Scuola Internazionale Superiore di Studi Avanzati

Area of Mathematics

The relaxed area of maps from the plane to the plane with a line discontinuity, and the role of semicartesian surfaces.

Ph.D. Thesis

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Introduction

In this thesis we address the problem of estimating the area of the graph of a map $\mathbf{u} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, which is discontinuous on a segment. From a geometrical point of view, the discontinuity produces a "hole" in graph($\mathbf{u}$) $\subset \mathbb{R}^4$, a two-dimensional manifold of codimension two, namely the boundary of graph($\mathbf{u}$) in $\Omega \times \mathbb{R}^2$ is non-empty. Usually the area of the graph of a discontinuous map is defined by relaxing the classical notion of non-parametric area of $\mathcal{C}^1$ maps with respect to the L^1-topology. The problem consists therefore in finding the "more convenient" way, in terms of area, to fill the hole in the graph. This led us to introduce a suitable class of surfaces in $\mathbb{R}^3$, which we have called *semicartesian surfaces*, that constitute a sort of intermediate object between general disk-type surfaces and graphs of scalar functions defined on a planar domain. In our analysis, we need also to introduce and study another relaxation of the area functional, made with respect to a sort of uniform convergence, stronger than the L^1-convergence.
The results reported in this thesis have been obtained during my Ph.D. at SISSA (International School for Advanced Studies) in Trieste, in collaboration with Giovanni Bellettini and Maurizio Paolini, and are contained in [6], [8], and [7].

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 1$; given a smooth map $\mathbf{v} \in \mathcal{C}^1(\Omega; \mathbb{R}^k)$, Chapter 1
$k \geq 1$, the area of the graph of $\mathbf{v}$ in Ω is given by the integral

$$\mathcal{A}(\mathbf{v}, \Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{v})| \, d\mathbf{x}, \qquad (0.1)$$

where $|\cdot|$ denotes the Euclidean norm and $\mathcal{M}(\zeta)$ is the vector whose components are all the minors up to the order $\min\{n, k\}$ of the $(k \times n)$ matrix ζ, see *e.g.* [4]. In the perspective of Calculus of Variations, it is convenient to provide a notion of non-parametric area also for non-regular maps. Following a well-established tradition (see for instance [18], [13], [14], [1]), we define for every map $\mathbf{v} \in L^1(\Omega; \mathbb{R}^k)$ the *relaxed area* on Ω as

$$\overline{\mathcal{A}}(\mathbf{v}, \Omega) := \inf \left\{ \liminf_{h \to +\infty} \mathcal{A}(\mathbf{v}_h, \Omega) : \ (\mathbf{v}_h) \subset \mathcal{C}^1(\Omega; \mathbb{R}^k), \, \mathbf{v}_h \to \mathbf{v} \text{ in } L^1(\Omega; \mathbb{R}^k) \right\},$$

that is the L^1-*lower semicontinuous envelope* of the functional $\mathcal{A}(\cdot, \Omega)$.
The behaviour of $\overline{\mathcal{A}}(\cdot, \Omega)$ strongly depends on the value of k. In the scalar case, *i.e.* $k = 1$, the relaxed area functional has been characterized (see *e.g.* [24], [12], [23]): we recall in particular that its domain is the space $\mathrm{BV}(\Omega)$ of functions with

bounded variation in Ω, and that $\overline{\mathcal{A}}(\mathbf{v},\Omega)$ can be represented as the integral over Ω of a suitable function depending on $\mathbf{v}$, for any $\mathbf{v} \in \mathrm{BV}(\Omega)$.
The study of the vectorial case, *i.e.* $k \geq 2$, is instead much more involved, and the issues concerning the representation of $\overline{\mathcal{A}}(\cdot,\Omega)$ and its domain are still open. A relevant reference for this problem and for this thesis is [1], where the authors analyse lower semicontinuity properties for the relaxed functional of the area and of more general polyconvex integrals. In [1] it is proven that the domain of the relaxed area functional is contained in $\mathrm{BV}(\Omega;\mathbb{R}^k)$. Moreover, the authors characterize the subset of $L^1(\Omega;\mathbb{R}^2)$, denoted here by $\mathcal{D}(\Omega;\mathbb{R}^k)$, where the functional $\mathbf{v} \to \int_\Omega |\mathcal{M}(\nabla\mathbf{v})|\,d\mathbf{x}$ is finite and lower semicontinuous, proving in particular that it contains the space $\mathcal{C}^1(\Omega;\mathbb{R}^k)$; this implies that the relaxed area functional valuated for a $\mathcal{C}^1$ map coincides with the classical notion of area of the graph, namely (0.1). We also remark that $\mathcal{D}(\Omega;\mathbb{R}^k)$ is strictly contained in the space $\mathcal{A}^1(\Omega;\mathbb{R}^k)$ of maps $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^k)$ such that all the components of $\mathcal{M}(\nabla\mathbf{v})$ are summable in Ω. This fact was one of the motivations for developing the theory of Cartesian currents (see for example [18], [21], [20]).
One of the few cases for which the value of $\mathcal{A}(\mathbf{v},\Omega)$ is known is when $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ is a piecewise constant map, without any triple or multiple point; this means that $\mathbf{v}$ maps Ω into a finite set of $\mathbb{R}^k$, and that for every $\mathbf{x} \in \Omega$ there exists a neighbourhood where $\mathbf{v}$ assumes no more than two distinct values. In this case (see [1, Theorem 3.14])

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) = \int_\Omega |\mathcal{M}(\nabla\mathbf{v})|\,d\mathbf{x} + |D^s\mathbf{v}|(\Omega), \tag{0.2}$$

where $\nabla\mathbf{v}$ and $D^s\mathbf{v}$ denote the absolutely continuous and the singular component with respect to the Lebesgue measure of the distributional derivative of $\mathbf{v}$, see [4]. Formula (0.2) implies that, under the previous hypotheses on $\mathbf{v}$, $\overline{\mathcal{A}}(\mathbf{v},\cdot)$ is a measure. In general this is not true and $\overline{\mathcal{A}}(\mathbf{v},\Omega)$ is only greater than or equal to the right hand side of (0.2), see [1, Theorem 3.7]. The dependence of the relaxed area functional on the domain is a very interesting problem. As conjectured by De Giorgi in [13], [14], and then proven by Acerbi and Dal Maso in [1], if $k \geq 2$ the relaxed area functional is non-subadditive; more specifically, there exist a map $\mathbf{v} \in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^k)$ and open bounded sets $\Omega_1,\Omega_2,\Omega_3 \subset \mathbb{R}^n$ such that $\Omega_3 \subset \Omega_1 \cup \Omega_2$ but $\overline{\mathcal{A}}(\mathbf{v},\Omega_3) > \overline{\mathcal{A}}(\mathbf{v},\Omega_1) + \overline{\mathcal{A}}(\mathbf{v},\Omega_2)$; therefore it is not possible to find an integral representation for the relaxed area, in terms of a local integrand. In [1, Theorems 4.1 and 5.1] the authors prove the non-subadditive behaviour of $\overline{\mathcal{A}}(\mathbf{v},\cdot)$ when $\mathbf{v}$ is either the triple point map $\mathbf{u}_T : \mathbb{R}^2 \to \mathbb{R}^2$, namely a piecewise constant map assuming three values in a neighbourhood of the origin, or the vortex map $\mathbf{u}_V : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\mathbf{u}_V(\mathbf{x}) := \frac{\mathbf{x}}{|\mathbf{x}|}$. We observe that the vortex map takes values in the sphere $\mathbb{S}^{n-1}$, and thus the determinant of $\nabla\mathbf{u}_V$ is identically zero. Incidentally, we recall that the maps $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^k$ with values in the sphere $\mathbb{S}^{k-1}$ have an important role also in the study of the so-called *distributional Jacobian*, or *distributional determinant* if $n = k$, (see *e.g.* [2] for an overview and a list of references). This functional appeared in different contexts, for example in the analysis of Sobolev and harmonic maps with singularities (see *e.g.* [9]), in the study of polyconvex functionals applied to nonlinear elasticity problems (see *e.g.* [17]), or also of Ginzburg-Landau type energies (see *e.g.* [29], [3]). We recall also that the maps with values in the spheres and bounded distributional Jacobian are, in some sense, the vectorial counterpart of finite perimeter sets (see *e.g.* [28], [2]). Moreover, we point out that in many cases the distributional Jacobian of a map $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^k$ can also be interpreted, in the perspective of Cartesian currents, as the projection onto $\mathbb{R}^n$ of the vertical part of a suitable Cartesian current (see *e.g.* [28]). Even if we shall come back to the example of the vortex map, the starting point of our analysis is more related to the piecewise constant maps examples. In order to prove the non-subadditivity of $\overline{\mathcal{A}}(\mathbf{u}_T, \cdot)$, Acerbi and Dal Maso provide suitable upper and lower estimates for $\overline{\mathcal{A}}(\mathbf{u}_T, \cdot)$; in [5] Bellettini and Paolini refine the upper bound of [1], estimating the singular contribution of the relaxed area functional for the triple point map by the area of a solution to a suitable non-parametric minimizing area problem in $\mathbb{R}^3$, whose boundary conditions depend on the traces of $\mathbf{u}_T$ on the jump set and on the presence of the triple point. The fact that this minimal surface can be written as a graph of a scalar map on a planar domain plays an important role in their computation, but, to be more precise, they use only the fact that the minimal surface can be parametrized by a map $\Phi : O \subset \mathbb{R}^2 \to \mathbb{R}^3$ whose first component is the identity in the first parameter (this is always verified by a graph-type surface). We will call the pair (O, Φ) a *semicartesian parametrization.*

In this thesis we aim to study the functional $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$, as well as the functional $\overline{\mathcal{A}}^\infty(\mathbf{u}, \Omega)$ that we are going to define later, for maps $\mathbf{u} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ jumping on a segment. Thus, from now on, $n = k = 2$. We shall compare the two functionals and characterize $\overline{\mathcal{A}}^\infty(\mathbf{u}, \Omega)$ in terms of a minimizing area problem among semicartesian surfaces satisfying suitable boundary conditions. We observe that the class of maps that we are considering can be seen as the natural generalization of the class of piecewise constant maps without any triple point.
Before stating the main results, we give some more detail on the semicartesian setting and on $\overline{\mathcal{A}}^\infty(\cdot, \Omega)$.

Let $\gamma^\pm \in \mathrm{Lip}([a, b]; \mathbb{R}^2)$ be two curves defined on a bounded closed interval $[a, b] \subset \mathbb{R}$, and let us denote by $\Gamma^\pm$ their graphs, that are curves contained in $[a, b] \times \mathbb{R}^2$. We set $\Gamma := \Gamma^- \cup \Gamma^+$ and we refer to Γ as to a *union of two graphs.* We observe that, by construction, the intersection of Γ with any plane of the form $\{t\} \times \mathbb{R}^2$, $t \in [a, b]$, is composed of only two (possibly coinciding) points, $(t, \gamma^-(t))$ and $(t, \gamma^+(t))$. Intuitively, a surface is said *semicartesian* and *spanning* Γ if its intersection with each of these planes is a curve, not necessarily simple, connecting $(t, \gamma^-(t))$ and $(t, \gamma^+(t))$. We notice that, depending on the values of $\gamma^\pm$ at $t = a$ and $t = b$, Γ can be either a closed curve, or the union of two open curves, or an open curve. We shall deal only with the first two situations. If Γ is union of two open curves, the boundary of any semicartesian surface spanning Γ is composed, besides Γ^- and Γ^+, of two other curves, not reduced to a single point, lying on the planes $\{a\} \times \mathbb{R}^2$ and $\{b\} \times \mathbb{R}^2$, respectively; we shall refer to these two curves as to the *free boundary.* If instead Γ is closed, a semicartesian surface spanning Γ has a possibly empty free boundary; in this case its intersection with the plane $\{a\} \times \mathbb{R}^2$ (resp. $\{b\} \times \mathbb{R}^2$) consists only in the point $(a, \gamma^\pm(a))$ (resp. $(b, \gamma^\pm(b))$). We are now in the position

Chapter 2

to define rigorously these two classes of semicartesian surfaces spanning Γ.

- Let us set $\mathrm{R} := (a,b) \times (-1,1) \subset \mathbb{R}^2_{(t,s)}$. The class of semicartesian maps spanning Γ with partially free boundary is defined as

$$\mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+) := \Big\{\Phi \in \mathcal{C}(\overline{\mathrm{R}};\mathbb{R}^3) \cap H^1(\mathrm{R};\mathbb{R}^3) : \Phi(t,s) = (t,\phi(t,s)), \\ \phi(t,\pm 1) = \gamma^\pm(t)\Big\}.$$

  We underline that there is no condition on the values assumed by a map $\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ on the vertical sides of the rectangle R.

- If Γ is closed, let us define $D := \{(t,s) \in \mathbb{R}^2 : t \in (a,b), s \in (\sigma^-(t),\sigma^+(t))\}$ for some $\sigma^\pm \in \mathrm{Lip}([a,b])$, $\sigma^- < 0$ and $\sigma^+ > 0$ on (a,b), and $\sigma^\pm(a) = 0 = \sigma^\pm(b)$. Then the class of semicartesian maps spanning Γ without free boundary is defined as

$$\mathrm{semicart}(D;\Gamma^-,\Gamma^+) := \Big\{\Phi \in \mathcal{C}(\overline{D};\mathbb{R}^3) \cap H^1(D;\mathbb{R}^3) : \Phi(t,s) = (t,\phi(t,s)), \\ \phi(t,\sigma^\pm) = \gamma^\pm(t)\Big\}.$$

We prove that $\mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ is non empty, and that, whenever Γ is closed, the same holds for $\mathrm{semicart}(D;\Gamma^-,\Gamma^+)$, see Lemma 2.10. Thus the problem of minimizing the area among semicartesian surfaces spanning Γ is well-posed, and we can define

$$m(\mathrm{R};\Gamma^-,\Gamma^+) := \inf_{\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)} \int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds, \tag{0.3}$$

and, if Γ is closed, also

$$m(D;\Gamma^-,\Gamma^+) := \inf_{\Phi \in \mathrm{semicart}(D;\Gamma^-,\Gamma^+)} \int_{D} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds. \tag{0.4}$$

We refer to the integral on the right hand side of (0.3) (resp. of (0.4)) as to the *area of the semicartesian parametrization* (R,Φ) (resp. (D,Φ)). We observe that, when Γ is closed, we trivially have $m(\mathrm{R};\Gamma^-,\Gamma^+) \le m(D;\Gamma^-,\Gamma^+)$, since any image of D through a map in $\mathrm{semicart}(D;\Gamma^-,\Gamma^+)$ can be also parametrized on R by a map in $\mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$.
An issue that immediately arises concerns the existence of an area minimizing semicartesian parametrization; we shall present later some results meant to address this problem, that is, to our knowledge, still open. We also underline that, by definition, every semicartesian surface, with or without free boundary, has the topology of the disk; thus, at least when Γ is a closed and simple curve, it seems quite natural to study the relations between $m(\mathrm{R};\Gamma^-,\Gamma^+)$, $m(D;\Gamma^-,\Gamma^+)$, and $a(\Gamma)$, where $a(\Gamma)$ denotes the area of an area minimizing solution of the classical Plateau's problem for the Jordan curve Γ. We recall that the Plateau's problem consists in finding an area minimizing surface, among all immersions of the disk mapping the boundary of the

disk monotonically onto Γ. It is not difficult to see that $a(\Gamma) \leq m(D;\Gamma^-,\Gamma^+)$, and we are able to prove also the opposite inequality under further regularity assumptions on $\Gamma^\pm$ (see Theorems 7.3 and 7.4). We expect anyway that removing these additional hypotheses should be possible. The situation for $m(\mathrm{R};\Gamma^-,\Gamma^+)$ is different. Due to the presence of the free boundary, there exist maps $\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ such that $\Phi(\mathrm{R})$ cannot be parametrized by any of the immersions considered in the classical Plateau's problem. Moreover, in Example 2.15 we show a situation where $m(\mathrm{R};\Gamma^-,\Gamma^+) < a(\Gamma)$ (and consequently also $m(\mathrm{R};\Gamma^-,\Gamma^+) < m(D;\Gamma^-,\Gamma^+)$).

Another natural question concerning problems (0.3) and (0.4) is their dependence on the initial data $\Gamma^\pm$. Proving suitable lower semicontinuity properties turns out to be crucial in order to provide estimates from below for the relaxed area functional of maps $\mathbf{u}$ jumping on a segment. The lower semicontinuity result that we would need in order to completely characterize $\overline{\mathcal{A}}(\mathbf{u},\Omega)$ would be that Chapter 3

$$m(\mathrm{R};\Gamma^-,\Gamma^+) \leq \liminf_{h\to+\infty} m(\mathrm{R};\Gamma_h^-,\Gamma_h^+), \tag{0.5}$$

whenever $(\gamma_h^\pm)$ converge to $\gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$, where $(\gamma_h^\pm) \subset \mathrm{Lip}([a,b];\mathbb{R}^2)$ are such that $\Gamma_h^\pm = \mathrm{graph}(\gamma_h^\pm)$ for every $h \in \mathbb{N}$, as well as an analogous result for $m(D;\cdot,\cdot)$. We are able to prove (0.5) only requiring a uniform bound on the L^∞-norm of the derivatives of $\gamma_h^\pm$. We do not know whether this further assumption can be removed in the semicontinuity result for $m(\mathrm{R};\cdot,\cdot)$; the examples in Chapter 5 suggest that the mere L^1-convergence does not suffice to prove the semicontinuity result for the problem without free boundary. Moreover, there exist (Γ_h) and Γ, Jordan curves union of two graphs, such that Γ_h converges[1] to Γ in L^1 and $a(\Gamma) > \liminf_{h\to+\infty} a(\Gamma_h)$, see Example 3.5.

This line of reasoning leads us to define, besides $\overline{\mathcal{A}}(\cdot,\Omega)$, the functional $\overline{\mathcal{A}}^\infty(\cdot,\Omega)$, obtained by relaxing $\mathcal{A}(\cdot,\Omega)$ with respect to a convergence that is stronger than the one induced by the L^1-topology, and that can be seen as a sort of uniform convergence. Given a map $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$, a closed set $J \subset \Omega$, and a sequence $(\mathbf{v}_h) \subset L^1(\Omega;\mathbb{R}^2)$, we say that *$(\mathbf{v}_h)$ converges uniformly out of J* to $\mathbf{v}$ if it converges uniformly to $\mathbf{v}$ in any compact set of $\Omega \setminus J$. The functional $\overline{\mathcal{A}}^\infty(\mathbf{v},\Omega)$ is then defined as Chapter 4

$$\overline{\mathcal{A}}^\infty(\mathbf{v},\Omega) := \inf\Big\{ \liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h,\Omega) : \ (\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^k), \mathbf{v}_h \to \mathbf{v} \text{ in } L^1(\Omega;\mathbb{R}^k), \\ \text{and uniformly out of } J_{\mathbf{v}} \Big\},$$

where $J_{\mathbf{v}}$ denotes the jump set of $\mathbf{v}$.

We are now ready to give a rigorous description of the class of maps we will take into consideration. We shall study the functionals $\overline{\mathcal{A}}(\mathbf{u},\Omega)$ and $\overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$ for maps $\mathbf{u} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ whose jump set $J_{\mathbf{u}}$ is the segment $[a,b]\times\{0\}$, and that satisfy one of the two following conditions:

- Ω and $\mathbf{u}$ satisfy condition I, that is $\mathbf{u} \in W^{1,\infty}(\Omega \setminus J_{\mathbf{u}};\mathbb{R}^2)$ and both the end points of $J_{\mathbf{u}}$ belong to $\partial\Omega$;

[1] This means that $\Gamma_h = \mathrm{graph}(\gamma_h^-) \cup \mathrm{graph}(\gamma_h^-)$, $\Gamma = \mathrm{graph}(\gamma^-) \cup \mathrm{graph}(\gamma^-)$, $(\gamma_h^\pm) \subset \mathrm{Lip}([a,b];\mathbb{R}^2)$, $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ and $\gamma_h^\pm \to \gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$.

- Ω and $\mathbf{u}$ satisfy condition II, that is $\mathbf{u} \in W^{1,\infty}(\Omega \setminus J_{\mathbf{u}}; \mathbb{R}^2)$ and $J_{\mathbf{u}} \subset\subset \Omega$.

We could as well consider maps jumping on a $\mathcal{C}^2$ simple and open curve, whose distinct end points either belong to $\partial\Omega$ or are internal points of Ω; in the Appendix we explain how to adapt to these cases some of the results that we are going to describe (in particular Theorems 4.7 and 4.9).
Both when Ω and $\mathbf{u}$ satisfy condition I and when they satisfy condition II, we can identify the plane containing Ω with the plane $\mathbb{R}^2_{(t,s)}$ where the domains R and D are defined, so that $\mathbf{u} = \mathbf{u}(t,s)$. Moreover, we can see the two traces of $\mathbf{u}$ on the two sides of $J_{\mathbf{u}}$ as two curves $\gamma^{\pm}[\mathbf{u}] \in \mathrm{Lip}((a,b); \mathbb{R}^2)$, that we can always assume to be defined also at $t = a$ and $t = b$; we set $\Gamma^{\pm}[\mathbf{u}] := \mathrm{graph}(\gamma^{\pm}[\mathbf{u}])$ and $\Gamma[\mathbf{u}] := \Gamma^-[\mathbf{u}] \cup \Gamma^+[\mathbf{u}]$. Notice that if Ω and $\mathbf{u}$ satisfy condition I, then $\Gamma[\mathbf{u}]$ will be in general composed of two open curves, while if Ω and $\mathbf{u}$ satisfy condition II, then $\Gamma[\mathbf{u}]$ will be a closed (not necessarily simple) curve.
With these notations, our results concerning the characterization of $\overline{\mathcal{A}}^{\infty}(\mathbf{u}, \Omega)$ are contained in the following theorems.

Theorem 0.1. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *Then*

$$\overline{\mathcal{A}}^{\infty}(\mathbf{u}, \Omega) = \int_{\Omega} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]). \tag{0.6}$$

Theorem 0.2. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* II. *Then*

$$\overline{\mathcal{A}}^{\infty}(\mathbf{u}, \Omega) = \int_{\Omega} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + m(D; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]). \tag{0.7}$$

The proof of both results is divided into two parts. In the first part (see Theorems 4.7 and 4.9) we show that $\overline{\mathcal{A}}^{\infty}(\Omega, \mathbb{R}^2)$ is less than or equal to the right hand side of (0.6) (resp. of (0.7)): since $\overline{\mathcal{A}}^{\infty}(\cdot, \Omega)$ is L^1-lower semicontinuous, the upper bound is obtained by exhibiting a sequence $(\mathbf{u}_h)$ of regular maps converging to $\mathbf{u}$ in $L^1(\Omega; \mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$ such that $\mathcal{A}(\mathbf{u}_h, \Omega)$ tends to the right hand side of (0.6) (resp. of (0.7)) as $h \to +\infty$. Omitting here all technical details, let us sketch our construction of the sequence $(\mathbf{u}_h)$ when Ω and $\mathbf{u}$ satisfy condition I. Let $\Phi \in \mathrm{semicart}(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}])$, and let $\phi \in H^1(\mathrm{R}; \mathbb{R}^2)$ be such that $\Phi(t,s) = (t, \phi(t,s))$. For every $c > 0$ we define $\mathrm{R}_c := (a,b) \times (-c, c)$; observe that $J_{\mathbf{u}} \subset \mathrm{R}_c$. Let (ε_h) be an infinitesimal sequence of positive numbers; for every $h \in \mathbb{N}$ the map $\mathbf{u}_h$ is suitably defined so that:

- $\mathbf{u}_h(t,s) = \mathbf{u}(t,s)$ in $\Omega \setminus \mathrm{R}_{2\varepsilon_h}$,
- $\mathbf{u}_h(t,s) = \phi(t, s/\varepsilon_h)$ in $\mathrm{R}_{\varepsilon_h}$.

The regularity of $\mathbf{u}$ out of the jump set implies that $\mathcal{A}(\mathbf{u}_h, \Omega \setminus \mathrm{R}_{\varepsilon_h})$ converges to $\int_{\Omega} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds$, as $h \to +\infty$, while using the semicartesian structure of Φ we can prove that

$$\lim_{h \to +\infty} \int_{\mathrm{R}_{\varepsilon_h}} |\mathcal{M}(\nabla \mathbf{u}_h)| \, dt \, ds = \int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds.$$

If the area of (R,Φ) equals $m(\mathrm{R};\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$, then the sequence $(\mathbf{u}_h)$ provides the desired upper bound for $\overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$. Otherwise we have to consider an area minimizing sequence of semicartesian parametrizations $((\mathrm{R},\Phi_h))$, and define $(\mathbf{u}_h)$ through a diagonal process. Similarly we get the upper bound also when Ω and $\mathbf{u}$ satisfy condition II. We underline that this strategy proves also

$$\overline{\mathcal{A}}(\mathbf{u},\Omega) \leq \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + m(\mathrm{R};\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]),$$

for Ω and $\mathbf{u}$ satisfying condition I, and

$$\overline{\mathcal{A}}(\mathbf{u},\Omega) \leq \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]),$$

for Ω and $\mathbf{u}$ satisfying condition II.
In order to conclude the proof of Theorems 0.1 and 0.2, we have to prove a lower bound for $\overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$. More precisely, we show that for any sequence $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of $J_\mathbf{u}$, the limit of $\mathcal{A}(\mathbf{u}_h,\Omega)$ is greater than or equal to the right hand side of (0.6) or (0.7) respectively (see Theorems 4.11 and 4.17). The proof of this lower bound is more delicate than the proof of the upper bound. First of all, we show that we can limit ourselves to prove the result for sequences $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and such that $\mathbf{u}_h$ coincides with $\mathbf{u}$ out of a neighbourhood of the jump, (see Propositions 4.13 and 4.19); hence, we can assume that there exists a decreasing sequence of neighbourhoods of $J_\mathbf{u}$ such that $\bigcap_{h\in\mathbb{N}} N_h = J_\mathbf{u}$ and $\mathbf{u}_h = \mathbf{u}$ in $\Omega \setminus N_h$, for every $h \in \mathbb{N}$. For such a sequence, we prove that $\mathcal{A}(\mathbf{u}_h, N_h)$ can be estimated from below by the area of a semicartesian parametrization spanning some $\Gamma_h^\pm = \mathrm{graph}(\gamma_h^\pm)$, where $\gamma_h^\pm$ depend on the traces of $\mathbf{u}_h$ on the boundary of ∂N_h. From the assumptions on $(\mathbf{u}_h)$, it turns out that $(\gamma_h^\pm)$ suitably converges to $\gamma^\pm[\mathbf{u}]$ and satisfies the hypotheses of the lower semicontinuity result for $m(\mathrm{R};\cdot,\cdot)$ (or $m(D;\cdot,\cdot)$). Thus (see Propositions 4.12 and 4.18) one gets

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, N_h) \geq \liminf_{h\to+\infty} m(\mathrm{R};\Gamma_h^-,\Gamma_h^+) \geq m(\mathrm{R};\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]),$$

(or an analogous statement for $m(D;\cdot,\cdot)$), concluding therefore the proof of Theorem 0.1 (and of Theorem 0.2). We stress that this latter part of the proof cannot be adapted to $\overline{\mathcal{A}}(\mathbf{u},\Omega)$ in place of $\overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$, since we lack the L^1-semicontinuity result for $m(\mathrm{R};\cdot,\cdot)$ and $m(D;\cdot,\cdot)$.
This fact seems not to be just a technical issue. Indeed, if on one hand we expect $\overline{\mathcal{A}}(\mathbf{u},\Omega) = \overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$ for Ω and $\mathbf{u}$ satisfying condition I (at least when $\Gamma^-[\mathbf{u}]\cap\Gamma^+[\mathbf{u}] = \emptyset$), on the other hand we exhibit examples of pairs $(\Omega,\mathbf{u})$ as in condition II such that $\overline{\mathcal{A}}(\mathbf{u},\Omega) < \overline{\mathcal{A}}^\infty(\mathbf{u},\Omega)$. This inequality is obtained by building sequences $(\mathbf{u}_h)$ of regular maps converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$, but *not* uniformly out of $J_\mathbf{u}$, such that Chapter 5

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) - \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds < m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]).$$

It is interesting to stress that our construction provides a sequence $(\mathbf{u}_h)$ converging to $\mathbf{u}$ uniformly out of some set J_{ext}, union of the jump $J_\mathbf{u}$ and of a simple curve connecting $J_{\mathbf{u}}$ to $\partial\Omega$; we call this curve *virtual jump*. How to choose the virtual jump in the "most convenient way" (and even how to define the regularity class of it) is an open and apparently not easy problem. Our examples reveal that the quantity $\overline{\mathcal{A}}(\mathbf{u},\Omega) - \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds$ depends not only on the traces of $\mathbf{u}$ on the two sides of $J_{\mathbf{u}}$, but also on the values of $\mathbf{u}$ far from the jump, and on the relative position of the jump with respect to $\partial\Omega$. These examples have been inspired by a construction presented in [1] (see also [21]) used to estimate the singular contribution of the area of the vortex map by the surface of a cylinder. We underline that these examples confirm the highly non-local behaviour of the functional $\overline{\mathcal{A}}(\cdot,\Omega)$, and moreover they justify the introduction of the functional $\overline{\mathcal{A}}^{\infty}(\cdot,\Omega)$.

Chapter 6 We also prove a result concerning the non-subadditivity of the relaxed area functional, see Theorem 6.1. As previously recalled, Acerbi and Dal Maso prove that $\overline{\mathcal{A}}(\mathbf{v},\cdot)$ is not subadditive when $\mathbf{v}$ is either the triple point map or the vortex map; in both cases, the non-subadditivity seems to be related to the behaviour of $\mathbf{v}$ at the origin. In this thesis we show that the same phenomenon happens also for completely different classes of maps. More precisely, we prove that $\overline{\mathcal{A}}(\mathbf{u},\cdot)$ is not subadditive for a map $\mathbf{u} = (u_1, u_2) : \Omega \to \mathbb{R}^2$, Ω and $\mathbf{u}$ satisfying condition I (therefore $\mathbf{u}$ jumping on a segment), where u_1 is regular and u_2 is a piecewise constant function assuming two values. We underline that this class of maps is the first generalization of the class of piecewise constant maps without triple points (for which we recall that the relaxed area functional is subadditive); this suggests that the non-subadditivity is a much general feature of $\overline{\mathcal{A}}$.

Chapter 7 In the last two chapters we go back to the problem of the existence of an area minimizing surface in the semicartesian context, following two different approaches. In Chapter 7, we consider curves $\Gamma^{\pm}$, graphs of $\gamma^{\pm} \in \mathcal{C}([a,b];\mathbb{R}^2) \cap \mathrm{Lip}_{\mathrm{loc}}((a,b),\mathbb{R}^2)$, such that their union Γ is a Jordan curve; this implies that the Plateau's problem for Γ is well-defined and admits a solution $Y \in \mathcal{C}^{\omega}(B;\mathbb{R}^3) \cap \mathcal{C}(\overline{B};\mathbb{R}^3)$, where B denotes the unit disk. Since in general the area $a(\Gamma)$ of Y is less than or equal to the area of any semicartesian surface spanning Γ without free boundary, our strategy consists in proving that $\Sigma_{\min} := Y(B)$ can be parametrized by a semicartesian map. We are able to prove this result under strong regularity assumptions on Γ, see Theorem 7.3. More specifically, if Γ is analytic and $\Gamma^{\pm}$ join in a non-degenerate way, then there exists a semicartesian map Φ defined on a domain O such that $\Phi(O) = \Sigma_{\min}$ and whose area is $a(\Gamma)$. Furthermore, by construction, Φ is conformal, and we are able to describe qualitatively the shape of O. We observe that this case does not fit into the previous setting, since, due to the analyticity of Γ, the curves $\gamma^{\pm}$ are not globally Lipschitz on $[a,b]$, and also the boundary of the domain O turns out to be analytic (differently from the boundary of the domain D previously defined). We adapt the proof to a curve Γ union of two $\mathcal{C}^{1,\alpha}$ graphs, $\alpha \in (0,1)$, that is instead a particular case of our general setting, for which $\mathrm{semicart}(D;\Gamma^-,\Gamma^+)$ and $m(D;\Gamma^-,\Gamma^+)$ are well-defined. In this case we prove a slightly different result, see Theorem 7.4: indeed, even though we do not provide a semicartesian parametrization for the whole $\Sigma_{\min}$, we show that there exists a sequence of semicartesian parametrizations spanning Γ whose area converges to $a(\Gamma)$, proving therefore that $m(D;\Gamma^-,\Gamma^+) = a(\Gamma)$.

Chapter 8 We conclude our discussion providing some results on the minimization of the Dirichlet functional among semicartesian parametrizations spanning a Jordan curve Γ union of two Lipschitz graphs. In [15, Chapter 4] the existence of a solution of the Plateau's problem for Γ is obtained by minimizing the Dirichlet functional among all immersions of the disk spanning Γ, and by proving that this solution is also an area minimizing surface. For this second step it is crucial to guarantee that the minimizing immersion is also $\mathcal{C}^2$-regular and conformal. Our aim is to adapt this strategy to the semicartesian context. Since the Dirichlet functional is not invariant with respect to reparametrizations, we are no more allowed to consider only semicartesian maps defined on the fixed domain D, but we have instead to let both the domains and the maps vary. Therefore we define the class semicart(Γ) containing all semicartesian parametrizations $([[\sigma^-, \sigma^+]], \Phi)$, where

$$[[\sigma^-, \sigma^+]] := \{(t,s); t \in (a,b),\ s \in (\sigma^-(t), \sigma^+(t))\}$$

for suitable functions $\sigma^\pm \in \mathrm{Lip}([a,b])$, and $\Phi \in \mathcal{C}(\overline{[[\sigma^-, \sigma^+]]}; \mathbb{R}^3) \cap H^1([[\sigma^-, \sigma^+]]; \mathbb{R}^3)$ is a semicartesian map spanning Γ. For every $\mathrm{S} > 0$, we also define the class semicart$_\mathrm{S}$(Γ) of all semicartesian parametrizations $([[\sigma^-, \sigma^+]], \Phi) \in$ semicart(Γ) such that $\mathrm{lip}(\sigma^\pm) \leq \mathrm{S}$. Due to the compactness properties of this second class, it is not difficult to prove the existence of a semicartesian parametrization $([[\sigma^-_\mathrm{S}, \sigma^+_\mathrm{S}]], \Phi_\mathrm{S})$ minimizing the Dirichlet functional in semicart$_\mathrm{S}$(Γ), see Theorem 8.4. The existence of a minimizer in the whole space semicart(Γ) is instead obtained only assuming the further hypothesis that

$$\Phi_\mathrm{S} \in \mathrm{Lip}(\overline{[[\sigma^-_\mathrm{S}, \sigma^+_\mathrm{S}]]}; \mathbb{R}^3) \tag{0.8}$$

for every S large enough, see Theorem 8.6. We are not aware whether this *a priori* regularity requirement can be removed. The proof of Theorem 8.4 follows proving that, for $\mathrm{S} > G := \max\{\mathrm{lip}(\gamma^-), \mathrm{lip}(\gamma^+)\}$ and assuming (0.8), Φ_S is conformal and $\mathrm{lip}(\sigma^\pm_\mathrm{S}) \leq G$, see Theorem 8.5.

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1. Preliminaries

Overview of the chapter

Sections 1.2-1.4 of this chapter are dedicated to introducing the relaxed area functional and the results present in the literature we will refer to in this thesis. In Sections 1.5 we collect some notations and definitions concerning the theory of Cartesian currents, that we shall need in the proof of Lemma 6.6 and of Theorem 6.1; in Sections 1.6 and 1.7 we present some results about the classical Plateau's problem and Morse theory that will be needed in Chapter 7.

1.1 Basic notation

In this thesis we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, endowed with the Euclidean norm $|\cdot|$; if necessary we explicit in the subscript the variables of the space, *e.g.* $\mathbb{R}^2_{(t,s)}$. The symbol $B_r(\mathbf{x})$ denotes the ball of radius r centred at $\mathbf{x}$; if $\mathbf{x} = 0$ we omit to specify the center and we write B_r, while $B := B_1$ is the unit disk centred in the origin. The symbol Ω always denotes an open and bounded subset of $\mathbb{R}^n$. We use the standard symbols $\mathcal{C}(\Omega;\mathbb{R}^k)$, $\mathcal{C}^m(\Omega;\mathbb{R}^k)$, $\mathcal{C}^{m,\alpha}(\Omega;\mathbb{R}^k)$ to denote the space of continuous maps, of m-times differentiable with continuity maps, and of m-times differentiable with continuity maps with m-th derivative Hölder continuous with exponent α. The symbol $\mathrm{Lip}(\Omega;\mathbb{R}^k)$ denotes the space of Lipschitz maps and for every $\mathbf{u} \in \mathrm{Lip}(\Omega;\mathbb{R}^k)$, we indicate by $\mathrm{lip}(\mathbf{u})$ its Lipschitz constant. $L^p(\Omega;R^k)$ for $p \in [1,+\infty]$ is the Lebesgue space of exponent p, and $W^{1,p}(\Omega;\mathbb{R}^k)$ is the space of maps belonging, with their distributional derivative, to $L^p(\Omega;\mathbb{R}^k)$; we set also $H^1(\Omega;\mathbb{R}^k) := W^{1,2}(\Omega;\mathbb{R}^k)$. If $k = 1$, we omit to indicate the target space (*e.g.* $\mathcal{C}(\Omega)$ in place of $\mathcal{C}(\Omega;\mathbb{R})$). We usually use the bold style to refer to a vectorial valued map, and the plain style with subscript for the components; *e.g.* $\mathbf{u} \in \mathrm{Lip}(\Omega;\mathbb{R}^2)$, $\mathbf{u} = (u_1, u_2)$. If not differently specified, we use ∂_t to indicate the partial derivative (in this case with respect to the variable t). For a function f depending on one variable we denote the derivative either as $\dot{f}$ or as f'. The n-dimensional Lebesgue and Hausdorff measure are denoted by $\mathcal{L}^n$ and $\mathcal{H}^n$ respectively.

1.2 Area of a graph

Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set; given a map $\mathbf{v} \in \mathcal{C}^1(\Omega;\mathbb{R}^k)$ the *(non-parametric) area functional* $\mathcal{A}(\cdot,\Omega)$ associates to $\mathbf{v}$ the $\mathcal{H}^n$-measure of its graph, taking into account possible multiplicities; it is thus defined as

$$\mathcal{A}(\mathbf{v},\Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\, d\mathbf{x}, \tag{1.1}$$

where for any $(k \times n)$ matrix ζ the symbol $\mathcal{M}(\zeta)$ denotes the n-vector whose components are minors up to order $\min\{n,k\}$ of $\zeta^{(1)}$.

The properties of $\mathcal{A}(\cdot,\Omega)$ change depending on the value of $\min\{n,k\}$; if $\min\{n,k\} = 1$, the functional turns out to be convex, while for $\min\{n,k\} > 1$ it is only polyconvex, see *e.g.* [11].

In order to study minimum problems involving the area functional, it is convenient to provide a notion of area also for less regular maps; we agree, *e.g.* [13], [14], [1], [21], [20], in extending $\mathcal{A}$ formally to the class $L^1(\Omega;\mathbb{R}^k)$ by setting $\mathcal{A}(\mathbf{v},\Omega) = +\infty$ for every $\mathbf{v} \in L^1(\Omega;\mathbb{R}^k) \setminus \mathcal{C}^1(\Omega;\mathbb{R}^k)$ and then considering the *relaxed area functional* $\overline{\mathcal{A}} : L^1(\Omega;\mathbb{R}^k) \to [0,+\infty]$, defined as the L^1-lower semicontinuous envelop of $\mathcal{A}(\cdot,\Omega)$.

Definition 1.1 (**Relaxed area functional**)**.** For every $\mathbf{v} \in L^1(\Omega;\mathbb{R}^k)$ we define

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) := \inf\left\{\liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h,\Omega)\right\}$$

where the infimum is taken over all sequences $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{v}$ in $L^1(\Omega;\mathbb{R}^2)$.

For more details on the relaxation technique we refer again to [11].

The interest of definition (1.1) is clearly seen in the scalar case, that is for functions $v : \Omega \to \mathbb{R}$, where this notion of extended area is useful for solving non-parametric minimal surface problems, under various type of boundary conditions (see for instance [23], [31], [21]). In this case the functional $\overline{\mathcal{A}}(v,\Omega)$ can be represented by an integral over Ω, whose integrand and measure depend both on $\mathbf{v}$; moreover its domain is well characterized and coincides with the space $\mathrm{BV}(\Omega)$ of functions with *bounded variation*. If $\min\{n,k\}$ is strictly greater than 1, instead, the situation is much more involved. In Section 1.4 we present some already known results and some open problems concerning the case of interest for this thesis, that is $n = k = 2$. Before this we recall some notations about the space $\mathrm{BV}(\Omega;\mathbb{R}^k)$.

1.3 The space $\mathrm{BV}(\Omega;\mathbb{R}^k)$

We refer to [4] for an exhaustive presentation of the theory of bounded variation maps. We recall here only some basic definitions and notations for the space $\mathrm{BV}(\Omega;\mathbb{R}^k)$.

$^{(1)}$The minor of order 0 is included and it is, by convention, equal to 1.

Definition 1.2 ($\mathrm{BV}(\Omega;\mathbb{R}^k)$)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The symbol $\mathrm{BV}(\Omega;\mathbb{R}^k)$ denotes the space of maps with bounded variation, that is those maps $\mathbf{v} \in L^1(\Omega;\mathbb{R}^k)$ whose distributional gradient $D\mathbf{v}$ is representable by a bounded Radon measure on Ω with values in the space of $(k \times n)$ matrices.

Since $D\mathbf{v}$ is a bounded Radon measure, we can univoquely decompose it into the *absolutely continuous* and the *singular* component with respect to $\mathcal{L}^n$, denoted by $\nabla\mathbf{v}$ and $D^s\mathbf{v}$ respectively. Thus for any Borel set $E \subset \Omega$ there holds $D\mathbf{v}(E) = \int_E \nabla\mathbf{v}\, d\mathbf{x} + D^s\mathbf{v}(E)$. With this notation and equipping the space of $(k \times n)$ matrices with the Hilbert-Schmidt norm defined by $||\zeta||^2 = \mathrm{tr}(\zeta\zeta^*)$, the total variation of any map $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ turns out to be

$$|D\mathbf{v}|(E) = \int_E ||\nabla\mathbf{v}||\, d\mathbf{x} + |D^s\mathbf{v}|(E).$$

For $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$, the measure $D^s\mathbf{v}$ can be further decomposed into the so called *jump part* $D^j\mathbf{v}$ and *Cantor part* $D^c\mathbf{v}$. In this thesis we shall deal only with maps without Cantor part, thus we limit ourselves to define $D^j\mathbf{v}$, referring to [4, Sections 3.6-3.9] for more details.

Definition 1.3 (**Approximate jump points**)**.** Let $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$. A point $\mathbf{x} \in \Omega$ is said to be an *approximate jump point* of $\mathbf{v}$ if there exist $p, q \in \mathbb{R}^k$, $p \neq q$ and $\nu \in \mathbb{S}^{n-1}$ such that

$$\lim_{r\downarrow 0^+} \fint_{B_r^+(\mathbf{x},\nu)} |\mathbf{v}(\mathbf{y}) - p|\, d\mathbf{y} = 0,$$
$$\lim_{r\downarrow 0^+} \fint_{B_r^-(\mathbf{x},\nu)} |\mathbf{v}(\mathbf{y}) - q|\, d\mathbf{y} = 0,$$

where $B_r^\pm(\mathbf{x},\nu)$ denote the sets of points of $B_r(\mathbf{x})$ such that $\langle \mathbf{y} - \mathbf{x}, \nu\rangle > 0$ and $\langle \mathbf{y} - \mathbf{x}, \nu\rangle < 0$ respectively.

Definition 1.4 (**Jump set and traces**)**.** Given a map $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^k)$ the set of approximate jump points is denoted by $J_\mathbf{v}$. For every $\mathbf{x} \in J_\mathbf{v}$ the triplet (p, q, ν) is uniquely determined by (1.3) (up to permutation and a change of sign of ν) and it is denoted by $(\mathbf{v}^+(\mathbf{x}), \mathbf{v}^-(\mathbf{x}), \nu_\mathbf{v}(\mathbf{x}))$. We refer to the maps $\mathbf{v}^\pm$ as the traces of $\mathbf{v}$ on the jump set $J_\mathbf{v}$.

Remark 1.5. As specified in Definition 4.3 and 4.4, with a small abuse of language we will indicate by $J_\mathbf{u}$ the *closure* of the set of approximate jump points.

1.4 The functional $\overline{\mathcal{A}}$: case $n = k = 2$

A fundamental reference for the study of the relaxed area functional (and of the relaxation of other polyconvex functionals) in general dimension and codimension is [1]. We collect here the main results on the functional $\overline{\mathcal{A}}(\cdot,\Omega)$ in the case $n = k = 2$, that are presented in [1] in larger generality. From now on Ω will denote a bounded open subset of $\mathbb{R}^2$, $\mathbf{x} := (x, y)$, and the integrand in (1.1) is

$$|\mathcal{M}(\nabla\mathbf{v})| = \sqrt{1 + |\nabla v_1|^2 + |\nabla v_2|^2 + (\partial_x v_1 \partial_y v_2 - \partial_y v_1 \partial_x v_2)^2},$$

where $\mathbf{v} = (v_1, v_1) \in \mathcal{C}^1(\Omega;\mathbb{R}^2)$.

1.4.1 Lower semicontinuity of $\int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy$

First of all the authors prove that $\overline{\mathcal{A}}(\cdot,\Omega)$ coincides with $\mathcal{A}(\cdot,\Omega)$ in $C^1(\Omega;\mathbb{R}^2)$, as conjectured by De Giorgi in [13]. More precisely:

Theorem 1.6 (Theorem 3.4, [1]). *The functional $\mathcal{A}(\cdot,\Omega)$ is lower semicontinuous on $\mathcal{C}^1(\Omega;\mathbb{R}^2) \cap L^1(\Omega;\mathbb{R}^2)$ with respect to the strong topology in $L^1(\Omega;\mathbb{R}^2)$. Thus, in particular*

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) = \mathcal{A}(\mathbf{v},\Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy, \tag{1.2}$$

for every $\mathbf{v} \in \mathcal{C}^1(\Omega;\mathbb{R}^2) \cap L^1(\Omega;\mathbb{R}^2)$.

We observe also that the right hand side of (1.2) is well defined in a space that is wider than $\mathcal{C}^1(\Omega;\mathbb{R}^2)$ (*i.e.* the space of maps $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^2)$ such that $\mathcal{M}(\nabla \mathbf{v})$ belongs to $L^1(\Omega;\mathbb{R}^6)$). There holds the following characterization of the subset of $\mathrm{BV}(\Omega;\mathbb{R}^2)$ where the relaxed area functional can be represented as in (1.2).

Theorem 1.7 (Theorem 6.4, [1]). *Let $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$. The following two conditions are equivalent:*

(i) $\overline{\mathcal{A}}(\mathbf{v},\Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy,$

(ii) $\mathbf{v} \in W^{1,1}(\Omega;\mathbb{R}^2)$, $\mathcal{M}(\nabla \mathbf{v}) \in L^1(\Omega;\mathbb{R}^6)$ *and there exists a sequence* $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$, $\mathbf{v}_h \to \mathbf{v}$ *in* $L^1(\Omega;\mathbb{R}^2)$ *such that* $(\mathcal{M}(\nabla \mathbf{v}_h))$ *converges to* $\mathcal{M}(\nabla \mathbf{v})$ *in* $L^1(\Omega;\mathbb{R}^6)$.

