedge dy$ cannot be deleted by the other ones (which can however delete each other).
- But if ω has *only* $\log t$ appearing in its expansion, then this term must be the one corresponding to $r(0, 0)$. Since it is a log, we know that $r(0, 0) = 0$, that is to say $w_1 + w_2 = 1$.
- Finally, $r(0, 0) = 0$ implies that f is non-degenerate. Indeed in this case, by the above remark, $d(f) = 1$ which implies that f is non-degenerate. □

3.2 General local case

We have already shown for quasi-homogeneous singularities that if the periods of a symplectic form are logarithmic then the singularity must be non-degenerate. This is true for general singularities:

Proposition 4.6. *Let ω be symplectic and whose partial period $J(t) = \int_{\{f=t\}} \frac{\omega}{df}$ has asymptotic expansion $J(t) \approx C_\pm \log(\pm t) + \text{smooth terms}$, with $C_\pm \neq 0$, for both $t \to 0^+$ and $t \to 0^-$. Then the singularity is non-degenerate.*

Proof. It follows from [56, Proposition 1.4, Part 3 and Proposition 3.1], together with Proposition 4.3, that in a appropriate system of coordinates (x, y) of $(\mathbb{R}^2, 0)$

$$\int_{\{f=t\}} \frac{\omega}{df} \approx t^{\frac{1}{d(f)}-1}(a + b \log t) + \text{higher order terms},$$

with a, b not both zero. Here $d(f)$ is the distance of the Newton polygon with respect to the coordinates (x, y). The hypothesis implies that $1/d(f) - 1 = 0$, therefore $d(f) = 1$ and the singularity is non-degenerate. □

3.3 Semi-local case

We can consider the semi-local situation as well:

Proposition 4.7. *Let (M, ω) be a 2-dimensional symplectic manifold and f an Hamiltonian function. Let $t \mapsto \delta(t) \subset \{f = t\}$ be a connected family of trajectories, and let $P_1, \dots, P_m$ be the singular points of f approached by $\delta(t)$ as $t \to 0^+$. Suppose $P_1, \dots, P_m$ are all of quasi-homogeneous type. If the asymptotic expansion of the period over δ is purely logarithmic, that is $\Pi_\omega(t) = C \log t + \mathbb{R}\{t\}$, then $P_1, \dots, P_m$ are non-degenerate saddles.*

Proof. For each singular point P_k let $r_k(i,j)$ denote its quasi-homogeneous weights (in a suitable system of coordinates). Let $C_{r_k(i,j)}$ denote the coefficient of the term $t^{r_k(i,j)}$ or $t^{r_k(i,j)}\log t$ in the partial period of $x^i y^j dx \wedge dy$ computed on the restriction of δ around P_k. We have

- $-1 < r_k(0,0) \leq 0$ for each i
- $r_k(0,0) = 0$ if and only if P_k is non-degenerate
- $C_{r_k(0,0)} > 0$ since ω is symplectic.

Suppose by absurd that $\widehat{r} := \min_k\{r_k(0,0)\} < 0$. Then in the asymptotic expansion of $\Pi_\omega(t)$ we will have the term $\widetilde{C}t^{\widehat{r}}$ appearing with a positive constant $\widetilde{C}$, and the asymptotic expansion is not purely logarithmic, in contrast with the hypothesis. This implies that all $r_k(0,0) = 0$, and we can conclude as in Proposition 4.5. □

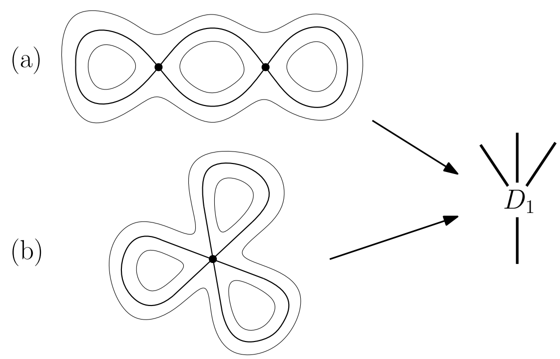


Figure 4.1

Remark 4.5. The above proposition can sometimes be used to obtain information about the topology of the semi-local fibration, given the Reeb graph. Consider for example the Reeb graph Γ of type D_1 (in the classification of [7]). It can correspond to two singular fibrations (see Figure 4.1). If we assume however that the periods on Γ are purely logarithmic, then all the critical points must be non-degenerate saddles, and possibility (b) is excluded.

Part II

Two degrees of freedom. Parabolic orbits and cuspidal tori

Chapter 5

Parabolic orbits and cuspidal tori

In this chapter we discuss normal forms and symplectic invariants of parabolic orbits and cuspidal tori in integrable Hamiltonian systems with two degrees of freedom. Such singularities appear in many integrable systems in geometry and mathematical physics and can be considered as the simplest example of degenerate singularities. We also suggest some new techniques which apparently can be used for studying symplectic invariants of degenerate singularities of more general type. These results have been obtained in a joint work with A. V. Bolsinov and E. A. Kudryavtseva.

1 Introduction

An integrable Hamiltonian system on a symplectic manifold (M^{2n}, Ω) is defined by n pairwise commuting functions $F_1, \dots, F_n$ which are independent on M^{2n} almost everywhere. We will consider the case $n = 2$ and denote such a pair of commuting functions by H and F (H is usually considered as the Hamiltonian and F as an additional first integral). Under the above assumptions, on M^4 we can introduce the structure of a singular Lagrangian fibration whose fibers are, by definition, common level surfaces $\mathcal{L}_{h,f} = \{H = h,\ F = f\}$, $(h, f) \in \mathbb{R}^2$ (or their connected components). We will assume that all the fibers are compact (unless we study local properties of a system). The functions H and F also define a Hamiltonian $\mathbb{R}^2$-action on M^4.

According to Liouville theorem, regular compact connected fibers are 2-dimensional Lagrangian tori of dimension 2 which coincide with orbits of the $\mathbb{R}^2$-action. We say that a fiber $\mathcal{L}_{h,f}$ is singular if it contains a singular point, i.e., a point P such that $dH(P)$ and $dF(P)$ are linearly dependent. Equivalently, we may say that $\mathcal{L}_{h,f}$ is singular if it contains an orbit of a non-maximal dimension, i.e., 1 or 0. A general problem of the theory of singularities of integrable systems is to describe the topology of singular fibers and their saturated neighborhoods (similarly for singular orbits). Notice that the fact that F and H commute makes this theory rather different as compared to the classical singularity theory for smooth maps.

Saying "describe" we may mean at least three different settings: topological, smooth and symplectic. For instance, saying that two given singularities (points, orbits or fibers) are symplectically equivalent we mean the existence of a fiberwise symplectomorphism between their neighborhoods. In the following, in addition, we will assume that all the objects we are working with are real (or complex) analytic.

In this chapter we discuss just one particular type of singularitites, namely parabolic orbits and cuspidal tori (speaking informally, a cuspidal torus is a compact singular fiber that contains one parabolic orbit and no other singular points).

Recall that typical (non-degenerate) singular orbits in integrable Hamiltonian systems can be of two different types: elliptic and hyperbolic. In integrable systems of two degrees of freedom, we may very often observe a transition from *elliptic* to *hyperbolic* in a smooth one-parameter family of singular orbits. At the very moment of transition, the orbit becomes degenerate and of parabolic type. This scenario is rather natural and *parabolic* can be viewed as the simplest possible type of degenerate singularities.

Another important property of parabolic orbits is their stability under small integrable perturbations [31]. This is one of the reasons why such orbits can be observed in many examples of integrable Hamiltonian systems: Kovalevskaya top [10], other integrable cases in rigid body dynamics including Steklov case, Clebsch case, Goryachev–Chaplygin–Sretenskii case, Zhukovskii case, Rubanovskii case and Manakov top on so(4) [7], as well as systems invariant w.r.t. rotations [38], [39], see also examples discussed in [23], [21]. Unlike non-degenerate singularities, however, in the literature on topology and singularities of integrable systems there are only few papers devoted to degenerate singularites including parabolic ones. We refer, first of all, to the following six — L. Lerman, Ya. Umanskii [44], V. Kalashnikov [37], N. T. Zung [61], H. Dullin, A. Ivanov [21], K. Efstathiou, A. Giacobbe [23] and Y. Colin de Verdière [15] — which we consider to be very important in the context of general classification programme for bifurcations occurring in integrable systems.

It is well known that from the smooth point of view, all parabolic orbits are equivalent, i.e. any two parabolic orbits admit fiberwise diffeomorphic neighborhoods (Lerman-Umanskii [43, 44], Kalashnikov [37]). The same is true for cuspidal tori [23]. The simplest model for a parabolic singularity is as follows.

Consider the direct product of $\mathbb{R}^3$ with coordinates x, y, λ and a circle S^1 parametrized by $\varphi \mod 2\pi$ and two functions on this product $\mathbb{R}^3 \times S^1$:

$$H = x^2 + y^3 + \lambda y \quad \text{and} \quad F = \lambda. \tag{5.1}$$

They commute with respect to the symplectic form

$$\Omega = dx \wedge dy + d\lambda \wedge d\varphi. \tag{5.2}$$

The curve $\gamma_0(t) = (0, 0, 0, t)$ is a parabolic orbit of an integrable Hamiltonian system defined by commuting functions H and F. However, in general, we cannot assume that these coordinates x, y, λ, φ are canonical (in other words, the formula for Ω could be different).

Our starting point is the following question. We know that elliptic and hyperbolic orbits have no (non-trivial) symplectic invariants [48]. In other words, for any elliptic or hyperbolic (with orientable or non-orientable separatrix diagram) orbit there exists a symplectic canonical form, one and the same for all orbits of a given type (see, e.g., [6]). Is the same true for parabolic orbits or they admit non-trivial symplectic invariants?

It appears that non-trivial symplectic invariants do exist (a very simple invariant is given by Proposition 5.10). Moreover, we show that all symplectic invariants of parabolic orbits can be expressed in terms of action variables (Theorem 5.4). The next natural step would be to extend a fiberwise symplectomorphism between tubular neighborhoods of two parabolic orbits to saturated neighborhoods of the cuspidal tori that contain these orbits. This is done in Section 6: Theorems 5.5 and 5.6 (see also Remark 5.7) give necessary and sufficient conditions for symplectic equivalence of cuspidal tori. The latter theorems basically say the only symplectic semi-local invariant of a cuspidal torus is the canonical integer affine structure on the base of the corresponding singular Lagrangian fibration. In other words, cuspidal tori satisfy the following principle formulated in [8]:

Let $\varphi : M \to B$ and $\varphi' : M' \to B'$ be two singular Lagrangian fibrations. If B and B' are affinely equivalent (as stratified manifolds with singular integer affine structures), then these Lagrangian fibrations are fiberwise symplectomorphic.

Also we would like to notice that although parabolic singularities are rather simple and specific, some techniques developed and used in this chapter are quite general and can be used for analysis of more complicated singularities. They also can be generalised to the case of many degrees of freedom.

2 Definition of parabolic singularities. Canonical form with no symplectic structure

We begin with the definition of parabolic orbits following [23]. Let H and F be a pair of Poisson commuting real-analytic functions on a real-analytic symplectic manifold (M^4, Ω). They define a Hamiltonian $\mathbb{R}^2$-action (perhaps local) on M^4. The dimension of the $\mathbb{R}^2$-orbit through a point $P \in M^4$ coincides with the rank of the differential of the momentum map $\mathcal{F} = (H, F) : M^4 \to \mathbb{R}^2$ at this point and we are interested in one-dimensional orbits. In the following we will assume, without loss of generality, that $dF(P) \neq 0$. Consider the restriction of H onto the three-dimensional level set of F through P, that is, $H_0 := H|_{\{F=F(P)\}}$. We assume that the rank of $d\mathcal{F}$ at the point P equals one. This is equivalent to any of the following:

- P is a critical point of H_0;
- there exists a unique $k \in \mathbb{R}$ such that $dH(P) = kdF(P)$, in particular, P is a critical point of $F - kH$.

These properties hold true for each singular point P of rank one of the momentum mapping $\mathcal{F} = (H, F)$ under the condition that $dF(P) \neq 0$.

Definition 5.1. A point P (and the corresponding $\mathbb{R}^2$-orbit through this point) is called **parabolic** if the following conditions hold:

i) the quadratic differential $d^2H_0(P)$ has rank 1;
ii) there exists a vector $v \in \operatorname{Ker} d^2H_0(P)$ such that $v^3H_0 \neq 0$ (by v^3H_0 we mean the third derivative of H_0 along the tangent vector v at P);
iii) the quadratic differential $d^2(H - kF)(P)$ has rank 3, where k is the real number determined by the condition $dH(P) = kdF(P)$.

Remark 5.1. In this definition, we use the third derivative of a function along a tangent vector which, in general, is not well defined. In our special case, however, this derivative makes sense as $dH_0(P) = 0$ and $v \in \operatorname{Ker} d^2 H_0(P)$. These two properties allows us to define it as follows:

$$v^3(H_0) = \frac{d^3}{dt^3}|_{t=0} H_0(\gamma(t)),$$

where $\gamma(t)$ is an arbitrary curve on the hypersurface $\{F = F(P)\}$ such that $\gamma(0) = P$, $\frac{d\gamma}{dt}(0) = v$. The result does not depend on the choice of $\gamma(t)$. Indeed,

$$\frac{d^3}{dt^3} H_0(\gamma(t)) = d^3 H_0(\gamma', \gamma', \gamma') + 3d^2 H_0(\gamma', \gamma'') + dH_0(\gamma''') = d^3 H_0(\gamma', \gamma', \gamma') = d^3 H_0(v, v, v),$$

as $dH_0 = 0$ and $\gamma' \in \operatorname{Ker} d^2 H_0$. This computation also shows that the third differential $d^3 H_0$ is a well-defined cubic form on $\operatorname{Ker} d^2 H_0$ so that condition (ii) is equivalent to the fact that the third differential $d^3 H_0$ does not vanish on $\operatorname{Ker} d^2 H_0$ (at the point P).

Remark 5.2. It can be checked that in Definition 5.1, we may replace H and F by any other independent functions $\widetilde{H} = \widetilde{H}(H, F)$, $\widetilde{F} = \widetilde{F}(H, F)$ such that $d\widetilde{F}(P) \neq 0$. In other words, the property of *being parabolic* refers to a singularity of the momentum mapping $\mathcal{F} : M^4 \to \mathbb{R}^2$ and does not depend on the choice of local coordinates in a neighborhood of $\mathcal{F}(P) \in \mathbb{R}^2$. Necessary details can be found in Appendix, see Proposition 5.14.

The following statement describes the structure of the singular Lagrangian fibration in a neighborhood of a parabolic point P. As we are mostly interested in this fibration (rather than specific commuting functions H and F), we allow ourselves to replace H with $\widetilde{H} = \widetilde{H}(H, F)$ where $\frac{\partial \widetilde{H}}{\partial H} \neq 0$ and to shift and change the sign of F, so that $\widetilde{H}$ and $\widetilde{F} = \pm F + \text{const}$ still commute and define the same Lagrangian fibration as H and F. Notice that according to Remark 5.2, P is parabolic for $\widetilde{H}$ and $\widetilde{F}$.

Proposition 5.1. *Locally in a neighborhood of P there exist a transformation*

$$\begin{aligned} &\widetilde{H} = \widetilde{H}(H, F), \quad \textit{with } \frac{\partial \widetilde{H}}{\partial H} \neq 0, \\ &\widetilde{F} = \pm F + \text{const}, \end{aligned} \tag{5.3}$$

and a local coordinate system x, y, λ, φ such that $(x, y, \lambda, \varphi)|_P = (0, 0, 0, 0)$ and

$$\widetilde{H} = \widetilde{H}(H, F) = x^2 + y^3 + \lambda y \quad \textit{and} \quad \widetilde{F} = \lambda. \tag{5.4}$$

Remark 5.3. We do not require that this coordinate system is canonical and, in this view, Proposition 5.1 describes a normal form of a parabolic singularity in the sense of Singularity Theory with no symplectic structure involved. This statement is local and we do not need to assume that the orbit through P is closed. Later on, the variable φ will be one of the angle variables defined modulo 2π, but here φ just belongs to a certain interval.

Proof. The proof of this statement if well known but we still want to briefly explain some of its steps to reveal important underlying phenomena. The first step is to find x, y, λ, φ without touching H and F.

Lemma 5.1. *Under the above assumptions, there exist local coordinates x, y, λ, φ such that $(x, y, \lambda, \varphi)|_P = (0, 0, F(P), 0)$ and*

$$H = \pm(x^2 + y^3 + b(\lambda)y + a(\lambda)), \quad F = \lambda, \tag{5.5}$$

where $a(\lambda)$ and $b(\lambda)$ are real-analytic functions with $b(F(P)) = 0$, $b'(F(P)) \neq 0$.

Proof. Without loss of generality, we assume that $H(P) = F(P) = 0$. First of all we need to kill one dimension using the fact that H and F Poisson commute. Since $dF(P) \neq 0$ we can choose a canonical coordinate system p_1, q_1, p_2, q_2 such that $F = q_2$. Since H and F commute, we conclude that H does not depend on p_2, i.e., $H = H(p_1, q_1, q_2)$. Thus, p_2 does not play any role, so we may forget about it and continue working with p_1, q_1, q_2.

Let us now think of H as a function of two variables q_1 and p_1 depending on $q_2 = \lambda$ as a parameter. We have $\partial H/\partial p_1|_P = \partial H/\partial q_1|_P = 0$ and, without loss of generality, $\partial^2 H/\partial p_1^2|_P \neq 0$. We are now in a quite standard situation in singularity theory.

By a parametric version of the Morse lemma, the function H can be written as $H = \pm(x^2 + f(q_1, \lambda))$, for some new local variable $x = x(p_1, q_1, \lambda)$ such that $x|_P = 0$ and $\partial x/\partial p_1 \neq 0$. Now, condition (ii) of the definition of a parabolic point is satisfied if and only if the function $f(q_1, 0)$ in one variable q_1 has order 3 at the point $q_1|_P$. Hence, this function can be written as $\hat{y}^3$ for some variable $\hat{y} = \hat{y}(q_1)$ with $\hat{y}(q_1(P)) = 0$.

Now the function $f(q_1, \lambda)$ is a 1-parameter "deformation" of the function $f(q_1, 0) = \hat{y}^3$ with the parameter λ. It follows from [3, Sec. 8.2, Theorem or Example] that the deformation $\hat{y}^3 + \lambda_2\hat{y} + \lambda_1$ is right-infinitesimally versal. By the versality theorem [3, Sec. 8.3], it is right-versal (for a definition of a versal deformation, see [3, Sec. 8.1]). Since any deformation is right-equivalent to a deformation induced from the right-versal one, we have $f(q_1, \lambda) = y^3 + b(\lambda)y + a(\lambda)$ for some real-analytic functions $y = y(\hat{y}, \lambda)$, $a(\lambda)$ and $b(\lambda)$ such that $y(\hat{y}, 0) = \hat{y}$, $a(0) = b(0) = 0$. Since, by assumption, the quadratic differential $d^2(H - kF)(P)$ has rank 3, we have $b'(0) \neq 0$. So, we obtain the representation (5.5). □

Later on we will need to rearrange leaves of our singular Lagrangian fibration by using some transformations of the form (the fibration itself remains unchanged)

$$H \mapsto \widetilde{H} = \widetilde{H}(H, F), \ F \mapsto \widetilde{F} = \widetilde{F}(H, F). \tag{5.6}$$

So we need to understand if such a transformation (acting on the base of the Lagrangian fibration) can be realised by a fiberwise analytic diffeomorphism upstairs. In other words, we want to know which of transformations (5.6) are liftable.

