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# Non-linear Schrödinger equations with singular perturbations and with rough magnetic potentials

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# Declaration

Il presente lavoro costituisce la tesi presentata da Raffaele Scandone, sotto la direzione del Prof. Alessandro Michelangeli, al fine di ottenere l'attestato di ricerca post-universitaria Doctor Philosophiae presso la SISSA, Curriculum in Analisi Matematica, Modelli e Applicazioni, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attesto é equipollente al titolo di Dottore di Ricerca in Matematica.

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# Abstract

In this thesis we discuss thoroughly a class of linear and non-linear Schrödinger equations that arise in various physical contexts of modern relevance.

First we work in the scenario where the main linear part of the equation is a singular perturbation of a symmetric pseudo-differential operator, which formally amounts to add to it a potential supported on a finite set of points.

A detailed discussion on the rigorous realisations and the main properties of such objects is given when the unperturbed pesudo-differential operator is the fractional Laplacian on $\mathbb{R}^d$.

We then consider the relevant special case of singular perturbations of the three-dimensional non-fractional Laplacian: we qualify their smoothing and scattering properties, and characterise their fractional powers and induced Sobolev norms.

As a consequence, we are able to establish local and global solution theory for a class of singular Schrödinger equations with convolution-type non-linearity.

As a second main playground, we consider non-linear Schrödinger equations with time-dependent, rough magnetic fields, and with local and non-local non-linearities.

We include magnetic fields for which the corresponding Strichartz estimates are not available. To this aim, we introduce a suitable parabolic regularisation in the magnetic Laplacian: by exploiting the smoothing properties of the heat-Schrödinger propagator and the mass/energy bounds, we are able to construct global solutions for the approximated problem.

Finally, through a compactness argument, we can remove the regularisation and deduce the existence of global, finite energy, weak solutions to the original equation.

# Contents

# Introduction and Overview

This thesis is devoted to the study of a class of Schrödinger equations, which naturally emerge in various physical contexts, and whose investigation has led to the development of deep tools in various mathematical fields, such as functional analysis, spectral theory, and harmonic analysis.

A classical example is the pure-power non-linear Schrödinger equation

$$\mathrm{i}\partial_t u = -\Delta_x u + |u|^{\gamma-1}u, \tag{0.1}$$

in the complex-valued unknown $u \equiv u(t,x)$, where $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$ are interpreted, respectively, as time and space variables.

In three spatial dimension, (0.1) is the effective evolution equation of an interacting Bose gas, and as such it can be derived in suitable scaling limits of infinitely many particles [**20, 81, 101**]: in this context the $|u|^{\gamma-1}u$ term with $\gamma = 3$ (resp., $\gamma = 5$) arises as the self-interaction term due to a two-body (resp., three-body) inter-particle interaction of short scale. On the other hand, (0.1) appears also in the study of small amplitude gravity waves, dynamics of quantum plasmas, non-linear optical fibers, and planar wave-guides.

From a purely analytical perspective, equation (0.1) has been studied extensively, and nowadays its local and global well-posedness, as well its long-time behavior are fully understood [**25, 50, 26, 27, 37**].

More generally, one can consider an equation of the form

$$\mathrm{i}\partial_t u = Lu + \mathcal{N}(u), \tag{0.2}$$

in the complex-valued unknown $u \equiv u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, where $L$ is a time-independent linear $L^2(\mathbb{R}^d)$-symmetric operator, and $\mathcal{N}(u)$ is a non-linear term.

In applications to quantum mechanics, typically $L$ arises as the Hamiltonian associated to the total energy of a quantum particle. A relevant example is $L = -(\nabla - \mathrm{i}A)^2 + V$, where $A : \mathbb{R}^d \to \mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ are interpreted, respectively, as external magnetic and electric potentials. Another important case is $L = \sqrt{-\Delta + m^2}$, which describes a semi-relativistic quantum particle of mass $m$. In order to provide an effective and unambiguous description of the physical system of interest, a fundamental requirement is to realise $L$ as a bounded below, self-adjoint operator on $L^2(\mathbb{R}^d)$.

Concerning the non-linear term, it naturally appears in many-body quantum systems as a self-interaction potential. Typical examples are the pure-power non-linearity $|u|^{\gamma-1}u$ discussed above, and the so called Hartree non-linearity $(w*|u|^2)u$, for some measurable $w : \mathbb{R}^d \to \mathbb{R}$, which arises as a two-body interaction of mean field type. In general, one assumes that the non-linearity satisfies the condition $\mathfrak{Im}\big(\mathcal{N}(u)\overline{u}\big) = 0$, which ensures that the *mass* $\mathcal{M}(u) := \|u\|^2_{L^2(\mathbb{R}^d)}$ is formally conserved in time along the solutions to (0.2).

A fundamental step in the study of equation (0.2) is to have a good control of the linear problem.

When $L$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$, it is the infinitesimal generator of the strongly continuous, one-parameter, unitary group $\{e^{-\mathrm{i}tL}\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$. It is natural to investigate the local and global smoothing effect of such unitary flow. In the case of the free Laplacian, we have the following fundamental result, known as *dispersive estimates*:

$$
\begin{aligned}
&\|e^{\mathrm{i}t\Delta} f\|_{L^p(\mathbb{R}^d)} \leqslant C\,|t|^{-\frac{d}{2}\left(1-\frac{2}{p}\right)}\|f\|_{L^{p'}(\mathbb{R}^d)}\,, \quad t \neq 0\,,\\
&\frac{1}{p}+\frac{1}{p'}=1, \quad p\in[2,+\infty].
\end{aligned} \tag{0.3}
$$

Estimates (0.3) provide, for a fixed time $t$, a non trivial gain of integrability for a solution of the free Schrödinger equation. Furthermore, the group $\{e^{\mathrm{i}t\Delta}\}_{t\in\mathbb{R}}$ exhibits a remarkable global-in-time smoothing effect, as shown by the so called *Strichartz estimates*:

$$
\begin{aligned}
&\|e^{\mathrm{i}t\Delta} f\|_{L^q(\mathbb{R}_t,L^p(\mathbb{R}^d_x))} \leqslant C\|f\|_{L^2(\mathbb{R}^3)}\\
&\frac{2}{q}+\frac{d}{p}=\frac{d}{2}, \qquad \begin{cases} p\in[2,+\infty] & d=1\\ p\in[2,+\infty) & d=2\\ p\in[2,\frac{2d}{d-2}] & d\geqslant 3.\end{cases}
\end{aligned} \tag{0.4}
$$

In dimension $d=1,2$, as well as in the *non-endpoint* case $(q,p)\neq(2,\frac{2d}{d-2})$ in dimension $d\geqslant 3$, (0.4) follows by the dispersive estimates (0.3) by means of a duality argument and fractional integration [**49, 110**]. The proof of the endpoint case in dimension $d\geqslant 3$ is more involved, and it was achieved by Keel and Tao [**70**] using a suitable atomic decomposition technique. Strichartz estimates are a crucial tool in the proof of local well-posedness of the non-linear Schrödinger equation (0.1). It is natural to ask whether similar esimates can be proved also for the unitary group generated by the self-adjoint operator $L$. There is a vast literature on this topic, in particular for Schrödinger operators of the form $L=-(\nabla-\mathrm{i}A)^2+V$ [**67, 111, 96, 52, 41, 94, 64, 51, 87, 113, 31, 32, 39, 40, 47, 104, 116**], and spectral properties of $L$ are known to play an important role.

Aiming at investigating the non-linear equation (0.2), another fundamental problem is to determine a class of Banach spaces which are invariant by the linear flow $e^{-\mathrm{i}tL}$. Indeed, in suitable "perturbative" regimes, one expects a local well-posedness result to hold in such spaces also for the non-linear problem. If $L$ is a bounded below, self-adjoint operator on $L^2(\mathbb{R}^d)$, with bottom $m(L)$, a natural class of invariant Banach spaces is provided by the following construction. For a given $s\in\mathbb{R}$, and $\lambda>m(L)$, we consider the fractional Sobolev space $H^s_L(\mathbb{R}^d)$ *adapted* to $L$:

$$
H^s_L(\mathbb{R}^d):=\mathcal{D}((L+\lambda)^{s/2}), \qquad \|f\|_{H^s_L(\mathbb{R}^d)}:=\|(L+\lambda)^{s/2}f\|_{L^2(\mathbb{R}^d)}, \tag{0.5}
$$

the case $s=0$ reproducing $L^2(\mathbb{R}^d)$. When $L$ is the self-adjoint Laplacian on $\mathbb{R}^d$, one recovers the classical Sobolev spaces $H^s(\mathbb{R}^d)$. It follows by basic results of spectral theory that

$$
\|e^{-\mathrm{i}tL}f\|_{H^s_L(\mathbb{R}^d)}=\|f\|_{H^s_L(\mathbb{R}^d)}, \qquad t\in\mathbb{R},\ f\in H^s_L(\mathbb{R}^d).
$$

Particularly relevant is the adapted *energy space* $H^1_L(\mathbb{R}^d)$, in which one defines the quadratic form $L[f]$ associated to $L$. Moreover, for typical non-linearities, including the pure-power and the Hartree (for suitable choices of the parameter $\gamma$ and the convolution potential $w$), it is possible to define in $H^1_L(\mathbb{R}^d)$ the energy functional associated to the non-linear equation (0.2). More precisely, if $\mathcal{N}$ is continuous from $H^1_L(\mathbb{R}^d)$ to $H^{-1}_L(\mathbb{R}^d)$, and $\mathcal{N}=\mathcal{P}'$ for some $\mathcal{P}:H^1_L(\mathbb{R}^d)\to\mathbb{R}$ of class $C^1$, then the *energy* $\mathcal{E}(u):=L[u]+\mathcal{P}(u)$ is well defined for $u\in H^1_L(\mathbb{R}^d)$, and it is formally conserved along the solutions of (0.2). When $\mathcal{P}(u)\geqslant 0$ for every $u\in H^1_L(\mathbb{R}^d)$, the non-linearity is called *defocusing*. Conservation of mass and energy is a key tool in order to construct global solutions to (0.2) in the adapted energy space. This is particularly clean in the case of defocusing non-linearities, as the quantity $\mathcal{M}(u)+\mathcal{E}(u)$ controls the $H^1_L(\mathbb{R}^d)$-norm of $u$.

Summarising so far, we have described a general scheme for the study of equation (0.2), which consists in the following main steps:

1. to realise $L$ as a bounded below, self-adjoint operator on $L^2(\mathbb{R}^d)$;
2. to characterise the adapted Sobolev space $H^s_L(\mathbb{R}^d)$;
3. to establish the smoothing properties of the unitary group generated by $L$, and to prove local well-posedness in $H^s_L(\mathbb{R}^d)$;
4. to exploit the conserved quantities of the equation in order to to extend, whenever it is possible, the solutions globally-in-time.

In this thesis we discuss in detail the case when $L$ is a *singular perturbation* of a pseudo-differential operator. In particular, we study the family of self-adjoint realisations of the fractional Laplacian $(-\Delta)^{s/2}$ perturbed with a 'pseudo-potential' virtually supported on a finite numbers of points in $\mathbb{R}^d$. In the physically relevant case of the three-dimensional Laplacian with point interactions we provide a thorough analysis of the adapted Sobolev spaces and of the smoothing properties of the associated unitary flow. As an application, we deduce local and global well-posedness results for a singular Schrödinger equation with Hartree non-linearity, which represents a fundamental step in order to investigate the effective dynamics of a many-body quantum system interacting with fixed impurities.

Another case taken into account is when $L$ is a three-dimensional Schrödinger operator with magnetic potentials, in a regime such that the adapted energy space is equivalent to $H^1(\mathbb{R}^3)$. In general, we deal with magnetic potentials for which the corresponding dispersive and Strichartz estimates are *not available* in the literature. To overcome this issue, we introduce in equation (0.2) a small regularisation term and we solve the approximating problem. This is achieved by obtaining suitable smoothing estimates for the linear dissipative evolution. The total mass and energy bounds allow one to extend the solutions globally in time. We then infer sufficient compactness properties in order to produce global-in-time finite energy weak solutions to the original equation. Our approach allows us to also consider the case of time-dependent magnetic potentials, which is significantly harder as one can not appeal to typical tools from functional analysis and operator theory.

An interesting research development is to combine the two cases discussed above, whence to study Schrödinger operators with a magnetic potential perturbed with a singular interaction.

## NLS with with singular potentials

A central topic in analysis and mathematical physics is the study of quantum systems subject to very short-range interactions, supported around a non-zero codimensional submanifold of the ambient space. A relevant situation occurs when the *singular* interaction is supported on a set of points in the Euclidian space $\mathbb{R}^d$. This leds to consider, formally, operators of the form

$$\text{“}-\Delta+\sum_{y\in Y}\mu_y\,\delta_y(\cdot)\text{”}, \tag{0.6}$$

where $Y$ is a discrete countable subset of $\mathbb{R}^d$, and $\mu_y$, $y\in Y$, are real coupling constants.

Heuristically, (0.6) is the Hamiltonian for a quantum particle moving under the influence of a “contact potentials”, created by “point sources” of strenghts $\mu_y$, located at $y$.

The first appearence of such Hamiltonians dates back to the celebrated paper of Kronig and Penney [**75**], who considered the case $d = 1$, $Y = \mathbb{Z}$ and $\mu_j$ independent on $y$ as a model for a non-relativistic electron moving in a fixed crystal lattice. Later, Bethe and Peierls [**23**] and Thomas [**107**], considered the case $d = 3$ and $Y = \{0\}$ as a model for a deuteron with idealized zero-range nuclear force between the nucleons, having introduced the center of mass and relative coordinates. In general such kind of models has found plenty of applications in nuclear, atomic, and solid state physic.

**Self-adjoint realisations.**

Following the general discussion above, one needs to rigorously realise the formal Hamiltonian (0.6) as a bounded below, self-adjoint operator on $L^2(\mathbb{R}^d)$. Since the interaction is supported at each point $y \in Y$, a natural approach is to consider the family of self-adjoint extensions of the restriction of $-\Delta$ to smooth functions supported away the centres of interactions. In dimension $d \geqslant 4$, the operator $(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^d \backslash Y)$ is essentially self-adjoint [**95**], hence (0.6) cannot be realised as a self-adjoint operator, except for the trivial case $\mu_y = 0$, $y \in Y$, which corresponds to an absence of point interactions, and one recovers the free Laplacian. In dimension $d = 1, 2, 3$, instead, there are infinitely many self-adjoint extension of $(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^d \backslash Y)$.

These classes of *Schrödinger operators* with *point interactions* are nowadays well known, since the first rigorous attempt [**21**] by Berezin and Faddeev, the seminal work of Albeverio, Fenstad, and Høegh-Krohn [**6**], and subsequent characterisation by many other authors [**6, 117, 56, 57, 29**] (see the monograph of Albeverio, Gesztesy, and Høegh-Krohn [**9**] and reference therein for a complete discussion).

The picture is particular clean in the case of one single point interaction, which can be assumed without loss of generality to be centred at the origin. In dimension $d = 3$, the non-negative symmetric operator $(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ has a one-parameter family of bounded below, self-adjoint extensions $\{-\Delta_\alpha\}_{\alpha \in \mathbb{R} \cup \infty}$, having introduced a natural parametrisations for which $\alpha = \infty$ corresponds to the free Laplacian on $L^2(\mathbb{R}^3)$. For $\alpha \in \mathbb{R}$, instead, an actual interaction occurs at the origin, and $-\Delta_\alpha$ is characterised by (see, e.g. [**9**, Chapter I.1])

$$
\begin{aligned}
\mathcal{D}(-\Delta_\alpha) \;&=\; \Big\{ g \in L^2(\mathbb{R}^3) \,\Big|\, g = F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, \mathsf{G}_\lambda \text{ with } F_\lambda \in H^2(\mathbb{R}^3) \Big\} \\
(-\Delta_\alpha + \lambda)\, g \;&=\; (-\Delta + \lambda)\, F_\lambda\,,
\end{aligned}
\tag{0.7}
$$

where $\lambda > 0$ is an arbitrarily fixed constant and

$$
\mathsf{G}_\lambda(x) \;:=\; \frac{e^{-\sqrt{\lambda}\,|x|}}{4\pi|x|}
\tag{0.8}
$$

is the Green function for the free Laplacian, that is, the distributional solution to $(-\Delta + \lambda)\mathsf{G}_\lambda = \delta$ in $\mathcal{D}'(\mathbb{R}^3)$. The decomposition in (0.7) is unique and is valid for every chosen $\lambda > 0$; it shows that a generic $g \in \mathcal{D}(-\Delta_\alpha)$ is the sum of a regular function and a more singular term, which is the signature of the singular interaction. Moreover, the two components are related by a boundary condition involving the evalution at zero of the regular part.

In dimension $d = 2$ the picture is completely analogous (see, e.g. [**9**, Chapter I.5]). There is a one-parameter family $\{-\Delta_\alpha\}_{\alpha \in \mathbb{R} \cup \infty}$ of self-adjoint realisations of the Hamiltonian of point interaction, among which $-\Delta_\infty$ is the free Laplacian. All others extensions are non-trivial, and a generic $g \in \mathcal{D}(-\Delta_\alpha)$ decomposes as the sum of an $H^2(\mathbb{R}^2)$-function and a singular term which exhibits a logarithmic divergence at the origin, according to the behavior of the two dimensional Green function of the Laplacian.

In dimension $d = 1$ the structure is more involved, for the symmetric operator $(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}\backslash\{0\})$ has a four-parameters family of self-adjoint extensions. Among these one finds the sub-family $\{-\Delta_\alpha\}_{\alpha\in\mathbb{R}\cup\infty}$, the analogous to those discussed in two and three dimensions. The other extensions are still such that a generic element of their domain decomposes as the sum of a regular and a singular term, however profound differences emerge:

- the singular term may diverge at the origin, even though the Green function of the one dimensional Laplacian is continuous;
- the boundary condition between the two components may also involve the evaluation at zero of the first derivative of the regular part, compatibly with the Sobolev embedding $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$.

In my recent work [**85**], in collaboration with A. Michelangeli and A. Ottolini, we consider the more general setting of *fractional* Schrödinger operators with a point interaction, that it, self-adjoint realisations on $L^2(\mathbb{R}^d)$ of the formal operator $(-\Delta)^{s/2} + \delta$, as well of its inhomogeneus variant $(\mathbb{1} - \Delta)^{s/2} + \delta$. Among other findings, our analysis allows us to give a rigorous interpretation to the Hamiltonian for a semi-relativistic quantum particle subject to a point-like impurity. We provided a detailed discussion on the existence and the properties of non-trivial self-adjoint extensions of the symmetric operator $(-\Delta)^{s/2} \upharpoonright C_0^\infty(\mathbb{R}\setminus\{0\})$, in terms of the dimension $d$ and of the fractional power $s$. In particular, we proved that the larger the number of classical derivatives allowed by Sobolev's embedding for $H^s(\mathbb{R}^d)$-functions, the richer and more complicated the structure of the class of self-adjoint extensions.

In the setting of a finite number $N$ of centres of interactions, the realisations of (0.6) exhibit additional features. In dimension $d = 2, 3$, the symmetric operator $(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^d\backslash Y)$ admits a $N^2$-parameter family of self-adjoint extensions, among which one distinguishes the $N$-parameter sub-family

$$\{-\Delta_{\alpha,Y} \bigm| \alpha \equiv (\alpha_1, \ldots, \alpha_N) \in (\mathbb{R} \cup \{\infty\})^N\}$$

of *local* extensions. In dimension $d = 1$, in addition to the analogous sub-family $\{-\Delta_{\alpha,Y}\}_{\alpha\in\mathbb{R}\cup\{\infty\}}$, there are a plethora of complicated self-adjoint extensions, which can mix a non-local behavior with a more involved structure of the singularities at the centres of interactions and of the boundary conditions relating them to the regular component.

Alternative approaches are possible in order to realise the formal Hamiltonian (0.6) as a self-adjoint operator on $L^2(\mathbb{R}^d)$, including local Dirichlet forms and non-standard analysis techniques (see [**9**]). One of particular interest is to obtain (0.6) as the limit of Schrödinger operators of the form

$$-\Delta + \sum_{y\in Y} V_\varepsilon^{(y)}(x - y),$$

where each potential $V_\varepsilon^{(y)}$, as $\varepsilon \to 0$, spikes up to create a delta-like profile, the support shrinking to the point $y$. This construction is nowadays well known and fully understoood in dimension $d = 1, 2, 3$, that is, all the dimensions in which non-trivial self-adjoint realisations exist (see [**9, 11**] for a comprehensive discussion). In my recent work [**86**], in collaboration with A. Michelangeli, we prove an analogous results in the more general setting of singular perturbations of the *fractional* Laplacian, in *arbitrary* dimension.

A relevant feature of Schrödinger operators with point interactions is that their resolvents have quite explicit and simple expressions. In particular, if $|Y| = N$, then the resolvent of $-\Delta_{\alpha,Y}$ is a rank-$N$ perturbation of the free resolvent. In a sense, Schrödinger operators with point interactions provide “solvable models” which approximate more realistic and complicated phenomena, governed by very short range interactions and whose study typically requires deep tools from perturbation theory.

**Adapted Sobolev spaces.**
Aiming at investigating non-linear problems of type (0.2), where the linear term $L$ is a singular perturbation of the Laplacian, one needs to extend the general scheme outlined above. Despite the solvable nature of Schrödinger operators with point interactions, the study of the corresponding adapted Sobolev spaces and of the smoothing properties of the associated unitary groups can be quite difficult, particularly in dimension three, even in the simplest case of a single point interaction centred at the origin.

As already commented, in three dimensions the domain of $-\Delta_\alpha$ exhibits a fairly complicated structure, which reflects on the behavior of the *singular* Sobolev spaces $H^s_\alpha(\mathbb{R}^3) := H^s_{-\Delta_\alpha}(\mathbb{R}^3)$, that is, the Sobolev space adapted to $-\Delta_\alpha$. In my recent work [**46**], in collaboration with V. Georgiev and A. Michelangeli, we provide an explicit characterisations of $H^s_\alpha(\mathbb{R}^3)$, where we restricted for the sake of concreteness to the case $\alpha \geqslant 0$ and $s \in [0,2]$. A considerably rich scenario emerges from our analysis, depending on the number $s$ of fractional derivatives.

- When $s \in (\frac{3}{2}, 2)$, $H^s_\alpha(\mathbb{R}^3)$ has the same structure of the operator domain (corresponding to the case $s = 2$), where now the regular part $F_\lambda$ belongs to $H^s(\mathbb{R}^3)$. The boundary value $F_\lambda(0)$ still make sense, owing to the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow C(\mathbb{R}^3)$.
- When $s \in (\frac{1}{2}, \frac{3}{2})$, a generic function in $H^s_\alpha(\mathbb{R}^3)$ still decomposes as the sum of a regular $H^s(\mathbb{R}^3)$-function and a singular part proportional to the Green function, but the link between the two components completely disappears.
- When $s \in [0, \frac{1}{2})$, the Green function itself belongs to $H^s(\mathbb{R}^3)$, which turns out to be equivalent to the singular Sobolev space $H^s_\alpha(\mathbb{R}^3)$.
- When $s = \frac{1}{2}$ or $s = \frac{3}{2}$, which are critical cases for the Sobolev embeddings, a characterisation is still available but is somewhat implicit.