We use the symbol $\mathcal{D}(\Omega;\mathbb{R}^2)$ to denote the set of maps in $\mathrm{BV}(\Omega;\mathbb{R}^2)$ satisfying condition (i) of Theorem 1.7. On $\mathcal{D}(\Omega;\mathbb{R}^2)$ we agree to write $\mathcal{A}$ in place of $\overline{\mathcal{A}}$.
We underline that $W^{1,p}(\Omega;\mathbb{R}^2) \subset \mathcal{D}(\Omega;\mathbb{R}^2)$ for every $p \in [2,+\infty]$.(2) This fact is interesting since we will use sequences in $H^1(\Omega;\mathbb{R}^2)$ (and not in $\mathcal{C}^1(\Omega;\mathbb{R}^2)$) in order to estimate from above the value of the relaxed area functional for maps jumping on a segment, see Theorems 4.7 and 4.9. Indeed we can prove that $\overline{\mathcal{A}}(\cdot,\Omega)$ can be seen also as the relaxation of the functional $\mathbf{v} \to \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy$ defined on $\mathcal{D}(\Omega;\mathbb{R}^2)$ with respect to the convergence in $L^1(\Omega;\mathbb{R}^2)$.

Lemma 1.8 (Appendix, [6]). *Let $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$. Then*

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) = \inf\left\{\liminf_{h\to+\infty} \int_\Omega |\mathcal{M}(\nabla \mathbf{v}_h)|\,dx\,dy,\ (\mathbf{v}_h) \subset \mathcal{D}(\Omega;\mathbb{R}^2),\ \mathbf{v}_h \overset{L^1(\Omega;\mathbb{R}^2)}{\longrightarrow} \mathbf{v}\right\}. \tag{1.3}$$

(2) The result is optimal, meaning that the functional $\mathbf{v} \to \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy$ is not lower semicontinuous in $W^{1,p}(\Omega;\mathbb{R}^2)$, $p < 2$, with respect to the L^1-topology, and thus it cannot coincide with $\overline{\mathcal{A}}(\cdot,\Omega)$, see[1, Remark 3.11].

Proof. Trivially $\overline{\mathcal{A}}(\mathbf{v},\Omega)$ is larger than or equal to the right hand side of (1.3), since $\mathcal{C}^1(\Omega;\mathbb{R}^2) \subset \mathcal{D}(\Omega;\mathbb{R}^2)$. In order to prove the opposite inequality, it is enough to exhibit a sequence $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$ such that $\lim_{h\to+\infty} \mathcal{A}(\mathbf{v}_h,\Omega)$ equals the right hand side of (1.3).
Let $(\mathbf{u}_h)$ be a sequence in $\mathcal{D}(\Omega;\mathbb{R}^2)$ such that

$$\begin{aligned}\lim_{h\to+\infty} &\int_\Omega |\mathcal{M}(\nabla \mathbf{u}_h)|\,dx\,dy \\ &= \inf\left\{\liminf_{h\to+\infty}\int_\Omega |\mathcal{M}(\nabla \mathbf{v}_h)|\,dx\,dy,\ (\mathbf{v}_h)\subset\mathcal{D}(\Omega;\mathbb{R}^2),\ \mathbf{v}_h \overset{L^1(\Omega;\mathbb{R}^2)}{\longrightarrow} \mathbf{v}\right\}.\end{aligned}$$

From Theorem 1.7, for each $h \in \mathbb{N}$ we can find a sequence $(\mathbf{u}_h^k)$ in $\mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}_h$ in $L^1(\Omega;\mathbb{R}^2)$ as $k \to +\infty$ such that

$$\mathcal{A}(\mathbf{u}_h^k,\Omega) = \int_\Omega \left|\mathcal{M}\left(\nabla \mathbf{u}_h^k\right)\right|\,dx\,dy \overset{k\to+\infty}{\longrightarrow} \int_\Omega |\mathcal{M}\left(\nabla \mathbf{u}_h\right)|\,dx\,dy.$$

Thus by a diagonal process we obtain a sequence $(\mathbf{v}_h) := \left(\mathbf{u}_h^{k(h)}\right) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{v}$ in $L^1(\Omega;\mathbb{R}^2)$ as $h \to +\infty$ such that the right hand side of (1.3) equals

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{v}_h,\Omega);$$

this concludes the proof. □

We recall also a result concerning the possibility of choosing a recovery sequence that is bounded in the $L^\infty(\Omega;\mathbb{R}^2)$ norm.

Lemma 1.9 (Lemma 3.3, [1]). *Let $\mathbf{v} \in L^\infty(\Omega;\mathbb{R}^2)$. Then there exists a sequence $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^2)$ bounded in $L^\infty(\Omega;\mathbb{R}^2)$ such that $\overline{\mathcal{A}}(\mathbf{v},\Omega) = \lim_{h\to+\infty}\mathcal{A}(\mathbf{v}_h,\Omega)$.*

1.4.2 The domain of $\overline{\mathcal{A}}(\cdot,\Omega)$

As in the case $k = 1$, the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ is contained in $\mathrm{BV}(\Omega;\mathbb{R}^2)$, and there holds the following lower bound.

Theorem 1.10 (Theorem 3.7, [1]). *For every $\mathbf{v} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ we have*

$$\overline{\mathcal{A}}(\mathbf{v},\Omega) \geq \int_\Omega |\mathcal{M}(\nabla \mathbf{v})|\,dx\,dy + |D^s\mathbf{v}|(\Omega).$$

Anyway, differently from the scalar case, the domain of $\overline{\mathcal{A}}(\cdot,\Omega)$ is *strictly* contained in $\mathrm{BV}(\Omega;\mathbb{R}^2)$. For example we can consider the map $\mathbf{v}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^{3/2}}$, for $\mathbf{x}$ in the unit disk $B_1((1,0)) \subset \mathbb{R}^2$ centred at $(1,0)$. Since $\mathbf{u} \in \mathcal{C}^1(B_1((1,0));\mathbb{R}^2) \cap W^{1,1}(B_1((1,0));\mathbb{R}^2)$, it belongs to $\mathrm{BV}(B_1((1,0));\mathbb{R}^2)$. Nevertheless $\det(\nabla\mathbf{u})$ is not integrable, and $\overline{\mathcal{A}}(\mathbf{u},B_1((1,0))) = +\infty$.

1.4.3 Piecewise constant maps

Differently from the scalar case, the problem of providing a representation for the relaxed area functional is still open. As we will point out in the next paragraph, it is impossible to find an integral representation, since it is known that $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ is possibly non-subadditive. In [1, Theorem 3.14] the authors compute the value of $\overline{\mathcal{A}}(\mathbf{v}, \Omega)$ for a piecewise constant map $\mathbf{v}$, without any triple or multiple point, that is a map $\mathbf{v}$ assuming no more than two values in a small enough neighbourhood of every $\mathbf{x} \in \Omega$. The complete statement of the result, for $n = k = 2$, is the following.

Theorem 1.11 (Theorem 3.14, [1]). *Let $(E_i)_{i\in I}$ be a finite partition of $\mathbb{R}^2$, such that each E_i is a locally finite perimeter set;*(3) *let $(\alpha_i)_{i\in I}$ be a finite family of points of $\mathbb{R}^2$, and let $\mathbf{v} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$ be the map defined as $\mathbf{v}(\mathbf{x}) = \alpha_i$ if $\mathbf{x} \in E_i$. Suppose that for every $\mathbf{x} \in \overline{\Omega}$ there exists $r > 0$ such that $\mathcal{L}^2(B_r(\mathbf{x}) \cap E_i) > 0$ for at most two distinct indices i. Then*

$$\begin{aligned} \overline{\mathcal{A}}(\mathbf{v}, \Omega) &= \mathcal{L}^2(\Omega) + \frac{1}{2} \sum_{i,j\in I} |\alpha_i - \alpha_j| \mathcal{H}^1(\partial^* E_i \cap \partial^* E_j \cap \Omega) \\ &= \int_\Omega |\mathcal{M}(\nabla \mathbf{v})| \, dx \, dy + |D^s \mathbf{v}|(\Omega), \end{aligned} \tag{1.4}$$

provided that $\mathcal{L}^2(\partial\Omega) = 0$ and $\mathcal{H}^1(\partial^* E_i \cap \partial\Omega) = 0$ for every $i \in I$.

We underline that, from (1.4), $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ is a measure, and thus subadditive, whenever $\mathbf{v}$ is as in the statement of Theorem 1.11. Nevertheless, as soon as we remove the hypothesis that there are no triple points, the non-subadditive behaviour of $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ arises, as explained in the following paragraph. In this thesis we will generalize the class of maps that are piecewise constant without triple points, studying maps $\mathbf{u} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ that are regular out of a segment. Even in this case, $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ turns out to be possibly non-subadditive, see Chapter 6.

1.4.4 Non-subadditivity: the triple point map

A relevant issue connected with the relaxed area is its non-subadditivity. This means that there exist maps $\mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ such that $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ is non-subadditive, namely there exist open sets $\Omega_1, \Omega_2, \Omega_3 \subset \Omega$ such that

$$\Omega_3 \subset \Omega_1 \cup \Omega_2 \qquad \text{and} \qquad \overline{\mathcal{A}}(\mathbf{v}, \Omega_3) > \overline{\mathcal{A}}(\mathbf{v}, \Omega_1) + \overline{\mathcal{A}}(\mathbf{v}, \Omega_2). \tag{1.5}$$

This fact has been conjectured by De Giorgi in [13], [14], and it has been proven in [1]. We underline that it implies that the functional $\overline{\mathcal{A}}$ cannot admit an integral representation, differently from what happens in the scalar case.

In [1, Theorem 4.1] the authors prove (1.5) with $\mathbf{v} = \mathbf{u}_T$, where the *triple point map* $\mathbf{u}_T : \mathbb{R}^2 \to \mathbb{R}^2$ is a piecewise constant map assuming three non-collinear values on three non-overlapping and non-degenerate angular regions with common vertex in

(3) For more details on the theory of finite perimeter sets we refer to [4]; in this statement ∂^* denotes the restricted boundary.

the origin. Just to fix the ideas, we can suppose that these regions are the ones represented in Figure 1.1 and that the three non-collinear values $\alpha_1, \alpha_2, \alpha_3$ assumed by $\mathbf{u}_T$ are the vertices of the equilateral triangle of side $L = 2$, depicted in Figure 1.1 too.

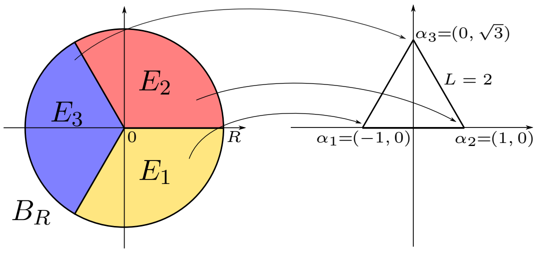

Figure 1.1: The map $\mathbf{u}_T$ assumes three non-collinear values on three non-overlapping circular sector with common vertex in the origin. The image of the ball B_R through the map $\mathbf{u}_T$ is composed of the three vertex of the equilateral triangle of side $L = 2$ depicted in the right of the picture.

In order to prove this Theorem, Acerbi and Dal Maso provide an upper bound for $\overline{\mathcal{A}}(\mathbf{u}_T, B_r)$ ([1, Lemma 4.2]) and a much more interesting and difficult lower bound ([1, Lemma 4.4]); more precisely for every $r > 0$ they show:

$$\begin{aligned} &\overline{\mathcal{A}}(\mathbf{u}_T, B_r) \leq \pi r^2 + 2r(|\alpha_1 - \alpha_3| + |\alpha_2 - \alpha_3|) = \pi r^2 + 4rL, \\ &\overline{\mathcal{A}}(\mathbf{u}_T, B_r) > \pi r^2 + r(|\alpha_1 - \alpha_2| + |\alpha_2 - \alpha_3| + |\alpha_3 - \alpha_1|) = \pi r^2 + 3rL. \end{aligned} \tag{1.6}$$

The proof of the lower bound uses the theory of Cartesian currents. Theorem 6.1 in in this thesis is largely inspired to it.
The upper bound in (1.6) has been refined by Bellettini and Paolini in [5], where the authors exhibit an approximating sequence (conjectured to be optimal, at least under symmetry assumptions) constructed by solving three (similar) Plateau-type problems entangled at the triple point. We report their result and we sketch their construction because it contains the ideas that led us to deal with area minimizing problems in the setting of semicartesian surfaces (see Chapter 2) in order to estimate the area functional for maps discontinuous on a segment.
The idea in [5] is considering the Neumann-Dirichlet problem

$$\min \int_{[0,R]\times[-1,1]} \sqrt{1 + |\nabla f|^2}\, dt\, ds := m, \tag{1.7}$$

where the minimum is taken among all scalar functions f continuous on the rectangle $[0, R] \times [-1, 1] \subset \mathbb{R}^2_{(t,s)}$ and $\mathcal{C}^2$ regular on $(0, R) \times (-1, 1)$, satisfying Dirichlet condition on $[0, R] \times \{\pm 1\}$ depending on the values of $\mathbf{u}_T$ near the radius connecting the origin and $(1, 0)$, Dirichlet condition on the side $\{0\} \times [-1, 1]$ depending on the presence of the triple point, and Neumann condition on the fourth side. Thus the result is the following.

Theorem 1.12 (**Theorem 1.1, [5]**). *Let* $\mathbf{u}_T : B_R \to \{\alpha_1, \alpha_2, \alpha_3\}$ *be the triple point map defined as before (see also Figure 1.1) and* m *be defined in* (1.7). *Then*

$$\overline{\mathcal{A}}(\mathbf{u}_T, B_R) \leq \pi R^2 + 3m.$$

Sketch of the proof. The problem (1.7) admits a unique solution $f_{\min}$; let us consider a suitable infinitesiamal sequence of positive numbers (ε_h), and define the map $\mathbf{u}_h$ as

$$\mathbf{u}_h(x,y) := \left(\frac{s}{\varepsilon_h}, f_{\min}\left(x, \frac{s}{\varepsilon_h}\right)\right) \tag{1.8}$$

for $x \in (0,R)$ and $|y| < \varepsilon_h$, similarly, by rotation, on neighbourhoods of the other radii where $\mathbf{u}_T$ jumps, and equal to $\mathbf{u}_T$ far from the jump set[(4)]. A direct computation proves that on each radius a singular contribution of area equal to m arises, and thus

$$\overline{\mathcal{A}}(\mathbf{u}_T, B_R) \leq \lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, B_R) = \int_{B_R} |\mathcal{M}(\nabla \mathbf{u}_T)|\, dx\, dy + 3m = \pi R^2 + 3m,$$

where the inequality follows from the definition of the relaxed area functional, and the last equality holds because $\mathbf{u}_T$ is piecewise constant, and therefore $\nabla \mathbf{u}_T = 0$ almost everywhere. □

It is worth to remark two facts. The first is that the singular contribution of area m is the area of a minimal surface satisfying suitable boundary conditions determined by the trace of $\mathbf{u}_T$ on its jump set. The second is that this minimal surface is a graph, and then it can be parametrized on the rectangle by a regular map $\Phi : [0,R] \times [-1,1] \to \mathbb{R}^3$ given by

$$\Phi(t,s) = (t, s, f_{\min}(t,s));$$

with this notation the components of the map $\mathbf{u}_h$ defined in (1.8) are the last two components of Φ. We obtain that $\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, [0,R] \times [-\varepsilon_h, \varepsilon_h]) = m$ due to the fact that the first component of the map Φ (that we do not consider to define $\mathbf{u}_h$) does not really bring information on the profile of the minimal surface, since it is just the identity on the first coordinate.
This argument leads us to think that in order to "glue" on a line the area of a surface, it is not necessary that it can be represented as the graph of a scalar function, but it is enough that it admits a parametrization whose first component is the identity on the first parameter. We will call such a map a *semicartesian map*, and we refer to Chapters 2 and 3 for definitions and properties of this class of surfaces.

1.4.5 Non-subadditivity: the vortex map

In [1, Theorem 5.1] the authors prove the non-subadditive behaviour of $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$, see (1.5), for $\mathbf{v} = \mathbf{u}_V$, where $\mathbf{u}_V(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$, $\mathbf{x} \in B_R \subset \mathbb{R}^n$ and thus $\mathbf{v} \in \mathrm{BV}(B_R; \mathbb{R}^n) \cap W^{1,p}(B_R; \mathbb{R}^n)$ for every $p < n$; we refer to $\mathbf{u}_V$ as to the vortex map. The authors prove this result for $n > 2$, but it is still true for $n = 2$. We comment about the proof, for $n = 2$, since our Proposition 5.1 can be seen as a generalization of this result.

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(4) Some more technical work near the origin is needed for the map $\mathbf{u}_h$ to be univocally defined and regular; anyway the area contribution concentrated over the triple point is negligible, and thus we omit the details and refer to [5].

As for the case of $\mathbf{v} = \mathbf{u}_T$, the proof of the non-subadditivity of $\overline{\mathcal{A}}(\mathbf{u}_V, \cdot)$ descends from two estimates ([1, Lemma 5.2] and [1, Lemma 5.3]). The first one is actually an equality, giving the value of $\overline{\mathcal{A}}(\mathbf{u}_T, B_R)$ for R large enough; more precisely

$$\overline{\mathcal{A}}(\mathbf{u}_V, B_R) = \int_{B_R} |\mathcal{M}(\nabla \mathbf{u}_V)|\, dx\, dy + \pi, \qquad R > R_2, \tag{1.9}$$

where R_2 depends only on the dimension $n = 2$. We observe that it is possible to prove that formula (1.9) holds with $\leq$ in place of the equal sign for any value of R; since $\overline{\mathcal{A}}(\cdot, B_R)$ is L^1-lower semicontinuous, this upper estimate is obtained by building a sequence $(\mathbf{u}_h)$ of regular maps converging to $\mathbf{u}_V$ in $L^1(B_R; \mathbb{R}^2)$, for which $\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, B_R)$ is the right hand side of (1.9). This sequence turns out to be such that $\mathbf{u}_h = \mathbf{u}_V$ out of balls B_{r_h}, where (r_h) is an infinitesimal sequence of positive radii (thus, referring to Definition 4.1 below, we say that $\mathbf{u}_h \to \mathbf{u}$ uniformly out of the origin).
The other estimate is the following bound from above, valid for every radius R:

$$\overline{\mathcal{A}}(\mathbf{u}_V, B_R) \leq \int_{B_R} |\mathcal{M}(\nabla \mathbf{u}_V)|\, dx\, dy + 2\pi R. \tag{1.10}$$

Again it is obtained by building a suitable sequence $(\mathbf{u}_h)$ converging to $\mathbf{u}_V$ in $L^1(B_R; \mathbb{R}^2)$ and then by using the lower semicontinuity of $\overline{\mathcal{A}}(\cdot, B_R)$.[(5)] In order to define this sequence, let us express the map $\mathbf{u}_V$ in polar coordinates (r, θ), without renaming it, that is $\mathbf{u}_V(r, \theta) = (\cos\theta, \sin\theta)$. Let (θ_h) and (r_h) be two infinitesimal sequences of positive numbers and let us define the maps

$$f_h(\theta) := \begin{cases} -\frac{\pi-\theta_h}{\theta_h}(\theta+\pi) & \theta \in [-\pi, -\pi+\theta_h), \\ \theta & \theta \in [-\pi+\theta_h, \pi-\theta_h), \\ -\frac{\pi-\theta_h}{\theta_h}(\theta-\pi) & \theta \in [\pi-\theta_h, \pi) \end{cases} \quad \text{and} \quad g_h(r) := \begin{cases} \frac{r}{r_h} & r \in (0, r_h), \\ 1 & r \in [r_h, R). \end{cases}$$

The sequence $(\mathbf{u}_h) \subset \mathrm{Lip}(B_R; \mathbb{R}^2)$ is defined by

$$\mathbf{u}_h(r, \theta) := \Big(\cos\big(g_h(r) f_h(\theta)\big), \sin\big(g_h(r) f_h(\theta)\big) \Big), \quad \forall n \in \mathbb{N};$$

notice that it converges to $\mathbf{u}_V$ in $L^1(B_R; \mathbb{R}^2)$ and that every $\mathbf{u}_h$ coincides with $\mathbf{u}_V$ out of a neighbourhood of the radius $\{(t, 0) : t \in (-R, 0)\}$ (thus using again the language of Definition 4.1, the sequence $(\mathbf{u}_h)$ converges to $\mathbf{u}_V$ uniformly out of that radius but not out of the origin). A direct computation shows that $\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, B_R) \leq \int_{B_R} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + 2\pi R$[(6)], and this implies (1.10). We also observe that for $R < \frac{1}{2}$ the upper bound in (1.10) is lower than the right hand side of (1.9).

(5) This construction appeared also in [19]. See also [21].

(6) It is possible to improve the estimate provided by [1, Lemma 5.3], obtaining, as singular contribution, the area of a catenoid in place of the area of a cylinder. Let us suppose $R > 0$ to be so small that there exists a catenary $c : (0, R) \to (0, +\infty)$ such that $c(0) = c(R) = 1$. Then, taken an infinitesimal sequence (ω_h), such that $\omega_h > \theta_h$ and $\omega_h/\theta_h \to 1$ as $h \to +\infty$, we define the map $\rho_h \in \mathrm{Lip}(B_R)$ such that $\rho_h(r, \theta) = c(r)$ if $\theta \in [-\pi, -\pi+\theta_h) \cup [\pi-\theta_h, \theta)$ and $\rho_h(r, \theta) = 1$ if $\theta \in (-\pi+\omega_h, \pi-\omega_h)$. Then the sequence

$$\mathbf{u}_h(r, \theta) := \rho_h(r, \theta) \Big(\cos\big(g_h(r) f_h(\theta)\big), \sin\big(g_h(r) f_h(\theta)\big) \Big)$$

provides the desired estimate.

1.5 On Cartesian Currents

The space of Cartesian Currents has been introduced by Giaquinta, Modica and Souček in [18] in order to generalize the notion of Cartesian graph.

We shall use the formalism of currents and some results about Cartesian currents in Lemma 6.6 and Theorem 6.1 in order to prove the non-subadditivity of $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ for a map $\mathbf{u}$ jumping on a segment. We report here some definitions and some results in order to fix the notation; we refer to [21] and [20] for a complete and detailed discussion and to [1] for a shorter presentation.

Given an open set $U \subset \mathbb{R}^m$, for $n \leq m$ the space of *n-currents in U* denoted by $\mathcal{D}_n(U)$ is the dual of the space $\mathcal{D}^n(U)$ of n-forms whose coefficients are smooth functions with compact support in U.

For any current $T \in \mathcal{D}_n(U)$, we define the *mass of T* as

$$\mathbf{M}(T) := \sup \big\{ T(\omega) : \omega \in \mathcal{D}^n(U), \, \omega(x) \leq 1 \; \forall z \in U \big\};$$

We point out that the mass is lower semicontinuous with respect to the weak convergence (see [21, Proposition 1, Section 2.3]).

The *boundary* of a n-current T in U, denoted by ∂T, is the $(n-1)$-current on U defined by

$$\partial T(\alpha) := T(d\alpha), \quad \forall \alpha \in \mathcal{D}^{n-1}(U),$$

where d denotes the external derivative.

Given an oriented n-rectifiable set $S \subset U$ with orientation ζ, and a $\mathcal{H}^n$-measurable and locally integrable multiplicity function $\theta : S \to \mathbb{N} \setminus \{0\}$, we define the current $T := \tau(S, \theta, \zeta)$ as

$$T(\omega) := \int_S \langle \zeta, \omega \rangle \, \theta \, d\mathcal{H}^n, \qquad \forall \omega \in \mathcal{D}^n(U),$$

and we call such a current an *integer multiplicity rectifiable current.* When the multiplicity is identically equal to 1, $T := \tau(S, 1, \zeta)$ reduces to be the integration over the oriented n-rectifiable set S, and we also indicate it by $[\![S]\!]$. In particular, given a map $\mathbf{v} \in \mathcal{C}^1(\Omega; \mathbb{R}^k)$, with $\Omega \subset \mathbb{R}^n$ an open bounded set, the integration over its graph is an n-dimensional integer multiplicity rectifiable current in $U := \Omega \times \mathbb{R}^k \subset \mathbb{R}^{n+k}$, denoted by $[\![G_{\mathbf{v}}]\!]$.

We are now in the position to define the space of Cartesian currents $\mathrm{cart}(\Omega; \mathbb{R}^k)$. In this definition we use the coordinates $\mathbf{x} = (x^1, \ldots, x^n)$ in $\Omega \subset \mathbb{R}^n$ and $\mathbf{y} = (y^1, \ldots, y^k)$ in the target space $\mathbb{R}^k$. We write also $\mathbf{z} := (\mathbf{x}, \mathbf{y})$.

Definition 1.13 (Cartiesian currents). The space $\mathrm{cart}(\Omega;\mathbb{R}^k)$ of Cartesian currents is the space of all integer multiplicity rectifiable n-currents T on $U := \Omega \times \mathbb{R}^k$ such that $\partial T = 0$, $\mathbf{M}(T) < +\infty$, and the following conditions hold:

- $p_\sharp T = [\![\Omega]\!]$, where $p_\sharp T(\omega) := T(p^\sharp \omega)$[7] for every $\omega \in \mathcal{D}^n(U)$, and $p : U \to \Omega$ is the canonical projection on $\mathbb{R}^n$,

- $T^{\hat{0}0} \geq 0$, where $T^{\hat{0}0}$ is the distribution defined by $T^{\hat{0}0}(f) := T(f dx^1 \wedge \cdots \wedge dx^n)$ for every $f \in \mathcal{C}^\infty_o(U)$,

- $\sup \left\{ T(|\mathbf{y}| f(\mathbf{x}, \mathbf{y}) dx^1 \wedge \cdots \wedge dx^n) : \, f \in \mathcal{C}^\infty_o(U), \, \sup |f| \leq 1 \right\} < +\infty.$

The definition is such that a current obtained as weak limit of graphs of $\mathcal{C}^1$ maps belongs to $\mathrm{cart}(\Omega;\mathbb{R}^k)$; we observed that if $T = [\![G_{\mathbf{v}}]\!]$, for some map $\mathbf{v} \in \mathcal{C}^1(\Omega;\mathbb{R}^k)$, the expression in the third assumption is just the L^1-norm of $\mathbf{v}$, and the mass of T is the n-dimensional area of its graph. We also underline that in general the graph of a non-regular map does not belong to $\mathrm{cart}(\Omega;\mathbb{R}^k)$, since its boundary in U is possibly non-zero.
For $T = \tau(S, \theta, \zeta) \in \mathrm{cart}(\Omega;\mathbb{R}^k)$, we define the set of regular points $S_r \subset S$ as the set of points $\mathbf{z} \in S$ such that the canonical projection p maps the tangent space $T_M(\mathbf{z})$ onto $\mathbb{R}^n$; the set of singular points S_s is then given by $S \setminus S_r$. We can then decompose the current T as $T = T_r + T_s$, where $T_r := \tau(S_r, \theta, \zeta)$ and $T_s := \tau(S_s, \theta, \zeta)$; notice that $\mathbf{M}(T) = \mathbf{M}(T_r) + \mathbf{M}(T_s)$.
We conclude this brief section with two results that we shall refer to in the proof of Theorem 6.1. We state them as in [1, Theorems 2.3 and 2.6], see also [1, Remark 2.4], but the original proof is contained in [18].

Theorem 1.14. *Let $T = \tau(S, \theta, \zeta) \in \mathrm{cart}(\Omega;\mathbb{R}^k)$ and let us set $\Omega_r := p(S_r)$ and $\Omega_s := p(S_s)$. Then $\mathcal{L}^n(S_s) = 0$ and there exists a map $\mathbf{v}_T : \Omega_r \to \mathbb{R}^k$ such that $G_{\mathbf{v}_T} \simeq S_r$ in the sense of $\mathcal{H}^n$. Moreover $\theta = 1$ $\mathcal{H}^n$-a.e. and thus $T_r = [\![G_{\mathbf{v}_T}]\!]$.*

Theorem 1.15. *Let $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega;\mathbb{R}^k)$ be a sequence converging to $\mathbf{v}$ in $L^1(\Omega;\mathbb{R}^k)$, and such that the graphs $G_{\mathbf{v}_h}$ have equibounded $\mathcal{H}^n$ measure. Then there exist a (non-relabeled) subsequence and a current $T \in \mathrm{cart}(\Omega;\mathbb{R}^k)$ such that*

- *$[\![G_{\mathbf{v}_h}]\!] \rightharpoonup T$ weakly in $\mathcal{D}_n(U)$;*

- *$\mathbf{v} = \mathbf{v}_T$ $\mathcal{L}^n$ a.e. (where $\mathbf{v}_T$ is as in Theorem 1.14);*

- *$D\mathbf{v}_h \rightharpoonup D\mathbf{v}$ in the weak * topology.*

If in addition $G_{\mathbf{v}}$ is countably n-rectifiable and the canonical projection $p : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ maps the tangent space $T_{G_{\mathbf{v}}}(\mathbf{z})$ onto $\mathbb{R}^n$ for $\mathcal{H}^n$ a.e. $\mathbf{z} \in G_{\mathbf{v}}$, then $T_r = [\![G_{\mathbf{v}}]\!]$.

[7] Given a function f we denote by $f^\sharp \omega$ the pull-back of the form ω through f.

1.6 On Plateau's problem - parametric approach

In this Section we briefly collect the main definitions on Plateau's problem and the results we need in the proof of Theorems 7.3 and 7.4.

Let $B \subset \mathbb{R}^2_{(u,v)}$ be the unit open disk and Γ be an oriented[(8)] rectifiable closed simple curve in $\mathbb{R}^3$. The Plateau's problem for Γ, [15], [33], is

$$\inf_{Y \in \mathcal{C}(\Gamma)} \int_B |\partial_u Y \wedge \partial_v Y| \, du\, dv := a(\Gamma) \tag{1.11}$$

where

$$\mathcal{C}(\Gamma) := \{Y \in H^1(B;\mathbb{R}^3) : Y_{|\partial B} \in \mathcal{C}(\partial B;\mathbb{R}^3) \text{ is a weakly monotonic parametrization of } \Gamma\}$$

Definition 1.16 (Area minimizing surface spanning Γ). A solution $Y \in \mathcal{C}(\Gamma)$ of problem (1.11) is called a *disk-type area minimizing solution of Plateau's problem* Γ. The image of B through Y is usually denoted by $\Sigma_{\min}$ and it is sometimes referred as area minimizing surface with a small abuse of language.

Concerning the existence of a solution of (1.11) the following holds.

Theorem 1.17 (Existence of minimizers and interior regularity). *Problem* (1.11) *admits a solution* $Y \in \mathcal{C}^2(B;\mathbb{R}^3) \cap \mathcal{C}(\overline{B};\mathbb{R}^3)$*, such that*

$$\Delta Y = 0 \qquad \text{in } B, \tag{1.12}$$

and the conformality relations hold:

$$|\partial_u Y|^2 = |\partial_v Y|^2 \qquad \text{and} \qquad \partial_u Y \cdot \partial_v Y = 0 \qquad \text{in } B. \tag{1.13}$$

Moreover the restriction $Y_{|\partial B}$ *is a (continuous) strictly monotonic map onto* Γ*.*

Proof. See for instance [15, Main Theorem 1, chapter 4, p. 270]. □

Remark 1.18 (Three points condition). One can impose on a minimizer Y the so-called three points condition: this means that we can fix three points ω_1, ω_2 and ω_3 on ∂B and three points P_1, P_2 and P_3 on Γ (in such a way that the orientation of Γ is respected) and find a solution Y of (1.11) such that $Y(\omega_j) = P_j$ for any $j = 1, 2, 3$.

Concerning the dependence of $a(\Gamma)$ on the boundary datum Γ, the following lower semicontinuity result holds.

Theorem 1.19 (Lower semicontinuity of $a(\cdot)$). *Let* (Γ_h) *be a sequence of Jordan and rectifiable curves converging to a Jordan curve* Γ *in the sense of Fréchet as* $h \to +\infty$*. Then*

$$a(\Gamma) \le \liminf_{h \to +\infty} a(\Gamma_h). \tag{1.14}$$

Moreover if $\sup_{h \in \mathbb{N}} \mathcal{H}^1(\Gamma_h) \le C < +\infty$*, then* (1.14) *holds with the equal sign.*

(8) The orientation is provided by fixing a homeomorphism from ∂B onto Γ.

Proof. See [33, §301, §305]. □

Concerning the regularity of a map $Y : B \to \mathbb{R}^3$ parametrizing a minimal surface, we cannot *a priori* avoid singular points, called *branch points*.

Definition 1.20 (**Branch point**). A point $\omega_0 \in B$ is called an *interior branch point* for a map Y satisfying (1.11) and (1.13) if

$$|\partial_u Y(\omega_0) \wedge \partial_v Y(\omega_0)| = 0. \tag{1.15}$$

If Y is differentiable on ∂B, and $\omega_0 \in \partial B$ is such that (1.15) holds, then ω_0 is called a *boundary branch point*.

Observe that if ω_0 is a branch point then $\partial_u Y(\omega_0) = \partial_v Y(\omega_0) = 0$. It is known that interior branch points can be excluded.

Theorem 1.21 (**Absence of interior branch points**). *Let Y be as in Theorem 1.17. Then Y has no interior branch points.*

Proof. See [34, Main Theorem]. □

Under the stronger assumption that Γ is analytic the classical Lewy's regularity theorem [30] guarantees that our solution of (1.11) is analytic on $\overline{B}$.

Theorem 1.22 (**Absence of boundary branch points**). *Let Γ be analytic and Y be as in Theorem 1.17. Then Y is analytic up to Γ and has no boundary branch points.*

Proof. See [26]. □

The following two results guarantee more regularity to Y up to the boundary, depending on the regularity of the curve Γ.

Theorem 1.23 (**$\mathcal{C}^{m,\alpha}$ extension**). *Consider a minimal surface $Y : B \to \mathbb{R}^3$ of class $\mathcal{C}^0(B \cup I; \mathbb{R}^3) \cap \mathcal{C}^2(B; \mathbb{R}^3)$ which maps an open subarc $I \subset \partial B$ into an open Jordan arc $Y(I) \subset \mathbb{R}^3$ which is a regular curve of class $\mathcal{C}^{m,\alpha}$ for some integer $m \geq 1$ and some $\alpha \in (0,1)$. Then Y is of class $\mathcal{C}^{m,\alpha}(B \cup I; \mathbb{R}^3)$. Moreover, if $Y(I)$ is a regular real analytic Jordan arc, then Y can be extended as a minimal surface across I.*

Proof. See [16, Theorem 1, chapter 2.3]. □

Thus in particular we get the following extension result.

Theorem 1.24 (**Analytic extension**). *Let Γ be analytic and Y be a minimal surface spanning Γ. Then Y can be extended as a minimal surface across Γ, that is there exist an open set $B^{\text{ext}} \supset \overline{B}$ and an analytic map $Y^{\text{ext}} : B^{\text{ext}} \to \mathbb{R}^3$ such that $Y^{\text{ext}} = Y$ in $\overline{B}$ and Y^{ext} satisfies (1.12) and (1.13) in B^{ext}.*

Proof. From Theorem 1.23 one can extend a minimal surface across an analytic subarc of Γ. We apply this result twice to two overlapping subarcs covering Γ. Where the two extensions overlap, they have to coincide due to analyticity. $\square$

The following classical result can be found in [15, p. 66].

Theorem 1.25 (**Local semicartesian parametrization**). *If a minimal surface Y is intersected by a family of parallel planes $\mathcal{P}$ none of which is tangent to the given surface and if each point of the surface belongs to some plane $\Pi \in \mathcal{P}$, then the intersection lines of these planes with the minimal surface form a family of curves which locally belong to a net of conformal parameters on the surface.*

1.7 On Morse Theory

In this short section we report a result from [32, Theorem 10] on critical points of Morse functions. The result holds in any dimension, but we need and state it only for $n = 2$.
Let U be a bounded open subset of $\mathbb{R}^2$ and let B be an open subset of U of class $\mathcal{C}^3$ with $\overline{\mathrm{B}} \subset U$. Suppose that

- $f : U \to \mathbb{R}$ is a Morse function;
- B contains all critical points of f;
- all critical points of the restriction $f_{|\partial \mathrm{B}}$ of f to $\partial \mathrm{B}$ are non-degenerate (*i.e.*, $f_{|\partial \mathrm{B}}$ is a Morse function).

Define

$$\partial_f^- \mathrm{B} := \{b \in \partial \mathrm{B} : \ \nabla f(b) \cdot \nu_{\mathrm{B}}(b) < 0\}, \tag{1.16}$$

where $\nu_{\mathrm{B}}(b)$ denotes the outward unit normal to $\partial \mathrm{B}$ at $b \in \partial \mathrm{B}$.
For $i = 0, 1, 2$, denote by $m_i(f, \mathrm{B})$ the number of critical points of index i of f in B and by $m_i(f_{|\partial_f^- \mathrm{B}})$ the number of critical points of index i of $f_{|\partial \mathrm{B}}$ on $\partial_f^- \mathrm{B}$, with $m_2(f_{|\partial_f^- \mathrm{B}}) := 0$. Define

$$M_i(f, \mathrm{B} \cup \partial \mathrm{B}) := m_i(f, \mathrm{B}) + m_i(f_{|\partial_f^- \mathrm{B}}), \qquad i = 0,\, 1\,, 2. \tag{1.17}$$

The following result holds.

Theorem 1.26. *We have*

$$M_0(f, \mathrm{B} \cup \partial \mathrm{B}) - M_1(f, \mathrm{B} \cup \partial \mathrm{B}) + M_2(f, \mathrm{B} \cup \partial \mathrm{B}) = \chi(\mathrm{B}),$$

where $\chi(\mathrm{B})$ is the Euler-Poincaré characteristic of B.

2. The semicartesian setting

Overview of the chapter

The main aim of this chapter is to introduce the two area minimizing problems (2.8) and (2.9) in the setting of semicartesian surfaces.
In few words, a surface is semicartesian if its first component is the identity on the first parameter. These surfaces are, in some sense, an intermediate step between graphs of scalar functions from a planar domain and generic immersions of the planar disk. Problem (2.8) is a sort of area minimizing problem with partially free boundary conditions, while in Problem (2.9) the boundary constraint is completely fixed.
In Section 2.1 we present the definitions and the first properties of the semicartesian setting; in Section 2.2 we introduce the area minimizing problems (2.8) and (2.9). The results of this chapter are contained in [8], but the notion of semicartesian parametrization previously appeared in [6].

2.1 First definitions and properties

Definition 2.1 (Union of two graphs). Let $\Gamma \subset \mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$; we say that Γ is union of two graphs on $[a,b]$ if $\Gamma = \Gamma^- \cup \Gamma^+$, where $\Gamma^\pm := \mathrm{graph}(\gamma^\pm)$ with $\gamma^\pm \in \mathcal{C}([a,b];\mathbb{R}^2) \cap \mathrm{Lip}_{\mathrm{loc}}((a,b);\mathbb{R}^2)$. We say that Γ is union of two Lipschitz graphs on $[a,b]$ if furthermore $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$.

Remark 2.2. Depending on the values of $\gamma^\pm$ at $t=a$ and $t=b$, Γ could be either a closed curve, or an open curve, or the union of two open curves. Notice that we do not exclude that $\gamma^-(t) = \gamma^+(t)$ for some $t \in (a,b)$.

Definition 2.3 (Semicartesian parameter domain). Let $O \subset \mathbb{R}^2 = \mathbb{R}^2_{(t,s)}$. We say that O is a semicartesian parameter domain if there exist $\sigma^\pm \in \mathcal{C}([a,b]) \cap \mathrm{Lip}_{\mathrm{loc}}((a,b))$ with $\sigma^- < \sigma^+$ in (a,b) such that

$$O := \{(t,s) \in \mathbb{R}^2 : \ \sigma^-(t) < s < \sigma^+(t), \ t \in (a,b)\},$$

If $\sigma^\pm \in \mathrm{Lip}([a,b])$, we say that O is a semicartesian Lipschitz parameter domain.

If we need to stress the dependence on the functions $\sigma^\pm$, we shall use the notation $O = [[\sigma^-, \sigma^+]]$.

Definition 2.4 (Semicartesian map). Let $O = [[\sigma^-, \sigma^+]]$ be a semicartesian parameter domain. A semicartesian map on O is a map $\Phi \in \mathcal{C}(\overline{O}; \mathbb{R}^3)$ of the form

$$\Phi(t,s) = (t, \phi(t,s)) = (t, \phi_1(t,s), \phi_2(t,s)), \qquad (t,s) \in O. \tag{2.1}$$

Definition 2.5 (Semicartesian parametrization spanning Γ). Given $\Gamma = \Gamma^- \cup \Gamma^+$ union of two graphs on $[a,b]$, $\Gamma^\pm := \mathrm{graph}(\gamma^\pm)$, a semicartesian parametrization spanning Γ is a pair (O, Φ), where $O = [[\sigma^-, \sigma^+]]$ is a semicartesian parameter domain and Φ is a semicartesian map on O satisfying the boundary condition

$$\Phi(t, \sigma^\pm(t)) = (t, \gamma^\pm(t)), \quad t \in [a,b]. \tag{2.2}$$

Sometimes we refer to the only map Φ as a semicartesian parametrization spanning Γ if it is clear what is the domain of definition.

We notice that if $\gamma^-(a) \neq \gamma^+(a)$ the domain $O = [[\sigma^-, \sigma^+]]$ of a semicartesian parametrization (O, Φ) spanning $\Gamma = \Gamma^- \cup \Gamma^+$ has to satisfy $\sigma^-(a) < \sigma^+(a)$, that is its boundary has to intersect the vertical line $\{t = a\}$ in a segment, not reduced to one single point. We stress the fact that the image of this vertical segment through Φ is not determined by the boundary conditions (2.2). Similarly, if $\gamma^-(b) \neq \gamma^+(b)$, necessarily $\sigma^-(b) < \sigma^+(b)$. On the other hand if $\gamma^-(a) = \gamma^+(a)$ we can in principle choose either a semicartesian parameter domain O such that $\sigma^-(a) = \sigma^+(a)$ or such that $\sigma^-(a) < \sigma^+(a)$. The intersection of $\Phi(\overline{O})$ with the plane $\{t = a\}$ is, in the first case, just the point $(a, \gamma^-(a))$, while, in the second case, a closed, not necessarily simple, curve.
In the following we will need more regularity requirements on the semicartesian parametrization. Indeed we want to write the area of a semicartesian parametrization, and we need it to be finite. Moreover, the regularity of the sequences $(\mathbf{u}_h)$ built in Theorems 4.7 and 4.9, used to estimate from above the relaxed area of a map jumping on a segment, depends on the regularity of the semicartesian maps we shall deal with. That is why we require also to Φ and its derivatives to be square integrable. This is in some sense coherent with the classical theory of Plateau's problem, see Section 1.6, where an area minimizing immersion of the disk is found by minimizing the Dirichlet functional, for which the space H^1 is natural.
We can then give the following definition.

Definition 2.6 (The class $\mathrm{semicart}(O; \Gamma^-, \Gamma^+)$). Let $\Gamma = \Gamma^- \cup \Gamma^+$ be union of two graphs on $[a,b]$, and let O be a semicartesian parameter domain. We set

$$\mathrm{semicart}(O; \Gamma^-, \Gamma^+) := \big\{\Phi \in H^1(O; \mathbb{R}^3) : (O, \Phi) \text{ is a semicartesian parametrization spanning } \Gamma\big\},$$

Remark 2.7 (Area integrand for semicartesian maps). For a semicartesian map Φ as in (2.1) belonging to $\mathrm{semicart}(O; \Gamma^-, \Gamma^+)$, we have

$$|\partial_t \Phi \wedge \partial_s \Phi| = \sqrt{|\partial_s \phi|^2 + (\partial_t \phi_1 \partial_s \phi_2 - \partial_t \phi_2 \partial_s \phi_1)^2}. \tag{2.3}$$

The area of a semicartesian parametrization is therefore

$$\int_O |\partial_t \Phi \wedge \partial_s \Phi| = \int_O \sqrt{|\partial_s \phi|^2 + (\partial_t \phi_1 \partial_s \phi_2 - \partial_t \phi_2 \partial_s \phi_1)^2}\, dt\, ds.$$

If in particular $\phi_1(t,s) = s$, the right hand side of 2.3 reduces obviously to the density of the area functional in the cartesian case, namely to $\sqrt{1 + |\partial_t \phi_2|^2 + |\partial_s \phi_2|^2}$.

Everywhere but in Chapters 7 and 8, we shall consider only curves $\Gamma = \Gamma^- \cup \Gamma^+$ that are union of Lipschitz graphs, and semicartesian maps defined either on R or on D, the semicartesian Lipschitz parameter domains given by the following definition.

Definition 2.8 (The domains R **and** D**).** Let $(a,b) \subset \mathbb{R}_t$ be a bounded interval. We set

$$\mathrm{R} := (a,b) \times (-1,1), \tag{2.4}$$

namely $\mathrm{R} = [[\sigma_{\mathrm{R}}^-, \sigma_{\mathrm{R}}^+]]$, with $\sigma_{\mathrm{R}}^\pm \equiv \pm 1$.
We also fix two maps $\sigma^\pm \in \mathrm{Lip}([a,b])$ so that $\sigma^+ > 0$ and $\sigma^- < 0$ on (a,b), and

- $\sigma^-(a) = \sigma^+(a) = 0$ and $\sigma^\pm(t) = \mathcal{O}(t-a)$, for $t \in (a, a+\delta)$, $\delta > 0$ small enough,
- $\sigma^-(b) = \sigma^+(b) = 0$ and $\sigma^\pm(t) = \mathcal{O}(b-t)$, for $t \in (b-\delta, b)$, $\delta > 0$ small enough,[(1)]

and we set

$$D := [[\sigma^-, \sigma^+]]. \tag{2.5}$$

Remark 2.9. If $\Gamma^\pm = \mathrm{graph}(\gamma^\pm)$ with either $\gamma^-(a) \neq \gamma^+(a)$ or $\gamma^-(b) \neq \gamma^+(b)$, then $\mathrm{semicart}(D; \Gamma^-, \Gamma^+) = \emptyset$.

We conclude this section with two Lemmas.
In Lemma 2.10 we prove that $\mathrm{semicart}(\mathrm{R}; \Gamma^-, \Gamma^+) \neq \emptyset$, for $\Gamma^\pm = \mathrm{graph}(\gamma^\pm)$, $\gamma^\pm \in \mathrm{Lip}([a,b]; \mathbb{R}^2)$; if $\Gamma = \Gamma^- \cup \Gamma^+$ is a closed curve, then also $\mathrm{semicart}(D; \Gamma^-, \Gamma^+) \neq \emptyset$.
In Lemma 2.12 we show that fixing the domains R and D is not restricting.

Lemma 2.10. *Let $\Gamma = \Gamma^- \cup \Gamma^+$ be union of two Lipschitz graphs on $[a,b]$, $\Gamma^\pm = \mathrm{graph}(\gamma^\pm)$. Then*

$$\mathrm{semicart}(\mathrm{R}; \Gamma^-, \Gamma^+) \neq \emptyset.$$

If in addition Γ is closed, then also $\mathrm{semicart}(D; \Gamma^-, \Gamma^+) \neq \emptyset$.