Let us look at the (local) bifurcation diagram (i.e. the set of critical values) of the map defined by H and F from (5.5). This bifurcation diagram is as follows, for a + sign in (5.5):

$$\Sigma = \left\{ \left(H - a(F)\right)^2 = -\frac{4}{27} b(F)^3 \right\} \subset \mathbb{R}^2(H, F),$$

and it has a cusp at the point $(H(P), F(P))$ that splits Σ into two smooth branches, Σ_{ell} and Σ_{hyp}, corresponding to one-parameter families of elliptic and hyperbolic orbits. The bifurcation diagram for $a(\lambda) = 0$ and $b(\lambda) = \lambda$ is shown on Figure 5.3. Notice that our choice of the $+$ sign in (5.5) simply means that the monotone function $H - kF|_{\Sigma}$ increases w.r.t. the orientation of Σ from Σ_{ell} to Σ_{hyp}.

It can be easily seen (see the proof of Proposition 5.2 below) that this bifurcation diagram allows us to reconstruct both functions $a(\lambda)$ and $b(\lambda)$. We will use this observation to prove the following

Proposition 5.2. *Assume we have two parabolic singularities defined by functions H, F at a point P and $\widetilde{H}, \widetilde{F}$ at a point $\widetilde{P}$ respectively. A map (local analytic diffeomorphism)*

$$\varphi : \mathbb{R}^2(H, F) \to \mathbb{R}^2(\widetilde{H}, \widetilde{F})$$

is liftable if and only if φ transforms the bifurcation diagram of (H, F) to that of $(\widetilde{H}, \widetilde{F})$, i.e. $\varphi(\Sigma) = \widetilde{\Sigma}$, together with its partition into elliptic and hyperbolic branches. In other words, the condition $\varphi(\Sigma) = \widetilde{\Sigma}$ is necessary and sufficient for the existence of a local analytic diffeomorphism Φ such that the diagram

$$\begin{array}{ccc} M^4 & \xrightarrow{\Phi} & \widetilde{M}^4 \\ \big\downarrow{\scriptstyle (H,F)} & & \big\downarrow{\scriptstyle (\widetilde{H},\widetilde{F})} \\ \mathbb{R}^2 & \xrightarrow{\varphi} & \mathbb{R}^2 \end{array}$$

is commutative.

Proof. The "only if" part is obvious.

Let us prove the "if" part. Denote $\varphi \circ (H, F)$ by (H_1, F_1). Clearly, φ transforms the bifurcation diagram of (H, F) to that of (H_1, F_1), together with their partitions into elliptic and hyperbolic branches. Hence, the bifurcation diagram Σ_1 of (H_1, F_1) coincides with the bifurcation diagram $\widetilde{\Sigma}$ of $(\widetilde{H}, \widetilde{F})$, together with its partition into elliptic and hyperbolic branches.

As shown above, under the condition that $d\widetilde{F}(\widetilde{P}) \neq 0$, the bifurcation diagram $\widetilde{\Sigma}$ of the mapping $\widetilde{\mathcal{F}} = (\widetilde{H}, \widetilde{F}) : \widetilde{M}^4 \to \mathbb{R}^2(h, f)$ is defined by

$$\widetilde{\Sigma} = \{(h, f) \in \mathbb{R}^2 \mid (h - \tilde{a}(f))^2 = -\frac{4}{27}\tilde{b}(f)^3\}$$

for some functions $a(\cdot)$ and $b(\cdot)$ determined by the canonical form (5.5). Hence $\widetilde{\Sigma}$ lies entirely in a half-plane $\{(h, f) \mid \widetilde{b}(f) \leq 0\} \subset \mathbb{R}^2(h, f)$ bounded by a line $\{f = \text{const}\}$ through the cusp point $(\widetilde{H}(\widetilde{P}), \widetilde{F}(\widetilde{P}))$. Since $\Sigma_1 = \widetilde{\Sigma}$, we conclude that $dF_1(P) \neq 0$ as well.

By Lemma 5.1, there exist local (real-analytic) coordinates $x_1, y_1, \lambda_1, \varphi_1$ in a neighborhood U_1 of P and coordinates $\widetilde{x}, \widetilde{y}, \widetilde{\lambda}, \widetilde{\varphi}$ in a neighborhood $\widetilde{U}$ of $\widetilde{P}$ such that

$$\begin{aligned} \eta_1 H_1 &= x_1^2 + y_1^3 + b_1(\lambda_1) y_1 + a_1(\lambda_1), \qquad & F_1 &= \lambda_1, \\ \widetilde{\eta}\widetilde{H} &= \widetilde{x}^2 + \widetilde{y}^3 + \widetilde{b}(\widetilde{\lambda})\widetilde{y} + \widetilde{a}(\widetilde{\lambda}), & \widetilde{F} &= \widetilde{\lambda}, \end{aligned} \tag{5.7}$$

for some signs $\eta_1, \widetilde{\eta} \in \{1, -1\}$. The elliptic and hyperbolic branches of $\widetilde{\Sigma}$ have the form

$$\widetilde{\Sigma}_{\text{ell}} = \left\{ (h, f) = \left(\widetilde{\eta} \left(\widetilde{a}(f) - 2(-\widetilde{b}(f)/3)^{3/2} \right), f \right) \,\Big|\, \widetilde{b}(f) < 0 \right\},$$

$$\widetilde{\Sigma}_{\text{hyp}} = \left\{ (h, f) = \left(\widetilde{\eta} \left(\widetilde{a}(f) + 2(-\widetilde{b}(f)/3)^{3/2} \right), f \right) \,\Big|\, \widetilde{b}(f) < 0 \right\},$$

in particular, $\widetilde{\eta}h - \widetilde{a}(f) > 0$ on the hyperbolic branch of $\widetilde{\Sigma}$ and < 0 on its elliptic branch. Since similar properties and formulae hold for the elliptic and hyperbolic branches $\Sigma_{1,\text{ell}}$ and $\Sigma_{1,\text{hyp}}$ of Σ_1, moreover $\Sigma_{1,\text{ell}} = \widetilde{\Sigma}_{\text{ell}}$ and $\Sigma_{1,\text{hyp}} = \widetilde{\Sigma}_{\text{hyp}}$, we obtain the equalities

$$\eta_1 = \widetilde{\eta}, \quad a_1(\lambda) = \widetilde{a}(\lambda), \quad b_1(\lambda) = \widetilde{b}(\lambda) \tag{5.8}$$

where the equalities of functions hold in a half-neighbourhood $\{\lambda \mid \widetilde{b}(\lambda) \leq 0\}$ of the point $\widetilde{F}(\widetilde{P}) \in \mathbb{R}$. Since all functions are real-analytic at this point, the equalities of functions in (5.8) hold in an entire neighbourhood.

Define a (real-analytic) diffeomorphism germ $\Phi : (U_1, P) \to (\widetilde{U}, \widetilde{P})$ given by the identity map in the local coordinates $(x_1, y_1, \lambda_1, \varphi_1)$ and $(\widetilde{x}, \widetilde{y}, \widetilde{\lambda}, \widetilde{\varphi})$. By (5.7) and (5.8), Φ transforms $(\widetilde{H}, \widetilde{F})$ to (H_1, F_1), so it has the desired property $\varphi \circ (H, F) = (H_1, F_1) = (\widetilde{H}, \widetilde{F}) \circ \Phi$. □

Proposition 5.2 implies the following

Corollary 5.1. *Let P be a parabolic point for an integrable Hamiltonian system with the momentum mapping $\mathcal{F} = (H, F) : M^4 \to \mathbb{R}^2$. Assume that the local bifurcation diagram $\Sigma \subset \mathbb{R}^2(H, F)$ of $\mathcal{F}$ takes the standard form*

$$\Sigma = \left\{ H^2 = -\frac{4}{27} F^3 \right\} \quad \textit{with} \quad \Sigma_{\text{ell}} = \Sigma \cap \{H < 0\}, \ \Sigma_{\text{hyp}} = \Sigma \cap \{H > 0\}. \tag{5.9}$$

Then in a neighborhood of a parabolic point there exists a local coordinate system (x, y, λ, φ) in which $H = x^2 + y^3 + \lambda y$ and $F = \lambda$.

Proof. It is sufficient to notice that the pair of functions $\widetilde{H} = x^2 + y^3 + \lambda y$, $\widetilde{F} = \lambda$ define a parabolic singular point with the standard bifurcation diagram (5.9). According to Proposition 5.2 any other parabolic singularity with the same bifurcation diagram is fiberwise diffeomorphic to this simplest model, moreover, the map $\varphi : \mathbb{R}^2(H, F) \to \mathbb{R}^2(\widetilde{H}, \widetilde{F})$ between the bases is defined by $\widetilde{H} = H$, $\widetilde{F} = F$. □

We are now able to complete the proof of Proposition 5.1. In view of Corollary 5.1, it is sufficient to show that by a suitable transformation (5.3) the bifurcation diagram, together with its partition into elliptic and hyperbolic branches, can be reduced to the standard form (5.9).

As shown above, for the original functions H and F the bifurcation diagram is defined by the equation

$$\Sigma = \left\{ \big(H - a(F)\big)^2 = -\frac{4}{27} b(F)^3 \right\}$$

(here we assume that H in (5.5) comes with $+$).

Let $F(P) = f_0$ so that $b(f_0) = 0$ and $b'(f_0) \neq 0$, then we can represent $b(\lambda)$ as $b(\lambda) = (\lambda - f_0)c(\lambda)$ with $c(f_0) \neq 0$ and rewrite the equation for Σ in the form

$$\left(\frac{H - a(F)}{|c(F)|^{3/2}} \right)^2 = -\eta_F \frac{4}{27} (F - f_0)^3$$

with $\eta_F = c(f_0)/|c(f_0)|$ or, equivalently,

$$\Sigma = \left\{ \widetilde{H}^2 = -\frac{4}{27}\widetilde{F}^3 \right\} \quad \text{with} \quad \Sigma_{\text{ell}} = \Sigma \cap \{\widetilde{H} < 0\},\ \Sigma_{\text{hyp}} = \Sigma \cap \{\widetilde{H} > 0\},$$

for $\widetilde{H} = \dfrac{H - a(F)}{|c(F)|^{3/2}}$ and $\widetilde{F} = \eta_F(F - f_0)$, which coincides with (5.9) as required. □

3 Description of a neighborhood of a parabolic orbit with symplectic structure

Our next goal is to describe the symplectic structure Ω near a parabolic orbit.

An important property of a parabolic orbit is the existence (in real-analytic case) of a free Hamiltonian S^1-action in its tubular neighborhood (N.T. Zung [61], compare Kalashnikov [37]). In other words, without loss of generality we may assume that one of the commuting functions, say F, generates this S^1-action, i.e., the Hamiltonian flow of F is 2π-periodic. From the viewpoint of singularity theory, this means that in our case the parameter of the versal deformation is essentially unique and is given by the Hamiltonian of the S^1-action (or in slightly different terms, by the action variable related to the cycle in the first homology group of fibers that corresponds to this S^1-action). The latter interpretation, in particular, means that one of two action variables is a real-analytic function defined on the whole neighborhood $U(\mathcal{L}_0)$ of $\mathcal{L}_0$ including singular fibers, where $\mathcal{L}_0$ denotes the singular fiber (cuspidal torus) containing the parabolic orbit γ_0. The action variable F is defined up to changing $F \to \pm F + \text{const}$, and we can (and will) choose F in such a way that $F(P) = 0$ and the bifurcation diagram Σ is located in the domain $\{F \le 0\}$.

Basically, what we want to do next is to reduce our Hamiltonian system w.r.t. this action. We shall think of F as a parameter and denote it by λ as above. In particular, now we can choose a coordinate system x, y, λ, φ in a tubular neighborhood $U(\gamma_0)$ of γ_0 in such a way that the Hamiltonian vector field of λ is $\frac{\partial}{\partial\varphi}$. Since H commutes with $F = \lambda$, we conclude that $H = H(x, y, \lambda)$ and we are in the situation discussed in the previous section. If we are only interested in the symplectic topology of the fibration, we are free in the choice of H (in contrast to F which is essentially unique), so according to Proposition 5.1 we may assume without loss of generality that $H = x^2 + y^3 + \lambda y$. However, these coordinates are not canonical, so that (in the tubular neighborhood $U(\gamma_0)$) the symplectic structure takes the following form (here we take into account the condition that Ω is closed and the Hamiltonian vector field of λ is $\frac{\partial}{\partial\varphi}$ or, equivalently, $i_{\partial/\partial\varphi}\Omega = d\lambda$):

$$\begin{aligned} \Omega &= f(x, y, \lambda) dx \wedge dy + d\lambda \wedge d\varphi + d\lambda \wedge \big(P(x, y, \lambda)dx + Q(x, y, \lambda)dy\big) = \\ &= \omega_\lambda + d\lambda \wedge d\varphi + \text{ (additional terms)}. \end{aligned} \tag{5.10}$$

The form $\omega_\lambda = f(x, y, \lambda)dx \wedge dy$ can be considered as the restriction of Ω onto the common level of λ and φ (we assume that $\varphi = 0$ but λ varies and is considered as a parameter). The other interpretation of ω_λ is that it is the one-parameter family of symplectic forms obtained from Ω by the reduction w.r.t. the Hamiltonian S^1-action (or, using old-style terminology, w.r.t. the cyclic variable φ). Here is a more formal statement.

Proposition 5.3. *In a tubular neighborhood of a parabolic orbit γ_0 we can choose a coordinate system x, y, λ, φ (with $\varphi \bmod 2\pi \in \mathbb{R}/2\pi\mathbb{Z}$) such that $(x, y, \lambda)|_{\gamma_0} = (0, 0, 0)$ and our singular Lagrangian fibration is given by two functions*

$$F = \lambda \quad \textit{and} \quad H = x^2 + y^3 + \lambda y$$

and the symplectic form

$$\Omega = f(x, y, \lambda) dx \wedge dy + d\lambda \wedge d\varphi + \text{ (additional terms)}$$

as in (5.10). □

Remark 5.4. Without loss of generality we may assume that $f(x, y, \lambda) > 0$ in (5.10). Indeed, in order for the latter property to be fulfilled, we only need to replace x with $-x$ if necessary. We also notice that since Ω is closed, formula (5.10) can be rewritten as

$$\Omega = dX(x, y, \lambda) \wedge dy + d\lambda \wedge d\widetilde{\varphi}$$

for a certain real-analytic function $X(x, y, \lambda)$ with $\frac{\partial X}{\partial x} > 0$ and $\widetilde{\varphi} = \varphi + R(x, y, \lambda)$ for some real-analytic function $R(x, y, \lambda)$.

It follows from Proposition 5.3 that the function F is uniquely defined (being a generator of the S^1-action), but H is not. However H cannot be chosen arbitrarily because the bifurcation diagram for F and H must be of a very special form, namely (5.9). If this condition is fulfilled then H is allowed and, using Corollary 5.1, we can modify Proposition 5.3 in the following way.

Proposition 5.4. *Consider a tubular neighborhood of a parabolic trajectory. Let H and F be two functions defining our fibration and satisfying the following conditions:*

i) the bifurcation diagram of (H, F) is canonical, i.e., as in (5.9)*;*
ii) F is 2π-periodic, i.e., is a generator of a free Hamiltonian S^1-action.

Then there exists a coordinate system (x, y, λ, φ) as in Proposition 5.3. □

Remark 5.5. It follows from Proposition 5.3 that if we are given two integrable systems with parabolic trajectories, we can always find a fiberwise real-analytic diffeomorphism between their tubular neighborhoods that respects the S^1-actions and corresponding periodic Hamiltonians. This means that without loss of generality we may assume that we are given just one single fibration defined by H and F having canonical form (5.4) with two different symplectic forms given by (5.10) (i.e. such that H and F commute and the Hamiltonian vector field of λ is $\frac{\partial}{\partial \varphi}$):

$$\Omega = \omega_\lambda + d\lambda \wedge d\varphi + \text{ (additional terms)} \tag{5.11}$$

and

$$\widetilde{\Omega} = \tilde{\omega}_\lambda + d\lambda \wedge d\varphi + \text{ (additional terms).} \tag{5.12}$$

We still have two different integrable systems but after the above "pre-identification" they have many common properties. Namely,

i) They have a common local coordinate system (x, y, λ, φ) from Proposition 5.3;

ii) $F = \lambda$ is a 2π-periodic integral for both systems;

iii) The S^1-actions defined by F for Ω and $\widetilde{\Omega}$ coincide (i.e., $X_F = \widetilde{X}_F = \frac{\partial}{\partial\varphi}$ where X_F and $\widetilde{X}_F$ denote the Hamiltonian vector fields generated by F w.r.t. Ω and $\widetilde{\Omega}$ respectively);

iv) The bifurcation diagrams of these two systems coincide;

v) The orientations and coorientations of the parabolic trajectory $\gamma_0(t)=(0,0,0,\varphi=t)$ induced by Ω and $\widetilde{\Omega}$ coincide (see Section 5, Theorem 5.2).

We need to find out whether Ω can be transformed to $\widetilde{\Omega}$ by a suitable fiberwise diffeomorphism Φ. First, we impose a stronger condition on Φ by requiring that Φ preserves not only the fibration but also each particular fiber, i.e. the functions H and F (in other words, rearrangements of fibers are temporarily forbidden, i.e. Φ induces the identity map on the base of the fibration).

The following statement reduces this 4-dim problem for Ω and $\widetilde{\Omega}$ to a similar problem for the reduced forms ω_λ and $\widetilde{\omega}_\lambda$ (in other words, we now reduce our "two-degrees-of-freedom" problem to a parametric "one-degree-of-freedom" problem).

Consider the singular fibration defined by the functions $H = x^2 + y^3 + \lambda y$ and $F = \lambda$. This fibration is obviously Lagrangian w.r.t. any of the symplectic structures (5.11) and (5.12) in a neighborhood of the parabolic orbit $\gamma_0 = \{x = y = \lambda = 0\}$.