**Smoothing and scattering properties.**
Concerning the global smoothing properties of the unitary group $\{e^{\mathrm{i}t\Delta_\alpha}\}_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^3)$, the analogous of dispersive and Strichartz estimates (0.3)-(0.4) cannot hold for the full range of exponents, as suggested by the typical local singularity of order $|x|^{-1}$ exhibited by a generic element in the domain of $-\Delta_\alpha$. In this respect, the reference work in the literature was the paper by D'Ancona, Pierfelice, and Teta [**33**], where the authors proved weighted $L^1 - L^\infty$ estimates, for a weight suitably chosen in order to compensate the local singularity. In my recent work [**61**], in collaboration with F. Iandoli, we proved that in the smaller regime $p \in [2,3)$, namely the range of $p$'s for which $\mathsf{G}_\lambda \in L^p(\mathbb{R}^3)$, $L^{p'} - L^p$ estimates hold without weights. As a consequence, we deduced also a class of Strichartz estimates for the dynamics generated by $-\Delta_\alpha$. Our proof is based on an explicit expression for the propagator $e^{\mathrm{i}t\Delta_\alpha}$, available in the case of a single centre of interaction [**100**], combined with a generalised Hausdorff-Young inequality.

A more general framework for the investigation of the dynamics generated by $-\Delta_\alpha$ through the Schrödinger equation $\mathrm{i}\partial_t u = -\Delta_\alpha u$ is provided by the study of the *wave operators* relative to the pair $(-\Delta_\alpha, -\Delta)$, which are defined as the strong limits

$$W^\pm_\alpha := \underset{t\to\pm\infty}{s\text{-}\lim}\, e^{-\mathrm{i}t\Delta_\alpha} e^{\mathrm{i}t\Delta}.$$

Since the resolvent of $-\Delta_\alpha$ is a finite-rank perturbation of the free resolvent, standard arguments from scattering theory [**97**] guarantee that the wave operators $W_\alpha^\pm$ exist in $L^2(\mathbb{R}^3)$ and are complete, meaning that

$$\operatorname{ran} W_\alpha^\pm \;=\; L^2_{\mathrm{ac}}(H_\alpha) \;=\; P_{\mathrm{ac}}(-\Delta_\alpha)L^2(\mathbb{R}^3)\,, \tag{0.9}$$

where $L^2_{\mathrm{ac}}(-\Delta_\alpha)$ denotes the absolutely continuous spectral subspace of $L^2(\mathbb{R}^3)$ for $-\Delta_\alpha$, and $P_{\mathrm{ac}}(-\Delta_\alpha)$ denotes the orthogonal projection onto $L^2_{\mathrm{ac}}(-\Delta_\alpha)$.

Wave operators are a fundamental tool for the study of the scattering governed by a "perturbed" Hamiltonian in comparison with a free "unperturbed" Hamiltonian [**76, 97**]. Owing to their completeness, $W_\alpha^+$ and $W_\alpha^-$ are unitary from $L^2(\mathbb{R}^3)$ onto $L^2_{\mathrm{ac}}(-\Delta_\alpha)$; moreover, they *intertwine* $-\Delta_\alpha P_{\mathrm{ac}}(-\Delta_\alpha)$ and $-\Delta$, viz., for any Borel function $f$ on $\mathbb{R}$ one has the identity

$$f(-\Delta_\alpha)P_{\mathrm{ac}}(-\Delta_\alpha) \;=\; W_\alpha^\pm\, f(-\Delta)\,(W_\alpha^\pm)^*\,. \tag{0.10}$$

Through such intertwining, the smoothing properties of $f(-\Delta)$ (which, upon Fourier transform, is the multiplication by $f(\xi^2)$) can be lifted to analogous properties for $f(-\Delta_\alpha)P_{\mathrm{ac}}(-\Delta_\alpha)$, provided that suitable mapping estimates of $W_\alpha^\pm$ are known.

In my recent work [**36**], in collaboration with G. Dell'Antonio, A. Michelangeli, and K. Yajima, we proved that $W_\alpha^\pm$ extend as bounded operators on $L^p(\mathbb{R}^3)$, for $p\in(1,3)$, but neither for $p=1$, nor $p\geqslant 3$. As a consequence of our result and of the intertwining formula (0.10), dispersive and Strichartz estimates for the propagator $e^{\mathrm{i}t\Delta_\alpha}$, in the regime $p\in(2,3]$, can be immediately recovered from the corresponding ones for the free Schrödinger dynamics.

Scattering theory allows one to also consider the more general case of Schrödinger operators with finitely many point interactions. The wave operators $W_{\alpha,Y}^\pm$ relative to the pair $(-\Delta_{\alpha,Y},-\Delta)$ still exist and are complete, as the resolvent of $-\Delta_{\alpha,Y}$ is a finite-rank perturbation of the free resolvent. However, the situation is more involved with respect to the single centre case. Indeed, it was recently proved by Cornean, Michelangeli, and Yajima [**28**], that for a sufficiently large number of centres, arranged under particular geometric configurations, and for "exceptional" choices of the parameter $\alpha$, the operator $-\Delta_{\alpha,Y}$ has a zero eigenvalue imbedded in the essential spectrum, a phenomen which does not occur in the single centre case. As is well known in the case of regular Schrödinger operators, a zero eigenvalue can be an obstruction for the bounded-mapping properties of the corresponding wave operators (see, e.g., Yajima [**113, 114, 115**], Erdoğan and Schlag [**41**], and Goldberg and Schlag [**52**]). In the "generic" case of absence of a zero eigenvalue, in [**36**] $L^p$-boundedness of the three-dimensional wave operators $W_{\alpha,Y}^\pm$ is proved for $p\in(1,3)$, and dispersive and Strichartz estimates in the same regime follow as consequence.

**NLS with point interaction.**

Having indentified the structure of singular Sobolev spaces and the smoothing properties of the linear evolution generated by Schrödinger operators with point interactions, one can finally approach non-linear problems whose linear part is a singular perturbation of the Laplacian.

In my recent work [**83**], in collaboration with A. Michelangeli and A. Olgiati, we consider the three-dimensional Schrödinger equation with a point interaction at the origin and a Hartree non-linearity. More precisely, for a given $s\in[0,2]$ and a real-valued convolution potential $w$, we consider the Cauchy problem

$$\begin{cases} \mathrm{i}\partial_t u \;=\; -\Delta_\alpha u + (w*|u|^2)u \\ u(0) \;=\; f \;\in\; H^s_\alpha(\mathbb{R}^3)\,. \end{cases} \tag{0.11}$$

We provide local well-posedness results both in a regime of low (i.e., $s \in [0, \frac{1}{2})$, intermediate (i.e., $s \in (\frac{1}{2}, \frac{3}{2})$), and high (i.e., $s \in (\frac{3}{2}, 2]$) regularity, under suitable integrability and regularity hypothesis for $w$. Then, exploiting the conservation of the mass and the energy, a *global* theory is deduced both in the mass space ($s = 0$) and in the energy space ($s = 1$).

Besides the relevance of our well-posedness results per se, they also represent the first fundamental step towards a rigorous derivation of the singuar Hartree equation as the effective equation for the dynamics of a Bose-Einstein condensate subject to a fixed impurity, in a mean-field type regime of two-body interaction between the particles.

A stricly related, yet more difficult problems is to obtain analogous results for pure power non-linearities.

Moreover, a challenging problem is the study non-linear Schrödinger equation with singular perturbation of the fractional Laplacian, which among the others requires a deep investigation of the scattering theory and of the smoothing properties of corresponding unitary propagator.

**NLS with magnetic fields**

As a second main playground, we consider a non-linear Schrödinger equation of the form

$$i\partial_t u = -(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \tag{0.12}$$

in the complex-valued unknown $u \equiv u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, where $\mathcal{N}(u)$ is a non-linear term, both of pure power and Hartree type, and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is interpreted as a magnetic potential.

The relevance of equation (0.12) is hard to underestimate, both for the interest it deserves per se, given the variety of techniques that have been developed for its study, and for the applications in various contexts in physics. Among the others, (0.12) arises as the effective evolution equation for the quantum dynamics of a Bose-Einstein condensate subject to an external magnetic field [**81, 90**], and for the dynamics of quantum plasmas [**60, 12**].

**Functional setting.**

With reference to the general scheme outlined previously, a first issue is to determine whether $-(\nabla - \mathrm{i}A)^2$ can be realised as a self-adjoint operator on $L^2(\mathbb{R}^3)$. A celebrated result by Leinfelder and Simander [**77**] asserts that, if $A \in L^4_{\mathrm{loc}}(\mathbb{R}^3)$ and $\operatorname{div} A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, then the non-negative, symmetric operator $-(\nabla - \mathrm{i}A)^2 \upharpoonright C_0^\infty(\mathbb{R}^3)$ is essentially self-adjoint on $L^2(\mathbb{R}^3)$, and therefore it admits a unique bounded below, self-adjoint extension.

A more general approach is available whenever $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ [**102**]. Under such assumption, for every $f \in L^2(\mathbb{R}^3)$ the *magnetic gradient* $(\nabla - \mathrm{i}A)f$ is well defined as a distribution on $\mathbb{R}^3$, and one can define the Banach space

$$\begin{aligned} H^1_A(\mathbb{R}^3) \;&:=\; \{ f \in L^2(\mathbb{R}^3) \mid (\nabla - \mathrm{i}\,A)f \in L^2(\mathbb{R}^3) \} \\ \|f\|^2_{H^1_A(\mathbb{R}^3)} \;&:=\; \|f\|^2_{L^2(\mathbb{R}^3)} + \|(\nabla - \mathrm{i}\,A)f\|^2_{L^2(\mathbb{R}^3)}, \end{aligned}$$

as well as the quadratic form

$$\mathcal{Q}_A[f,g] \;:=\; \int_{\mathbb{R}^3} \overline{(\nabla - \mathrm{i}\,A)f} \cdot (\nabla - \mathrm{i}\,A)g \,\mathrm{d}x\,, \qquad f,g \in H^1_A(\mathbb{R}^3).$$

$\mathcal{Q}_A$ is a closed and positive form on $L^2(\mathbb{R}^3)$, and its associated self-adjoint operator, denoted by $-\Delta_A$, is a realisation of the magnetic Laplacian $-(\nabla - \mathrm{i}A)^2$. It is worth noticing that the Banach space $H^1_A(\mathbb{R}^3)$ is equivalent to the adapted energy space for $-\Delta_A$. In general, $H^1_A(\mathbb{R}^3)$ has an involved structure, and it can even have trivial intersection with the classical Sobolev space $H^1(\mathbb{R}^3)$.

A fundamental tool in the context of magnetic Schrödinger operators is Kato's celebrated *diamagnetic inequality* [**78**]

$$|(\nabla|f|)(x)| \leqslant |((\nabla - \mathrm{i}\,A)f)(x)| \quad \text{for a.e. } x \in \mathbb{R}^3\,, \tag{0.13}$$

valid for any $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and $f \in H^1_A(\mathbb{R}^3)$. When $A \in L^p(\mathbb{R}^3)$, with $p \geqslant 3$, it follows from the diamagnetic inequality and the Sobolev embedding that the magnetic Sobolev space $H^1_A(\mathbb{R}^3)$ is equivalent to $H^1(\mathbb{R}^3)$.

In the same regime of $p$'s, global-in-time magnetic Strichartz estimates were established by various authors under suitable spectral assumptions (absence of zero-energy resonances) on the magnetic Laplacian $A$ [**39, 40, 31**], or alternatively under suitable smallness of the so called non-trapping component of the magnetic field [**32**]. Explicit counterxamples at critical scaling $|A(x)| \sim |x|^{-1}$ were obtained by Fanelli and Garcia [**42**], adapting to the magnetic case the Strichartz counterexample for eletric potential by Goldberg, Vega, and Visciglia [**53**].

Beyond the regime of Strichartz-controllable magnetic fields very few is known, despite the extreme topicality of the problem in applications with potentials $A$ that are rough, have strong singularities locally in space, and have a very mild decay at spatial infinity, virtually a $L^\infty$-behaviour. This generic case can be actually covered, and global well-posedness for (0.12) was indeed established by Michelangeli [**82**], by means of energy methods, as an alternative to the lack of magnetic Strichartz estimates.

However, such an approach requires the non-linearity to be locally Lipschitz in the energy space, and is only applicable to a suitable class of Hartree equations, power-type non-linearities being instead way less regular and hence escaping this method. The same feature indeed allows one to prove global well-posedness for the Maxwell-Schrödinger system in higher regularity spaces (see Nakamura and Wada [**89**]).

**The parabolic regularisation approach.**

In my recent work [**13**], in collaboration with P. Antonelli and A. Michelangeli, we consider (0.12) with a potential $A \in L^p(\mathbb{R}^3)$, $p > 3$, and with a non-linearity $\mathcal{N}$ which is continuous from $H^1(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^3)$. This way we cover quite an ample generality, including the scenario where neither are the external magnetic fields Strichartz-controllable, nor can the non-linearity be handled with energy methods.

Our approach consists of adding a small dissipation term in equation (0.12), which amount to consider the approximating problem

$$i\partial_t u \;=\; -(1-\mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u). \tag{0.14}$$

Similar parabolic regularisation procedures are commonly used in PDEs, see for example the vanishing viscosity approximation in fluid dynamics or in systems of conservation laws, and in fact this was also exploited in a similar context by Guo, Nakamitsu, and Strauss to study on the existence of finite energy weak solutions to the Maxwell-Schrödinger system [**59**].

By exploiting the parabolic regularisation, one can regard $-(1-\mathrm{i}\,\varepsilon)\Delta u$ as the main linear part and treat $(1-\mathrm{i}\,\varepsilon)(2\,\mathrm{i}\,A\cdot\nabla u + |A|^2 u) + \mathcal{N}(u)$ as a perturbation.

Evidently, this cannot be done in the original equation (0.12). Indeed, the term $A \cdot \nabla u$ is not a Kato perturbation of the free Laplacian and the whole derivative Schrödinger equation must be considered as the principal part [**104**].

One can instead establish the local well-posedness in the energy space $H^1(\mathbb{R}^3)$ for the approximated equation (0.14). The key step is to obtain suitable Strichartz-type and smoothing estimates for the regularised magnetic semi-group. This can be done by exploiting the smoothing effect of the heat-Schrödinger semi-group $t \mapsto e^{(i+\varepsilon)t\Delta}$ and by inferring the same space-time bounds also for the regularised magnetic evolution, in a similar fashion as in Naibo and Stefanov [**88**], and Yajima [**116**], where scalar (electric) potentials are treated as perturbations of the free Schrödinger evolution.

In the case of a defocusing non-linearity, the a priori bounds on the mass and the energy allow us to extend the solution of the regularised problem globally in time. Moreover, the mass/energy bounds turn out to be uniform in the regularising parameter $\varepsilon > 0$. By means of a compactness argument, it is then possible to remove the regularisation and to show the existence of a global, finite energy weak solution to the original problem (0.12), at the obvious price of loosing the uniqueness, as well as its continuous dependence on the initial data.

The parabolic regularisation technique is quite an efficient tool in the context of semi-linear PDE's, and in [**13**] also time-dependent magnetic fields are taken into account. More precisely, for a class of potentials $A \in AC_{\mathrm{loc}}(\mathbb{R}, L^p(\mathbb{R}^3))$, $p > 3$, and of defocusing non-linearities, global existence of finite energy, weak solutions to equation (0.12) is proved. An interesting and challenging open question is to cover also the endpoint case $p = 3$.

## Structure of the thesis

In Chapter 1 we study the rigorous construction and the main properties of singular perturbations of fractional powers of the Laplacian. We first consider three-dimensional Schrödinger operators with finitely many point interactions, with a particular emphasis on their spectral properties. Then we discuss the fractional case in arbitrary dimension.

Chapter 2 is devoted to the realisation of singular perturbations of the fractional Laplacian as limit of regular operators with regular potentials spiking up to a delta-like profile and shrinking around the centres of interactions. After presenting the general strategy, we treat in details the 3D and the 1D cases.

The time-dependent scattering theory for three-dimensional Schrödinger operators with point interactions is the object of Chapter 3. We provide a suitable integral representation of the *singular* wave operators relative to the pair $(-\Delta_{\alpha,Y}, -\Delta)$, based on explicit resolvent formula for $-\Delta_{\alpha,Y}$. Then we appeal to the theory of Calderón-Zygmund singular integrals in order to deduce the $L^p$-boundedness of the singular wave operators, for $p \in (1,3)$, and we prove as well that such range is optimal. Last, we compare the wave operators of $-\Delta_\alpha$ with those relative to the corresponding approximating Schrödinger operators.

In Chapter 4 we study the smoothing effect of the unitary evolution generated by a Schrödinger operator with point interactions. We first discuss weighted $L^1 - L^\infty$ estimates. Then, as a consequence of the $L^p$-boundedness of the singular wave operators, we prove non-weighted dispersive and Strichartz estimates in a suitable regime of exponents. We also provide an alternative, simpler proof in the single centre case.

Chapter 5 is devoted to the study of the singular Sobolev space $H^s_\alpha(\mathbb{R}^3)$, that is, the adapted Sobolev space for $-\Delta_\alpha$. We provide an explicit characterisation for $s \in [0,2]$, as well as the mutual control with the classical Sobolev norm. We provide also useful formula for the action of the fractional powers on a generic element of $H^s_\alpha(\mathbb{R}^3)$.

In Chapter 6 we establish a solution theory the singular Hartree equation on $\mathbb{R}^3$. We provide, for every $s \in [0,2]$, local well posedness result under suitable regularity assumption for $w$. Moreover, exploiting conservation of mass and energy, we also provide a global theory in $L^2(\mathbb{R}^3)$ and in $H^1_\alpha(\mathbb{R}^3)$.

Last, in Chapter 7 we study the non-linear magnetic Schrödinger equation in $\mathbb{R}^3$, for a wide class of magnetic potentials that are not Strichartz-controllable. Using a parabolic regularisation technique, we are able to prove existence of global, finite energy, weak solutions in $H^1(\mathbb{R}^3)$.

# Notation

We write $\mathbb{C}$ for the complex plane and $\mathbb{C}^+$ for the open upper half plane. For $z \in \mathbb{C}\setminus[0,+\infty)$, $\sqrt{z}$ is chosen in $\mathbb{C}^+$. By $\delta_{j,\ell}$ we denote the Kronecker delta, namely the quantity 1 for $j=\ell$ and 0 otherwise. We shall write $\langle\lambda\rangle \equiv (1+\lambda^2)^{\frac{1}{2}}$ for $\lambda\in\mathbb{R}$. For $u,v \in L^2(\mathbb{R}^3)$, we shall write $|u\rangle\langle v|$ to denote the rank-1 operator $f\mapsto u\langle v,f\rangle$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $L^2(\mathbb{R}^3)$, anti-linear in the first entry and linear in the second. Given $p\in[1,+\infty]$, we denote by $p'$ its Hölder conjugate exponent, defined via $p^{-1}+{p'}^{-1}=1$. For sequences and convergence of sequences, we write $(u_n)_n$ and $u_n\to u$ for $(u_n)_{n\in\mathbb{N}}$ and $u_n\to u$ as $n\to+\infty$.

For an operator $T$ on a Hilbert space, $\mathcal{D}(T)$ denotes its operator domain. When $T$ is self-adjoint, $\mathcal{D}[T]$ denotes its form domain and $E^{(T)}(\mathrm{d}\lambda)$ denotes its spectral measure. We shall denote by $\mathbb{1}$, resp., by $\mathbb{O}$, the identity and the null operator on any of the considered Hilbert spaces.

We shall indicate the Fourier transform by $\widehat{\phi}$ or $\mathcal{F}\phi$ with the convention

$$\widehat{\phi}(p) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\mathrm{i}px}\phi(x)\mathrm{d}x.$$

For two positive quantities $P$ and $Q$, we write $P\lesssim Q$ to mean that $P\leqslant CQ$ for some positive constant $C$ independent of the variables or of the parameters which $P$ and $Q$ depend on, unless explicitly declared; in the latter case we write, self-explanatorily, $P\lesssim_\alpha Q$, and the like. We write $f\leqslant_{|\,\cdot\,|} g$ when $|f|\leqslant|g|$.

We use the symbols div, $\nabla$ and $\Delta$ to denote derivations in the spatial variables only. When referring to a vector field $A:\mathbb{R}^3\to\mathbb{R}^3$, conditions like $A\in L^p(\mathbb{R}^3)$ are to be understood as $A\in L^p(\mathbb{R}^3,\mathbb{R}^3)$.

CHAPTER 1

# Singular perturbations of the fractional Laplacian

In this Chapter we study the rigorous construction and the main properties of singular perturbations of fractional powers of the Laplacian.

For given $d \in \mathbb{N}$ and $s \in \mathbb{R}$, the $d$-dimensional *fractional Laplacian* $(-\Delta)^{s/2}$ can be defined via functional calculus as a non-negative self-adjoint operator on $L^2(\mathbb{R}^d)$, with domain $H^s(\mathbb{R}^d)$. Its action is obvious in terms of the corresponding power of the Fourier multiplier for $-\Delta$:

$$((-\Delta)^{s/2} f)(x) \;=\; (|p|^s \widehat{f})^\vee(x) \qquad x, p \in \mathbb{R}^d.$$

A singular perturbation of the fractional Laplacian heuristically amounts to add to it a potential virtually supported at a finite numbers of points in $\mathbb{R}^d$.

A rigorous realisation can be atteined using a restriction-extension argument: first one restricts $(-\Delta)^{s/2}$ to sufficiently smooth functions supported away from the centres of interaction, and then one builds an operator extension of such restriction that is self-adjoint on $L^2(\mathbb{R}^d)$. The extension step is based upon the classical theory of Kreĭn and von Neumann, whose basic facts are introduced below (for a detailed discussion we refer to [**95**, Chapter X]).

Let $S$ be a closed, densely defined, symmetric operator on a Hilbert space $\mathcal{H}$. Assume moreover that $S$ is bounded below, and let

$$m(S) := \inf \{\langle f, Sf \rangle \mid f \in \mathcal{D}(S)\,,\, \|f\|_{\mathcal{H}} = 1\}$$

its bottom.

(i) The quantity $\dim\ker\big(S^* + \lambda\mathbb{1}\big)$ is constant for $\lambda > -m(S)$. It is called the *deficiency index* of $S$, and we denote it by $\mathcal{J}(S)$.

(ii) If $\mathcal{J}(S) = 0$, then $S$ is self-adjoint. Otherwise, it admits infinitely many self-adjoint extensions, which can be parametrized by $\alpha \in (\mathbb{R} \cup \{\infty\})^{\mathcal{J}^2}$.

Among all the extensions, a distinguished one is the Friedrichs extension $S_F$, whose bottom coincides with the one of $S$, which is characterised by being the only self-adjoint extension whose domain is entirely contained in the form domain of $S$, and which has the property to be the largest among all self-adjoint extensions of $S$, in the sense of operator ordering for self-adjoint operators. An explicit and convenient way to characterise all the self-adjoint extensions of $S$ is provided by the Kreĭn-Višik-Birman theory (see, e.g., [**45**, Section 3], and references therein). In particular, once the Friedrichs extension $S_F$ is known, it is possible to associate, in a canonical way, every extension to a self-adjoint operator acting on a Hilbert subspace of $\ker\big(S^* + \lambda\mathbb{1}\big)$, for some $\lambda > -m(S)$.

In the non-fractional case, the literature on the self-adjoint realisations of singular perturbations is vast [**6, 117, 56, 57, 29**], see also the monograph [**9**] for a comprehensive discussion.

In the fractional setting the picture is much less devoleped. In my recent work [**85**], in collaboration with A. Michelangeli and A. Ottolini, we discuss in details the case of a single point interaction centred at the origin. More precisely, we exploit the Kreĭn-Višik-Birman scheme in order to classify all the self-adjoint extensions of the symmetric operator $(-\Delta)^{s/2} \upharpoonright C_0^\infty(\mathbb{R}^d\backslash\{0\})$.