Proof. Let us define the following $\mathbb{R}^2$-valued Lipschitz continuous linear interpolating map:

$$\ell_{\mathrm{R}}(t,s) := \frac{1-s}{2}\gamma^-(t) + \frac{1+s}{2}\gamma^+(t), \qquad (t,s) \in \overline{\mathrm{R}}.$$

By construction $\ell_{\mathrm{R}}(t, \pm 1) = \gamma^\pm(t)$; since $\ell_{\mathrm{R}} \in \mathrm{Lip}(\mathrm{R}; \mathbb{R}^2)$, the map $\Phi_{\ell_{\mathrm{R}}}$ defined by $\Phi_{\ell_{\mathrm{R}}}(t,s) := (t, \ell_{\mathrm{R}}(t,s))$ belongs to $\mathrm{Lip}(\mathrm{R}; \mathbb{R}^3)$ and thus also to $\mathrm{semicart}(\mathrm{R}; \Gamma^-, \Gamma^+)$.

(1) These growth assumptions are needed in order to prove that the linear interpolating map Φ_{ℓ_D} defined in Lemma 2.10 has the right regularity. Lemma 2.12 shows that this assumption is not restrictive.

Now, suppose that $\gamma^-(a) = \gamma^+(a)$ and $\gamma^-(b) = \gamma^+(b)$ (*i.e.*, Γ is closed) and define

$$\ell_D(t,s) := \begin{cases} \frac{\sigma^+(t)-s}{\sigma^+(t)-\sigma^-(t)}\gamma^-(t) + \frac{s-\sigma^-(t)}{\sigma^+(t)-\sigma^-(t)}\gamma^+(t) & (t,s) \in D, \\ \gamma^\pm(t) & s = \sigma^\pm(t),\ t \in [a,b]. \end{cases}$$

Thus for $(t,s) \in D$

$$\begin{aligned} \partial_t \ell(t,s) = & \frac{\sigma^+(t) - s}{\sigma^+(t) - \sigma^-(t)}\dot\gamma^-(t) + \frac{s - \sigma^-(t)}{\sigma^+(t) - \sigma^-(t)}\dot\gamma^+(t) \\ & + \frac{(\gamma^+(t) - \gamma^-(t))(\dot\sigma^+(t)\sigma^-(t) - \sigma^+(t)\dot\sigma^-(t))}{(\sigma^+(t) - \sigma^-(t))^2} \\ & - \frac{s(\gamma^+(t) - \gamma^-(t))(\dot\sigma^+(t) - \dot\sigma^-(t))}{(\sigma^+(t) - \sigma^-(t))^2}, \end{aligned}$$

$$\partial_s \ell(t,s) = \frac{\gamma^+(t) - \gamma^-(t)}{\sigma^+(t) - \sigma^-(t)}.$$

Since $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, the properties on $\sigma^\pm$ in Definition 2.8 ensure that

$$\left| \frac{\gamma^+(t) - \gamma^-(t)}{\sigma^+(t) - \sigma^-(t)} \right| \le C < +\infty \qquad t \in (a, a+\delta);$$

similarly for $t \in (b-\delta, b)$. Noticing also that $|\dot\sigma^+\sigma^- - \sigma^+\dot\sigma^-| \le C(\sigma^+ - \sigma^-)$, and recalling that $\gamma^\pm$ and $\sigma^\pm$ are Lipschitz, we get that $\partial_t \ell_D$ and $\partial_s \ell_D$ are bounded. Thus the map

$$\Phi_{\ell_D}(t,s) := (t, \ell_D(t,s))$$

belongs to $\mathrm{Lip}(D;\mathbb{R}^3)$, and in particular to $\mathrm{semicart}(D;\Gamma^-,\Gamma^+)$. □

Remark 2.11. If $\gamma^+(t) = \gamma^-(t)$ for some $t \in (a,b)$, the map $\Phi_{\ell_{\mathrm{R}}}$ (or Φ_{ℓ_D}) sends the segment $\{t\} \times [-1,1]$ (or $\{t\} \times [\sigma^-(t), \sigma^+(t)]$) into the point $(t, \gamma^+(t))$, hence it is not injective. More generally a semicartesian map could be possibly not injective even if $\gamma^-(t) \neq \gamma^+(t)$, for every $t \in (a,b)$.

Lemma 2.12. *Let $\Gamma = \Gamma^- \cup \Gamma^+$ be union of two Lipschitz graphs on $[a,b]$, $\Gamma^\pm = \mathrm{graph}(\gamma^\pm)$. Let $O_1 = [[\sigma_1^-, \sigma_1^+]]$ be a semicartesian Lipschitz parameter domain such that $\sigma_1^-(a) < \sigma_1^+(a)$ and $\sigma_1^-(b) < \sigma_1^+(b)$. If (O_1, Ψ) is a semicartesian parametrization spanning Γ such that $\Psi \in H^1(O_1;\mathbb{R}^3)$, then there exists a semicartesian map $\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ such that*

$$\int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt\, ds = \int_{O_1} |\partial_t \Psi \wedge \partial_s \Psi| \, dt\, ds. \tag{2.6}$$

If moreover Γ is closed, $O_2 = [[\sigma_2^+, \sigma_2^+]]$ is a semicartesian Lipschitz parameter domain such that $\sigma_2^-(a) = \sigma_2^+(a)$ and $\sigma_2^-(b) = \sigma_2^+(b)$, and (O_2, χ) is a semicartesian parametrization spanning Γ such that $\chi \in H^1(O_2;\mathbb{R}^3)$, then there exists a map $\Phi \in \mathrm{semicart}(D;\Gamma^-,\Gamma^+)$ such that

$$\int_D |\partial_t \Phi \wedge \partial_s \Phi| \, dt\, ds = \int_{O_2} |\partial_t \chi \wedge \partial_s \chi| \, dt\, ds. \tag{2.7}$$

Proof. Let us define the map $T_1 : \mathrm{R} \to O_1$ as

$$T_1(t,s) := \left(t, \frac{1-s}{2}\sigma_1^-(t) + \frac{1+s}{2}\sigma_1^+(t)\right).$$

Since $\sigma_1^\pm \in \mathrm{Lip}([a,b])$, we have that $T \in \mathrm{Lip}(\mathrm{R}; O_1)$ and thus the map $\Phi := \Psi \circ T_1$ belongs to semicart$(\mathrm{R}; \Gamma^-, \Gamma^+)$; moreover T_1 is injective and thus (2.6) holds.
Let us suppose that Γ is closed. Recall that $D = [[\sigma^-, \sigma^+]]$. We define the map $T_2 : D \to O_2$ as

$$T_2(t,s) := \left(t, \frac{\sigma^+(t)-s}{\sigma^+(t)-\sigma^-(t)}\sigma_2^-(t) + \frac{s-\sigma^-(t)}{\sigma^+(t)-\sigma^-(t)}\sigma_2^+(t)\right).$$

One can show that $T_2 \in \mathrm{Lip}(D; O_2)$ with computation similar to those in Lemma 2.10, and thus the map $\Phi := \chi \circ T_2$ belongs to semicart$(D; \Gamma^-, \Gamma^+)$. The injectivity of T_2 implies (2.7). □

2.2 Two area minimizing problems

As a consequence of Lemma 2.10, we can introduce the following quantities.

Definition 2.13. Let $\Gamma = \Gamma^- \cup \Gamma^-$ be union of two Lipschitz graphs on $[a,b]$. We define

$$m(\mathrm{R}; \Gamma^-, \Gamma^+) := \inf_{\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)} \int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds. \tag{2.8}$$

If Γ is closed we define also

$$m(D; \Gamma^-, \Gamma^+) := \inf_{\Phi \in \mathrm{semicart}(D;\Gamma^-,\Gamma^+)} \int_{D} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds. \tag{2.9}$$

It is worthwhile to observe that (2.8) is a problem among surfaces with partially free boundary on the planes $\{t = a\} \times \mathbb{R}^2$ and $\{t = b\} \times \mathbb{R}^2$.
We notice that trivially if Γ is closed, then

$$m(\mathrm{R}; \Gamma^-, \Gamma^+) \le m(D; \Gamma^-, \Gamma^+).$$

Indeed we can suppose without loss of generality that $|\sigma^\pm| < 1$ (see Lemma 2.12) and find, for any $\Psi \in$ semicart$(D; \Gamma^-, \Gamma^+)$, a map $\Phi \in$ semicart$(\mathrm{R}; \Gamma^-, \Gamma^+)$ having the same area, just defining Φ as

$$\Phi(t,s) := \begin{cases} \Psi(t,s) & \text{in } D, \\ \gamma^+(t) & \text{in } \{(t,s) : t \in (a,b),\, s \in (\sigma^+(t), 1)\}, \\ \gamma^-(t) & \text{in } \{(t,s) : t \in (a,b),\, s \in (-1, \sigma^-(t))\}. \end{cases}$$

The problem of the existence of a minimum in (2.8) and (2.9) is open and require further investigation (see also Chapters 7 and 8).
Here we limit ourselves to some comments and examples. If $\Gamma = \Gamma^- \cup \Gamma^+$ is a closed simple curve, it is natural to compare $m(D; \Gamma^-, \Gamma^+)$ with the area $a(\Gamma)$ of a solution

to the classical Plateau's problem for Γ, that is an area minimizing immersion of the disk mapping the boundary of the disk onto Γ monotonically, see Section 1.6 and reference therein. It is possible[(2)] to see that

$$m(D;\Gamma^-,\Gamma^+) \geq a(\Gamma). \tag{2.10}$$

In Chapter 7 it is proven that if Γ is an analytic curve with further non-degeneracy properties at $(a,\gamma^{\pm}(a))$ and $(b,\gamma^{\pm}(b))$ then there exists a solution to the classical Plateau's problem that admits a semicartesian parametrization.[(3)] Even if we do not have such a result for curves Γ that are union of Lipschitz graphs, this fact suggests that it is still possible that (2.10) holds with equal sign (in Theorem 7.4 we prove the equality when $\gamma^{\pm} \in \mathcal{C}^{1,\alpha}([a,b];\mathbb{R}^2)$, $\alpha \in (0,1)$), and that $m(D;\Gamma^-,\Gamma^+)$ is actually a minimum.
On the other hand, for what concerns the semicartesian maps defined on the rectangle R, even assuming that Γ is a closed simple curve, the existence of an area minimizing semicartesian parametrization in semicart$(\mathrm{R};\Gamma^-,\Gamma^+)$ does not follow from the existence of a solution to the Plateau's problem. Indeed since the class of surfaces parametrized by maps in semicart$(\mathrm{R};\Gamma^-,\Gamma^+)$ strictly contains (due to the free boundary on the planes $\{t=a\}\times\mathbb{R}^2_{(\xi,\eta)}$ and $\{t=b\}\times\mathbb{R}^2_{(\xi,\eta)}$) the ones parametrized by maps in semicart$(D;\Gamma^-,\Gamma^+)$, we could expect that in general it contains also the class of surfaces considered in the classical setting. Moreover we prove that possibly $a(\Gamma) > m(\mathrm{R};\Gamma^-,\Gamma^+)$: in Example 2.14 we build a semicartesian parametrization whose image is not in the class of surfaces considered for the classical Plateau's problem; in Example 2.15 we exhibit $\gamma^{\pm} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ such that the union of their graphs is a Jordan curve for which the semicartesian parametrization built as in Example 2.14 has area strictly less than $a(\Gamma)$.
The next example is also strictly related to the construction made in Section 5.1.

Example 2.14 (Partial free boundary at $\{t=b\}\times\mathbb{R}^2$). Let $\gamma^{\pm} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ and suppose that $\gamma^-(a)=\gamma^+(a)$; let us denote by C the set $\gamma^-([a,b])\cup\gamma^+([a,b]) \subset \mathbb{R}^2_{(\xi,\eta)}$; notice that C is connected. In Figure 2.1 we depict a case when γ^+ is open and not injective.
We want to define the semicartesian parametrization $\Phi \in$ semicart$(\mathrm{R};\Gamma^-,\Gamma^+)$ which, for every $t\in(a,b)$, maps the segment $\{t\}\times[-1,1]\subset \mathrm{R}$ onto the portion of $\{t\}\times \mathrm{C}$ bounded by the points $(t,\gamma^-(t))$ and $(t,\gamma^+(t))$, and containing $(t,\gamma^-(a))$.
If for convenience we parametrize C by a curve $\gamma \in \mathrm{Lip}([-1,1];\mathbb{R}^2)$, defined by

$$\gamma(\lambda) := \begin{cases} \gamma^-(-(b-a)\lambda+a) & \lambda\in[-1,0],\\ \gamma^+((b-a)\lambda+a) & \lambda\in(0,1],\end{cases}$$

so that $\gamma(-1)=\gamma^-(b)$, $\gamma(0)=\gamma^-(a)=\gamma^+(a)$ and $\gamma(1)=\gamma^+(b)$, then $\Phi(\{t\}\times[-1,1])$ must be equal to $\left\{(t,\gamma(\lambda));\lambda\in\left[-\frac{t-a}{b-a},\frac{t-a}{b-a}\right]\right\}$.

(2) For instance, as a consequence of the Riemann mapping theorem, see *e.g.* [35].

(3) Note that the analyticity of Γ leads to consider semicartesian parametrizations whose domain $O=[[\sigma_O^-,\sigma_O^+]]$ is such that $\sigma_O^{\pm}\in \mathrm{Lip}_{\mathrm{loc}}((a,b))\setminus \mathrm{Lip}([a.b])$.

Thus we can define explicitly $\Phi \in \text{semicart}(\mathrm{R}; \Gamma^-, \Gamma^+)$ as

$$\Phi(t,s) := \left(t, \gamma\left(\frac{t-a}{b-a}s\right)\right), \quad (t,s) \in \mathrm{R}. \tag{2.11}$$

We observe that, if $\Gamma := \text{graph}(\gamma^-) \cup \text{graph}(\gamma^+)$ is a closed simple curve, the surface $\Phi(\mathrm{R})$ is not the image of any immersion of the disk mapping the boundary of the disk monotonically onto Γ, because its boundary is $\Gamma \cup (\{b\} \times \mathrm{C})$. Therefore

$$m(\mathrm{R}; \Gamma^-, \Gamma^+) \leq \int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds \leq (b-a) \int_a^b \left(|\dot{\gamma}^-| + |\dot{\gamma}^+|\right) \, dt.$$

Notice that, if $\gamma^\pm$ are injective, we have that $\Phi(\mathrm{R})$ lies on the lateral part of the surface of the cylinder $(a,b) \times \mathrm{C}$.

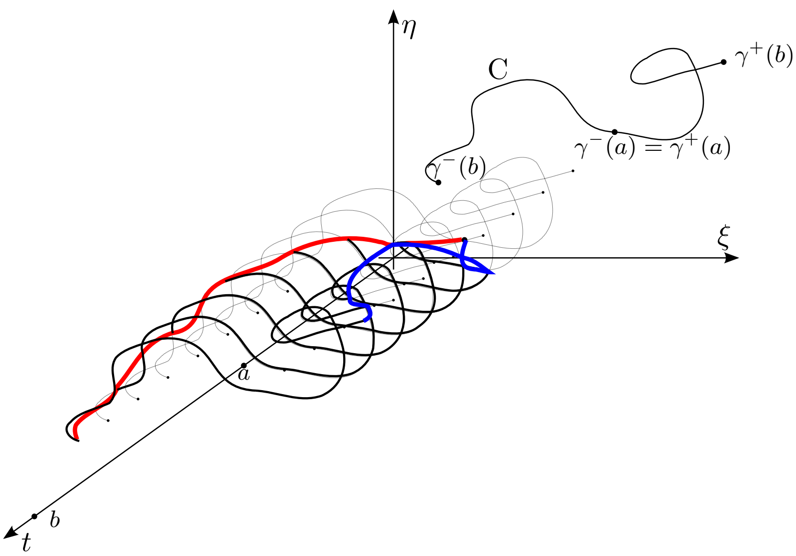

Figure 2.1: In the plane $\{0\} \times \mathbb{R}^2_{(\xi,\eta)}$ the curve C is represented; C is the projection of the curves $\Gamma^\pm$ (in red and blue) on the plane $\{0\} \times \mathbb{R}^2_{(\xi,\eta)}$. In light grey we draw the copies of C in the planes $\{t\} \times \mathbb{R}^2_{(\xi,\eta)}$, $t \in [a,b]$. The surface $\Phi(\mathrm{R})$ is the union of all portions of $\{t\} \times \mathrm{C}$ bounded by $(t, \gamma^-(t))$ and $(t, \gamma^+(t))$, when t varies in $[a,b]$.

We now exhibit maps $\gamma^\pm \in \text{Lip}([a,b]; \mathbb{R}^2)$ so that $\Gamma := \text{graph}(\gamma^-) \cup \text{graph}(\gamma^+)$ is a closed simple curve and the previous construction proves that

$$m(\mathrm{R}; \Gamma^-, \Gamma^+) < a(\Gamma).$$

We shall refer to the following example also in Example, 3.5, in Example 5.4, and in Section 5.2.

Example 2.15 ($m(\mathrm{R}; \Gamma^-, \Gamma^+) < a(\Gamma)$)**.** Let ρ be a positive real number with

$$\rho > 2(b-a). \tag{2.12}$$

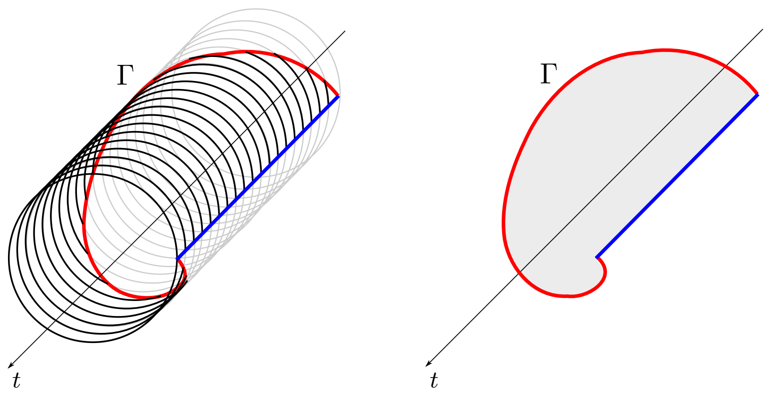

Figure 2.2: The curve Γ defined in Example 2.15. The left picture is the image of the semicartesian parametrization (R, Φ) spanning Γ built as in Example 2.14; we notice that it lies on the lateral surface of the cylinder of base the disk of radius ρ and height $b-a$ and that its free boundary is non-empty. The right picture represents the image of an embedding of the disk mapping the boundary of the disk onto Γ: the area of such a surface is greater or equal the area of its projection on a plane orthogonal to the t-axis, that is a disk of radius ρ.

Let us define the maps $\gamma^{\pm} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ as follows: if $t \in [a,b]$,

$$\begin{aligned} \gamma^{-}(t) &:= (1,0), \\ \gamma^{+}(t) &:= \rho\,(\cos(\theta(t)), \sin(\theta(t))) + (1-\rho, 0), \end{aligned}$$

where $\theta : [a,b] \to [0, 2\pi]$ is given by

$$\theta(t) := \frac{2\pi(t-a)}{b-a}, \qquad t \in [a,b], \tag{2.13}$$

see Figure 2.2. We observe that

$$\Gamma := \mathrm{graph}(\gamma^{-}) \cup \mathrm{graph}(\gamma^{+})$$

is a Lipschitz closed simple curve: any disk-type surface spanning Γ has area greater than or equal to the area $\pi\rho^2$ of its projection (a disk of radius ρ) on the coordinate plane $\mathbb{R}^2_{(\xi,\eta)}$, hence

$$a(\Gamma) \geq \pi\rho^2.$$

On the other hand the image of the semicartesian parametrization (R, Φ) defined in (2.11) has area strictly less than $2\pi\rho(b-a)$ (see the first picture in Figure 2.2). From our choice (2.12), it then follows

$$m(\mathrm{R}; \Gamma^{-}, \Gamma^{+}) < a(\Gamma).$$

3. Semicontinuity properties in the semicartesian setting

Overview of the chapter

In this chapter we study some lower semicontinuity properties for $m(\mathrm{R};\cdot,\cdot)$ and for $m(D;\cdot,\cdot)$. Even though the problem is interesting in itself, in the perspective of this thesis these results are crucial in order to estimate from below the relaxed area functional for a map jumping on a segment, see Theorems 4.11 and 4.17.
What we would want to prove is that given $\gamma^{\pm} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, $(\gamma_h^{\pm}) \subset \mathrm{Lip}([a,b];\mathbb{R}^2)$, and denoting by $\Gamma^{\pm}$ and $\Gamma_h^{\pm}$ their graphs respectively, then

$$m(\mathrm{R};\Gamma^-,\Gamma^+) \leq \liminf_{h\to+\infty} m(\mathrm{R};\Gamma_h^-,\Gamma_h^+) \tag{3.1}$$

whenever $\gamma_h^{\pm} \to \gamma^{\pm}$ in $L^1((a,b);\mathbb{R}^2)$; unfortunately, we are able to prove (3.1) only with a further hypothesis on the L^{∞}-norm of the derivatives of $\gamma^{\pm}$. Our first result is indeed the following.

Lemma 3.1 (**Lower semicontinuity of** $m(\mathrm{R};\cdot,\cdot)$). *Let* $(\gamma_h^{\pm}) \subset \mathrm{Lip}([a,b];\mathbb{R}^2)$, *and* $\gamma^{\pm} \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ *be such that:*

(i) there exists $C_1 > 0$ *such that* $||\dot{\gamma}_h^{\pm}||_{L^{\infty}((a,b);\mathbb{R}^2)} \leq C_1$ *for every* $h \in \mathbb{N}$,

(ii) $\gamma_h^{\pm} \to \gamma^{\pm}$ *in* $L^1((a,b);\mathbb{R}^2)$ *as* $h \to +\infty$.

Then, setting $\Gamma_h^{\pm} := \mathrm{graph}(\gamma_h^{\pm})$, $\Gamma^{\pm} := \mathrm{graph}(\gamma^{\pm})$, *we have*

$$m(\mathrm{R};\Gamma^-,\Gamma^+) \leq \liminf_{h\to+\infty} m(\mathrm{R};\Gamma_h^-,\Gamma_h^+).$$

As we shall see in Chapter 4, the fact that we are not able to remove hypothesis (i) in Lemma 3.1 induces us to define the functional $\overline{\mathcal{A}}^{\infty}(\cdot,\Omega)$, that is the relaxation of $\mathcal{A}(\cdot,\Omega)$ with respect to a convergence stronger than the L^1-convergence.
In this chapter we also adapt Lemma 3.1 to prove some lower semicontinuity properties of $m(D;\cdot,\cdot)$, see Lemma 3.3. For this case we need some further technical assumptions, due mainly to the fact that, even though $\Gamma = \Gamma^- \cup \Gamma^+$ is a closed curve, the union of L^1-perturbations $\Gamma_h^{\pm}$ is not necessarily a closed curve; thus, *a priori*, $m(D;\Gamma_h^-,\Gamma_h^+)$ is not even defined.

We do not know whether our hypotheses can be relaxed; however in Example 3.5 we show that they seem to be coherent with the known lower semicontinuity results for the classical Plateau's problem. Indeed we exhibit a sequence (Γ_h) of Jordan curves, union of graphs, converging only in L^1 to a Jordan curve $\Gamma^{(1)}$ such that

$$a(\Gamma) > \liminf_{h\to+\infty} a(\Gamma_h),$$

where $a(\cdot)$ is defined in (1.11). Moreover the examples in Chapter 5, coupled with Remark 4.15, confirm that it is not possible to prove a semicontinuity result for $m(D;\cdot,\cdot)$ assuming only the L^1-convergence of the boundary data.
The results presented in this chapter are contained in [8].

3.1 Lower semicontinuity properties for $m(\mathrm{R};\cdot,\cdot)$

Before proving Lemma 3.1, we provide, given $\alpha,\beta \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, an upper estimate for $m(\mathrm{R};\Gamma_\alpha,\Gamma_\beta)$ depending on the L^∞-norm of $\dot\alpha$ and $\dot\beta$, where Γ_α and Γ_β are the graphs of α and β respectively.

Lemma 3.2. *Let $\alpha,\beta \in \mathrm{Lip}([a,b];\mathbb{R}^2)$. Let $\Gamma_\alpha = \mathrm{graph}(\alpha)$, $\Gamma_\beta = \mathrm{graph}(\beta)$. Then*

$$m(\mathrm{R};\Gamma_\alpha,\Gamma_\beta) \le C\|\alpha-\beta\|_{L^1((a,b);\mathbb{R}^2)}\left(1+\max\left\{\|\dot\alpha\|_{L^\infty((a,b);\mathbb{R}^2)}, \|\dot\beta\|_{L^\infty((a,b);\mathbb{R}^2)}\right\}\right), \tag{3.2}$$

where the constant C does not depend on α and β.

Proof. Let us define the map $\ell \in W^{1,\infty}(\mathrm{R};\mathbb{R}^2)$ linearly interpolating α and β, as in Lemma 2.10; that is

$$\ell(t,s) := \frac{1-s}{2}\alpha(t) + \frac{1+s}{2}\beta(t).$$

Defining $\Phi_\ell := (t,\ell(t,s))$, we get

$$\partial_t\Phi_\ell(t,s) = \left(1, \frac{1-s}{2}\dot\alpha(t) + \frac{1+s}{2}\dot\beta(t)\right),$$
$$\partial_s\Phi_\ell(t,s) = \left(0, \frac{\beta(t)-\alpha(t)}{2}\right).$$

Thus:

$$|\partial_t\Phi_\ell \wedge \partial_s\Phi_\ell| = \frac{1}{2}\sqrt{|\beta-\alpha|^2 + \left[\left(\frac{1-s}{2}\dot\alpha + \frac{1+s}{2}\dot\beta\right)\cdot(\beta-\alpha)^\perp\right]^2}, \tag{3.3}$$

where $\mathbf{v}^\perp := (-v_2, v_1)$.

(1) This means that $\Gamma_h = \mathrm{graph}(\gamma_h^-) \cup \mathrm{graph}(\gamma_h^+)$, $\Gamma = \mathrm{graph}(\gamma^-) \cup \mathrm{graph}(\gamma^+)$ for $\gamma^\pm, \gamma_h^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ and $\gamma_h^\pm \to \gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$.

Hence we deduce

$$\int_{\mathrm{R}} |\partial_t \Phi_\ell \wedge \partial_s \Phi_\ell|\,dt\,ds \le C||\beta-\alpha||_{L^1((a,b);\mathbb{R}^2)}\Big(1+\max\Big\{||\dot\alpha||_{L^\infty((a,b);\mathbb{R}^2)}, ||\dot\beta||_{L^\infty((a,b);\mathbb{R}^2)}\Big\}\Big), \tag{3.4}$$

and, since $\Phi_\ell \in \mathrm{semicart}(\mathrm{R};\Gamma_\alpha,\Gamma_\beta)$, also (3.2). □

We are now in the position to prove Lemma 3.1.

Proof of Lemma 3.1. For any h, let $\big(\Phi^h_k\big) \subset \mathrm{semicart}(\mathrm{R};\Gamma^-_h,\Gamma^+_h)$ be such that

$$\lim_{k\to+\infty}\int_{\mathrm{R}} |\partial_t \Phi^h_k \wedge \partial_s \Phi^h_k|\,dt\,ds = m(\mathrm{R};\Gamma^-_h,\Gamma^+_h).$$

Let us denote by $\ell^+_h, \ell^-_h : \mathrm{R}\to\mathbb{R}^2$ the linear interpolating maps, such that

$$\begin{aligned} \ell^+_h(t,-1) &= \gamma^+_h(t), \qquad & \ell^+_h(t,1) &= \gamma^+(t),\\ \ell^-_h(t,-1) &= \gamma^-_h(t), \qquad & \ell^-_h(t,1) &= \gamma^-(t). \end{aligned}$$

Following the notation of Lemma 2.10 we also write $\Phi_{\ell^\pm_h}(t,s) := (t,\ell^\pm_h(t,s))$. We define the maps $(\Psi^h_k) \subset \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ as

$$\Psi^h_k(t,s) := \begin{cases} \Phi_{\ell^+_h}(t,4s-3) & \text{if } t\in(a,b), s\in(1/2,1),\\ \Phi^h_k(t,2s) & \text{if } t\in(a,b), s\in(-1/2,1/2),\\ \Phi_{\ell^-_h}(t,-4s-3) & \text{if } t\in(a,b), s\in(-1,-1/2). \end{cases}$$

Thus, by construction we have

$$\begin{aligned} &\int_{\mathrm{R}} |\partial_t \Psi^h_k \wedge \partial_s \Psi^h_k|\,dt\,ds\\ &= \int_{\mathrm{R}} |\partial_t \Phi^h_k \wedge \partial_s \Phi^h_k|\,dt\,ds + \int_{\mathrm{R}} |\partial_t \Phi_{\ell^+_h} \wedge \partial_s \Phi_{\ell^+_h}|\,dt\,ds + \int_{\mathrm{R}} |\partial_t \Phi_{\ell^-_h} \wedge \partial_s \Phi_{\ell^-_h}|\,dt\,ds\\ &\overset{k\to+\infty}{\longrightarrow} m(\mathrm{R};\Gamma^-_h,\Gamma^+_h) + \int_{\mathrm{R}} |\partial_t \Phi_{\ell^+_h} \wedge \partial_s \Phi_{\ell^+_h}|\,dt\,ds + \int_{\mathrm{R}} |\partial_t \Phi_{\ell^-_h} \wedge \partial_s \Phi_{\ell^-_h}|\,dt\,ds. \end{aligned}$$

Now, recalling computation (3.4), we have

$$\begin{aligned} &\int_{\mathrm{R}} |\partial_t \Phi_{\ell^+_h} \wedge \partial_s \Phi_{\ell^+_h}|\,dt\,ds + \int_{\mathrm{R}} |\partial_t \Phi_{\ell^-_h} \wedge \partial_s \Phi_{\ell^-_h}|\,dt\,ds\\ &\le C\left(\|\gamma^+_h-\gamma^+\|_{L^1((a,b);\mathbb{R}^2)} + \|\gamma^-_h-\gamma^-\|_{L^1((a,b);\mathbb{R}^2)}\right)(1+C_1), \end{aligned}$$

and the right hand side is infinitesimal as $h\to+\infty$ by assumption. Hence, we can suitably choose a subsequence (k_h) and obtain a sequence $(\Psi_h) := \left(\Psi^h_{k_h}\right) \subset \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$ so that

$$\liminf_{h\to+\infty}\int_{\mathrm{R}} |\partial_t \Psi_h \wedge \partial_s \Psi_h|\,dt\,ds = \liminf_{h\to+\infty} m(\mathrm{R};\Gamma^-_h,\Gamma^+_h).$$

The inclusion $\Psi_h \in \mathrm{semicart}(\mathrm{R}; \Gamma^-, \Gamma^+)$ implies that

$$\int_{\mathrm{R}} |\partial_t \Psi_h \wedge \partial_s \Psi_h| \, dt \, ds \geq m(R; \Gamma^-, \Gamma^+),$$

and the assertion follows. □

3.2 Lower semicontinuity properties for $m(D; \cdot, \cdot)$

In this section we suitably modify Lemma 3.1 in order to provide a lower semicontinuity result for $m(D; \cdot, \cdot)$ (we refer to Lemma 3.3 in this way with a small abuse of language, as explained in Remark 3.4).

Before stating the result let us fix some notation. For any $\varepsilon \in \left(0, \frac{b-a}{2}\right)$, let $\lambda_\varepsilon : \mathbb{R}_t \to \mathbb{R}_t$ be defined as

$$\lambda_\varepsilon(t) := \frac{b-a-2\varepsilon}{b-a+2\varepsilon}(t-(a-\varepsilon)) + a + \varepsilon,$$

so that $\lambda_\varepsilon((a-\varepsilon, b+\varepsilon)) = (a+\varepsilon, b-\varepsilon)$. The map $\Lambda_\varepsilon : \mathbb{R}^2_{(t,s)} \to \mathbb{R}^2_{(t,s)}$ is, instead, defined as

$$\Lambda_\varepsilon(t,s) = (\lambda_\varepsilon(t), s).$$

We set $O_\varepsilon := [[\sigma_\varepsilon^-, \sigma_\varepsilon^+]]$, where $\sigma_\varepsilon^\pm \in \mathrm{Lip}([a-\varepsilon, b+\varepsilon])$ are such that $\sigma_\varepsilon^-(a-\varepsilon) = \sigma_\varepsilon^+(a-\varepsilon)$ and $\sigma_\varepsilon^-(b+\varepsilon) = \sigma_\varepsilon^+(b+\varepsilon)$, and such that $\Lambda_\varepsilon(O_\varepsilon) \subset\subset D$. We shall also require ∂O_ε without any horizontal cusp (see Figure 3.1).

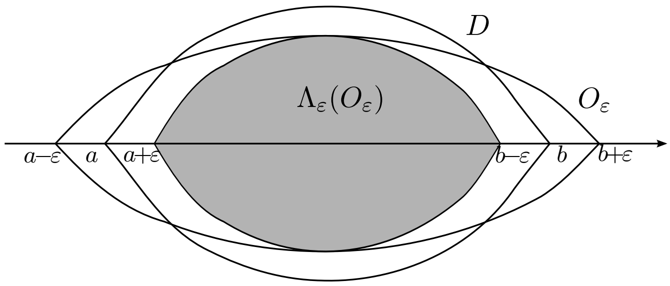

Figure 3.1: For any $\varepsilon > 0$ small enough, the domain $O_\varepsilon = [[\sigma_\varepsilon^-, \sigma_\varepsilon^+]]$ is such that its image through the map Λ_ε is compactly contained in the fixed domain D.

Lemma 3.3. *Let $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$ be such that $\gamma^-(a) = \gamma^+(a)$ and $\gamma^-(b) = \gamma^+(b)$. Let (ε_h) be an infinitesimal sequence of positive numbers and let $\gamma_h^\pm \in \mathrm{Lip}([a-\varepsilon_h, b+\varepsilon_h];\mathbb{R}^2)$ be maps with the following properties:*

(i) $\gamma_h^-(a-\varepsilon_h) = \gamma_h^+(a-\varepsilon_h)$ *and* $\gamma_h^-(b+\varepsilon_h) = \gamma_h^+(b+\varepsilon_h)$ *for every* $h \in \mathbb{N}$;

(ii) $\lim_{h\to+\infty} \gamma_h^-(a-\varepsilon_h) = \gamma^-(a)$ *and* $\lim_{h\to+\infty} \gamma_h^-(b+\varepsilon_h) = \gamma^-(b)$;

(iii) $\lim_{h\to+\infty} ||\gamma_h^\pm \circ \lambda_{\varepsilon_h}^{-1} - \gamma^\pm||_{L^1((a+\varepsilon_h, b-\varepsilon_h);\mathbb{R}^2)} = 0.$

Moreover we also suppose:

(iv) there exists a constant $C_1 > 0$ *such that* $||\dot\gamma_h^\pm||_{L^\infty((a-\varepsilon_h,b+\varepsilon_h);\mathbb{R}^2)} \leq C_1$ *for every* $h \in \mathbb{N}$.

Then

$$m(D;\Gamma^-,\Gamma^+) \leq \liminf_{h\to+\infty} m(O_{\varepsilon_h};\Gamma_h^-,\Gamma_h^+), \tag{3.5}$$

where $\Gamma^\pm := \mathrm{graph}(\gamma^\pm)$, $\Gamma_h^\pm := \mathrm{graph}(\gamma_h^\pm)$.[(2)]

Remark 3.4. For a closed curve $\Gamma = \mathrm{graph}(\gamma^-) \cup \mathrm{graph}(\gamma^+)$, $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, we cannot expect to prove that $m(D;\Gamma^-,\Gamma^+) \leq \liminf_{h\to+\infty} m(D;\Gamma_h^-,\Gamma_h^+)$ for $\Gamma_h^\pm = \mathrm{graph}(\gamma_h^\pm)$, $\gamma_h^\pm \to \gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$, $||\dot\gamma_h^\pm||_{L^\infty((a,b);\mathbb{R}^2)}$ equibounded. This because the union of L^1-perturbations of the curves $\Gamma^\pm$ is not necessarily a closed curve. The choice of considering maps $\gamma_h^\pm$ defined on an interval strictly containing $[a,b]$ will be motivated by the proof of Proposition 4.18. In few words, Lemma 3.1 will be used to estimate the area of a map whose jump set $(a,b)\times\{0\}$ has both the end points on the boundary of the domain of definition; the role of $\gamma_h^\pm$ shell then be taken by the traces of a regular map on $[a,b]\times\{\pm\varepsilon_h\}$ for an infinitesimal sequence (ε_h). Lemma 3.3 will be instead invoked in the case of a map whose jump set $[a,b]\times\{0\}$ is compactly contained in the domain; $\gamma_h^\pm$ will be chosen as the traces of the map on the boundary of a neighbourhood of the jump strictly containing it.

Proof. Let Ψ_h be a semicartesian map in $H^1(O_{\varepsilon_h};\mathbb{R}^3)$ spanning Γ_h such that

$$\int_{O_{\varepsilon_h}} |\partial_t\Psi_h \wedge \partial_s\Psi_h|\, dt\, ds \leq m(O_{\varepsilon_h};\Gamma_h^-,\Gamma_h^+) + \varepsilon_h, \tag{3.6}$$

with $\Psi_h(t,s) = (t,\psi_h(t,s))$. Let us define $\Phi_h \in H^1(\Lambda_{\varepsilon_h}(O_{\varepsilon_h});\mathbb{R}^3)$ as

$$\Phi_h(t,s) := (t,\psi_h(\lambda_{\varepsilon_h}^{-1}(t),s)) := (t,\phi_h(t,s)).$$

In words, we start from a point in $\Lambda_{\varepsilon_h}(O_{\varepsilon_h})$, we take its image in O_{ε_h} through the dilatation $\Lambda_{\varepsilon_h}^{-1}$, we pass to its image through the semicartesian map Ψ_h, and we contract in the t-direction through the map $(t,\xi,\eta) \to (\lambda_{\varepsilon_h}(t),\xi,\eta)$. Recalling (3.6) and since the determinant of the Jacobian of Λ_{ε_h} tends to 1 as $h \to +\infty$, we get

$$\int_{\Lambda_{\varepsilon_h}(O_{\varepsilon_h})} |\partial_t\Phi_h \wedge \partial_s\Phi_h|\, dt\, ds = m(O_{\varepsilon_h};\Gamma_h^-,\Gamma_h^+) + \mathcal{O}(\varepsilon_h). \tag{3.7}$$

Recalling that $\Lambda_{\varepsilon_h}(O_{\varepsilon_h}) \subset\subset D$, we can extend Φ_h to a semicartesian map in $\mathrm{semicart}(D;\Gamma^-,\Gamma^+)$. If $\Lambda(O_{\varepsilon_h}) := [[\sigma_h^-,\sigma_h^+]]$, we define Φ_h in $S_h^+ := \{(t,s) \in D : t \in (a+\varepsilon_h, b-\varepsilon_h),\ s \in (\sigma_h^+(t),\sigma^+(t))\}$ as the linear interpolation, that is

$$\Phi_h(t,s) := \left(t, \frac{s-\sigma_h^+(t)}{\sigma^+(t)-\sigma_h^+(t)}\gamma^+(t) - \frac{\sigma^+(t)-s}{\sigma^+(t)-\sigma_h^+(t)}\phi_h(t,\sigma_h^+(t))\right).$$

(2) We denote by $m(O_{\varepsilon_h};\Gamma_h^-,\Gamma_h^+)$ the infimum of the area in $\mathrm{semicart}(O_{\varepsilon_h};\Gamma_h^-,\Gamma_h^+)$.

Similarly we define $\Phi_h(t,s)$ on $S_h^- := \{(t,s)\in D : t\in(a+\varepsilon_h, b-\varepsilon_h), s\in(\sigma^-(t), \sigma_h^-(t))\}$. Thanks to hypothesis (iv) and recalling Lemma 3.2, we deduce

$$\int_{S_h^- \cup S_h^+} |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt\, ds \overset{h\to+\infty}{\longrightarrow} 0. \tag{3.8}$$

We have just to define Φ_h on the curved triangles $T_h^a := \{(t,s) \in D : t \in (a, a+\varepsilon_h)\}$ and $T_h^b := \{(t,s) \in D : t \in (b-\varepsilon_h, b)\}$. Let us define $f_h^a \in \mathrm{Lip}([a, a+\varepsilon_h]; \mathbb{R}^2)$ as

$$f_h^a(t) := \frac{\phi_h(a+\varepsilon_h, 0) - \gamma^+(a)}{\varepsilon_h}(t-a) + \gamma^+(a),$$

so that its graph is the segment joining $(a, \gamma^+(a))$ and $(a+\varepsilon_h, \phi_h(a+\varepsilon_h, 0))$. Next, for $(t,s) \in T_h^a$, we define

$$\Phi_h(t,s) := \begin{cases} \left(t, \frac{s}{\sigma^+(t)}\gamma^+(t) + \frac{\sigma^+(t)-s}{\sigma^+(t)} f_h^a(t)\right) & t \in (a, a+\varepsilon_h),\ s \geq 0, \\ \left(t, \frac{s}{\sigma^-(t)}\gamma^-(t) + \frac{\sigma^-(t)-s}{\sigma^-(t)} f_h^a(t)\right) & t \in (a, a+\varepsilon_h),\ s < 0. \end{cases}$$

Similarly, we define Φ_h on T_h^b. Again Lemma 3.2 and hypotheses (ii) and (iii) imply

$$\int_{T_h^a \cup T_h^b} |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt\, ds \overset{h\to+\infty}{\longrightarrow} 0. \tag{3.9}$$

Thus, using (3.7), (3.8) and (3.9) we obtain, for any $h \in \mathbb{N}$,

$$m(D; \Gamma^-, \Gamma^+) \leq \int_D |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt\, ds \leq m(O_{\varepsilon_h}; \Gamma_h^-, \Gamma_h^+) + \mathcal{O}(\varepsilon_h).$$

Passing to the limit we get (3.5). □

As underlined in the introduction of this chapter, it could be interesting to prove Lemma 3.1 without assuming hypothesis (i) and Lemma 3.3 without assuming hypothesis (iv), see also Remark 4.15. We do not know if the lower semicontinuity result for $m(\mathrm{R}; \cdot, \cdot)$ would be still true. On the other hand the examples in Chapter 5 indirectly prove that we cannot expect Lemma 3.3 to be true assuming only hypotheses $(i) - (iii)$. Moreover, in the following Example we observe that the semicontinuity result requiring only the L^1-convergence is false for the classical Plateau's problem.

Example 3.5. Given a Jordan rectifiable curve $\Gamma \subset \mathbb{R}^3$, we denote by $a(\Gamma)$ the area of the solution of the calssical Plateau's problem, that is the minimum of the area among all immersion of the planar disk B mapping monotonically ∂B onto Γ, see Section 1.6. It is known that $a(\cdot)$ is lower semicontinuous with respect to the Fréchet convergence of curves (see Theorem 1.19). We show here that if the Jordan curve Γ is union of the graphs of $\gamma^\pm \in \mathrm{Lip}([a,b]; \mathbb{R}^2)$, it may happen that

$$a(\Gamma) > \liminf_{h\to+\infty} a(\Gamma_h), \tag{3.10}$$

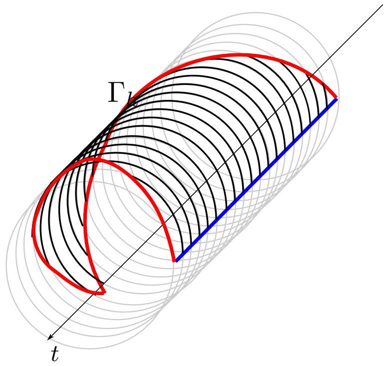

Figure 3.2: The curve Γ_h is one element of the sequence defined in Example 3.5 that approximates in L^1 sense the curve Γ defined in Example 2.15 and represented in Figure 2.2. It is also depicted a disk-type surface with boundary Γ_h that lies on the lateral surface of the cylinder of base the disk of radius ρ and height $b-a$, having therefore area controlled by $2\pi\rho(b-a)$. The sequence (Γ_h) provides a counterexample to the lower semicontinuity of $a(\Gamma)$ with L^1-convergence.

where (Γ_h) is a sequence of Jordan curves, $\Gamma_h = \mathrm{graph}(\gamma_h^-) \cup \mathrm{graph}(\gamma_h^+)$, and $\gamma_h^\pm \to \gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$.
We choose the maps $\gamma^\pm$ as in Example 2.15. Let $(\gamma_h^\pm) \subset \mathrm{Lip}([a,b];\mathbb{R}^2)$ be the sequences converging to $\gamma^\pm$ in $L^1((a,b);\mathbb{R}^2)$ defined as

$$\gamma_h^-(t) = (1,0) = \gamma^-(t),$$
$$\gamma_h^+(t) = \rho\Big(\cos\big(\theta_h(t)\big), \sin\big(\theta_h(t)\big)\Big) + (1-\rho,0),$$

where the function θ_h is given by

$$\theta_h(t) := \begin{cases} \theta(t) & \text{if } t \in [a, b-h^{-1}], \\ -2\pi\frac{b-a-h^{-1}}{b-a}h(t-b) & \text{if } t \in (b-h^{-1}, b], \end{cases}$$

$\theta(\cdot)$ defined in Example 2.15. Hence, in the short interval $(b-h^{-1},b)$, the path made by γ_h^+ is the same as the path it makes in $(a, b-h^{-1})$, with reversed orientation. The curve Γ_h is represented in the Figure 3.2. For any h there exists an immersion of the disk, mapping the boundary of the disk onto Γ_h whose image lies on the cylinder $[a,b]\times B_\rho((1-\rho,0))$; hence for any h we have $a(\Gamma_h) \le 2\pi\rho(b-a)$ that, for $\rho > 2(b-a)$, provides (3.10), see Example 2.15.

We notice that this example does not exclude that $m(\mathrm{R};\Gamma^-,\Gamma^+)$ may be L^1-lower semicontinuous. Indeed the limit of the area of the surfaces represented in Figure 3.2 tends to the area of the semicartesian surface represented in the first picture of Figure 2.2.

4. The functional $\overline{\mathcal{A}}^{\infty}$

Overview of the chapter

The purpose of this thesis is to study $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$ for maps $\mathbf{u}$ in $\mathrm{BV}(\Omega; \mathbb{R}^2)$ regular enough out of a segment; in few words, we are looking for the more "convenient" way (in terms of area) to fill the "hole" that appears in $\mathrm{graph}(\mathbf{u}) \subset \mathbb{R}^4$ due to the presence of the discontinuity. The computation for the triple point map presented in [5] (see also Section 1.4.4) suggests that the optimal profile could be described by a semicartesian surface in $\mathbb{R}^3$.
As it will be clear from the proofs of Theorems 4.11 and 4.17 (see also Remark 4.15), the semicontinuity properties of $m(\mathrm{R}; \cdot, \cdot)$ and $m(D; \cdot, \cdot)$ play an important role in characterizing the singular contribution of the area. Since we do not have the L^1-lower semicontinuity result, that is we are not able to remove hypothesis (i) in Lemma 3.1 and hypothesis (iv) in Lemma 3.3, we are forced to define another relaxation of the functional $\mathcal{A}(\cdot, \Omega)$, denoted by $\overline{\mathcal{A}}^{\infty}(\cdot, \Omega)$, with respect to a stronger notion of convergence, that we are going to describe.

Definition 4.1 (Uniform convergence out of a closed set). Let $\mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ and $J \subset \Omega$ be a closed set with zero Lebesgue measure. A sequence $(\mathbf{v}_h) \subset L^1(\Omega; \mathbb{R}^2)$ is said to converge to $\mathbf{v}$ uniformly out of J if $\mathbf{v}_h \to \mathbf{v}$ uniformly in any compact set of $\Omega \setminus J$, as $h \to +\infty$.