Proposition 5.5. *The following two statements are equivalent.*

i) In a tubular neighborhood of the parabolic orbit γ_0 there is a (real-analytic) diffeomorphism Φ such that

- *Φ preserves H and F;*
- *$\Phi^*(\widetilde{\Omega}) = \Omega$.*

ii) There exists a one-parameter family of local diffeomorphisms $\psi_\lambda(x, y)$ (real-analytic in x, y and λ) leaving fixed the origin in $\mathbb{R}^2(x, y)$ at $\lambda = 0$ and such that, for each $\lambda \in \mathbb{R}$ close enough to 0,

- *ψ_λ preserves $H(x, y, \lambda)$;*
- *$\psi_\lambda^*(\widetilde{\omega}_\lambda) = \omega_\lambda$.*

Roughly speaking, this statement says that the additional terms in (5.11) and (5.12) are not important and can be ignored. We also remark that we can replace the conditions that H and F are preserved by saying that the fibration is preserved.

Proof. The fact that (i) implies (ii) is almost obvious. Indeed, since Ω and $\widetilde{\Omega}$ are of quite special form, $\Phi^*(\widetilde{\Omega}) = \Omega$ and $F = \lambda$ is preserved, then in local coordinates x, y, λ, φ, the diffeomorphism Φ takes the following form:

$$\begin{aligned}
\tilde{x} &= \tilde{x}(x, y, \lambda),\\
\tilde{y} &= \tilde{y}(x, y, \lambda),\\
\tilde{\lambda} &= \lambda,\\
\tilde{\varphi} &= \varphi + R(x, y, \lambda),
\end{aligned}$$

then if we consider the first two functions as a family of diffeomorphisms $\psi_\lambda(x, y)$, then we will immediately see that (ii) holds. Since Φ preserves H and F, it leaves invariant the set of such points (x, y, λ, φ) that $dH(x, y, \lambda, \varphi)$ and $dF(x, y, \lambda, \varphi)$ are proportional. But for $\lambda = 0$ this set coincides with γ_0, so Φ maps γ_0 to itself. Therefore $\psi_0(0, 0) = (0, 0)$.

The proof of the converse statement consists of two steps. Assuming that $\psi_\lambda(x, y)$ satisfies the conditions from (ii), we define Φ_1 as follows:

$$(\tilde{x}, \tilde{y}) = \psi_\lambda(x, y),$$
$$\tilde{\lambda} = \lambda,$$
$$\tilde{\varphi} = \varphi.$$

It is easily checked that, for this Φ_1, the symplectic forms $\Phi_1^*(\widetilde{\Omega})$ and Ω coincide up to additional terms, that is

$$\Phi_1^*(\widetilde{\Omega}) - \Omega = d\lambda \wedge \big(P(x, y, \lambda)dx + Q(x, y, \lambda)dy\big). \tag{5.13}$$

Hence, our goal is to show that these additional terms do not play any essential role and can be killed by an appropriate shift $\varphi \mapsto \varphi - R(x, y, \lambda)$ (without changing the other coordinates). In other words, we need to find $R(x, y, \lambda)$ such that $d\lambda \wedge dR(x, y, \lambda) = d\lambda \wedge \big(P(x, y, \lambda)dx + Q(x, y, \lambda)dy\big)$. The existence of such a function follows immediately from the closedness of the form $d\lambda \wedge \big(P(x, y, \lambda)dx + Q(x, y, \lambda)dy\big)$ (this form is the difference of two closed forms $\Phi_1^*(\widetilde{\Omega})$ and Ω). Finally, we define Φ as the composition of Φ_1 and the above shift, and we get $\Phi^*(\widetilde{\Omega}) = \Phi_1^*(\widetilde{\Omega}) - d\lambda \wedge dR(x, y, \lambda) = \Omega$ due to (5.13).

It remains to notice that, since $\psi_0(0, 0) = (0, 0)$ and $\gamma_0 = \{x = y = \lambda = 0\}$, we have $\Phi(\gamma_0) = \gamma_0$, thus Φ is defined in a neighborhood of γ_0 as required. □

Our next observation is that symplectic invariants do exist, in other words, the desired map Φ (or, equivalently, the family ψ_λ) may not exist. Moreover, the existence of just one map ψ_0 implies rather strong condition. To show this, we treat the case $\lambda = 0$ in detail.

4 The case $\lambda = 0$, one-degree of freedom problem

In this Section, for notational convenience, we use a different sign in the definition of H. Consider the function $H = y^3 - x^2$ (in a neighborhood of the origin) and two symplectic forms ω_0 and $\widetilde{\omega}_0$ (all of our objects are real-analytic). We want to know necessary and sufficient conditions for the existence of a local diffeomorphism ψ_0 satisfying $\psi_0^*\widetilde{\omega}_0 = \omega_0$ and (two versions):

- either preserving H (strong condition);
- or preserving the (singular) fibration defined by H (weaker condition) (more formally, $\psi_0^*(H) = h(H)$ where $h(H)$ is real-analytic and $h'(0) \neq 0$).

The complex version of the first problem was studied in [26], in this Section we adapt some of these results to the real case we are considering. In the following, $\mathbb{R}\{H\}$ and $\mathbb{C}\{H\}$ will denote, respectively, real-analytic germs and complex-analytic germs in the variable H at 0, i.e., convergent power series in the respective fields. Consider $H = y^3 - x^2$ as a holomorphic function, we have:

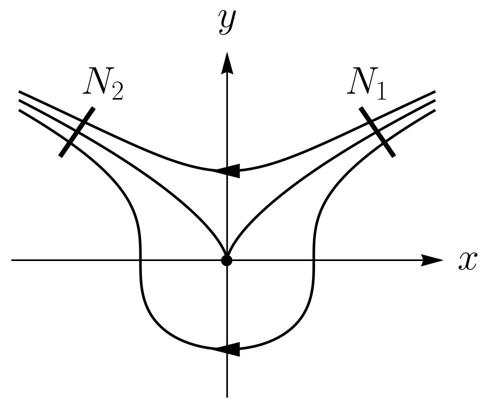


Figure 5.1: Two cross-sections N_1, N_2 to the fibration defined by $H = y^3 - x^2$

Proposition 5.6 ([26, Theorems 2.3 and 3.0])**.** *Any holomorphic 2-form ω_0 can be decomposed, in a sufficiently small neighborhood U of $0 \in \mathbb{C}^2$, as follows*

$$\omega_0 = \alpha(H)dx \wedge dy + \beta(H)ydx \wedge dy + dH \wedge d\eta \tag{5.14}$$

for some holomorphic germ $\eta(x, y)$, and unique $\alpha, \beta \in \mathbb{C}\{H\}$.

Remark 5.6. If ω_0 is symplectic, then $\alpha(0) \neq 0$.

In our case we are dealing with real objects ω_0 and H, in this case $\alpha(H), \beta(H)$ are real-analytic, and $\eta(x, y)$ can be chosen to be real-analytic, as can be shown by taking the real part of Equation (5.14).

Choose two one-dimensional cross-sections N_1, N_2 to the fibration defined by H as shown in Figure 5.1. Each non-singular leaf τ_H of this fibration (with a given value of H) now will be interpreted as a trajectory of the Hamiltonian vector field $X_H = \omega_0^{-1}(dH)$ with respect to the symplectic form ω_0. For each trajectory τ_H we can measure the *passage time* $\Pi(H)$ from N_1 to N_2. This function can be expressed as

$$\Pi(H) = \int_{N_1}^{N_2} \frac{\omega_0}{dH} \tag{5.15}$$

(integral taken along the trajectory τ_H) where ω_0/dH is the *Gelfand-Leray form* associated to the pair (ω_0, H), i.e., any 1-form γ defined in the region $dH \neq 0$ and such that $dH \wedge \gamma = \omega_0$ (the form γ is not uniquely defined, but its restriction to the level-sets $H = \text{const}$ is unique).

We can similarly consider the *area function* $\mathsf{area}(H)$ defined as the integral of ω_0 over the subset of $\{0 \leq H(x, y) \leq H\}$ bounded by the sections N_1, N_2. As a consequence of Fubini's theorem, one has

$$\frac{d\mathsf{area}(H)}{dH} = \Pi(H). \tag{5.16}$$

Clearly, $\Pi(H)$ is a real-analytic function defined for all (small) H. As H tends to 0, the passage time $\Pi(H)$ tends to infinity and it is natural to look at the asymptotic behaviour of $\Pi(H)$ at zero.

Lemma 5.2. *The function* $\Pi(H)$ *for* $H > 0$ *can be written as*

$$\Pi(H) = a(H)H^{-1/6} + b(H)H^{1/6} + c(H), \quad H > 0, \tag{5.17}$$

where $a, b, c \in \mathbb{R}\{H\}$*. Moreover,* $a(H) = C_0\alpha(H)$ *and* $b(H) = C_1\beta(H)$ *for some non-zero constants* $C_0, C_1 \in \mathbb{R}$*, with* $C_0 > 0$ *and* $C_1 < 0$*.*

Before proving the lemma, we give some remarks:

- The functions in this representations are uniquely defined, i.e., if

$$a(H)H^{-1/6} + b(H)H^{1/6} + c(H) = \tilde{a}(H)H^{-1/6} + \tilde{b}(H)H^{1/6} + \tilde{c}(H),$$

  then $a(H) = \tilde{a}(H)$, $b(H) = \tilde{b}(H)$ and $c(H) = \tilde{c}(H)$.
- If we change the sections N_1 and N_2 by a deformation in the class of such sections, then the function $\Pi(H)$ changes by adding a certain analytic function, given by the passage time between the old and the new sections. However, if we replace N_1 and N_2 by each other, then the function $\Pi(H)$ will be replaced by $-\Pi(H)$. This shows that the functions $a(H)$ and $b(H)$ (up to multiplying with -1 simultaneously) do not depend on the choice of the cross-sections N_1 and N_2. Since we are working with a symplectic form, we have $\alpha(0) \neq 0$ and $a(0) \neq 0$, and we can be more specific: the functions $a(H)$ and $b(H)$ with $a(0) > 0$ do not depend on the choice of the cross-sections N_1 and N_2.
- In a similar way, we can define the functions $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ for the second symplectic structure $\widetilde{\omega}_0$. If ψ preserves H and transforms ω_0 to $\widetilde{\omega}_0$, then the Hamiltonian vector field X_H will be transformed to the Hamiltonian vector field $\widetilde{X}_H = \widetilde{\omega}_0^{-1}(dH)$ (with the same Hamiltonian H). Since ψ does not preserve the cross-sections N_1 and N_2, the passage time $\widetilde{\Pi}(H)$ will, in general, differ from $\Pi(H)$ by adding some analytic functions (and, possibly, by multiplying with -1), which shows that the functions $a(H)$ and $b(H)$ with $a(0) > 0$ remain invariant under ψ, i.e. $a(H) = \tilde{a}(H)$, $b(H) = \tilde{b}(H)$, provided that $\tilde{a}(0) > 0$ too. In other words, $a(H)$ and $b(H)$ with $a(0) > 0$ are symplectic invariants (under the condition that ψ preserves H).
- It is easy to give an example of two symplectic structures producing two different pairs of functions a and b in the asymptotic decomposition (5.17).

Proof of Lemma 5.2. Consider the decomposition (5.14). Taking the integral of the Gelfand-Leray form we get:

$$\Pi(H) = \alpha(H)\int_{N_1}^{N_2} \frac{dx \wedge dy}{dH} + \beta(H)\int_{N_1}^{N_2} \frac{ydx \wedge dy}{dH} + N_2^*\eta(H) - N_1^*\eta(H)$$

(the coefficients can be taken outside of the integral, since we integrate along a trajectory τ_H where H is constant). The last two terms give a real-analytic contribution. To finish the proof it is sufficient to show that, for $H > 0$

$$\int_{N_1}^{N_2} \frac{dx \wedge dy}{dH} - C_0 H^{-1/6} \in \mathbb{R}\{H\}, \qquad \int_{N_1}^{N_2} \frac{ydx \wedge dy}{dH} - C_1 H^{1/6} \in \mathbb{R}\{H\}, \tag{5.18}$$

for some non-zero real constants C_0, C_1, so that $a(H) = C_0\alpha(H)$ and $b(H) = C_1\beta(H)$. We can assume that $N_1 = \{x = 1\}$ and $N_2 = \{x = -1\}$. We have

$$\frac{y^j dx \wedge dy}{dH} = -\frac{dx}{3y^{2-j}}, \quad j = 0, 1.$$

Hence, we are reduced to compute, for $j = 0, 1$, the integral:

$$\begin{aligned} J_j(H) &= -\frac{1}{3}\int_1^{-1} y(H,x)^{j-2}\, dx = \frac{2}{3}\int_0^1 (H + x^2)^{\frac{j-2}{3}}\, dx \\ &= \frac{2}{3}H^{\frac{j-2}{3}}\int_0^1 \left(1 + \tfrac{x^2}{H}\right)^{\frac{j-2}{3}}\, dx = \frac{1}{3}H^{\frac{j-2}{3}}\int_0^1 t^{\frac{1}{2}-1}\left(1 + \tfrac{t}{H}\right)^{\frac{j-2}{3}}\, dt \\ &= \frac{2}{3}H^{\frac{j-2}{3}} F\left(\tfrac{2-j}{3}, \tfrac{1}{2}, \tfrac{3}{2}; -\tfrac{1}{H}\right) \end{aligned}$$

where $F(p, q, r; z)$ is the hypergeometric function. In this case we can use the connection formula ([42, Eq. (9.5.9)])

$$\begin{aligned} F(p,q,r;z) = \quad & c_1(-z)^{-p}F(p, 1+p-r, 1+p-q; 1/z) + \\ & + c_2(-z)^{-q}F(q, 1+q-r, 1+q-p; 1/z) \end{aligned}$$

where

$$c_1 = \frac{\Gamma(r)\Gamma(q-p)}{\Gamma(r-p)\Gamma(q)}, \quad c_2 = \frac{\Gamma(r)\Gamma(p-q)}{\Gamma(r-q)\Gamma(p)}.$$

This gives:

$$\begin{aligned} J_j(H) &= \tfrac{2}{3}H^{\frac{j-2}{3}}\left(c_1 H^{\frac{2-j}{3}}F(\tfrac{2-j}{3}, \tfrac{1-2j}{6}, \tfrac{7-2j}{6}; -H) + c_2 H^{1/2}F(\tfrac{1}{2}, 0, \tfrac{5+2j}{6}; -H)\right) \\ &= \tfrac{2}{3}c_1 F(\tfrac{2-j}{3}, \tfrac{1-2j}{6}, \tfrac{7-2j}{6}; -H) + \tfrac{2}{3}c_2 H^{\frac{2j-1}{6}} \\ &= C_j H^{\frac{2j-1}{6}} + d_j(H), \qquad d_j \in \mathbb{R}\{H\}, \end{aligned}$$

where $C_0 = \frac{\sqrt{\pi}}{3}\frac{\Gamma(1/6)}{\Gamma(2/3)}$ and $C_1 = \frac{\sqrt{\pi}}{3}\frac{\Gamma(-1/6)}{\Gamma(1/3)}$. This proves (5.18) as required. □

For $r \in \mathbb{Q}$, consider the operator $\varphi_r : \mathbb{R}\{H\} \to \mathbb{R}\{H\}$ defined by $\varphi_r : A(H) \mapsto A'(H)H + rA(H)$. If $r \notin \mathbb{Z}$ then φ_r is bijective.

Corollary 5.2. *The function* $\mathsf{area}(H)$ *for* $H \geq 0$ *can be written as*

$$\mathsf{area}(H) = A(H)H^{5/6} + B(H)H^{7/6} + C(H), \quad H \geq 0, \tag{5.19}$$

where $A, B, C \in \mathbb{R}\{H\}$ *are the unique real-analytic germs such that*

$$a(H) = A'(H)H + \tfrac{5}{6}A(H), \quad b(H) = B'(H)H + \tfrac{7}{6}B(H), \quad c(H) = C'(H),\ C(0) = 0,$$

in other words $A = \varphi^{-1}_{5/6}(a)$, $B = \varphi^{-1}_{7/6}(b)$. □

Theorem 5.1. *Let* $\omega_0, \widetilde{\omega}_0$ *be two real-analytic symplectic forms. Suppose that* $\omega_0 - \widetilde{\omega}_0 = dH \wedge d\eta$ *for some real-analytic function germ* $\eta(x, y)$ *at* $0 \in \mathbb{R}^2$. *Then there is a local diffeomorphism* ψ *at* $0 \in \mathbb{R}^2$ *such that* $\psi^* H = H$ *and* $\psi^*\widetilde{\omega}_0 = \omega_0$.

Proof. The proof follows [26, Theorem 2.1]. See Theorem 2.4 in Chapter 2. □

In the following we will specify as a subscript the symplectic structure ω_0 in the notation for α, β, a, b and A, B, that is, writing $\alpha_{\omega_0}, \beta_{\omega_0}$, $a_{\omega_0}, b_{\omega_0}$ and $A_{\omega_0}, B_{\omega_0}$. In the rest of the Section, for the reasons explained in the second and third remarks below Lemma 5.2, we will consider symplectic forms inducing a fixed orientation. In this regard we can consider, without loss of generality, only symplectic forms ω satisfying $\alpha_\omega(0) > 0$. Such a symplectic form is said to be *positively-oriented.*

In the above setting and notation we come to the following statement:

Proposition 5.7. *Let $\omega_0, \widetilde{\omega}_0$ be positively-oriented symplectic forms. An H-preserving map ψ such that $\psi^*\widetilde{\omega}_0 = \omega_0$ exists, if and only if the following conditions hold:*

$$\alpha_{\omega_0}(H) = \alpha_{\widetilde{\omega}_0}(H) \text{ and } \beta_{\omega_0}(H) = \beta_{\widetilde{\omega}_0}(H)$$

or, equivalently, $a_{\omega_0}(H) = a_{\widetilde{\omega}_0}(H)$ and $b_{\omega_0}(H) = b_{\widetilde{\omega}_0}(H)$ or $A_{\omega_0}(H) = A_{\widetilde{\omega}_0}(H)$ and $B_{\omega_0}(H) = B_{\widetilde{\omega}_0}(H)$.

Proof. Sufficiency: suppose $\alpha_{\omega_0} = \alpha_{\widetilde{\omega}_0}$ and $\beta_{\omega_0} = \beta_{\widetilde{\omega}_0}$, then $\omega_0 - \widetilde{\omega}_0 = dH \wedge d\eta$ for some real-analytic germ η, and Theorem 5.1 proves the assertion. From Lemma 5.2 and Corollary 5.2 we know that equalities of any of these invariants are equivalent.