Our discussion on the singular perturbations of the fractional Laplacian will have two main focus:

1. to qualify the nature of the perturbation in the resolvent sense;
2. to qualify the natural decomposition of the domain of the considered operators into a *regular component* and a *singular component*, and to determine the boundary condition constraining such two components.

The first issue is central for deducing an amount of properties from the unperturbed to the perturbed operators. The second issue also arises naturally, as one can see heuristically that the considered operators must act in an ordinary way on those functions supported away from the perturbation centres, and therefore their domains must contain a subspace of $H^s$-regular functions, where $s$ is the considered power, next to a more singular component that is the signature of the perturbation.

The Chapter is organised as follows. In Section 1 we consider singular perturbations of the three dimensional Laplacian. The fractional case in arbitrary dimension is discussed in Section 2.

## 1.1. Non-fractional case in dimension three

In this Section we study the self-adjoint realisations of singular perturbations of the three dimensional Laplacian. We start our analysis with the single-centre case, which is simpler but retains most of the the main ideas, then we consider the general multi-centre scenario. We discuss the basic features of the self-adjoint realisations, such as the decomposition of their operator and form domains into a regular and a singular part, and the explicit formulas for their resolvents. In the last Subsection, we introduce a low-energy expansion for the resolvent, and the important concept of *zero energy resonance*.

**1.1.1. One centre case.** It is not restrictive to fix the origin as the centre of interaction. Consider

$$\mathring{\mathsf{k}} := \overline{(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^3\backslash\{0\})} \tag{1.1}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$. It is a densely defined, closed, non-negative, symmetric operator on $L^2(\mathbb{R}^3)$.

Define, for $z \in \mathbb{C}$,

$$\mathcal{G}_z(x) := \frac{e^{\mathrm{i}z|x|}}{4\pi|x|}, \qquad x \in \mathbb{R}^3 . \tag{1.2}$$

A straightforward computation shows that $(-\Delta - z^2)\,\mathcal{G}_z = \delta$ holds as a distributional identity in $\mathbb{R}^3$. As a consequence, $\mathcal{G}_z$ is the convolution kernel of the resolvent of the free Laplacian, namely

$$(-\Delta - z^2\mathbb{1})^{-1} f = \mathcal{G}_z * f, \qquad \mathfrak{Im} z > 0,\ f \in L^2(\mathbb{R}^3). \tag{1.3}$$

Consider also, for $\lambda \geqslant 0$, the function $\mathsf{G}_\lambda$ defined by

$$\mathsf{G}_\lambda(x) := \mathcal{G}_{\mathrm{i}\sqrt{\lambda}} = \frac{e^{-\sqrt{\lambda}\,|x|}}{4\pi|x|} . \tag{1.4}$$

For a given $\lambda > 0$, one has

$$\ker\big(\mathring{\mathsf{k}}^* + \lambda\mathbb{1}\big) = \mathrm{span}\big\{\mathsf{G}_\lambda\big\}. \tag{1.5}$$

It follows that $\mathring{\mathsf{k}}$ has deficiency index one, hence it admits a one-parameter family of self-adjoint extensions $(-\Delta_\alpha)_{\alpha\in\mathbb{R}\cup\{\infty\}}$. They can be characterized by the following Theorem (see [**9**, I.1.1]).

THEOREM 1.1.1.

(i) *The extension* $-\Delta_{\alpha=\infty}$ *is the Friedrichs extension of* $\mathring{\mathsf{k}}$, *and is precisely the self-adjoint realization of* $-\Delta$ *on* $L^2(\mathbb{R}^3)$ *with domain* $H^2(\mathbb{R}^3)$. *All other extensions are given, for arbitrary* $\lambda > 0$, *with* $\sqrt{\lambda} \neq -4\pi\alpha$, *by*

$$\mathcal{D}(-\Delta_\alpha) \;=\; \Big\{ g \in L^2(\mathbb{R}^3) \,\Big|\, g = F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda \text{ with } F_\lambda \in H^2(\mathbb{R}^3) \Big\} \tag{1.6}$$

*and*

$$(-\Delta_\alpha + \lambda)\, g \;=\; (-\Delta + \lambda)\, F_\lambda\,, \tag{1.7}$$

*The above decomposition of a generic* $g \in \mathcal{D}(-\Delta_\alpha)$ *is unique and holds true for every chosen* $\lambda$. *The same formulas are valid also for* $\lambda = -z^2$, *with* $z \in \mathbb{C}$, $\mathfrak{Im} z > 0$, $z \neq -4\pi\alpha \mathrm{i}$.

(ii) *For each* $\alpha \in \mathbb{R}$, *the quadratic form of the extension* $-\Delta_\alpha$ *is given by*

$$\mathcal{D}[-\Delta_\alpha] \;=\; H^1(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\} \tag{1.8}$$

$$\begin{aligned}(-\Delta_\alpha)[F_\lambda + \kappa_\lambda \mathsf{G}_\lambda] \;=&\; \|\nabla F_\lambda\|^2_{L^2(\mathbb{R}^3)} - \lambda \|F_\lambda + \kappa_\lambda \mathsf{G}_\lambda\|^2_{L^2(\mathbb{R}^3)} \\ &+ \lambda \|F_\lambda\|^2_{L^2(\mathbb{R}^3)} + \big(\alpha + \tfrac{\sqrt{\lambda}}{4\pi}\big)|\kappa_\lambda|^2\end{aligned} \tag{1.9}$$

*for arbitrary* $\lambda > 0$, *with* $\sqrt{\lambda} \neq -4\pi\alpha$.

(iii) *For each* $\alpha \in \mathbb{R}$, *the resolvent of* $-\Delta_\alpha$ *is given by*

$$(-\Delta_\alpha + \lambda\mathbb{1})^{-1} \;=\; (-\Delta + \lambda\mathbb{1})^{-1} + \frac{1}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, |\mathsf{G}_\lambda\rangle\langle\overline{\mathsf{G}_\lambda}|\,. \tag{1.10}$$

*for arbitrary* $\lambda > 0$, $\sqrt{\lambda} \neq -4\pi\alpha$. *The same formula is valid also for* $\lambda = -z^2$, *with* $z \in \mathbb{C}$, $\mathfrak{Im} z > 0$, $z \neq -4\pi\alpha\mathrm{i}$.

(iv) *Each extension is semi-bounded from below, and*

$$\begin{aligned}&\sigma_{\mathrm{ess}}(-\Delta_\alpha) \;=\; \sigma_{\mathrm{ac}}(-\Delta_\alpha) \;=\; [0,+\infty)\,, \qquad \sigma_{\mathrm{sc}}(-\Delta_\alpha) \;=\; \emptyset\,, \\ &\sigma_{\mathrm{disc}}(-\Delta_\alpha) \;=\; \begin{cases} \emptyset & \text{if } \alpha \geqslant 0 \\ \{E_\alpha\} & \text{if } \alpha < 0\,, \end{cases}\end{aligned} \tag{1.11}$$

*where the eigenvalue* $E_\alpha := -(4\pi\alpha)^2$ *is simple, the (non-normalised) eigenfunction being* $\mathsf{G}_{|E_\alpha|}$.

It follows from (1.6) that a generic element $g \in \mathcal{D}(-\Delta_\alpha)$ decomposes as the sum of the $H^2$-function $F_\lambda$ and a less regular term. The Sobolev embedding guarantees the continuity of $F_\lambda$, whence the boundary condition for $g$ reads

$$g(x) \approx F_\lambda(0)(1 + (4\pi\alpha + \sqrt{\lambda})^{-1}|x|^{-1}) \qquad \text{as } x \to 0,$$

and hence also, owing to the arbitrariness of $\lambda > 0$,

$$g(x) \underset{x\to 0}{\sim} \frac{1}{4\pi} \cdot \Big(\frac{1}{|x|} - \frac{1}{a}\Big)\,, \qquad a := -(4\pi\alpha)^{-1}. \tag{1.12}$$

The latter condition is the short-range asymptotics typical of the low-energy bound state of a potential with almost zero support and $s$-wave scattering length $a$, as was first recognised by Bethe and Peierls [**23**], whence the name of *Bethe-Peierls contact condition*.

Clear consequences of (1.6)-(1.7) above are: on $H^2$-functions vanishing at $x = 0$ the operator $-\Delta_\alpha$ acts precisely as $-\Delta$; moreover, the only singularity that the elements of $\mathcal{D}(-\Delta_\alpha)$ may display at $x = 0$ is of the form $|x|^{-1}$. This suggests that $-\Delta g$ fails to be in $L^2(\mathbb{R}^3)$ by a distributional contribution removing which yields $-\Delta_\alpha g$. This is precisely what can be proved:

$$-\Delta_\alpha g \;=\; -\Delta g - \Big(\lim_{x\to 0} |x| g(x)\Big)\,\delta_0\,, \qquad g \in \mathcal{D}(-\Delta_\alpha)\,. \tag{1.13}$$

Identity (1.13) indicates that $-\Delta_\alpha g$ may be thought of a suitable *renormalisation* of $-\Delta g$: in fact, in the r.h.s. there is a difference of two distributions which gives eventually a $L^2$-function.

Another relevant form of the boundary condition for $g \in \mathcal{D}(-\Delta_\alpha)$ is available in Fourier transform. The following limit is finite

$$\xi \;=\; \lim_{R\to+\infty} \frac{1}{4\pi R}\int_{\substack{p\in\mathbb{R}^3\\ |p|<R}} \widehat{g}(p)\,\mathrm{d}p \tag{1.14}$$

and is customarily referred to as the *charge* of $g$, in terms of which one has the asymptotics

$$\int_{\substack{p\in\mathbb{R}^3\\ |p|<R}} \widehat{g}(p)\,\mathrm{d}p \;=\; 4\pi\xi\,(R+2\pi^2\alpha)+o(1) \qquad \text{as} \qquad R\to+\infty\,. \tag{1.15}$$

The latter is the so-called *Ter-Martyrosyan–Skornyakov condition*, originally identified by Ter-Martyrosyan and Skornyakov [**106**] (see also [**84**] for a recent discussion), and is in practice the Fourier counterpart of (1.12). One can show that imposing the Ter-Martyrosyan–Skornyakov condition at given $\alpha$ to the functions in the domain of the adjoint of $\mathring{\mathsf{k}}$ selects precisely $\mathcal{D}(-\Delta_\alpha)$. The action of $-\Delta_\alpha$ in Fourier transform reads

$$(\widehat{-\Delta_\alpha g})(p) \;=\; p^2\widehat{g}(p)-\xi \;=\; p^2\widehat{g}(p)-\lim_{R\to+\infty}\frac{1}{4\pi R}\int_{\substack{p\in\mathbb{R}^3\\ |p|<R}} \widehat{g}(p)\,\mathrm{d}p\,, \tag{1.16}$$

which is the Fourier counterpart of (1.13).

Moreover, the following equivalent characterisation of $-\Delta_\alpha$ has the virtue of showing explicitly that the two operators $-\Delta_\alpha$ and $-\Delta$ only differ on the subspace of spherically symmetric functions. The canonical decomposition

$$L^2(\mathbb{R}^3) \;\cong\; \bigoplus_{\ell=0}^{\infty} L^2(\mathbb{R}^+, r^2\,\mathrm{d}r)\otimes \mathrm{span}\{Y_{\ell,-\ell},\dots,Y_{\ell,\ell}\} \;\equiv\; \bigoplus_{\ell=0}^{\infty} L^2_\ell(\mathbb{R}^3) \tag{1.17}$$

(where the $Y_{\ell,m}$'s are the spherical harmonics on $\mathbb{S}^2$) reduces $-\Delta_\alpha$ and for each $\ell\geqslant 1$ one has $-\Delta_\alpha|_{L^2_\ell} = -\Delta|_{L^2_\ell}$. On the sector $\ell=0$, namely the Hilbert space

$$L^2_{\ell=0}(\mathbb{R}^3) \;=\; U^{-1}L^2(\mathbb{R}^+\,\mathrm{d}r)\otimes\mathrm{span}\Big\{\frac{1}{4\pi}\Big\}\,, \tag{1.18}$$

where $U : L^2(\mathbb{R}^+, r^2\,\mathrm{d}r) \xrightarrow{\cong} L^2(\mathbb{R}^+,\mathrm{d}r)$, $(Uf)(r) = rf(r)$, one has

$$-\Delta_\alpha|_{L^2_{\ell=0}} \;=\; (U^{-1}h_{0,\alpha}\,U)\otimes\mathbb{1}\,, \tag{1.19}$$

and $h_{0,\alpha}$ is self-adjoint on $L^2(\mathbb{R}^+\,\mathrm{d}r)$ with

$$\begin{aligned} h_{0,\alpha} \;&=\; -\frac{\mathrm{d}^2}{\mathrm{d}r^2} \\ \mathcal{D}(h_{0,\alpha}) \;&=\; \left\{ g\in L^2(0,+\infty) \;\middle|\; \begin{array}{c} g,g'\in AC_{\mathrm{loc}}((0,+\infty)) \\ g''\in L^2((0,+\infty)) \\ -4\pi\alpha\,g(0^+)+g'(0^+)=0 \end{array} \right\}. \end{aligned} \tag{1.20}$$

Let us analyse the quadratic form associated to $-\Delta_\alpha$, characterised by (1.8) and (1.9). Analogously to the operator domain, also for the functions in the form domain the highest local singularity is $|x|^{-1}$, since $\mathsf{G}_\lambda \in H^{\frac{1}{2}-}(\mathbb{R}^3)\setminus H^{\frac{1}{2}}(\mathbb{R}^3)$, while $F_\lambda\in H^1(\mathbb{R}^3)$. Instead, as typical when passing from the domain of a self-adjoint operator to its (larger) form domain, the characteristic boundary condition of $\mathcal{D}(-\Delta_\alpha)$ is lost in $\mathcal{D}[-\Delta_\alpha]$ and no constraint between regular and singular component remains (actually regular components of functions in $\mathcal{D}(-\Delta_\alpha)$ are not necessarily continuous).

Last, let us comment on the spectral properties of $-\Delta_\alpha$ identified in Theorem 1.1.1(iv). Identity (1.10) provides an explicit formula for the resolvent of $-\Delta_\alpha$, which turns out to be a rank-one perturbation of the free resolvent. As a consequence, the spectrum of $-\Delta_\alpha$ is completely characterized. In particular, only a simple negative eigenvalue can occur.

**1.1.2. Finitely many center in three dimension.** We fix a natural number $N \geqslant 1$ and the set $Y = \{y_1, \dots, y_N\} \subseteq \mathbb{R}^3$ of centres of the singular interactions. Consider

$$\mathring{\mathsf{k}}_Y \;:=\; \overline{(-\Delta) \upharpoonright C_0^\infty(\mathbb{R}^3 \backslash \{Y\})} \tag{1.21}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^3)$. It is a densely defined, closed, non-negative, symmetric operator on $L^2(\mathbb{R}^3)$.

For $x, y \in \mathbb{R}^3$, $z \in \mathbb{C}$, $\lambda \geqslant 0$, we set

$$\mathcal{G}_z^y(x) \;:=\; \mathcal{G}_z(x-y) = \;\frac{e^{\mathrm{i}z|x-y|}}{4\pi|x-y|} \tag{1.22}$$

$$\mathsf{G}_\lambda^y(x) := \mathcal{G}_{i\sqrt{\lambda}}^y(x) = \;\frac{e^{-\sqrt{\lambda}|x-y|}}{4\pi|x-y|}. \tag{1.23}$$

A straightforward computation shows that, for $\lambda > 0$,

$$\ker\big((\mathring{\mathsf{k}}_Y)^* + \lambda\mathbb{1}\big) \;=\; \mathrm{span}\big\{\mathsf{G}_\lambda^{y_1}, \dots \mathsf{G}_\lambda^{y_N}\big\}. \tag{1.24}$$

It follows that $\mathring{\mathsf{k}}_Y$ has deficiency index $N$. Hence, it admits a $N^2$-parameter family of self-adjoint extensions. The most relevant sub-class of them is the $N$-parameter family

$$\{-\Delta_{\alpha,Y} \;\big|\; \alpha \equiv (\alpha_1, \dots, \alpha_N) \in (\mathbb{R} \cup \{\infty\})^N\} \tag{1.25}$$

of so-called *local* extensions, namely extensions of $\mathring{\mathsf{k}}_Y$ whose domain of self-adjointness is only qualified by certain local boundary conditions at the singularity centres.

More precisely, we shall see that the elements of $\mathcal{D}(-\Delta_{\alpha,Y})$ satisfy, at each centre of the point interaction, the 'physical' Bethe-Peierls boundary condition introduced in (1.12) above, namely

$$g(x) \underset{x \to y_j}{\sim} \frac{q_j}{4\pi} \cdot \Big(\frac{1}{|x-y_j|} - \frac{1}{a_j}\Big), \qquad a_j := -(4\pi\alpha_j)^{-1}, \tag{1.26}$$

for suitable constants $q_j \in \mathbb{C}$.

If for some $j \in \{1, \dots, N\}$ one has $\alpha_j = \infty$, then no actual interaction is present at the point $y_j$ (no boundary condition as $x \to y_j$) and in practice things are as if one discards the point $y_j$. In particular, when $\alpha = \infty$, we recover the the Friedrichs extension of $\mathring{\mathsf{k}}_Y$, namely the self-adjoint realisation of $-\Delta$ on $L^2(\mathbb{R}^3)$.

Owing to the discussion above, we may henceforth assume, without loss of generality, that $\alpha$ runs over $\mathbb{R}^N$.

We review the basic properties of $-\Delta_{\alpha,Y}$, from [**9**, Section II.1.1] and [**93**] (see also [**34, 33**]). We introduce first some notation.

For $z \in \mathbb{C}$ and $x, y, y' \in \mathbb{R}^3$, set

$$\mathcal{G}_z^{yy'} \;:=\; \begin{cases} \dfrac{e^{\mathrm{i}z|y-y'|}}{4\pi|y-y'|} & \text{if } y' \neq y \\ \quad 0 & \text{if } y' = y\,, \end{cases} \tag{1.27}$$

and

$$\Gamma_{\alpha,Y}(z) \;:=\; \Big(\Big(\alpha_j - \frac{\mathrm{i}z}{4\pi}\Big)\delta_{j,\ell} - \mathcal{G}_z^{y_j y_\ell}\Big)_{j,\ell=1,\dots,N}. \tag{1.28}$$

Thus, the function $z \mapsto \Gamma_{\alpha,Y}(z)$ has values in the $N \times N$ symmetric matrices and is clearly entire, and $z \mapsto \Gamma_{\alpha,Y}(z)^{-1}$ is meromorphic in $z \in \mathbb{C}$. It is known that $\Gamma_{\alpha,Y}(z)^{-1}$ has at most $N$ poles in the open upper half-plane $\mathbb{C}^+$, which are all located along the positive imaginary semi-axis. We denote by $\mathcal{E}$ the set of such poles.

The following facts are known.

THEOREM 1.1.2.

(i) *The domain of $-\Delta_{\alpha,Y}$ has the following representation, for any $z \in \mathbb{C}^+ \backslash \mathcal{E}$:*

$$\mathcal{D}(-\Delta_{\alpha,Y}) \;=\; \Big\{ g = F_z + \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk}\, F_z(y_k) \mathcal{G}_z^{y_j} \,\Big|\, F_z \in H^2(\mathbb{R}^3) \Big\}\,. \tag{1.29}$$

*The summands in the decomposition of each $g \in \mathcal{D}(-\Delta_{\alpha,Y})$ depend on the chosen $z$, however, $\mathcal{D}(-\Delta_{\alpha,Y})$ does not. Equivalently, for any $z \in \mathbb{C}^+ \backslash \mathcal{E}$,*

$$\mathcal{D}(-\Delta_{\alpha,Y}) \;=\; \left\{ g = F_z + \sum_{j=1}^{N} q_j\, \mathcal{G}_z^{y_j} \;\middle|\; \begin{array}{c} F_z \in H^2(\mathbb{R}^3) \\ (q_1, \dots, q_N) \in \mathbb{C}^N \\ \begin{pmatrix} F_z(y_1) \\ \vdots \\ F_z(y_N) \end{pmatrix} = \Gamma_{\alpha,Y}(z) \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} \end{array} \right\}. \tag{1.30}$$

*At fixed $z$, the decompositions above are unique.*

(ii) *With respect to the decompositions* (1.29)-(1.30)*, one has*

$$(-\Delta_{\alpha,Y} - z^2 \mathbb{1})\, g \;=\; (-\Delta - z^2 \mathbb{1})\, F_z\,. \tag{1.31}$$

*Moreover, $-\Delta_{\alpha,Y}$ has the following locality property: if $g \in \mathcal{D}(-\Delta_{\alpha,Y})$ is such that $g|_{\mathcal{U}} \equiv 0$ for some open $\mathcal{U} \subset \mathbb{R}^3$, then $(-\Delta_{\alpha,Y} g)|_{\mathcal{U}} \equiv 0$.*

(iii) *For $z \in \mathbb{C}^+ \backslash \mathcal{E}$, we have the resolvent identity*

$$(-\Delta_{\alpha,Y} - z^2 \mathbb{1})^{-1} - (-\Delta - z^2 \mathbb{1})^{-1} \;=\; \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(z)^{-1})_{jk}\, |\mathcal{G}_z^{y_j}\rangle \langle \overline{\mathcal{G}_z^{y_k}}|\,. \tag{1.32}$$

Parts (i) and (ii) of Theorem 1.1.2 above originate from [**57**] and are discussed in [**9**, Theorem II.1.1.3], in particular (1.30) is highlighted in [**34**]. Part (iii) was first proved in [**56, 57**] – see also the discussion in [**9**, equation (II.1.1.33)].

By exploiting the boundary condition in (1.30) between the regular and the singular part of a generic $g \in \mathcal{D}(-\Delta_{\alpha,Y})$, it is straightforward to see that

$$\lim_{r_j \downarrow 0} \Big( \frac{\partial (r_j g)}{\partial r_j} - 4\pi \alpha_j r_j g \Big) \;=\; 0\,, \quad r_j := |x - y_j|\,, \quad j \in \{1, \dots, N\}\,, \tag{1.33}$$

whence also

$$\lim_{x \to y_j} \Big( g(x) - \frac{q_j}{4\pi |x - y_j|} - \alpha_j q_j \Big) \;=\; 0, \qquad j \in \{1, \dots, N\}\,, \tag{1.34}$$

which is equivalent to the above mentioned Bethe-Peierls boundary condition (1.26).

In fact, $\mathcal{D}(-\Delta_{\alpha,Y})$ is nothing but the space of those $L^2$-functions $g$ such that the distribution $\Delta g$ belongs to $L^2(\mathbb{R}^3 \setminus Y)$ and the boundary condition (1.34) is satisfied.

We record a simple consequence of Theorem 1.1.2 which will turn out to be useful in our discussion.

LEMMA 1.1.3. *The operator $-\Delta_{\alpha,Y}$ is a* real *self-adjoint operator, that is, for a real-valued function $g \in \mathcal{D}(-\Delta_{\alpha,Y})$, $-\Delta_{\alpha,Y} g$ is also real-valued.*

PROOF. Let $z = \mathrm{i}\lambda$, $\lambda > 0$, be such that $\mathrm{i}\lambda \notin \mathcal{E}$ and let $g$ be a real-valued function in $\mathcal{D}(-\Delta_{\alpha,Y})$. Then, with the notation of the decomposition (1.30) of $g$, the asymptotics (1.34) show that the coefficients $q_1, \dots, q_N$ are all real. The entries of $\Gamma_{\alpha,Y}(\mathrm{i}\lambda)$ are real too, because $\mathfrak{Re}\, z > 0$. Then (1.30) implies that $F_z$ is real-valued and so must be $-\Delta_{\alpha,Y} g + \lambda^2 g$, owing to (1.31). $\square$

Let us discuss the spectral properties of $-\Delta_{\alpha,Y}$, whose resolvent is characterised by (1.32) as an explicit rank-$N$ perturbation of the free resolvent. Contrary to the single centre case, the discrete spectrum of $-\Delta_{\alpha,Y}$ may include a zero eigenvalue imbedded in the essential spectrum.