We are now in the position to define the functional $\overline{\mathcal{A}}(\cdot, \Omega)$.

Definition 4.2 (The functional $\overline{\mathcal{A}}^{\infty}(\cdot, \Omega)$). For any $\mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ we define

$$\overline{\mathcal{A}}^{\infty}(\mathbf{v}, \Omega) := \inf \left\{ \liminf_{h \to +\infty} \mathcal{A}(\mathbf{v}_h, \Omega) \right\},$$

where the infimum is taken among all sequences $(\mathbf{v}_h) \subset \mathcal{C}^1(\Omega; \mathbb{R}^2)$ converging to $\mathbf{v}$ in $L^1(\Omega; \mathbb{R}^2)$ and uniformly out of $J_{\mathbf{v}}$.

It is clear that

$$\overline{\mathcal{A}}^{\infty}(\mathbf{v}, \Omega) \geq \overline{\mathcal{A}}(\mathbf{v}, \Omega), \qquad \mathbf{v} \in \mathrm{BV}(\Omega; \mathbb{R}^2).$$

For every $\mathbf{v}$ in the domain of the functional $\overline{\mathcal{A}}(\cdot, \Omega)$ it is also worth to define

$$\overline{\mathcal{A}}_s(\mathbf{v}, \Omega) := \overline{\mathcal{A}}(\mathbf{v}, \Omega) - \int_{\Omega} |\mathcal{M}(\nabla \mathbf{v})| \, dx \, dy,$$

$$\overline{\mathcal{A}}_s^{\infty}(\mathbf{v}, \Omega) := \overline{\mathcal{A}}^{\infty}(\mathbf{v}, \Omega) - \int_{\Omega} |\mathcal{M}(\nabla \mathbf{v})| \, dx \, dy.$$

In this chapter we shall characterize $\overline{\mathcal{A}}_s^\infty(\mathbf{u},\Omega)$ in two rather different cases: $(\Omega,\mathbf{u})$ satisfying condition I and $(\Omega,\mathbf{u})$ satisfying condition II (see Definitions 4.3 and 4.4, respectively). Condition I takes into account maps $\mathbf{u}\in W^{1,\infty}(\Omega\setminus J_{\mathbf{u}};\mathbb{R}^2)$ having as jump set $J_{\mathbf{u}}$ a horizontal segment with both end-points belonging to $\partial\Omega$; namely, the fracture "traverses" the whole domain Ω. Condition II deals with maps $\mathbf{u}\in W^{1,\infty}(\Omega\setminus J_{\mathbf{u}};\mathbb{R}^2)$ with $J_{\mathbf{u}}\subset\subset\Omega$. In the first case $\overline{\mathcal{A}}_s^\infty(\mathbf{u},\Omega)$ comes out to be $m(\mathrm{R};\Gamma^-,\Gamma^+)$ (*i.e.* the minimum problem with partially free boundary conditions) for suitable $\Gamma^\pm$, while in the second case it is represented by $m(D;\Gamma^-,\Gamma^+)$.
Before stating the results, let us fix rigorously the hypotheses on the pairs $(\Omega,\mathbf{u})$.

Let $\Omega\subset\mathbb{R}^2_{(x,y)}$ be a bounded open connected and simply connected set. Let $\mathbf{u}:\Omega\to\mathbb{R}^2_{(\xi,\eta)}$ be a map belonging to $\mathrm{BV}(\Omega;\mathbb{R}^2)\cap W^{1,\infty}(\Omega\setminus J_{\mathbf{u}};\mathbb{R}^2)$ where $J_{\mathbf{u}}\subset\Omega$ is a $\mathcal{C}^2$ simple curve parametrized by an arc-length parametrization $\alpha:(a,b)\subset\mathbb{R}_t\to\mathbb{R}^2_{(x,y)}$. We underline that we could have two cases, $J_{\mathbf{u}}\subset\subset\Omega$ or $\overline{J_{\mathbf{u}}}\cap\partial\Omega\neq\emptyset$.
We denote by $\mathbf{u}^\pm$ the two Lipschitz traces on the two sides of the jump, and we define $\gamma^\pm[\mathbf{u}]\in\mathrm{Lip}([a,b];\mathbb{R}^2)$[(1)] as

$$\gamma^\pm[\mathbf{u}](t):=\mathbf{u}^\pm(\alpha(t)).$$

In accordance with the notation used in Chapter 2, we denote by $\Gamma^\pm[\mathbf{u}]$ the graph of $\gamma^\pm[\mathbf{u}]$ in the space $\mathbb{R}^3=\mathbb{R}_t\times\mathbb{R}^2_{(\xi,\eta)}$. When there is no ambiguity, we will write $\gamma^\pm$ and $\Gamma^\pm$ in place of $\gamma^\pm[\mathbf{u}]$ and $\Gamma^\pm[\mathbf{u}]$ respectively.
In Chapters 4-6 we will always deal with Ω and $\mathbf{u}$ satisfying one of the two conditions, either I or II, specified in Definitions 4.3 and 4.4. If Ω and $\mathbf{u}$ satisfy either condition I or II, the jump $J_{\mathbf{u}}$ is a horizontal segment; this assumption allows to identify the plane $\mathbb{R}^2_{(x,y)}$ (containing the domain of $\mathbf{u}$) with the space of the semicartesian parameters $\mathbb{R}^2_{(t,s)}$, thus simplifying the presentation. When $J_{\mathbf{u}}$ is not a segment but is a simple curve of class $\mathcal{C}^2$ this identification cannot be done, however, we expect that our results could be generalized; in Appendix A we show how to prove the upper bound results in Sections 4.1 and 4.2 in the general case.
From now on R and D are as in (2.4) and (2.5) and we set

$$\mathrm{R}^-:=(a,b)\times(-1,0),\qquad \mathrm{R}^+:=(a,b)\times(0,1).$$

Definition 4.3 (**Condition** I)**.** We say that Ω and $\mathbf{u}\in\mathrm{BV}(\Omega;\mathbb{R}^2)$ satisfy condition I if $\Omega=\mathrm{R}$, $J_{\mathbf{u}}=(a,b)\times\{0\}$, and $\mathbf{u}\in\mathrm{Lip}(\mathrm{R}^-;\mathbb{R}^2)\cap\mathrm{Lip}(\mathrm{R}^+;\mathbb{R}^2)$.

Definition 4.4 (**Condition** II)**.** We say that Ω and $\mathbf{u}\in\mathrm{BV}(\Omega;\mathbb{R}^2)$ satisfy condition II if $J_{\mathbf{u}}:=[a,b]\times\{0\}\subset\subset\Omega$, $\mathbf{u}\in W^{1,\infty}(\Omega\setminus J_{\mathbf{u}};\mathbb{R}^2)$, and there exist the pointwise limits (still denoted with $\mathbf{u}^\pm$) of $\mathbf{u}$ at all points of $J_{\mathbf{u}}$.

Our results are the following.

Theorem 4.5. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *Then*

$$\overline{\mathcal{A}}^\infty(\mathbf{u},\mathrm{R})=\int_{\mathrm{R}}|\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds+m(\mathrm{R};\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]).$$

[(1)] In the case where $\alpha(a)$ and $\alpha(b)$ belong to $\partial\Omega$, $\gamma^\pm$ are rigorously defined only in the open interval (a,b); anyway they are Lipschitz, and thus we can extend them also to $t=a$ and $t=b$.

Theorem 4.6. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* II. *Then*

$$\overline{\mathcal{A}}^{\infty}(\mathbf{u},\Omega) = \int_{\Omega} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + m(D;\Gamma^{-}[\mathbf{u}],\Gamma^{+}[\mathbf{u}]).$$

We split the proof of each theorem into the proof of an upper bound (Sections 4.1 and 4.2) and a lower bound (Sections 4.3 and 4.4). These results are contained in [8], even if the strategy to prove the upper bound is already used in [6], for a more general jump curve.

4.1 Upper bound - condition I

In this Section we prove the following upper bound that, together with the results in Section 4.3 gives Theorem 4.5.

Theorem 4.7 (**Upper bound - condition** I)**.** *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *Then there exists a sequence* $(\mathbf{u}_h) \subset H^1(\mathrm{R};\mathbb{R}^2)$ *converging to* $\mathbf{u}$ *in* $L^1(\mathrm{R};\mathbb{R}^2)$ *and uniformly out of* $J_{\mathbf{u}}$*, such that*

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\mathrm{R}) = \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + m(\mathrm{R};\Gamma^{-}[\mathbf{u}],\Gamma^{+}[\mathbf{u}]). \tag{4.1}$$

Therefore

$$\overline{\mathcal{A}}_s^{\infty}(\mathbf{u},\mathrm{R}) \le m(\mathrm{R};\Gamma^{-}[\mathbf{u}],\Gamma^{+}[\mathbf{u}]). \tag{4.2}$$

Proof. The existence of a sequence in $H^1(\mathrm{R};\mathbb{R}^2)$ satisfying (4.1) implies (4.2) because, as shown in Lemma 1.8 for $\overline{\mathcal{A}}$, we can obtain $\overline{\mathcal{A}}^{\infty}(\mathbf{u},\mathrm{R})$ relaxing the functional $\mathbf{v} \to \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{v})|\, dt\, ds$ defined on $\mathcal{D}(\mathrm{R};\mathbb{R}^2)$ with respect to the L^1 and uniform out of $J_{\mathbf{u}}$ convergence. The proof reduces thus to exhibit a sequence $(\mathbf{u}_h)$ as in the statement of the Theorem.

Let $(\Phi_h) \subset \mathrm{semicart}(\mathrm{R};\Gamma^{-},\Gamma^{+})$ be an area minimizing sequence, that is

$$\int_{\mathrm{R}} |\partial_t \Phi_h \wedge \partial_s \Phi_h|\, dt\, ds \to m(\mathrm{R};\Gamma^{-},\Gamma^{+}) \qquad \text{as } h \to +\infty.$$

For any h, let $\phi_h = (\phi_{h,1},\phi_{h,2}) \in H^1(\mathrm{R};\mathbb{R}^2)$ be such that $\Phi_h(t,s) = (t,\phi_h(t,s))$. Given any positive number c, let us denote by R_c the rectangle $(a,b)\times(-c,c)$. Let (ε_k) be an infinitesimal sequence of positive numbers, $\varepsilon_k < 1/2$ for every $k \in \mathbb{N}$; for every fixed $h \in \mathbb{N}$, let us define the sequence $(\mathbf{u}_k^h) \subset H^1(\mathrm{R};\mathbb{R}^2)$ as

$$\mathbf{u}_k^h(t,s) := \begin{cases} \mathbf{u}(t,s) & (t,s) \in \mathrm{R}\setminus \mathrm{R}_{2\varepsilon_k}, \\ \mathbf{u}(t,2(s-\varepsilon_k)) & (t,s) \in (a,b)\times(\varepsilon_k,2\varepsilon_k), \\ \mathbf{u}(t,2(s+\varepsilon_k)) & (t,s) \in (a,b)\times(-2\varepsilon_k,-\varepsilon_k), \\ \phi_h(t,\frac{s}{\varepsilon_k}) & (t,s) \in \overline{\mathrm{R}_{\varepsilon_k}}. \end{cases}$$

Let us compute the limit of the area contribution in each region separately.

Let us start from the inner rectangle $\mathrm{R}_{\varepsilon_k}$. Since in this region

$$\partial_t \mathbf{u}_k^h(t,s) = \partial_t \phi_h\left(t, \frac{s}{\varepsilon_k}\right) \qquad \text{and} \qquad \partial_s \mathbf{u}_k^h(t,s) = \frac{1}{\varepsilon_k}\partial_s \phi_h\left(t, \frac{s}{\varepsilon_k}\right),$$

then

$$\mathcal{A}(\mathbf{u}_k^h, \mathrm{R}_{\varepsilon_k}) = \\ = \int_a^b \int_{-\varepsilon_k}^{\varepsilon_k} \sqrt{1 + |\partial_t \phi_h|^2 + \varepsilon_k^{-2}|\partial_s \phi_h|^2 + \varepsilon_k^{-2}\left(\partial_t \phi_{h,1}\partial_s \phi_{h,2} - \partial_t \phi_{h,2}\partial_s \phi_{h,1}\right)^2}\, dt\, ds,$$

where ϕ_h is valuated in $\left(t, \frac{s}{\varepsilon_k}\right)$. Performing the change of variable $s \to \frac{s}{\varepsilon_k}$, without renaming it, we get:

$$\begin{aligned} &\mathcal{A}(\mathbf{u}_k^h, \mathrm{R}_{\varepsilon_k}) = \\ &= \int_a^b \int_{-1}^1 \sqrt{\varepsilon_k^2\left(1 + |\partial_t \phi_h|^2\right) + |\partial_s \phi_h|^2 + \left(\partial_t \phi_{h,1}\partial_s \phi_{h,2} - \partial_t \phi_{h,2}\partial_s \phi_{h,1}\right)^2}\, dt\, ds, \\ &\overset{k\to+\infty}{\longrightarrow} \int_a^b \int_{-1}^1 \sqrt{|\partial_s \phi_h|^2 + \left(\partial_t \phi_{h,1}\partial_s \phi_{h,2} - \partial_t \phi_{h,2}\partial_s \phi_{h,1}\right)^2} \\ &= \int_{\mathrm{R}} |\partial_t \Phi_h \wedge \partial_s \Phi_h|\, dt\, ds, \end{aligned} \tag{4.3}$$

where we have applied the Lebesgue dominated convergence Theorem in order to pass to the limit, and (2.3) in the last equality.
Recalling that $\mathbf{u}_{|\mathrm{R}^\pm} \in \mathrm{Lip}(\mathrm{R}^\pm; \mathbb{R}^2)$ we get that

$$|\mathcal{M}(\nabla \mathbf{u}_k^h)| \le C \qquad \text{in } \mathrm{R}^+ \cup \mathrm{R}^-,$$

where C depends only on the Lipschitz constants $\mathrm{lip}(\mathbf{u}_{|\mathrm{R}^+})$ and $\mathrm{lip}(\mathbf{u}_{|\mathrm{R}^-})$. Therefore we get

$$\begin{aligned} \mathcal{A}(\mathbf{u}_k^h, (a,b)\times(\varepsilon_k, 2\varepsilon_k)) &\le C\varepsilon_k(b-a) \\ \mathcal{A}(\mathbf{u}_k^h, (a,b)\times(-2\varepsilon_k, -\varepsilon_k)) &\le C\varepsilon_k(b-a), \end{aligned} \tag{4.4}$$

and thus the area contribution on the two strips $(a,b)\times(-2\varepsilon_k,-\varepsilon_k)$ and $(a,b)\times(\varepsilon_k, 2\varepsilon_k)$ is negligible as $k \to +\infty$.
Finally we observe that, from the definition of $(\mathbf{u}_k^h)$, there holds $\mathcal{A}(\mathbf{u}_k^h, \mathrm{R}\setminus \mathrm{R}_{2\varepsilon_k}) = \mathcal{A}(\mathbf{u}, \mathrm{R}\setminus \mathrm{R}_{2\varepsilon_k})$, and thus

$$\mathcal{A}(\mathbf{u}_k^h, \mathrm{R}\setminus \mathrm{R}_{2\varepsilon_k}) \to \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds \qquad \text{as } k \to +\infty. \tag{4.5}$$

Using (4.3), (4.4) and (4.5) we get that for every $h \in \mathbb{N}$ there holds

$$\lim_{k\to+\infty} \mathcal{A}(\mathbf{u}_k^h, \mathrm{R}) = \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + \int_{\mathrm{R}} |\partial_t \Phi_h \wedge \partial_s \Phi_h|\, dt\, ds.$$

By a diagonalization process, we can choose a suitable sequence (k_h) such that $(\mathbf{u}_h) := (\mathbf{u}_{k_h}^h)$ satisfies

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) = \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]).$$

$\square$

Remark 4.8. We notice that, since the sequence $(\mathbf{u}_h)$ built in Theorem 4.7 converges to $\mathbf{u}$ in $L^1(\mathrm{R};\mathbb{R}^2)$, (4.1) implies also that $\overline{\mathcal{A}}_s(\mathbf{u},\mathrm{R}) \le m(\mathrm{R};\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$.

4.2 Upper bound - condition II

We prove here the counterpart of Theorem 4.7 for $(\Omega,\mathbf{u})$ satisfying condition II, that, with the results in Section 4.4, prove Theorem 4.6.

Theorem 4.9 (**Upper bound - condition** II)**.** *Let Ω and $\mathbf{u}$ satisfy condition* II*. Then there exists a sequence $(\mathbf{u}_h) \subset H^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$, such that*

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) = \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds + m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]). \tag{4.6}$$

Therefore

$$\overline{\mathcal{A}}_s^\infty(\mathbf{u},\Omega) \le m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]). \tag{4.7}$$

Remark 4.10. First of all we observe that, since $J_{\mathbf{u}} = [a,b]\times\{0\} \subset \Omega$ and $\mathbf{u} \in W^{1,\infty}(\Omega\setminus J_{\mathbf{u}};\mathbb{R}^2)$, the space semicart$(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$ is non-empty. Indeed the regularity assumption on $\mathbf{u}$ implies that there exist the limits of $\mathbf{u}(t,s)$ for $(t,s) \in \Omega\setminus J_{\mathbf{u}}$ converging either to $(a,0)$ or $(b,0)$ and that

$$\lim_{\Omega\setminus J_{\mathbf{u}}\ni(t,s)\to(a,0)} \mathbf{u}(t,s) = \gamma^-[\mathbf{u}](a) = \gamma^+[\mathbf{u}](a),$$
$$\lim_{\Omega\setminus J_{\mathbf{u}}\ni(t,s)\to(b,0)} \mathbf{u}(t,s) = \gamma^-[\mathbf{u}](b) = \gamma^+[\mathbf{u}](b);$$

therefore the curve $\Gamma[\mathbf{u}] = \Gamma^-[\mathbf{u}]\cup\Gamma^+[\mathbf{u}]$ is a closed (not necessarily simple) curve.

Proof. As in the case Ω, $\mathbf{u}$ satisfying condition I, (4.6) implies (4.7)[(2)].
Thanks to Lemma 2.12, we can suppose without loss of generality that $D = [[\sigma^-,\sigma^+]]$ with $|\sigma^\pm| < 1$, and thus that $D \subset \mathrm{R}$. For every positive number c we denote by D_c the set $\{(t,s) : t \in (a,b),\ s \in (c\sigma^-(t), c\sigma^+(t))\}$; notice that $D_c \subset \mathrm{R}_c$, R_c defined as in the proof of Theorem 4.7.
Let $(\Phi_h) \subset$ semicart$(D;\Gamma^-,\Gamma^+)$ be an area minimizing sequence, that is

$$\lim_{h\to+\infty}\int_D |\partial_t\Phi_h \wedge \partial_s\Phi_h|\,dt\,ds = m(D;\Gamma^-,\Gamma^+)$$

and let (ε_k) be an infinitesimal sequence of positive numbers such that $\mathrm{R}_{\varepsilon_k} \subset\subset \Omega$ for every $k \in \mathbb{N}$.
For any k, we define the map $T_{\varepsilon_k} : \mathrm{R}_{\varepsilon_k}\setminus\overline{D_{\varepsilon_k}} \to \mathrm{R}_{\varepsilon_k}\setminus(a,b)\times\{0\}$ as follows:

$$T_{\varepsilon_k}(t,s) := \begin{cases} \left(t, \frac{s-\varepsilon_k\sigma^+(t)}{1-\sigma^+(t)}\right) & (t,s) \in (\mathrm{R}_{\varepsilon_k}\setminus\overline{D_{\varepsilon_k}})\cap\{s>0\}, \\ \left(t, \frac{s-\varepsilon_k\sigma^-(t)}{1+\sigma^-(t)}\right) & (t,s) \in (\mathrm{R}_{\varepsilon_k}\setminus\overline{D_{\varepsilon_k}})\cap\{s<0\}. \end{cases} \tag{4.8}$$

(2) We remark that again (4.6) implies also $\overline{\mathcal{A}}_s(\mathbf{u},\Omega) \le m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$.

Notice that T_{ε_k} is the identity on $\partial \mathrm{R}_{\varepsilon_k}$.
Then let us consider the sequence $(\mathbf{u}_k^h) \subset H^1(\Omega; \mathbb{R}^2)$ given by

$$\mathbf{u}_k^h(t,s) := \begin{cases} \mathbf{u}(t,s) & (t,s) \in \Omega \setminus \mathrm{R}_{\varepsilon_k}, \\ \mathbf{u}(T_{\varepsilon_k}(t,s)) & (t,s) \in \mathrm{R}_{\varepsilon_k} \setminus \overline{D_{\varepsilon_k}}, \\ \phi_h\left(t, \frac{s}{\varepsilon_k}\right) & (t,s) \in \overline{D_{\varepsilon_k}}, \end{cases}$$

where ϕ_h is such that $\Phi_h(t,s) = (t, \phi_h(t,s))$.
With a computation analogous to (4.3) we get

$$\lim_{k \to +\infty} \mathcal{A}(\mathbf{u}_k^h, D_{\varepsilon_k}) = \int_D |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt \, ds. \tag{4.9}$$

In order to prove that the area contribution on $\mathrm{R}_{\varepsilon_k} \setminus \overline{D_{\varepsilon_k}}$ is negligible as $k \to +\infty$, we observe that

$$\begin{aligned} \partial_t \mathbf{u}_k^h(t,s) &= \partial_t \mathbf{u}(T_{\varepsilon_k}(t,s)) + \partial_t T_{\varepsilon_k,2}(t,s) \partial_s \mathbf{u}(T_{\varepsilon_k}(t,s)), \\ \partial_s \mathbf{u}_k^h(t,s) &= \partial_s T_{\varepsilon_k,2}(t,s) \partial_s \mathbf{u}(T_{\varepsilon_k}(t,s)), \end{aligned}$$

where $T_{\varepsilon_k,2}$ denotes the second component of the map T_{ε_k}; since $|\nabla T_{\varepsilon_k,2}| \le \widetilde{C}$, $\widetilde{C}$ depending only on $\mathrm{lip}(\sigma^-)$ and $\mathrm{lip}(\sigma^+)$, then

$$|\mathcal{M}(\nabla \mathbf{u}_k^h)| \le C \qquad \text{in } \mathrm{R}_{\varepsilon_k} \setminus \overline{D_{\varepsilon_k}},$$

where C depends only on $\mathrm{lip}(\sigma^-)$, $\mathrm{lip}(\sigma^+)$, and on the $L^\infty(\Omega \setminus J_{\mathbf{u}}; \mathbb{R}^2)$ norms of $\partial_t \mathbf{u}$ and $\partial_s \mathbf{u}$. Therefore

$$\lim_{k \to +\infty} \mathcal{A}(\mathbf{u}_k^h, \mathrm{R}_{\varepsilon_k} \setminus \overline{D_{\varepsilon_k}}) = 0. \tag{4.10}$$

Finally we observe that by the definition of $(\mathbf{u}_k^h)$

$$\lim_{k \to +\infty} \mathcal{A}(\mathbf{u}_h^k, \Omega \setminus \overline{\mathrm{R}_{\varepsilon_k}}) = \lim_{k \to +\infty} \mathcal{A}(\mathbf{u}, \Omega \setminus \overline{\mathrm{R}_{\varepsilon_k}}) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds,$$

that, together with (4.9) and (4.10), provides

$$\lim_{k \to +\infty} \mathcal{A}(\mathbf{u}_k^h, \Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + \int_D |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt \, ds.$$

Thus we can choose (k_h) such that the sequence $(\mathbf{u}_h) := (\mathbf{u}_{k_h}^h) \subset H^1(\Omega; \mathbb{R}^2)$ converges to $\mathbf{u} \in L^1(\Omega; \mathbb{R}^2)$ and satisfies (4.6). $\square$

4.3 Lower bound - condition I

In this section we conclude the proof of Theorem 4.5 proving the following lower bound result.

Theorem 4.11 (**Lower bound - condition** I)**.** *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *For every sequence* $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ *converging to* $\mathbf{u}$ *in* $L^1(\mathrm{R};\mathbb{R}^2)$ *and uniformly out of* $J_{\mathbf{u}}$ *there holds*

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) \geq \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]).$$

Therefore

$$\overline{\mathcal{A}}_s^{\infty}(\mathbf{u}, \mathrm{R}) \geq m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]). \tag{4.11}$$

We split the theorem into two steps. Firstly we prove the result for maps satisfying the further condition that, for every h, $\mathbf{u}_h = \mathbf{u}$ out of a neighbourhood N_h of $J_{\mathbf{u}}$, (N_h) converging monotonically to $J_{\mathbf{u}}$. That is:

Proposition 4.12. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *If* $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ *converges to* $\mathbf{u}$ *in* $L^1(\mathrm{R};\mathbb{R}^2)$ *and*

$$\mathbf{u}_h = \mathbf{u} \qquad \textit{in } \mathrm{R} \setminus N_h,$$

for some decreasing sequence (N_h) *of neighbourhoods of* $J_{\mathbf{u}}$ *such that* $\bigcap_h N_h = J_{\mathbf{u}}$*, then*

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) \geq \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]).$$

Then we prove that we can reduce any sequence satisfying the hypotheses of Theorem 4.11 to a sequence as in Proposition 4.12, namely:

Proposition 4.13. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* I. *Let* $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ *be a sequence converging to* $\mathbf{u}$ *in* $L^1(\mathrm{R};\mathbb{R}^2)$ *and uniformly out of* $J_{\mathbf{u}}$. *Then there exists a sequence* $(\mathbf{v}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ *satisfying the hypotheses of Proposition 4.12 and such that*

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) \geq \liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h, \mathrm{R}). \tag{4.12}$$

The proof of Theorem 4.11 follows immediately from Propositions 4.12 and 4.13.

Proof of Theorem 4.11. Let $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ be any sequence converging to $\mathbf{u}$ in $L^1(\mathrm{R};\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$; let $(\mathbf{v}_h)$ be the sequence provided by Proposition 4.13. Then

$$\begin{aligned}\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) &\overset{\text{Prop. 4.13}}{\geq} \liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h, \mathrm{R}) \\ &\overset{\text{Prop. 4.12}}{\geq} \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + m(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}]).\end{aligned}$$

Since this holds for any sequence satisfying the hypotheses of the theorem, (4.11) is implied. □

Thus we have to prove Propositions 4.12 and 4.13.

Proof of Proposition 4.12

The proof of Proposition 4.12 follows from the lower semicontinuity result for $m(\mathrm{R};\cdot,\cdot)$, Lemma 3.1, coupled with the following Lemma.

Lemma 4.14 (Lower bound on a strip). *Let* $\varepsilon \in (0,1)$ *and* $\mathrm{R}_\varepsilon := (a,b)\times(-\varepsilon,\varepsilon)$. *Given a map* $\mathbf{v} \in \mathrm{Lip}(\mathrm{R}_\varepsilon;\mathbb{R}^2)$, *let* $\Gamma_\varepsilon^\pm$ *denote the graphs on* $[a,b]$ *of the sections* $\mathbf{v}(\cdot,\pm\varepsilon) \in \mathrm{Lip}([a,b];\mathbb{R}^2)$. *Then*

$$\mathcal{A}(\mathbf{v},\mathrm{R}_\varepsilon) \geq m(\mathrm{R};\Gamma_\varepsilon^-,\Gamma_\varepsilon^+).$$

Proof. We denote by v_1 and v_2 the two components of the map $\mathbf{v}$. By neglecting the constant 1 and the term $|\partial_t \mathbf{v}|^2$ in the expression of $|\mathcal{M}(\nabla\mathbf{v})|$, we deduce

$$\mathcal{A}(\mathbf{v},\mathrm{R}_\varepsilon) \geq \int_{\mathrm{R}_\varepsilon} \sqrt{|\partial_s\mathbf{v}|^2 + (\partial_t v_1 \partial_s v_2 - \partial_s v_1 \partial_t v_2)^2}\, dt\, ds. \tag{4.13}$$

On the other hand we can define the map $\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma_\varepsilon^-,\Gamma_\varepsilon^+)$ as

$$\Phi(t,s) = (t, \mathbf{v}(t,\varepsilon s)),$$

and (2.3) shows that $\int_{\mathrm{R}} |\partial_t\Phi \wedge \partial_s\Phi|\, dt\, ds$ equals the right hand side of (4.13). Hence

$$\mathcal{A}(\mathbf{v},\mathrm{R}_\varepsilon) \geq \int_{\mathrm{R}} |\partial_t\Phi \wedge \partial_s\Phi|\, dt\, ds \geq m(\mathrm{R};\Gamma_\varepsilon^-,\Gamma_\varepsilon^+).$$

□

Now, we can prove Proposition 4.12.

Proof of Proposition 4.12. Recalling the properties of the sequence $(\mathbf{u}_h)$, we can choose an infinitesimal sequence (ε_h) of positive numbers such that $\mathrm{R}_{\varepsilon_h} := (a,b) \times (-\varepsilon_h,\varepsilon_h) \supseteq N_h$. We have

$$\mathcal{A}(\mathbf{u}_h,\mathrm{R}) = \mathcal{A}(\mathbf{u},\mathrm{R}\setminus\mathrm{R}_{\varepsilon_h}) + \mathcal{A}(\mathbf{u}_h,\mathrm{R}_{\varepsilon_h});$$

Let $\gamma_h^\pm(\cdot) := \mathbf{u}_h(\cdot,\pm\varepsilon_h)$ and $\gamma^\pm := \gamma^\pm[\mathbf{u}]$. We observe that, by hypothesis, $\gamma_h^\pm = \mathbf{u}(\cdot,\pm\varepsilon_h)$ and thus $\gamma_h^\pm$ and γ satisfy the hypotheses of Lemma 3.1. Hence, applying also Lemma 4.14, we get

$$\begin{aligned}\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\mathrm{R}) &\geq \liminf_{h\to+\infty} \left[\mathcal{A}(\mathbf{u},\mathrm{R}\setminus\mathrm{R}_{\varepsilon_h}) + m(\mathrm{R};\Gamma_h^-,\Gamma_h^+)\right] \\ &\geq \int_{\mathrm{R}} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + m(\mathrm{R};\Gamma^-,\Gamma^+),\end{aligned}$$

that is the thesis. □

Remark 4.15 (About semicontinuity). The strategy of the proof of Proposition 4.12 would provide the lower bound (4.11) for any sequence $(\mathbf{u}_h) \subset \mathcal{C}^1(\mathrm{R};\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\mathrm{R};\mathbb{R}^2)$, if we would be able to remove the hypothesis (i) on the L^∞-norm of $\dot\gamma_h^\pm$ in Lemma 3.1. Indeed, as a consequence of Fubini's Theorem, the convergence of $\mathbf{u}_h$ to $\mathbf{u}$ in $L^1(\mathrm{R};\mathbb{R}^2)$ implies that $\mathbf{u}_h(\cdot,\varepsilon) \to \mathbf{u}(\cdot,\varepsilon)$ in $L^1((a,b);\mathbb{R}^2)$ for almost every level $\varepsilon \in (0,1)$.

Proof of Proposition 4.13

In order to prove Proposition 4.13 we need the following technical result, inspired by [1, Proposition 7.3], that provides a way to interpolate two maps on a strip, by controlling the amount of area of the interpolating map in dependence on the thickness of the strip itself. We recall that $\mathrm{R}^+ = (a,b) \times (0,1)$.

Lemma 4.16 (**Interpolation - condition** I)**.** *Let* $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R}^+;\mathbb{R}^2)$*,* $\mathbf{u} \in \mathrm{Lip}(\mathrm{R}^+;\mathbb{R}^2)$ *and let* $\mathbf{u}_h \to \mathbf{u}$ *in* $L^1(\mathrm{R}^+;\mathbb{R}^2)$*. Let* $\varepsilon_o \in (0,1)$ *be fixed, such that* $\partial_t \mathbf{u}(t,s)_{|s=\varepsilon_o}$ *exists almost everywhere in* (a,b)*. Let* $\varepsilon_i \in (0,\varepsilon_o)$ *be such that:*

(i) $||\mathbf{u}_h(\cdot,\varepsilon_i) - \mathbf{u}(\cdot,\varepsilon_i)||_{L^\infty((a,b);\mathbb{R}^2)} \to 0$ *as* $h \to +\infty$*;*

(ii) $\partial_t \mathbf{u}_h(t,s)_{|s=\varepsilon_i}$ *exists almost everywhere in* (a,b) *for any* $h \in \mathbb{N}$*;*

(iii) $\liminf_{h\to+\infty} ||\partial_t \mathbf{u}_h(\cdot,\varepsilon_i)||_{L^1((a,b);\mathbb{R}^2)} \le M$*, where the constant* M *may depend on* ε_i*.*

Then the sequence $(\mathbf{v}_h) \subset \mathrm{Lip}(\mathrm{R}^+;\mathbb{R}^2)$ *defined as*

$$\mathbf{v}_h(t,s) := \begin{cases} \mathbf{u}(t,s) & t \in (a,b)\,, s > \varepsilon_o, \\ \frac{\varepsilon_o - s}{\varepsilon_o - \varepsilon_i}\mathbf{u}_h(t,\varepsilon_i) + \frac{s-\varepsilon_i}{\varepsilon_o-\varepsilon_i}\mathbf{u}(t,\varepsilon_o) & t \in (a,b),\ s \in [\varepsilon_i,\varepsilon_o], \\ \mathbf{u}_h(t,s) & t \in (a,b)\,, s < \varepsilon_i \end{cases} \tag{4.14}$$

satisfies

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h,(a,b)\times(\varepsilon_i,\varepsilon_o)) \le C[1+M]|\varepsilon_o - \varepsilon_i|, \tag{4.15}$$

where $C > 0$ *depends on* $\mathrm{lip}(\mathbf{u})$ *and* $b-a$*, and it is independent of* ε_o *and* ε_i*.*

Proof. Let us denote by $S_{\varepsilon_i}^{\varepsilon_o}$ the strip $(a,b)\times(\varepsilon_i,\varepsilon_o)$. The Jacobian matrix of $\mathbf{v}_h$ at almost every $(t,s) \in S_{\varepsilon_i}^{\varepsilon_o}$ is:

$$\frac{1}{\varepsilon_o - \varepsilon_i}\left((\varepsilon_o - s)\partial_t \mathbf{u}_h(t,\varepsilon_i) + (s-\varepsilon_i)\partial_t \mathbf{u}(t,\varepsilon_o) \;\middle|\; \mathbf{u}(t,\varepsilon_o) - \mathbf{u}_h(t,\varepsilon_i) \right).$$

We control the area of $\mathbf{v}_h$ as

$$\mathcal{A}(\mathbf{v}_h, S_{\varepsilon_i}^{\varepsilon_o}) \le C \int_{S_{\varepsilon_i}^{\varepsilon_o}} [1 + |\partial_t \mathbf{v}_h| + |\partial_s \mathbf{v}_h| + |\det \nabla \mathbf{v}_h|]\, dt\, ds$$

where $C > 0$ is an absolute positive constant. We estimate each of the four integrals on the right hand side as follows.

- The first term is obviously controlled by $(\varepsilon_o - \varepsilon_i)(b-a)$.
- For the second term we have

$$\begin{aligned} \int_{\varepsilon_i}^{\varepsilon_o}\int_a^b |\partial_t \mathbf{v}_h|\, dt\, ds &\le (\varepsilon_o - \varepsilon_i)\int_a^b (|\partial_t \mathbf{u}_h(t,\varepsilon_i)| + |\partial_t \mathbf{u}(t,\varepsilon_o)|)\, dt \\ &\le (\varepsilon_o - \varepsilon_i)\left[\int_a^b |\partial_t \mathbf{u}_h(t,\varepsilon_i)|\, dt + \mathrm{lip}(\mathbf{u})(b-a)\right]. \end{aligned} \tag{4.16}$$

- Similarly for the third term we have

$$\begin{aligned}&\int_{\varepsilon_i}^{\varepsilon_o}\int_a^b |\partial_s \mathbf{v}_h(t,s)|\,dt\,ds\\ &\leq \int_a^b |\mathbf{u}(t,\varepsilon_o)-\mathbf{u}(t,\varepsilon_i)| + |\mathbf{u}(t,\varepsilon_i)-\mathbf{u}_h(t,\varepsilon_i)|\,dt\\ &\leq \mathrm{lip}(\mathbf{u})(b-a)(\varepsilon_o-\varepsilon_i) + ||\mathbf{u}(\cdot,\varepsilon_i)-\mathbf{u}_h(\cdot,\varepsilon_i)||_{L^1((a,b);\mathbb{R}^2)}.\end{aligned} \tag{4.17}$$

- Finally for the term with the determinant, there holds:

$$\begin{aligned}&\int_{\varepsilon_i}^{\varepsilon_o}\int_a^b |\det \nabla \mathbf{v}_h(t,s)|\ dtds\\ \leq& 2\int_a^b |\mathbf{u}(t,\varepsilon_o)-\mathbf{u}_h(t,\varepsilon_i)|\big(|\partial_t \mathbf{u}_h(t,\varepsilon_i)| + |\partial_t \mathbf{u}(t,\varepsilon_o)|\big)\,dt\\ =& 2\int_a^b |\mathbf{u}(t,\varepsilon_o)-\mathbf{u}_h(t,\varepsilon_i)||\partial_t \mathbf{u}(t,\varepsilon_o)|\,dt + 2\int_a^b |\mathbf{u}(t,\varepsilon_o)-\mathbf{u}_h(t,\varepsilon_i)||\partial_t \mathbf{u}_h(t,\varepsilon_i)|\,dt\\ =:& \ \mathrm{I}_h + \mathrm{II}_h.\end{aligned}$$

We have

$$\begin{aligned}\mathrm{I}_h \leq& 2\mathrm{lip}(\mathbf{u})\int_a^b \left(|\mathbf{u}(t,\varepsilon_o)-\mathbf{u}(t,\varepsilon_i)| + |\mathbf{u}(t,\varepsilon_i)-\mathbf{u}_h(t,\varepsilon_i)|\right)\,dt\\ \leq& 2(\mathrm{lip}(\mathbf{u}))^2(b-a)(\varepsilon_o-\varepsilon_i) + 2\mathrm{lip}(\mathbf{u})||\mathbf{u}(\cdot,\varepsilon_i)-\mathbf{u}_h(\cdot,\varepsilon_i)||_{L^1((a,b);\mathbb{R}^2)}.\end{aligned} \tag{4.18}$$

Next

$$\begin{aligned}\mathrm{II}_h \leq& 2\int_a^b |\mathbf{u}(t,\varepsilon_o)-\mathbf{u}(t,\varepsilon_i)||\partial_t \mathbf{u}_h(t,\varepsilon_i)|\,dt\\ &+ 2\int_a^b |\mathbf{u}(t,\varepsilon_i)-\mathbf{u}_h(t,\varepsilon_i)||\partial_t \mathbf{u}_h(t,\varepsilon_i)|\,dt\\ \leq& 2\big(\mathrm{lip}(\mathbf{u})(\varepsilon_o-\varepsilon_i)\\ &+ ||\mathbf{u}(\cdot,\varepsilon_i)-\mathbf{u}_h(\cdot,\varepsilon_i)||_{L^\infty((a,b);\mathbb{R}^2)}\big)\int_a^b |\partial_t \mathbf{u}_h(t,\varepsilon_i)|\,dt.\end{aligned} \tag{4.19}$$

Finally, using (4.16), (4.17), (4.18), (4.19) we get:

$$\begin{aligned}\mathcal{A}(\mathbf{v}_h, S_{\varepsilon_i}^{\varepsilon_o}) \leq& C(\varepsilon_o-\varepsilon_i)\Big[1 + ||\mathbf{u}(\cdot,\varepsilon_i)-\mathbf{u}_h(\cdot,\varepsilon_i)||_{L^1((a,b);\mathbb{R}^2)}\\ &+ (1 + ||\mathbf{u}(\cdot,\varepsilon_i)-\mathbf{u}_h(\cdot,\varepsilon_i)||_{L^\infty((a,b);\mathbb{R}^2)})\int_a^b |\partial_t \mathbf{u}_h(t,\varepsilon_i)|\,dt\Big].\end{aligned}$$

Using hypotheses $(i)-(iii)$, and passing to the limit, we get

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h, S_{\varepsilon_i}^{\varepsilon_o}) \leq C[1+M](\varepsilon_o-\varepsilon_i),$$

where C depends just on $b-a$ and $\mathrm{lip}(\mathbf{u})$. □

We are now in the position to prove Proposition 4.13.

Proof of Proposition 4.13. We can suppose that $\mathcal{A}(\mathbf{u}_h, \mathrm{R})$ is uniformly bounded, otherwise the result is trivial. Moreover, passing to a not relabeled subsequence, we can suppose also that there exist

$$\begin{aligned}
&\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) < +\infty,\\
&\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}^+) < +\infty,\\
&\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}^-) < +\infty.
\end{aligned}$$

Now, we want to select an infinitesimal sequence (ε_k) of positive numbers, and a subsequence $(\mathbf{u}_{h_j})$, such that for any k the hypotheses of $(i) - (iii)$ of Proposition 4.16 are verified by $\mathbf{u}_{h_j}(\cdot, \varepsilon_k)$.
Since $\mathbf{u}_h \to \mathbf{u}$ uniformly on every compact set of R^+, hypothesis (i) is verified for any choice of the level $\varepsilon_1 \in (0,1)$. Using Fatou's Lemma we get

$$\int_0^1 \liminf_{h\to+\infty} \left(\int_a^b |\partial_t \mathbf{u}_h(t,s)|\, dt \right) ds \le \liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}^+) < +\infty.$$

Thus we can select a level $\varepsilon_1 \in (0,1)$, a subsequence $(\mathbf{u}_{h_j})$ and a constant $M(\varepsilon_1)$ both depending on ε_1, such that

$$\lim_{j\to+\infty} \int_a^b |\partial_t \mathbf{u}_{h_j}(t, \varepsilon_1)|\, dt \le M(\varepsilon_1).$$

Repeating the argument a countably number of times and using the same procedure on R^-, we can select a subsequence $(\mathbf{u}_{h_j})$ of $(\mathbf{u}_h)$, and a sequence of positive levels $\varepsilon_k \to 0$ such that $\mathbf{u}_{h_j}(\cdot, \pm\varepsilon_k)$ satisfies the hypotheses (i), (ii), and (iii) of Proposition 4.16.
Let us choose now a sequence (δ_k) of positive numbers such that

$$\delta_k M(\varepsilon_k) \overset{k\to+\infty}{\longrightarrow} 0$$

and such that $\partial_t \mathbf{u}(t,s)_{|s=\pm(\varepsilon_k+\delta_k)}$ exists for almost every $t \in (a,b)$.
Now for any k we define the maps, as in (4.14):

$$\mathbf{v}^k_{h_j}(t,s) := \begin{cases}
\mathbf{u}(t,s) & t\in(a,b), |s| > \varepsilon_k + \delta_k,\\
\frac{\varepsilon_k+\delta_k-s}{\delta_k}\mathbf{u}_{h_j}(t,\varepsilon_k) + \frac{s-\varepsilon_k}{\delta_k}\mathbf{u}(t,\varepsilon_k+\delta_k) & t\in(a,b),\\
& \varepsilon_k \le s \le \varepsilon_k+\delta_k,\\
\frac{\varepsilon_k+\delta_k+s}{\delta_k}\mathbf{u}_{h_j}(t,-\varepsilon_k) + \frac{-s-\varepsilon_k}{\delta_k}\mathbf{u}(t,-(\varepsilon_k+\delta_k)) & t\in(a,b),\\
& -(\varepsilon_k+\delta_k) \le s \le -\varepsilon_k,\\
\mathbf{u}_{h_j}(t,s) & t\in(a,b), |s| < \varepsilon_k.
\end{cases}$$

We want to prove that for any k there holds

$$\liminf_{j\to+\infty} \mathcal{A}(\mathbf{v}^k_{h_j}, \mathrm{R}) \le \lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathrm{R}) + C\,[1 + M(\varepsilon_k)]\,\delta_k, \tag{4.20}$$

where C is given by Proposition 4.16. Without loss of generality we can suppose that there exists $\lim_{h\to+\infty}\mathcal{A}(\mathbf{u}_h, \mathrm{R}\setminus\mathrm{R}_{\varepsilon_k+\delta_k})$; using the same notation as that in Proposition 4.16 we get:[3]

$$\begin{aligned}\mathcal{A}(\mathbf{v}^k_{h_j},\mathrm{R}^+) &= \mathcal{A}(\mathbf{u},\mathrm{R}^+\setminus\mathrm{R}^+_{\varepsilon_k+\delta_k}) + \mathcal{A}(\mathbf{v}^k_{h_j}, S^{\varepsilon_k+\delta_k}_{\varepsilon_k}) + \mathcal{A}(\mathbf{u}_{h_j},\mathrm{R}^+_{\varepsilon_k})\\ &\leq \mathcal{A}(\mathbf{u},\mathrm{R}^+\setminus\mathrm{R}^+_{\varepsilon_k+\delta_k}) - \mathcal{A}(\mathbf{u}_{h_j},\mathrm{R}^+\setminus\mathrm{R}^+_{\varepsilon_k+\delta_k}) + \mathcal{A}(\mathbf{v}^k_{h_j}, S^{\varepsilon_k+\delta_k}_{\varepsilon_k}) + \mathcal{A}(\mathbf{u}_{h_j},\mathrm{R}^+).\end{aligned}$$

We perform the same construction in R^-. Passing to the limit, recalling that $\mathcal{A}(\cdot,\mathrm{R}\setminus\mathrm{R}_{\varepsilon_k+\delta_k})$ is lower semicontinuous, and also (4.15), we get (4.20). Finally it is possible to choose a sequence (k_{h_j}) so that defining $\mathbf{v}_{h_j} := \mathbf{v}^{k_{h_j}}_{h_j}$, the sequence $(\mathbf{v}_{h_j})$ satisfies (4.12). □

4.4 Lower bound - condition II

In this section we prove the following lower bound result that coupled with Theorem 4.9 provides Theorem 4.6.

Theorem 4.17 (Lower bound - condition II). *Let Ω and $\mathbf{u}$ satisfy condition II. For every sequence $(\mathbf{u}_h)\subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$ there holds*

$$\lim_{h\to+\infty}\mathcal{A}(\mathbf{u}_h,\Omega) \geq \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds + m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]).$$

Therefore

$$\overline{\mathcal{A}}^{\infty}_s(\mathbf{u},\Omega) \geq m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]).$$

As in Section 4.3, the proof of Theorem 4.17 reduces to the following two Propositions.

Proposition 4.18. *Let Ω and $\mathbf{u}$ satisfy condition II. If $(\mathbf{u}_h)\subset\mathrm{Lip}(\Omega;\mathbb{R}^2)$ converges to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and*

$$\mathbf{u}_h = \mathbf{u} \qquad \text{in } \Omega\setminus N_h,$$

for some decreasing sequence (N_h) of neighbourhoods of $J_{\mathbf{u}}$ such that $\bigcap_h N_h = J_{\mathbf{u}}$, then

$$\liminf_{h\to+\infty}\mathcal{A}(\mathbf{u}_h,\Omega) \geq \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds + m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]).$$

Proposition 4.19. *Let Ω and $\mathbf{u}$ satisfy condition II. Let $(\mathbf{u}_h)\subset\mathrm{Lip}(\Omega;\mathbb{R}^2)$ be a sequence converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$. Then there exists a sequence $(\mathbf{v}_h)\subset\mathrm{Lip}(\Omega;\mathbb{R}^2)$ satisfying the hypotheses of Proposition 4.18 and such that*

$$\liminf_{h\to+\infty}\mathcal{A}(\mathbf{u}_h,\Omega) \geq \liminf_{h\to+\infty}\mathcal{A}(\mathbf{v}_h,\Omega).$$

As for the case of Ω and $\mathbf{u}$ satisfying condition I, in the proof of Proposition 4.18 we shall need the lower semicontinuity result for $m(D;\cdot,\cdot)$, Lemma 3.3, while the proof of Proposition 4.19 is based on Lemma 4.20.