Necessity: suppose ψ exists, let us prove that the invariants coincide. Since ψ preserves H and sends $\widetilde{\omega}_0$ to ω_0, we conclude (due to Lemma 5.2 and the third remark below it) that $a_{\omega_0}(H) = a_{\widetilde{\omega}_0}(H)$ and $b_{\omega_0}(H) = b_{\widetilde{\omega}_0}(H)$, which implies $\alpha_{\omega_0}(H) = \alpha_{\widetilde{\omega}_0}(H)$ and $\beta_{\omega_0}(H) = \beta_{\widetilde{\omega}_0}(H)$. □

It follows from the above proposition, together with the first remark below Lemma 5.2, that:

Corollary 5.3. *Let $\omega_0, \widetilde{\omega}_0$ be positively-oriented symplectic forms. An H-preserving map ψ such that $\psi^*\widetilde{\omega}_0 = \omega_0$ exists if and only if $\Pi_{\omega_0}(H) - \Pi_{\widetilde{\omega}_0}(H)$, which is defined on $\{H > 0\}$, extends to a real-analytic function in a neighborhood of $H = 0$.* □

We can also reformulate this result in terms of *normal forms.*

Proposition 5.8. *For $H = y^3 - x^2$ and $\omega_0 = f(x,y)dx \wedge dy$ there is a real-analytic local coordinate system u, v and germs $\alpha, \beta \in \mathbb{R}\{H\}$ such that*

$$H = v^3 - u^2 \quad \text{and} \quad \omega_0 = \alpha(H) \cdot du \wedge dv + \beta(H) \cdot v\, du \wedge dv.$$

For positively-oriented symplectic forms, the functions $\alpha(H)$ and $\beta(H)$ are uniquely defined (the coordinates u, v are not). □

Let us now see what happens if ψ does not preserve H, but transforms it to a function of the form $h(H)$, $h'(0) \neq 0$ (in fact $h'(0) > 0$). Let $\omega_0, \widetilde{\omega}_0$ be positively-oriented symplectic forms. We consider necessary and sufficient conditions for the existence of a local diffeomorphism ψ

such that $\psi^*\widetilde{\omega}_0 = \omega_0$ and $\psi^* H = h(H)$ with $h'(0) > 0$, i.e. a local symplectomorphism ψ making the following diagram commutative:

$$\begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{\psi} & (\mathbb{R}^2, 0) \\ \downarrow H & & \downarrow H \\ (\mathbb{R}, 0) & \xrightarrow{h} & (\mathbb{R}, 0). \end{array}$$

Lemma 5.3. *Suppose there exists ψ such that $\psi^*\widetilde{\omega}_0 = \omega_0$ and $\psi^* H = h(H)$ with $h(H) = H \cdot g(H)$, $g(0) > 0$. Then we have the following relations:*

i)

$$\begin{cases} A_{\omega_0}(H) = g(H)^{5/6} A_{\widetilde{\omega}_0}(h(H)), \\ B_{\omega_0}(H) = g(H)^{7/6} B_{\widetilde{\omega}_0}(h(H)), \end{cases}$$

ii)

$$\begin{cases} \alpha_{\omega_0}(H) = g(H)^{-1/6} \left(g'(H)H + g(H)\right) \alpha_{\widetilde{\omega}_0}(h(H)), \\ \beta_{\omega_0}(H) = g(H)^{1/6} \left(g'(H)H + g(H)\right) \beta_{\widetilde{\omega}_0}(h(H)), \end{cases}$$

iii)

$$\begin{cases} a_{\omega_0}(H) = g(H)^{-1/6} \left(g'(H)H + g(H)\right) a_{\widetilde{\omega}_0}(h(H)), \\ b_{\omega_0}(H) = g(H)^{1/6} \left(g'(H)H + g(H)\right) b_{\widetilde{\omega}_0}(h(H)). \end{cases}$$

For proving Lemma 5.3, we need the following:

Lemma 5.4. *For any real-analytic map $h(H)$ with $h(0) = 0$ and $h'(0) > 0$ there exists ψ (local real-analytic diffeomorphism) such that*

$$H(\psi(x, y)) = h(H(x, y)).$$

In other words, any local diffeomorphism germ $H \mapsto h(H)$ at 0 with $h'(0) > 0$ is liftable.

Proof. Let $h(H) = H \cdot g(H)$. Define $r_h(x, y) := \left(g(H(x, y))^{1/2} x, g(H(x, y))^{1/3} y\right)$, then

$$H(r_h(x, y)) = g(H(x, y))H(x, y) = h(H(x, y)). \qquad \square$$

Proof of Lemma 5.3. (i) The integrals below are taken over subsets bounded by the sections N_1, N_2. For any $H \geq 0$, we have

$$\begin{aligned} \mathsf{area}_{\omega_0}(H) &= \int_{0 \leq H(x,y) \leq H} \omega_0 = \int_{0 \leq H(x,y) \leq H} \psi^*\widetilde{\omega}_0 \\ &= \int_{\psi(0 \leq H(x,y) \leq H)} \widetilde{\omega}_0 + D(H) = \int_{0 \leq H(x',y') \leq h(H)} \widetilde{\omega}_0 + D(H) \\ &= \mathsf{area}_{\widetilde{\omega}_0}(h(H)) + D(H), \qquad D \in \mathbb{R}\{H\}. \end{aligned}$$

Substituting in (5.19) and comparing coefficients gives (i).

(*ii*) Put $\Psi := \psi \circ r_h^{-1}$, then $\Psi^* \widetilde{\omega}_0 = (r_h^{-1})^* \omega_0$ and $H \circ \Psi = H$. Therefore by Proposition 5.7 we have

$$\widetilde{\omega}_0 - (r_h^{-1})^* \omega_0 = dH \wedge d\eta$$

or again

$$r_h^* \widetilde{\omega}_0 - \omega_0 = dH \wedge d\left(\frac{dh(H)}{dH} r_h^* \eta\right).$$

We want to understand the relationship between $\alpha_{\omega_0}, \beta_{\omega_0}$ and $\alpha_{\widetilde{\omega}_0}, \beta_{\widetilde{\omega}_0}$. The Jacobian matrix of the transformation r_h is given by:

$$J_{r_h} = \begin{pmatrix} \frac{1}{2} g(H)^{-1/2} g'(H) \frac{\partial H}{\partial x} x + g(H)^{1/2} & \frac{1}{2} g(H)^{-1/2} g'(H) \frac{\partial H}{\partial y} x \\ \frac{1}{3} g(H)^{-2/3} g'(H) \frac{\partial H}{\partial x} y & \frac{1}{3} g(H)^{-2/3} g'(H) \frac{\partial H}{\partial y} y + g(H)^{1/3} \end{pmatrix}$$

and

$$|J_{r_h}| = \det J_{r_h} = g(H)^{-1/6} \left(g'(H) H + g(H)\right).$$

This means that:

$$r_h^* \widetilde{\omega}_0 = \alpha_{\widetilde{\omega}_0}(h(H)) |J_{r_h}| dx \wedge dy + \beta_{\widetilde{\omega}_0}(h(H)) |J_{r_h}| g(H)^{1/3} y dx \wedge dy + dh(H) \wedge d(r_h^* \widetilde{\eta}).$$

The last term can be written as $dH \wedge d\eta'$ with $\eta' = \frac{dh(H)}{dH} r_h^* \widetilde{\eta}$. By uniqueness of the characteristic series we get the relations.

(*iii*) Follows from (*ii*) and Lemma 5.2. ☐

Let's discuss the opposite statement:

Proposition 5.9. *Consider two positively-oriented symplectic forms $\omega_0, \widetilde{\omega}_0$. Suppose there exists a real-analytic function $h(H) = H \cdot g(H)$ such that $g(0) > 0$ and one of the three relations (i), (ii), (iii) of Lemma 5.3 is satisfied. Then there exists an H-fibration preserving map ψ such that $\psi^* \widetilde{\omega}_0 = \omega_0$.*

Proof. We assume (*ii*) is satisfied, the other two cases are equivalent. Consider the map r_h from the proof of Lemma 5.4. It satisfies $r_h^* H = h(H)$ and (trivially) transforms $\widetilde{\omega}_0$ to $r_h^* \widetilde{\omega}_0$, therefore by Lemma 5.3

$$\alpha_{r_h^* \widetilde{\omega}_0}(H) = g^{-1/6}(H) \left(g'(H) H + g(H)\right) \alpha_{\widetilde{\omega}_0}(h(H)) = \alpha_{\omega_0}(H),$$
$$\beta_{r_h^* \widetilde{\omega}_0}(H) = g^{1/6}(H) \left(g'(H) H + g(H)\right) \beta_{\widetilde{\omega}_0}(h(H)) = \beta_{\omega_0}(H).$$

This shows that ω_0 and $r_h^* \widetilde{\omega}_0$ have the same charateristic series, therefore

$$\omega_0 - r_h^* \widetilde{\omega}_0 = dH \wedge d\eta,$$

for some real-analytic germ η. As we know from the case of H-preserving maps, this equation implies the existence of a H-preserving diffeomorphism φ such that $\varphi^* r_h^* \widetilde{\omega}_0 = \omega_0$. In conclusion, $\psi = r_h \circ \varphi$ is the map we are looking for. ☐

Finally we show how one symplectic invariant survives in the case of H-fibration-preserving maps. Consider a positively-oriented symplectic form ω_0. Suppose we can solve the equation

$$\alpha_{\omega_0}(H) = Kg(H)^{-1/6}\left(g'(H)H + g(H)\right), \qquad K \in \mathbb{R}, \tag{5.20}$$

for $g(H)$. Then the rescaling map r_h, with $h(H) = H \cdot g(H)$, transforms H to $h(H)$ and the symplectic form $\widetilde{\omega}_0 = (r_h^{-1})^*\omega_0$ to ω_0, where $\alpha_{\widetilde{\omega}_0}(H) = K$. After this, the other characteristic series $\beta_{\widetilde{\omega}_0}(H)$ survives as an invariant in the usual sense (of H-preserving local diffeomorphisms).

Lemma 5.5. *The invariant $a_{\omega_0}(H)$ can be reduced to a constant.*

Proof. By assumption we have $a_{\omega_0}(0) > 0$. It follows from Corollary 5.2 that $A_{\omega_0}(0) > 0$ as well. Setting $g(H) = A_{\omega_0}(H)^{6/5}$ and $\widetilde{\omega}_0 = (r_h^{-1})^*\omega_0$, we obtain from Lemma 5.3 that $A_{\widetilde{\omega}_0}(H) = 1$. With this choice of $h(H) = H \cdot g(H)$, we have $A_{\omega_0}(H) = g(H)^{5/6}$, therefore by Corollary 5.2

$$\begin{aligned} a_{\omega_0}(H) &= A'_{\omega_0}(H)H + \tfrac{5}{6}A_{\omega_0}(H) \\ &= \tfrac{5}{6}g(H)^{-1/6}\left(g'(H)H + g(H)\right), \end{aligned}$$

so that $a_{\widetilde{\omega}_0}(H)$ is constant (and $\alpha_{\widetilde{\omega}_0}(H)$ as well). □

Proposition 5.10. *A real-analytic singular Lagrangian fibration with one degree of freedom is symplectomorphic, in a neighborhood of an A_2 singularity, to (one of) the following model:*

$$H = y^3 - x^2, \quad \omega_0 = dx \wedge dy + f(y^3 - x^2) \cdot y\, dx \wedge dy.$$

Or, equivalently, in a neighborhood of an A_2 singularity (with one degree of freedom) we can always find local coordinates x and y such that the fibration is defined by the function $H = y^3 - x^2$ and the symplectic structure takes the form $\omega_0 = dx \wedge dy + f(y^3 - x^2) \cdot y\, dx \wedge dy$.

In this representation, the real-analytic function $f(H)$ is uniquely defined. □

This proposition says that as a complete symplectic invariant of an A_2 singular fibration with one degree of freedom we may consider one (real-analytic) function in one variable. Since such a fibration appears as a symplectic reduction of the Lagrangian fibration near a parabolic orbit (for $\lambda = 0$), we conclude that parabolic orbits possess non-trivial symplectic invariants and the next section is aimed at describing "all of them".

5 Parametric version

Our next step is a parametric version of the above construction. We now assume that H depends on λ as a parameter:

$$H(x, y, \lambda) = H_\lambda(x, y) = x^2 + y^3 + \lambda y$$

and for each value of λ we consider a symplectic structure $\omega_\lambda = f(x, y, \lambda)dx \wedge dy,\ f > 0$.

We first give necessary and sufficient conditions for the existence of a family of maps ψ_λ from Proposition 5.5.

Following the same idea as before, we choose two 2-dimensional sections $\mathcal{N}_1$ and $\mathcal{N}_2$ analogous to the above sections N_1 and N_2 (but now for all values of λ) and define the passage time

$$\Pi(H,\lambda) = \int_{\mathcal{N}_1(H,\lambda)}^{\mathcal{N}_2(H,\lambda)} \frac{\omega_\lambda}{dH_\lambda}$$

for each trajectory with parameters H and λ, $(H,\lambda) \notin \Sigma_{\text{hyp}}$, see Figure 5.2. Also we see that for each $\lambda < 0$ we have a family of closed trajectories also parametrized by H and λ. Let us denote by $\Pi_\circ(H,\lambda)$ the period of these trajectories[1]. We can compute these functions for both forms ω_λ and $\widetilde{\omega}_\lambda$. For $\widetilde{\omega}_\lambda$, we denote them by $\widetilde{\Pi}(H,\lambda)$ and $\widetilde{\Pi}_\circ(H,\lambda)$.

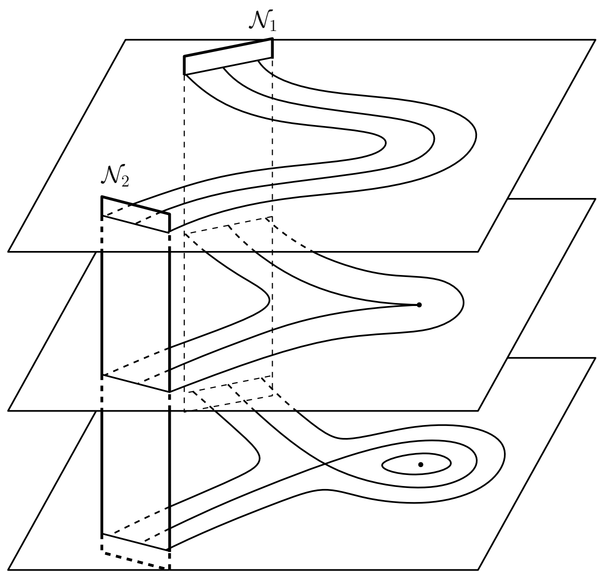


Figure 5.2: Two cross-sections $\mathcal{N}_1, \mathcal{N}_2$ to the fibration near a parabolic orbit

Proposition 5.11. *A family of local diffeomorphisms ψ_λ from Proposition 5.5 exists if and only if*

i) $\Pi(H,\lambda) - \widetilde{\Pi}(H,\lambda)$ *extends to a real-analytic function in a neighborhood of the point* $H = 0, \lambda = 0$,

ii) $\Pi_\circ(H,\lambda) = \widetilde{\Pi}_\circ(H,\lambda)$.

Proof. We need to justify the "if" part only. First of all we notice that, for each λ (if we consider each slice $\{\lambda = \text{const}\}$ separately), a map ψ_λ exists. Indeed, for $\lambda > 0$, there are no obstructions for the existence of ψ_λ at all, since our fibration is regular. For $\lambda = 0$, the existence of ψ_λ was proved in Corollary 5.3. As for $\lambda < 0$, this property follows from non-degeneracy of singular points (see [19]).

[1] Alternatively we may compute the area $\mathsf{area}_\circ(H,\lambda) = 2\pi I_\circ(H,\lambda)$ enclosed by such a trajectory. This function can be understood as the action variable corresponding to this family of closed cycles. Notice that $\Pi_\circ$ and $I_\circ$ are related by differentiation: $\Pi_\circ(H,\lambda) = 2\pi \frac{\partial}{\partial H} I_\circ(H,\lambda)$, comp. (5.16).

We only need to "combine" all these maps into one single $\Psi(x, y, \lambda) = \psi_\lambda(x, y)$ in such a way that Ψ is real-analytic with respect to all variables (including λ).

To that end, we notice first of all that the maps ψ_λ can be chosen in such a way that each section $N_{1,\lambda} = \{(x, y, \lambda) \in \mathcal{N}_1 \text{ with } \lambda \text{ fixed}\}$ (i.e. the intersection of $\mathcal{N}_1$ with the corresponding λ-slice) is mapped to itself, i.e., $\psi_\lambda|_{N_{1,\lambda}} = \mathrm{id}$. This choice (of the initial data) makes our construction unique. In other words, we may assume without loss of generality that Ψ leaves $\mathcal{N}_1$ fixed.

Let σ^t and $\widetilde{\sigma}^t$ denote the Hamiltonian flows of H_λ w.r.t. ω_λ and $\widetilde{\omega}_\lambda$ respectively. Since H is preserved and $\psi_\lambda^*(\widetilde{\omega}_\lambda) = \omega_\lambda$, we conclude that ψ_λ sends the Hamiltonian flow of H w.r.t ω_λ to that w.r.t. $\widetilde{\omega}_\lambda$, i.e., the following relation holds

$$\psi_\lambda \circ \sigma^t = \widetilde{\sigma}^t \circ \psi_\lambda.$$

This relation implies a simple "explicit" formula for ψ_λ (for those points Q which can be obtained from $\mathcal{N}_1$ by shifting along the flow σ^t). Namely, let $Q = \sigma^{t(Q)}(Q_0)$ with $Q_0 \in \mathcal{N}_1$. Then applying the above relation to the point Q with $t = -t(Q)$ we get

$$\psi_\lambda \circ \sigma^{-t(Q)}(Q) = \widetilde{\sigma}^{-t(Q)} \circ \psi_\lambda(Q)$$

or, equivalently,

$$\psi_\lambda(Q) = \widetilde{\sigma}^{t(Q)} \circ \psi_\lambda \circ \sigma^{-t(Q)}(Q)$$

and, using that $\psi_\lambda \circ \sigma^{-t(Q)}(Q) = \psi_\lambda(Q_0) = Q_0 = \sigma^{-t(Q)}(Q)$, we finally get:

$$\psi_\lambda(Q) = \widetilde{\sigma}^{t(Q)} \circ \sigma^{-t(Q)}(Q), \tag{5.21}$$

where the time $t(Q)$ is chosen in such a way that $\sigma^{-t(Q)}(Q) \in \mathcal{N}_1$. Notice that the family ψ_λ so defined automatically satisfies the required conditions (ii) from Proposition 5.5 and is locally analytic w.r.t. all the variables (including the parameter λ) everywhere where it makes sense. The problem, however, is that (5.21) works neither at the singular points nor at the points lying on "small" closed trajectories that appear for $\lambda < 0$ (the reason is obvious: the Hamiltonian flow σ^t starting from $\mathcal{N}_1$ does not reach them).

Below, we will use a slightly different version of formula (5.21). Notice that Q can also be obtained from $Q_0 \in \mathcal{N}_1$ by shifting along the other Hamiltonian flow $\widetilde{\sigma}^t$, that is, $Q = \widetilde{\sigma}^{\tilde{t}(Q)}(Q_0)$ for some $\tilde{t}(Q) \in \mathbb{R}$. Hence,

$$\psi_\lambda(Q) = \widetilde{\sigma}^{t(Q)} \circ \sigma^{-t(Q)}(Q) = \widetilde{\sigma}^{t(Q)-\tilde{t}(Q)} \circ \widetilde{\sigma}^{\tilde{t}(Q)} \circ \sigma^{-t(Q)}(Q) = \widetilde{\sigma}^{t(Q)-\tilde{t}(Q)} \circ \widetilde{\sigma}^{\tilde{t}(Q)}(Q_0) = \widetilde{\sigma}^{t(Q)-\tilde{t}(Q)}(Q),$$

or, finally,

$$\psi_\lambda(Q) = \widetilde{\sigma}^{r(Q)}(Q), \qquad \text{where } r(Q) = t(Q) - \tilde{t}(Q). \tag{5.22}$$

This formula has a very natural meaning. If Q can be obtained by shifting a certain point $Q_0 \in \mathcal{N}_1$ along the flows σ^t and $\widetilde{\sigma}^t$, then ψ_λ simply shifts Q along $\widetilde{\sigma}^t$ by time $r(Q) = t(Q) - \tilde{t}(Q)$, where $t(Q)$ (resp. $\tilde{t}(Q)$) is the time necessary for the flow σ^t (resp. $\tilde{\sigma}^t$) to reach Q starting from the section $\mathcal{N}_1$.