THEOREM 1.1.4. *The spectrum $\sigma(-\Delta_{\alpha,Y})$ of $-\Delta_{\alpha,Y}$ consists of at most $N$ non-positive eigenvalues and the absolutely continuous part $\sigma_{\mathrm{ac}}(-\Delta_{\alpha,Y}) = [0, \infty)$, the singular continuous spectrum is absent.*

(i) *There is a one to one correspondence between the poles $\mathrm{i}\lambda \in \mathcal{E}$ of $\Gamma_{\alpha,Y}(z)^{-1}$ and the negative eigenvalues $-\lambda^2$ of $-\Delta_{\alpha,Y}$, counting the multiplicity. The eigenfunctions associated to the eigenvalue $-\lambda^2 < 0$ have the form*

$$\psi \;=\; \sum_{j=1}^{N} c_j\, \mathcal{G}_{\mathrm{i}\lambda}^{y_j},$$

*where $(c_1, \dots, c_N)$ is an eigenvector with eigenvalue zero of $\Gamma_{\alpha,Y}(\mathrm{i}\lambda)$. The ground state, if it exists, is non-degenerate.*

(ii) *In a neighbourhood of $z = 0$, the meromorphic matrix-valued function $\Gamma_{\alpha,Y}(z)^{-1}$ has the expansion*

$$\Gamma_{\alpha,Y}(z)^{-1} \;=\; \frac{\Theta_e}{z^2} + \frac{\Theta_r}{z} + \Gamma_{\alpha,Y}^{(\mathrm{reg})}(z) \tag{1.35}$$

*for some constant matrices $\Theta_e$, $\Theta_r$ and some analytic matrix-valued function $\Gamma_{\alpha,Y}^{(\mathrm{reg})}(z)$. Moreover, $\Theta_e \neq 0$ if and only if zero is an eigenvalue for $-\Delta_{\alpha,Y}$.*

Part (i) of Theorem (1.1.4) is an extension, proved in [**9**, Theorem II.1.1.4], of some of the corresponding results established in [**57**]. The proof of part (ii) follows the very same scheme identified in [**28**] for the two dimensional case. In the same paper, explicit examples of the occurrence of a zero eigenvalue are provided for the first time. It is worth noticing that such occurrence is, in a suitable sense, "exceptional". Indeed, following the discussion in [**28**], the self-adjoint operator $-\Delta_{\alpha,Y}$ may have a zero eigenvalue only for a sufficiently large number $N$ of centres, arranged in very specific geometric configurations, and for a measure-zero set of values of the parameter $\alpha \in \mathbb{R}^N$.

**1.1.3. Low energy expansion for the resolvent.** Expansion (1.35) suggests that, in addition to a possible eigenvalue, another kind of obstruction can occur at the bottom of the essential spectrum. In order to clarify the situation, we preliminary recall a version of the celebrated *Limiting Absorption Principle* for the free Laplacian [**4, 76**].

Given $\sigma > 0$, we consider the Banach space

$$\mathbf{B}_\sigma := \mathcal{B}(L^2(\mathbb{R}^3, \langle x\rangle^{1+\sigma} dx); L^2(\mathbb{R}^3, \langle x\rangle^{-1-\sigma} dx)) \tag{1.36}$$

We have the following result.

THEOREM 1.1.5 (Limiting Absorption Principle). *Fix $\sigma, \varepsilon > 0$ and $\gamma < 0$. The following bound holds true:*

$$\|(-\Delta + (\gamma + \mathrm{i}\varepsilon)\mathbb{1})^{-1}\|_{\mathbf{B}_\sigma} \leqslant C,$$

*where the constant $C$ depends only on $\sigma$ and $\gamma$. Moreover, in the norm operator topology of $\mathbf{B}_\sigma$, there exists the limit*

$$\lim_{\varepsilon\to 0}(-\Delta+(\gamma+\mathrm{i}\varepsilon)\mathbb{1})^{-1} \;=:\; (-\Delta+\gamma\mathbb{1})^{-1},$$

*which can be interpreted as the boundary value of the free resolvent on the negative half-line $\gamma<0$. The $\mathbf{B}_\sigma$-valued map $\gamma\mapsto(-\Delta+\gamma\mathbb{1})^{-1}$ is Hölder continuous.*

REMARK 1.1.6. By a direct inspection, it is possible to show that the $\mathbf{B}_\sigma$-valued map $\gamma\mapsto(-\Delta+\gamma\mathbb{1})^{-1}$ can be continuously extended also at $\gamma=0$.

Owing to the resolvent formula (1.32), it is easy to show that the Limiting Absorption Principle holds true also for $-\Delta_{\alpha,Y}$. Nevertheless, unlike the case of the free Laplacian, we cannot expect in general to continuously extend $(-\Delta_{\alpha,Y}+\gamma\mathbb{1})^{-1}$ at $\gamma=0$. In fact, it follows from (1.35) that in a neighborhood of $\gamma=0$ we have the expansion

$$(-\Delta_{\alpha,Y}+\gamma\mathbb{1})^{-1}=\frac{A_e}{\gamma}+\frac{A_r}{\sqrt{\gamma}}+A^{(reg)}(\gamma)\,, \tag{1.37}$$

where $A_e,A_r\in\mathbf{B}_\sigma$ and $A^{(reg)}(\gamma)$ is a continuous $\mathbf{B}_\sigma$-valued map. In addition, $A_e\neq 0$ if and only if zero is an eigenvalue for $-\Delta_{\alpha,Y}$. If $A_r\neq 0$, or equivalently if $\Theta_r\neq 0$ in (1.35), we say that $-\Delta_{\alpha,Y}$ is *zero energy resonant*.

In the single centre case we can give a clean description of the occurrences of an obstruction at the bottom of the essential spectrum. In fact, we already saw in Theorem 1.1.1 that $-\Delta_\alpha$ cannot have a non-negative eigenvalue. Moreover, it is easy to show that, for $\alpha\neq 0$, neither $-\Delta_\alpha$ is zero energy resonant. Instead, for $\alpha=0$, we have in a neighborhood of $\gamma=0$ the expansion

$$(-\Delta_0+\gamma\mathbb{1})^{-1}=\frac{4\pi}{\sqrt{\gamma}}|G_0\rangle\langle G_0|+A_0(\gamma),$$

where

$$A_0(\gamma):=(-\Delta+\gamma\mathbb{1})^{-1}+\frac{4\pi}{\sqrt{\gamma}}(|G_\gamma\rangle\langle G_\gamma|-|G_0\rangle\langle G_0|)$$

is continuous at $\gamma=0$. It follows that $-\Delta_0$ is zero energy resonant.

In the general multi-centre case, both obstructions can occur for $-\Delta_{\alpha,Y}$. As already mentioned, a zero eigenvalue is possible only under specific geometric configurations of the centres and a measure zero set of choiches of the parameter $\alpha$. For an arbitrary configurations of the centres, $-\Delta_{\alpha,Y}$ can be zero energy resonant, but this is an exceptional behaviour that holds only when $\alpha$ is chosen in a measure zero set. The *regular* case, that is, when the resolvent can be extended continuously at the origin, is the "generic" one.

REMARK 1.1.7. For actual Schrödinger operators of the form $-\Delta+V$ the Limiting Absorption Principle and the low-energy resolvent expansion analogous to (1.37) can be proved under suitable short-range Assuption on $V$ [**4, 64**]. For such a class of operators, the zero-energy resonant condition can be equivalently phrased as the existence of a generalized eigenfunction (a *zero-energy resonance* for $-\Delta+V$), namely a function $\psi\in L^2(\mathbb{R}^3,\langle x\rangle^{-1-\sigma}dx)\setminus L^2(\mathbb{R}^3)$, for every $\sigma>0$, which satisfies $-\Delta+V=0$ as a distributional identity on $\mathbb{R}^3$ (we will use a similar definition in Chapter 2, in the more general context of fractional Schrödinger operators). In a sense, this definition could be applied also to $-\Delta_{\alpha,Y}$. Consider, for example, the zero-energy resonant operator $-\Delta_0$. The negative eigenvalue of $-\Delta_\alpha$, when $\alpha<0$, vanishes as $\alpha\uparrow 0$ and the corresponding eigenfunction converge pointwise a.e. to $\mathsf{G}_0\in L^2(\mathbb{R}^3,\langle x\rangle^{-1-\sigma}dx)\setminus L^2(\mathbb{R}^3)$, for every $\sigma>0$, which can be considered as the zero-energy resonance for $-\Delta_0$.

## 1.2. Fractional case in arbitrary dimension

In this Section, using the Kreĭn-Višik-Birman theory, we study the family of self-adjoint realisations of singular perturbations of the fractional Laplacian $(-\Delta)^{s/2}$ in $\mathbb{R}^d$, for arbitrary $d$ and $s$. For the sake of concreteness, we restrict our attention to the case of a single point interaction, which can be assumed to be centred at the origin.

Consider

$$\mathring{\mathsf{k}}^{(s/2)} \;:=\; \overline{(-\Delta)^{s/2}\upharpoonright C^\infty_0(\mathbb{R}^d\backslash\{0\})} \tag{1.38}$$

as an operator closure with respect to the Hilbert space $L^2(\mathbb{R}^d)$. Moreover, for chosen $d\in\mathbb{N}$, $\lambda>0$, and $s\in\mathbb{R}$ we set

$$\mathsf{G}_{s,\lambda}(x) \;:=\; \frac{1}{(2\pi)^{\frac{d}{2}}}\Big(\frac{1}{|p|^s+\lambda}\Big)^{\vee}(x)\,,\qquad x,p\in\mathbb{R}^d\,, \tag{1.39}$$

whence

$$((-\Delta)^{s/2}+\lambda)\,\mathsf{G}_{s,\lambda} \;=\; \delta(x) \tag{1.40}$$

as a distributional identity on $\mathbb{R}^d$. In three dimensions, $\mathring{\mathsf{k}}^{(1)}$ and $\mathsf{G}_{2,\lambda}$ coincides, respectively, with $\mathring{\mathsf{k}}$ and $\mathsf{G}_\lambda$ introduced in Section 1.1.

The domain of $\mathring{\mathsf{k}}^{(s/2)}$, as a consequence of the operator closure in (1.38), is a space of functions with $H^s$-regularity and vanishing conditions at $x=0$ for each function and its partial derivatives. The amount of vanishing conditions depends on $d$ and $s$, to classify which we introduce the intervals

$$I_n^{(d)} \;:=\; \begin{cases} (0,\frac{d}{2}) & n=0 \\ (\frac{d}{2}+n-1,\frac{d}{2}+n) & n=1,2,\dots \end{cases} \tag{1.41}$$

For our purposes it is convenient to use momentum coordinates to express the vanishing conditions that qualify the domain of $\mathring{\mathsf{k}}^{(s/2)}$. With the notation $p\equiv(p_1,\dots,p_d)\in\mathbb{R}^d$, one can prove (see [**85**, Appendix A])

(1.42)

$$\begin{aligned}
\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)}) \;&=\; H^s_0(\mathbb{R}^d\backslash\{0\}) \;=\; \overline{C^\infty_0(\mathbb{R}^d\backslash\{0\})}^{\|\,\|_{H^s}} \\
&=\; \begin{cases} H^s(\mathbb{R}^3) & \text{if } s\in I_0^{(d)} \\[2ex] \left\{\begin{array}{c} f\in H^s(\mathbb{R}^3) \text{ such that} \\ \int_{\mathbb{R}^d} p_1^{\gamma_1}\cdots p_d^{\gamma_d}\widehat{f}(p)\,\mathrm{d}p=0 \\ \gamma_1,\dots,\gamma_d\in\mathbb{N}_0\,,\ \sum_{j=1}^d\gamma_j\leqslant n-1 \end{array}\right\} & \text{if } s\in I_n^{(d)}\,,\ n=1,2,\dots \end{cases}
\end{aligned}$$

Clearly, $\int_{\mathbb{R}^d} p_1^{\gamma_1}\cdots p_d^{\gamma_d}\widehat{f}(p)\,\mathrm{d}p=0$ is the same as $\big(\frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}\cdots\frac{\partial^{\gamma_d}}{\partial x_d^{\gamma_d}}f\big)(0)=0$, with the notation $x\equiv(x_1,\dots,x_d)\in\mathbb{R}^d$.

The expression of $\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$ for the endpoint values $s=\frac{d}{2}+n$ requires an amount of extra analysis: we do not discuss it here, an omission that does not affect the conceptual structure of our presentation.

Being densely defined, closed, and non-negative, either the symmetric operator $\mathring{\mathsf{k}}^{(s/2)}$ is already self-adjoint on $L^2(\mathbb{R}^d)$, or it admits a $\mathcal{J}(s,d)^2$-parameter family of self-adjoint extensions, where

$$\mathcal{J}(s,d) \;:=\mathcal{J}(\mathring{\mathsf{k}}^{(s/2)}) =\; \dim\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big) \tag{1.43}$$

for one, and hence for all $\lambda>0$. The self-adjointness of $\mathring{\mathsf{k}}^{(s/2)}$ is equivalent to $\mathcal{J}(s,d)=0$.

We saw in Section 1 that $\mathcal{J}(2,3)=1$. It is not difficult to compute $\mathcal{J}(s,d)$ for generic values of $d$ and $s$ and to identify a natural basis of the $\mathcal{J}(s,d)$-dimensional space $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)$.

LEMMA 1.2.1. *For given $d\in\mathbb{N}$ and $s>0$,*

$$s\in I_n^{(d)}\quad\Rightarrow\quad\mathcal{J}(s,d)\;=\;\binom{d+n-1}{d}\,. \tag{1.44}$$

*In particular, when $s\in I_n^{(d)}$ for some $n\in\mathbb{N}$, then*

$$\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)\;=\;\mathrm{span}\Big\{u^\lambda_{\gamma_1,\dots,\gamma_d}\,\Big|\,\gamma_1,\dots,\gamma_d\in\mathbb{N}_0\,,\;\sum_{j=1}^d\gamma_j\leqslant n-1\Big\}\,, \tag{1.45}$$

*where*

$$\widehat{u^\lambda}_{\gamma_1,\dots,\gamma_d}(p)\;:=\;\frac{p_1^{\gamma_1}\cdots p_d^{\gamma_d}}{|p|^s+\lambda}\,. \tag{1.46}$$

It is worth noticing, comparing (1.39) and (1.46), that

$$u^\lambda_{0,\dots,0}\;=\;(2\pi)^{\frac{d}{2}}\,\mathsf{G}_{s,\lambda}\,. \tag{1.47}$$

In the particular case $s=2$, one gets $\mathcal{J}(2,1)=2$, $\mathcal{J}(2,2)=\mathcal{J}(2,3)=1$ and $\mathcal{J}(2,d)=0$ for $d\geqslant 4$. Hence, non-trivial point perturbations of the free Laplacian exist only in dimension $d=1,2$ or 3.

PROOF OF LEMMA 1.2.1. When $s\in I_0^{(d)}$, we see from (1.42) that $\mathring{\mathsf{k}}^{(s/2)}$ is self-adjoint: then $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)$ is trivial and $\mathcal{J}(s,d)=0$, consistently with (1.44). When $s\in I_n^{(d)}$, $n=1,2,\dots$, then $u\in\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)=\mathrm{ran}\big(\mathring{\mathsf{k}}^{(s/2)}+\lambda\mathbb{1}\big)^\perp$ is equivalent to

$$0\;=\int_{\mathbb{R}^3}\widehat{u}(p)(|p|^s+\lambda)\widehat{f}(p)\mathrm{d}p\qquad\forall f\in\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$$

and one argues from (1.42) that $\mathring{\mathsf{k}}^{(s/2)}$ is spanned by linearly independent functions of the form $u^\lambda_{\gamma_1,\dots,\gamma_d}$. Such functions are as many as the linearly independent monomials in $d$ variables with degree at most equal to $n-1$, and therefore their number equals $\binom{d+n-1}{d}$. □

The knowledge of $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)$ and of the inverse of the Friedrichs extension of $\mathring{\mathsf{k}}^{(s/2)}$ are the two inputs for the Kreĭn-Višik-Birman extension theory, by means of which we can produce the whole family of self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$.

Such a construction is particularly clean in the case, relevant in applications, of deficiency index one: the comprehension of this case is instructive to understand the case of higher deficiency index. Moreover, as we shall see, in this case the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ turn out to be rank-one perturbations, in the resolvent sense: we will use the jargon $\mathcal{J}=1$ or 'rank one' interchangeably.

We discuss in detail the $\mathcal{J}(s,d)=1$ scenario when $s\in I_1^{(d)}$, deferring to Subsection 1.2.3 an outlook on the high-$\mathcal{J}$ scenario. This corresponds to analysing the regimes $s\in(\frac{1}{2},\frac{3}{2})$ when $d=1$, $s\in(1,2)$ when $d=2$, $s\in(\frac{3}{2},\frac{5}{2})$ when $d=3$, etc.

The construction of the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ in any such regimes is technically the very same, irrespectively of $d$, except for a noticeable peculiarity when $d=1$, as opposite to $d=2,3,\dots$

Indeed, when $s \in I_1^{(d)}$ and hence $\mathscr{J}(s,d)=1$, we know from Lemma 1.2.1 and (1.47) that $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)=\mathrm{span}\{\mathsf{G}_{s,\lambda}\}$, and the function $\mathsf{G}_{s,\lambda}$ may or may not have a *local singularity* as $x\to 0$. As follows from the $d$-dimensional distributional identity

$$2^{\frac{s}{2}}\Gamma(\tfrac{s}{2})\,\frac{1}{|p|^s} \;=\; 2^{\frac{d-s}{2}}\Gamma(\tfrac{d-s}{2})\,\Big(\widehat{\frac{1}{|x|^{d-s}}}\Big)\,,\qquad s\in(0,d)\,,$$

$\mathsf{G}_{s,\lambda}$ has a singularity $\sim |x|^{-(d-s)}$ when $s<d$, it has a logarithmic singularity when $s=d$, and it is continuous at $x=0$ when $s>d$. More precisely,

$$\mathsf{G}_{s,\lambda}(x)\;\xrightarrow{x\to 0}\;\mathsf{G}_{s,\lambda}(0)\;=\;\big(2^{d-1}\pi^{\frac{d}{2}-1}\Gamma(\tfrac{d}{2})\,\lambda^{\frac{s-d}{s}}\,s\sin\tfrac{\pi d}{s}\big)^{-1},\quad s>d\,, \tag{1.48}$$

$$\mathsf{G}_{s,\lambda}(x)\;=\;\frac{\Lambda_s^{(d)}}{|x|^{(d-s)}}+\mathsf{J}_{s,\lambda}(x),\qquad s\in(0,d), \tag{1.49}$$

with

$$\begin{aligned}\Lambda_s^{(d)}\;&:=\;\frac{\Gamma(\frac{d-s}{2})}{(2\pi)^{\frac{d}{2}}\,2^{s-\frac{d}{2}}\,\Gamma(\frac{d}{2})}\,,\\ \mathsf{J}_{s,\lambda}\;&:=\;-\frac{\lambda}{(2\pi)^{\frac{d}{2}}}\Big(\frac{1}{|p|^s(|p|^s+\lambda)}\Big)^{\vee}\;\in\;C_\infty(\mathbb{R}^d)\,.\end{aligned} \tag{1.50}$$

Now, all the considered regimes $s\in(1,2)$ when $d=2$, $s\in(\frac{3}{2},\frac{5}{2})$ when $d=3$, etc. lie *below* the transition value $s=d$ between the local singular and the local regular behaviour of $\mathsf{G}_{s,\lambda}$, whereas the regime $s\in(\frac{1}{2},\frac{3}{2})$ when $d=1$ lies *across* the transition value $s=1$.

The same type of distinction clearly occurs for the spanning functions (1.45)-(1.46) of $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)$ for higher deficiency index $\mathscr{J}(s,d)$.

In the present context, the peculiarity described above when $d=1$ results in certain different steps of the construction of the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ and ultimately in the type of parametrisation of such extensions, as we shall see.

Therefore, we articulate our discussion on the extensions of $\mathring{\mathsf{k}}^{(s/2)}$ when the deficiency index is one discussing first the three-dimensional case (Subection 1.2.1) and then the one-dimensional case (Subsection 1.2.2). As commented already, for generic $d\geqslant 2$ the discussion and the final results are completely analogous to $d=3$.

**1.2.1. Rank-one singular perturbations in dimension three.** In terms of the general discussion above, we consider here the operator $\mathring{\mathsf{k}}^{(s/2)}$ on $L^2(\mathbb{R}^3)$ when $s\in(\frac{3}{2},\frac{5}{2})$. $\mathring{\mathsf{k}}^{(s/2)}$ acts as the fractional Laplacian $(-\Delta)^{s/2}$ on the domain

$$\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})\;=\;\Big\{f\in H^s(\mathbb{R}^3)\,\Big|\,\int_{\mathbb{R}^3}\widehat{f}(p)\,\mathrm{d}p=0\Big\} \tag{1.51}$$

and its deficiency index is 1.

We start by producing all the self-adjoint extension in the Birman scheme. We first obtain a paramatresion which depends on the shift $\lambda$. Then, we will produce the natural $\alpha$-representation. One has the following construction.