---

[3] $\mathrm{R}^+_c := (a,b)\times(0,c)$, for every $c>0$.

Proof of Proposition 4.18

Proof of Proposition 4.18. Let us suppose that $(\mathbf{u}_h)$ satisfies (4.18). Let (ε_h) and $(\tilde{\varepsilon}_h)$ be two infinitesimal sequences of positive numbers such that

$$N_h \subset\subset \left\{(t,s) : \left(t, \frac{s}{\tilde{\varepsilon}_h}\right) \in O_{\varepsilon_h}\right\} \subset \Omega,$$

where $O_{\varepsilon_h} = [[\sigma^-_{\varepsilon_h}, \sigma^+_{\varepsilon_h}]]$ is defined as in Lemma 3.3.
Let $\gamma^\pm_h \in \mathrm{Lip}((a-\varepsilon_h, b+\varepsilon_h); \mathbb{R}^2)$ be defined as

$$\gamma^\pm_h(t) := \mathbf{u}_h(t, \tilde{\varepsilon}_h \sigma^\pm_{\varepsilon_h}(t)) = \mathbf{u}(t, \tilde{\varepsilon}_h \sigma^\pm_{\varepsilon_h}(t))$$

Following the same computation as in Lemma 4.14 we get

$$\mathcal{A}\left(\mathbf{u}_h, \left\{(t,s) : \left(t, \frac{s}{\tilde{\varepsilon}_h}\right) \in O_{\varepsilon_h}\right\}\right) \geq m(O_{\varepsilon_h}; \Gamma^-_h, \Gamma^+_h),$$

where $\Gamma^\pm_h := \mathrm{graph}(\gamma^\pm_h)$. Due to the regularity assumptions on $\mathbf{u}$, the sequences $(\gamma^\pm_h)$ satisfy hypotheses of Lemma 3.3, and thus we can conclude that

$$\begin{aligned}\liminf_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \Omega) &\geq \int_\Omega |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + \liminf_{h\to+\infty} m(O_{\varepsilon_h}; \Gamma^-_h, \Gamma^+_h) \\ &\geq \int_\Omega |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + m(D; \Gamma^-, \Gamma^+).\end{aligned}$$

□

Proof of Proposition 4.19

In order to prove Proposition 4.19, we need a suitable version of Lemma 4.16.
Let us fix some notation. For any $d > 0$, define

$$J^d_{\mathbf{u}} := \{(t,s) \in \mathbb{R}^2 : \mathrm{dist}((t,s), J_{\mathbf{u}}) < d\}.$$

We parametrize the curve $\{s > 0\} \cap \partial J^d_{\mathbf{u}}$ on the interval $I := (a - \frac{\pi}{2}, b + \frac{\pi}{2})$ by the map β^+_d defined by

$$\beta^+_d(\theta) := \begin{cases} (a + d\sin(\theta - a), d\cos(\theta - a)) & \text{if } \theta \in (a - \pi/2, a), \\ (\theta, d) & \text{if } \theta \in (a,b), \\ (b + d\sin(\theta - b), d\cos(\theta - b)) & \text{if } \theta \in (b, b+\pi/2). \end{cases}$$

Similarly we can define the parametrization β^-_d for $\{s < 0\} \cap \partial J^d_{\mathbf{u}}$. We can now define the coordinates (θ, r) in $\mathbb{R}^2 \setminus \{s = 0\}$ such that

$$\begin{aligned} (t,s) &= \beta^+_r(\theta) \quad \text{if } s > 0, \\ (t,s) &= \beta^-_r(\theta) \quad \text{if } s < 0. \end{aligned}$$

We also set $\Omega^+ := \Omega \cap \{s > 0\}$.

Lemma 4.20 (**Interpolation,** II)**.** *Let* $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega^+;\mathbb{R}^2)$*,* $\mathbf{u} \in \mathrm{Lip}(\Omega^+;\mathbb{R}^2)$*, and suppose that* $\mathbf{u}_h \to \mathbf{u}$ *in* $L^1(\Omega^+;\mathbb{R}^2)$ *as* $h \to +\infty$*. Let* $\varepsilon_o > 0$ *be fixed so that* $\overline{J^{\varepsilon_o}_{\mathbf{u}}} \cap \{s > 0\} \subset \Omega^+$*. For any* $\varepsilon \in (0, \varepsilon_o]$ *we define* $\gamma^\varepsilon_h := \mathbf{u}_h \circ \beta^+_\varepsilon$ *and* $\gamma^\varepsilon := \mathbf{u} \circ \beta^+_\varepsilon$*. Let us suppose that* $\dot\gamma^{\varepsilon_o}$ *exists almost everywhere in* I*, and let* $\varepsilon_i \in (0, \varepsilon_o)$ *be such that:*

(i) $||\gamma^{\varepsilon_i}_h - \gamma^{\varepsilon_i}||_{L^\infty(I;\mathbb{R}^2)} \to 0$ *as* $h \to +\infty$*;*

(ii) $\dot\gamma^{\varepsilon_i}_h$ *exists almost everywhere in* I *for any* $h \in \mathbb{N}$*;*

(iii) $\liminf_{h\to+\infty} ||\dot\gamma^{\varepsilon_i}_h||_{L^1(I;\mathbb{R}^2)} \le M$ *where the constant* M *may depend on* ε_i*.*

Let us define the sequence $(\mathbf{v}_h) \subset \mathrm{Lip}(\Omega^+;\mathbb{R}^2)$ *as* $\mathbf{v}_h := \mathbf{u}$ *on* $\Omega^+ \setminus J^{\varepsilon_o}_{\mathbf{u}}$*,* $\mathbf{v}_h := \mathbf{u}_h$ *in* $\Omega^+ \cap J^{\varepsilon_i}_{\mathbf{u}}$*, and such that its representation in the* (θ, r) *coordinates in the strip* $\Omega^+ \cap (J^{\varepsilon_o}_{\mathbf{u}} \setminus J^{\varepsilon_i}_{\mathbf{u}})$ *is*

$$\tilde{\mathbf{v}}_h(\theta, r) := \frac{\varepsilon_o - r}{\varepsilon_o - \varepsilon_i}\gamma^{\varepsilon_i}_h(\theta) + \frac{r - \varepsilon_i}{\varepsilon_o - \varepsilon_i}\gamma^{\varepsilon_o}(\theta).$$

Then

$$\liminf_{h\to+\infty} \mathcal{A}(\mathbf{v}_h, \Omega^+ \cap (J^{\varepsilon_o}_{\mathbf{u}} \setminus J^{\varepsilon_i}_{\mathbf{u}})) \le C(1 + M)|\varepsilon_o - \varepsilon_i|, \tag{4.21}$$

where $C = C(\mathrm{lip}(\mathbf{u}))$*.*

Proof. Let us denote by $S^{\varepsilon_o}_{\varepsilon_i}$ the set $\Omega^+ \cap (J^{\varepsilon_o}_{\mathbf{u}} \setminus J^{\varepsilon_i}_{\mathbf{u}})$. We observe that the estimate (4.21) for $\mathcal{A}(\mathbf{v}_h, S^{\varepsilon_o}_{\varepsilon_i} \cap \{t \in (a, b)\})$ is provided by Lemma 4.16, since in this set we have $\theta(t, s) = t$ and $r(t, s) = s$.
We prove the estimate only for $\mathcal{A}(\mathbf{v}_h, S^{\varepsilon_o}_{\varepsilon_i} \cap \{t < a\})$, since the computations in $\mathcal{A}(\mathbf{v}_h, S^{\varepsilon_o}_{\varepsilon_i} \cap \{t > b\})$ are similar. We have:

$$\begin{aligned}\mathcal{A}(\mathbf{v}_h, S^{\varepsilon_o}_{\varepsilon_i} \cap \{t < a\}) &= \int_{\varepsilon_i}^{\varepsilon_o} \int_{a-\pi/2}^{a} \sqrt{r^2 + |\partial_\theta \tilde{\mathbf{v}}_h|^2 + r^2|\partial_r \tilde{\mathbf{v}}_h|^2 + (\det\nabla_{\theta,r}\tilde{\mathbf{v}}_h)^2}\, d\theta\, dr \\ &\le C \int_{\varepsilon_i}^{\varepsilon_o} \int_{a-\pi/2}^{a} [r + |\partial_\theta \tilde{\mathbf{v}}_h| + r|\partial_r \tilde{\mathbf{v}}_h| + |\det\nabla_{\theta,r}\tilde{\mathbf{v}}_h|]\, d\theta\, dr\end{aligned}$$

where $\nabla_{\theta,r}$ denotes the Jacobian with respect to the coordinates (θ, r), and C is an absolute constant.
Again we estimate the right hand side as in Proposition 4.16.

- The first term is $\int_{\varepsilon_i}^{\varepsilon_o} \int_{a-\pi/2}^{\pi} r\, dr\, d\theta = \pi/2(\varepsilon_o^2 - \varepsilon_i^2)$.
- Concerning the second term, there holds

$$\begin{aligned}\int_{\varepsilon_i}^{\varepsilon_o} \int_{a-\pi/2}^{a} |\partial_\theta \tilde{\mathbf{v}}_h|\, d\theta\, dr &\le \int_{\varepsilon_i}^{\varepsilon_o} \int_{a-\pi/2}^{a} \left[|\dot\gamma^{\varepsilon_o}| + |\dot\gamma^{\varepsilon_i}_h|\right]\, d\theta\, dr \\ &\le \pi\varepsilon_o \mathrm{lip}(\mathbf{u})(\varepsilon_o - \varepsilon_i) + (\varepsilon_o - \varepsilon_i)M,\end{aligned} \tag{4.22}$$

  where the second inequality follows from the Lipschitzianity of $\mathbf{u}$ and hypothesis (iii).

- Similarly for the third term we have

$$
\begin{aligned}
\int_{\varepsilon_i}^{\varepsilon_o}\int_{a-\pi/2}^{a} r|\partial_r \tilde{\mathbf{v}}_h|\,d\theta\,dr &= \int_{\varepsilon_i}^{\varepsilon_o}\int_{a-\pi/2}^{a} r\frac{|\gamma^{\varepsilon_o}-\gamma_h^{\varepsilon_i}|}{\varepsilon_o-\varepsilon_i}\,d\theta\,dr \\
&\le \int_{\varepsilon_i}^{\varepsilon_o}\int_{a-\pi/2}^{a} r\frac{|\gamma^{\varepsilon_o}-\gamma^{\varepsilon_i}|+|\gamma^{\varepsilon_i}-\gamma_h^{\varepsilon_i}|}{\varepsilon_o-\varepsilon_i}\,d\theta\,dr \\
&\le \frac{\pi}{4}\mathrm{lip}(\mathbf{u})(\varepsilon_o^2-\varepsilon_i^2)+\frac{\varepsilon_o+\varepsilon_i}{2}\int_{a-\pi/2}^{a}|\gamma^{\varepsilon_i}-\gamma_h^{\varepsilon_i}|\,d\theta.
\end{aligned}
\tag{4.23}
$$

- Finally we estimate the term containing the determinant:

$$
\begin{aligned}
\int_{\varepsilon_i}^{\varepsilon_o}\int_{a-\pi/2}^{a} |\det\nabla_{\theta,r}\tilde{\mathbf{v}}_h|\,d\theta\,dr &\le 2\int_{\varepsilon_i}^{\varepsilon_o}\int_{a-\pi/2}^{a}|\partial_r\tilde{\mathbf{v}}_h||\partial_\theta\tilde{\mathbf{v}}_h|\,d\theta\,dr \\
&\le 2\int_{a-\pi/2}^{a}|\gamma^{\varepsilon_o}-\gamma_h^{\varepsilon_i}||\dot\gamma^{\varepsilon_o}+\dot\gamma_h^{\varepsilon_i}|\,d\theta \\
&\le \int_{a-\pi/2}^{a}|\dot\gamma^{\varepsilon_o}||\gamma^{\varepsilon_o}-\gamma_h^{\varepsilon_i}|\,d\theta+\int_{a-\pi/2}^{a}|\dot\gamma_h^{\varepsilon_i}||\gamma^{\varepsilon_o}-\gamma_h^{\varepsilon_i}|\,d\theta \\
&:= \mathrm{I}_h+\mathrm{II}_h,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}_h &\le \varepsilon_o\mathrm{lip}(\mathbf{u})(\mathrm{lip}(\mathbf{u})|\varepsilon_o-\varepsilon_i|+||\gamma^{\varepsilon_i}-\gamma_h^{\varepsilon_i}||_{L^1(I,\mathbb{R}^2)}),\\
\mathrm{II}_h &\le M\left(\mathrm{lip}(\mathbf{u})|\varepsilon_o-\varepsilon_i|+||\gamma^{\varepsilon_i}-\gamma_h^{\varepsilon_i}||_{L^\infty(I;\mathbb{R}^2)}\right).
\end{aligned}
\tag{4.24}
$$

Using *(i)-(ii)*, (4.22), (4.23), and (4.24) and we get

$$
\liminf_{h\to+\infty}\mathcal{A}(\mathbf{v}_h,S_{\varepsilon_i}^{\varepsilon_o}\cap\{t<a\})\le C(1+M)|\varepsilon_o-\varepsilon_i|,
$$

where $C=C(\mathrm{lip}(\mathbf{u}))$. □

We conclude proving Proposition 4.19.

Proof of Proposition 4.19. The proof follows exactly that of Theorem 4.19, where the possibility to choose an infinitesimal sequence (ε_k) of positive numbers satisfying hypotheses *(i)-(iii)* in Lemma 4.20 is guaranteed by the Fatou's Lemma applied to the expression of the area functional in the (θ,r) coordinates. □

5. $\overline{\mathcal{A}}(\mathbf{u},\Omega) < \overline{\mathcal{A}}^{\infty}(\mathbf{u},\Omega)$: examples

Overview of the chapter

In this chapter we exhibit some examples (appeared in [8]) of pairs $(\Omega,\mathbf{u})$ satisfying condition II for which $\overline{\mathcal{A}}(\mathbf{u},\Omega) < \overline{\mathcal{A}}^{\infty}(\mathbf{u},\Omega)$. This is proven by showing that, in certain circumstances, sequences converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$, but not uniformly out of $J_{\mathbf{u}}$, provide an upper bound lower than $m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$. These examples suggest that we can extend in some way the jump, adding to $J_{\mathbf{u}}$ a sort of *virtual* jump, and build sequences converging uniformly to $\mathbf{u}$ out of this extension of the jump. How to choose the virtual jump is a difficult issue. We present different possibilities that confirm the strong non-local behaviour of the functional $\overline{\mathcal{A}}(\mathbf{u},\cdot)$. The results and the examples in Sections 5.1 and 5.2 have been inspired by the construction used in [1] in order to estimate the area of the vortex map, see also Section 1.4.5.

5.1 Virtual jump starting from an end point of $J_{\mathbf{u}}$

We adapt the strategy of [1, Lemma 5.3], see Section 1.4.5, to the case of Ω and $\mathbf{u}$ satisfying condition II. This procedure allows to build a sequence $(\mathbf{u}_h) \subset \mathrm{Lip}(\mathrm{R};\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of a curve strictly containing $J_{\mathbf{u}}$ and having an end point on $\partial\Omega$. In this case the virtual jump is therefore connecting one end point of $J_{\mathbf{u}}$ and $\partial\Omega$. The singular contribution

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) - \int_{\Omega} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds$$

can be interpreted as the area of a suitable semicartesian parametrization, see Remark 5.2, and it is possibly lower than $m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$, see Example 5.4.

Proposition 5.1. *Let* Ω *and* $\mathbf{u}$ *satisfy condition* II*, with the further conditions that* $\Omega\cap\{s=0\} = (a_1, b+\delta)\times\{0\}$ *for some* $a_1 < a$ *and* $\delta > 0$*, and*

$$\mathbf{u} \in \mathcal{C}^1\big(\overline{\Omega\cap\{s>0\}};\mathbb{R}^2\big) \cap \mathcal{C}^1\big(\overline{\Omega\cap\{s<0\}};\mathbb{R}^2\big). \tag{5.1}$$

Then there exists a sequence $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ *converging to* $\mathbf{u}$ *in* $L^1(\Omega;\mathbb{R}^2)$ *and*

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) \le \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds + (b+\delta-a)\left\{\int_a^b \Big[|\dot\gamma^-|+|\dot\gamma^+|\Big]\,dt + 2\int_b^{b+\delta} |\partial_t \mathbf{u}(t,0)|\,dt\right\}. \tag{5.2}$$

Hence

$$\overline{\mathcal{A}}_s(\mathbf{u},\Omega) \le (b+\delta-a)\left\{\int_a^b \Big[|\dot\gamma^-|+|\dot\gamma^+|\Big]\,dt + 2\int_b^{b+\delta} |\partial_t \mathbf{u}(t,0)|\,dt\right\}.$$

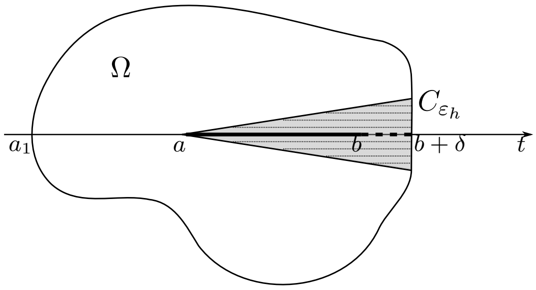

Figure 5.1: The set Ω and, in grey, the triangle C_{ε_h} built in Proposition 5.1. The map $\mathbf{u}_h$ defined in (5.1) is constant on the horizontal segments in C_{ε_h}. The sequence $(\mathbf{u}_h)$ converges to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of the segment $J_{\text{ext}} = (a, b+\delta)\times\{0\}$, union of $J_{\mathbf{u}}$ (the bold segment) and of the *virtual jump* $[b, b+\delta]\times\{0\}$, represented by a bold dotted line.

Proof. Given an infinitesimal sequence of positive numbers (ε_h), define

$$C_{\varepsilon_h} := \{(t,s)\in\Omega:\ t>a,\ |s|<\varepsilon_h(t-a)\},$$

and $r_{\varepsilon_h}:\Omega\to\Omega\setminus C_{\varepsilon_h}$ as

$$r_{\varepsilon_h}(t,s) = \begin{cases} (t,s) & (t,s)\in\Omega\setminus C_{\varepsilon_h}, \\ \left(\frac{s}{\varepsilon_h}+a, s\right) & (t,s)\in C_{\varepsilon_h},\ s\ge 0, \\ \left(-\frac{s}{\varepsilon_h}+a, s\right) & (t,s)\in C_{\varepsilon_h},\ s<0, \end{cases} \tag{5.3}$$

that is the retraction mapping each point $(t,s)\in C_{\varepsilon_h}$ into the point of $\partial C_{\varepsilon_h}\cap\Omega$ having s as second coordinate.
The sequence $(\mathbf{u}_h)\subset\mathrm{Lip}(\Omega;\mathbb{R}^2)$ is thus defined as

$$\mathbf{u}_h(t,s) := \mathbf{u}(r_{\varepsilon_h}(t,s)), \qquad (t,s)\in\Omega.$$

We observe that $(\mathbf{u}_h)$ converges to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of the segment $J_{\text{ext}} := (a, b+\delta)\times\{0\}$, (but not out of $J_{\mathbf{u}}$), that we interpret as an extension of the jump, that is the union of $J_{\mathbf{u}}$ itself and of the virtual jump $(b, b+\delta)\times\{0\}$.

Denoting by $\partial_1\mathbf{u}$ and $\partial_2\mathbf{u}$ the derivatives with respect to the first and second variable of $\mathbf{u}$, let us compute the contribution of area of $\mathbf{u}_h$ on $C_{\varepsilon_h}\cap\{s\geq 0\}$[(1)]

$$\begin{aligned}\mathcal{A}(\mathbf{u}_h, C_{\varepsilon_h}\cap\{s\geq 0\}) &= \int_a^{b+\delta}\int_0^{\varepsilon_h(t-a)}\sqrt{1+\frac{1}{\varepsilon_h^2}\left|\partial_1\mathbf{u}\left(\frac{s}{\varepsilon_h}+a,s\right)\right|^2+\left|\partial_2\mathbf{u}\left(\frac{s}{\varepsilon_h}+a,s\right)\right|^2}\,ds\,dt\\ &= \int_a^{b+\delta}\int_0^{\varepsilon_h(t-a)}\frac{1}{\varepsilon_h}\sqrt{\left|\partial_1\mathbf{u}\left(\frac{s}{\varepsilon_h}+a,s\right)\right|^2+\mathcal{O}(\varepsilon_h^2)}\,ds\,dt\\ &= \int_a^{b+\delta}\int_a^{t}\sqrt{|\partial_1\mathbf{u}(\tau,\varepsilon_h(\tau-a))|^2+\mathcal{O}(\varepsilon_h^2)}\,d\tau\,dt\\ &\leq (b+\delta-a)\int_a^{b+\delta}\sqrt{|\partial_1\mathbf{u}(\tau,\varepsilon_h(\tau-a))|^2+\mathcal{O}(\varepsilon_h^2)}\,d\tau.\end{aligned}$$

Similarly

$$\mathcal{A}(\mathbf{u}_h, C_{\varepsilon_h}\cap\{s<0\}) \leq (b+\delta-a)\int_a^{b+\delta}\sqrt{|\partial_1\mathbf{u}(\tau,-\varepsilon_h(\tau-a))|^2+\mathcal{O}(\varepsilon_h^2)}\,d\tau.$$

Thus, noticing that $|\partial_1\mathbf{u}(\tau,\pm\varepsilon_h(\tau-a))|$ are uniformly bounded, $\partial_1\mathbf{u}(\tau,\pm\varepsilon_h(\tau-a))$ converges to $\dot\gamma^{\pm}[\mathbf{u}](\tau)$ pointwise in (a,b), and $\partial_1\mathbf{u}(\tau,\pm\varepsilon_h(\tau-a)) = \partial_1\mathbf{u}(\tau,0)$ in $(b,b+\delta)$, possibly passing to a subsequence, we get (5.2) $\square$

Remark 5.2. The term $(b+\delta-a)\left\{\int_a^b\left[|\dot\gamma^-|+|\dot\gamma^+|\right]dt+2\int_b^{b+\delta}|\partial_t\mathbf{u}(t,0)|\,dt\right\}$ in the right hand side of (5.2) can be interpreted as the area of the semicartesian parametrization built as in Example 2.14, with $b+\delta$ in place of b and with $\mathrm{C} := \gamma([0,2(b+\delta-a)])$, where $\gamma:[0,2(b+\delta-a)]\to\mathbb{R}^2$ is defined as follows:

$$\gamma(t) := \begin{cases}\mathbf{u}(b+\delta-t,0) & t\in(0,\delta),\\ \gamma^+(b+\delta-t) & t\in(\delta,b+\delta-a),\\ \gamma^-(t+2a-b-\delta) & t\in(b+\delta-a,2(b-a)+\delta),\\ \mathbf{u}(t+2a-b-\delta,0) & t\in(2(b-a)+\delta,2(b-a+\delta)).\end{cases}$$

We notice also that this construction can be done even if Ω and $\mathbf{u}$ satisfy condition I and $\gamma^-(a)=\gamma^+(a)$ (or, symmetrically, if $\gamma^-(b)=\gamma^+(b)$). In this case, $J_{\mathrm{ext}} = J_{\mathbf{u}}$ and therefore the sequence $(\mathbf{u}_h)$ built in Proposition 5.1 would converge to $\mathbf{u}$ in $L^1(\mathrm{R};\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$, and $\lim_{h\to+\infty}\mathcal{A}(\mathbf{u}_h,\mathrm{R})-\int_{\mathrm{R}}|\mathcal{M}(\nabla\mathbf{u})|\,dt\,ds \geq \overline{\mathcal{A}}_s^\infty(\mathbf{u},\mathrm{R}) = m(\mathrm{R};\Gamma^-,\Gamma^+)$.

Remark 5.3. Even if Proposition 5.1 has been inspired by the construction in [1, Lemma 5.3] described in Section 1.4.5, it is worth to underline some important differences. When $\Omega = B_R$ and $\mathbf{u}=\mathbf{u}_V$, we could interpret the origin as a "collapsed" jump, and the radius $\{(t,0): t\in(-R,0)\}$ as the virtual jump. Differently from the case where Ω and $\mathbf{u}$ satisfy condition II, the vortex map (which belongs

(1) The computation is done supposing that the triangle $\{(t,s): t\in(a,b+\delta), |t|\leq\varepsilon_h(t-a)\}$ is contained in Ω, as depicted in Figure 5.1, but it can be easily arranged to a more general situation.

to $W^{1,p}(B_R;\mathbb{S}^1)$ for any $p \in [1,2)$) does not admit any limit as $(t,s) \to (0,0)$, $(t,s) \in B_R \setminus \{(t,0)\ t \in (-R,0)\}$. Intuitively if we would interpret the bound $\overline{\mathcal{A}}_s(\mathbf{u}_V, B_R) \le 2\pi R$ in terms of the area of semicartesian parametrizations, this lack of continuity in $(0,0)$ of $\mathbf{u}_V$ would force us to consider only surfaces having as trace on the plane $\{t=0\} \times \mathbb{R}^2_{(\xi,\eta)}$ the unit circumference, covered with the right orientation. Moreover, we notice that also in [1, Lemma 5.3] the sequence $(\mathbf{u}_h)$ is defined as the composition of $\mathbf{u}_V$ and a suitable retraction from the whole B_R into $B_R \setminus C_{\varepsilon_h}$, where (C_{ε_h}) is a decreasing family of circular sectors containing $\{(t,0) : t \in (-R,0)\}$ and converging to it. In [1] the image of C_{ε_h} through this retraction covers the whole $B_R \setminus C_{\varepsilon_h}$, while in our case it is contained in $\partial C_{\varepsilon_h}$, see (5.3); this difference is due to the fact that, contrary to our case, the trace of the vortex map on $\partial C_{\varepsilon_h}$ is not continuous, due to the discontinuity in the origin.

In the following example we exhibit a map $\mathbf{u}$ for which the sequence built in Proposition 5.1 provides an upper bound that is lower than $m(D;\Gamma^-,\Gamma^+)$. This example shows that possibly $\overline{\mathcal{A}}(\mathbf{u},\Omega) < \overline{\mathcal{A}}^{\infty}(\mathbf{u},\Omega)$.

Example 5.4. Let Ω be as in Proposition 5.1, and define(2)

$$\mathbf{u}(t,s) := \begin{cases} (1,0) & \text{in } \{(t,s)\in\Omega : s \ge 0\}, \\ (1,0) & \text{in } \{(t,s)\in\Omega : s<0, t<a \text{ or } t>b\}, \\ \rho\Big(\cos\big(\theta(t)\big), \sin\big(\theta(t)\big)\Big) + (1-\rho,0) & \text{in } \{(t,s)\in\Omega : s<0, a\le t\le b\}, \end{cases}$$

where

$$\rho > 2(b+\delta-a), \tag{5.4}$$

and $\theta : [a,b] \to [0,2\pi]$ is defined in (2.13). We observe that γ^+ is constant, γ^- covers once the circumference centered at $(1-\rho,0)$ and with radius ρ, and $\partial_t \mathbf{u}(t,0) = 0$ for $t \in (b,b+\delta)$. Formula (5.2) provides in this case

$$\overline{\mathcal{A}}_s(\mathbf{u},\Omega) \le (b+\delta-a)\int_a^b |\dot{\gamma}^-|\,dt = (b+\delta-a)2\pi\rho. \tag{5.5}$$

On the other hand we have already observed in Example 2.15 that $a(\Gamma) \ge \pi\rho^2$. Thus, from (5.4) we obtain that $a(\Gamma)$ is strictly greater than the right hand side of (5.5). Since in general $m(D;\Gamma^-,\Gamma^+) \ge a(\Gamma)$, see (2.10), we have

$$\overline{\mathcal{A}}_s(\mathbf{u},\Omega) < m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}]) = \overline{\mathcal{A}}_s^{\infty}(\mathbf{u},\Omega).$$

Remark 5.5 (Joining two components of $J_{\mathbf{u}}$). Example 5.4 suggests also that, if the jump set of a discontinuous map $\mathbf{u} : \widetilde{\Omega} \to \mathbb{R}^2$ is not connected, it could be convenient (as far as only the L^1-convergence is involved) considering sequences of regular maps $(\mathbf{u}_h)$ that converge to $\mathbf{u}$ in $L^1(\widetilde{\Omega};\mathbb{R}^2)$ and uniformly out of a connected curve containing the jump. Let Ω be as in Example 5.4 and let $\widetilde{\Omega}$ be the union of Ω

(2) The map $\mathbf{u}$ defined in this way does not satisfy (5.1); anyway the fact that it does not depend on s in $\Omega \cap \{s>0\}$ and in $\Omega \cap \{s<0\}$ allows to obtain (5.2).

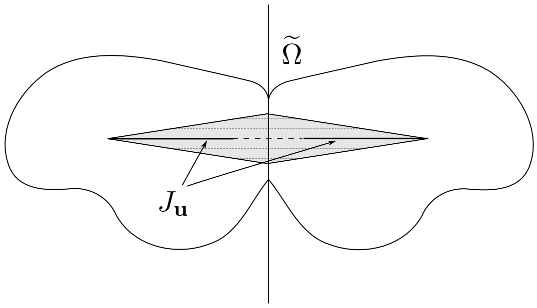

Figure 5.2: Remark 5.5. The domain $\widetilde{\Omega}$ is built by reflecting with respect to the axis in the figure the domain Ω considered in Proposition 5.1. The map $\mathbf{u} : \widetilde{\Omega} \to \mathbb{R}^2$ is defined again by reflecting the map in Example 5.4, so that $J_{\mathbf{u}}$ has two connected components. It is then possible to build a sequence of regular maps $(\mathbf{u}_h)$, again by reflection, converging to $\mathbf{u}$ in $L^1(\widetilde{\Omega}; \mathbb{R}^2)$ and uniformly out of the curve composed of $J_{\mathbf{u}}$ and of the bold dotted segment joining the two components of the jump. This sequence provides an upper bound for $\overline{\mathcal{A}}(\mathbf{u}, \widetilde{\Omega})$ lower than the one obtained by any sequence converging to $\mathbf{u}$ uniformly out of $J_{\mathbf{u}}$.

and of its symmetrized with respect to the axis $\{t = c\}$, for some $c \in (b, b + \delta)$, see Figure 5.2. Let us define $\mathbf{u}$ and $(\mathbf{u}_h)$ in $\widetilde{\Omega}$ as in Example 5.4 for $(t, s) \in \Omega$, and by reflection elsewhere in $\widetilde{\Omega}$. Then

$$\lim_{h \to +\infty} \mathcal{A}(\mathbf{u}_h, \widetilde{\Omega}) - \int_{\widetilde{\Omega}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds \leq 4\rho\pi(c - a). \tag{5.6}$$

If $\rho > 2(c - a)$, the right hand side of (5.6) is strictly less than $2m(D; \Gamma^-, \Gamma^+)$ (where $\Gamma^\pm$ are the graph of the traces of $\mathbf{u}$ on $(a, b) \times \{0\}$), that would be the bound obtained reasoning as in Theorem 4.9 in distinct neighbourhoods of the two connected components of $J_{\mathbf{u}}$. Moreover, if $J_{\mathbf{u}}$ is far from $\partial\widetilde{\Omega}$, the right hand side of (5.6) is also smaller than the upper bound obtained by connecting each component of $J_{\mathbf{u}}$ with $\partial\widetilde{\Omega}$, using the construction in Proposition 5.1.

5.2 Virtual jump starting from an interior point of $J_{\mathbf{u}}$

In this section we build, for a particular pair $(\Omega, \mathbf{u})$ satisfying condition II, a sequence of maps $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega; \mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega; \mathbb{R}^2)$ and uniformly out of J_{ext}, union of $J_{\mathbf{u}}$ and a virtual jump connecting an interior point of $J_{\mathbf{u}}$ to $\partial\Omega$; again, this sequence provides an upper bound to $\overline{\mathcal{A}}_s(\mathbf{u}, \Omega)$ lower than $m(D; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}])$.

Let the bounded connected open domain Ω be such that $\{(\bar{t}, s),\ s > 0\} \cap \Omega = \{\bar{t}\} \times (0, \delta)$ for some $\bar{t} \in (a, b)$. Without loss of generality we can suppose $\bar{t} = 0$, $a = -b$, and $\Omega := (-L, L) \times (-1, \delta)$, for $L > b > 0$, $\delta \in (0, 1)$, see Figure 5.3. For $\rho > 2(\delta + b)$ we define the map $\mathbf{u}$ on Ω as

$$\mathbf{u}(t, s) := \begin{cases} \rho \left(\cos\left(\frac{\pi}{b} t + \pi\right), \sin\left(\frac{\pi}{b} t + \pi\right) \right) + (1 - \rho, 0) & |t| \leq b,\ s < 0, \\ (1, 0) & \text{otherwise} ; \end{cases}$$

we observe that it is essentially the same map of Example 5.4.

We want to build a sequence $(\mathbf{u}_h) \subset \mathrm{Lip}(\Omega;\mathbb{R}^2)$ of maps converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of

$$J_{\text{ext}} := J_{\mathbf{u}} \cup (\{0\} \times (0,\delta)),$$

where $J_{\text{ext}} \setminus J_{\mathbf{u}} = \{0\} \times (0,\delta)$ takes the role of the virtual jump, and such that

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})|\, dt\, ds + 2\pi\rho(b+\delta). \tag{5.7}$$

For any $\varepsilon \in (0, \min\{\delta/2, b/2\})$, set (see Figure 5.3)

$$\mathbb{T}_\varepsilon := \{(t,s) \in \Omega : \ |t| < b, s > 0, \ \mathrm{dist}((t,s), J_{\text{ext}}) < \varepsilon\},$$

$$\mathbb{T}_\varepsilon^- := \mathbb{T}_\varepsilon \cap \{t < 0\} \qquad \text{and} \qquad \mathbb{T}_\varepsilon^+ := \mathbb{T}_\varepsilon \cap \{t > 0\}.$$

The definitions will be given on $\mathbb{T}_\varepsilon^+$ and next extended on the whole $\mathbb{T}_\varepsilon$ by reflection with respect to the s-axis.
Let us parametrize the closure of the curve $\Omega \cap \{t > 0\} \cap \partial\mathbb{T}_\varepsilon$ by the arc-length parametrization $\lambda_\varepsilon \in \mathrm{Lip}([0,\delta+2b-\varepsilon];\mathbb{R}^2)$ such that $\lambda_\varepsilon(0) = (\varepsilon,\delta)$ and $\lambda_\varepsilon(\delta+2b-\varepsilon) = (0,0)$. Let us also parametrize the closure of the set $J_{\text{ext}}^+ := \{0\} \times [0,\delta) \cup [0,b) \times \{0\}$ by the arc-length parametrization $\tilde\alpha$ defined on $[0,\delta+b]$, such that $\tilde\alpha(0) = (0,\delta)$ and $\tilde\alpha(b+\delta) = (b,0)$.

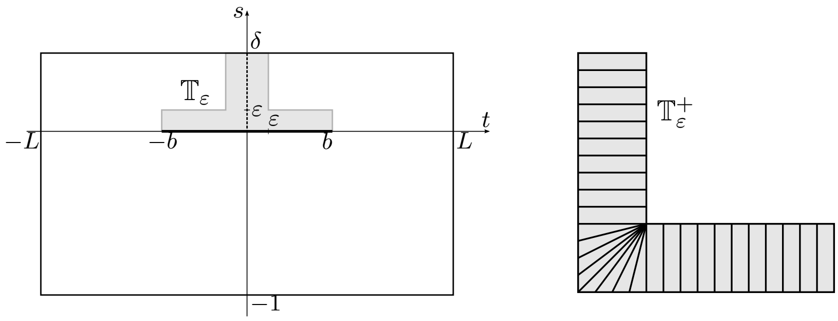

Figure 5.3: In the left picture the domain Ω and, in bold, the jump set $(-b,b) \times \{0\}$ of $\mathbf{u}$. The virtual jump $\{0\} \times (0,\delta)$ is represented by a dotted line, while the set $\mathbb{T}_\varepsilon$ is in grey. In the right picture we show the details of $\mathbb{T}_\varepsilon^+$ and of the retraction r_ε. For each of the point of $\mathbb{T}_\varepsilon^+$ we define the coordinates (p,d); points belonging to the same segment have the same p-coordinate, the d-coordinate being the distance from the point of the segment belonging to $J_{\text{ext}} := J_{\mathbf{u}} \cap (\{0\} \times (0,\delta))$.

On $\mathbb{T}_\varepsilon^+$ we consider the coordinates $(p,d) : \mathbb{T}_\varepsilon^+ \to S_\varepsilon^+$ defined as[3]:

- $p(t,s) \in (0,\delta+b)$ is the image through $\tilde\alpha^{-1}$ of the end point on J_{ext}^+ of the segment represented in Figure 5.3 passing through (t,s);

- $d(t,s)$ is the distance between (t,s) and $\tilde\alpha(p(t,s))$.

[3] Even if we do not write it explicitly the coordinates (p,d) depend on ε.

The set S_ε^+ is therefore $\{(p,d)\in\mathbb{R}^2 : p\in(0,\delta+b), d\in(0,d_{\max}(p))\}$ where $d_{\max}$ depends on ε and is given by

$$d_{\max}(p)=\begin{cases}\varepsilon & p\in(0,\delta-\varepsilon],\\ \sqrt{\varepsilon^2+(p-\delta+\varepsilon)^2} & p\in(\delta-\varepsilon,\delta],\\ \sqrt{\varepsilon^2+(-p+\delta+\varepsilon)^2} & p\in(\delta,\delta+\varepsilon],\\ \varepsilon & p\in(\delta+\varepsilon,\delta+b).\end{cases}$$

We define now a map $\ell_\varepsilon : S_\varepsilon^+ \to (0,\delta+2b-\varepsilon)$, such that ℓ_ε is linear on each segment $\{p\}\times(0,d_{\max}(p))$ and:

- for $p\in(0,\delta-\varepsilon]$: $\ell_\varepsilon(p,0)=\delta+2b-\varepsilon$, $\ell_\varepsilon(p,d_{\max}(p))=p$;
- for $p\in(\delta-\varepsilon,\delta]$: $\ell_\varepsilon(p,0)=\delta+2b-\varepsilon$, $\ell_\varepsilon(p,d_{\max}(p))=\delta-\varepsilon$;
- for $p\in(\delta,\delta+\varepsilon]$: $\ell_\varepsilon(p,0)=2\delta+2b-\varepsilon-p$, $\ell_\varepsilon(p,d_{\max}(p))=\delta-\varepsilon$;
- for $p\in(\delta+\varepsilon,\delta+b)$: $\ell_\varepsilon(p,0)=2\delta+2b-\varepsilon-p$, $\ell_\varepsilon(p,d_{\max}(p))=p-2\varepsilon$.

Thus we can define the retraction $r_\varepsilon : \Omega\to\Omega\setminus\mathbb{T}_\varepsilon$ as:

- $r_\varepsilon(t,s)=(t,s)$ if $(t,s)\in\Omega\setminus\mathbb{T}_\varepsilon$;
- $r_\varepsilon(t,s)=\lambda_\varepsilon(\ell_\varepsilon(p(t,s),d(t,s)))$, if $(t,s)\in\mathbb{T}_\varepsilon^+$;
- $r_\varepsilon(t,s)=(0,0)$, if $t=0$ and $s\in[0,\delta)$;
- $r_\varepsilon(t,s)=(-r_{\varepsilon,1}(-t,s),r_{\varepsilon,2}(-t,s))$, if $(t,s)\in\mathbb{T}_\varepsilon^-$, where $r_{\varepsilon,1}$ and $r_{\varepsilon,2}$ denote the first and the second component of r_ε respectively.

In words, on $\mathbb{T}_\varepsilon^+$ the map r_ε sends each segment in Figure 5.3 into $\partial\mathbb{T}_\varepsilon\cap\{t>0\}$ in such a way that:

- if both the end points of the segment lie on $\partial\mathbb{T}_\varepsilon\cap\{t>0\}$, the image of the segment is the portion of $\partial\mathbb{T}_\varepsilon\cap\{t>0\}$ bounded by the two end points;
- if only one of the end points lies on $\partial\mathbb{T}_\varepsilon\cap\{t>0\}$, the image of the segment is the portion of $\partial\mathbb{T}_\varepsilon\cap\{t>0\}$ bounded by that end point and $(0,0)$.

Let $A_\varepsilon\subset\Omega$ be the set sent by r_ε in $J_{\mathbf{u}}$. We observe that the image of $A_\varepsilon\cap\mathbb{T}_\varepsilon^+$ through the coordinates (p,d) is the subset of S_ε^+ given by

$$\{(p,d) : p\in(0,\delta+b),\, d\in(0,d_{J_{\mathbf{u}}}(p)]\},$$

where

$$d_{J_{\mathbf{u}}}(p):=\begin{cases}\frac{b}{\delta+2b-\varepsilon-p}\varepsilon & p\in(0,\delta-\varepsilon],\\ \frac{d_{\max}}{2} & p\in(\delta-\varepsilon,0],\\ \frac{\delta+b-p}{\delta+2b-p}d_{\max} & p\in(0,\delta+\varepsilon],\\ \frac{\delta+b-p}{2\delta+2b-2p+\varepsilon}\varepsilon & p\in(\delta+\varepsilon,\delta+b).\end{cases}$$

We are now in position to define $(\mathbf{u}_h)$ as

$$\mathbf{u}_h(t,s) := \begin{cases} \mathbf{u}(r_{\varepsilon_h}(t,s)) & (t,s) \in \Omega \setminus A_{\varepsilon_h}, \\ \gamma^-[\mathbf{u}](r_{\varepsilon_h,1}(t,s)) & (t,s) \in A_{\varepsilon_h}. \end{cases}$$

where (ε_h) is an infinitesimal sequence of numbers, $0 < \varepsilon_h < \min\{\delta/2, b/2\}$ for every $h \in \mathbb{N}$. We observe that since $\partial_2 \mathbf{u} = 0$ almost everywhere[(4)], we have

$$\mathcal{A}(\mathbf{u}_h, \Omega) := \int_\Omega \sqrt{1 + |\partial_1 \mathbf{u}(r_{\varepsilon_h})|^2 \left[(\partial_t r_{\varepsilon_h})^2 + (\partial_s r_{\varepsilon_h})^2\right]}\, dt\, ds.$$

We observe also that $\mathbb{T}_{\varepsilon_h} \setminus A_{\varepsilon_h}$ is sent by r_{ε_h} into $\partial\mathbb{T}_{\varepsilon_h} \setminus J_{\mathbf{u}}$, where also $\partial_1 \mathbf{u} = 0$. Let us compute the area of $\mathbf{u}_h$ on $\mathbb{T}^+_{\varepsilon_h} \cap \{s > \varepsilon_h\}$, on $\mathbb{T}^+_{\varepsilon_h} \cap \{t > \varepsilon_h\}$, and on $(0, \varepsilon_h) \times (0, \varepsilon_h)$ separately.

On $\mathbb{T}^+_{\varepsilon_h} \cap \{s > \varepsilon_h\}$ we have $p(t,s) = \delta - s$ and $d(t,s) = t$. Thus

$$A_{\varepsilon_h} \cap (\mathbb{T}^+_{\varepsilon_h} \cap \{s > \varepsilon_h\}) = \left\{(t,s) \in \mathbb{T}^+_{\varepsilon_h} :\ s \in (\varepsilon, \delta),\, t \in \left(0, \frac{b\varepsilon_h}{2b - \varepsilon_h + s}\right)\right\}.$$

and for (t,s) in this set $\mathbf{u}_h(t,s) = \gamma^-[\mathbf{u}]\left(\frac{2b - \varepsilon_h + s}{\varepsilon_h} t\right)$.
Therefore

$$\begin{aligned} &\mathcal{A}(\mathbf{u}_h, \mathbb{T}^+_{\varepsilon_h} \cap \{s > \varepsilon_h\}) \\ &= \mathcal{A}(\mathbf{u}_h, (\mathbb{T}^+_{\varepsilon_h} \setminus A_{\varepsilon_h}) \cap \{s > \varepsilon_h\}) + \mathcal{A}(\mathbf{u}_h, A_{\varepsilon_h} \cap \mathbb{T}^+_{\varepsilon_h} \cap \{s > \varepsilon_h\}) \\ &= \int_{\varepsilon_h}^{\delta} \int_{\frac{b\varepsilon_h}{2b+s-\varepsilon_h}}^{\varepsilon_h} 1\, dt\, ds \\ &\quad + \int_{\varepsilon_h}^{\delta} \int_0^{\frac{b\varepsilon_h}{2b+s-\varepsilon_h}} \sqrt{1 + \left|\dot\gamma^-\left(\frac{2b+s-\varepsilon_h}{\varepsilon_h} t\right)\right|^2 \left(\frac{(2b+s-\varepsilon_h)^2}{{\varepsilon_h}^2} + \frac{t^2}{{\varepsilon_h}^2}\right)}\, dt\, ds \\ &= \varepsilon_h \int_{\varepsilon_h}^{\delta} \left(\frac{b + s - \varepsilon_h}{sb + s - \varepsilon_h}\right) ds + \int_{\varepsilon_h}^{\delta} \int_0^b \sqrt{\mathcal{O}({\varepsilon_h}^2) + |\dot\gamma^-(\tau)|^2 (1 + \mathcal{O}({\varepsilon_h}^2))}\, d\tau\, ds \\ &\xrightarrow{h \to +\infty} \delta \int_0^a |\dot\gamma^-(\tau)|\, d\tau = \pi\rho\delta. \end{aligned}$$

On $\mathbb{T}^+_{\varepsilon_h} \cap \{t > \varepsilon_h\}$ we have $p(t,s) = \delta + t$ and $d(t,s) = s$. Thus

$$A_{\varepsilon_h} \cap (\mathbb{T}^+_{\varepsilon_h} \cap \{t > \varepsilon_h\}) = \left\{(t,s) \in \mathbb{T}^+_{\varepsilon} :\ t \in (\varepsilon_h, b),\, s \in \left(0, \frac{(b-t)\varepsilon_h}{2b - 2t + \varepsilon_h}\right)\right\};$$

(4) Again ∂_1 and ∂_2 denote the derivative with respect to the first and the second variable respectively.

for (t,s) in this set $\mathbf{u}_h(t,s) = \gamma^-[\mathbf{u}]\left(\frac{2b-2t+\varepsilon_h}{\varepsilon_h}s + t\right)$. Hence

$$
\begin{aligned}
&\mathcal{A}(\mathbf{u}_h, \mathbb{T}^+_{\varepsilon_h} \cap \{t > \varepsilon_h\}) \\
&= \mathcal{A}(\mathbf{u}_h, (\mathbb{T}^+_{\varepsilon_h} \setminus A_{\varepsilon_h}) \cap \{t > \varepsilon_h\}) + \mathcal{A}(\mathbf{u}_h, A_{\varepsilon_h} \cap \mathbb{T}^+_{\varepsilon_h} \cap \{t > \varepsilon_h\}) \\
&= \int_{\varepsilon_h}^{b} \int_{\frac{(b-t)\varepsilon_h}{2b-2t+\varepsilon_h}}^{\varepsilon_h} 1 \, dt \, ds \\
&+ \int_{\varepsilon_h}^{b} \int_0^{\frac{(b-t)\varepsilon_h}{2b-2t+\varepsilon_h}} \sqrt{1 + \left|\dot\gamma^-\left(t + \frac{2b-2t+\varepsilon_h}{\varepsilon_h}s\right)\right|^2 \left(\left(1 - \frac{2s}{\varepsilon_h}\right)^2 + \frac{(2b-2t+\varepsilon_h)^2}{{\varepsilon_h}^2}\right)} \, ds \, dt \\
&= \varepsilon_h \int_{\varepsilon_h}^{b} \left(\frac{b-t+\varepsilon_h}{2b-2t+\varepsilon_h}\right) dt + \int_{\varepsilon_h}^{b} \int_0^{b} \sqrt{\mathcal{O}({\varepsilon_h}^2) + |\dot\gamma^-(\tau)|^2(1 + \mathcal{O}({\varepsilon_h}^2))} \, d\tau \, dt \\
&\overset{h \to +\infty}{\longrightarrow} b \int_0^b |\dot\gamma^-(\tau)| \, d\tau = \pi\rho b.
\end{aligned}
$$

In order to estimate the area contribution on $(0,\varepsilon_h) \times (0,\varepsilon_h)$, is it enough to notice that $|\nabla r_{\varepsilon_h,1}| = \mathcal{O}(\varepsilon_h^{-1})$. And thus

$$
\begin{aligned}
\mathcal{A}(\mathbf{u}_h, (0,\varepsilon_h) \times (0,\varepsilon_h)) &= \int_0^{\varepsilon_h} \int_0^{\varepsilon_h} \sqrt{1 + |\partial_1 \mathbf{u}(r_{\varepsilon_h})|^2 \left((\partial_t r_{\varepsilon_h})^2 + (\partial_s r_{\varepsilon_h})^2\right)} \, dt \, ds \\
&\leq \varepsilon_h^{-1} \int_0^{\varepsilon_h} \int_0^{\varepsilon_h} \sqrt{\mathcal{O}(\varepsilon_h^2) + \mathcal{O}(1)} \overset{h \to +\infty}{\longrightarrow} 0.
\end{aligned}
$$

Since by symmetry $\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathbb{T}^+_{\varepsilon_h}) = \lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \mathbb{T}^-_{\varepsilon_h})$, we get (5.7). This implies that $\overline{\mathcal{A}}_s(\mathbf{u},\Omega) \leq 2\pi\rho(\delta + b)$. Due to our choice of ρ, we can conclude, as in Example 5.4, that $\overline{\mathcal{A}}_s(\mathbf{u},\Omega) < \overline{\mathcal{A}}_s^\infty(\mathbf{u},\Omega) = m(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$.