Our goal is to show that this formula extends to a neighborhood of the parabolic point up to a well defined real-analytic map (in the sense of all the variables x, y and λ).

To that end, we will use a "complexification trick". Since all the objects under consideration are real-analytic we can naturally complexify them, that is, we may think of x, y, λ as complex variables, H and F as complex functions, ω and $\widetilde{\omega}$ as complex symplectic forms, etc. We will also assume that the section $\mathcal{N}_1$ is given by an analytic equation like $f(x,y) = 0$, so that the same equation defines a (local) complex hypersurface that is transversal to all complexified leaves $\mathcal{L}_{\varepsilon_1,\varepsilon_2} = \{H = \varepsilon_1, F = \varepsilon_2\}$, $(\varepsilon_1, \varepsilon_2) \subset \mathbb{C}^2$, for small enough $|\varepsilon_1| + |\varepsilon_2|$. We are now looking for a local holomorphic map $\Psi(x, y, \lambda) = \psi_\lambda(x, y)$ which preserves H and F and transforms $\widetilde{\omega}_\lambda$ to ω_λ.

We want this map to be the "complexification" of the family ψ_λ defined above (in particular, the complex section $\mathcal{N}_1$ does not move under the action of ψ_λ). We keep the same notations for all the objects, but now we think of them from the complex viewpoint. In particular, the parameter t for the flows σ^t and $\widetilde{\sigma}^t$ is complex and plays the role of "complex time". Similarly, $t(Q)$, $\tilde{t}(Q)$ and $r(Q)$ are complex functions which, by construction, are locally holomorphic.

One of the advantages of the complexified picture is that all the leaves $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ (both regular and singular) are now connected, each of them intersects the section $\mathcal{N}_1$ at exactly one point and, moreover, every regular point Q (even if it belongs to a singular leaf) can be joint with $\mathcal{N}_1$ by a continuous path lying on the leaf. Notice that the regular part of each leaf $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ can be understood as a complex trajectory of the complex flow σ^t or $\widetilde{\sigma}^t$.

The problem coming with "complexification" is that $t(Q)$ and $\tilde{t}(Q)$ are not uniquely defined anymore. Indeed, the complex leaf $\mathcal{L}_{\varepsilon_1,\varepsilon_2} = \{H = \varepsilon_1, F = \varepsilon_2\}$ is now a two-dimensional surface with a non-trivial topology. In particular, the first homology group of $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ is non-trivial and this leads to the fact that Q can be reached from $\mathcal{N}_1$ in many different ways, e.g., $\sigma^{t_1}(Q_0) = \sigma^{t_2}(Q_0) = Q$. So we need to make sure that the choice of t_i does not affect the final result of (the complex version of) (5.21) and (5.22).

Let us discuss this issue in more detail. Consider one particular leaf $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ (not necessarily regular). It intersects the section $\mathcal{N}_1$ at exactly one point Q_0. For $Q \in \mathcal{L}_{\varepsilon_1,\varepsilon_2}$, consider a path $\gamma(s)$ connecting this point with Q_0 so that $\gamma(0) = Q_0$ and $\gamma(1) = Q$. Each point of this path can be written as $\gamma(s) = \sigma^{t(s)}(Q_0) = \widetilde{\sigma}^{\tilde{t}(s)}(Q_0)$ with $t(s) \in \mathbb{C}$, $t(s)$ continuous and $t(0) = 0$. In this way, we set $t(Q) = t(1)$ and $\tilde{t}(Q) = \tilde{t}(1)$. It is easy to see that deforming $\gamma(s)$ continuously does not change $t(Q)$ and $\tilde{t}(Q)$. Thus, this construction shows that $t(Q)$ and $\tilde{t}(Q)$ (and consequently $r(Q) = t(Q) - \tilde{t}(Q)$) are uniquely defined if we fix the homotopy type of a curve connecting Q_0 and Q. If we choose two homotopically different curves γ_1 and γ_2, then, in general, $t_1(Q) \neq t_2(Q)$ and $\tilde{t}_1(Q) \neq \tilde{t}_2(Q)$.

The condition we need is $r_1(Q) = t_1(Q) - \tilde{t}_1(Q) = t_2(Q) - \tilde{t}_2(Q) = r_2(Q)$ or, equivalently, $t_1(Q) - t_2(Q) = \tilde{t}_1(Q) - \tilde{t}_2(Q)$. The latter has a very simple geometric meaning. Namely, $Q = \sigma^{t_1(Q)}(Q_0) = \sigma^{t_2(Q)}(Q_0)$ means that $\sigma^{t_1(Q)-t_2(Q)}(Q) = Q$. In other words, $t_1(Q) - t_2(Q)$ is the period of $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ as a "complex trajectory" of the flow σ^t, which corresponds to the (homotopy class of the) loop formed by the curves γ_1 and $-\gamma_2$. In other words, the condition that we need can be formulated as follows: for each loop γ on $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$, the corresponding periods of the Hamiltonian flows generated by H_λ w.r.t. the symplectic forms ω_λ and $\widetilde{\omega}_\lambda$ coincide. These periods can be computed explicitly as (compare with (5.15)):

$$\Pi_\gamma(H, \lambda) = \oint_\gamma \frac{\omega_\lambda}{dH_\lambda} \quad \text{and} \quad \widetilde{\Pi}_\gamma(H, \lambda) = \oint_\gamma \frac{\widetilde{\omega}_\lambda}{dH_\lambda},$$

so that the required condition takes the following form:

$$\Pi_\gamma(H,\lambda) = \widetilde{\Pi}_\gamma(H,\lambda) \tag{5.23}$$

for any closed loop γ on $\mathcal{L}_{\varepsilon_1,\varepsilon_2} = \mathcal{L}_{H,\lambda}$ (equivalently, for any cycle of the first homology group).

Let us assume that this condition holds true (we will below explain why, under our assumptions, this is indeed the case) and make the next step of our construction. As just shown, (5.23) guarantees that the function $r(Q)$ is well defined for any point Q that can be reached by the flows σ^t and $\widetilde{\sigma}^t$ starting from the section $\mathcal{N}_1$. Since (after complexification) every regular point satisfies this property, $r(Q)$ is defined everywhere except for singular points and is locally holomorphic by construction. But the set of singular points,

$$\left\{\frac{\partial H}{\partial x} = 0,\ \frac{\partial H}{\partial y} = 0\right\} = \left\{x = 0,\ 3y^2 + \lambda = 0\right\},$$

is an algebraic variety of (complex) codimension 2, and therefore by the second Riemann extension theorem ([34, Theorem 4.4] or [32, Theorem 7.2]), $r(Q)$ can be extended up to a holomorphic function defined everywhere in the considered domain. In particular, this function is bounded and therefore, by taking a smaller neighborhood of the parabolic point, we may assume that the flow σ^t is well defined for all t satisfying $|t| \le \max|r(Q)|$.

After this, our formula (5.22) can be applied to every point from this neighborhood giving a well defined holomorphic map Ψ with required properties. It remains to return to the real world (i.e., restrict ψ_λ to the real part of our complex neighborhood) and we are done[2].

To complete the proof we need to explain why condition (5.23) is fulfilled in our case. First we notice that the first homology group of complex leaves $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ is generated by 2 cycles (topologically, $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$ is a torus with one hole if $(\varepsilon_1,\varepsilon_2) \notin \Sigma^{\mathbb{C}} = \{\varepsilon_1^2 = -\frac{4}{27}\varepsilon_2^3\}$, a 2-disk with one hole if $(\varepsilon_1,\varepsilon_2) = (0,0)$, and a pinched torus with one hole otherwise, where one of the basic cycles is pinched to a point). Consider the (real) "swallow-tail domain" $\{(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^2 \mid \varepsilon_1^2 < -\frac{4}{27}\varepsilon_2^3\} \subset \{\lambda < 0\}$. Then one of these two cycles can be chosen real. Such a cycle is shown in Figure 5.2 as a small loop, whose periods w.r.t. ω_λ and $\widetilde{\omega}_\lambda$ were denoted by $\Pi_\circ(H,\lambda)$ and $\widetilde{\Pi}_\circ(H,\lambda)$. By our assumption $\Pi_\circ(H,\lambda) = \widetilde{\Pi}_\circ(H,\lambda)$, i.e., one of the required conditions coincides with the second condition (ii) of Proposition 5.11.

Now consider condition (i) for $\{(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^2 \mid \varepsilon_1^2 < -\frac{4}{27}\varepsilon_2^3\} \subset \{\lambda < 0\}$. When approaching a hyperbolic singular leaf, the functions $\Pi(H,\lambda)$ and $\widetilde{\Pi}(H,\lambda)$ both have logarithmic singularity. This is a well known property of non-degenerate hyperbolic points ([19, 9]), in other words, they have the following asymptotics[3]:

$$\Pi(H,\lambda) = \alpha(H,\lambda)\ln\left|3\sqrt{3}H - 2(-\lambda)^{3/2}\right| + \beta(H,\lambda),$$
$$\widetilde{\Pi}(H,\lambda) = \widetilde{\alpha}(H,\lambda)\ln\left|3\sqrt{3}H - 2(-\lambda)^{3/2}\right| + \widetilde{\beta}(H,\lambda)$$

[2] A more technical proof of this part can be deduced from [33] (see also [28]). It follows from [33, Propositions 2.3 and 3.1], that if a 3-form β satisfies $[\beta/dH \wedge d\lambda] = 0 \in H^1(\mathcal{L}_{\varepsilon_1,\varepsilon_2},\mathbb{C})$ on every regular fiber $\mathcal{L}_{\varepsilon_1,\varepsilon_2}$, then $\beta = dH \wedge d\lambda \wedge d\eta(x,y,\lambda)$, η defined in a neighborhood of zero. Applying this to $\beta = d\lambda \wedge (\Omega - \widetilde{\Omega})$, with $\Omega, \widetilde{\Omega}$ from (5.11), (5.12), we deduce $\widetilde{\omega}_\lambda - \omega_\lambda = dH \wedge d\eta(x,y,\lambda)$. Now we can apply Moser path method, as in Theorem 5.1, to each slice $\lambda = \text{const}$.

[3] In the domain $\{(\varepsilon_1,\varepsilon_2) \in \mathbb{R}^2 \mid \varepsilon_1^2 > -\frac{4}{27}\varepsilon_2^3\}$, similar asymptotics for $\Pi(H,\lambda)$ and $\widetilde{\Pi}(H,\lambda)$ hold, where the coefficients $\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}$ are replaced by $2\alpha, \delta, 2\widetilde{\alpha}, \widetilde{\delta}$ for some real-analytic functions $\delta, \widetilde{\delta}$ in a neighbourhood of $\Sigma_{\rm hyp}$.

for some real-analytic functions $\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta}$ in a neighborhood of $\Sigma_{\rm hyp} = \Sigma \cap \{H > 0\}$. Condition (i) of Proposition 5.11, therefore, implies that $\alpha(H,\lambda) = \widetilde{\alpha}(H,\lambda)$. For hyperbolic points, this coefficient in front of logarithm is known to be proportional to the period of the second (invisible in the real setting) cycle on the complex leaf $\mathcal{L}_{H,\lambda}$ (see Proposition 5.15 and discussion in Appendix).

Thus, for real $\lambda < 0$ and real $H \in \big(-2(-\lambda)^{3/2}/(3\sqrt{3}), 2(-\lambda)^{3/2}/(3\sqrt{3})\big)$ the required conditions (5.23) are fulfilled. Since the periods Π_γ and $\widetilde{\Pi}_\gamma$ are locally holomorphic (we cannot consider them as single-valued functions because of the monodromy phenomenon) and coincide on an open real domain, we conclude that (5.23) is fulfilled identically, which completes the proof of Proposition 5.11. □

We now return to our discussion on symplectic invariants of parabolic trajectories that we started in Section 3. According to Proposition 5.3, this problem can be reduced to the situation explained in Remark 5.5.

Namely, we consider two functions $H = x^2 + y^3 + \lambda y$ and $F = \lambda$ that commute simultaneously with respect to two symplectic forms Ω and $\widetilde{\Omega}$ defined by (5.11) and (5.12) with $\omega_\lambda = f(x,y,\lambda) dx \wedge dy$ and $\tilde{\omega}_\lambda = \tilde{f}(x,y,\lambda) dx \wedge dy$ and $f, \tilde{f} > 0$. Combining Proposition 5.5 and Proposition 5.11, we obtain the following

Proposition 5.12. *The following two statements are equivalent.*

i) In a tubular neighborhood of the parabolic orbit $\gamma_0(t) = (0,0,0,\varphi{=}t)$ there is a (real-analytic) diffeomorphism Φ such that
- *Φ preserves H and F;*
- *$\Phi^*(\widetilde{\Omega}) = \Omega$.*

ii) The functions $\Pi, \Pi_\circ, \widetilde{\Pi}, \widetilde{\Pi}_\circ$ (real-analytic in the complement of the bifurcation diagram) satisfy the relations
- *$\Pi(H,\lambda) - \widetilde{\Pi}(H,\lambda)$ is real-analytic (in a neighborhood of the point $H = 0, \lambda = 0$),*
- *$\Pi_\circ(H,\lambda) = \widetilde{\Pi}_\circ(H,\lambda)$.* □

In fact, the functions $\Pi(H,\lambda)$ and $\Pi_\circ(H,\lambda)$ are not independent. Indeed, as $H \to 2(-\lambda)^{3/2}/(3\sqrt{3})$ (i.e., when the real disconnected regular fiber approaches the hyperbolic singular one) these two functions have a logarithmic singularity with the same logarithmic coefficient, that is, we have the following asymptotics:

$$\Pi(H,\lambda) = \alpha(H,\lambda) \ln \left| 3\sqrt{3}H - 2(-\lambda)^{3/2} \right| + \beta(H,\lambda),$$
$$\Pi_\circ(H,\lambda) = \alpha(H,\lambda) \ln \left| 3\sqrt{3}H - 2(-\lambda)^{3/2} \right| + \beta_\circ(H,\lambda).$$

In other words, the functions $\beta(H,\lambda)$ and $\beta_\circ(H,\lambda)$ are different and not related to each other in any sense, but the coefficients $\alpha(H,\lambda)$ are the same for both functions. According to Proposition 5.12, however, the regular part $\beta(H,\lambda)$ of $\Pi(H,\lambda)$ does not play any role, so that the only important information for us is the coefficient $\alpha(H,\lambda)$ which, as we have just explained, can be "obtained" from $\Pi_\circ(H,\lambda)$. Hence we conclude that $\Pi_\circ(H,\lambda)$ contains all the information we need for symplectic characterisation of a parabolic trajectory.

We also note that the period $\Pi_\circ(H,\lambda)$ of closed trajectories can naturally be interpreted in terms of the action variables of our integrable system. Indeed, the family of *small closed trajectories* shown on Figure 5.2 corresponds to a family of "narrow" two-dimensional Liouville tori (recall that a four-dimensional neighborhood $U(\gamma_0)$ of the parabolic orbit γ_0 is the product (Figure 5.2)$\times S^1$). For this family, we can naturally define two action variables I_1 and I_2. The first of them corresponds to the free Hamiltonian S^1-action on $U(\gamma_0)$ generated by $F = \lambda$, that is, $I_1 = \lambda$. The other $I_2(H,\lambda)$ corresponds to the family of vanishing cycles shown in Figure 5.2 as *small closed trajectories*. We re-denote this function as $I_2(H,\lambda) = I_\circ(H,\lambda)$. Without loss of generality we will assume that

$$I_\circ > 0 \quad \text{and} \quad I_\circ \to 0 \quad \text{as} \quad (H,\lambda) \to (H(\gamma_0), F(\gamma_0)), \tag{5.24}$$

i.e., as we approach the singular fiber. Notice that, in a coordinate system (x,y,λ,φ), $I_\circ(H,\lambda)$ can be defined by an explicit formula. Fixing H and λ, we define a unique closed cycle (see Figure 5.2). This cycle bounds a certain domain $V_{H,\lambda} \subset \mathbb{R}^2(x,y)$ on the corresponding layer $\{\lambda = \text{const}\}$. Then

$$I_\circ(H,\lambda) = \frac{1}{2\pi}\mathsf{area}_\circ\big(V_{H,\lambda}\big) = \frac{1}{2\pi}\int_{V_{H,\lambda}} \omega_\lambda.$$

It is well-known that $I_\circ(H,\lambda)$ and $\Pi_\circ(H,\lambda) > 0$ are related in the following very simple way:

$$\Pi_\circ(H,\lambda) = 2\pi \frac{\partial}{\partial H} I_\circ(H,\lambda),$$

which shows that $\Pi_\circ(H,\lambda)$ can be reconstructed from $I_\circ(H,\lambda)$, so that we finally come to the following equivalent version of Proposition 5.12.

Proposition 5.13. *In the same assumptions as in Proposition 5.12, the following two statements are equivalent.*

i) In a tubular neighborhood of the parabolic orbit γ_0 there is a (real-analytic) diffeomorphism Φ such that

- *Φ preserves H and F;*
- *$\Phi^*(\widetilde{\Omega}) = \Omega$.*

ii) The actions (real-analytic on the "swallow-tail domain" corresponding to a family of "narrow" Liouville tori) corresponding to the family of vanishing cycles (cf. (5.24)*) coincide, $I_\circ(H,F) = \widetilde{I}_\circ(H,F)$.* □

We now want to give one more version of the criterion for the existence of Φ by omitting the condition $F = \lambda$ which, in particular, means that F is a 2π-periodic integral (equivalently, the action variable I_1) simultaneously for both integrable systems.

Consider H and F commuting with respect to Ω and $\widetilde{\Omega}$ in a tubular neighborhood of a parabolic orbit γ_0. Notice that now we are not allowed to assume that these two integrable systems share the same canonical coordinate system (x,y,λ,φ) as we did in Propositions 5.12 and 5.13.

Let $dF|_{\gamma_0} \neq 0$. We will say that Ω and $\widetilde{\Omega}$ *induce*

- *the same orientation* of γ_0 if the Hamiltonian flows of F w.r.t. Ω and $\widetilde{\Omega}$ induce the same orientation of γ_0;
- *the same coorientation* of γ_0 if (the restrictions of) Ω and $\widetilde{\Omega}$ induce the same orientation of a (local) 2-dimensional surface in $\{F = F(\gamma_0)\}$ transversal to γ_0 (i.e., on a 2-dim Poincaré section).