THEOREM 1.2.2. *Let $s\in(\frac{3}{2},\frac{5}{2})$ and $\lambda>0$.*

(i) *The self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\tau^{(s/2)})_{\tau\in\mathbb{R}\cup\{\infty\}}$, where $\mathsf{k}_\infty^{(s/2)}$ is its Friedrichs extension, namely the self-adjoint fractional Laplacian $(-\Delta)^{s/2}$, and all other extensions are given* *by*

$$\mathcal{D}\big(\mathsf{k}_\tau^{(s/2)}\big) \;:=\; \left\{ g\in L^2(\mathbb{R}^3) \left| \begin{array}{c} \widehat{g}(p)=\widehat{f^\lambda}(p)+\dfrac{\tau\,\xi}{(|p|^s+\lambda)^2}+\dfrac{\xi}{|p|^s+\lambda} \\ \xi\in\mathbb{C}\,,\;\; f^\lambda\in H^s(\mathbb{R}^3)\,,\;\; \int_{\mathbb{R}^3}\widehat{f^\lambda}(p)\,\mathrm{d}p=0 \end{array} \right.\right\} \tag{1.52}$$
$$= \;\Big\{ g=F^\lambda+\tfrac{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)}\,F^\lambda(0)\,\mathsf{G}_{s,\lambda}\,\Big|\,F^\lambda\in H^s(\mathbb{R}^3)\Big\}\,,$$

*and*

$$\big(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1}\big)g \;:=\; \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)F^\lambda. \tag{1.53}$$

(ii) *Each extension is semi-bounded from below and*

$$\begin{aligned} \inf\sigma(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1}) \;\geqslant\; 0 \quad &\Leftrightarrow \quad \tau\;\geqslant\;0 \\ \inf\sigma(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1}) \;>\; 0 \quad &\Leftrightarrow \quad \tau\;>\;0 \\ (\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1}) \textit{ is invertible} \quad &\Leftrightarrow \quad \tau\neq 0\,. \end{aligned} \tag{1.54}$$

(iii) *For each $\tau\in\mathbb{R}$ the quadratic form of the extension $\mathsf{k}_\tau^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] \;=\; H^{\frac{s}{2}}(\mathbb{R}^3)\dotplus\mathrm{span}\{\mathsf{G}_{s,\lambda}\} \tag{1.55}$$
$$\begin{aligned} \mathsf{k}_\tau^{(s/2)}[F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}] \;=\; & \||\nabla|^{\frac{s}{2}}F^\lambda\|^2_{L^2(\mathbb{R}^3)}-\lambda\|F^\lambda+\kappa_\lambda\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)} \\ & +\lambda\|F^\lambda\|^2_{L^2(\mathbb{R}^3)}+\frac{\tau(s-3)}{2\pi s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}|\kappa_\lambda|^2 \end{aligned} \tag{1.56}$$

*for any $F^\lambda\in H^{s/2}(\mathbb{R}^3)$ and $\kappa_\lambda\in\mathbb{C}$.*

(iv) *For $\tau\neq 0$, one has the resolvent identity*

$$(\mathsf{k}_\tau^{(s/2)}+\lambda\mathbb{1})^{-1} \;=\; ((-\Delta)^{s/2}+\lambda\mathbb{1})^{-1}+\frac{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)}\,|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|\,. \tag{1.57}$$

PROOF. The whole construction is based upon the Kreĭn-Višik-Birman self-adjoint extension scheme. Since $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big) = \mathrm{span}\{\mathsf{G}_{s,\lambda}\}$ and the Friedrichs extension of $\mathring{\mathsf{k}}^{(s/2)}+\lambda\mathbb{1}$ is $(-\Delta)^{s/2}+\lambda\mathbb{1}$, one has the following formula for the adjoint (see, e.g., Ref. [**45**], Theorem 2.2):

$$\mathcal{D}\big((\mathring{\mathsf{k}}^{(s/2)})^*\big) \;=\; \left\{ g\in L^2(\mathbb{R}^3) \left| \begin{array}{c} \widehat{g}(p)=\widehat{f^\lambda}(p)+\dfrac{\eta}{(|p|^s+\lambda)^2}+\dfrac{\xi}{|p|^s+\lambda} \\ \eta,\xi\in\mathbb{C}\,,\;\; f^\lambda\in H^s(\mathbb{R}^3)\,,\;\; \int_{\mathbb{R}^3}\widehat{f^\lambda}(p)\,\mathrm{d}p=0 \end{array} \right.\right\}$$
$$\big((\mathring{\mathsf{k}}^{(s/2)})^*+\lambda\mathbb{1}\big)g \;=\; \mathcal{F}^{-1}\Big((|p|^s+\lambda)\Big(\widehat{f^\lambda}+\frac{\eta}{(|p|^s+\lambda)^2}\Big)\Big)\,.$$

Each element of the one-parameter family of self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ is identified (see, e.g., Ref [**45**], Theorem 3.4) by the *Birman self-adjointness condition*

$$\eta=\tau\xi \quad \text{for some } \tau\in\mathbb{R}\cup\{\infty\}.$$

This establishes the first line of (1.52). Setting $\widehat{F^\lambda}:=\widehat{f^\lambda}+(|p|^s+\lambda)^{-2}\tau\xi$, the boundary condition between $F^\lambda$ and $\xi$ in Fourier transform reads

$$\int_{\mathbb{R}^3}\widehat{F^\lambda}(p)\,\mathrm{d}p \;=\; \xi\,\tfrac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}\,. \tag{*}$$

Then, from $F^\lambda(0)=(2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3}\widehat{F^\lambda}\mathrm{d}p$, and using (1.39) with $d=3$, the second line of (1.52) follows. Since $\mathsf{k}_\tau^{(s/2)}$ is a restriction of $(\mathring{\mathsf{k}}^{(s/2)})^*$, from the above action of the adjoint one deduces (1.53). This completes the proof of part (i).

Part (ii) lists standard facts of the Kreĭn-Višik-Birman theory – see Ref. [**45**], Theorems 3.5 and 5.1.

The quadratic form is characterised in the extension theory (Ref. [**45**], Theorem 3.6) by the formulas $\mathcal{D}[\mathsf{k}_\tau^{(s/2)}] = \mathcal{D}[\mathsf{k}_F^{(s/2)}] \dotplus \ker\big((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}\big)$ ('$F$' stands for Friedrichs), whence (1.55), and $(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] = ((-\Delta)^{s/2} + \lambda\mathbb{1})[F^\lambda] + \tau|\kappa_\lambda|^2 \|\mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)}$, whence (1.56). The proof of part (iii) is completed.

Kreĭn's resolvent formula for deficiency index 1 (Ref. [**45**], Theorem 6.6) prescribes

$$f(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})^{-1} \;=\; ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} + \beta_{\lambda,\tau}\, |\mathsf{G}_{s,\lambda}\rangle\langle \mathsf{G}_{s,\lambda}|$$

for some scalar $\beta_{\lambda,\tau}$ to be determined, whenever $(\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})$ is invertible, hence for $\tau \neq 0$. Thus, for a generic $h \in L^2(\mathbb{R}^3)$, the element $g := (\mathsf{k}_\tau^{(s/2)} + \lambda\mathbb{1})^{-1} h \in \mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ reads, in view of (1.52) and of the resolvent formula above,

$$\begin{aligned} \widehat{g}(p) \;&= \widehat{F^\lambda}(p) + \frac{\xi_\lambda}{|p|^s + \lambda} \\ \widehat{F^\lambda}(p) \;&:= \frac{\widehat{h}(p)}{|p|^s + \lambda}\,, \qquad \xi_\lambda \;:=\; \frac{\beta_{\lambda,\tau}}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\widehat{h}(q)}{|q|^s + \lambda}\,\mathrm{d}q\,. \end{aligned}$$

The boundary condition (*) for $F^\lambda$ and $\xi_\lambda$ then implies $1 = \beta_{\lambda,\tau} \frac{\tau(s-3)}{2\pi s^2 \sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}$, which determines $\beta_{\lambda,\tau}$ and proves (1.57), thus completing also the proof of (iv). $\square$

The $\tau$-parametrisation of the family $(\mathsf{k}_\tau^{(s/2)})_{\tau \in \mathbb{R}\cup\{\infty\}}$ depends on the initially chosen shift $\lambda > 0$, meaning that with a different choice $\lambda' > 0$ the same self-adjoint realisation previously identified by $\tau$ with shift $\lambda$ is now selected by a different extension parameter $\tau'$. It is more convenient to switch onto a natural parametrisation that identifies one extension irrespectively of the infinitely many different pairs $(\lambda, \tau)$ attached to it by the parametrisation of Theorem 1.2.2. We shall do it in the next Theorem: observe that indeed, as compared to Theorem 1.2.2, here below $\lambda > 0$ is arbitrary.

THEOREM 1.2.3. *Let $s \in (\frac{3}{2}, \frac{5}{2})$.*

(i) *The self-adjoint extensions in $L^2(\mathbb{R}^3)$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\alpha^{(s/2)})_{\alpha \in \mathbb{R}\cup\{\infty\}}$, where $\mathsf{k}_\infty^{(s/2)}$ is its Friedrichs extension, namely the self-adjoint fractional Laplacian $(-\Delta)^{s/2}$, and all other extensions are given, for arbitrary $\lambda > 0$, by*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_\alpha^{(s/2)}) \;&=\; \left\{ g = F^\lambda + \frac{F^\lambda(0)}{\alpha - \frac{\lambda^{\frac{3}{s}-1}}{2\pi s \sin(\frac{3\pi}{s})}}\, \mathsf{G}_{s,\lambda} \;\middle|\; F^\lambda \in H^s(\mathbb{R}^3) \right\} \\ (\mathsf{k}_\alpha^{(s/2)} + \lambda)\, g \;&=\; ((-\Delta)^{s/2} + \lambda)\, F^\lambda\,. \end{aligned} \tag{1.58}$$

(ii) *For each $\alpha \in \mathbb{R}$ the quadratic form of the extension $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\alpha^{(s/2)}] \;=\; H^{\frac{s}{2}}(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} \tag{1.59}$$

$$\begin{aligned} \mathsf{k}_\alpha^{(s/2)}[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] \;&=\; \big\||\nabla|^{\frac{s}{2}} F^\lambda\big\|^2_{L^2(\mathbb{R}^3)} - \lambda \|F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R}^3)} \\ &\quad + \lambda \|F^\lambda\|^2_{L^2(\mathbb{R}^3)} + \big(\alpha - \tfrac{\lambda^{\frac{3}{s}-1}}{2\pi s \sin(\frac{3\pi}{s})}\big)|\kappa_\lambda|^2 \end{aligned} \tag{1.60}$$

*for arbitrary $\lambda > 0$.*

(iii) *The resolvent of $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\begin{aligned} (\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1} \;&=\; ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} \\ &\quad + \big(\alpha - \tfrac{\lambda^{\frac{3}{s}-1}}{2\pi s \sin(\frac{3\pi}{s})}\big)^{-1} |\mathsf{G}_{s,\lambda}\rangle\langle \mathsf{G}_{s,\lambda}| \end{aligned} \tag{1.61}$$

*for arbitrary $\lambda > 0$.*

(iv) *Each extension is semi-bounded from below, and*

$$\begin{aligned} \sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)}) \;&=\; \sigma_{\mathrm{ac}}(\mathsf{k}_\alpha^{(s/2)}) \;=\; [0,+\infty)\,, \qquad \sigma_{\mathrm{sc}}(\mathsf{k}_\alpha^{(s/2)}) \;=\; \emptyset\,, \\ \sigma_{\mathrm{disc}}(\mathsf{k}_\alpha^{(s/2)}) \;&=\; \begin{cases} \emptyset & \text{if } \alpha \geqslant 0 \\ \{E_\alpha^{(s)}\} & \text{if } \alpha < 0\,, \end{cases} \end{aligned} \tag{1.62}$$

*where the eigenvalue* $E_\alpha^{(s)}$ *is non-degenerate and is given by*

$$E_\alpha^{(s)} \;=\; -\big(2\pi|\alpha|\, s\, \sin(-\tfrac{3\pi}{s})\big)^{\frac{s}{3-s}}\,, \tag{1.63}$$

*the (non-normalised) eigenfunction being* $\mathsf{G}_{s,\lambda=|E_\alpha^{(s)}|}$.

Remark 1.2.4. When $s = 2$, Theorem 1.2.3 produce exactly the family of self-adjoint operators $-\Delta_\alpha$ introduced in Section 1.1.

Proof of Theorem 1.2.3. We seek for the relation $\tau = \tau(\lambda)$ that ensures that all the pairs $(\lambda, \tau(\lambda))$, with $\lambda > 0$, preserve the decomposition (1.52)-(1.53) and thus label the same element of the family of extensions.

For chosen $\lambda$ and $\tau$, a function $g \in \mathcal{D}(\mathsf{k}_\tau^{(s/2)})$ decomposes uniquely as

$$\widehat{g} \;=\; \widehat{F^\lambda} + \frac{\xi}{|p|^s+\lambda}\,, \qquad \begin{array}{c} F^\lambda \in H^s(\mathbb{R}^3) \\ \xi \in \mathbb{C} \end{array}\,, \qquad \int_{\mathbb{R}^3} \widehat{F^\lambda}\mathrm{d}p \;=\; \xi\,\frac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}\,.$$

Let now $\lambda' > 0$ and $\tau' \in \mathbb{R}$ be such that for the *same* function $g$ in the domain of the *same* self-adjoint realisation $\mathsf{k}_\tau^{(s/2)}$ one also has

$$\widehat{g} \;=\; \widehat{F^{\lambda'}} + \frac{\xi'}{|p|^s+\lambda'}\,, \qquad \begin{array}{c} F^{\lambda'} \in H^s(\mathbb{R}^3) \\ \xi' \in \mathbb{C} \end{array}\,, \qquad \int_{\mathbb{R}^3} \widehat{F^{\lambda'}}\mathrm{d}p \;=\; \xi'\,\frac{4\pi^2\tau'(s-3)}{s^2\lambda'^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}\,.$$

The new splitting of $g$ is equivalent to

$$\xi' \;=\; \xi\,, \qquad F^{\lambda'} \;=\; F^\lambda + \frac{\xi}{|p|^s+\lambda} - \frac{\xi'}{|p|^s+\lambda'}\,,$$

and the boundary condition for $F^{\lambda'}$ and $\xi'$ is equivalent to

$$\xi\,\frac{4\pi^2\tau(s-3)}{s^2\lambda^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})} + \int_{\mathbb{R}^3}\Big(\frac{\xi'}{|p|^s+\lambda} - \frac{\xi'}{|p|^s+\lambda'}\Big)\mathrm{d}p \;=\; \xi'\,\frac{4\pi^2\tau'(s-3)}{s^2\lambda'^{2-\frac{3}{s}}\sin(\frac{3\pi}{s})}\,. \tag{*}$$

Let us analyze the integral in (*). Both summands in the integrand diverge, with two identical divergences that cancel out. Thus, by means of the identity $r^2(r^s+\lambda)^{-1} = r^{2-s} - \lambda r^{2-s}(r^s+\lambda)^{-1}$, one has

$$\begin{aligned} \int_{\mathbb{R}^3}\Big(\frac{1}{|p|^s+\lambda} - \frac{1}{|p|^s+\lambda'}\Big)\mathrm{d}p \;&=\; 4\pi \lim_{R\to+\infty}\Big(\int_0^R \frac{r^2}{r^s+\lambda}\,\mathrm{d}r - \int_0^R \frac{r^2}{r^s+\lambda'}\,\mathrm{d}r\Big) \\ &=\; 4\pi \lim_{R\to+\infty}\Big(\int_0^R \frac{\lambda'\,\mathrm{d}r}{r^{s-2}(r^s+\lambda')} - \int_0^R \frac{\lambda\,\mathrm{d}r}{r^{s-2}(r^s+\lambda)}\Big) \\ &=\; \frac{4\pi^2}{\lambda^{1-\frac{3}{s}}s\sin(\frac{3\pi}{s})} - \frac{4\pi^2}{\lambda'^{1-\frac{3}{s}}s\sin(\frac{3\pi}{s})}\,. \end{aligned}$$

Plugging the result of the above computation into (*) yields

$$\frac{\tau(3-s)-s\lambda}{\lambda^{2-\frac{3}{s}}} \;=\; \frac{\tau'(3-s)-s\lambda'}{\lambda'^{2-\frac{3}{s}}}\,,$$

which shows that all pairs $(\lambda,\tau)$ such that

$$-\frac{\tau(3-s)-s\lambda}{2\pi s^2\sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}} \;=:\; \alpha \tag{**}$$

indeed label the same extension (the pre-factor $-2\pi s^2 \sin(\frac{3\pi}{s})$ having being added for convenience). Thus, $\alpha \in \mathbb{R}\cup\{\infty\}$ defined in (**) is the natural parametrisation we were aiming for (and the Friedrichs case $\tau \to +\infty$ corresponds to $\alpha \to +\infty$).

Upon replacing $\frac{2\pi s^2 \sin(\frac{3\pi}{s})\lambda^{2-\frac{3}{s}}}{\tau(s-3)} = \big(\alpha - \frac{1}{2\pi s \sin(\frac{3\pi}{s})\lambda^{1-\frac{3}{s}}}\big)^{-1}$ in the formulas of Theorem 1.2.2 we deduce at once all formulas of parts (i), (ii), and (iii), together of course with the certainty, proved above, that the decompositions are now $\lambda$-independent.

Since the deficiency index is 1, and hence all extensions are a rank-one perturbation, in the resolvent sense, of the self-adjoint fractional Laplacian, then all extensions have the same essential spectrum $[0, +\infty)$ of the latter, and additionally may have *at most* one negative non-degenerate eigenvalue, in any case all extensions are semi-bounded from below – all these being general facts of the extension theory, see, e.g., Ref. [**45**], Theorem 5.9 and Corollary 5.10. This proves, in particular, the first line in (1.62).

The occurrence of a negative eigenvalue $E_\alpha = -\lambda$ of an extension $\mathsf{k}_\alpha^{(s/2)}$, for some $\lambda > 0$, can be read out from the resolvent formula (1.61) as the pole of $(\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1}$, that is, imposing

$$\alpha - \frac{1}{2\pi s \sin(\frac{3\pi}{s})\lambda^{1-\frac{3}{s}}} = 0\,,$$

i.e.,

$$\alpha = -\lambda^{\frac{3-s}{s}}\big(2\pi s \sin(-\tfrac{3\pi}{s})\big)^{-1}.$$

The identity above can be only satisfied by some $\lambda > 0$ when $\alpha < 0$, because $\sin(-\frac{3\pi}{s}) > 0$, in which case

$$\lambda = \big(2\pi|\alpha|\, s \sin(-\tfrac{3\pi}{s})\big)^{\frac{s}{3-s}}.$$

Alternatively, one can argue from (1.52)-(1.53) that the eigenvalue $-\lambda$ must correspond to the eigenfunction $(\frac{1}{|p|^s+\lambda})^\vee$, that is, an element of the domain with only singular component, and to the parameter $\tau = 0$, hence with $f^\lambda \equiv 0$ in the notation therein. Then, setting $\tau = 0$ in (**) yields the same condition above on $\alpha$ and $\lambda$. This proves (1.63) and the second line in (1.62) when $\alpha < 0$, and it also qualifies the eigenfunction.

When such a bound state is absent, and therefore when $\alpha \geqslant 0$, for what argued before one has $\sigma(\mathsf{k}_\alpha^{(s/2)}) = \sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)}) = [0, +\infty)$. This proves the second line in (1.62) when $\alpha \geqslant 0$, and completes the proof of part (iv). $\square$

The elements of the domain $\mathsf{k}_\alpha^{(s/2)}$ split into a regular $H^s$-component plus a singular component, whit the singularity of $\mathsf{G}_{s,\lambda}$, namely $|x|^{-(3-s)}$ for all powers $s \in (\frac{3}{2}, \frac{5}{2})$. Moreover, a local boundary condition constrains regular and singular components: working out the asymptotics as $x \to 0$ in (1.58) by means of (1.49) we find

$$g(x) \sim \alpha + \Lambda_s^{(3)}|x|^{-(3-s)} \quad \text{as } x \to 0\,, \quad g \in \mathcal{D}(\mathsf{k}_\alpha^{(s/2)})\,,\ s \in (\tfrac{3}{2}, \tfrac{5}{2})\,, \tag{1.64}$$

where $\Lambda_s^{(3)}$ is defined in (1.49). Asymptotics (1.64) is the fractional version of the Bethe-Peierls contact condition (1.12) (observe that $\Lambda_2^{(3)} = (4\pi)^{-1}$, consistently).

**1.2.2. Rank-one singular perturbations in dimension one.** We consider here the operator $\mathring{\mathsf{k}}^{(s/2)}$ on $L^2(\mathbb{R})$ when $s \in (\frac{1}{2}, \frac{3}{2})$. The conceptual scheme is the very same as in the 3D case, except for an amount of extra technicalities due to the somewhat more involved structure of the Friedrichs extensions for high regularity $s \in (1, \frac{3}{2})$. For simplicity, we do not treat the tranistion case $s = 1$. Details can be found in [**85**, Section V] and we content ourselves here to only state the main results.

We start with qualifying the Friedrichs extension $\mathsf{k}_F^{(s/2)}$ of $\mathring{\mathsf{k}}^{(s/2)}$. Unlike the three-dimensional case, the structure of $\mathsf{k}_F^{(s/2)}$ depends on whether $s<1$ or $s>1$.

PROPOSITION 1.2.5. *Let $s \in (\frac{1}{2}, \frac{3}{2})$, $s \neq 1$.*

(i) *The quadratic form of the Friedrichs extension $\mathsf{k}_F^{(s/2)}$ of $\mathring{\mathsf{k}}^{(s/2)}$ is*

$$\begin{aligned} \mathcal{D}[\mathsf{k}_F^{(s/2)}] \;&=\; \begin{cases} H^{s/2}(\mathbb{R}) & \text{if } s \in (\frac{1}{2},1) \\ H_0^{s/2}(\mathbb{R}\backslash\{0\}) & \text{if } s\in(1,\frac{3}{2}) \end{cases} \\ \mathsf{k}_F^{(s/2)}[f,g] \;&=\; \langle\, |\nabla|^{\frac{s}{2}} f, \nabla|^{\frac{s}{2}} g\rangle\,. \end{aligned} \tag{1.65}$$

(ii) *When $s\in(\frac{1}{2},1)$, one has*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_F^{(s/2)}) \;&=\; H^s(\mathbb{R}) \\ \mathsf{k}_F^{(s/2)} f \;&=\; (-\Delta)^{\frac{s}{2}} f\,. \end{aligned} \tag{1.66}$$

(iii) *When $s\in(1,\frac{3}{2})$, for every $\lambda>0$ one has*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_F^{(s/2)}) \;&=\; \left\{ f = \phi - \frac{\phi(0)}{\mathsf{G}_{s,\lambda}(0)}\,\mathsf{G}_{s,\lambda} \,\middle|\, \phi \in H^s(\mathbb{R}) \right\} \\ (\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1}) f \;&=\; ((-\Delta)^{\frac{s}{2}} + \lambda)\phi\,. \end{aligned} \tag{1.67}$$

*In particular, $\mathcal{D}(\mathsf{k}_F^{(s/2)}) \subset H^s(\mathbb{R}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\}$. In this regime of $s$, $(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})$ has an everywhere defined and bounded inverse on $L^2(\mathbb{R})$ with*

$$(\mathsf{k}_F^{(s/2)} + \lambda\mathbb{1})^{-1} \;=\; ((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})^{-1} - \frac{1}{G_{s,\lambda}(0)}\,|G_{s,\lambda}\rangle\langle G_{s,\lambda}|\,. \tag{1.68}$$

We can now state the one-dimensional analogue of Theorem 1.2.3.