6. Non-subadditivity of $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$

Overview of the chapter

In [14] De Giorgi conjectured that the relaxed area is, in general, non-subadditive, namely there exist maps $\mathbf{v} \in L^1_{\rm loc}(\mathbb{R}^2; \mathbb{R}^2)$ such that $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ is not a measure on $\mathbb{R}^2$. Acerbi and Dal Maso proved the conjecture for $\mathbf{v} = \mathbf{u}_T$, *i.e.* the triple point map, and for $\mathbf{v} = \mathbf{u}_V$, *i.e.* the vortex map, see [1, Theorems 4,1 and 5.1] and Sections 1.4.4 and 1.4.5. We observe that in both these cases the non-subadditivity seems to descend from the behaviour of $\mathbf{v}$ in a neighbourhood of a single point, the origin. As observed in [6], the question arises as to whether the relaxed area functional $\overline{\mathcal{A}}(\mathbf{v}, \cdot)$ is non-subadditive also for qualitatively different maps $\mathbf{v}$. In this chapter we report a result contained in [8], affirmatively answering to this question. Indeed we prove that there exists a class of maps $\mathbf{u}$ defined on R, satisfying condition I, such that $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ is non-subadditive. We also observe that this class is really the simplest generalization of piecewise constant maps assuming two values, since the second component of $\mathbf{u}$ is piecewise constant, and the first component is regular; moreover, the norm of the difference between the two traces remains constant on $J_{\mathbf{u}}$.

Let us suppose that Ω and $\mathbf{u}$ satisfy condition I, and take

$$\mathbf{u}(t,s) := \begin{cases} (f(t), 0) & \text{if } (t,s) \in \mathrm{R}^-, \\ (f(t), 1) & \text{if } (t,s) \in \mathrm{R}^+, \end{cases} \tag{6.1}$$

where $f \in \mathrm{Lip}([a,b])$ is a piecewise $\mathcal{C}^1$-function with $f(a) = 0$. Clearly $\gamma^-(t) = (f(t), 0)$ and $\gamma^+(t) = (f(t), 1)$.
The aim of this chapter is to prove the following result.

Theorem 6.1 (Non-subadditivity of $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$). *Let* $\mathbf{u}$ *be as in* (6.1) *for a non-constant function* f. *Then* $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ *is not subadditive.*

Remark 6.2. It is worth to recall that in [1] it is proven that if f is constant (and hence $\mathbf{u}$ is a piecewise constant map without any triple point) then $\overline{\mathcal{A}}_s(\mathbf{u}, \cdot) = |D^s\mathbf{u}|(\cdot)$, and thus $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ is subadditive, see Theorem 1.11.

Note that in order to prove Theorem 6.1 we do not use any result from Chapters 4 and 5 except for Theorem 4.7.

6.1 Proof of Theorem 6.1

Before proving Theorem 6.1, we need some intermediate results: Proposition 6.3, that provides an estimate from above of $\overline{\mathcal{A}}_s(\mathbf{u}, \mathrm{R})$, in terms of the area of a suitable semicartesian parametrization (see Figure 6.1); Proposition 6.4, where we show that, if $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ were subadditive, than $\overline{\mathcal{A}}_s(\mathbf{u}, \cdot)$ would be forced to coincide with $|D^s\mathbf{u}|(\cdot)$ (see (6.3)); Proposition 6.6 that provides a characterization of suitable "vertical" bidimensional currents in $\mathbb{R}^4$, whose mass is controlled from above. We also stress that from Proposition 6.4 it follows that there exist *coplanar* curves $\Gamma^\pm$ such that the image of an area minimizing semicartesian parametrized surface (if it exists) is *not* planar, see Remark 6.5: this is a consequence of the fact that the area of the rectangle E having Γ^- and Γ^+ as edges can be larger than the sum of the area of its orthogonal projection on the $t\eta$-plane and the areas of the two triangles $\mathcal{T}_0$ and $\mathcal{T}_1$ (see inequality (6.4) and Figure 6.1).

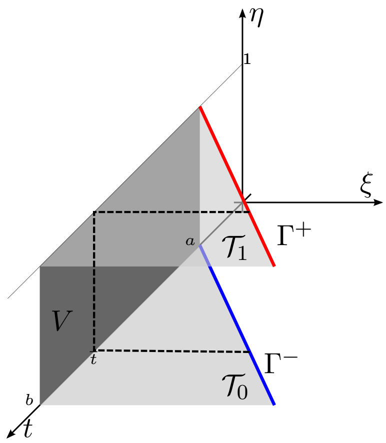

Figure 6.1: The curves $\Gamma^- = \{(t, f(t), 0), : t \in [a, b]\}$, $\Gamma^+ = \{(t, f(t), 1), : t \in [a, b]\}$ for a linear non-constant function f and the sets V, $\mathcal{T}_0$, and $\mathcal{T}_1$. In this case $\mathcal{T}_0$ and $\mathcal{T}_1$ are triangles at different height. The broken dotted curve is the image throught the map Φ of the vertical segment $\mathrm{R} \cap \{t = \bar{t}\}$.

Proposition 6.3. *Let* $\mathbf{u}$ *be as in* (6.1)*. Then*

$$\overline{\mathcal{A}}(\mathbf{u}, \mathrm{R}) \leq \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + |D^s\mathbf{u}|(\mathrm{R}) + \mathrm{lip}(f)(b-a)^2,$$

where $\mathrm{lip}(f)$ *is the Lipschitz constant of* f*.*

Proof. Since $|\mathbf{u}^+(t, 0) - \mathbf{u}^-(t, 0)| = 1$ for any $t \in (a, b)$, we have

$$|D^s\mathbf{u}|(\mathrm{R}) = b - a = \mathcal{H}^2(V),$$

where V is the rectangle $V := [a, b] \times \{0\} \times [0, 1] \subset \mathbb{R}^3_{(t,\xi,\eta)}$. Let

$$\mathcal{T}_0 := \{(t, \xi, 0) : \; t \in (a, b), \; \xi \in (0, f(t)) \text{ if } 0 \leq f(t), \; \xi \in (f(t), 0) \text{ otherwise}\}$$

and $\mathcal{T}_1 := \mathcal{T}_0 + (0,0,1)$. We observe that, since $f(a) = 0$,

$$\mathcal{H}^2(\mathcal{T}_0) = \mathcal{H}^2(\mathcal{T}_1) \leq \frac{\mathrm{lip}(f)}{2}(b-a)^2,$$

and hence, if

$$\Sigma := \mathcal{T}_0 \cup V \cup \mathcal{T}_1, \tag{6.2}$$

we have

$$\mathcal{H}^2(\Sigma) \leq |D^s\mathbf{u}|(\mathrm{R}) + \mathrm{lip}(f)(b-a)^2$$

Recalling Theorem 4.7, the result follows if we prove that Σ can be parametrized by an injective map $\Phi \in \mathrm{semicart}(\mathrm{R};\Gamma^-,\Gamma^+)$. This is true, by considering for example the map $\Phi(t,s) := (t,\phi(t,s))$, with $\phi \in H^1(\mathrm{R};\mathbb{R}^2)$ defined by

$$\phi(t,s) := \begin{cases} -\frac{s+1/3}{2/3}\gamma^-(t) & \text{in } \mathrm{R}\cap\{-1 < s < -1/3\}, \\ \frac{s+1/3}{2/3}(0,1) & \text{in } \mathrm{R}\cap\{-1/3 \leq s < 1/3\}, \\ \frac{1-s}{2/3}(0,1) + \frac{s-1/3}{2/3}\gamma^+(t) & \text{in } \mathrm{R}\cap\{1/3 \leq s < 1\}, \end{cases}$$

which satisfies $\Phi(R) = \Sigma$ and $\int_{\mathrm{R}} |\partial_t\Phi \wedge \partial_s\Phi|\, dt\, ds = \mathcal{H}^2(\Sigma)$. □

Proposition 6.4. *Let* $\mathbf{u}$ *be as in* (6.1) *for a non-constant function* f*. If the functional* $\overline{\mathcal{A}}(\mathbf{u},\cdot)$ *were subadditive, then*

$$\overline{\mathcal{A}}_s(\mathbf{u},\mathrm{R}) = |D^s\mathbf{u}|(\mathrm{R}). \tag{6.3}$$

Proof. Fix $\delta \in (0,(b-a)/2)$ and let $N(\delta) \in \mathbb{N}$ be such that $a + N(\delta)\delta < b \leq a + (N(\delta)+1)\delta$. Define

$$\mathrm{R}_i := [(a+i\delta, a+(i+1)\delta)\cap(a,b)]\times(-1,1), \qquad \text{for } i = 0,\dots,N(\delta),$$

$$\mathrm{P}_i := [(a+i\delta-\delta^2, a+i\delta+\delta^2)\cap(a,b)]\times(-1,1), \quad \text{for } i = 1,\dots,N(\delta).$$

From Proposition 6.3, applied with R_i and P_i in place of R, it follows

$$\overline{\mathcal{A}}(\mathbf{u},\mathrm{R}_i) \leq \int_{\mathrm{R}_i} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + |D^s\mathbf{u}|(\mathrm{R}_i) + \mathrm{lip}(f)\delta^2,$$

$$\overline{\mathcal{A}}(\mathbf{u},\mathrm{P}_i) \leq \int_{\mathrm{P}_i} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + |D^s\mathbf{u}|(\mathrm{P}_i) + 4\mathrm{lip}(f)\delta^4.$$

If $\overline{\mathcal{A}}(\mathbf{u},\cdot)$ were subadditive, we would get

$$\begin{aligned}\overline{\mathcal{A}}(\mathbf{u},\mathrm{R}) &\leq \sum_{i=0}^{N(\delta)} \overline{\mathcal{A}}(\mathbf{u},\mathrm{R}_i) + \sum_{i=1}^{N(\delta)} \overline{\mathcal{A}}(\mathbf{u},\mathrm{P}_i) \\ &\leq \int_{\mathrm{R}} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + |D^s\mathbf{u}|(\mathrm{R}) + \sum_{i=1}^{N(\delta)} \int_{\mathrm{P}_i} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds \\ &\quad + \sum_{i=1}^{N(\delta)} |D^s\mathbf{u}|(\mathrm{P}_i) + \mathcal{O}(\delta) + \mathcal{O}(\delta^3) \\ &\leq \int_{\mathrm{R}} |\mathcal{M}(\nabla\mathbf{u})|\, dt\, ds + |D^s\mathbf{u}|(\mathrm{R}) + \mathcal{O}(\delta).\end{aligned}$$

Since by Theorem 1.10 we have

$$\overline{\mathcal{A}}(\mathbf{u}, \mathrm{R}) \geq \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + |D^s \mathbf{u}|(\mathrm{R}),$$

the thesis follows from the arbitrariness of δ. □

Remark 6.5. Let us consider a map as in (6.1), for a linear function $f(t) = c(t-a)$, $c > 0$. We observe that the curves $\Gamma^{\pm}$ in this case are coplanar. The map $\Psi \in$ semicart$(\mathrm{R}; \Gamma^-, \Gamma^+)$ defined by

$$\Psi(t, s) := \left(t, c(t-a), \frac{s+1}{2}\right), \qquad t \in (a, b), \;\; s \in (-1, 1),$$

parametrizes the rectangle E bounded by Γ^- and Γ^+ and the two vertical segments $\{(a, 0, \eta) : \eta \in (0, 1)\}$, and $\{(b, c(b-a), \eta) : \eta \in (0, 1)\}$. We have

$$\mathcal{H}^2(E) = (b-a)\sqrt{1+c^2} = \int_{\mathrm{R}} |\partial_t \Psi \wedge \partial_s \Psi| \, dt \, ds.$$

On the other hand, in this case the sets $\mathcal{T}_0$ and $\mathcal{T}_1$ built in Proposition 6.4 are two triangles of area $\frac{c(b-a)^2}{2}$, so that $\mathcal{H}^2(\Sigma) = c(b-a)^2 + (b-a)$, with Σ defined as in (6.2). A simple computation shows that

$$\mathcal{H}^2(E) > \mathcal{H}^2(\Sigma), \tag{6.4}$$

provided that $b - a \in \left(0, \frac{\sqrt{1+c^2}-1}{c}\right)$. Interestingly, this implies that the area minimizing semicartesian surface (if it exists) spanning two coplanar curves *is not necessarily planar.*

The next proposition is a modification of [1, Lemma 4.8]. We refer to Section 1.5, and to [1] and [21] for notations and results concerning Cartesian currents.

Proposition 6.6. *Let $T = \tau(S_T, \theta_T, \zeta_T)$ be a 2-dimensional integer rectifiable current with bounded support in $U := \mathrm{R} \times \mathbb{R}^2 \subset \mathbb{R}^2_{(t,s)} \times \mathbb{R}^2_{(\xi,\eta)}$. Denote by $p : U \to \mathbb{R}^2_{(t,s)}$ the orthogonal projection. Suppose that*

(i) $\mathcal{L}^2(p(S_T)) = 0$,

(ii) $\partial T = [\![\{(t, 0, f(t), 1)\}_{t \in (a,b)}]\!] - [\![\{(t, 0, f(t), 0)\}_{t \in (a,b)}]\!]$ *in U, where $f \in \mathrm{Lip}((a, b))$ is piecewise $\mathcal{C}^1$.*

Then

$$f \text{ non-constant} \quad \Rightarrow \quad \mathbf{M}_U(T) > \int_a^b |(f(t), 1) - (f(t), 0)| \, dt = b - a.$$

Proof. Assume by contradiction that $\mathbf{M}_U(T) \leq b-a$. Let $\pi : U \to \mathrm{R} \times \mathbb{R}_\eta$ and $q : \mathrm{R}\times\mathbb{R}_\eta \to \mathrm{R}$ be the orthogonal projections, so that $p = q\circ\pi$. Since T has bounded support in U, then $\partial(\pi_\sharp T) = \pi_\sharp(\partial T)$ (see [21, Sec 2.3]) and thus, from (ii), $\partial(\pi_\sharp T) = [\![\{(t,0,1)\}_{t\in(a,b)}]\!] - [\![\{(t,0,0)\}_{t\in(a,b)}]\!] =: [\![(a,0),(b,0)]\!] \times [\![1]\!] - [\![(a,0),(b,0)]\!] \times [\![0]\!]$ in $\mathrm{R}\times\mathbb{R}_\eta$. Since T is rectifiable, also $\pi_\sharp T$ is[(1)], and $\pi_\sharp T = \tau(S_{\pi_\sharp T}, \theta_{\pi_\sharp T}, \zeta_{\pi_\sharp T})$. Moreover $S_{\pi_\sharp T} \subseteq \pi(S_T)$ and thus, applying the projection q and recalling assumption (i), we get $\mathcal{L}^2(q(S_{\pi_\sharp T})) = 0$. Applying the one-codimensional result in [1, Lemma 4.7], we deduce that $\pi_\sharp T = [\![(a,0),(b,0)]\!]\times[\![0,1]\!]$. In particular $S_{\pi_\sharp T} = (a,b)\times\{0\}\times\{0\}\times[0,1]$, $\theta_{\pi_\sharp T} = 1$, $\zeta_{\pi_\sharp T} = (1,0,0,0)\wedge(0,0,0,1)$.
We now use the assumption on the mass of the current T, obtaining

$$b-a = \mathcal{H}^2(S_{\pi_\sharp T}) \leq \mathcal{H}^2(\pi(S_T)) \leq \mathcal{H}^2(S_T) \leq \mathbf{M}_U(T) \leq b-a;$$

hence the above inequalities are indeed equalities and in particular

$$\pi(S_T) \simeq (a,b)\times\{0\}\times\{0\}\times[0,1]$$

in the sense of $\mathcal{H}^2$. Moreover, since $\mathcal{H}^2(\pi(S_T)) = \mathcal{H}^2(S_T) = \mathbf{M}_U(T)$, it follows that $\theta_T = 1$ $\mathcal{H}^2$-almost everywhere on S_T, and $\zeta_T = e_t \wedge \varepsilon_\eta$, where $e_t = (1,0,0,0)$ and $\varepsilon_\eta = (0,0,0,1)$, see [1, Lemma 4.8].
Thus, if we write any smooth 2-form ω compactly supported in U as

$$\begin{aligned}\omega := {}&\omega^{t,s}(1,0,0,0)\wedge(0,1,0,0)+\omega^{t,\xi}(1,0,0,0)\wedge(0,0,1,0)+\omega^{t,\eta}(1,0,0,0)\wedge(0,0,0,1)\\ &+\omega^{s,\xi}(0,1,0,0)\wedge(0,0,1,0)+\omega^{s,\eta}(0,1,0,0)\wedge(0,0,0,1)+\omega^{\xi,\eta}(0,0,1,0)\wedge(0,0,0,1),\end{aligned}$$

we have

$$T(\omega) = \int_{S_T} \omega^{t,\eta}\, d\mathcal{H}^2(t,s,\xi,\eta).$$

Let $\alpha := \alpha^t dt + \alpha^s ds + \alpha^\xi d\xi + \alpha^\eta d\eta$ be a smooth 1-form compactly supported in U. Then

$$\partial T(\alpha) = T(d\alpha) = \int_{S_T} (\partial_t \alpha^\eta - \partial_\eta \alpha^t)\, d\mathcal{H}^2(t,s,\xi,\eta). \tag{6.5}$$

On the other hand from assumption (ii) it follows

$$\partial T(\alpha) = \int_a^b \left[\alpha^t(t,0,f(t),1) - \alpha^t(t,0,f(t),0)\right] + f'(t)\left[\alpha^\xi(t,0,f(t),1) - \alpha^\xi(t,0,f(t),0)\right] dt. \tag{6.6}$$

Now, we choose a 1−form α such that $\alpha^t = 0$, $\alpha^\eta = 0$, so that the right hand side of (6.5) vanishes, and the right hand side of (6.6) reduces to

$$\partial T(\alpha) = \int_a^b f'(t)\left[\alpha^\xi(t,0,f(t),1) - \alpha^\xi(t,0,f(t),0)\right] dt.$$

Since f is not constant and piecewise $\mathcal{C}^1$, there exists a non-empty open interval $I \subset (a,b)$ where f' is either positive or negative. It is then sufficient to choose α^ξ compactly supported in I and such that $\alpha^\xi(t,0,f(t),1) \neq \alpha^\xi(t,0,f(t),0)$ in I, to obtain that the right hand side of (6.6) is non-zero, which is a contradiction. □

[(1)] Again $\pi_\sharp T(\omega) := T(\pi^\sharp \omega)$, $\forall \omega \in \mathcal{D}^2(\mathrm{R}\times\mathbb{R}_\eta)$.

Now we are in a position to prove Theorem 6.1

Proof of Theorem 6.1. Let us suppose by contradiction that $\overline{\mathcal{A}}(\mathbf{u}, \cdot)$ is subadditive. As a consequence of Proposition 6.4 and Lemma 1.9, we can select a sequence $(\mathbf{u}_h) \subset \mathcal{C}^1(\mathrm{R}; \mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\mathrm{R}; \mathbb{R}^2)$, bounded in $L^\infty(\mathrm{R}; \mathbb{R}^2)$, and such that

$$\mathcal{A}(\mathbf{u}_h, \mathrm{R}) \to \overline{\mathcal{A}}(\mathbf{u}, \mathrm{R}) = \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + |D^s \mathbf{u}|(\mathrm{R}) \qquad \text{as } h \to +\infty.$$

From Theorem 1.15 it follows that the sequence of the graphs $[\![G_{\mathbf{u}_h}]\!]$ of the maps $\mathbf{u}_h$ converges weakly in the distributional sense to a Cartesian 2-current $T = \tau(S, \theta, \zeta)$ in $U := \mathrm{R} \times \mathbb{R}^2$, decomposable in its regular part $T_r = [\![G_{\mathbf{u}}]\!]$ and its singular part $T_s = \tau(S_s, \theta, \zeta)$; recall that $\mathcal{L}^2(p(S_s)) = 0$, see Theorem 1.14. Moreover, by the lower semicontinuity of the mass we have

$$\mathbf{M}(T) \leq \liminf_{h \to +\infty} \mathbf{M}([\![G_{\mathbf{u}_h}]\!]) = \overline{\mathcal{A}}(\mathbf{u}, \mathrm{R})$$

and

$$\mathbf{M}(T) = \mathbf{M}(T_r) + \mathbf{M}(T_s) = \mathbf{M}([\![G_{\mathbf{u}}]\!]) + \mathbf{M}(T_s) = \int_{\mathrm{R}} |\mathcal{M}(\nabla \mathbf{u})| \, dt \, ds + \mathbf{M}(T_s).$$

In addition, the support of T_s is bounded in U, since $(\mathbf{u}_h)$ is bounded in $L^\infty(\mathrm{R}; \mathbb{R}^2)$, and $\partial T_s = -\partial T_r = -\partial [\![G_{\mathbf{u}}]\!] = [\![\{(t, 0, f(t), 1)\}_{t \in (a,b)}]\!] - [\![\{(t, 0, f(t), 0)\}_{t \in (a,b)}]\!]$ in U. Therefore T_s satisfies all hypotheses of Proposition 6.6, which implies that f has to be constant, providing a contradiction. □

7. Classical Plateau's problem and semicartesian surfaces

Overview of the chapter

In this chapter we report a result contained in [6] that guarantees the existence of a solution $\Sigma_{\min}$ to the Plateau's problem for the Jordan curve Γ admitting a semicartesian parametrization, whenever Γ is union of two graphs and satisfies suitable regularity hypotheses. More precisely, we prove that if Γ is analytic, union of two analytic graphs, joining in a non-degenerative way (see Definition 7.1), then there exists a semicartesian parametrization (O, Φ) spanning Γ such that

$$\begin{aligned} &\int_O |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds = a(\Gamma), \\ &\Phi(O) = \Sigma_{\min}. \end{aligned} \tag{7.1}$$

We refer to Section 1.6 for the notation and the results on the Plateau's problem we shall use in this chapter, and to [33], [15], [16] for an exhaustive discussion.
Before stating Theorem 7.3 let us give the following definition.

Definition 7.1 (Condition (A)). We say that a curve Γ union of two graphs satisfies condition (A) if there exists an injective *analytic* map

$$g = (g_1, g_2, g_3) : \partial B \to \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$$

such that

$$\Gamma = g(\partial B),$$

where the following properties are satisfied:

$$\begin{aligned} &|g'(\theta)| \neq 0, \qquad \theta \in [0, 2\pi), \\ &g_1' < 0 \text{ in } (\theta_{\rm n}, \theta_{\rm s} + 2\pi), \\ &g_1' > 0 \text{ in } (\theta_{\rm s}, \theta_{\rm n}), \\ &g_1''(\theta_{\rm s}) > 0, \qquad g_1''(\theta_{\rm n}) < 0. \end{aligned} \tag{7.2}$$

Remark 7.2. In Definition 7.1 we have denoted for simplicity by g the composition $g \circ \mathbf{b}$, where, $\mathbf{b}$ is an arc-length parametrization of ∂B and $\theta_{\rm s}, \theta_{\rm n} \in [0, 2\pi)$ are so that $\theta_{\rm s} < \theta_{\rm n}$, $\mathbf{b}(\theta_{\rm s}) = (0, -1)$, and $\mathbf{b}(\theta_{\rm n}) = (0, 1)$. We use the prime for differentiating with respect to θ.

Note carefully that the last three conditions in (7.2) involve the first component of g only.
Our result is the following.

Theorem 7.3 (Existence of semicartesian parametrizations). *Let $\Gamma = \Gamma^- \cup \Gamma^+ \subset \mathbb{R}^3$ be a Jordan curve union of the two graphs on $[a,b]$, $\Gamma^\pm = \mathrm{graph}(\gamma^\pm)$. If Γ satisfies condition* (A) *then*

(i) there exists a semicartesian parametrization (O,Φ), $O = [[\sigma_O^-, \sigma_O^+]]$, satisfying (7.1)*; moreover Φ is conformal and free of interior branch points and of boundary branch points;*

(ii) the Lipschitz constant of $\sigma_O^\pm$ on a relatively compact subinterval of (a,b) is bounded by the Lipschitz constant of the restriction of $\gamma^\pm$ on the same subinterval;

(iii) near the point $(a,0)$ (similarly near the point $(b,0)$), the curve ∂O is of the form $\{(\tau(s),s)\}$ for $|s|$ small enough, with

$$\tau(s) = a + \lambda_2 s^2 + o(s^2), \tag{7.3}$$

and $\lambda_2 > 0$.

We present the proof of Theorem 7.3 as it appeared in [6], even if, as observed by Brian White, it can be shortened, see Remark 7.7. He also suggested that, plausibly, a solution to the Plateau's problem for Γ should admit a semicartesian parametrization whenever Γ is union of two graphs. This hint led us to prove the result in Section 7.3, that is

Theorem 7.4 (Γ union of two $\mathcal{C}^{1,\alpha}$ graphs: $m(D;\Gamma^-,\Gamma^+) = a(\Gamma)$). *Let $\gamma^\pm \in \mathcal{C}^{1,\alpha}([a,b];\mathbb{R}^2)$, be such that $\Gamma := \Gamma^- \cup \Gamma^+$ is a Jordan curve, with $\Gamma^\pm := \mathrm{graph}(\gamma^\pm)$. Then there exists a sequence (O_h) of Lipschitz semicartesian parameter domains on $[a,b]$ and a sequence of semicartesian maps (Φ_h), $\Phi_h \in \mathrm{semicart}(O_h;\Gamma^-,\Gamma^+)$, such that*

$$\lim_{h\to+\infty} \int_{O_h} |\partial_t \Phi_h \wedge \partial_s \Phi_h| \, dt \, ds = a(\Gamma). \tag{7.4}$$

Remark 7.5. Let Γ and (O,Φ) as in the statement of Theorem 7.3. We underline that, Theorem 7.3 does not directly imply the existence of a minimum for $m(D;\Gamma^-,\Gamma^+)$. Indeed one could define the semicartesian parametrization (D,Ψ) spanning Γ with $\Psi := \Phi \circ T$ where $T : D \to O$ is defined following the same strategy as for the maps T_1 and T_2 in Lemma 2.12, but in general $\Psi \notin H^1(D;\mathbb{R}^3)$, since $\sigma_O^\pm \notin \mathrm{Lip}([a,b])$.
On the other hand if Γ is union of two $\mathcal{C}^{1,\alpha}$ graphs, Theorem 7.4 implies that $m(D;\Gamma^-,\Gamma^+) = a(\Gamma)$, even if it does not provide a minimum for $m(D;\Gamma^-,\Gamma^+)$. Indeed since O_h is a semicartesian Lipschitz parameter domain and $\Phi_h \in H^1(O_h;\mathbb{R}^3)$, one can always find a parametrization $(D,\Psi_h) \in \mathrm{semicart}(D;\Gamma^-,\Gamma^+)$ having the same area of (O_h,Φ_h), see Lemma 2.12; this proves that $a(\Gamma) \geq m(D;\Gamma^-,\Gamma^+)$, while the other inequality is given in (2.10).

In Section 7.1 we prove the assertion (i) of Theorem 7.3; in Section 7.2 we show also (ii) and (iii) of Theorem 7.3, concerning the shape of the semicartesian parameter domain O. In Section 7.3 we demonstrate Theorem 7.4.

7.1 Γ analytic: proof of (i) in Theorem 7.3

In this section we prove assertion (i) of Theorem 7.3.
Let us start by fixing some notation. Given Γ as in the statement of Theorem 7.3, let us denote by Y the solution of the Plateau's problem provided by Theorem 1.17; we underline the fact that Y is conformal and, since Γ satisfies condition 7.1, Y belongs also to $\mathcal{C}^\omega(\overline{B};\mathbb{R}^3)$ and it is free of both internal and boundary branch points (see Theorems 1.24, 1.21, 1.22). We can also suppose (see Remark 1.18) that

$$\begin{aligned} Y(0,-1) &= (a,\gamma^+(a)) = (a,\gamma^-(a)) =: S \\ Y(0,1) &= (b,\gamma^+(b)) = (b,\gamma^-(b)) =: N, \end{aligned} \tag{7.5}$$

and we fix a third condition as we wish (respecting the monotonicity on the boundary parametrization), for definitiveness $Y(1,0) = ((a+b)/2, \gamma^+((a+b)/2))$.

Before proving the existence of a global semicartesian parametrization for $\Sigma_{\min} := Y(B)$, we prove a transversality result that, coupled with Theorem 1.25, provides the existence of a *local* semicartesian parametrization.

Theorem 7.6 (**Transversality**). *Let $\mathcal{P}$ be the family of parallel planes orthogonal to the unit vector $e_t = (1,0,0)$, that is the planes in the form*

$$\Big\{(t,\xi,\eta) \in \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)} : t = \text{const}\Big\}.$$

Let Γ satisfies the same hypotheses as in Theorem 7.3. Then none of the planes of $\mathcal{P}$ is tangent to $\Sigma_{\min}$.

Proof. We have to show that the normal direction to $\Sigma_{\min}$ at a point of $\Sigma_{\min}$ is never parallel to $(1,0,0)$; at self-intersection points of $\Sigma_{\min}$, the statement refers to all normal directions.
Our strategy is to introduce a height function having the planes of the family $\mathcal{P}$ as level sets, namely the function given by the first coordinate t in $\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{R}^2_{(\xi,\eta)}$, restricted to an extension of $\Sigma_{\min}$. The proof consists then in proving that the only critical points of the height function are the minimum and the maximum corresponding to points S and N (see (7.5)).

Since $\partial\Sigma_{\min} = \Gamma$ is non-empty, in order to deal with boundary critical points, it is convenient to extend $\Sigma_{\min}$ across Γ. By condition (A) the curve Γ is analytic; therefore (Theorem 1.24) we can extend $\Sigma_{\min}$ to an analytic minimal surface Σ^{ext} across Γ; Σ^{ext} can be parametrized on a bounded smooth simply connected open set $B^{\text{ext}} \supset B$ through an analytic map $Y^{\text{ext}} = (Y_1^{\text{ext}}, Y_2^{\text{ext}}, Y_3^{\text{ext}})$ which coincides with Y on $\overline{B}$, is harmonic, *i.e.* $\Delta Y^{\text{ext}} = 0$ in B^{ext}, and satisfies the conformality relations $|\partial_u Y^{\text{ext}}|^2 = |\partial_v Y^{\text{ext}}|^2$, $\partial_u Y^{\text{ext}} \cdot \partial_v Y^{\text{ext}} = 0$ in B^{ext}. In addition, from Theorems 1.21

and 1.22, Y^{ext} has no interior (*i.e.*, in B) and no boundary (*i.e.*, on ∂B) branch points. Hence, possibly reducing B^{ext}, we can suppose that Y^{ext} has no branch points in B^{ext}.
Therefore, the Gauss map

$$\mathcal{N} : (u,v) \in B^{\text{ext}} \to \mathcal{N}(u,v) := \frac{\partial_u Y^{\text{ext}}(u,v) \wedge \partial_v Y^{\text{ext}}(u,v)}{|\partial_u Y^{\text{ext}}(u,v) \wedge \partial_v Y^{\text{ext}}(u,v)|} \tag{7.6}$$

is well-defined in B^{ext}(1).
Let us define

$$h : (u,v) \in B^{\text{ext}} \to h(u,v) := Y_1^{\text{ext}}(u,v) \in \mathbb{R}_t.$$

Observe that $(u_0, v_0) \in B^{\text{ext}}$ is a critical point for h if and only if the plane $\{(t,\xi,\eta) \in \mathbb{R}^3 : t = Y_1^{\text{ext}}(u_0,v_0)\}$ is tangent to Σ^{ext} at $Y^{\text{ext}}(u_0,v_0)$. Indeed, criticality implies $\partial_u Y_1^{\text{ext}}(u_0,v_0) = \partial_v Y_1^{\text{ext}}(u_0,v_0) = 0$, and one checks from (7.6) that

$$\mathcal{N}(u_0,v_0) = (1,\, 0,\, 0). \tag{7.7}$$

On the other hand, if $\mathcal{N}(u_0,v_0) = (1,0,0)$, the image of any vector of $\mathbb{R}^2_{(u,v)}$ through the differential of Y at (u_0,v_0) is orthogonal to $(1,0,0)$. In particular, if we consider the image of $e_u = (1,0)$ and $e_v = (0,1)$, we obtain $\partial_u Y_1(u_0,v_0) = 0 = \partial_v Y_1(u_0,v_0)$.
From the above observation, it follows that the proof of the theorem reduces to show that the function h has no critical points in $\overline{B}$, except for $(0,\pm 1)$, for which we shall prove separately that $\mathcal{N}(0,\pm 1) \neq (1,0,0)$.

At first, we shall show that the thesis of the theorem holds true up to a small rotation of Σ^{ext} around a line in the orthogonal space to $(1,0,0)$ that takes a direction in a suitable set to become $(1,0,0)$; moreover this set of directions is dense in a small neighbourhood of $(1,0,0)$.
In the last step we will show that the statements holds true *without applying this rotation.*

`step 1`. Up to a suitable rotation in $\mathbb{R}^3$, the function h has no degenerate critical points.

We notice that any degenerate critical point of h is a critical point also for the Gauss map. Indeed let $(u_0,v_0) \in B^{\text{ext}}$ be critical: using (7.7) we have, for the coefficients of the second fundamental form,

$$\begin{aligned}
\partial^2_{uu} Y^{\text{ext}} \cdot \mathcal{N} &= \partial^2_{uu} Y_1^{\text{ext}} = \partial^2_{uu} h,\\
\partial^2_{uv} Y^{\text{ext}} \cdot \mathcal{N} &= \partial^2_{uv} Y_1^{\text{ext}} = \partial^2_{uv} h,\\
\partial^2_{vv} Y^{\text{ext}} \cdot \mathcal{N} &= \partial^2_{vv} Y_1^{\text{ext}} = \partial^2_{vv} h.
\end{aligned}$$

If in addition (u_0,v_0) is degenerate, then the determinant of the Hessian of h at (u_0,v_0) vanishes, and this implies that also the determinant of the second fundamental form is zero. That is, (u_0,v_0) is a critical point for the Gauss map.

(1) $\mathcal{N}$ is also harmonic and satisfies the conformality relations, see [15, Chapter 1.2].

From Sard's Lemma, it follows that we can find a rotation around a line in the orthogonal space to $(1,0,0)$, as close as we want to the identity, so that the t-direction does not belong to the set of critical values of the Gauss map. Moreover such a rotation can be freely chosen in a set that is dense in a neighbourhood of the identity. We also remark that, for a sufficiently small rotation, condition (A) remains valid, although the values $\theta_{\rm n}$ and $\theta_{\rm s}$ of the parameter leading to maximal and minimal value of the t-component are perturbed of a small amount.
Therefore, from now on we assume that

$$\text{all critical points of } h \text{ in } B^{\rm ext} \text{ are nondegenerate.}$$

`step 2`. The height function h has no critical points on ∂B.

Suppose first by contradiction that there exists $(u,v) \in \partial B \setminus \{(0,\pm 1)\}$ such that $\nabla h(u,v) = 0$, namely (u,v) is a critical point of h different from $(0,\pm 1)$. We claim that if $\tau_{\partial B} \in \mathbb{R}^2$, $|\tau_{\partial B}| = 1$, $\tau_{\partial B}$ tangent to ∂B at (u,v), then for some $\lambda \neq 0$

$$Y_{\tau_{\partial B}}(u,v) = \lambda \tau_\Gamma(u,v),$$

where $\tau_\Gamma(u,v)$ is a tangent unit vector to Γ at $Y(u,v)$ and $Y_{\tau_{\partial B}}$ is the derivative of Y along $\tau_{\partial B}$. Indeed, since Y is smooth up to ∂B, it follows that $Y_{\tau_{\partial B}}(u,v)$ is tangent to Γ at $Y(u,v)$. Now write $\tau_{\partial B} = \alpha e_u + \beta e_v$, $\alpha^2 + \beta^2 = 1$ and $e_u = (1,0)$, $e_v = (0,1)$. Since

$$Y_{\tau_{\partial B}}(u,v) = \alpha \partial_u Y(u,v) + \beta \partial_v Y(u,v),$$

the conformality relations imply

$$|Y_{\tau_{\partial B}}(u,v)|^2 = (\alpha^2 + \beta^2)|\partial_u Y(u,v)|^2.$$

Then the absence of boundary branch points guarantees that $|Y_{\tau_{\partial B}}(u,v)|^2 \neq 0$. Hence $Y_{\tau_{\partial B}}(u,v)$ is a non-zero vector parallel to $\tau_\Gamma(u,v)$ and the claim follows. Observe now that, by assumption, $\tau_\Gamma(u,v)$ has non-zero t-component, so that

$$\alpha \partial_u Y^1(u,v) + \beta \partial_v Y^1(u,v) \neq 0, \tag{7.8}$$

which contradicts the criticality of (u,v) for h. Thus (7.8) shows that h has no critical points on $\partial B \setminus \{(0,\pm 1)\}$. In order to exclude that S (and similarly N) is a critical point for h, we observe that condition (A) implies that the convex hull of Γ, and hence the convex hull of $\Sigma_{\min}$(2), is contained in a wedge having the tangent to Γ at its lowest point as ridge and the two slopes are strictly increasing starting from the ridge. Thus the normal vector to $\Sigma^{\rm ext}$ in S cannot be parallel to $(1,0,0)$.

As a consequence of `step 2` we can suppose that all critical points of h are contained in B.

`step 3`. The function h has neither local maxima nor local minima in B.

(2) Any connected minimal surface X with a parameter domain D is contained in the convex hull of $X_{|\partial D}$. See [16, Theorem 1, chapter 4.1].

Indeed, assume by contradiction that $p = Y(u_0, v_0) \in \Sigma_{\min}$, where $(u_0, v_0) \in B$ is a local minimum point for h. Then locally the surface $\Sigma_{\min}$ is contained in a half-space delimited by the tangent plane $\{(t, \xi, \eta) : t = Y_1(u_0, v_0)\}$, the intersection with this tangent plane being locally only the point p. We now construct a competitor surface Σ' as follows: we remove from $\Sigma_{\min}$ a small portion locally around p, obtained by cutting $\Sigma_{\min}$ locally with a plane at a level slightly higher than the minimal value. We fill the removed portion with a portion of plane, and this givs Σ'[(3)]. Then the area of Σ' is strictly smaller than the area of $\Sigma_{\min}$, a contradiction. A similar argument holds for a local maximum point and therefore the proof of `step 3` is concluded.

Employing the notation of Section 1.7, we have therefore

$$m_0(h, B) = m_2(h, B) = 0.$$

The next step is a consequence of the monotonicity and non-degeneracy assumptions expressed in (7.2), and of the conformality and analyticity of $\Sigma_{\min}$.

`step 4`. The restriction $h_{|\partial B}$ of h to ∂B is a Morse function; moreover $m_0^-(h_{|\partial_h^- B}) = 1$ and $m_1^-(h_{|\partial_h^- B}) = 0$.

We observe that condition (A) implies that there exists a parametrization of Γ on ∂B whose first components is a Morse function. We have to show that also the parametrization induced by the area minimizing surface Y has the same property. As already done for the function g, we denote by $Y^{\text{ext}}_{|\partial B}$ and by $h_{|\partial B}$ the composition $Y^{\text{ext}} \circ \mathbf{b}$ and $h \circ \mathbf{b}$ respectively (see Remark 7.2) and we use the prime for differentiation with respect to θ. At first, we observe that out of branch points, all the directional derivatives of Y^{ext} are non-zero. Thus in particular, from the absence of boundary branch points on ∂B, we deduce that

$$|(Y^{\text{ext}}_{|\partial B})'(\theta)| \neq 0, \quad \theta \in [0, 2\pi).$$

On the other hand, since g is analytic with differentiable inverse, there exists a $\mathcal{C}^1$ function ψ from $[0, 2\pi]$ in itself such that $\psi(2\pi) = \psi(0) + 2\pi$ and

$$Y^{\text{ext}}_{|\partial B}(\theta) = g(\psi(\theta)), \quad \theta \in [0, 2\pi).$$

Differentiating the last expression and remembering from (7.2) that $|g'| \neq 0$, we get that also ψ' never vanishes, indeed:

$$0 \neq |(Y^{\text{ext}}_{|\partial B})'(\theta)| = |g'(\psi(\theta))||\psi'(\theta)|.$$

From the semicartesian form of Γ, $h_{|\partial B}$ has just a minimum and a maximum in correspondence of $N = (0, 1)$ and $S = (0, -1)$. From the properties of g we infer that $\psi(\theta_s)$ is the value of θ corresponding to S, and similarly for N. Since

$$h''_{|\partial B}(\theta) = g_1''(\psi(\theta))(\psi(\theta))^2 + g_1'(\psi(\theta))\psi''(\theta),$$

[(3)]If the cut level is close enough to the critical level, Σ' is the image of a map in $\mathcal{C}(\Gamma)$.

computing for the values corresponding to S and N we get that the first addendum is non-zero, while the second vanishes. We have thus proven that $h_{|\partial B}$ is a Morse function, with a maximum in $(0,1)$ and a minimum in $(0,-1)$.

Following once more Section 1.7 (see (1.16)), we now set

$$\partial_h^- B := \{(u,v) \in \partial B : \nabla h(u,v) \cdot \nu_B(u,v) < 0\},$$

where $\nu_B(u,v)$ denotes the outward unit normal to ∂B at $(u,v) \in \partial B$.
We prove that

$$(0,-1) \in \partial_h^- B \qquad \text{and} \qquad (0,1) \notin \partial_h^- B.$$

Indeed if $\nabla h(0,-1) \cdot \nu_B(0,-1) \geq 0$, we get a contradiction from the same argument used in `step 2` to prove that $(0,-1)$ is not critical for h. Similarly $(0,1) \notin \partial_h^- B$.

We have thus obtained that

$$m_0^-(h_{|\partial_h^- B}) = 1, \qquad m_1^-(h_{|\partial_h^- B}) = 0.$$

`step 5`. The function h has no saddle points in B.

The Morse function h (`step 1`) has no points of index zero (minima) in B and no points of index two (maxima) in B by `step 3`: again following the notation of Section 1.7 (see (1.17)), we have

$$M_0(h, B \cup \partial B) = 1, \qquad M_2(h, B \cup \partial B) = 0.$$

In addition, using `steps 2` and `4`, we can apply Theorem 1.26, and obtain, being $\chi(B) = 1$,

$$M_1(h) = M_0(h, B \cup \partial B) + M_2(h, B \cup \partial B) - \chi(B) = 0.$$

`step 6.` It is not necessary to apply any rotation.

It is sufficient to show that the direction given by $(1,0,0)$ is actually not critical for the Gauss map. At first we can assume that Γ is not contained in a plane. Indeed if it were planar, necessarily

$$\mathcal{N}(u,v) = \nu_0, \quad (u,v) \in B$$

for some constant unit vector $\nu_0 \neq (1,0,0)$, since Γ is union of two graphs.
Assuming that Γ is non-planar, we reason by contradiction and suppose that there is a degenerate critical point $p = Y(u_0, v_0)$ for the height function h in the relative interior of $\Sigma_{\min}$. This means that (u_0, v_0) is a critical point for the Gauss map, that is the product $\kappa_1\kappa_2$ of the two principal curvatures is 0; because of the minimality of $\Sigma_{\min}$ we get that p is an umbilical point, with $\kappa_1 = 0 = \kappa_2$. Recalling that in a non-planar minimal surface the umbilical points are isolated (see for example [15, Remark 2, chapter 5.2]), we can find a direction in a small neighbourhood of $(1,0,0)$ that is normal to $\Sigma_{\min}$ in a neighbourhood of the degenerate critical point p and is not a critical value for the Gauss map. If we rotate $\Sigma_{\min}$ taking this direction to become vertical, we have a non-degenerate critical point for the height function, which is a contradiction in view of the previous steps. □

Remark 7.7. The fact that h has no critical points in B (that is `step 3` and `step 5`) can be obtained as a consequence of Rado's Lemma,[(4)] see for example [15, Lemma 2, Section 4.9], that we state here for the sake of completeness.

Lemma 7.8 (**Rado's Lemma**). *Let $f : \overline{B} \to \mathbb{R}$ be harmonic in B and continuous on $\overline{B}$ and not identically zero. If there exists a point $z_0 \in B$ where f and its derivatives of order $1, \ldots, m$ vanish, then f changes its sign on ∂B at least $2(m+1)$ times.*

Indeed, since the map h defined in the proof of Theorem 7.3 satisfies the assumptions of Rado's Lemma, if there were a critical point ω_0 for h, then we should find at least 4 points $p_i \in \partial B$, $i = 1, \ldots 4$ such that $h(\omega_0) = h(p_i)$. But this is impossible due to the definition of h and to the fact that Γ is union of two graphs. Anyway, we remark that even if we use Rado's Lemma, requiring that Γ satisfies condition (A) seems still necessary in order to obtain the transversality result up to the boundary.

We can apply Theorem 1.25 to Σ^{ext} with the family of planes of Theorem 7.6, obtaining a *local* semicartesian parametrization. More precisely, for any point $p \in \Sigma^{\text{ext}}$ there exists an open domain $O_p \subset \mathbb{R}^2_{(t,s)}$ and an analytic[(5)], conformal semicartesian map Φ_p parametrizing an open neighbourhood of p on Σ^{ext}:

$$\begin{aligned} \Phi_p : \quad & O_p \to \Sigma^{\text{ext}}, \\ & (t, s_p) \to (t, \phi_p(t, s_p)). \end{aligned} \tag{7.9}$$

We are now in position to prove the existence of a semicartesian parametrization for $\Sigma_{\min}$.