Without loss of generality, we may (and will) assume that Ω and $\widetilde{\Omega}$ induce the same orientation and the same coorientation of γ_0. Indeed, we can easily achieve this condition by using additional maps $(x, y, \lambda, \varphi) \mapsto (x, y, \lambda, -\varphi)$ and $(x, y, \lambda, \varphi) \mapsto (-x, y, \lambda, \varphi)$ (written in a canonical coordinate system from Proposition 5.3) that change respectively the orientation and coorientation without changing the functions F and H.

As above we can define two natural action variables for each of these two integrable systems $I(H, F)$, $I_\circ(H, F)$ and $\widetilde{I}(H, F)$, $\widetilde{I}_\circ(H, F)$. Here $I(H, F)$ and $\widetilde{I}(H, F)$ are smooth on a certain neighborhood $U(\gamma_0)$ and are generators of the Hamiltonian S^1-actions w.r.t. Ω and $\widetilde{\Omega}$ respectively.

Alternatively, we may define $I(H, F)$ by

$$I(H, F) = \frac{1}{2\pi} \oint_\gamma \varkappa, \quad \text{where } d\varkappa = \Omega$$

and $\gamma = \gamma_{H,F}$ is a closed cycle on the fiber $\mathcal{L}_{H,F}$ that is homotopic to γ_0 (recall that locally our fibration can be understood as the direct product of S^1 and a three-dimensional foliated domain $\mathcal{V}$ shown in Figure 5.2, then $\gamma_{H,F}$ can be taken of the form $S^1 \times \{P\}$ where $P \in \mathcal{V}$ is a point lying on the corresponding fiber).

The other action variable $I_\circ(H, F)$ is only defined on the family of "narrow" Liouville tori corresponding to *small oriented loops* $\mu_\circ = \mu_\circ(H, F)$ shown in Figure 5.2:

$$I_\circ(H, F) = \frac{1}{2\pi} \oint_{\mu_\circ} \varkappa, \quad \text{where } d\varkappa = \Omega.$$

In other words, $I_\circ(H, F)$ is a function defined on the "swallow-tail" domain on $\mathbb{R}^2(H, F)$ bounded by the bifurcation diagram Σ (this definition coincides with (5.24) up to, perhaps, changing the sign).

The actions $\widetilde{I}(H, F)$ and $\widetilde{I}_\circ(H, F)$ for the second system are defined in a similar way by integrating $\widetilde{\varkappa}$, $d\widetilde{\varkappa} = \widetilde{\Omega}$, over the same cycles γ and $\mu_\circ$ with the same orientations.

Theorem 5.2. *Suppose that the singular fibration defined by the functions H and F is Lagrangian w.r.t. both the symplectic forms Ω and $\widetilde{\Omega}$. Suppose that Ω and $\widetilde{\Omega}$ induce the same orientation and the same coorientation of a parabolic orbit γ_0. Then the following two statements are equivalent.*

i) In a tubular neighborhood of the parabolic orbit γ_0 there is a (real-analytic) diffeomorphism Φ such that

- *Φ preserves H and F;*
- *$\Phi^*(\widetilde{\Omega}) = \Omega$.*

ii) These two integrable systems have common action variables described above, i.e.,

$$I(H, F) = \widetilde{I}(H, F) + \text{const} \quad \textit{and} \quad I_\circ(H, F) = \widetilde{I}_\circ(H, F).$$

Proof. Suppose (ii) holds true. First of all we replace the functions F and H by new functions $\hat{F}$ and $\hat{H}$ satisfying the following conditions (cf. Proposition 5.1 and Remark 5.2):

- $\hat{F} = \pm I(H,F) + \text{const}$ where $\pm$ and const are chosen in such a way that $\hat{F} = 0$ on the parabolic trajectory γ_0 and $\hat{F} < 0$ on the swallow-tail domain of the bifurcation diagram;
- $\hat{H}$ is chosen in such a way that the bifurcation diagram of $\hat{\mathcal{F}} = (\hat{F}, \hat{H})$ takes the standard form (5.9).

After this we apply Proposition 5.4 which says that formulas from Proposition 5.3 holds true exactly for the functions $\hat{F}$ and $\hat{H}$. In other words, we can introduce *two different* "good" coordinate systems (x, y, λ, φ) and $(\tilde{x}, \tilde{y}, \tilde{\lambda}, \tilde{\varphi})$ as in Proposition 5.3 for $(\hat{H}, \hat{F}, \Omega)$ and $(\hat{H}, \hat{F}, \widetilde{\Omega})$ respectively (notice that $\lambda = \tilde{\lambda}$ automatically as both λ and $\tilde{\lambda}$ coincide with $\hat{F}$).

The next step is to consider the map $\Psi : (x, y, \lambda, \varphi) \mapsto (\tilde{x}, \tilde{y}, \tilde{\lambda}, \tilde{\varphi})$ and after this continue working with the forms Ω and $\Psi^*(\widetilde{\Omega})$. Now (x, y, λ, φ) is a common "good" coordinate system for both systems and the conditions of Theorem 5.2 are still fulfilled for Ω and $\Psi^*(\widetilde{\Omega})$. After this, it remains to apply Proposition 5.13 for the integrable systems $(\hat{H}, \hat{F}, \Omega)$ and $(\hat{H}, \hat{F}, \Psi^*(\widetilde{\Omega}))$.

The fact that (i) implies (ii) follows from the assumption that the symplectic forms Ω and $\widetilde{\Omega}$ induce the same orientation and coorientation on γ_0. Indeed this implies that Φ preserves the homology class of γ and $\mu_\circ$ on each "narrow" torus. Therefore if we set $\varkappa = \Phi^* \widetilde{\varkappa}$ in the definition of the actions $I(H,F)$ and $I_\circ(H,F)$, then $I(H,F) = \widetilde{I}(H,F)$ and $I_\circ(H,F) = \widetilde{I}_\circ(H,F)$. □

In fact, we do not even need to mention H and F in the statement of Theorem 5.2 at all. We may simply say:

Theorem 5.3. *Consider a singular fibration with a parabolic orbit γ_0 which is Lagrangian with respect to two symplectic structures Ω and $\widetilde{\Omega}$. Suppose that Ω and $\widetilde{\Omega}$ induce the same orientation and the same coorientation of γ_0. The necessary and sufficient condition for the existence of a (real-analytic) diffeomorphism Φ in a tubular neighborhood of γ_0 sending each fiber to itself and such that $\Phi^*(\widetilde{\Omega}) = \Omega$ is that these two systems have common action variables in the sense that for every closed cycle τ on any "narrow" torus we have*

$$\oint_\tau \varkappa = \oint_\tau \widetilde{\varkappa}, \quad \textit{for } d\varkappa = \Omega, \ d\widetilde{\varkappa} = \widetilde{\Omega},$$

where $\varkappa$ and $\widetilde{\varkappa}$ are chosen in such a way that $\oint_{\gamma_0} \varkappa = \oint_{\gamma_0} \widetilde{\varkappa} = 0$. □

Notice that due to analyticity it is sufficient to compare the actions only on the family of "narrow" tori, although I and $\tilde{I}$ are defined on the whole neighborhood $U(\gamma_0)$.

Finally, we want to relax the condition that each fiber goes to itself (indeed, this assumption makes no sense at all if we want to compare parabolic orbits for two different integrable systems).

Assume that we are given two integrable systems with parabolic orbits γ_0 and $\widetilde{\gamma}_0$, respectively. For both systems we consider the bifurcation diagrams (or bifurcation complexes), Σ and $\widetilde{\Sigma}$ respectively, and the "swallow-tail domains" corresponding to the families of "narrow"

Liouville tori. On each of these domains we have two actions I and $I_\circ$ (as functions of H and F) and correspondingly $\widetilde{I}$ and $\widetilde{I}_\circ$ (as functions of $\widetilde{H}$ and $\widetilde{F}$) defined as above. Without loss of generality we will assume that these action variables are "normalised" in such a way that

- all of them vanish at the corresponding cusp point,
- $I_\circ$ and $\widetilde{I}_\circ$ are positive on the corresponding "swallow-tail" domains,
- I and $\tilde{I}$ are negative on the corresponding "swallow-tail" domains.

Combining Theorem 5.2 with Proposition 5.2 we obtain

Theorem 5.4. *The necessary and sufficient condition for the existence of a real-analytic fiber-wise symplectomorphism $\Phi : U(\gamma_0) \to \widetilde{U}(\widetilde{\gamma}_0)$ between some tubular neighborhoods $U(\gamma_0), \widetilde{U}(\widetilde{\gamma}_0)$ of the parabolic orbits $\gamma_0, \widetilde{\gamma}_0$ is that these two systems have common action variables in the sense that there is a real-analytic diffeomorphism*

$$\varphi : (H, F) \mapsto (\widetilde{H}, \widetilde{F}) \tag{5.25}$$

between some neighborhoods of the cusp points $(H(\gamma_0), F(\gamma_0))$ and $(\widetilde{H}(\widetilde{\gamma}_0), \widetilde{F}(\widetilde{\gamma}_0))$ in $\mathbb{R}^2$ that

- *respects the bifurcation diagrams together with their partitions into hyperbolic and elliptic branch*[4]*:*
$$\varphi(\Sigma) = \widetilde{\Sigma}, \quad \textit{moreover} \quad \varphi(\Sigma_{\text{ell}}) = \widetilde{\Sigma}_{\text{ell}} \quad \textit{and} \quad \varphi(\Sigma_{\text{hyp}}) = \widetilde{\Sigma}_{\text{hyp}},$$
- *and preserves the action variables described above: $I = \widetilde{I} \circ \varphi$ and $I_\circ = \widetilde{I}_\circ \circ \varphi$, i.e., for the action variables defined on the "swallow-tail domains" we have*
$$I(H, F) = \widetilde{I}(\widetilde{H}(H, F), \widetilde{F}(H, F)) \quad \textit{and} \quad I_\circ(H, F) = \widetilde{I}_\circ(\widetilde{H}(H, F), \widetilde{F}(H, F)). \qquad \square$$

The latter conclusion basically means that the only symplectic invariants of hyperbolic orbits are *action variables*. This conclusion does not provide any tools to decide whether a suitable map (5.25) (making the actions equal) exists or not, but some necessary conditions can be easily found. Some of them have been already described in Section 4, e.g., the function $f(\cdot)$ from Proposition 5.10. This function is a symplectic invariant of a parabolic singularity which "corresponds" to the level $\lambda = 0$, where λ, as above, denotes the first action variable $I(H, F)$. We now want to describe another non-trivial symplectic invariant which will be a function $h(\lambda)$, $\lambda < 0$.

Since $\lambda = \lambda(H, F)$ is a real-analytic function, we can consider it as a parameter on the hyperbolic branch Σ_{hyp} of the bifurcation diagram Σ. Consider $I_\circ(H, \lambda)$ as a function of H (with λ as a parameter). This function is defined on the interval

$$\left(-2(-\lambda)^{3/2}/(3\sqrt{3}), 2(-\lambda)^{3/2}/(3\sqrt{3})\right),$$

is strictly increasing from 0 to its maximum attained on the hyberbolic branch. We denote it by $h(\lambda) = \max_H I_\circ(H, \lambda)$. Obviously, $h(\lambda)$ does not depend on the choice of commuting

[4]Equivalently, we may say that φ defines a (local) homeomorphism between the corresponding bifurcation complexes.

functions H and F defining the Lagrangian fibration, so that $h(\lambda)$ can be considered as a symplectic invariant of a parabolic singularity.

The problem of an explicit description of a complete set of symplectic invariants is equivalent, as shown above, to the analysis of the asymptotics of the function $I_\circ(H,\lambda)$. More precisely, we should describe invariants of such functions under (real-analytic) transformations of the form $(H,\lambda) \mapsto \big(\widetilde{H}(H,\lambda), \widetilde{\lambda} = \lambda\big)$.

6 Semi-local invariants of cusp singularities

Finally, we want to describe semi-local invariants of cusp singularities. In other words, we now consider a saturated neighborhood of a compact singular fiber $\mathcal{L}_0$ containing a parabolic orbit, i.e., cuspidal torus. We assume that this fiber contains no other critical points, so that the topology of the fibration in a neighborhood of $\mathcal{L}_0$ is standard and illustrated in Figure 5.3. This figure also shows the bifurcation complex, i.e., the base of this fibration, which consists of two 2-dimensional strata (attached to each other along Σ_{hyp}, one of the branches of the bifurcation diagram Σ that corresponds to the family of hyperbolic orbits). Each stratum represents a family of Liouville tori and therefore we can naturally assign a pair of action variables to each of them. Our goal is to show that fibrations with the same actions are symplectomorphic.

In a neighborhood $U(\mathcal{L}_0)$ of the singular fiber $\mathcal{L}_0$, on all neighboring Liouville tori we can choose a natural basis of cycles in the first homology group of $H_1(T^2_{F,H},\mathbb{Z})$ where $T^2_{H,F}$ is the Liouville torus defined by fixing the values of the integrals F and H respectively. These cycles are shown in Figure 5.3. One of them corresponds to the S^1-action defined on $U(\mathcal{L}_0)$ (in Figure 5.3, this cycle γ is denoted by S^1). The other cycle can be obtained by considering a global 3-dimensional cross-section to this S^1-action. Since this S^1-action (and the corresponding S^1-fibration) is topologically trivial, such a cross section exists. It is illustrated on the left in Figure 5.3 and denoted by V so that we may think of $U(\mathcal{L}_0)$ as the direct product $V \times S^1 = U(\mathcal{L}_0)$. Each Liouville torus $T^2_{F,H}$ intersects V along a closed curve (these curves are shown in Figure 5.3) and this curve is taken as the second basis cycle μ in $H_1(T^2_{F,H},\mathbb{Z})$. More precisely, we need to take into account that for a point (F,H) from the swallow-tail zone, we will have two disjoint Liouville tori. The corresponding cycles will be denoted by μ and $\mu_\circ$, where $\mu_\circ$ is used for the vanishing cycle on the family of "narrow" tori, the other, i.e., μ, corresponds to a "wide" torus.

Notice that the first cycle γ is uniquely defined by the S^1-action. The cycle $\mu_\circ$ is also well defined by the topology of the fibration (as a vanishing cycle). The other cycle μ is not. It is easy to see that μ is defined up to the transformation of the form $\mu \mapsto \mu + k\gamma$, $k \in \mathbb{Z}$. This is caused by ambiguity in the choice of the cross-section V which can be chosen in many homotopically different ways (this phenomenon is discussed and explained in details in [7]).

Summarizing, on each stratum of the bifurcation complex, we have a pair of action variables I_γ, I_μ and $I_\gamma, I_{\mu_\circ}$ (the latter for the swallow-tail stratum). Each of these functions can be treated as a real-analytic function of H and F. In fact, we have already considered the actions I_γ and $I_{\mu_\circ}$ in the previous Section 5, where they were denoted by $I(H,F)$ and $I_\circ(H,F)$. We will keep this notation here, i.e., we set $I_\gamma = I$, $I_{\mu_\circ} = I_\circ$. The remaining action will be denoted by I_μ so that we have 3 action variables $I, I_\circ$ and I_μ. The first two of them are well-defined, but I_μ is defined modulo transformation $I_\mu \mapsto I_\mu + kI$.

Also notice that $I(H,F)$ is real-analytic everywhere (strictly speaking we need to distinguish this action for the families of "narrow" and "wide" tori, but due to real-analyticity $I(H,F)$, as function of H and F, is the same for both families). The function $I_\mu(H,F)$ is defined and is real-analytic everywhere except for the hyperbolic branch Σ_{hyp} of the bifurcation diagram. When approaching Σ_{hyp} the function tends to certain finite limits, but these limits from above and from below are different. The function $I_\circ$ is defined on the swallow-tail domain and is continuous on its closure.

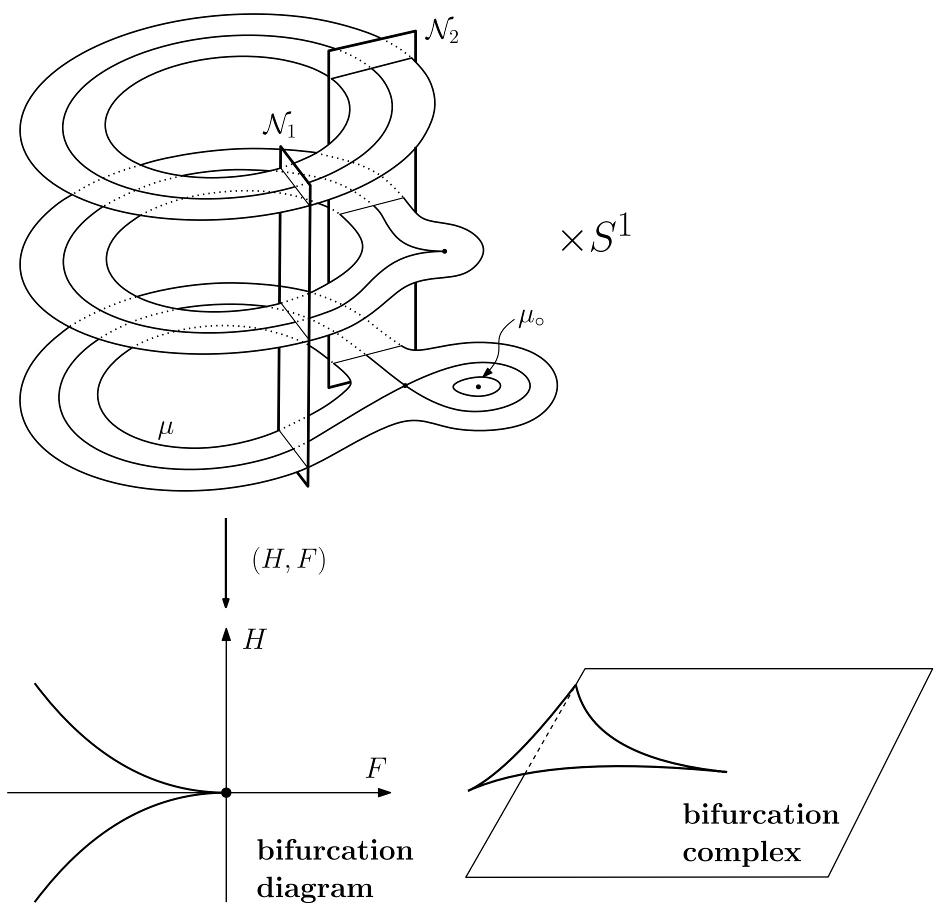


Figure 5.3: Singular fibration near a cuspidal torus

Our final result basically states that the systems with equal actions are symplectomorphic. We will give two versions of this result. Consider two integrable Hamiltonian systems $\big(H,F,\Omega,U(\mathcal{L}_0)\big)$ and $\big(\widetilde{H},\widetilde{F},\widetilde{\Omega},\widetilde{U}(\widetilde{\mathcal{L}}_0)\big)$ defined on some neighborhoods[5] of cuspidal tori $\mathcal{L}_0$ and $\widetilde{\mathcal{L}}_0$.