THEOREM 1.2.6. *Let $s\in(\frac{1}{2},\frac{3}{2})$, $s\neq 1$. We set*

$$\begin{aligned} \Theta(s,\lambda) \;&:=\; \big(\lambda^{1-\frac{1}{s}} s \sin\big(\tfrac{\pi}{s}\big)\big)^{-1}, \qquad \lambda>0\,, \\ \theta_s \;&:=\; \begin{cases} 0 & \text{if } s<1 \\ 1 & \text{if } s>1\,. \end{cases} \end{aligned} \tag{1.69}$$

(i) *The self-adjoint extensions in $L^2(\mathbb{R})$ of the operator $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\alpha^{(s/2)})_{\alpha\in\mathbb{R}\cup\{\infty\}}$, where for arbitrary $\lambda>0$*

$$\begin{aligned} \mathcal{D}(\mathsf{k}_\alpha^{(s/2)}) \;&=\; \left\{ \begin{array}{c} g = F^\lambda + \dfrac{F^\lambda(0)}{\alpha - \Theta(s,\lambda)}\,\mathsf{G}_{s,\lambda} \\ F^\lambda \in H^s(\mathbb{R}) \end{array} \right\} \\ (\mathsf{k}_\alpha^{(s/2)} + \lambda)\, g \;&=\; ((-\Delta)^{s/2} + \lambda)\, F^\lambda\,. \end{aligned} \tag{1.70}$$

*The Friedrichs extension $\mathsf{k}_F^{(s/2)}$, already qualified in Proposition 1.2.5, corresponds to $\alpha=\infty$ when $s\in(\frac{1}{2},1)$ and to $\alpha=0$ when $s\in(1,\frac{3}{2})$. For generic $s$, the extension with $\alpha=\infty$ is the ordinary self-adjoint fractional Laplacian $(-\Delta)^{s/2}$ on $L^2(\mathbb{R})$.*

(ii) *For each $\alpha \in \mathbb{R}$ the quadratic form of the extension $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\mathcal{D}[\mathsf{k}_\alpha^{(s/2)}] = \begin{cases} H^{s/2}(\mathbb{R}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (\tfrac{1}{2},1) \\ H_0^{s/2}(\mathbb{R}\backslash\{0\}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} & \text{if } s \in (1,\tfrac{3}{2}),\ \alpha \neq 0 \end{cases} \tag{1.71}$$

$$\begin{aligned} \mathsf{k}_\alpha^{(s/2)}[F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}] &= \||\nabla|^{\frac{s}{2}} F^\lambda\|^2_{L^2(\mathbb{R})} - \lambda \|F^\lambda + \kappa_\lambda \mathsf{G}_{s,\lambda}\|^2_{L^2(\mathbb{R})} \\ &\quad + \lambda \|F^\lambda\|^2_{L^2(\mathbb{R})} + \Big(\frac{\theta_s}{\Theta(s,\lambda)} + \frac{1}{\alpha - \Theta(s,\lambda)}\Big)^{-1} |\kappa_\lambda|^2 \end{aligned} \tag{1.72}$$

*for arbitrary $\lambda > 0$.*

(iii) *The resolvent of $\mathsf{k}_\alpha^{(s/2)}$ is given by*

$$\begin{aligned} (\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1} &= ((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} \\ &\quad + \big(\alpha - \Theta(s,\lambda)\big)^{-1} |\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}| \end{aligned} \tag{1.73}$$

*for arbitrary $\lambda > 0$.*

(iv) *For each $\alpha \in \mathbb{R}$ the extension $\mathsf{k}_\alpha^{(s/2)}$ is semi-bounded from below, and*

$$\sigma_{\mathrm{ess}}(\mathsf{k}_\alpha^{(s/2)}) = \sigma_{\mathrm{ac}}(\mathsf{k}_\alpha^{(s/2)}) = [0,+\infty), \qquad \sigma_{\mathrm{sc}}(\mathsf{k}_\alpha^{(s/2)}) = \emptyset, \tag{1.74}$$

$$\sigma_{\mathrm{disc}}(\mathsf{k}_\alpha^{(s/2)}) = \begin{cases} \emptyset & \text{if } (s-1)\,\alpha \leqslant 0 \\ \{-E_\alpha^{(s)}\} & \text{if } (s-1)\,\alpha > 0 \end{cases} \tag{1.75}$$

*where the eigenvalue $-E_\alpha^{(s)}$ is non-degenerate and is given by*

$$E_\alpha^{(s)} = \big(\alpha s \sin(\tfrac{\pi}{s})\big)^{\frac{s}{1-s}} \tag{1.76}$$

*the (non-normalised) eigenfunction being $\mathsf{G}_{s,\lambda=|E_\alpha^{(s)}|}$.*

In the regime $s \in (1,\frac{3}{2})$ Theorem 1.2.6(ii) can be re-phrased in the following even more natural formulation, which shows that $\mathsf{k}_\alpha^{(s/2)}$ can be equivalently qualified as a *form perturbation* of $(-\Delta)^{\frac{s}{2}}$.

PROPOSITION 1.2.7. *Let $s \in (1,\frac{3}{2})$. The self-adjoint extensions in $L^2(\mathbb{R})$ of $\mathring{\mathsf{k}}^{(s/2)}$ form the family $(\mathsf{k}_\alpha^{(s/2)})_{\alpha\in\mathbb{R}\cup\{\infty\}}$, where $\alpha = 0$ labels the Friedrichs extension given by (1.65), $\alpha = \infty$ labels the ordinary self-adjoint fractional Laplacian $(-\Delta)^{s/2}$, and for $\alpha \in \mathbb{R}\setminus\{0\}$ one has*

$$\begin{aligned} \mathcal{D}[\mathsf{k}_\alpha^{(s/2)}] &= H_0^{s/2}(\mathbb{R}\backslash\{0\}) \dotplus \mathrm{span}\{\mathsf{G}_{s,\lambda}\} = H^{s/2}(\mathbb{R}) \\ \mathsf{k}_\alpha^{(s/2)}[g] &= \||\nabla|^{\frac{s}{2}} g\,\|^2_{L^2(\mathbb{R})} - \frac{1}{\alpha}\,|g(0)|^2 \end{aligned} \tag{1.77}$$

*for every $\lambda > 0$.*

The quadratic form (1.77) can be also used in higher deficiency index regimes to define a sub-family of the whole class of self-adjoint extension of $\mathring{\mathsf{k}}^{(s/2)}$

**1.2.3. High deficiency index (high fractional power) scenario.** Let us outline in this Section how the previous constructions of the self-adjoint extensions of $\mathring{\mathsf{k}}^{(s/2)}$ get modified when $s > \frac{d}{2}+1$.

We recall from Lemma 1.2.1 that when $s \in I_n^{(d)}(\frac{d}{2}+n-1,\frac{d}{2}+n)$, $n \in \mathbb{N}$, one has

$$\ker\big((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}\big) = \mathrm{span}\Big\{u^\lambda_{\gamma_1,\dots,\gamma_d}\,\Big|\,\gamma_1,\dots,\gamma_d \in \mathbb{N}_0\,,\ \sum_{j=1}^d \gamma_j \leqslant n-1\Big\}, \tag{1.78}$$

having defined

$$\widehat{v^\lambda}_{\gamma_1,\dots,\gamma_d}(p) := \frac{p_1^{\gamma_1}\cdots p_d^{\gamma_d}}{(p^2+\lambda)^{\frac{s}{2}}}\,. \tag{1.79}$$

The same extension scheme applied in Section 1.2.1 provides an analogous classification of all the self-adjoint extensions in the case of generic deficiency index $\mathcal{J}(s,d)$, where now *each* extension of $\mathring{\mathsf{k}}^{(s/2)}$ is an operator $\mathsf{k}_T^{(s/2)}$ labelled by a self-adjoint operator $T$ in some subspace $\mathcal{D}(T)$ of $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}\big) \cong \mathbb{C}^{\mathcal{J}(s,d)}$, hence labelled by some $N\times N$ hermitian matrix, $1 \leqslant N \leqslant \mathcal{J}(s,d)$.

Explicitly (see, e.g., Theorem 3.4 in Ref. [**45**]),

$$
\begin{aligned}
\mathcal{D}(\mathsf{k}_T^{(s/2)}) \;&=\; \left\{ g\in L^2(\mathbb{R}^d) \;\middle|\; \begin{array}{c} g = f + (\mathring{\mathsf{k}}_F^{(s/2)} + \lambda\mathbb{1})^{-1}(Tu+w) + u \\ \text{where } f\in H_0^s(\mathbb{R}^d\backslash\{0\})\,,\ u\in\mathcal{D}(T)\,, \\ w\in\ker\big((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}\big)\cap\mathcal{D}(T)^\perp \end{array} \right\} \\
(\mathsf{k}_T^{(s/2)} + \lambda\mathbb{1})g \;&=\; ((-\Delta)^{s/2} + \lambda)F_\lambda \\
&\qquad F_\lambda \;:=\; f + (\mathring{\mathsf{k}}_F^{(s/2)} + \lambda\mathbb{1})^{-1}(Tu+w) \;\in\; H^s(\mathbb{R}^d)\,,
\end{aligned}
\tag{1.80}
$$

with $\mathsf{k}_F^{(s/2)}$ denoting the Friedrichs extension.

The theory provides also a counterpart classification of the the quadratic forms of the extensions (see Ref. [**45**], Theorem 3.6).

The above formulas show that for high powers $s$ the operator $\mathring{\mathsf{k}}^{(s/2)}$ has a richer variety (a $\mathcal{J}(s,d)^2$-parameter family) of self-adjoint extensions. The parametrising matrix $T$ determines a more complicated set of 'boundary conditions' between the regular part of a generic element of the extension domain, which has $H^s$-regularity, and the singular part, and the resulting constraint involves the evaluation at $x=0$ of some number of partial derivatives of the regular part.

This construction produces finite-rank perturbations in the resolvent sense, hence extensions that are all semi-bounded from below and may admit a (finite) number of negative eigenvalues, up to $\mathcal{J}(s,d)$, counting the multiplicity.

Unlike the case of deficiency index 1, depending on the extension parameter $T$ the large-$p$ vanishing behaviour in momentum space of the singular component may be milder than that of the Green function, and therefore the local singularity of $g$ in position space may be more severe than the behaviour of the Green function as $x\to 0$.

Let us comment on how the *worst leading singularity* at $x=0$ of a generic function $g\in\mathcal{D}(\mathsf{k}_T^{(s/2)})$ depends on $s$ and $d$.

As expressed by (1.80), such a singularity is due to the singular component of $g$, namely to those functions of type $u^\lambda_{\gamma_1,\dots,\gamma_d}$ that span $\mathcal{D}(T)$. When $s\in I_n^{(d)}$ the worst local singularity occurs when such functions decrease at infinity in momentum coordinates with the slowest possible vanishing rate compatible with $s$ and $d$, that is, when $\gamma_1+\cdots+\gamma_d = n-1$.

Let $u$ be any such most singular function, which then behaves as $|\widehat{u}(p)|\approx |p|^{-(s+1-n)}$ as $|p|\to+\infty$. Then $|u(x)|\approx|x|^{-(d-1+n-s)}$ as $x\to 0$. Since the map

$$
I_n^{(d)} \;\ni\; s \;\mapsto\; d-1+n-s
$$

is monotone decreasing and takes values in $(\frac{d}{2}-1,\frac{d}{2})$, if the extension $\mathsf{k}_T^{(s/2)}$ is such that $\mathcal{D}(T)\ni u$, then the functions in $\mathcal{D}(\mathsf{k}_T^{(s/2)})$ display a local singularity that ranges from $|x|^{-\frac{d}{2}}$ to $|x|^{-\frac{d-1}{2}}$ as long as $s$ increases in $I_n^{(d)}$, precisely as (1.64) when $s$ increases in $I_1^{(3)}$.

Noticeably, at the transition values $s\in\mathbb{N}+\frac{1}{2}$ the above picture undergoes a discontinuity in $s$, due to the further control of one more derivative in $\mathcal{D}(\mathring{\mathsf{k}}^{(s/2)})$, as a consequence of Sobolev's Lemma, and consequently to emergence in $\ker\big((\mathring{\mathsf{k}}^{(s/2)})^* + \lambda\mathbb{1}\big)$ of elements that in momentum coordinates vanish more slowly at infinity.

CHAPTER 2

# Approximation by means of regular Schrödinger operators

In this Chapter we study the construction of singular perturbations of the fractional Laplacian as a limit of more regular operators. For the sake of concreteness, we consider the case of a single point interaction centred at the origin.

The extension theory approach, discussed in Chapter 1, is surely satisfactory from the point of view of the interpretation of the output operator. However, it obfuscates an amount of physical meaning, since it does not provide information, as the intuition would make one expect instead, on how the actual singular perturbation is approximatively realised as a genuine pseudo-differential operator $(-\Delta)^{s/2} + V(x)$ with a regular potential $V$ centred around $x = 0$, with sufficiently short range and strong magnitude.

For the non-fractional Laplacian $-\Delta$ in $L^2(\mathbb{R}^d)$, the realisation of a singular perturbation at the origin by means of approximating Schrödinger operators $-\Delta + V_\varepsilon$ with regular potentials $V_\varepsilon$ spiking up and shrinking around $x = 0$ at a spatial scale $\varepsilon^{-1}$ in the limit $\varepsilon \downarrow 0$ is known since long for dimension $d = 1$ [**10**], $d = 2$ [**8**], and $d = 3$ [**7**] (we also refer to [**9, 11**] for a comprehensive overview), that is, all the dimensions in which non-trivial singular perturbations exist. In the fractional setting, instead, it was a recent achievement [**86**], which rises up the conceptually new issue of how a *local* potential $V_\varepsilon$ can be suitably re-scaled so as to produce the desired perturbation of the *non-local* operator $(-\Delta)^{s/2}$.

For concreteness of the presentation, we consider the case of deficiency index 1 only, and for simplicity we omit further the explicit discussion of the 'endpoint' values of $s$, namely the largest possible value, at given $d$, compatible with $\mathscr{J}(s,d) = 1$. As expressed by (1.44), this amounts to analysing the regime $s \in (\frac{1}{2}, \frac{3}{2})$ in $d = 1$, $s \in (1,2)$ in $d = 2$, $s \in (\frac{3}{2}, \frac{5}{2})$ in $d = 3$, etc. We shall refer to such cases as the '$\mathscr{J} = 1$ scenario'. For this scenario we then discuss how to realise the corresponding extensions in the limit of Schrödinger operators with fractional Laplacian and shrinking potentials, say, $(-\Delta)^{s/2} + V_\varepsilon$ as $\varepsilon \downarrow 0$.

We distinguish two possibilities, according to the local behavior of the Green function $\mathsf{G}_{s,\lambda}$ defined in (1.39).

- RESONANCE-DRIVEN CASE: $s < d$, that is, the regime for which the Green function has local singularity.
- RESONANCE-INDEPENDENT CASE: $s > d$, that is, the regime for which the Green function is regular.

We mention also that an analogous dichotomy occurs when the deficiency index of $(-\Delta)^{s/2}|_{C_0^\infty(\mathbb{R}^d \setminus \{0\})}$ is larger than 1: the singular (non-$H^s$) component of the elements in the domain of the considered self-adjoint extension may or may not display a local singularity as $x \to 0$. The 'resonance' jargon has to do with how the limit of shrinking potentials must be organised in order to reach a self-adjoint extension of $(-\Delta)^{s/2}|_{C_0^\infty(\mathbb{R}^d \setminus \{0\})}$ in one case or in the other. Extensions in the locally regular case can be reached as $\varepsilon \downarrow 0$ through suitably rescaled versions $V_\varepsilon$ of a given potential $V$ with no further prescription on $V$ but those technical assumptions ensuring that the limit itself is well-posed. Instead, extensions in the locally singular case can only be reached if the unscaled operator $(-\Delta)^{s/2}+V$ admits a *zero-energy resonance*, a spectral behaviour at the bottom of its essential spectrum which we shall define in due time and roughly speaking amounts to the existence of a suitably decaying, non square-integrable, $L^2_{\mathrm{loc}}$-solution $f$ to $((-\Delta)^{s/2}+V)f=0$. In a sense that we shall make precise, this difference is due to the fact that a zero-energy resonance is needed in the approximating fractional Schrödinger operator in order to reproduce in the limit the locally singular behaviour in the domain of the considered self-adjoint extension.

In view of the above alternative, we make the following presentational choice. Since in all dimensions $d$ but $d=1$ the interval $s\in(\frac{d}{2},\frac{d}{2}+1)$ corresponding to deficiency index 1 lies strictly below the transition value $s=d$ that separates the locally regular from the locally singular regime, as a representative of any such value of $d$ for concreteness we choose $d=3$: the discussion on the limit of shrinking potentials would then be *immediately exportable to any other* $d\geqslant 2$. Next to that, we also discuss the case $d=1$, where instead the interval $s\in(1,\frac{3}{2})$ corresponding to deficiency index 1 contains the transition value $s=1$. As is evident from the discussion of Chapter 1, the self-adjoint extensions of $(-\Delta)^{s/2}|_{C^\infty_0(\mathbb{R}\setminus\{0\})}$ exhibits different features depending on whether $s<1$ or $s>1$, which reflect into the different types of approximation. The $s=1$ case, albeit technically more involved, could be covered as well. For simplicity we will ignore it in our discussion.

The material of this Chapter is organised as follows. In Section 2.1 we present the approximation scheme in three dimensions in terms of fractional Schrödinger operators with regular, shrinking potentials. In Section 2.2 we present the one-dimensional analogue, with the two distinct approximation schemes, for the resonance-driven and the resonance-independent cases. Section 2.3 contains the proof of the three-dimensional limit. Section 2.4 contains the proof of the one-dimensional limit in the resonance-driven case. From the technical point of view, the argument here is completely analogous to that of Section 2.3, as the 3D case too is resonance-driven. Section 2.5 contains instead the proof of the one-dimensional limit in the resonance-independent case.

## 2.1. Approximation scheme in dimension three

Our goal is to qualify each of the three-dimensional extensions $\mathsf{k}_\alpha^{(s/2)}$ identified in Theorem 1.2.3, as suitable limits of approximating fractional Schrödinger operators with finite range potentials.

It is convenient to introduce the class $\mathcal{R}_{s,d}$, $d\in\mathbb{N}$, $s\in(\frac{d}{2},d)$, of measurable functions $V:\mathbb{R}^d\to\mathbb{C}$ such that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}x\,\mathrm{d}y\,\frac{|V(x)|\,|V(y)|}{|x-y|^{2(d-s)}}\;=:\;\|V\|^2_{\mathcal{R}_{s,d}}\;<\;+\infty\,. \tag{2.1}$$

$\mathcal{R}_{2,3}$ is the well-known Rollnick class on $\mathbb{R}^3$. Clearly, $\mathcal{R}_{s,d}\supset C^\infty_0(\mathbb{R}^d)$.

For each $s\in(\frac{3}{2},\frac{5}{2})$ we make the following assumption.

**Assumption (I$_s$).**

(i) $V:\mathbb{R}^3\to\mathbb{R}$ is a measurable function in $L^1(\mathbb{R}^3,\langle x\rangle^{2s-3}\mathrm{d}x)\cap\mathcal{R}_{s,3}$.

(ii) $\eta:\mathbb{R}\to\mathbb{R}^+$ is a continuous function satisfying $\eta(0)=\eta(1)=1$ and

$$\eta(\varepsilon)\;=\;1+\eta_s\,\varepsilon^{3-s}+o(\varepsilon^{3-s})\qquad\text{as }\varepsilon\downarrow 0$$

for some $\eta_s\in\mathbb{R}$ that we call the *strength* of the *distortion factor* $\eta$.

LEMMA 2.1.1. *Let* $V:\mathbb{R}^3\to\mathbb{R}$ *belong to* $L^1(\mathbb{R}^3)\cap\mathcal{R}_{s,3}$ *for some* $s\in(\frac{3}{2},3)$. *Then:*

(i) *for every* $\lambda \geqslant 0$, $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$ *is a Hilbert-Schmidt operator on* $L^2(\mathbb{R}^3)$*;*
(ii) $|V|^{\frac{1}{2}} \ll (-\Delta)^{\frac{s}{4}}$ *in the sense of infinitesimally bounded operators;*
(iii) *the operator* $(-\Delta)^{\frac{s}{2}}+V$ *defined as a form sum is self-adjoint on* $L^2(\mathbb{R}^3)$*, and* $\sigma_{\mathrm{ess}}((-\Delta)^{\frac{s}{2}}+V)=[0,+\infty)$*.*

PROOF. (i) $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$ acts as an integral operator with kernel

$$\mathcal{K}_{s,\lambda}(x,y) \;:=\; |V(x)|^{\frac{1}{2}}\mathsf{G}_{s,\lambda}(x-y)|V(y)|^{\frac{1}{2}}\,,$$

and its Hilbert-Schmidt norm is estimated as

$$\begin{aligned}\Big\||V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}\Big\|^2_{\mathrm{H.S.}} \;&=\; \iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,|\mathcal{K}_{s,\lambda}(x,y)|^2\\ \leqslant\; 2(\Lambda_s^{(3)})^2\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,&\frac{|V(x)|\,|V(y)|}{|x|^{2(3-s)}}+2\|\mathsf{J}_{s,\lambda}\|^2_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,|V(x)|\,|V(y)|\\ \leqslant\; 2(\Lambda_s^{(3)})^2\|V\|^2_{\mathcal{R}_{s,3}}&+2\|\mathsf{J}_{s,\lambda}\|^2_{L^\infty}\|V\|^2_{L^1}\;<\;+\infty\,,\end{aligned}$$

having used (1.49)-(1.50) in the second step.

(ii) The map $\lambda\mapsto|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$ is continuous from $(0,+\infty)$ to the space of Hilbert-Schmidt operators, and by dominated convergence

$$\lim_{\lambda\to+\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,|V(x)|\,|\mathsf{G}_{s,\lambda}(x-y)|^2\,|V(y)|\;=\;0\,.$$

Therefore, for arbitrary $\varepsilon>0$ it is possible to find $\lambda_\varepsilon>0$ large enough such that

$$\begin{aligned}\varepsilon\;&\geqslant\;\Big\||V|^{\frac{1}{2}}\big((-\Delta)^{\frac{s}{2}}+\lambda_\varepsilon\mathbb{1}\big)^{-1}|V|^{\frac{1}{2}}\Big\|^2_{\mathrm{H.S.}}\\ &=\;\Big\|\big((-\Delta)^{\frac{s}{2}}+\lambda_\varepsilon\mathbb{1}\big)^{-\frac{1}{2}}\,|V|\,\big((-\Delta)^{\frac{s}{2}}+\lambda_\varepsilon\mathbb{1}\big)^{-\frac{1}{2}}\Big\|^2_{\mathrm{H.S.}}\\ &\geqslant\;\Big\|\big((-\Delta)^{\frac{s}{2}}+\lambda_\varepsilon\mathbb{1}\big)^{-\frac{1}{2}}\,|V|\,\big((-\Delta)^{\frac{s}{2}}+\lambda_\varepsilon\mathbb{1}\big)^{-\frac{1}{2}}\Big\|^2_{\mathrm{op}}\,,\end{aligned}$$

which implies, for some $b_\varepsilon>0$,

$$|\langle\varphi,V\varphi\rangle_{L^2}|\;\leqslant\;\varepsilon\,\langle\varphi,(-\Delta)^{\frac{s}{2}}\varphi\rangle_{L^2}+b_\varepsilon\|\varphi\|^2_{L^2}\qquad\forall\varphi\in\mathcal{D}[(-\Delta)^{\frac{s}{2}}]=H^{\frac{s}{2}}(\mathbb{R}^3)\,,$$

and hence $|V|^{\frac{1}{2}}\ll(-\Delta)^{\frac{s}{4}}$.

(iii) The statement follows at once from (ii). □

For given $V$ and $\eta$ satisfying Assumption ($\mathrm{I}_s$), let us set

$$h_\varepsilon^{(s/2)}\;:=\;(-\Delta)^{s/2}+V_\varepsilon\,,\qquad V_\varepsilon(x)\;:=\;\frac{\eta(\varepsilon)}{\varepsilon^s}\,V(\tfrac{x}{\varepsilon})\,,\qquad\varepsilon>0\,. \tag{2.2}$$

For every $\varepsilon>0$ the operator $h_\varepsilon^{(s/2)}$, defined as a form sum, is self-adjoint on $L^2(\mathbb{R}^3)$ and $\sigma_{\mathrm{ess}}(h_\varepsilon^{(s/2)})=[0,+\infty)$, as it follows from Lemma 2.1.1(iii).

The spectral properties of the unscaled operator $(-\Delta)^{s/2}+V$ at the bottom of the essential spectrum are crucial for the limit $\varepsilon\downarrow 0$ in $h_\varepsilon^{(s/2)}$. In the next Theorem we qualify the zero-energy behaviour of $(-\Delta)^{s/2}+V$.