Proposition 7.9 (**Global semicartesian parametrization for $\Sigma_{\min}$**). *In the same hypotheses of Theorem 7.3, assertion (i) is verified.*

Proof. The local parametrization in (7.9) is unique up to an additive constant: $s_p \mapsto s_p + \rho$. Indeed, if t_p is the t-coordinate of p, the direction of $\partial_{s_p}\Phi_p$ is given by the intersection of the tangent plane to Σ^{ext} and the plane $\{t = t_p\}$, since its t-component is zero. The vector $\partial_t \Phi_p$ is then uniquely determined by being in the tangent plane to Σ^{ext}, orthogonal to $\partial_{s_p}\Phi_p$ and having 1 as t-component. This in turn determines the norm of $\partial_{s_p}\Phi_p$ and hence $\partial_{s_p}\Phi_p$ itself (up to a choice of the orientation of Σ^{ext})[(6)]. Function $\phi_p(t, s_p)$ can now be obtained by integrating the vector field $\partial_{s_p}\Phi_p$ along the curve $\{t = t_p\} \cap \Sigma^{\text{ext}}$ and transported as constant along the curves $\{s = \text{const}\}$. Now we can cover $\Sigma_{\min} \cup \Gamma$ with a finite number of such neighbourhoods (local charts) having connected pairwise intersection, and we can choose the constant in such a way that on the intersection of two neighbourhoods the different parametrizations coincide. In this way we can "transport" the parametrization from a fixed chart along a chain of pairwise intersecting charts. This definition

[(4)] We are grateful to Brian White for having pointed us this fact.

[(5)] From the proof of Theorem 1.25 one infers that the regularity of the local semicartesian map is the same as the surface.

[(6)] Incidentally, we note here that $|\partial_{s_p}\Phi_p| = |\partial_t \Phi_p| \geq 1$ (which excludes branch points).

is well-posed if we can prove that the transported parametrization is independent of the actual chain, or equivalently that transporting the parametrization along a closed chain of charts produces the original parametrization. This is a consequence of the simple connectedness of the surface, indeed we can take a closed curve that traverses the original chain of charts and let it shrink until it is contained in a single chart.

Thus we can construct a *global* semicartesian parametrization Φ defined on a open domain $O^{\rm ext} \subset \mathbb{R}^2$. We observe that from the properties on Φ_p, Φ is analytic on $O^{\rm ext}$ and free of branch points. Eventually

$$\overline{O} := \Phi^{-1}(\Sigma_{\min} \cup \Gamma) \tag{7.10}$$

is a closed bounded (connected and simply connected) set such that the intersection with the line $\{t = k\}$, for $k \in (a, b)$, is an interval (not reduced to a point); indeed if the intersection were composed by two (or more) connected components, there would be at least 4 points on the intersection of Γ with the plane $\{t = k\}$, and this is impossible since Γ is union of two graphs. $\square$

Remark 7.10. We observe that the domain O is in the form

$$O = [[\sigma_O^-, \sigma_O^+]] = \{(t, s) : t \in (a, b),\, s \in (\sigma_O^-(t), \sigma_O^+(t))\}, \tag{7.11}$$

for $\sigma_O^\pm \in \mathcal{C}([a, b]) \cap {\rm Lip}_{\rm loc}((a, b))$, $\sigma_O^- < \sigma_O^+$ in the open interval (a, b), $\sigma_O^-(a) = \sigma_O^+(a)$ and $\sigma_O^-(b) = \sigma_O^+(b)$.

7.2 Γ analytic: proof of (*ii*) and (*iii*) in Theorem 7.3

Let us starting observing that the boundary of O is analytic.(7)

Lemma 7.11 (Analyticity of ∂O). *The domain O defined in* (7.10) *has analytic boundary.*

Proof. The boundary of O is the image of an analytic map defined on ∂B. This latter fact follows directly from the analyticity of the map $Y : B \to \Sigma_{\min}$ and of the map $\Phi : O \to \Sigma_{\min}$. The fact that $\Sigma_{\min}$ can have self-intersections is not a problem here because the preimages of points (in either B or O) in a self-intersection are well-separated, so that we can restrict to small patches of the surface and reason locally. $\square$

We are now in a position to specify a further property of ∂O.

Proposition 7.12 (Bound for ${\rm lip}(\sigma_O^\pm)$ on compact subintervals). *In the hypotheses and with the notation of Proposition 7.9, the two functions $\sigma^\pm : [a, b] \to \mathbb{R}$ satisfy condition (ii) of Theorem 7.3.*

(7)The analyticity of ∂O implies that we cannot have a global Lipschitz constant for $\sigma_O^\pm$; thus O is not a Lipschitz semicartesian parameter domain.

Proof. Since O satisfy (7.11), up to translation we can suppose also $\sigma_O^+(a) = 0 = \sigma_O^-(a)$.
Let $(t,s) \in \partial O$ and let $p = \Phi(t,s) \in \Gamma$. Let us suppose that $s = \sigma_O^-(t)$ (the case $s = \sigma_O^+(t)$ being similar) and let us write σ in place of σ_O^- for simplicity. We have to show that

$$|\sigma'(t)| \leq |\gamma'(t)|. \tag{7.12}$$

Let $\vartheta(t,s) \in [-\pi/2, \pi/2]$ be the angle between the tangent line to Γ at p (spanned by $\frac{\Gamma'(t)}{|\Gamma'(t)|}$) and the direction of $\partial_t\Phi(t,s)$. Note that if $\vartheta(t,s) \in (-\pi/2, \pi/2)$ we have

$$\mathrm{tg}(\vartheta(t,s)) = \sigma'(t). \tag{7.13}$$

Indeed, take a vector ℓ generating the tangent line to ∂O at (t,s), for instance $\ell = (\sigma'(t), 1)$. Using also the conformality of Φ, the derivative Φ_ℓ of Φ along the direction of ℓ is given by $\Phi_\ell(t,s) = \sigma'(t)\partial_s\Phi(t,s) + \partial_t\Phi(t,s)$, and is a vector generating the tangent line to Γ at p, and (7.13) follows.
Let now $\Theta(t,s) \in [0, \pi/2]$ be the angle between the tangent line to Γ at p and the line generated by $e_t = (1,0,0)$. If $\Theta(t,s) \in [0, \pi/2)$ we have, writing γ in place of γ^-,

$$\mathrm{tg}(\Theta(t,s)) = |\gamma'(t)|.$$

Hence, to show (7.12), it is sufficient to show that $\vartheta(t,s) \leq \Theta(t,s)$, or equivalently

$$\frac{\pi}{2} - \vartheta(t,s) \geq \frac{\pi}{2} - \Theta(t,s). \tag{7.14}$$

Consider $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ as a point on $\mathbb{S}^2 \subset \mathbb{R}^3$ and think of e_t as the vertical direction (Figure 7.1(b)). We have that $\frac{\pi}{2} - \Theta(t,s)$ is the latitude of $\frac{\Gamma'(t)}{|\Gamma'(t)|}$. On the other hand, remembering that $\partial_s\Phi(t,s)$ is orthogonal to e_t, we have that $\frac{\pi}{2} - \vartheta(t,s)$ (the angle between $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ and $\partial_s\Phi(t,s)$ by conformality) is the geodesic distance (on $\mathbb{S}^2$) between $\frac{\Gamma'(t)}{|\Gamma'(t)|}$ and the point obtained as the intersection between $T_p(\Sigma_{\min})$ and the equatorial plane. Hence inequality (7.14) holds true. □

In order to conclude the proof of Theorem 7.3, we need to study the behaviour of ∂O near $(a,0)$ and $(b,0)$.

Proposition 7.13 (Shape of O near $(a,0)$). *Assertion (iii) of Theorem 7.3 holds.*

Proof. Let us consider the point $(a,0)$. From the analyticity of ∂O (Lemma 7.11) and the fact that $(a,0)$ minimizes the t-component in ∂O, we can express it locally in a neighbourhood of $(a,0)$ as the graph $(\tau(s), s)$ of a function $\tau : (s^-, s^+) \to \mathbb{R}$ defined in a neighbourhood (s^-, s^+) of the origin that can be Taylor expanded as

$$\tau(s) = a + \lambda_2 s^2 + \alpha_3 s^3 + \alpha_4 s^4 + o(s^4), \qquad s \in (s^-, s^+),$$

with $\lambda_2 \geq 0$.
Assume by contradiction that (7.3) does not hold, and therefore

$$\lambda_2 = 0.$$

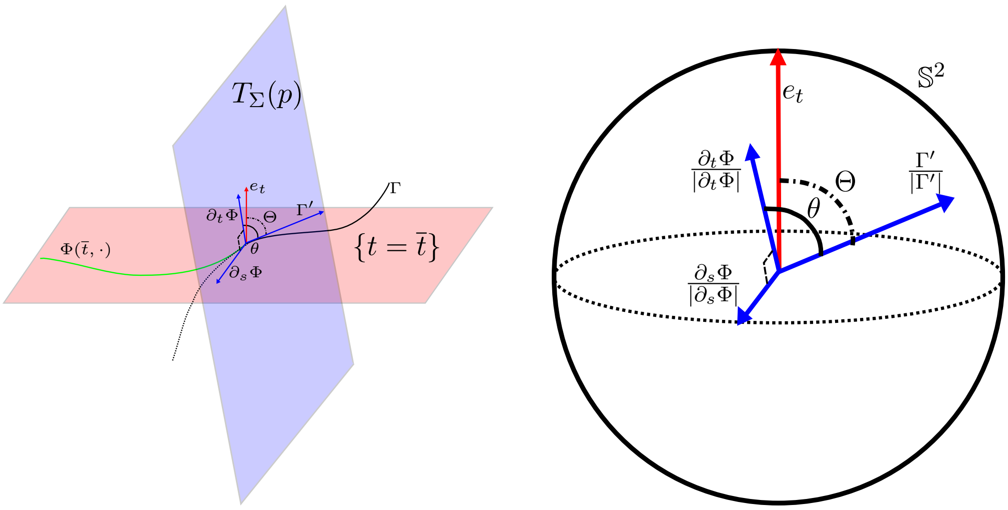

Figure 7.1: (a): The red vector e_t is perpendicular to the plane $\{t=\bar{t}\}$ (in pink) on which we have represented a part of the curve $\{\Phi(\bar{t},s) : s \in [\sigma_O^-(t), \sigma_O^+(t)]\}$ (in green). Γ is also drawn, and passes through the plane $\{t=\bar{t}\}$ transversally. The blue plane is the tangent plane to $\Sigma_{\min}$ at $p=\Phi(\bar{t},\sigma^-(\bar{t}))$ and the three vectors are the conformal basis of the tangent plane span$\{\partial_t\Phi, \partial_s\Phi\}$ and the vector $\Gamma'(\bar{t})$. The angles θ and Θ are also displayed. (b): the same vectors normalized and represented on the sphere $\mathbb{S}^2$.

Since O is contained in the half-plane $\{t \geq a\}$ it follows that

$$\alpha_3 = 0 \quad \text{and} \quad \alpha_4 \geq 0.$$

We shall now compute the area $A(\varepsilon)$ of

$$\Sigma_{\min}^\varepsilon := \Sigma_{\min} \cap \{t < a+\varepsilon\} = \Phi(O \cap S_\varepsilon)$$

for small positive values of ε, where $S_\varepsilon := \{(t,s) : a \leq t < a+\varepsilon\}$. Using the conformal map Φ we need to integrate the area element over the set $O \cap S_\varepsilon$. However the integrand is the modulus of the external product of the two derivatives of Φ with respect to t and to s, which is always greater than or equal to 1, so that, integrating, we get

$$A(\varepsilon) \geq \mathcal{L}^2(O \cap S_\varepsilon) \geq c\varepsilon^{1+1/4} \tag{7.15}$$

for some positive constant c independent of ε.

We now want to show that the minimality of $\Sigma_{\min}$ entails that $\mathcal{H}^2(\Sigma_{\min}^\varepsilon) \leq c\varepsilon^{1+1/2}$, which is in contradiction with (7.15). Indeed we can compare the area of $\Sigma_{\min}$ with the competitor surface

$$\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4,$$

where (see Figure 7.2):

- Σ_1 is the parabolic sector delimited by the osculating parabola to Γ in the minimum point and by the plane $\{t = a+\varepsilon\}$;

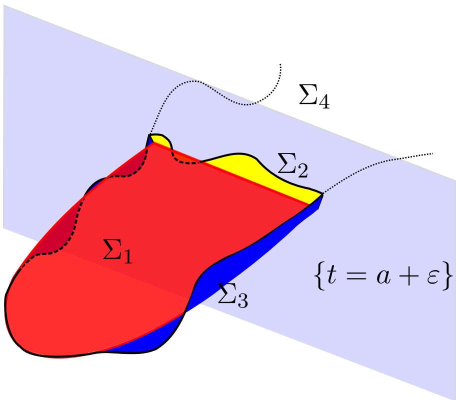

Figure 7.2: The competitor surface Σ. Σ_1, Σ_2 and Σ_3 are the light red, yellow and blue surface respectively.

- Σ_2 is the portion of the plane $\{t = a+\varepsilon\}$ between the curve $\Sigma_{\text{min}} \cap \{t = a+\varepsilon\}$ and the boundary of Σ_1;

- Σ_3 is obtained by connecting linearly each point of the osculating parabola with the point of Γ having the same t-coordinate;

- $\Sigma_4 := \Sigma_{\text{min}} \cap \{a+\varepsilon \le t \le b\}$.

Notice that Σ is a Lipschitz surface and $\partial\Sigma = \Gamma$. Moreover $\Sigma_{\text{min}} = \Sigma^{\varepsilon}_{\text{min}} \cup \Sigma_4$ with $\mathcal{H}^2(\Sigma^{\varepsilon}_{\text{min}} \cap \Sigma_4) = 0$. Thus, using also the minimality of Σ_{min}, we get

$$\mathcal{H}^2(\Sigma_{\text{min}}) = A(\varepsilon) + \mathcal{H}^2(\Sigma_4) \le \mathcal{H}^2(\Sigma) \le \sum_{i=1}^{4} \mathcal{H}^2(\Sigma_i),$$

which implies $A(\varepsilon) \le \mathcal{H}^2(\Sigma_1)+\mathcal{H}^2(\Sigma_2)+\mathcal{H}^2(\Sigma_3)$. Now, we notice that, for a constant c independent of ε:

- $\mathcal{H}^2(\Sigma_1) \le c\varepsilon^{1+1/2}$, since it is a parabolic sector,

- $\mathcal{H}^2(\Sigma_2) \le c\varepsilon^{1+1/2}$ because Σ_{min} is bounded by the two planes of the wedge,

- $\mathcal{H}^2(\Sigma_3) = o(\varepsilon^{1+1/2})$ because $\Sigma^{\varepsilon}_{\text{min}}$ is contained in the inside of a cylindrical shape obtained by translation of Γ in the direction orthogonal to both the tangent vector to Γ in its minimum point and the vector $(1, 0, 0)$.

Thus we get the contradicting relation:

$$c_1\varepsilon^{1+1/2} \ge A(\varepsilon) \ge c_2\varepsilon^{1+1/4},$$

where c_1 and c_2 are two positive constants independent of ε. $\square$

7.3 Γ union of two $\mathcal{C}^{1,\alpha}$ graphs: proof of Theorem 7.4

In this Section we prove Theorem 7.4, showing that if Γ is a Jordan curve, union of two $\mathcal{C}^{1,\alpha}$ graphs on $[a,b]$, $\alpha \in (0,1)$, then $a(\Gamma)$ can be obtained as limit of the area of semicartesian parametrizations spanning Γ.
We recall, see Theorem 1.23, that under this regularity assumptions on Γ there exists a solution of the Plateau's problem $Y \in \mathcal{C}(\Gamma)$ such that $Y \in \mathcal{C}^\omega(B;\mathbb{R}^3) \cap \mathcal{C}^0(\overline{B};\mathbb{R}^2) \cap \mathcal{C}^{1,\alpha}(B \cup I;\mathbb{R}^3)$ for every open arc $I \subset \partial B$ such that $Y(I) \subset \Gamma \setminus \{(a,\gamma^-(a)),(b,\gamma^-(b))\}$. We set $\Sigma_{\min} := Y(B)$.
As in the proof of 7.3, we shall use a transversality property to guarantee (through Theorem 1.25) the existence of local semicartesian parametrizations; we deduce it from the following Theorem contained in [27].

Theorem 7.14 (Theorem 4.1, [27]). *Let $\Sigma \subset \mathbb{R}^3_{(t,\xi,\eta)}$ be a minimal surface of the topology of the disk contained in the cylinder $\mathbb{R}_t \times B_r$, $B_r \subset \mathbb{R}^2_{(\xi,\eta)}$. Let $\Gamma = \partial\Sigma$. Suppose that $\Gamma \cap \{\alpha < t < \beta\}$ is composed of two components, each of which is a $\mathcal{C}^1$ curve whose tangent vector forms with the vector $(1,0,0)$ an angle that is, in modulus, less or equal $\pi - \delta$, for some $\delta > 0$. Let $\Sigma_\varepsilon := \Sigma \cap \{\alpha+\varepsilon < t < \beta - \varepsilon\}$, with $\varepsilon > 0$. Then*

- *(i) Σ_ε has no branch points;*
- *(ii) for any point $p \in \Sigma_\varepsilon$, the normal vector to $\nu(p)$ forms with $(1,0,0)$ an angle that is, in modulus, in $(\overline{\delta}, \pi - \overline{\delta})$, where*
$$\overline{\delta} := \min\left\{\delta, \frac{\varepsilon}{2r}\right\}.$$
- *(iii) for any point $p \in \Sigma_\varepsilon$, the norm of the second fundamental form of Σ_ε at p is bounded by*
$$\frac{C}{\operatorname{dist}(p, \partial\Sigma_\varepsilon)},$$
where the costant C depends only on δ and ε/r.

As we shall see from the proof, we use this result to guarantee the existence of a semicartesian parametrization $(O_\varepsilon, \Phi_\varepsilon)$ on $[a+\varepsilon, b-\varepsilon]$ for $\Sigma_{\min} \cap \{a+\varepsilon < t < b-\varepsilon\}$ for any $\varepsilon > 0$ small enough. In order to prove Theorem 7.4 it remains to extend $(O_\varepsilon, \Phi_\varepsilon)$ to a semicartesian parametrization $(O_\varepsilon^{\mathrm{ext}}, \Phi_\varepsilon^{\mathrm{ext}})$ on $[a,b]$, such that the area contribution of $\Phi_\varepsilon^{\mathrm{ext}}$ on $O_\varepsilon^{\mathrm{ext}} \setminus O_\varepsilon$ is negligible as $\varepsilon \to 0^+$.

Proof of Theorem 7.4. We observe that the semicartesian structure of the curve Γ and the regularity assumption on $\gamma^\pm$ imply that $\Sigma_{\min}$ satisfies the hypotheses of Theorem 7.14 with $\alpha = a$, $\beta = b$ and $\delta = \delta(\gamma^+, \gamma^-) > 0$. For any $\varepsilon > 0$ small enough, we set
$$\Sigma_\varepsilon := \Sigma_{\min} \cap \{a+\varepsilon < t < b-\varepsilon\}.$$
Since $\partial\Sigma_{\varepsilon/2} \cap \Gamma$ is composed of two $\mathcal{C}^{1,\alpha}$-regular curves, Y is $\mathcal{C}^{1,\alpha}$ up to the boundary of the preimage of $\Sigma_{\varepsilon/2}$. Thus we can extend the map Y to a $\mathcal{C}^{1,\alpha}$ map $Y_{\varepsilon/2}^{\mathrm{ext}}$ defined

on an open set $B^{\rm ext}_{\varepsilon/2}$ that strictly contains the preimage of the set Σ_ε, see Theorem 1.23. Moreover, thanks to the uniform bound on the slope of the tangent planes of $\Sigma_{\varepsilon/2}$ provided by assertion (ii) of Theorem 7.14, we can suppose that in every point of the surface $Y^{\rm ext}_{\varepsilon/2}(B^{\rm ext}_{\varepsilon/2})$ the tangent plane is not orthogonal to $(1,0,0)$. Thus, see Theorem 1.25, for any point of this surface we can find a local semicartesian parametrization (as in (7.9)). Since the preimage of Σ_ε is compactly contained in $B^{\rm ext}_{\varepsilon/2}$, following the strategy of the proof of Proposition 7.9, we can cover it with a finite number of charts and thus define a semicartesian parametrization for Σ_ε, denoted by $(O_\varepsilon, \Phi_\varepsilon)$. Following Proposition 7.12, there exist $\sigma^\pm_\varepsilon \in {\rm Lip}([a+\varepsilon, b-\varepsilon])$, ${\rm lip}(\sigma^\pm_\varepsilon) \le {\rm lip}(\gamma^\pm)$ such that

$$O_\varepsilon = \{(t,s) \in \mathbb{R}^2 : t \in (a+\varepsilon, b-\varepsilon),\ s \in (\sigma^-_\varepsilon(t), \sigma^+_\varepsilon(t))\}.$$

We can also suppose $\sigma^+_\varepsilon > 0$ and $\sigma^-_\varepsilon < 0$ on $(a+\varepsilon, b-\varepsilon)$.[(8)] In order to prove the thesis of Theorem 7.4, it is enough to provide, for any $\varepsilon > 0$, a semicartesian parametrization $(O^{\rm ext}_\varepsilon, \Phi^{\rm ext}_\varepsilon)$, such that $O^{\rm ext}_\varepsilon$ is a semicartesian Lipschitz parameter domain on $[a,b]$ and

$$\begin{aligned} &O^{\rm ext}_\varepsilon \cap \{a+\varepsilon < t < b-\varepsilon\} = O_\varepsilon, \\ &\Phi^{\rm ext}_\varepsilon = \Phi_\varepsilon, \text{ on } O_\varepsilon, \end{aligned}$$

and satisfying

$$\lim_{\varepsilon\to 0^+} \int_{O^{\rm ext}_\varepsilon \setminus O_\varepsilon} |\partial_t \Phi^{\rm ext}_\varepsilon \wedge \partial_s \Phi^{\rm ext}_\varepsilon|\, dt\, ds = 0. \tag{7.16}$$

For any ε, let ε' be in $(0,\varepsilon)$ to be chosen later. Let us extend (without renaming them) the function σ^+_ε to the interval $[a,b]$ as follows:

$$\sigma^+_\varepsilon(t) = \begin{cases} \frac{\sigma^+_\varepsilon(a+\varepsilon)}{\varepsilon'}(t-a) & t \in [a, a+\varepsilon'], \\ g\left(\frac{t-(a+\varepsilon')}{\varepsilon-\varepsilon'}\right) + \sigma^+_\varepsilon(a+\varepsilon) & (a+\varepsilon', a+\varepsilon], \\ \sigma^+_\varepsilon(t) & t \in (a+\varepsilon, b-\varepsilon], \\ g\left(\frac{t-(b-\varepsilon)}{\varepsilon-\varepsilon'}\right) + \sigma^+_\varepsilon(b-\varepsilon) & t \in (b-\varepsilon, b-\varepsilon'], \\ \frac{\sigma^+_\varepsilon(a+\varepsilon)}{\varepsilon'}(t-a) & t \in (b-\varepsilon', b], \end{cases}$$

where $g \in {\rm Lip}([0,1])$ satisfies $g(0) = 0 = g(1)$ and $g(t) > 0$ in the open interval $(0,1)$. Similar definition for σ^-_ε. We denote by $O^{\rm ext}_\varepsilon$ the semicartesian set $[[\sigma^-_\varepsilon, \sigma^+_\varepsilon]]$ on $[a,b]$.

(8) Let us define the Steiner symmetrization $S : O_\varepsilon \to O^{\rm sym}_\varepsilon$ as $S(t,s) := \left(t, s - \frac{\sigma^+_\varepsilon(t) - \sigma^-_\varepsilon(t)}{2}\right)$ and the map $\Phi^{\rm sym}_\varepsilon := \Phi_\varepsilon \circ S^{-1}$. The pair $(O^{\rm sym}_\varepsilon, \Phi^{\rm sym}_\varepsilon)$ is a semicartesian parametrization on $(a+\varepsilon, b-\varepsilon)$ spanning Γ, the domain $O^{\rm sym}_\varepsilon$ has the same qualitative properties of the domain O_ε, and the segment $(a+\varepsilon, b-\varepsilon)$ is contained in $O^{\rm sym}_\varepsilon$; we notice that in general $\Phi^{\rm sym}_\varepsilon$ is not conformal on $O^{\rm sym}_\varepsilon$, even though Φ_ε is conformal on O_ε.

Now we define $\Phi_\varepsilon^{\rm ext}$ on $O_\varepsilon^{\rm ext} \cap \{a < t < a+\delta\}$ as follows

$$\Phi_\varepsilon^{\rm ext}(t,s) = \begin{cases} \frac{s-\sigma_\varepsilon^-(t)}{\sigma_\varepsilon^+-\sigma_\varepsilon^-}\gamma^+(t) + \frac{\sigma_\varepsilon^+(t)-s}{\sigma_\varepsilon^+-\sigma_\varepsilon^-}\gamma^-(t) & (t,s)\in E^1_{\varepsilon,\varepsilon'}, \\ \frac{t-(a+\varepsilon')}{\varepsilon-\varepsilon'}\Phi_\varepsilon(a+\varepsilon,s) + \frac{(a+\varepsilon)-t}{\varepsilon-\varepsilon'}\Phi_\varepsilon^{\rm ext}(a+\varepsilon',s) & (t,s)\in E^2_{\varepsilon,\varepsilon'}, \\ \frac{\sigma_\varepsilon^+(t)-s}{\sigma_\varepsilon^+(t)-\sigma_\varepsilon^+(a+\varepsilon)}\Phi_\varepsilon^{\rm ext}(t,\sigma_\varepsilon^+(a+\varepsilon)) + \frac{s-\sigma_\varepsilon^+(a+\varepsilon)}{\sigma_\varepsilon^+(t)-\sigma_\varepsilon^+(a+\varepsilon)}\gamma^+(t) & (t,s)\in E^3_{\varepsilon,\varepsilon'}, \\ \frac{\sigma_\varepsilon^-(t)-s}{\sigma_\varepsilon^-(t)-\sigma_\varepsilon^-(a+\varepsilon)}\Phi_\varepsilon^{\rm ext}(t,\sigma_\varepsilon^-(a+\varepsilon)) + \frac{s-\sigma_\varepsilon^-(a+\varepsilon)}{\sigma_\varepsilon^-(t)-\sigma_\varepsilon^-(a+\varepsilon)}\gamma^-(t) & (t,s)\in E^4_{\varepsilon,\varepsilon'}, \end{cases} \tag{7.17}$$

where

$$\begin{aligned} E^1_{\varepsilon,\varepsilon'} &:= \{(t,s)\in O_\varepsilon^{\rm ext},\, t\in(a,a+\varepsilon']\}, \\ E^2_{\varepsilon,\varepsilon'} &:= \{(t,s)\in O_\varepsilon^{\rm ext},\, t\in(a+\varepsilon',a+\varepsilon],\, s\in[\sigma_\varepsilon^-(a+\varepsilon),\sigma_\varepsilon^+(a+\varepsilon)]\} \\ E^3_{\varepsilon,\varepsilon'} &:= \{(t,s)\in O_\varepsilon^{\rm ext},\, t\in(a+\varepsilon',a+\varepsilon),\, s\in(\sigma_\varepsilon^+(a+\varepsilon),\sigma_\varepsilon^+(t))\} \\ E^4_{\varepsilon,\varepsilon'} &:= \{(t,s)\in O_\varepsilon^{\rm ext},\, t\in(a+\varepsilon',a+\varepsilon),\, s\in(\sigma_\varepsilon^-(t),\sigma_\varepsilon^-(a+\varepsilon))\}. \end{aligned}$$

Similarly we define in $\Phi_\varepsilon^{\rm ext}$ on $O^{\rm ext} \cap \{b-\varepsilon < t < b\}$.
Now we have to check (7.16) for a suitable choice of ε'.

- The image of $E^1_{\varepsilon,\varepsilon'}$ through $\Phi_\varepsilon^{\rm ext}$ is a piece of the ruled surface obtained by linearly interpolating the points $(t,\gamma^-(t))$ and $(t,\gamma^+(t))$, with $t\in(a,b)$. Since this surface has finite area (see (3.3)), the area contribution on $(a,a+\varepsilon')$ vanishes as $\varepsilon'\to 0^+$, and thus as $\varepsilon\to 0^+$.

- The image of $E^2_{\varepsilon,\varepsilon'}$ through $\Phi_\varepsilon^{\rm ext}$ is the ruled surface obtained by linearly interpolating the points $\Phi_\varepsilon(a+\varepsilon,s)$ and $\Phi_\varepsilon^{\rm ext}(a+\varepsilon',s)$, for $s\in(\sigma_\varepsilon^-(a+\varepsilon),\sigma_\varepsilon^+(a+\varepsilon))$. The area of this surface depends on $||\partial_s\Phi_\varepsilon(a+\varepsilon,s)||_{L^\infty((\sigma_\varepsilon^-(a+\varepsilon),\sigma_\varepsilon^+(a+\varepsilon));\mathbb{R}^2)}$, on $||\partial_s\Phi_\varepsilon^{\rm ext}(a+\varepsilon',s)||_{L^\infty((\sigma_\varepsilon^-(a+\varepsilon),\sigma_\varepsilon^+(a+\varepsilon));\mathbb{R}^2)}$, and on the thickness $\varepsilon-\varepsilon'$. The second norm depends just on the velocity of $\gamma^\pm$, and thus is independent of ε and ε'. The first norm, instead, depends on ε and, a priory, it can blow up as $\varepsilon\to 0^+$. Thus we can choose $\varepsilon'=\varepsilon'(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0^+}\int_{E^2_{\varepsilon,\varepsilon'}} |\partial_t\Phi_\varepsilon^{\rm ext}\wedge\partial_s\Phi_\varepsilon^{\rm ext}|\,dt\,ds = 0.$$

- The image of $E^3_{\varepsilon,\varepsilon'}$ (and similarly of $E^4_{\varepsilon,\varepsilon'}$) through $\Phi_\varepsilon^{\rm ext}$ is again a piece of a ruled surface. Its area depends just on the velocity of γ^+ (of γ^-), and thus its area contribution vanishes as $\varepsilon-\varepsilon'\to 0^+$.

Similar arguments hold for the area contribution on $O_\varepsilon^{\rm ext}\cap\{b-\varepsilon<t<b\}$.
Thus we can choose suitable infinitesimal sequences (ε_h), (ε'_h), $\varepsilon_h>0$, $\varepsilon'_h\in(0,\varepsilon_h)$, such that (7.4) is satisfied for $(O_h,\Phi_h) := (O^{\rm ext}_{\varepsilon_h},\Phi^{\rm ext}_{\varepsilon_h})$. □

8. Dirichlet energy minimizing semicartesian surfaces

Overview of the chapter

In this chapter we suppose that $\Gamma := \Gamma^- \cup \Gamma^+$ is a Jordan curve, union of two Lipschitz graphs, and we address the problem of the existence of a minimum for $m(D; \Gamma^-, \Gamma^+)$ with a different strategy with respect to Chapter 7. It is well known that the classical Plateau's problem (1.11) can be solved by finding a conformal minimizer for the Dirichlet functional in the class $\mathcal{C}(\Gamma)$ (see also Section 1.6), and then proving that it is indeed a solution also for the area minimizing problem (1.11); in this last step conformality plays a crucial role. We refer to [15, Chapter 4] for all details.
We would like to adapt this strategy in the context of semicartesian parametrizations. Differently from the area, the Dirichlet functional is *not* invariant under change of coordinates. This implies that we are no more allowed to fix the semicartesian parameter domain, but indeed we have to minimize the functional among all semicartesian parametrizations (O, Φ) spanning Γ, letting both the domain O and the map Φ vary.
Before stating the results of this chapter, that are contained in [7], let us fix some notation and give some definitions.

Definition 8.1 (**The class** semicart(Γ))**.** Let $\Gamma = \Gamma^- \cup \Gamma^+$ be union of two Lipschitz graphs on $[a, b]$. We set

$$\mathrm{semicart}(\Gamma) := \bigcup \mathrm{semicart}(O; \Gamma^-, \Gamma^+),$$

where the union is considered on all semicartesian Lipschitz parameter domain $O = [[\sigma^-, \sigma^+]]$[(1)] such that $\sigma^-(a) = \sigma^+(a) = 0$[(2)] and $\sigma^-(b) = \sigma^+(b)$. We call such an O an *admissible* (Lipschitz semicartesian parameter) domain.

In the following we need also to control the Lipschitz constant of the maps $\sigma^\pm$

[(1)]We remark that in this chapter the symbols $\sigma^\pm$ denote general functions in $\mathrm{Lip}([a, b])$, differently from Definition 2.8.
[(2)]We can suppose $\sigma^\pm(a) = 0$ without loss of generality; this condition is necessary in the proof of Theorem 8.4 in order to avoid the non-compactness of domains of a minimizing sequence due to translation invariance.

defining an admissible semicartesian parameter domain. Thus we give the following definition.

Definition 8.2 (The class $\mathrm{semicart_S}(\Gamma)$**).** Let $\Gamma = \Gamma^- \cup \Gamma^+$ be union of two Lipschitz graphs on $[a,b]$. For $\mathrm{S} > 0$ we set

$$\mathrm{semicart_S}(\Gamma) := \big\{(O,\Phi) \text{ semicartesian parametrizations spanning } \Gamma : \\ O := [[\sigma^-,\sigma^+]] \text{ is an admissible domain with } \mathrm{lip}(\sigma^\pm) \le \mathrm{S}\big\}.$$

We call a domain O satisfying $\mathrm{lip}(\sigma^\pm) \le \mathrm{S}$ an S-*admissible* domain; if $\mathrm{lip}(\sigma^\pm) < \mathrm{S}$, then O is said to be an S-*strictly admissible* domain.

Remark 8.3. We observe that, reasoning as in Lemma 2.10, $\mathrm{semicart_S}(\Gamma) \neq \emptyset$ for every $\mathrm{S} > 0$. This implies that also $\mathrm{semicart}(\Gamma) \neq \emptyset$, since

$$\mathrm{semicart}(\Gamma) := \bigcup_{\mathrm{S}>0} \mathrm{semicart_S}(\Gamma). \tag{8.1}$$

We also stress that the subscript S refers to the Lipschitz constants of the maps defining the domain of the semicartesian parametrization, and does not give any information about the regularity of the semicartesian map.

Given a bounded open set $U \subset \mathbb{R}^2$, and $f \in H^1(U;\mathbb{R}^2)$, we denote by $\mathrm{Dir}(f,\Omega)$ the Dirichlet functional of f on U, cf. [15], namely

$$\mathrm{Dir}(f,U) := \frac{1}{2}\int_U |\nabla f|^2 \, dt\, ds,$$

where $|\nabla f|^2$ is the sum of the squares of the entries of the matrix ∇f. Given $(O,\Phi) := ([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart}(\Gamma)$, we have that $|\nabla\Phi|^2 = 1 + |\partial_t\phi|^2 + |\partial_s\phi|^2$, where ϕ is such that $\Phi(t,s) = (t,\phi(t,s))$; hence

$$\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]) = \frac{1}{2}\int_a^b \int_{\sigma^-(t)}^{\sigma^+(t)} \left[1 + |\partial_t\phi|^2 + |\partial_s\phi|^2\right] ds\, dt.$$

The main results of this chapter are the following.

Theorem 8.4 (Existence of minima in $\mathrm{semicart_S}(\Gamma)$**).** *Let Γ be union of two Lipschitz graphs on $[a,b]$ and $\mathrm{S} > 0$. Then the problem*

$$\min\big\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]) : ([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart_S}(\Gamma)\big\} \tag{8.2}$$

admits a solution $([[\sigma^-_\mathrm{S},\sigma^+_\mathrm{S}]],\Phi_\mathrm{S})$, and Φ_S is harmonic in $[[\sigma^-_\mathrm{S},\sigma^+_\mathrm{S}]]$ and continuous on $\overline{[[\sigma^-_\mathrm{S},\sigma^+_\mathrm{S}]]}$.

Notice that from (8.1), we deduce that for every $\mathrm{S} > 0$

$$\mathrm{Dir}(\Phi_\mathrm{S},[[\sigma^-_\mathrm{S},\sigma^+_\mathrm{S}]]) \ge \inf\{\mathrm{Dir}(\Phi,[[\sigma^-,\sigma^+]]),\ ([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart}(\Gamma)\},$$

thus, in general, $([[\sigma_S^-, \sigma_S^+]], \Phi_S)$ does not minimize the Dirichlet functional in the whole space semicart(Γ). In order to prove the existence of a minimum for the Dirichlet functional in semicart(Γ), we need to prove some conformality properties of the map Φ_S provided by Theorem 8.4. Unfortunately we are able to get the result only requiring *a priori* that Φ_S is Lipschitz up to the boundary of $[[\sigma_S^-, \sigma_S^+]]$. We comment more on this issues in Remark 8.8. Our conformality result turns out to be the following.

Theorem 8.5 (**Conformality of minima in** semicart$_S(\Gamma)$). *For* $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, *let* $\mathrm{G} > 0$ *be such that* $\mathrm{lip}(\gamma^\pm) \le G$. *Let* Γ *be the union of the two Lipschitz graphs of* $\gamma^\pm$. *Fix* $\mathrm{S} > 0$ *and suppose that* $([[\sigma_S^-, \sigma_S^+]], \Phi_S)$ *is a solution of* (8.2) *with*

$$\Phi_S \in \mathrm{Lip}(\overline{[[\sigma_S^-, \sigma_S^+]]}; \mathbb{R}^3).$$

The following assertions hold:

(i) if $[[\sigma_S^-, \sigma_S^+]]$ *is* S*-strictly admissible, then* Φ_S *is conformal;*

(ii) if $\mathrm{S} > \mathrm{G}$, *then* $\mathrm{lip}(\sigma_S^\pm) \le \mathrm{G} < \mathrm{S}$, *that is* $[[\sigma_S^-, \sigma_S^+]]$ *is* S*-strictly admissible.*

We observe that assertions (i) and (ii) imply that if $\mathrm{S} > \mathrm{G}$ and $([[\sigma_S^-, \sigma_S^+]], \Phi_S) \in$ semicart$_S(\Gamma)$ is a solution of (8.2) such that $\Phi_S \in \mathrm{Lip}(\overline{[[\sigma_S^-, \sigma_S^+]]}; \mathbb{R}^3)$, then Φ_S is conformal. Moreover, the fact that under the hypotheses of Theorem 8.5 the minimum belongs to semicart$_G(\Gamma)$ for every $\mathrm{S} > \mathrm{G}$, implies the following result of existence of a minimizer of the Dirichlet energy in semicart(Γ).

Theorem 8.6 (**Existence of a minimum in** semicart(Γ)). *For* $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, *let* $\mathrm{G} > 0$ *be such that* $\mathrm{lip}(\gamma^\pm) \le G$. *Let* Γ *be the union of the two Lipschitz graphs of* $\gamma^\pm$. *Suppose that there exists* $\mathrm{M} > 0$ *such that* $\Phi_S \in \mathrm{Lip}\big(\overline{[[\sigma_S^-, \sigma_S^+]]}; \mathbb{R}^3\big)$ *for every* $\mathrm{S} > \mathrm{M}$. *Then the problem*

$$\min \big\{ \mathrm{Dir}(\Phi, [[\sigma^-, \sigma^+]]) : ([[\sigma^-, \sigma^+]], \Phi) \in \mathrm{semicart}(\Gamma) \big\} \tag{8.3}$$

has a solution $([[\sigma^-, \sigma^+]], \Phi) \in$ semicart$_G(\Gamma)$ *and* Φ *is harmonic and conformal on* $[[\sigma^-, \sigma^+]]$, *and continuous on* $\overline{[[\sigma^-, \sigma^+]]}$.

Remark 8.7. Let $([[\sigma^-, \sigma^+]], \Phi)$ be a solution to (8.3). Then Φ is harmonic in $[[\sigma^-, \sigma^+]]$. This can be seen, as in [15, section 2.1] by exploiting the equality

$$0 = \frac{d}{d\varepsilon} \mathrm{Dir}(X_\varepsilon, [[\sigma^-, \sigma^+]])_{|\varepsilon=0}$$

where $X_\varepsilon(t,s) := \Phi(t,s) + \varepsilon X(t,s)$ for $X \in \mathcal{C}_c^\infty(\Omega;\mathbb{R}^3)$ of the form $X(t,s) = (0, \psi(t,s))$.

In section 8.1 we prove Theorem 8.4. In section 8.2 we prove Theorem 8.5, dividing the statement and the proof into Proposition 8.9, Lemma 8.11, and Proposition 8.12. In section 8.3 we prove Theorem 8.6.

8.1 Existence of minima in $\mathrm{semicart}_{\mathrm{S}}(\Gamma)$

In this section we prove Theorem 8.4. For simplicity the minimum will be denoted by $([[\sigma^-, \sigma^+]], \Phi)$, in place of $([[\sigma^-_{\mathrm{S}}, \sigma^+_{\mathrm{S}}]], \Phi_{\mathrm{S}})$.

Proof of Theorem 8.4. Let $(([[\sigma^-_h, \sigma^+_h]], \Phi_h)) \subset \mathrm{semicart}_{\mathrm{S}}(\Gamma)$ be a minimizing sequence for problem (8.2), namely

$$\begin{aligned} \lim_{h\to+\infty} &\mathrm{Dir}(\Phi_h, [[\sigma^-_h, \sigma^+_h]]) = \\ &\inf\left\{\mathrm{Dir}(\Phi, [[\sigma^-, \sigma^+]]) : ([[\sigma^-, \sigma^+]], \Phi) \in \mathrm{semicart}_{\mathrm{S}}(\Gamma)\right\} < +\infty, \end{aligned} \tag{8.4}$$

and write $\Phi_h(t,s) = (t, \phi_h(t,s))$ for any $(t,s) \in [[\sigma^-_h, \sigma^+_h]]$. From the definition of the space $\mathrm{semicart}_{\mathrm{S}}(\Gamma)$, the sequences $(\sigma^\pm_h)$ are equibounded and equicontinuous; thus, by Ascoli-Arzelà Theorem, we have, up to subsequences, that

$$\sigma^\pm_h \to \sigma^\pm, \qquad \text{uniformly as } h \to +\infty.$$

Therefore the limit functions satisfy:

$$\begin{aligned} &\sigma^\pm \in \mathrm{Lip}([a,b]), \\ &\mathrm{lip}(\sigma^\pm) \le \mathrm{S}, \\ &\sigma^-(a) = \sigma^+(a) = 0, \\ &\sigma^-(b) = \sigma^+(b) \end{aligned}$$

In order to guarantee that $[[\sigma^-, \sigma^+]]$ is S-admissible, we need to check that

$$\sigma^-(t) < \sigma^+(t), \qquad \forall t \in (a,b). \tag{8.5}$$

Suppose by contradiction that there exists $t_0 \in (a,b)$ such that $\sigma^-(t_0) = \sigma^+(t_0)$. The uniform convergence of $(\sigma^\pm_h)$ to $(\sigma^\pm)$ implies in particular that

$$\lim_{h\to+\infty} (\sigma^-_h(t_0) - \sigma^+_h(t_0)) = 0.$$

Let us select $\delta > 0$ such that

$$a < t_0 - \delta < t_0 + \delta < b.$$

Using the Cauchy-Schwarz's inequality and the boundary conditions $\phi_h(t, \sigma^\pm_h(t)) = \gamma^\pm(t)$, we have

$$\begin{aligned} \mathrm{Dir}(\Phi_h, [[\sigma^-_h, \sigma^+_h]]) \ge& \frac{1}{2}\int_{t_0-\delta}^{t_0+\delta} \left(\int_{\sigma^-_h(t)}^{\sigma^+_h(t)} |\partial_s \phi_h(t,s)|^2\, ds\right) dt \\ \ge& \frac{1}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{1}{|\sigma^+_h(t) - \sigma^-_h(t)|} \left(\int_{\sigma^-_h(t)}^{\sigma^+_h(t)} |\partial_s \phi_h(t,s)|\, ds\right)^2 dt \\ \ge& \frac{1}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{|\gamma^+(t) - \gamma^-(t)|^2}{|\sigma^+_h(t) - \sigma^-_h(t)|}\, dt \\ \ge& \frac{C}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{1}{|\sigma^+_h(t) - \sigma^-_h(t)|}\, dt \\ \ge& \frac{C}{2}\int_{t_0-\delta}^{t_0+\delta} \frac{1}{2\mathrm{S}|t-t_0| + |\sigma^+_h(t_0) - \sigma^-_h(t_0)|}\, dt, \end{aligned}$$

where the constant $C = C(t_0, \delta) \in (0, +\infty)$ is found by using $\gamma^-(t) \neq \gamma^+(t)$ for $t \in (a, b)$, and the last inequality follows from the bound $\text{lip}(\sigma_h^\pm) \leq \text{S}$. Since the right-hand side blows up as $h \to +\infty$, while the left-hand side is uniformly bounded with respect to h, from (8.4), we get a contradiction, and this proves (8.5). We conclude that $[[\sigma^-, \sigma^+]]$ is a S-admissible domain.

Now we have to look for a limit semicartesian map Φ defined in $[[\sigma^-, \sigma^+]]$. Without loss of generality, following an argument similar to [15, Theorem 1, section 4.3], we can suppose that ϕ_h is harmonic in its domain $[[\sigma_h^-, \sigma_h^+]]$ for any $h \in \mathbb{N}$. Indeed, if not, we can replace it with the unique solution of the system

$$\begin{cases} \Delta\psi_h = 0 & \text{in } [[\sigma_h^-, \sigma_h^+]], \\ \psi_h(t, \sigma_h^\pm(t)) = \gamma^\pm(t) & t \in [a, b], \end{cases}$$

which minimizes the Dirichlet functional among all functions with the same trace. We observe also that, since $[[\sigma_h^-, \sigma_h^+]]$ satisfies the exterior cone condition, then $\psi_h \in \mathcal{C}\left(\overline{[[\sigma_h^-, \sigma_h^+]]}; \mathbb{R}^2\right) \cap \mathcal{C}^2\left([[\sigma_h^-, \sigma_h^+]]; \mathbb{R}^2\right)$, see [22, Theorem 6.13 and Problem 6.3].