[5]We do not specify the sizes of these neighborhoods, but assume that they are sufficiently small. In other words, we are talking about germs of fibrations and germs of maps.

Theorem 5.5. *Assume that there is a fiberwise diffeomorphism* $\Psi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$ *that preserves the actions in the sense that for every cycle* $\tau \subset \mathcal{L}_{H,F}$ *we have*

$$\oint_{\Psi(\tau)} \widetilde{\varkappa} = \oint_{\tau} \varkappa$$

for some 1-forms $\varkappa$ *and* $\widetilde{\varkappa}$ *satisfying* $d\varkappa = \Omega$, $d\widetilde{\varkappa} = \widetilde{\Omega}$. *Then there exists a fiberwise symplectomorphism* $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$.

Remark 5.7. The converse statement is obviously true, since a fiberwise symplectomorphism Φ preserves the actions:

$$\oint_{\Phi(\tau)} \widetilde{\varkappa} = \oint_{\tau} \varkappa$$

where $\varkappa$ and $\widetilde{\varkappa}$ are related by $\Phi^* \widetilde{\varkappa} = \varkappa$.

A stronger version is as follows. For each system we compute the actions $I(H, F)$, $I_\circ(H, F)$ and $I_\mu(H, F)$ and respectively $\widetilde{I}(\widetilde{H}, \widetilde{F})$, $\widetilde{I}_\circ(\widetilde{H}, \widetilde{F})$ and $\widetilde{I}_{\widetilde{\mu}}(\widetilde{H}, \widetilde{F})$ for the second system as explained above.

Theorem 5.6. *Assume that there is a local real-analytic diffeomorphism* $\varphi : (H, F) \mapsto (\widetilde{H}, \widetilde{F})$, $\widetilde{H} = \widetilde{H}(H, F)$ *and* $\widetilde{F} = \widetilde{F}(H, F)$ *that*

- *respects the bifurcation diagrams together with their partitions into hyperbolic and elliptic branches:*

$$\varphi(\Sigma) = \widetilde{\Sigma}, \quad \textit{moreover} \quad \varphi(\Sigma_{\text{ell}}) = \widetilde{\Sigma}_{\text{ell}} \quad \textit{and} \quad \varphi(\Sigma_{\text{hyp}}) = \widetilde{\Sigma}_{\text{hyp}},$$

- *and makes the actions equal (for some choice of* μ *and* $\widetilde{\mu}$*):*

$$\begin{aligned} I(H, F) &= \widetilde{I}(\widetilde{H}(H, F), \widetilde{F}(H, F)), \\ I_\circ(H, F) &= \widetilde{I}_\circ(\widetilde{H}(H, F), \widetilde{F}(H, F)) \textit{ and} \\ I_\mu(H, F) &= \widetilde{I}_{\widetilde{\mu}}(\widetilde{H}(H, F), \widetilde{F}(H, F)). \end{aligned}$$

Then there exists a fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$.

Remark 5.8. Notice that the converse statement is also true: a fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$ induces a diffeomorphism φ between the bases of the fibrations which automatically satisfies the properties above (where the choice of $\widetilde{\mu}$ is induced by Φ and μ).

Remark 5.9. We can rewrite this statement in a slightly different and shorter way. For each of the above integrable systems we may consider the momentum map $\pi : U(\mathcal{L}_0) \to B \subset \mathbb{R}^2(H, F)$ and $\pi : \widetilde{U}(\widetilde{\mathcal{L}}_0) \to \widetilde{B} \subset \mathbb{R}^2(\widetilde{H}, \widetilde{F})$, where B and $\widetilde{B}$ are some neighborhoods of the corresponding cusp points of the bifurcation diagrams. Then we can think of the actions as functions on B (more precisely on the corresponding domains defined by the bifurcation diagrams). Then Theorem 5.6 can be rephrased as follows:

Assume that there exists a local real-analytic diffeomorphism $\varphi : B \to \widetilde{B}$ *respecting the bifurcation diagrams* Σ *and* $\widetilde{\Sigma}$ *and such that* $I = \widetilde{I} \circ \varphi$, $I_\circ = \widetilde{I}_\circ \circ \varphi$ *and* $I_\mu = \widetilde{I}_{\widetilde{\mu}} \circ \varphi$. *Then there exist a fiberwise symplectomorphism* $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$.

The proof of this theorem is based on the following lemma. Consider two (non-singular) integrable systems (H, F, Ω) and $(\widetilde{H}, \widetilde{F}, \widetilde{\Omega})$ defined in some neighborhoods $T^2 \times B$ and $\widetilde{T}^2 \times \widetilde{B}$ of regular Liouville tori. Here B and $\tilde{B}$ are 2-dimensional discs viewed as the bases of the corresponding (regular) Lagrangian fibrations endowed with induced integer affine structures (action variables). The functions (H, F) and $(\widetilde{H}, \widetilde{F})$ are treated as smooth functions on B and $\widetilde{B}$ respectively. We also consider the Hamiltonian $\mathbb{R}^2$-actions $\sigma^{(t_1,t_2)}$ and $\widetilde{\sigma}^{(t_1,t_2)}$, $(t_1, t_2) \in \mathbb{R}^2$ generated by the commuting functions (H, F) and $(\tilde{H}, \tilde{F})$. Here $\sigma^{(t_1,t_2)}$ denotes the composition of the Hamiltonian shifts along vector fields X_H and X_F by time t_1 and time t_2 respectively. Similarly for $\tilde{\sigma}^{(t_1,t_2)}$.

Lemma 5.6. *Assume that we have a real-analytic diffeomorphism*

$$\varphi : B \to \tilde{B},$$

which provides an (integer) affine equivalence between B and $\widetilde{B}$. Let $H = \widetilde{H} \circ \varphi$ and $F = \widetilde{F} \circ \varphi$ and consider two Liouville tori $T_p = T^2 \times \{p\}$ and $\widetilde{T}_{\varphi(p)} = \widetilde{T}^2 \times \{\varphi(p)\}$ where $p \in B$ is an arbitrary point (in other words, these tori correspond to each other under the map $\varphi : B \to \widetilde{B}$). Let $x \in T_p$ and $\tilde{x} \in \widetilde{T}_{\varphi(p)}$ be arbitrary two points from these fibers.

Then $\sigma^{(t_1,t_2)}(x) = x$ (or more generally $\sigma^{(t_1,t_2)}(x) = \sigma^{(t_1',t_2')}(x)$) if and only if $\widetilde{\sigma}^{(t_1,t_2)}(\tilde{x}) = \tilde{x}$ (respectively $\widetilde{\sigma}^{(t_1,t_2)}(\tilde{x}) = \widetilde{\sigma}^{(t_1',t_2')}(\tilde{x})$).

Proof. We will give a proof of this statement in the case of n degrees of freedom. Recall that B and $\widetilde{B}$ are endowed with integer affine structures induced by the action variables. By definition, $\varphi : B \to \widetilde{B}$ is an (integer) affine equivalence if φ sends "actions to actions". More precisely, let $\widetilde{I}_1, \dots, \widetilde{I}_n$ be action variables for $\tilde{B}$, which means that these functions define the Hamiltonian action of the *standard* torus $\mathbb{R}^n/\Gamma_0$ where $\Gamma_0 = \mathbb{Z}^n$ is the standard integer lattice in $\mathbb{R}^n$. We say that $\varphi : B \to \widetilde{B}$ is an affine equivalence, if $I_1 = \widetilde{I}_1 \circ \varphi, \dots, I_n = \widetilde{I}_n \circ \varphi$ are action variables on B.

If in Lemma 5.6 instead of (H, F) and $(\widetilde{H}, \widetilde{F})$ we consider (I_1, I_2) and $(\widetilde{I}_1, \widetilde{I}_2)$, then the statement is obvious: both relations $\sigma^{(t_1,t_2)}(x) = x$ and $\widetilde{\sigma}^{(t_1,t_2)}(\widetilde{x}) = \widetilde{x}$ simply mean that (t_1, t_2) belongs to the standard integer lattice, i.e., $t_1, t_2 \in \mathbb{Z}$.

Let us see what happens if take arbitrary functions (H, F) or, more generally, $(F_1, F_2, \dots F_n)$ in the case of n degrees of freedom. The relation $\sigma^{(t_1,\dots,t_n)}x = x$ means that $(t_1, \dots, t_n)$ belongs to the period lattice $\Gamma \subset \mathbb{R}^n$ which is the stationary subgroup of x in the sense of the Hamiltonian $\mathbb{R}^n$-action generated by $F_1, F_2, \dots F_n$. Since this lattice is the same for any point x from a fixed torus T_p, $p \in B$, we may denote it by $\Gamma(T_p)$. This lattice is not standard anymore and it depends on two things, the torus T_p (or just a point $p \in B$) and the generators $F_1, F_2, \dots F_n$ of the Hamiltonian $\mathbb{R}^n$-action.

If we know the expressions of $F_1, \dots, F_n$ in terms of the actions $I_1, \dots, I_n$, then the lattice $\Gamma(T_p)$ is easy to describe. Namely:

$$\Gamma(T_p) = \Gamma_0 \cdot J^{-1}(p),$$

where Γ_0 is the standard integer lattice and $J(p)$ denotes the Jacobi matrix $J(p) = \left(J^i_j = \frac{\partial F_i}{\partial I_j}|_p \right)$. In more details,

$$(t_1, \dots, t_n) \in \Gamma(T_p) \quad \text{if and only if} \quad (t_1, \dots, t_n) = (k_1, \dots, k_n) \cdot J^{-1}(p)$$

for $(k_1, \dots, k_n) \in \Gamma_0$, i.e., for some vector with integer components $k_i \in \mathbb{Z}$.

The same, of course, holds for $\widetilde{x} \in \widetilde{T}_{\varphi(p)}$, that is

$$(t_1, \dots, t_n) \in \Gamma(\widetilde{T}_{\varphi(p)}) \quad \text{if and only if} \quad (t_1, \dots, t_n) = (k_1, \dots, k_n) \cdot \widetilde{J}^{-1}(\varphi(p))$$

where $\widetilde{J}(\varphi(p)) = \left(\widetilde{J}^i_j = \frac{\partial \widetilde{F}_i}{\partial \widetilde{I}_j}|_{\varphi(p)}\right)$. It remains to notice that under our assumptions these matrices coincide. The reason is obvious: since $I_k = \widetilde{I}_k \circ \varphi$ and also $F_i = \widetilde{F}_i \circ \varphi$, we see that $F_i = f_i(I_1, \dots, I_n)$ implies that $\widetilde{F} = f_i(\widetilde{I}_1, \dots, \widetilde{I}_n)$, i.e., F_i depends on $I_1, \dots, I_n$ exactly in the same way as $\widetilde{F}_i$ depends on $\widetilde{I}_1, \dots, \widetilde{I}_n$ so that the corresponding partial derivatives (being computed at p and $\varphi(p)$, i.e., at those points for which $(I_1, \dots, I_n) = (\widetilde{I}_1, \dots, \widetilde{I}_n)$) obviously coincide. In other words, we have proved that $\Gamma(T_p) = \Gamma(\widetilde{T}_{\varphi(p)})$, which is equivalent to our statement. □

This lemma implies the following two extension results.

Under the assumptions and notation from Lemma 5.6, assume that N and $\widetilde{N}$ are Lagrangian (real-analytic) sections of the Lagrangian fibrations $\pi : T^2 \times B \to B$ and $\widetilde{\pi} : \widetilde{T}^2 \times \widetilde{B} \to \widetilde{B}$ respectively. Since the sections N and $\widetilde{N}$ can be naturally identified with the bases B and $\widetilde{B}$, the map $\varphi : B \to \widetilde{B}$ induces a natural map between N and $\tilde{N}$ which we denote by the same letter $\varphi : N \to \tilde{N}$. For any point $y \in T^2 \times B$ we can find (not uniquely!) $(t_1(y), t_2(y)) \in \mathbb{R}^2$ such that $x = \sigma^{(t_1(y), t_2(y))}(y) \in N$. Consider the map $\Phi : T^2 \times B \to \widetilde{T}^2 \times \widetilde{B}$ defined by

$$\Phi(y) = \widetilde{\sigma}^{(-t_1(y), -t_2(y))}(\varphi(x)), \quad \text{where } x = \sigma^{(t_1(y), t_2(y))}(y) \in N.$$

Corollary 5.4. *The map $\Phi(y)$ is well defined and is a fiber-wise real-analytic diffeomorphism satisfying $\Phi^*(\widetilde{\Omega}) = \Omega$.*

Proof. The fact that Φ is well defined (i.e., does not depend on the choice of $(t_1, t_2) \in \mathbb{R}^2$ with the property $\sigma^{(t_1, t_2)}(y) \in N$) follows from Lemma 5.6. To show that Φ is symplectomorphism, i.e., $\Phi^*(\widetilde{\Omega}) = \Omega$, we notice that the position of each point $y \in T^2 \times B$ is defined by the values of H, F (which can be understood as coordinates on B) and t_1, t_2 (which can be understood as coordinates on the torus T^2 with the "origin" $(0, 0)$ located on N). These four functions define a canonical coordinate system, i.e.,

$$\Omega = dH \wedge dt_1 + dF \wedge dt_2.$$

A similar canonical coordinate system $\tilde{H}, \tilde{F}, \tilde{t}_1, \tilde{t}_2$ can be defined on $\widetilde{T}^2 \times \widetilde{B}$ by using the action $\widetilde{\sigma}$ and the Lagrangian section $\widetilde{N}$. It remains to notice that our map Φ in these coordinate systems, by construction, takes the form $\widetilde{H} = H$, $\widetilde{F} = F$, $\widetilde{t}_1 = t_1$, $\widetilde{t}_2 = t_2$. □

Let $U \subset T^2 \times B$ be an open subset such that the intersection of U with each fiber is connected and non-empty. Let $\Phi_{\text{loc}} : U \to \widetilde{U}$ be a real-analytic fiber-wise diffeomorphism with a certain open subset $\widetilde{U} \subset \widetilde{T}^2 \times \widetilde{B}$ such that $\Phi^*_{\text{loc}}(\widetilde{\Omega}) = \Omega$. Since Φ_{loc} is fiberwise and U intersects each fiber, Φ_{loc} induces a real-analytic map φ between the bases B and $\widetilde{B}$.

Corollary 5.5. *Φ_{loc} can be extended up to a real-analytic fiber-wise diffeomorphism $\Phi : T^2 \times B \to \widetilde{T}^2 \times \widetilde{B}$ with the property $\Phi^*(\widetilde{\Omega}) = \Omega$ if and only $\varphi : B \to \widetilde{B}$ is an integer affine equivalence.*

Proof. First of all we notice that such an extension (if it exists) is always unique. Indeed, since Φ is a symplectomorphism, we have

$$\Phi \circ \sigma^{(t_1,t_2)} = \widetilde{\sigma}^{(t_1,t_2)} \circ \Phi,$$

where σ and $\widetilde{\sigma}$ are Hamiltonian $\mathbb{R}^2$-actions generated by H, F and $\widetilde{H} = H \circ \Phi^{-1}$, $\widetilde{F} = F \circ \Phi^{-1}$ respectively. Therefore for any $y \in T^2 \times B$, its image $\Phi(y)$ is uniquely defined by:

$$\Phi(y) = \widetilde{\sigma}^{(t_1,t_2)} \circ \Phi_{\rm loc} \circ \sigma^{(-t_1,-t_2)}(y), \tag{5.26}$$

where (t_1, t_2) are chosen in such a way that $\sigma^{(-t_1,-t_2)}(y) \in U$ (such $(t_1, t_2) \in \mathbb{R}^2$ exists as each orbit of the action σ has a non-trivial intersection with U). Moreover, this formula can be understood as an explicit formula for the required extension. In a neighborhood of every point y, the expression $\widetilde{\sigma}^{(t_1,t_2)} \circ \Phi_{\rm loc} \circ \sigma^{(-t_1,-t_2)}$ (with fixed (t_1, t_2)) is a composition of three real-analytic fiberwise symplectomorphisms. So the only condition we need to check is that formula (5.26) is well defined, i.e., does not depend on the choice of $(t_1, t_2) \in \mathbb{R}^2$.

Assume that

$$y = \sigma^{(t_1,t_2)}(x) = \sigma^{(t_1',t_2')}(x') \quad \text{with } x, x' \in U.$$

We need to check that

$$\widetilde{\sigma}^{(t_1,t_2)} \circ \Phi_{\rm loc} \circ \sigma^{(-t_1,-t_2)}(y) = \widetilde{\sigma}^{(t_1',t_2')} \circ \Phi_{\rm loc} \circ \sigma^{(-t_1',-t_2')}(y) \tag{5.27}$$

or, equivalently,

$$\widetilde{\sigma}^{(t_1,t_2)} \circ \Phi_{\rm loc}(x) = \widetilde{\sigma}^{(t_1',t_2')} \circ \Phi_{\rm loc}(x'). \tag{5.28}$$

By our assumption, the intersection of U with each torus (interpreted now as an orbit of σ) is connected, therefore there exists a continuous curve $(\varepsilon_1(s), \varepsilon_2(s))$, $s \in [0,1]$ and $\varepsilon_1(0) = \varepsilon_2(0) = 0$ such that

$$\sigma^{(\varepsilon_1(s),\varepsilon_2(s))}(x) \in U \quad \text{for all } s \in [0,1] \quad \text{and} \quad \sigma^{(\varepsilon_1(1),\varepsilon_2(1))}(x) = x'.$$

Since $\Phi_{\rm loc}$ is a fiberwise symplectomorphism, we have

$$\Phi_{\rm loc} \circ \sigma^{(\varepsilon_1(s),\varepsilon_2(s))}(x) = \widetilde{\sigma}^{(\varepsilon_1(s),\varepsilon_2(s))} \circ \Phi_{\rm loc}(x)$$

for any s and, in particular,

$$\Phi_{\rm loc}(x') = \widetilde{\sigma}^{(\varepsilon_1(1),\varepsilon_2(1))} \circ \Phi_{\rm loc}(x).$$

Hence (5.28) can be rewritten as

$$\widetilde{\sigma}^{(t_1,t_2)}\big(\Phi_{\rm loc}(x)\big) = \widetilde{\sigma}^{(t_1'+\varepsilon_1(1),t_2'+\varepsilon_2(1))}\big(\Phi_{\rm loc}(x)\big). \tag{5.29}$$

On the other hand, since $\sigma^{(t_1,t_2)}(x) = \sigma^{(t_1',t_2')}(x')$, we also have

$$\sigma^{(t_1,t_2)}(x) = \sigma^{(t_1'+\varepsilon_1(1),t_2'+\varepsilon_2(1))}(x). \tag{5.30}$$

According to Lemma 5.6, if φ is an affine equivalence then (5.30) implies (5.29) and therefore (5.27), as needed.