THEOREM 2.1.2. *Let* $s\in(\frac{3}{2},\frac{5}{2})$, $V\in L^1(\mathbb{R}^3,\langle x\rangle^{2s-3}\mathrm{d}x)\cap\mathcal{R}_{s,3}$*, real-valued. Let* $v:=|V|^{\frac{1}{2}}$ *and* $u:=|V|^{\frac{1}{2}}\mathrm{sign}(V)$*.*

(i) *The operator* $u(-\Delta)^{-\frac{s}{2}}v$ *is compact on* $L^2(\mathbb{R}^3)$*.*

*Assume in addition that*

$$u(-\Delta)^{-\frac{s}{2}}v\,\phi\;=\;-\phi\qquad\textit{for some }\phi\in L^2(\mathbb{R}^3)\setminus\{0\} \tag{2.3}$$

*and define*

$$\psi\;:=\;(-\Delta)^{-\frac{s}{2}}v\,\phi\,. \tag{2.4}$$

*Then:*

(ii) $\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ *and* $\big((-\Delta)^{s/2}+V\big)\psi = 0$ *in the sense of distributions,*

(iii) $\langle v,\phi\rangle_{L^2} \;=\; -\displaystyle\int_{\mathbb{R}^3}\mathrm{d}x\,V(x)\psi(x)$,

(iv) $\psi \in L^2(\mathbb{R}^3) \Leftrightarrow \langle v,\phi\rangle_{L^2}=0$*, in which case* $\psi\in\mathcal{D}((-\Delta)^{s/2}+V)$.

Proof. The fact that for a real-valued $V\in L^1(\mathbb{R}^3)\cap\mathcal{R}_{s,3}$ the operator $u(-\Delta)^{\frac{s}{2}}v$ is Hilbert-Schmidt follows from Lemma 2.1.1(i), thus part (i) is proved.

Let us split

$$\begin{aligned}\psi(x) \;&=\; ((-\Delta)^{-\frac{s}{2}}v\phi)(x) \;=\; \int_{\mathbb{R}^3}\mathrm{d}y\,\frac{\Lambda_s^{(3)}}{|x-y|^{3-s}}\,v(y)\phi(y)\\ &=\; \frac{\Lambda_s^{(3)}\langle v,\phi\rangle_{L^2}}{|x|^{3-s}}+\Lambda_s^{(3)}\int_{\mathbb{R}^3}\mathrm{d}y\,\Big(\frac{1}{|x-y|^{3-s}}-\frac{1}{|x|^{3-s}}\Big)v(y)\phi(y) \qquad \text{(a)}\\ &\equiv\; \frac{\Lambda_s^{(3)}\langle v,\phi\rangle_{L^2}}{|x|^{3-s}}+\psi_1(x)\,,\end{aligned}$$

where $\Lambda_s$ is the constant defined in (1.50). We show now that $\psi_1\in L^2(\mathbb{R}^3)$. To this aim, we observe that setting $\widehat{y}:=\frac{y}{|y|}$ one has

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(\frac{1}{|x-y|^{3-s}}-\frac{1}{|x|^{3-s}}\Big)^2\\ &\qquad=\; |y|^{2s-3}\int_{\mathbb{R}^3}\mathrm{d}x\Big(\frac{|x-\widehat{y}\,|^{3-s}-|x|^{3-s}}{|x-\widehat{y}\,|^{3-s}|x|^{3-s}}\Big)^2\\ &\qquad\lesssim\; |y|^{2s-3}\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(\frac{\langle x\rangle^{2-s}}{|x-\widehat{y}\,|^{3-s}|x|^{3-s}}\Big)^2,\end{aligned}$$

having used the change of variable $x\mapsto|y|\,x$ in the first step and the uniform bound $\big||x-\widehat{y}\,|^{3-s}-|x|^{3-s}\big|\lesssim\langle x\rangle^{2-s}$ in the last step. Since $s\in(\frac{3}{2},\frac{5}{2})$, the last integral above is finite, thus we deduce

$$\int_{\mathbb{R}^3}\mathrm{d}x\,\Big(\frac{1}{|x-y|^{3-s}}-\frac{1}{|x|^{3-s}}\Big)^2\;\lesssim\;|y|^{2s-3}\,. \qquad \text{(b)}$$

As a consequence,

$$\begin{aligned}\|\psi_1\|^2_{L^2(\mathbb{R}^3)}\;&\lesssim\;\int_{\mathbb{R}^3}\mathrm{d}x\,\Big|\int_{\mathbb{R}^3}\mathrm{d}y\,\Big(\frac{1}{|x-y|^{3-s}}-\frac{1}{|x|^{3-s}}\Big)v(y)\phi(y)\Big|^2\\ &\leqslant\;\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,\Big(\frac{1}{|x-y|^{3-s}}-\frac{1}{|x|^{3-s}}\Big)^2|V(y)|\\ &\lesssim\;\int_{\mathbb{R}^3}\mathrm{d}y\,|V(y)|\,|y|^{2s-3}\;<\;+\infty\,,\end{aligned}$$

as follows from a Cauchy-Schwartz inequality in the second step, from the bound (b) in the third step, and from the assumption $V\in L^1(\mathbb{R}^3,\langle x\rangle^{2s-3}\mathrm{d}x)$ in the last step.

Since $|x|^{-(3-s)}\in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, because $s>\frac{3}{2}$, then identity (a) implies that $\psi\in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Moreover, from (2.3) and (2.4) one finds

$$V\psi\;=\;vu(-\Delta)^{-\frac{s}{2}}v\phi\;=\;-v\phi\;=\;-(-\Delta)^{\frac{s}{2}}\psi\,,$$

whence $((-\Delta)^{\frac{s}{2}}+V)\psi=0$ distributionally. This completes the proof of part (ii).

Using (2.4) and the distributional identity proved in part (ii) one finds

$$\langle v,\phi\rangle_{L^2}\;=\;\int_{\mathbb{R}^3}\mathrm{d}x\,v(x)\,\phi(x)\;=\;\int_{\mathbb{R}^3}\mathrm{d}x\,((-\Delta)^{\frac{s}{2}}\psi)(x)\;=\;-\int_{\mathbb{R}^3}\mathrm{d}x\,V(x)\,\psi(x)\,,$$

which proves part (iii).

Last, the identity (a) also implies that $\psi \in L^2(\mathbb{R}^3)$ is equivalent to $\langle v, \phi\rangle_{L^2} = 0$. When this is the case, the identity $((-\Delta)^{\frac{s}{2}} + V)\psi = 0$ holds in the $L^2$-sense, implying that $\psi \in \mathcal{D}((-\Delta)^{\frac{s}{2}} + V)$. This completes the proof of part (iv). $\square$

When a $L^2$-function $\phi$ exists that satisfies (2.3) and the corresponding function $\psi$ defined by (2.4) belongs to $L^2_{\mathrm{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)$ we say that $(-\Delta)^{s/2} + V$ is *zero-energy resonant* and that $\psi$ is a *zero-energy resonance* for $(-\Delta)^{s/2} + V$. If for the zero-energy resonant operator $(-\Delta)^{s/2} + V$ the eigenvalue $-1$ of $u(-\Delta)^{-\frac{s}{2}} v$ is non-degenerate, then we say that the resonance is *simple*. Of course, if $\psi \in L^2(\mathbb{R}^3)$, then $\psi$ is an eigenfunction of $(-\Delta)^{s/2} + V$ with eigenvalue zero.

In [**85**] a wide class of zero-energy resonant operators $(-\Delta)^{s/2} + V$ is exhibited. By means of a more refined discussion, in the same spirit of [**73**], one could also identify the threshold coupling parameter $\lambda \in \mathbb{R}$, for a given potential $V$ in a suitable class, for which $(-\Delta)^{s/2} + \lambda V$ is zero-energy resonant.

Let us now formulate our main result for dimension three. It is the control of the approximation, in the norm resolvent sense, of the singular perturbation operator $\mathsf{k}_\alpha^{(s/2)}$ by means of Schrödinger operators with the $\frac{s}{2}$-th fractional Laplacian and shrinking potentials $V_\varepsilon$ around the origin. We shall prove it in Section 2.3.

THEOREM 2.1.3. *Let $s \in (\frac{3}{2}, \frac{5}{2})$. Given a potential $V$ and a distortion factor $\eta$ with strength $\eta_s$ satisfying Assumption* (I$_s$)*, for every $\varepsilon > 0$ let $h_\varepsilon^{(s/2)} = (-\Delta)^{s/2} + V_\varepsilon$ be the corresponding self-adjoint Schrödinger operator defined in* (2.2) *with the $\frac{s}{2}$-th fractional Laplacian and the shrinking potential $V_\varepsilon$.*

(i) *If $(-\Delta)^{s/2} + V$ is not zero-energy resonant, then $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon\downarrow 0} (-\Delta)^{s/2}$ in the norm-resolvent sense on $L^2(\mathbb{R}^3)$.*
(ii) *If $(-\Delta)^{s/2} + V$ admits a* simple *zero-energy resonance $\psi$, then for*

$$\alpha := -\eta_s \Big| \int_{\mathbb{R}^3} \mathrm{d}x\, V(x)\psi(x) \Big|^{-2}$$

*one has $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon\downarrow 0} \mathsf{k}_\alpha^{(s/2)}$ in the norm-resolvent sense on $L^2(\mathbb{R}^3)$.*

The two possible alternatives in Theorem 2.1.3 are the manifestation of the locally singular, resonant-driven nature of the limit: the limit is well-posed for a generic class of potentials $V$, but it is non-trivial only if additionally $(-\Delta)^{s/2} + V$ is zero-energy resonant.

By a simple scaling argument one sees that $(-\Delta)^{s/2} + V_\varepsilon$ remains zero-energy resonant for any $\varepsilon > 0$ if the scaling is 'purely geometric', namely with trivial distortion factor, $\eta(\varepsilon) \equiv 1$. In this case, the signature of the resonance is particularly transparent: as stated in Theorem 2.1.3(ii), the limit $\varepsilon \downarrow 0$ with $\eta(\varepsilon) \equiv 1$ produces the extension parametrised by $\alpha = 0$ and we see from Theorem 1.2.3(iv) that the negative eigenvalue of $\mathsf{k}_\alpha^{(s/2)}$ when $\alpha < 0$ converges to 0 as $\alpha \uparrow 0$, with the corresponding eigenfunction $\mathsf{G}_{s,\lambda=|E_\alpha^{(s)}|}$ converging pointwise to $\mathsf{G}_{s,0}(x) = \frac{\Lambda_s^{(3)}}{|x|^{(3-s)}}$ (see (1.49)-(1.50) and (1.63) above). In fact, as already discussed in Remark 1.1.7 for the special case $s = 2$, the $L^2_{\mathrm{loc}} \setminus L^2$-function $\mathsf{G}_{s,0}$ can be actually regarded as a *zero-energy resonance* for $\mathsf{k}_{\alpha=0}^{(s/2)}$ (the local square-integrability following from $s \in (\frac{3}{2}, \frac{5}{2})$).

## 2.2. Approximation scheme in dimension one

In this Section we qualify each of the one-dimensional extensions $\mathsf{k}_\alpha^{(s/2)}$, identified in Theorem 1.2.6, as suitable limits of approximating fractional Schrödinger operators with finite range potentials. Unlike the 3D setting, here the defect-one regime $s \in (\frac{1}{2}, \frac{3}{2})$ is separated by the transition value $s = 1$, below which we are in the *locally singular case* for the Green function $G_{s,\lambda}$, and above which we are in the *locally regular case*. This will result in different assumptions on the approximating potentials and different schemes for the resolvent limit.

We therefore proceed by splitting our discussion into the two above-mentioned cases.

### 2.2.1. Locally singular, resonance-driven case.

This is the regime $s \in (\frac{1}{2}, 1)$. The Green function $\mathsf{G}_{s,\lambda}$ has a local singularity (see (1.49)-(1.50)). For each $s \in (\frac{1}{2}, 1)$ we make the following assumption (the class $R_{s,d}$ was introduced in (2.1)).

**Assumption ($\mathbf{I}_s^-$).**

(i) $V : \mathbb{R} \to \mathbb{R}$ is a measurable function in $L^1(\mathbb{R}, \langle x\rangle^{2s-1}\mathrm{d}x) \cap \mathcal{R}_{s,1}$;

(ii) $\eta : \mathbb{R} \to \mathbb{R}^+$ is a continuous function satisfying $\eta(0) = \eta(1) = 1$ and

$$\eta(\varepsilon) \;=\; 1 + \eta_s\, \varepsilon^{1-s} + o(\varepsilon^{1-s}) \qquad \text{as } \varepsilon \downarrow 0$$

for some $\eta_s \in \mathbb{R}$ that we call the *strength* of the *distortion factor* $\eta$.

For given $V$ and $\eta$ satisfying Assumption ($\mathrm{I}_s^-$), let us set

$$h_\varepsilon^{(s/2)} \;:=\; (-\Delta)^{s/2} + V_\varepsilon\,, \qquad V_\varepsilon(x) \;:=\; \frac{\eta(\varepsilon)}{\varepsilon^s}\, V(\tfrac{x}{\varepsilon})\,, \qquad \varepsilon > 0\,. \tag{2.5}$$

For every $\varepsilon > 0$ the operator $h_\varepsilon^{(s/2)}$, defined as a form sum, is self-adjoint on $L^2(\mathbb{R}^3)$ and $\sigma_{\mathrm{ess}}(h_\varepsilon^{(s/2)}) = [0, +\infty)$ (Lemma 2.4.2(iii)).

The zero-energy spectral behaviour of $(-\Delta)^{s/2} + V$, which is crucial for the limit $\varepsilon \downarrow 0$ in $h_\varepsilon^{(s/2)}$, is characterised by the following result, whose proof proceeds along the same lines as in Theorem 2.1.2.

THEOREM 2.2.1. *Let $s \in (\frac{1}{2}, 1)$, $V \in L^1(\mathbb{R}, \langle x\rangle^{2s-1}\mathrm{d}x) \cap \mathcal{R}_{s,1}$, real-valued. Let $v := |V|^{\frac{1}{2}}$ and $u := |V|^{\frac{1}{2}}\mathrm{sign}(V)$.*

(i) *The operator $u(-\Delta)^{-\frac{s}{2}}v$ is compact on $L^2(\mathbb{R})$.*

*Assume in addition that*

$$u(-\Delta)^{-\frac{s}{2}}v\,\phi \;=\; -\phi \qquad \text{for some } \phi \in L^2(\mathbb{R}) \setminus \{0\} \tag{2.6}$$

*and define*

$$\psi \;:=\; (-\Delta)^{-\frac{s}{2}}v\,\phi\,. \tag{2.7}$$

*Then:*

(ii) *$\psi \in L^2_{\mathrm{loc}}(\mathbb{R})$ and $\big((-\Delta)^{s/2} + V\big)\psi = 0$ in the sense of distributions,*

(iii) $\langle v, \phi\rangle_{L^2} \;=\; -\displaystyle\int_{\mathbb{R}} \mathrm{d}x\, V(x)\psi(x)$,

(iv) *$\psi \in L^2(\mathbb{R}) \Leftrightarrow \langle v, \phi\rangle_{L^2} = 0$, in which case $\psi \in \mathcal{D}((-\Delta)^{s/2} + V)$.*

With the same terminology of Section 2.1, we say that $(-\Delta)^{s/2} + V$ is *zero-energy resonant* and that $\psi$ is a *zero-energy resonance* for $(-\Delta)^{s/2} + V$ when there exists a non-zero $L^2$-function $\phi$ satisfying (2.6) and the corresponding function $\psi$ defined by (2.7) belongs to $L^2_{\mathrm{loc}}(\mathbb{R})\setminus L^2(\mathbb{R})$. If, for the zero-energy resonant operator $(-\Delta)^{s/2}+V$, the eigenvalue $-1$ of $u(-\Delta)^{-\frac{s}{2}}v$ is non-degenerate, then the resonance is *simple*. In [**85**], explicit examples of zero-energy resonant operators $(-\Delta)^{s/2}+V$ on $L^2(\mathbb{R})$ are exhibited.

Here below is our first main result in dimension one, relative to the resonance-driven regime.

THEOREM 2.2.2. *Let $s \in (\frac{1}{2}, 1)$. Given a potential $V$ and a distortion factor $\eta$ with strength $\eta_s$ satisfying Assumption* ($\mathrm{I}_s^-$), *for every $\varepsilon > 0$ let $h_\varepsilon^{(s/2)} = (-\Delta)^{s/2} + V_\varepsilon$ be the corresponding self-adjoint Schrödinger operator defined in* (2.5) *with the $\frac{s}{2}$-th fractional Laplacian and the shrinking potential $V_\varepsilon$.*

(i) *If $(-\Delta)^{s/2} + V$ is not zero-energy resonant, then $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon \downarrow 0} (-\Delta)^{s/2}$ in the norm-resolvent sense on $L^2(\mathbb{R})$.*
(ii) *If $(-\Delta)^{s/2} + V$ admits a* simple *zero-energy resonance $\psi$, then for*

$$\alpha \;:=\; -\eta_s \left| \int_{\mathbb{R}} \mathrm{d}x\, V(x)\psi(x) \right|^{-2}$$

*one has $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon \downarrow 0} \mathsf{k}_\alpha^{(s/2)}$ in the norm-resolvent sense on $L^2(\mathbb{R})$.*

We shall prove Theorem 2.2.2 in Section 2.4.

The alternative in Theorem 2.2.2 is completely analogous to that of Theorem 2.1.3, due to the the locally singular, resonant-driven nature of both limits: only for zero-energy resonant operators $(-\Delta)^{s/2} + V$ is the limit non-trivial.

The signature of the resonance is particularly transparent in the absence of distortion factor: when $\eta(\varepsilon) \equiv 1$ by scaling one sees that $(-\Delta)^{s/2} + V_\varepsilon$ remains zero-energy resonant for any $\varepsilon > 0$, and we may regard the limit operator $\mathsf{k}_{\alpha=0}^{(s/2)}$ too as zero-energy resonant, for the negative eigenvalue of $\mathsf{k}_\alpha^{(s/2)}$ when $|\alpha| \neq 0$ vanishes as $|\alpha| \to 0$ and the corresponding eigenfunctions becomes (proportional to) the $L^2_{\mathrm{loc}} \backslash L^2$-function $|x|^{-(1-s)}$ (see (1.76) above).

### 2.2.2. Locally regular, resonance-independent case.

This is the regime $s \in (1, \frac{3}{2})$. In contrast with the resonance-driven regime, *no spectral requirement* is now needed on the unscaled fractional operator $(-\Delta)^{s/2} + V$ and the scaling in $V_\varepsilon$ is *independent* of $s$. Thus, we make the following assumption.

**Assumption ($\mathbf{I}_s^+$).**

(i) $V : \mathbb{R} \to \mathbb{R}$ is a measurable function in $L^1(\mathbb{R})$.
(ii) $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a smooth function satisfying $\eta(1) = 1$.

Correspondingly, we set

$$h_\varepsilon^{(s/2)} \;:=\; (-\Delta)^{s/2} + V_\varepsilon\,, \qquad V_\varepsilon(x) \;:=\; \frac{\eta(\varepsilon)}{\varepsilon}\, V\big(\tfrac{x}{\varepsilon}\big)\,, \qquad \varepsilon > 0\,. \tag{2.8}$$

For every $\varepsilon > 0$ the operator $h_\varepsilon^{(s/2)}$, defined as a form sum, is self-adjoint on $L^2(\mathbb{R}^3)$ and $\sigma_{\mathrm{ess}}(h_\varepsilon^{(s/2)}) = [0, +\infty)$ (Lemma 2.5.1(iii)).

Here below is our second main result in dimension one, which, as opposite to Theorem 2.2.2, takes the following form.

THEOREM 2.2.3. *Let $s \in (1, \frac{3}{2})$. For every $\varepsilon > 0$ let $h_\varepsilon^{(s/2)} = (-\Delta)^{s/2} + V_\varepsilon$ be defined according to Assumption ($I_s^+$) and* (2.8)*. Then $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon \downarrow 0} \mathsf{k}_\alpha^{(s/2)}$ in the norm-resolvent sense on $L^2(\mathbb{R})$, where*

$$\alpha \;:=\; -\Big(\eta(0) \int_{\mathbb{R}} \mathrm{d}x\, V(x)\Big)^{-1}.$$

We shall prove Theorem 2.2.3 in Section 2.5.

## 2.3. Convergence of the 3D limit

The goal of this Section is to prove Theorem 2.1.3.

For chosen $s \in (\frac{3}{2}, \frac{5}{2})$, $\varepsilon > 0$, and $V$ and $\eta$ satisfying Assumption ($\mathrm{I}_s$), let us recall from (2.2) that $V_\varepsilon(x) = \frac{\eta(\varepsilon)}{\varepsilon^s} V(\frac{x}{\varepsilon})$ and let us define

$$
\begin{aligned}
v(x) &:= |V(x)|^{\frac{1}{2}}, & u(x) &:= |V(x)|^{\frac{1}{2}} \operatorname{sign}(V(x)), \\
v_\varepsilon(x) &:= |V_\varepsilon(x)|^{\frac{1}{2}}, & u_\varepsilon(x) &:= |V_\varepsilon(x)|^{\frac{1}{2}} \operatorname{sign}(V_\varepsilon(x)).
\end{aligned} \tag{2.9}
$$

Thus,

$$
v_\varepsilon(x) = \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{s/2}} v(\tfrac{x}{\varepsilon}), \qquad u_\varepsilon(x) = \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{s/2}} u(\tfrac{x}{\varepsilon}), \qquad v_\varepsilon u_\varepsilon = V_\varepsilon. \tag{2.10}
$$

The Hamiltonian $h_\varepsilon^{(s/2)} = (-\Delta)^{s/2} + V_\varepsilon$ defined in (2.2) as a form sum is self-adjoint on $L^2(\mathbb{R}^3)$, as guaranteed by Lemma 2.1.1(iii). An expression for its resolvent that is convenient in the present context is the Konno-Kuroda identity [**74**]. One has the following.

Lemma 2.3.1. *Let $V : \mathbb{R}^3 \to \mathbb{R}$ belong to $L^1(\mathbb{R}^3) \cap \mathcal{R}_{s,3}$ for some $s \in (\frac{3}{2}, \frac{5}{2})$. Then*

$$
\begin{aligned}
\big(h_\varepsilon^{(s/2)} + \lambda\mathbb{1}\big)^{-1} &= \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} - \\
&- \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} v_\varepsilon \Big(\mathbb{1} + u_\varepsilon \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} v_\varepsilon\Big)^{-1} u_\varepsilon \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1}
\end{aligned} \tag{2.11}
$$

*for every $\varepsilon > 0$ and every $-\lambda < 0$ in the resolvent set of $h_\varepsilon^{(s/2)}$, as an identity between bounded operators on $L^2(\mathbb{R}^3)$.*

Proof. The statement is precisely the application of the Konno-Kuroda resolvent identity, for which we follow the formulation presented in [**9**, Theorem B.1(b)], to the operator $(-\Delta)^{s/2} + v_\varepsilon u_\varepsilon$. For the validity of such identity two conditions are needed: the compactness of $u_\varepsilon((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} v_\varepsilon$ and the infinitesimal bound $|V|^{\frac{1}{2}} \ll (-\Delta)^{\frac{s}{4}}$. Both conditions are guaranteed by Lemma 2.1.1. □

Observe that the invertibility of $\mathbb{1} + u_\varepsilon((-\Delta)^{s/2} + \lambda\mathbb{1})^{-1} v_\varepsilon$ (with bounded inverse) is part of the statement of the Konno-Kuroda formula (2.11).