Now, let $\text{R}_M := (a, b) \times (-M, M)$ for M large enough such that $[[\sigma_h^-, \sigma_h^+]] \subseteq \text{R}_M$ for all h, and $[[\sigma^-, \sigma^+]] \subseteq \text{R}_M$. We extend ϕ_h, and consequently also Φ_h, defining

$$\begin{aligned} \phi_h(t, s) &:= \gamma^-(t), \quad (t, s) \in \text{R}_M \cap \{s < \sigma_h^-(t)\} \\ \phi_h(t, s) &:= \gamma^+(t), \quad (t, s) \in \text{R}_M \cap \{s > \sigma_h^+(t)\}. \end{aligned} \tag{8.6}$$

We observe that $\text{Dir}(\Phi_h, \text{R}_M) \leq C$ for some constant $C > 0$ independent of h, and thus, using also a Poincaré's type inequality (see [15, Theorem 1, section 4.6, p. 277]), we get that (ϕ_h) is a bounded sequence in $H^1(\text{R}_M; \mathbb{R}^2)$. We can then extract a (not relabeled) subsequence converging weakly to some $\phi \in H^1(\text{R}_M; \mathbb{R}^2)$. We notice also that the convergence is pointwise, due to the regularity of ϕ_h and that the restriction of ϕ to $[[\sigma^-, \sigma^+]]$ is harmonic. We define $\Phi(t, s) := (t, \phi(t, s))$ for any $(t, s) \in \text{R}_M$. In order to verify that the pair $([[\sigma^-, \sigma^+]], \Phi)$ is a semicartesian parametrization, we have to guarantee that $\Phi(t, \sigma^\pm(t)) = (t, \gamma^\pm(t))$ (for almost all $t \in (a, b)$). This is evident since, if we pass to the limit in (8.6), we get

$$\begin{aligned} \phi(t, s) &:= \gamma^-(t), \quad (t, s) \in \text{R}_M \cap \{s < \sigma^-(t)\} \\ \phi(t, s) &:= \gamma^+(t), \quad (t, s) \in \text{R}_M \cap \{s > \sigma^+(t)\}. \end{aligned}$$

We recall that the Dirichlet functional $\text{Dir}(\cdot, \text{R}_M)$ is lower semicontinuous with respect to the weak convergence in $H^1(\text{R}_M; \mathbb{R}^3)$ (see [15, Theorem 1, section 4.6]). Thus

$$\text{Dir}(\Phi, \text{R}_M) \leq \lim_{h \to +\infty} \text{Dir}(\Phi_h, \text{R}_M).$$

Noticing also that, since $\gamma^\pm$ are Lipschitz and $(\sigma_h^\pm)$ are equiLipschitz, we have

$$\lim_{h \to +\infty} \text{Dir}(\Phi_h, \text{R}_M \setminus [[\sigma_h^-, \sigma_h^+]]) = \text{Dir}(\Phi, \text{R}_M \setminus [[\sigma^-, \sigma^+]]),$$

and thus

$$\text{Dir}(\Phi, [[\sigma^-, \sigma^+]]) \leq \lim_{h \to +\infty} \text{Dir}(\Phi_h, [[\sigma_h^-, \sigma_h^+]]).$$

Hence, $([[\sigma^-,\sigma^+]],\Phi)$ is a minimizer for (8.2), and Φ is harmonic in $[[\sigma^-,\sigma^+]]$ and continuous[(3)] on $\overline{[[\sigma^-,\sigma^+]]}$. □

Remark 8.8 (**Boundary regularity of Φ_S**)**.** We are not able to guarantee that $\Phi_S \in \mathrm{Lip}\big(\overline{[[\sigma_S^-,\sigma_S^+]]};\mathbb{R}^3\big)$, where $([[\sigma_S^-,\sigma_S^+]],\Phi_S)$ is the minimizer of problem (8.2) provided by Theorem 8.4. The fact that this result is not easy to reach can be supported by the following two observations:

- there exist a Lipschitz domain $U \subset \mathbb{R}^2$ and a function $g \in \mathrm{Lip}(U)$ such that the (weak) solution u of
$$\begin{cases} \Delta u = 0 & \text{in } U,\\ u = g & \text{on } \partial U \end{cases}$$
is not globally Lipschitz on $\overline{U}$ (see [25] for an example);

- we cannot even guarantee Lipschitzianity up to the boundary for graph-type minimal surfaces on a Lipschitz domain U with Lipschitz boundary datum g, that is solutions of
$$\begin{cases} \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } U,\\ u = g & \text{on } \partial U; \end{cases}$$
see [23] for more (an example can be found also in [5]).

However, this does not exclude the possibility that $\Phi_S \in \mathrm{Lip}\big(\overline{[[\sigma_S^-,\sigma_S^+]]};\mathbb{R}^3\big)$, because the minimum problem in (8.2) is on the *pair* $([[\sigma^-,\sigma^+]],\Phi)$ and not only on the functions with a fixed domain. This larger degree of freedom makes possible to hope for a global Lipschitzianity of Φ_S.

8.2 Conformality of minima in $\mathrm{semicart}_S(\Gamma)$

In this section we prove Theorem 8.5. Proposition 8.9 proves assertion *(i)* of Theorem 8.5, Lemma 8.11 shows that conformality implies that the domain is indeed G-admissible, and Proposition 8.12 shows assertion *(ii)* of Theorem 8.5.

Proposition 8.9 (**Conformality of strictly admissible and Lipschitz minimizers in $\mathrm{semicart}_S(\Gamma)$**)**.** *For $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$, let $G > 0$ be such that $\mathrm{lip}(\gamma^\pm) \le G$. Let Γ be the union of the two Lipschitz graphs of $\gamma^\pm$. Fix $S > 0$ and suppose that $([[\sigma_S^-,\sigma_S^+]],\Phi_S)$ is a solution of (8.2) with $\Phi_S \in \mathrm{Lip}\big(\overline{[[\sigma_S^-,\sigma_S^+]]};\mathbb{R}^3\big)$. Suppose moreover that $[[\sigma_S^-,\sigma_S^+]]$ is S-strictly admissible. Then Φ_S is conformal.*

Proof. Again for simplicity we denote the minimum by $([[\sigma^-,\sigma^+]],\Phi)$ and we also set $O := [[\sigma^-,\sigma^+]]$. Let $\mu \in \mathrm{Lip}(\mathbb{R}^2) \cap \mathcal{C}^1(O)$[(4)]; following [15], with the difference

(3) Still by [22, Theorem 6.13 and Problem 6.3].

(4) Equivalently we can consider $\mu \in \mathrm{Lip}(O) \cap \mathcal{C}^1(O)$; indeed every Lipschitz function on a Lipschitz domain admits a Lipschitz extension.

that they consider $\mathcal{C}^1$-regular vector field with values in $\mathbb{R}^2$, we take an *internal* variation of the form

$$\begin{aligned} &T_\varepsilon : O \mapsto \mathbb{R}^2 \\ &(t,s) \in O \mapsto (t, \sigma_\varepsilon(t,s)), \qquad \text{with } \sigma_\varepsilon(t,s) := s - \varepsilon\mu(t,s), \end{aligned} \tag{8.7}$$

for $(t,s) \in O$ and $\varepsilon \in \mathbb{R}$. If $|\varepsilon| < \frac{1}{2\|\partial_s\mu\|_{L^\infty(O)}}$, this map is invertible and the inverse $s_\varepsilon(t,\sigma)$ is Lipschitz, by [10, Theorem 1], on the deformed domain

$$O_\varepsilon := \{(t,\sigma) = (t, \sigma_\varepsilon(t,s)) : \ (t,s) \in O\}.$$

We observe that, since $\mu \in \mathrm{Lip}(O)$, the functions

$$\sigma_\varepsilon^\pm(t) := \sigma_\varepsilon(t, \sigma^\pm(t)) \tag{8.8}$$

are Lipschitz, and the strict S-admissibility of O entails that there exists $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the deformed domain O_ε is also S-admissible. We define the function Φ_ε on the deformed domain O_ε as

$$\Phi_\varepsilon(t,\sigma) := \Phi(t, s_\varepsilon(t,\sigma)) \quad (t,\sigma) \in O_\varepsilon,$$

and, for $|\varepsilon| < \varepsilon_0$, we compute

$$F(\varepsilon) := \mathrm{Dir}(\Phi_\varepsilon, O_\varepsilon) = \frac{1}{2}\int_{O_\varepsilon} |\nabla(\Phi\,(t, s_\varepsilon(t,\sigma)))|^2 \; dt\, d\sigma.$$

Since $\sigma_\varepsilon(t, s_\varepsilon(t,\sigma)) = \sigma$, using (8.7),

$$s_\varepsilon(t,\sigma) - \varepsilon\mu(t, s_\varepsilon(t,\sigma)) = \sigma;$$

differentiating with respect to t and σ (see [4, Theorem 3.101] for a chain rule for Lipschitz functions) we get for almost every $(\tau,\sigma) \in O_\varepsilon$,

$$\begin{aligned} \partial_t s_\varepsilon(t,\sigma) &= \frac{\varepsilon\partial_t\mu(t, s_\varepsilon(t,\sigma))}{1 - \varepsilon\partial_s\mu(t, s_\varepsilon(t,\sigma))}, \\ \partial_\sigma s_{\varepsilon,}(t,\sigma) &= \frac{1}{1 - \varepsilon\partial_s\mu(t, s_\varepsilon(t,\sigma))}. \end{aligned}$$

If $(t,\sigma) \in O \cap O_\varepsilon$, recalling that $\mu \in \mathcal{C}^1(O)$, we get also

$$\begin{aligned} \partial_t s_\varepsilon(t,\sigma) &= \varepsilon\partial_t\mu(t,\sigma) + o(\varepsilon) \qquad && \text{as } \varepsilon \to 0, \\ \partial_\sigma s_\varepsilon(t,\sigma) &= 1 + \varepsilon\partial_s\mu(t,\sigma) + o(\varepsilon) \qquad && \text{as } \varepsilon \to 0. \end{aligned} \tag{8.9}$$

We have:

$$\begin{aligned} \frac{d}{d\varepsilon}F(\varepsilon)_{|\varepsilon=0} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\mathrm{Dir}(\Phi_\varepsilon, O_\varepsilon) - \mathrm{Dir}(\Phi, O)\right] \\ &= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\left\{\int_{O_\varepsilon} |\nabla\Phi_\varepsilon|^2 \, dt\, d\sigma - \int_O |\nabla\Phi|^2 \, dt\, ds\right\} \end{aligned} \tag{8.10}$$

For almost every $(t,\sigma) \in O_\varepsilon$, the derivatives of Φ_ε are given by:

$$\partial_t \Phi_\varepsilon(t,\sigma) = \partial_t \Phi(t, s_\varepsilon(t,\sigma)) + \partial_s \Phi(t, s_\varepsilon(t,\sigma))\partial_t s_\varepsilon(t,\sigma),$$
$$\partial_\sigma \Phi_\varepsilon(t,\sigma) = \partial_s \Phi(t, s_\varepsilon(t,\sigma))\partial_\sigma s_\varepsilon(t,\sigma).$$

Thus, recalling also that the absolute value of the determinant of the Jacobian of the change of variables T_ε is given by

$$\left| \begin{matrix} 1 & 0 \\ -\varepsilon\partial_t\mu & 1-\varepsilon\partial_s\mu \end{matrix} \right| = 1 - \varepsilon\partial_s\mu,$$

the limit in (8.10) becomes:

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_O \Big\{ \big[|\partial_t\Phi|^2 + |\partial_s\Phi|^2(\partial_t s_\varepsilon^2 + \partial_\sigma s_\varepsilon^2) + 2\partial_t s_\varepsilon \partial_t\Phi\cdot\partial_s\Phi\big](1-\varepsilon\partial_s\mu) \\ -|\partial_t\Phi|^2 - |\partial_s\Phi|^2 \Big\}\, dt\, ds, \tag{8.11}$$

Now, recalling that for $(t,\sigma) \in O_\varepsilon \cap O$ we have (8.9) for the derivatives of s_ε, it is convenient to compute the integral on $O = T_\varepsilon^{-1}(O_\varepsilon)$ as the contribution of the integral on $T_\varepsilon^{-1}(O_\varepsilon \cap O)$ and on $T_\varepsilon^{-1}(O_\varepsilon \setminus O)$ separately.
On $T_\varepsilon^{-1}(O_\varepsilon \cap O)$ we have

$$|\partial_t\Phi|^2 + |\partial_s\Phi|^2(\partial_t s_\varepsilon^2 + \partial_\sigma s_\varepsilon^2) + 2\partial_t s_\varepsilon\partial_t\Phi\cdot\partial_s\Phi \\ = 1 + |\partial_t\phi|^2 + |\partial_s\phi|^2 + 2\varepsilon[\partial_t\mu\partial_t\phi\cdot\partial_s\phi + \partial_s\mu|\partial_s\phi|^2] + o(\varepsilon);$$

thus (8.11) on $T_\varepsilon^{-1}(O_\varepsilon \cap O)$ is:

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{T_\varepsilon^{-1}(O_\varepsilon\cap O)} \Big\{\big[1+|\partial_t\phi|^2+|\partial\phi|^2+2\varepsilon(\partial_t\mu\partial_t\phi\cdot\partial_s\phi + \partial_s\mu|\partial_s\phi|^2) + o(\varepsilon)\big](1-\varepsilon\partial_s\mu) \\ -(1+|\partial_t\phi|^2+|\partial_s\phi|^2)\Big\}\, dt\, ds$$
$$= \lim_{\varepsilon\to 0}\int_{T_\varepsilon^{-1}(O_\varepsilon\cap O)} \big[(\partial_t\mu\partial_t\phi\cdot\partial_s\phi + \partial_s\mu|\partial_s\phi|^2) - \partial_s\mu(1+|\partial_t\phi|^2+|\partial_s\phi|^2) + o(\varepsilon)\big]\, dt\, ds$$
$$= \int_O [A\partial_t\mu + B\partial_s\mu]\, dt\, ds,$$

where

$$A = \partial_t\phi\cdot\partial_s\phi, \qquad B = \frac{1}{2}\left(|\partial_s\phi|^2 - |\partial_t\phi|^2 - 1\right) \tag{8.12}$$

are harmonic and bounded in O(5). On the other hand, the integral in (8.11) on $T_\varepsilon^{-1}(O_\varepsilon \setminus O)$ is asymptotically negligible. Indeed we have:

$$\lim_{\varepsilon\to 0}\int_{T_\varepsilon^{-1}(O_\varepsilon\cap O)} \frac{1}{2\varepsilon}\left\{|\partial_s\phi|^2\left(\frac{\varepsilon^2\partial_t\mu^2+1}{(1-\varepsilon\partial_s\mu)^2} - 1\right) + 2\varepsilon\frac{\partial_t\mu}{1-\varepsilon\partial_s\mu}\partial_t\phi\cdot\partial_s\phi \right. \\ \left. -\varepsilon\partial_s\mu\left[|\partial_t\phi|^2 + |\partial_s\phi|^2\frac{\varepsilon^2\partial_t\mu^2+1}{(1-\varepsilon\partial_s\mu)^2} + 2\varepsilon\frac{\partial_t\mu}{1-\varepsilon\partial_s\mu}\partial_t\phi\cdot\partial_s\phi\right]\right\}\, dt\, ds = 0,$$

(5) Please note that our A, B correspond to a and b of [15, section 4.5] through $a = -2B$, $b = 2A$.

since the integrand is bounded while the measure of the domain of integration tends to 0. Thus, using the minimality of (O, Φ) we get:

$$0 = \frac{d}{d\varepsilon} F(\varepsilon) = \int_O [A\partial_t \mu + B\partial_s \mu] \, dt \, ds. \tag{8.13}$$

with (A, B) defined in (8.12). The vector field (A, B) is irrotational in O (and also divergence free), because

$$-\partial_t B + \partial_s A = -\partial_s \phi \cdot \partial^2_{ts} \phi + \partial_t \phi \cdot \partial^2_{tt} \phi + \partial_t \phi \cdot \partial^2_{ss} \phi + \partial_s \phi \cdot \partial^2_{ts} \phi = \partial_t \phi \cdot \Delta \phi = 0$$

where the last equality follows from the harmonicity of ϕ (see Remark 8.7). Thus, since O is simply connected, there exists a $f \in \mathcal{C}^1(O)$ such that

$$\nabla f = (A, B);$$

since (A, B) is bounded, we can extend f up to ∂O and $f \in \mathrm{Lip}(O)$. Therefore we can choose

$$\mu = f$$

and get from (8.13) that

$$\int_O (A^2 + B^2) \, dt \, ds = 0;$$

this implies $A = 0 = B$, that is, ϕ is conformal. □

Remark 8.10. In [15], the authors are able to prove conformality of critical points of the Dirichlet functional in the class $\mathcal{C}(\Gamma)$ without any assumption on the boundary regularity. We cannot use the same techniques due to the greater rigidity of the semicartesian setting.

The following lemma provides a bound on the Lipschitz constant of the functions defining the domain of a semicartesian parametrization, under conformality hypothesis.

Lemma 8.11. *For $\gamma^\pm \in \mathrm{Lip}([a, b]; \mathbb{R}^2)$, let $\mathrm{G} > 0$ be such that $\mathrm{lip}(\gamma^\pm) \leq G$. Let Γ be the union of the two Lipschitz graphs of $\gamma^\pm$. Let $([[\rho^-, \rho^+]], \Psi) \in \mathrm{semicart}_\mathrm{S}(\Gamma)$ for some $\mathrm{S} > 0$. Suppose that $\Psi \in \mathrm{Lip}\big(\overline{[[\rho^-, \rho^+]]}; \mathbb{R}^3\big)$ and that it is conformal, that is for almost every $(t, s) \in [[\rho^-, \rho^+]]$*

$$\partial_t \psi \cdot \partial_s \psi = 0 \quad \text{and} \quad 1 + |\partial_t \psi|^2 = |\partial_s \psi|^2. \tag{8.14}$$

Then $\mathrm{lip}(\rho^\pm) \leq \mathrm{G}$.

Proof. From the equality

$$\gamma^\pm(t) = \psi(t, \rho^\pm(t)),$$

using the conformality relations (8.14) and differentiating , we get

$$\dot{\gamma}^\pm(t) = \partial_t \psi(t, \rho^\pm(t)) + \partial_s \psi(t, \rho^\pm(t)) \dot{\rho}^\pm(t), \quad a.e.\ t \in [a, b].$$

Hence, taking the squared norm and using the conformality relations,

$$\begin{aligned}|\dot\gamma^\pm(t)|^2 &= |\partial_t\psi(t,\rho^\pm(t))|^2 + |\partial_s\psi(t,\rho^\pm(t))|^2\dot\rho^\pm(t)^2\\ &= \dot\rho^\pm(t)^2 + |\partial_t\psi(t,\dot\rho^\pm(t))|^2(1+\dot\rho^\pm(t)^2)\\ &\geq \dot\rho^\pm(t)^2,\end{aligned}$$

for almost every $t \in [a,b]$. □

Proposition 8.9 and Lemma 8.11 imply that if a solution $([[\sigma^-,\sigma^+]],\Phi)$ of (8.2) is such that $\Phi \in \mathrm{Lip}\big(\overline{[[\sigma^-,\sigma^+]]};\mathbb{R}^3\big)$ and $\mathrm{lip}(\sigma^\pm) < \mathrm{S}$, then

$$([[\sigma^-,\sigma^+]],\Phi) \in \mathrm{semicart}_{\mathrm{G}}(\Gamma).$$

In the next proposition, we prove that the same result holds , without supposing *a priori* that $\mathrm{lip}(\sigma^\pm) < \mathrm{S}$.

Proposition 8.12 (Conformality of Lipschitz minimizers). *Given two curves* $\gamma^\pm \in \mathrm{Lip}([a,b];\mathbb{R}^2)$*, let* $\mathrm{G} > 0$ *be such that* $\mathrm{lip}(\gamma^\pm) \leq G$*. Let* Γ *be the union of the two Lipschitz graphs of* $\gamma^\pm$*. Let* $([[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]],\Phi_{\mathrm{S}})$ *be a solution of* (8.2) *with* $\Phi_{\mathrm{S}} \in \mathrm{Lip}\big(\overline{[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]};\mathbb{R}^3\big)$*. Then*

$$\mathrm{lip}(\sigma^\pm) \leq \mathrm{G}.$$

In particular, $[[\sigma_{\mathrm{S}}^-,\sigma_{\mathrm{S}}^+]]$ *is* S*-strictly admissible for any* $\mathrm{S} > \mathrm{G}$.

Proof. Again the minimum is denoted simply by $([[\sigma^-,\sigma^+]],\Phi)$. Suppose by contradiction that one of the two Lipschitz functions σ^+,σ^- has a Lipschitz constant strictly larger than G. From Lemma 8.11 it follows that Φ is not conformal, hence

$$\int_{[[\sigma^-,\sigma^+]]} (A^2+B^2)\,dt\,ds > 0, \tag{8.15}$$

where A and B are defined in (8.12). In the following, we will employ the notation already used in the proof of Proposition 8.9. The aim is to find a deformation of $[[\sigma^-,\sigma^+]]$ that does not increase the Lipschitz constants of $\sigma^\pm$ and that strictly decreases the value of the Dirichlet functional.
A natural choice is the variation defined in (8.7), with

$$\mu = -f,$$

so that $\nabla\mu = (-A,-B)$. Contrary to Proposition 8.9, in this case $[[\sigma^-,\sigma^+]]$ is not a S-strictly admissible domain; thus we have to check that the domain of the parametrizations $(O_\varepsilon,\Phi_\varepsilon)$ obtained by inner variation with the field $\mu = -f$ are admissible at least for $\varepsilon > 0$ small enough. If this is the case, from (8.15), we would reach the inequality

$$\frac{d}{d\varepsilon}F(\varepsilon)_{|\varepsilon=0^+} = -\int_{[[\sigma^-,\sigma^+]]} (A^2+B^2)\,dt\,ds < 0,$$

and thus a contradiction.
Thus the proof is reduced to prove that if $|\dot\sigma^+(t)| \geq M >$ G, then $|\dot\sigma^+_\varepsilon(t)| < |\dot\sigma^+(t)|$ ($\sigma^\pm_\varepsilon$ defined as in (8.8)), or similar relations for $|\dot\sigma^-|$ and $|\dot\sigma^-_\varepsilon|$, guaranteeing that $O_\varepsilon = [[\sigma^-_\varepsilon, \sigma^+_\varepsilon]]$ is admissible for $\varepsilon > 0$ small enough. For simplicity we consider the case $\dot\sigma^+(t) > M$.
We have already observed that, due to the lipschitzianity of ϕ, also f is Lipschitz. Thus the following computations are meaningful almost everywhere in $[a, b]$, where $\gamma^+(\cdot)$, $\sigma^+(\cdot)$ and $f(\cdot, \sigma^\pm(\cdot))$ are differentiable. Recalling (8.7), we get:

$$\dot\sigma^+_\varepsilon(t) = \dot\sigma^+(t) - \varepsilon\nabla\mu(t, \sigma^+(t)) \cdot (1, \dot\sigma^+(t)) + o(\varepsilon).$$

Let us prove that

$$\nabla\mu(t, \sigma^+(t)) \cdot \tau^+(t) > 0,$$

where $\tau^+(t) = (1, \dot\sigma^+)$ is the tangent vector to the graph of σ^+. Recalling that $\nabla\mu = (-A, -B)$, we get:

$$\nabla\mu \cdot \tau^+(t) = -A - \dot\sigma^+ B = -\partial_t\phi \cdot \partial_s\phi - \frac{1}{2}\dot\sigma^+(|\partial_s\phi|^2 - |\partial_t\phi|^2 - 1).$$

Differentiating $\phi(t, \sigma^+(t)) = \gamma^+(t)$, we get $\partial_t\phi = \dot\gamma^+ - \dot\sigma^+\partial_s\phi$; substituting it in the previous equation, we get

$$\begin{aligned}
&\nabla\mu \cdot \tau^+(t) \\
&\quad = -(\dot\gamma^+ - \dot\sigma^+\partial_s\phi) \cdot \partial_s\phi - \frac{1}{2}\dot\sigma^+ \left(|\partial_s\phi|^2 - (\dot\gamma^+ - \dot\sigma^+\partial_s\phi) \cdot (\dot\gamma^+ - \dot\sigma^+\partial_s\phi) - 1\right) \\
&\quad = -\dot\gamma^+ \cdot \partial_s\phi + \frac{1}{2}\dot\sigma^+|\partial_s\phi|^2 + \frac{1}{2}\dot\sigma^+|\dot\gamma^+|^2 + \frac{1}{2}(\dot\sigma^+)^3|\partial_s\phi|^2 - (\dot\sigma^+)^2\dot\gamma^+ \cdot \partial_s\phi + \frac{1}{2}\dot\sigma^+ \\
&\quad = I + II + III + IV + V + VI.
\end{aligned}$$

The first term can be bounded in absolute value using the Young inequality by

$$|I| = |\dot\gamma^+ \cdot \partial_s\phi| \leq \frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2 + \frac{1}{2}\dot\sigma^+|\partial_s\phi|^2,$$

(recall that $\dot\sigma^+ \geq M > 0$). Similarly the fifth term can be bounded as

$$|V| = |(\dot\sigma^+)^2\dot\gamma^+ \cdot \partial_s\phi| \leq \frac{1}{2}\dot\sigma^+|\dot\gamma^+|^2 + \frac{1}{2}(\dot\sigma^+)^3|\partial_s\phi|^2$$

so that

$$I + II \geq -\frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2$$

and

$$III + IV + V \geq 0.$$

Hence we obtain

$$\nabla\mu \cdot \tau^+(t) \geq -\frac{1}{2}(\dot\sigma^+)^{-1}|\dot\gamma^+|^2 + \frac{1}{2}\dot\sigma^+ \geq \frac{1}{2}(\dot\sigma^+)^{-1}\left((\dot\sigma^+)^2 - |\dot\gamma^+|^2\right) > 0,$$

where the last strict inequality holds since $\dot{\sigma}^+ > M \geq |\dot{\gamma}^+|$.
Thus we have proved that the domains of the parametrizations $(O_\varepsilon, \Phi_\varepsilon)$ obtained through inner variation with the scalar field $\mu = -f$ are admissible, for $\varepsilon > 0$ small enough, and thus we have found a contradiction. Hence every solution $([[\sigma^-, \sigma^+]], \Phi)$ of (8.2) such that $\Phi \in \mathrm{Lip}\big(\overline{[[\sigma^-, \sigma^+]]}; \mathbb{R}^3\big)$ satisfies

$$\mathrm{lip}(\sigma^\pm) \leq \mathrm{G};$$

moreover, if $\mathrm{S} > \mathrm{G}$, we have that $[[\sigma^-, \sigma^+]]$ is S-strictly admissible, and therefore, applying Proposition 8.9, Φ is also conformal. □

8.3 Existence of a minimum in $\mathrm{semicart}(\Gamma)$

Proof of Theorem 8.6. Let us fix $\mathrm{S} > \mathrm{M} > \mathrm{G}$ (M is the constant defined in the statement of this Theorem) and let $([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) \in \mathrm{semicart}_\mathrm{S}(\Gamma)$ be the solution provided by Theorem 8.4; from the hypotheses and applying Theorem 8.5 we get that $([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) \in \mathrm{semicart}_\mathrm{G}(\Gamma)$. Let us suppose by contradiction that this semicartesian parametrization does not minimize the Dirichlet functional in $\mathrm{semicart}(\Gamma)$, that is let us suppose that there exists a semicartesian parametrization $([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N})$ belonging to $\mathrm{semicart}_\mathrm{N}(\Gamma)$ for some $\mathrm{N} > \mathrm{S}$ such that

$$\mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) > \mathrm{Dir}([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N}).$$

From our hypotheses and Theorem 8.5, there exists also a parametrization

$$([[\sigma_{\mathrm{N,min}}^-, \sigma_{\mathrm{N,min}}^+]], \Phi_{\mathrm{N,min}}) \in \mathrm{semicart}_\mathrm{G}(\Gamma)$$

minimizing the Dirichlet functional in $\mathrm{semicart}_\mathrm{N}(\Gamma)$, and thus also in $\mathrm{semicart}_\mathrm{S}(\Gamma)$. Thus we get the contradictory inequalities chain

$$\begin{aligned}\mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}) &> \mathrm{Dir}([[\sigma_\mathrm{N}^-, \sigma_\mathrm{N}^+]], \Phi_\mathrm{N}) \\ &\geq \mathrm{Dir}([[\sigma_{\mathrm{N,min}}^-, \sigma_{\mathrm{N,min}}^+]], \Phi_{\mathrm{N,min}}) \\ &\geq \mathrm{Dir}([[\sigma_\mathrm{S}^-, \sigma_\mathrm{S}^+]], \Phi_\mathrm{S}).\end{aligned}$$

□

A. Upper bound for $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$, $\mathbf{u}$ jumping on a $\mathcal{C}^2$ regular curve

In [6] we studied the relaxed area functional for a suitable map $\mathbf{u} : \Omega \to \mathbb{R}^2$ jumping on a line compactly contained in Ω, such that the corresponding curve $\Gamma[\mathbf{u}]$ is a Jordan curve; we obtained an upper bound for $\overline{\mathcal{A}}(\mathbf{u}, \Omega)$ slightly different from that provided in Theorem 4.9. First of all, we proved the limit in formula (4.6) with $a(\Gamma[\mathbf{u}])$ in place of $m(D; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}])$. In order to obtain this limit, we had to prove the existence of a solution to the Plateau's problem for $\Gamma[\mathbf{u}]$ admitting a semicartesian parametrization; since this result is guaranteed by Theorem 7.3, we were forced to consider maps less regular than $W^{1,\infty}(\Omega \setminus J_{\mathbf{u}}; \mathbb{R}^2)$(1), so that the curve $\Gamma[\mathbf{u}]$ was analytic and satisfied condition (A) (see Definition 7.1). This regularity assumptions on $\mathbf{u}$ implied in particular very careful computations in small neighbourhoods of the two end points of the jump. We refer to [6] for the details.
The second difference was that we considered maps $\mathbf{u}$ such that $J_{\mathbf{u}} \subset\subset \Omega$ was a general $\mathcal{C}^2$ simple open curve (not necessarily a segment). In this Appendix we show how to adapt Theorems 4.7 and 4.9 to general $\mathcal{C}^2$ jumps. To this aim, we need to distinguish the plane containing the domain of definition of $\mathbf{u}$ and the plane containing the semicartesian parameter domains. Thus we shall use (x, y) to denote the variables of $\mathbf{u}$ and we shall maintain (t, s) for the semicartesian parameters.

Let us fix some notation.
Let $J \subset \mathbb{R}^2_{(x,y)}$ be a simple, open and $\mathcal{C}^2$ regular curve, with $\mathcal{H}^1(J) = b - a$. For any J satisfying these requirements, we denote by $\alpha : [a, b] \subset \mathbb{R}_t \to \mathbb{R}^2_{(x,y)}$ an arc-length parametrization of $\overline{J}$. Moreover there exists $\delta > 0$ (depending on J) and an open set containing J of the form $\Lambda(\mathrm{R}_\delta)$, where $\mathrm{R}_\delta := (a, b) \times (-\delta, \delta) \subset \mathbb{R}^2_{(t,s)}$ and $\Lambda \in \mathcal{C}^1(\mathrm{R}_\delta; \Lambda(\mathrm{R}_\delta))$ is the diffeomorphism defined by(2)

$$\Lambda(t, s) := \alpha(t) + s\dot{\alpha}(t)^{\perp}, \qquad (t, s) \in \mathrm{R}_\delta.$$

If $\Lambda^{-1} : \Lambda(\mathrm{R}_\delta) \to \mathrm{R}_\delta$ is the inverse of Λ, we have $\Lambda^{-1}(x, y) = (t(x, y), s(x, y))$, where

(1)We considered $\mathbf{u} = (u_1, u_2)$ with $u_2 \in W^{1,2}(\Omega \setminus J; \mathbb{R}^2)$ and $u_1 \in W^{1,\infty}(\Omega \setminus (J_{\mathbf{u}} \cup B^a_r \cup B^b_r); \mathbb{R}^2)$ for every radius $r > 0$ small enough, where B^a_r and B^b_r are two balls centred in the two end points of the jump with radius r; we required also a bound on the derivative of the traces of u_1 on ∂B^a_r and ∂B^b_r (see [6, Section 4.1]).

(2)The symbol $\perp$ denote the counterclockwise rotation of $\pi/2$ in $\mathbb{R}^2_{(x,y)}$.

- $s(x,y) = d(x,y)$ is the distance of (x,y) from J on one side of the jump and minus the distance of (x,y) from J on the other side,

- $t(x,y)$ is so that $\alpha(t(x,y)) = (x,y) - d(x,y)\nabla d(x,y)$ is the unique point of J nearest to (x,y).

Since J is of class $\mathcal{C}^2$, we have that d is of class $\mathcal{C}^2$ and t is of class $\mathcal{C}^1$ on $\Lambda(\overline{\mathrm{R}_\delta})$. Using these notations, we are in the position to give the following generalization of Definition 4.4.

Definition A.1 (**Generalized condition** II)**.** We say that Ω and $\mathbf{u} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ satisfy the generalized condition II if $J_{\mathbf{u}}$ is a simple, open and $\mathcal{C}^2$ regular curve, with $\mathcal{H}^1(J) = b - a$, such that $J_{\mathbf{u}} := \alpha([a,b]) \subset\subset \Omega$, $\mathbf{u} \in W^{1,\infty}(\Omega \setminus J_{\mathbf{u}};\mathbb{R}^2)$, and there exist the pointwise limits (still denoted with $\mathbf{u}^\pm$) of $\mathbf{u}$ at all points of $J_{\mathbf{u}}$.

We define $\gamma^\pm[\mathbf{u}] := \mathbf{u}^\pm \circ \alpha$, where $\mathbf{u}^\pm$ are the Lipschitz traces of $\mathbf{u}$ on $J_{\mathbf{u}}$, and $\Gamma^\pm[\mathbf{u}] := \mathrm{graph}(\gamma^\pm[\mathbf{u}])$.
We prove the following result.

Lemma A.2 (**Generalization of Theorem 4.9**)**.** *Let Ω and $\mathbf{u}$ satisfy the generalized condition* II *and let $\Phi \in \mathrm{semicart}(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$, $\Phi(t,s) := (t,\phi(t,s))$. Then there exists a sequence $(\mathbf{u}_h) \subset H^1(\Omega;\mathbb{R}^2)$ converging to $\mathbf{u}$ in $L^1(\Omega;\mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$ such that*

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h,\Omega) = \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dx\,dy + \int_D |\partial_t\Phi \wedge \partial_s\Phi|\,dt\,ds. \tag{A.1}$$

If we consider an area minimizing sequence $(\Phi_k) \subset \mathrm{semicart}(D;\Gamma^-[\mathbf{u}],\Gamma^+[\mathbf{u}])$, applying Lemma A.2 for any k and then using a diagonal technique, we can prove that for Ω and $\mathbf{u}$ satisfying the generalized condition II we get

$$\overline{\mathcal{A}}(\mathbf{u},\Omega) \le \int_\Omega |\mathcal{M}(\nabla\mathbf{u})|\,dx\,dy + m(D;\Gamma^-,\Gamma^+).$$

We can also generalize the case of Ω and $\mathbf{u}$ satisfying condition I, giving the following Definition and statement.

Definition A.3 (**Generalized condition** I)**.** We say that Ω and $\mathbf{u} \in \mathrm{BV}(\Omega;\mathbb{R}^2)$ satisfy the generalized condition I if

- $J_{\mathbf{u}}$ is a simple, open and $\mathcal{C}^2$ regular curve, with $\mathcal{H}^1(J) = b - a$, such that $\alpha(a), \alpha(b) \in \partial\Omega$;

- $\Lambda(\mathrm{R}_\delta) = \{(x,y) \in \Omega : \mathrm{dist}((x,y),J_{\mathbf{u}}) < \delta\}$;(3)

- denoting by Ω^+ and Ω^- the two connected components of $\Omega\setminus J_{\mathbf{u}}$, $\mathbf{u} \in \mathrm{Lip}(\Omega^+;\mathbb{R}^2) \cap \mathrm{Lip}(\Omega^-;\mathbb{R}^2)$.

(3) This condition also implies that the jump reaches $\partial\Omega$ perpendicularly; with some more technicalities, we could also further relax this hypothesis.

Lemma A.4 (**Generalization of Theorem 4.7**). *Let* Ω *and* $\mathbf{u}$ *satisfy the generalized condition* I *and let* $\Phi \in \mathrm{semicart}(\mathrm{R}; \Gamma^-[\mathbf{u}], \Gamma^+[\mathbf{u}])$, $\Phi(t,s) := (t, \phi(t,s))$. *Then there exists a sequence* $(\mathbf{u}_h) \subset H^1(\Omega; \mathbb{R}^2)$ *converging to* $\mathbf{u}$ *in* $L^1(\Omega; \mathbb{R}^2)$ *and uniformly out of* $J_{\mathbf{u}}$ *such that*

$$\lim_{h \to +\infty} \mathcal{A}(\mathbf{u}_h, \Omega) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})| \, dx \, dy + \int_{\mathrm{R}} |\partial_t \Phi \wedge \partial_s \Phi| \, dt \, ds.$$

Since the proof of this second lemma is almost the same of that of Lemma A.2, we prove the result only for Ω and $\mathbf{u}$ satisfying the generalized condition II.

Proof of Lemma A.2. Since δ depends only on $J_{\mathbf{u}}$, we can suppose without loss of generality[(4)] that $\delta = 1$, and thus that $\mathrm{R}_\delta = \mathrm{R}$; moreover we assume Ω so that $\Lambda(\mathrm{R}) \subset \Omega$. Thanks to Lemma 2.12, we also require that $|\sigma^\pm(t)| < 1$, for every $t \in (a,b)$, where $D = [[\sigma^-, \sigma^+]]$ as in Definition 2.8. For any $\varepsilon \in (0,1)$ let us set $\mathrm{R}_\varepsilon := (a,b) \times (-\varepsilon, \varepsilon)$, and $D_\varepsilon := [[\varepsilon\sigma^-, \varepsilon\sigma^+]]$.
Let $(\varepsilon_h) \subset (0,1)$ be an infinitesimal sequence and let us define $\mathbf{u}_h$ as

$$\mathbf{u}_h(x,y) := \begin{cases} \mathbf{u}(x,y) & (x,y) \in \Omega \setminus \Lambda(\mathrm{R}_{\varepsilon_h}), \\ \mathbf{u}\left(Q_h(x,y)\right) & (x,y) \in \Lambda(\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h}), \\ \phi\left(t(x,y), \frac{s(x,y)}{\varepsilon_h}\right) & (x,y) \in \Lambda(D_{\varepsilon_h}), \end{cases}$$

where $Q_h := \Lambda \circ T_h \circ \Lambda^{-1}$ with T_h defined in (4.8) (with h in place of k), see Figure A.1.

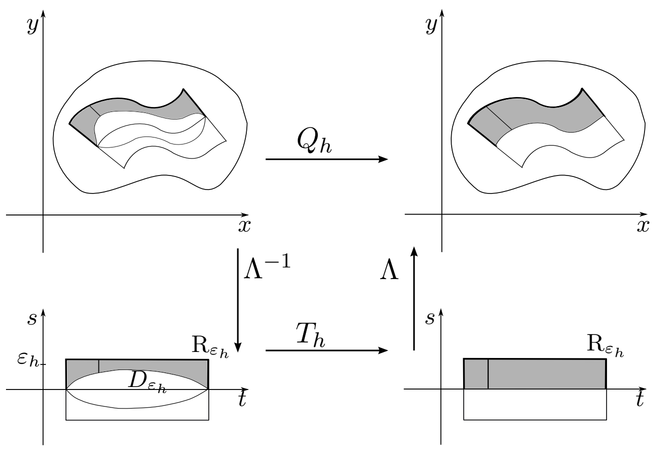

Figure A.1: The action of the map Q_h on the set $\Lambda((\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h}) \cap \{s > 0\})$. Any oblique small segment on the top left is mapped in the parallel longer segment reaching the fracture, on the top right. Observe that Q_h is the identity on $\partial(\Lambda(\mathrm{R}_{\varepsilon_h}))$.

We observe that $(\mathbf{u}_h)$ converges to $\mathbf{u}$ in $L^1(\Omega; \mathbb{R}^2)$ and uniformly out of $J_{\mathbf{u}}$. In order to prove (A.1), let us estimate $\mathcal{A}(\mathbf{u}_h, \Omega)$ on the three regions $\Omega \setminus \Lambda(\mathrm{R}_{\varepsilon_h})$, $\Lambda(\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h})$, and $\Lambda(D_{\varepsilon_h})$.

(4) Otherwise it is enough to replace, in what follows, R by $\mathrm{R}_\delta := (a,b) \times (-\delta, \delta)$ and R_ε by $\mathrm{R}_{\varepsilon\delta} := (a,b) \times (-\varepsilon\delta, \varepsilon\delta)$.

- Due to the regularity of **u**, we can prove, as in Theorem 4.9, that

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \Omega \setminus \Lambda(\mathrm{R}_{\varepsilon_h})) = \lim_{h\to+\infty} \mathcal{A}(\mathbf{u}, \Omega \setminus \Lambda(\mathrm{R}_{\varepsilon_h})) = \int_\Omega |\mathcal{M}(\nabla \mathbf{u})|\, dx\, dy.$$

- On the intermediate region $\Lambda(\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h})$ the area contribution is negligible. Indeed, denoting by $Q_{h,1}$ and $Q_{h,2}$ the components of the map Q and by ∂_1 and ∂_2 the derivatives with respect to the first and the second variable, we get

$$\begin{aligned}\partial_x \mathbf{u}_h &= \partial_1 \mathbf{u}\, \partial_x Q_{h,1} + \partial_2 \mathbf{u}\, \partial_x Q_{h,2},\\ \partial_y \mathbf{u}_h &= \partial_1 \mathbf{u}\, \partial_y Q_{h,1} + \partial_2 \mathbf{u}\, \partial_y Q_{h,2}.\end{aligned}$$

Observing that

$$\nabla Q_h = \nabla \Lambda(T_h(t(x,y), s(x,y))^T \nabla T_h(t(x,y), s(x,y)) \nabla \Lambda^{-1}(x,y),$$

and recalling that both the components of the Jacobian of Λ and the components of the Jacobian of T_h are bounded, we obtain

$$\mathcal{A}(\mathbf{u}_h, \Lambda(\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h})) \le C\mathcal{L}^2(\Lambda(\mathrm{R}_{\varepsilon_h} \setminus D_{\varepsilon_h})) \stackrel{h\to+\infty}{\longrightarrow} 0.$$

- Finally we want to prove that

$$\lim_{h\to+\infty} \mathcal{A}(\mathbf{u}_h, \Lambda(D_{\varepsilon_h})) = \int_D |\partial_t \Phi \wedge \partial_s \Phi|\, dt\, ds.$$

We observe that

$$\begin{aligned}\partial_x \mathbf{u}_h &= \partial_t \phi\, \partial_x t + \frac{1}{\varepsilon_h} \partial_s \phi\, \partial_x s,\\ \partial_y \mathbf{u}_h &= \partial_t \phi\, \partial_y t + \frac{1}{\varepsilon_h} \partial_y \phi\, \partial_x s,\end{aligned}$$

where t and s are evaluated in (x,y) while $\partial_t\phi$ and $\partial_s\phi$ in $\left(t(x,y), \frac{s(x,y)}{\varepsilon_h}\right)$. By explicit computation we get that

$$\begin{aligned}|\mathcal{M}(\nabla_{(x,y)}\mathbf{u}_h)|^2 = 1 &+ |\partial_t\phi|^2 |\nabla_{(x,y)} t|^2 + \frac{2}{\varepsilon_h}(\partial_t\phi \cdot \partial_s\phi)(\nabla_{(x,y)} t \cdot \nabla_{(x,y)} s)\\ &+ \frac{1}{\varepsilon_h^2}\left[|\partial_s\phi|^2 |\nabla_{(x,y)} s|^2 + (\det \nabla_{(t,s)}\phi)^2 |\nabla_{(x,y)} t \cdot \nabla_{(x,y)} s^\perp|^2\right]\end{aligned} \tag{A.2}$$

where again ϕ and its derivatives are computed in $\left(t(x,y), \frac{s(x,y)}{\varepsilon_h}\right)$, and we have explicitly indicate whether the Jacobian is with respect to the variables (x,y) or (t,s). In what follows we use the symbol ∇ both for $\nabla_{(x,y)}$ and $\nabla_{(t,s)}$.

We recall that $|\nabla s|^2 = |\nabla d|^2 = 1$, since d is the signed distance function from $J_{\mathbf{u}}$ and thus it satisfies the eikonal equation, and that $|\nabla t|^2$ is uniformly bounded with respect to h, since $J_{\mathbf{u}}$ is of class $\mathcal{C}^2$.

Notice that if $(x, y) \in \Lambda(D_{\varepsilon_h})$ then the vector $\nabla d^{\perp}(x, y) = \nabla d^{\perp}(\pi(x, y))$ is tangent to $J_{\mathbf{u}}$ at $\pi(x, y)^{(5)}$, and has unit length. In addition, t is constant along the normal direction to $J_{\mathbf{u}}$, so that if $(x, y) \in \Lambda(D_{\varepsilon_h})$ then $\nabla t(x, y) = \nabla t(\pi(x, y)) + \mathcal{O}(\varepsilon_h)$, and $\nabla t(\pi(x, y))$ is also tangent to $J_{\mathbf{u}}$, where

$$|\mathcal{O}(\varepsilon)| \leq c\|\kappa\|_{L^\infty(J_{\mathbf{u}})} \max_{t\in[a,b]} \varepsilon_h(\sigma^+ - \sigma^-),$$

κ being the curvature of $J_{\mathbf{u}}$, for a positive constant c independent of ε_h.

Since α is an arc-length parametrization of $J_{\mathbf{u}}$, it follows that $|\nabla t| = 1$ on $J_{\mathbf{u}}$. Therefore $\nabla t \cdot \nabla s^{\perp} = 1 + \mathcal{O}(\varepsilon_h)$ on $\Lambda(D_{\varepsilon_h})$.

Thus from (A.2) we get, on $\Lambda(D_{\varepsilon_h})$

$$\begin{aligned} |\mathcal{M}(\nabla \mathbf{u})|^2 &= 1 + |\partial_t\phi|^2|\nabla t|^2 + \frac{2}{\varepsilon_h}(\partial_t\phi \cdot \partial_s\phi)(\nabla t \cdot \nabla s) \\ &\quad + \frac{1}{\varepsilon_h^2}\left[|\partial_s\phi|^2 + (\det\nabla\phi)^2(1 + \mathcal{O}(\varepsilon_h))\right] \\ &= \sqrt{G_0 + \frac{2}{\varepsilon_h}G_1 + \frac{1}{\varepsilon_h^2}G_2}; \end{aligned}$$

The area formula implies that

$$\begin{aligned} \mathcal{A}(\mathbf{u}_h, \Lambda(D_{\varepsilon_h})) &= \int_{\Lambda(D_{\varepsilon_h})} \sqrt{G_0 + \frac{2}{\varepsilon_h}G_1 + \frac{1}{\varepsilon_h^2}G_2}\, dx\, dy \\ &= \int_{D_{\varepsilon_h}} \sqrt{\widehat{G}_0 + \frac{2}{\varepsilon_h}\widehat{G}_1 + \frac{1}{\varepsilon_h^2}\widehat{G}_2}|\det(\nabla\Lambda)|\, dt\, ds, \end{aligned}$$

where $\widehat{G}_i$, i=0,1,2, equals G_i with (x, y) replaced by $\Lambda^{-1}(x, y) = (t, s)$; in particular ϕ is evaluated in $\left(t, \frac{s}{\varepsilon_h}\right)$. Remember also that $|\det(\nabla\Lambda)| = |1 - \kappa s|$, κ being the curvature of $J_{\mathbf{u}}$. Making the change of variable $s/\varepsilon_h \to s$ we finally get

$$\begin{aligned} \mathcal{A}(\mathbf{u}_h, \Lambda(D_{\varepsilon_h})) &= \int_D \sqrt{\mathcal{O}(\varepsilon_h) + |\partial_s\phi|^2 + (\det\nabla\phi)^2(1 + \mathcal{O}(\varepsilon_h))}|1 - \varepsilon_h\kappa s|\, dt\, ds \\ &\overset{h\to+\infty}{\longrightarrow} \int_D |\partial_t\Phi \wedge \partial_s\Phi|\, dt\, ds. \end{aligned}$$

□

(5) π denotes the projection on $J_{\mathbf{u}}$.

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E in ultimo, GRAZIE a Matteo. Trieste é stata la nostra prima casa. E discutere questa tesi é un po' come prendere in mano le chiavi della prossima.

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