The necessity of the condition that $\varphi : B \to \widetilde{B}$ is an integer affine equivalence is obvious: every fiberwise symplectomorphism induces an affine equivalence between B and $\widetilde{B}$. $\square$

We now use Corollary 5.5 to prove Theorem 5.6.

Proof. First we apply Theorem 5.4 which guarantees the existence of a real-analytic fiberwise diffeomorphism $\Phi_{\rm loc}$ [6] between some neighborhoods of parabolic trajectories $\gamma_0 \subset \mathcal{L}_0$ and $\widetilde{\gamma}_0 \subset \widetilde{\mathcal{L}}_0$. We now need to extend $\Phi_{\rm loc}$ up to the desired fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$.

According to Corollary 5.5 such an extension exists for all Liouville tori (more precisely, we only need to consider "wide" Liouville tori because all "narrow" Liouville tori are already contained in the domain of $\Phi_{\rm loc}$). Thus, it remains to explain why this map can be extended by continuity to each singular fiber.

On Figure 5.3 we can see the domain U on which $\Phi_{\rm loc}$ is already defined and the complementary domain W to which $\Phi_{\rm loc}$ should be extended. Without loss of generality we may assume that both domains are bounded by the sections $\mathcal{N}_1$ and $\mathcal{N}_2$. Namely, U is located to the right of $\mathcal{N}_1$ and $\mathcal{N}_2$ and contains all singular orbits including the parabolic one. The complimentary domain W is located to the left of $\mathcal{N}_1$ and $\mathcal{N}_2$ and contains no singularities at all.

Let $y \in W$ be an arbitrary point located on one of singular fibers and $V(y)$ be a sufficiently small neighborhood of y. Then there exists $(t_1, t_2) \in \mathbb{R}^2$ such that $\sigma^{(-t_1,-t_2)}(V(y)) \subset U$ and we may apply our extension formula (5.27) to define Φ on $V(y)$. Obviously, this formula defines a real-analytic fiberwise (local) symplectomorphism from $V(y)$ to its image in $\widetilde{U}(\widetilde{\mathcal{L}}_0)$ and moreover, due to the uniqueness of such an extension, this map coincides with Φ that has been already defined on non-singular fibers (Liouville tori). This is equivalent to saying that Φ can be naturally extended (by continuity) from Liouville tori to all singular fibers. This completes the proof. □

Remark 5.10. Our final remark is that the statement of Theorem 5.6 given in Remark 5.9 can be also understood in terms of natural affine structures defined on B and $\widetilde{B}$.

A necessary and sufficient condition for the existence of a *semi-local* fiberwise symplectomorphism $\Phi : U(\mathcal{L}_0) \to \widetilde{U}(\widetilde{\mathcal{L}}_0)$ between neighborhoods of two cuspidal tori $\mathcal{L}_0$ and $\widetilde{\mathcal{L}}_0$ is that the corresponding bases B and $\widetilde{B}$ are *locally* equivalent as manifolds with singular integer affine structures. Moreover, every affine equivalence $\varphi : B \to \widetilde{B}$ can be lifted up to a fiberwise symplectomorphism Φ.

This gives a partial answer to Problem 27 from the collection [8] of open problems in the theory of finite-dimensional integrable systems.

7 Appendix

In this appendix we give a formal proof of the statement made in Remark 5.2, namely we prove the following

Proposition 5.14. *Let P be a parabolic point of a momentum mapping $\mathcal{F} = (F, H) : M^4 \to$*

[6] In Theorem 5.4, this map was denoted by Φ.

$\mathbb{R}^2$ in the sense of Definition 5.1 we (in particular, $dF(P) \neq 0$) and

$$\widetilde{H} = \widetilde{H}(H, F), \quad \widetilde{F} = \widetilde{F}(H, F) \tag{5.31}$$

be a non-degenerate transformation such that $d\widetilde{F}(P) \neq 0$. Then P is still parabolic w.r.t. $\widetilde{H}$ and $\widetilde{F}$.

Proof. First of all, we notice that in Definition 5.1 we can replace H by $H - \text{const}F$ and, in particular, by $H - kF$ where $k \in \mathbb{R}$ is chosen in such a way that $d(H - kF) = 0$. In other words, without loss of generality we may assume that $dH(P) = 0$ and similarly for $\widetilde{H}$. Under this additional assumption, the quadratic differential $d^2H(P)$ makes sense on the whole tangent space T_PM^4. Taking into account that the tangent space to the hypersurface $\{F = F(P)\}$ coincides with the kernel of the differential $d\mathcal{F}$ (here we use the fact that $\operatorname{Ker} d\mathcal{F}(P) = \operatorname{Ker} dF(P) = T_P\{F = F(P)\}$), we can reformulate the first condition (i) as follows:

i) the rank of the restriction $d^2H(P)|_{\operatorname{Ker} d\mathcal{F}}$ equals 1.

The advantage of such a reformulation is that now this condition does not depend on the choice of F at all. Therefore to verify the invariance of Condition (i) w.r.t. transformation (5.31), it is sufficient to prove

Lemma 5.7. *Let $\widetilde{H} = \widetilde{H}(H, F)$ and $d\widetilde{H}(P) = 0$, then the forms $d^2H(P)|_{\operatorname{Ker} d\mathcal{F}}$ and $d^2\widetilde{H}(P)|_{\operatorname{Ker} d\mathcal{F}}$ are proportional with a non-zero factor.*

Proof. It is sufficient to compare the Taylor expansions of H and $\tilde{H}$ at the point P up to second order terms. Let

$$\Delta\widetilde{H} \simeq a_1\Delta H + a_2\Delta F + a_{11}\Delta H^2 + 2a_{12}\Delta H\Delta F + a_{22}\Delta F^2 \dots$$

Since $d\widetilde{H}(P) = dH(P) = 0$, we conclude that $a_2 = 0$ and $a_1 \neq 0$ and obtain:

$$\Delta\widetilde{H} \simeq \frac{a_1}{2} d^2H(\Delta x, \Delta x) + a_{22}\,(dF(\Delta x))^2 + \dots \tag{5.32}$$

(all the other terms are of order ≥ 3 and we omit them) or equivalently:

$$d^2\widetilde{H} = a_1 d^2H + 2a_{22} dF \otimes dF \tag{5.33}$$

(the differentials and second differentials are taken at the point P). Restricting to $\operatorname{Ker} dF = \operatorname{Ker} d\mathcal{F}$, we get $d^2\widetilde{H}(P)|_{\operatorname{Ker} d\mathcal{F}} = a_1 d^2\widetilde{H}(P)|_{\operatorname{Ker} d\mathcal{F}}$ with $a_1 \neq 0$, as required. □

Suppose that Condition (i) holds. Then the invariance of Condition (iii) w.r.t. transformation (5.31) amounts to the following

Lemma 5.8. *Let $\widetilde{H} = \widetilde{H}(H, F)$, $d\widetilde{H}(P) = 0$ and $\operatorname{rank}\,(d^2H)|_{\operatorname{Ker} d\mathcal{F}} = 1$. Then the conditions $\operatorname{rank} d^2H(P) = 3$ and $\operatorname{rank} d^2\widetilde{H}(P) = 3$ are equivalent.*

Proof. It is sufficient to use formula (5.33), namely:

$$d^2\widetilde{H} = a_1 d^2H + 2a_{22} dF \otimes dF.$$

In general, these two forms $d^2\widetilde{H}$ and d^2H do not necessarily have the same rank, but under the condition that rank $(d^2H)|_{\mathrm{Ker}\, d\mathcal{F}} = 1$ (using $\mathrm{Ker}\, d\mathcal{F} = \mathrm{Ker}\, dF$), the statements rank $d^2\widetilde{H} = 3$ and rank $d^2H = 3$ become equivalent (simple exercise in Matrix Algebra). Indeed, if we choose a basis e_1, e_2, e_3, e_4 in such a way that e_1, e_2, e_3 span $\mathrm{Ker}\, dF$ and e_2, e_3 span $\mathrm{Ker}\, d^2H|_{\mathrm{Ker}\, dF}$ and, in addition, $dF(e_4) = 1$ we will see that in matrix terms, the above formula (5.33) can be rewritten as

$$d^2H = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \delta \\ \beta & \gamma & \delta & \lambda \end{pmatrix}, \qquad d^2\widetilde{H} = a_1 \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \delta \\ \beta & \gamma & \delta & \lambda \end{pmatrix} + 2a_{22} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now it is easy to see that both statements rank $d^2H = 3$ and rank $d^2\widetilde{H} = 3$ are equivalent to the condition $(\gamma, \delta) \neq (0, 0)$, which completes the proof. □

Finally, we need to verify the invariance of Condition (ii) w.r.t. transformation (5.31). We first show that this condition does not change if we change F. Consider a new function $\widetilde{F} = \widetilde{F}(F, H)$. Since scaling $F \mapsto \mathrm{const} \cdot F$ does not affect the surface $\{F = F(P)\}$, we may assume that $\frac{\partial \widetilde{F}}{\partial F}|_{\mathcal{F}(P)} = 1$.

According to the definition of v^3H_0 we need to differentiate the same function H but along two different curves $\gamma(t) \subset \{F = F(P)\}$ and $\tilde{\gamma}(t) \subset \{\widetilde{F} = \widetilde{F}(P)\}$. Let us choose local coordinates $x_1, \dots, x_n$ on M in such a way that $x_1 = F$ and $P = (0, \dots, 0)$. It is easy to see that if we set $\tilde{x}_1 = \widetilde{F}$, then still $\tilde{x}_1, x_2, \dots, x_n$ is a good coordinate system. Moreover, the Jacobi matrix of the corresponding transformation at P is the identity. In coordinates $x_1, \dots, x_n$, the curve $\gamma(t)$ can be defined as $\gamma(t) = (0, x_2(t), \dots, x_n(t))$. The curve $\tilde{\gamma}(t)$ in the same coordinate system will be defined as

$$\tilde{\gamma}(t) = (x_1(t), x_2(t), \dots, x_n(t)) = \gamma(t) + (x_1(t), 0, \dots, 0)$$

(all functions are the same except for $x_1(t)$ which should be chosen in such a way that $\widetilde{F} = 0$ along the curve. In other words, x_1 can be found as a function of the other variables $x_2, \dots, x_n$ from the implicit relation

$$0 = \widetilde{F}(F, H) = \widetilde{F}\big(x_1, H(x_1, x_2, \dots, x_n)\big) \quad \Leftrightarrow \quad x_1 = g(x_2, \dots, x_n)$$

and correspondingly $x_1(t) = g\big(x_2(t), \dots, x_n(t)\big)$.

Our goal is to show that

$$\frac{d^3}{dt^3}|_{t=0} H(\tilde{\gamma}(t)) = \frac{d^3}{dt^3}|_{t=0} H(\gamma(t)). \tag{5.34}$$

It is easy to see that the Taylor expansion of $x_1 = g(x_2, \dots, x_n)$ starts with quadratic terms. Indeed, if (w.l.o.g. we assume that $F(P) = 0$ and $H(P) = 0$ so that $\Delta F = F$ and $\Delta H = H$)

$$\widetilde{F}(F, H) = F + b_2H + b_{11}F^2 + 2b_{12}FH + b_{22}H^2 + \dots$$

then we need to resolve the equation (with respect to x_1)

$$0 = x_1 + b_2H + b_{11}x_1^2 + \dots = x_1 + b_2\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} x_i x_j + b_{11}x_1^2 + \dots \tag{5.35}$$

These quadratic terms are sufficient to reconstruct the quadratic terms of the Taylor expansion of the function $x_1 = g(x_2, \dots, x_n)$. This can be done by using the implicit function theorem, but we can also use the substitution

$$x_1 = g(x_2, \dots, x_n) = \sum_{i,j=2}^{n} c_{ij} x_i x_j + \dots$$

into (5.35). If we collect (after substitution) all quadratic terms we obtain:

$$0 = \sum_{i,j=2}^{n} c_{ij} x_i x_j + b_2 \frac{1}{2} \sum_{i,j=2}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} x_i x_j + \dots$$

which means that up to a constant factor the quadratic expansions of g and H_0 coincide:

$$g(x_2, \dots, x_n) = -\frac{b_2}{2} \sum_{i,j=2}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} x_i x_j + \dots$$

or

$$d^2 g(\xi, \xi) = -b_2 d^2 H(\xi, \xi) \quad \text{for any } \xi \in \operatorname{Ker} dF = \operatorname{Ker} d\mathcal{F}. \tag{5.36}$$

Now we are ready to verify (5.34). We have:

$$\frac{d^3}{dt^3}|_{t=0} H(\tilde\gamma(t)) = d^3 H(\tilde\gamma', \tilde\gamma', \tilde\gamma') + 3d^2 H(\tilde\gamma', \tilde\gamma'') + dH(\tilde\gamma'''),$$

and

$$\frac{d^3}{dt^3}|_{t=0} H(\gamma(t)) = d^3 H(\gamma', \gamma', \gamma') + 3d^2 H(\gamma', \gamma'') + dH(\gamma''').$$

Since $\gamma' = \tilde\gamma' = v$ and $dH(P) = 0$, we only need to compare the middle terms. In the second relation this term vanishes because $\gamma'' \in \operatorname{Ker} dF$ (as the first component of $\gamma(t)$ identically vanishes) and γ' belongs to the kernel of $(d^2 H)|_{\operatorname{Ker} dF}$.

Consider the difference between $d^2 H(\tilde\gamma', \tilde\gamma'')$ and $d^2 H(\gamma', \gamma'')$. Since $\tilde\gamma' = \gamma'$, we have

$$d^2 H(\tilde\gamma', \tilde\gamma'') - d^2 H(\gamma', \gamma'') = d^2 H(\gamma', (\tilde\gamma - \gamma)'').$$

But $\tilde\gamma - \gamma = (g(x_2(t), \dots, x_n(t)), 0, \dots, 0)$ so that for the potentially non-zero component of $\tilde\gamma - \gamma$ we get

$$\frac{d^2}{dt^2}|_{t=0} g(x_2(t), \dots, x_n(t)) = d^2 g(\gamma', \gamma') = (\text{see } (5.36)) = -b_2\, d^2 H(\gamma', \gamma').$$

It remains to notice that $d^2 H(\gamma', \gamma') = 0$ as $v = \gamma' \in \operatorname{Ker}(d^2 H)|_{\operatorname{Ker} d\mathcal{F}}$. Thus, $(\tilde\gamma - \gamma)'' = 0$.

Thus, we have shown that condition (iii) does not depend on the choice of F (keeping H fixed). The last step is to show that (iii) does not depend on the choice of H (keeping F fixed). Again we use the Taylor expansion (5.32), but now up to third order terms

$$\Delta\widetilde{H} = a_1 \Delta H + 2a_{12} \Delta H \Delta F + a_{22} (\Delta F)^2 + a_{222} (\Delta F) + \dots, \quad a_1 \neq 0.$$

We do not need other terms (like $(\Delta H)^2$ for example) as in local coordinates ΔH starts with quadratic terms. This formula shows that under the condition $\Delta F = 0$, the Taylor expansions of H and $\widetilde{H}$ (up to cubic terms) are proportional with a non-zero factor. In particular for any curve $\gamma(t)$ lying on the surface $\{F = F(P)\} = \{\Delta F = 0\}$ we have

$$\frac{d^3}{dt^3}|_{t=0}\widetilde{H}(\gamma(t)) = a_1 \frac{d^3}{dt^3}|_{t=0} H(\gamma(t)) \quad \text{or equivalently} \quad v^3 \widetilde{H}_0 = a_1 v^3 H_0,$$

as needed. This shows that (ii) is invariant under transformations (5.31) completing the proof of Proposition 5.14. □

We also want to explain one important phenomenon mentioned in the proof of Proposition 5.11: for hyperbolic points, this coefficient in front of logarithm is known to be proportional to the period of the second (invisible in the real setting) cycle on the complex leaf $\mathcal{L}_{H,\lambda}$. More rigorously, this statement can be formulated as follows.

Consider an analytic integrable system with one degree of freedom with the Hamiltonian of the form $H = xy$ and symplectic structure $\omega = f(x, y)dx \wedge dy$.

Thinking of x and y as real variables, consider one-parameter family of curves

$$\gamma_H = \{xy = H,\ 0 < x \le 1,\ 0 < y \le 1\}$$

and the function (cf. Section 4)

$$\Pi(H) = \int_{\gamma_H} \frac{\omega}{dH}\,.$$

It can be easily checked by an explicit computation that this function has the following asymptotics at zero:

$$\Pi(H) = a(H) \ln H + b(H),$$

where $a(H)$ and $b(H)$ are both real-analytic in a neighborhood of zero. Similar to Section 4, here we integrate along a trajectory of the Hamiltonian flow between two sections $N_1 = \{y = 1\}$ and $N_2 = \{x = 1\}$. If we change these sections, then $b(H)$ changes too whereas $a(H)$ remains the same so that $a(H)$ has an invariant meaning, i.e., does not depend on the choice of local coordinates (x, y).

On the other hand if we think of x and y as complex variables, then the level $\{xy = H\}$, from the real viewpoint, is a surface locally homeomorphic to a cylinder which contains a non-trivial cycle of the form

$$\hat{\gamma}_H = \{x(t) = He^{it},\ y(t) = e^{-it}\},$$

so that we can introduce another function

$$\hat{\Pi}(H) = \int_{\hat{\gamma}_H} \frac{\omega}{dH}.$$

This function is analytic in H. The relation between $\Pi(H)$ and $\hat{\Pi}(H)$ is given by the following

Proposition 5.15. *We have* $a(H) = \pm\frac{1}{2\pi i}\hat{\Pi}(H)$ *or equivalently,*

$$\Pi(H) = \pm\frac{1}{2\pi i}\hat{\Pi}(H)\ln H + b(H)$$

with $b(H)$ *being analytic*[7]*.*

Proof. One can proof this fact by using monodromy arguments (which is nice and conceptual), but we will use a well known "isochore Morse lemma" [16] which allows to get this result by an explicit computation: there exists a local coordinate system such that

$$H = xy \qquad \text{and} \qquad \omega = f(H)dx \wedge dy.$$

The form $\dfrac{\omega}{dH}$ can be replaced by the form $f(H)y^{-1}dy$ and we obtain:

$$\hat{\Pi}(H) = \int_{\hat{\gamma}_H} \frac{\omega}{dH} = \int_{\hat{\gamma}_H} f(H)y^{-1}dy = f(H)\int_0^{2\pi}(e^{-it})^{-1}d(e^{-it}) = -2\pi i f(H)$$

On the other hand, γ_H can be parametrised as $y = t$, $x = Ht^{-1}$, $t \in [H, 1]$ and we get:

$$\Pi(H) = \int_{\gamma_H} \frac{\omega}{dH} = \int_{\gamma_H} f(H)y^{-1}dy = f(H)\int_H^1 \frac{dt}{t} = -f(H)\ln H.$$

Comparing these formulas for $\hat{\Pi}(H)$ and $\Pi(H)$ gives the required result. □

[7] The sign $\pm$ reflects the fact that both $\Pi(H)$ and $\hat{\Pi}(H)$ depend on the choice of orientations on the curves γ_H and $\hat{\gamma}_H$.

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