It is convenient to manipulate the identity (2.11) further so as to isolate terms in the r.h.s. which are easily controllable in the limit $\varepsilon \downarrow 0$. To this aim, let us introduce for each $\varepsilon > 0$ the unitary scaling operator $U_\varepsilon : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ defined by

$$
(U_\varepsilon f)(x) := \frac{1}{\varepsilon^{3/2}} f(\tfrac{x}{\varepsilon}). \tag{2.12}
$$

Its adjoint clearly acts as $(U_\varepsilon^* f)(x) = \varepsilon^{3/2} f(\varepsilon x)$. $U_\varepsilon$ induces the scaling transformations

$$
\begin{aligned}
U_\varepsilon^* v_\varepsilon U_\varepsilon &= \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{s/2}} v, \qquad U_\varepsilon^* u_\varepsilon U_\varepsilon = \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{s/2}} u, \\
U_\varepsilon^* \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} U_\varepsilon &= \varepsilon^s \big((-\Delta)^{s/2} + \lambda\varepsilon^s \mathbb{1}\big)^{-1},
\end{aligned} \tag{2.13}
$$

whose proof is straightforward.

Let us also introduce, for each $\varepsilon > 0$ and for each $\mu > 0$ such that $-\mu^s$ belongs to the resolvent set of $h_\varepsilon^{(s/2)}$, the operators

$$
\begin{aligned}
A_\varepsilon^{(s)} &:= \varepsilon^{-\frac{3-s}{2}} \big((-\Delta)^{s/2} + \mu^s \mathbb{1}\big)^{-1} (\eta(\varepsilon))^{-\frac{1}{2}} v_\varepsilon U_\varepsilon \\
B_\varepsilon^{(s)} &:= \eta(\varepsilon)\, u \big((-\Delta)^{s/2} + (\mu\varepsilon)^s \mathbb{1}\big)^{-1} v \\
C_\varepsilon^{(s)} &:= U_\varepsilon^* u_\varepsilon (\eta(\varepsilon))^{-\frac{1}{2}} \big((-\Delta)^{s/2} + \mu^s \mathbb{1}\big)^{-1} \varepsilon^{-\frac{3-s}{2}}.
\end{aligned} \tag{2.14}
$$

We shall see in a moment (Lemma 2.3.3) that $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$ are Hilbert-Schmidt operators on $L^2(\mathbb{R}^3)$. Most importantly for our purposes, the resolvent of $h_\varepsilon^{(s/2)}$ takes the following convenient form.

LEMMA 2.3.2. *Under the present assumptions,*

$$\big(h_\varepsilon^{(s/2)}+\mu^s\mathbb{1}\big)^{-1} \;=\; \big((-\Delta)^{s/2}+\mu^s\mathbb{1}\big)^{-1}-A_\varepsilon^{(s)}\varepsilon^{3-s}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)} \tag{2.15}$$

*for every $\varepsilon>0$ and every $\mu>0$ such that $-\mu^s$ belongs to the resolvent set of $h_\varepsilon^{(s/2)}$.*

PROOF. In formula (2.11) we set $\lambda=\mu^s$ and we insert $\mathbb{1}=U_\varepsilon U_\varepsilon^*$ in the second summand of the r.h.s. right after $((-\Delta)^{s/2}+\lambda\mathbb{1})^{-1}v_\varepsilon$. We then commute $U_\varepsilon^*$ all the way through by means of the scaling transformations (2.13): this way, we reproduce the product $A_\varepsilon^{(s)}\varepsilon^{3-s}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)}$. □

The limit $\varepsilon\downarrow 0$ can be monitored explicitly for $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$.

LEMMA 2.3.3. *For every $\varepsilon>0$, $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$ are Hilbert-Schmidt operators on $L^2(\mathbb{R}^3)$ with limit*

$$\lim_{\varepsilon\downarrow 0}A_\varepsilon^{(s)} \;=\; |\mathsf{G}_{s,\mu^s}\rangle\langle v| \tag{2.16}$$

$$\lim_{\varepsilon\downarrow 0}B_\varepsilon^{(s)} \;=\; B_0^{(s)} \;=\; u\,(-\Delta)^{-\frac{s}{2}}v \tag{2.17}$$

$$\lim_{\varepsilon\downarrow 0}C_\varepsilon^{(s)} \;=\; |u\rangle\langle\mathsf{G}_{s,\mu^s}| \tag{2.18}$$

*in the Hilbert-Schmidt operator norm.*

PROOF. By construction, see (2.10), (2.12), and (2.14) above,

$$\begin{aligned}(A_\varepsilon^{(s)}f)(x) \;&=\; \varepsilon^{-\frac{3-s}{2}}\,\varepsilon^{-\frac{s}{2}}\,\varepsilon^{-\frac{3}{2}}\int_{\mathbb{R}^3}\mathsf{G}_{s,\mu^s}(x-y)\,v(\tfrac{y}{\varepsilon})f(\tfrac{y}{\varepsilon})\,\mathrm{d}y\\ &=\;\int_{\mathbb{R}^3}\mathsf{G}_{s,\mu^s}(x-\varepsilon y)\,v(y)f(y)\,\mathrm{d}y \qquad\forall f\in L^2(\mathbb{R}^3)\,,\end{aligned}$$

that is, $A_\varepsilon^{(s)}$ acts as an integral operator with kernel $\mathsf{G}_{s,\mu^s}(x-\varepsilon y)v(y)$. The latter is clearly a function in $L^2(\mathbb{R}^3\times\mathbb{R}^3,\mathrm{d}x\,\mathrm{d}y)$ uniformly in $\varepsilon$, and dominated convergence implies

$$\|A_\varepsilon^{(s)}\|^2_{\mathrm{H.S.}} \;=\; \iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,|\mathsf{G}_{s,\mu^s}(x-\varepsilon y)v(y)|^2 \;\xrightarrow{\varepsilon\downarrow 0}\; \|\mathsf{G}_{s,\mu^s}\|^2_{L^2}\|V\|_{L^1}$$

as well as

$$\begin{aligned}\langle g,A_\varepsilon^{(s)}f\rangle_{L^2} \;&=\; \iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{d}x\,\mathrm{d}y\,\overline{g(x)}\,\mathsf{G}_{s,\mu^s}(x-\varepsilon y)\,v(y)f(y)\\ &\xrightarrow{\varepsilon\downarrow 0}\; \langle g,\mathsf{G}_{s,\mu^s}\rangle_{L^2}\langle v,f\rangle_{L^2} \qquad\forall f,g\in C_0^\infty(\mathbb{R}^3)\,.\end{aligned}$$

As a consequence, as $\varepsilon\downarrow 0$, $A_\varepsilon^{(s)}\to|\mathsf{G}_{s,\mu^s}\rangle\langle v|$ *weakly* in the operator topology, *and* the Hilbert-Schmidt norm of $A_\varepsilon^{(s)}$ converges to the Hilbert-Schmidt norm of its limit. By a well-known feature of compact operators [**103**, Theorem 2.21], the combination of these two properties implies that $A_\varepsilon^{(s)}\to|\mathsf{G}_{s,\mu^s}\rangle\langle v|$ in the Hilbert-Schmidt topology. This proves (2.16).

The discussion for $C_\varepsilon^{(s)}$ is analogous: its integral kernel is $u(x)\mathsf{G}_{s,\mu^s}(\varepsilon x-y)$ and (2.18) is proved by the very same type of argument.

Concerning $B_\varepsilon^{(s)}$, its integral kernel is $\eta(\varepsilon)u(x)\mathsf{G}_{s,(\mu\varepsilon)^s}(x-y)v(y)$ and the integral kernel of $B_0^{(s)}$ is $u(x)\mathsf{G}_{s,0}(x-y)v(y)$: owing to Lemma 2.1.1(i) both operators are Hilbert Schmidt, and moreover by dominated convergence $B_\varepsilon^{(s)}\to B_0^{(s)}$ weakly in the operator topology and $\|B_\varepsilon^{(s)}\|^2_{\mathrm{H.S.}} \to \|B_0^{(s)}\|^2_{\mathrm{H.S.}}$ as $\varepsilon \downarrow 0$. By the same property [**103**, Theorem 2.21] the limit (2.17) then holds in the Hilbert-Schmidt norm. □

It is evident from (2.15) that, in order for the limits (2.16)–(2.18) above to qualify the behaviour of the resolvent of $h_\varepsilon^{(s/2)}$ as $\varepsilon \downarrow 0$, one needs additional information on the possible failure of invertibility in $L^2(\mathbb{R}^3)$ of the operator $\mathbb{1}+B_0^{(s)}$. By the Fredholm alternative, since $B_0^{(s)}$ is compact, $(\mathbb{1}+B_0^{(s)})^{-1}$ exists everywhere defined and bounded, in which case (2.15) implies at once $\big(h_\varepsilon^{(s/2)}+\mu^s\mathbb{1}\big)^{-1} \to \big((-\Delta)^{s/2}+\mu^s\mathbb{1}\big)^{-1}$ as $\varepsilon\downarrow 0$, *unless* $B_0^{(s)}$ admits an eigenvalue $-1$.

Let us then assume that the latter circumstance does occurs, namely condition (2.3) of Theorem 2.1.2. More precisely, we make the following assumption.

**Assumption (II$_s$).** Assumption (I$_s$) holds. $B_0^{(s)}$ has eigenvalue $-1$, which is non-degenerate. $\phi \in L^2(\mathbb{R}^3)$ is a non-zero function such that $B_0^{(s)}\phi = -\phi$ and, in addition, $\langle\widetilde{\phi},\phi\rangle_{L^2} = -1$, where $\widetilde{\phi} := (\mathrm{sign}V)\phi$.

Since $\langle\widetilde{\phi},\phi\rangle_{L^2} = -\langle(\mathrm{sign}V)\phi,(\mathrm{sign}V)v(-\Delta)^{-\frac{s}{2}}v\phi\rangle_{L^2} = -\|(-\Delta)^{-\frac{s}{4}}v\phi\|^2_{L^2}$, the normalisation $\langle\widetilde{\phi},\phi\rangle_{L^2} = -1$ is always possible.

Under Assumption (II$_s$), $(\mathbb{1}+B_\varepsilon^{(s)})^{-1}$ becomes singular in the limit $\varepsilon\downarrow 0$, with a singularity that now competes with the vanishing factor $\varepsilon^{3-s}$ of (2.15). To resolve this competing effect, we need first an expansion of $B_\varepsilon^{(s)}$ around $\varepsilon = 0$ to a further order, than the limit (2.17). This expansion holds irrespectively of Assumption (II$_s$).

LEMMA 2.3.4. *Let $s \in (\frac{3}{2},\frac{5}{2})$ and $\lambda > 0$.*

(i) *For every $x \in \mathbb{R}^3\setminus\{0\}$*

$$\lim_{\lambda\downarrow 0}\frac{\mathsf{G}_{s,\lambda}(x)-\mathsf{G}_{s,0}(x)}{(2\pi s\sin(\frac{3\pi}{s}))^{-1}\lambda^{\frac{3}{s}-1}} \;=\; 1\,. \tag{2.19}$$

(ii) *In the norm operator topology one has*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{(\mu\varepsilon)^{3-s}}\big(B_\varepsilon^{(s)}-B_0^{(s)}\big) \;=\; \frac{\eta_s}{\mu^{3-s}}\,B_0^{(s)} + \frac{1}{2\pi s\sin(\frac{3\pi}{s})}\,|u\rangle\langle v|\,. \tag{2.20}$$

*Here $\mu > 0$ is the constant chosen in the definition* (2.14) *of $B_\varepsilon^{(s)}$ and $\eta_s \in \mathbb{R}$ is the constant that is part of Assumption* (I$_s$).

PROOF. (i) From (1.39) we write

$$\begin{aligned}\frac{\mathsf{G}_{s,\lambda}(x)-\mathsf{G}_{s,0}(x)}{\lambda^{\frac{3}{s}-1}} \;&=\; \frac{1}{\lambda^{\frac{3}{s}-1}(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\; e^{\mathrm{i}x\cdot p}\,\frac{-\lambda}{(2\pi)^{\frac{3}{2}}|p|^s(|p|^s+\lambda)}\\ &=\; -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}p\; e^{\mathrm{i}\,\lambda^{1/s}x\cdot p}\,\frac{1}{|p|^s(|p|^s+1)}\,,\end{aligned}$$

whence

$$\frac{\mathsf{G}_{s,\lambda}(x)-\mathsf{G}_{s,0}(x)}{\lambda^{\frac{3}{s}-1}} \;\xrightarrow{\lambda\downarrow 0}\; -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{1}{|p|^s(|p|^s+1)} \;=\; \frac{1}{2\pi s\sin(\frac{3\pi}{s})}$$

by dominated convergence, since $p\mapsto(|p|^s(|p|^s+1))^{-1}$ is integrable when $s\in(\frac{3}{2},3)$.

(ii) The Hilbert-Schmidt operator

$$\frac{1}{(\mu\varepsilon)^{3-s}}\big(B_\varepsilon^{(s)}-B_0^{(s)}\big) - \frac{\eta_s}{\mu^{3-s}}\,B_0^{(s)} - \frac{1}{2\pi s\sin(\frac{3\pi}{s})}\,|u\rangle\langle v|$$

has integral kernel

$$(*)\qquad \begin{aligned} &u(x)\Big(\frac{\eta(\varepsilon)-1}{(\mu\varepsilon)^{3-s}}-\frac{\eta_s}{\mu^{3-s}}\Big)\,\mathsf{G}_{s,(\mu\varepsilon)^s}(x-y)\,v(y)\,+\\ &\qquad\qquad+u(x)\Big(\frac{\mathsf{G}_{s,(\mu\varepsilon)^s}(x-y)-\mathsf{G}_{s,0}(x-y)}{(\mu\varepsilon)^{3-s}}-\frac{1}{2\pi s\sin(\frac{3\pi}{s})}\Big)\,v(y)\,. \end{aligned}$$

The first summand in (*) vanishes as $\varepsilon\downarrow 0$ for a.e. $x,y\in\mathbb{R}^3$ as a consequence of Assumption $(\mathrm{I}_s)$(ii), and so does the second summand in (*) as a consequence of (2.19), where we take $\lambda=(\mu\varepsilon)^s$. Moreover, each such summand belongs to $L^2(\mathbb{R}^3\times\mathbb{R}^3,\mathrm{d}x\,\mathrm{d}y)$ uniformly in $\varepsilon$, thanks to the assumption $(\mathrm{I}_s)$(i) on the potentials $v$ and $u$. Thus, by dominated convergence, the function (*) vanishes in $L^2(\mathbb{R}^3\times\mathbb{R}^3,\mathrm{d}x\,\mathrm{d}y)$ as $\varepsilon\downarrow 0$, and this proves the limit (2.20) in the Hilbert-Schmidt norm. $\square$

We can now monitor the competing effect in $\varepsilon^{3-s}(\mathbb{1}+B_\varepsilon^{(s)})^{-1}$ as $\varepsilon\downarrow 0$.

Lemma 2.3.5. *Under the Assumptions* $(\mathrm{I}_s)$ *and* $(\mathrm{II}_s)$ *one has*

$$\lim_{\varepsilon\downarrow 0}\,(\mu\varepsilon)^{3-s}\big(\mathbb{1}+B_\varepsilon^{(s)}\big)^{-1}\;=\;\Big(\frac{\eta_s}{\mu^{3-s}}+\frac{|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)^{-1}\,|\phi\rangle\langle\widetilde{\phi}| \tag{2.21}$$

*in the operator norm topology.*

Proof. We re-write (2.20) in the form of the expansion

$$B_\varepsilon^{(s)}\;=\;B_0^{(s)}+(\mu\varepsilon)^{3-s}\mathcal{B}^{(s)}+o(\varepsilon^{3-s}) \tag{i}$$

where, for short,

$$\mathcal{B}^{(s)}\;:=\;\frac{\eta_s}{\mu^{3-s}}\,B_0^{(s)}+\frac{1}{2\pi s\sin(\frac{3\pi}{s})}\,|u\rangle\langle v|\,,$$

whence also

$$(\mathrm{ii})\qquad \begin{aligned} &(\mu\varepsilon)^{3-s}\big(\mathbb{1}+B_\varepsilon^{(s)}\big)^{-1}\;=\\ &\qquad=\;\Big(\mathbb{1}+(\mu\varepsilon)^{3-s}\big(\mathbb{1}+(\mu\varepsilon)^{3-s}+B_0^{(s)}\big)^{-1}\big(\mathcal{B}^{(s)}-\mathbb{1}+o(1)\big)\Big)^{-1}\,\times\\ &\qquad\qquad\times\;(\mu\varepsilon)^{3-s}\big(\mathbb{1}+(\mu\varepsilon)^{3-s}+B_0^{(s)}\big)^{-1}\,. \end{aligned}$$

The $o(\varepsilon^a)$-remainders in (i) and (ii) above are clearly meant in the Hilbert-Schmidt norm.

The operator $(\mu\varepsilon)^{3-s}(\mathbb{1}+(\mu\varepsilon)^{3-s}+B_0^{(s)})^{-1}$ that appears twice in (ii) is of the form

$$z(\mathbb{1}+T+z\mathbb{1})^{-1}\,,\qquad z\in\mathbb{C}\setminus\{0\}\,,$$

for a closed operator $T$ with *isolated* eigenvalue $-1$; this is a general setting for which a well-known expansion by Kato is available as $z\to 0$ [**68**, Sec. 3.6.5], which in the present context (in complete analogy with the argument of the proof of [**9**, Lemma I.1.2.4]) reads

$$(\mu\varepsilon)^{3-s}\big(\mathbb{1}+(\mu\varepsilon)^{3-s}+B_0^{(s)}\big)^{-1}\;=\;-|\phi\rangle\langle\widetilde{\phi}|+O(\varepsilon^{3-s}) \tag{iii}$$

as $\varepsilon\downarrow 0$ in the operator norm topology. In practice, $(\mathbb{1}+(\mu\varepsilon)^{3-s}+B_0^{(s)})^{-1}$ remains bounded also in the limit $\varepsilon\downarrow 0$ when restricted to the orthogonal complement of the eigenspace $-1$ of $B_0^{(s)}$, whereas it becomes singular when restricted to such eigenspace; the magnitude of the singularity is precisely $(\mu\varepsilon)^{-(3-s)}$, which is cancelled exactly by the pre-factor $(\mu\varepsilon)^{3-s}$ in the l.h.s. of (iii). In fact, by assumption of non-degeneracy, the eigenspace $-1$ is spanned by $\phi$ and $P:=-|\phi\rangle\langle\widetilde{\phi}|$ projects onto $\mathrm{span}\{\phi\}$ with $P\phi=\phi$, as follows from the normalisation $\langle\widetilde{\phi},\phi\rangle_{L^2}=-1$.

Combining (ii) and (iii) above yields

$$\text{(iv)}\qquad (\mu\varepsilon)^{3-s}\big(\mathbb{1}+B_\varepsilon^{(s)}\big)^{-1} \;=\; \big(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1})+O(\varepsilon^{3-s})\big)^{-1}(P+O(\varepsilon^{3-s}))$$

as $\varepsilon\downarrow 0$ in the operator norm topology.

Next, in order to see that the limit $\varepsilon\downarrow 0$ in the r.h.s. of (iv) exists and is a bounded operator, we write explicitly

$$\text{(v)}\qquad \begin{aligned} \mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1}) \;&=\; \mathbb{1}-|\phi\rangle\langle\widetilde{\phi}|\Big(\frac{\eta_s}{\mu^{3-s}}\,u(-\Delta)^{-\frac{s}{2}}v+\frac{1}{2\pi s\sin(\frac{3\pi}{s})}\,|u\rangle\langle v|-\mathbb{1}\Big) \\ &=\; \mathbb{1}+\frac{\eta_s}{\mu^{3-s}}|\phi\rangle\langle\widetilde{\phi}|-\frac{\overline{\langle v,\phi\rangle_{L^2}}}{2\pi s\sin(\frac{3\pi}{s})}\,|\phi\rangle\langle v|+|\phi\rangle\langle\widetilde{\phi}|\,, \end{aligned}$$

where we used the identities $\langle\widetilde{\phi},u\rangle_{L^2}=\langle\phi,v\rangle_{L^2}$ and

$$\begin{aligned} \big\langle\widetilde{\phi},u(-\Delta)^{-\frac{s}{2}}vf\big\rangle_{L^2} \;&=\; \big\langle v(-\Delta)^{-\frac{s}{2}}u\,\widetilde{\phi},f\big\rangle_{L^2} \;=\; \big\langle(\mathrm{sign}V)u(-\Delta)^{-\frac{s}{2}}v\phi,f\big\rangle_{L^2} \\ &=\; -\langle\widetilde{\phi},f\rangle_{L^2}\qquad \forall f\in L^2(\mathbb{R}^3)\,. \end{aligned}$$

Setting the constants

$$\begin{aligned} a \;&:=\; \Big(\frac{\eta_s}{\mu^{3-s}}+1\Big)\Big(\frac{\eta_s}{\mu^{3-s}}+\frac{|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)^{-1} \\ b \;&:=\; -\frac{\overline{\langle v,\phi\rangle_{L^2}}}{2\pi s\sin\frac{3\pi}{s}}\Big(\frac{\eta_s}{\mu^{3-s}}+\frac{|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)^{-1}, \end{aligned}$$

the expression (v) allows one to compute explicitly (using again $\langle\widetilde{\phi},\phi\rangle_{L^2}=-1$)

$$\big(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1})\big)\,\big(\mathbb{1}+a\,|\phi\rangle\langle\widetilde{\phi}|+b\,|\phi\rangle\langle v|\big) \;=\; \mathbb{1}$$

and therefore to deduce that $(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1}))^{-1}$ exists and is bounded. This fact allows one to deduce from (iv) that

$$\text{(vi)}\qquad \lim_{\varepsilon\downarrow 0}\,(\mu\varepsilon)^{3-s}\big(\mathbb{1}+B_\varepsilon^{(s)}\big)^{-1} \;=\; \big(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1}))\big)^{-1}P$$

in the operator norm topology.

Last, from (v), using $\langle\widetilde{\phi},\phi\rangle_{L^2}=-1$ and $\langle\widetilde{\phi},u\rangle_{L^2}=\langle\phi,v\rangle_{L^2}$, one finds

$$\big(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1})\big)\phi \;=\; -\Big(\frac{\eta_s}{\mu^{3-s}}+\frac{|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)\,\phi$$

and hence

$$\big(\mathbb{1}+P(\mathcal{B}^{(s)}-\mathbb{1})\big)^{-1}\phi \;=\; -\Big(\frac{\eta_s}{\mu^{3-s}}+\frac{|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)^{-1}\,\phi\,.$$

Plugging the latter identity into (vi) yields finally (2.21) as a limit in the operator norm. □

We are now in the condition to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. Owing to (2.15) we need to determine the limit of

$$-A_\varepsilon^{(s)}\varepsilon^{3-s}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)}$$

as $\varepsilon\downarrow 0$. As observed already, if $u(-\Delta)^{-\frac{s}{2}}v$ has no eigenvalue $-1$, then the above expression vanishes with $\varepsilon$ and

$$\big(h_\varepsilon^{(s/2)}+\mu^s\mathbb{1}\big)^{-1} \;\xrightarrow{\varepsilon\downarrow 0}\; \big((-\Delta)^{s/2}+\mu^s\mathbb{1}\big)^{-1}$$

in the operator norm. If instead $u(-\Delta)^{-\frac{s}{2}}v$ does admit a simple eigenvalue $-1$, be $(-\Delta)^{\frac{s}{2}}+V$ zero-energy resonant or not, we are under the Assumption $(\mathrm{I}_s)$ and

($\mathrm{II}_s$) of the present Section and we can therefore apply the limits (2.16), (2.18), and (2.21). This yields

$$(*)\qquad \begin{aligned} &-A_\varepsilon^{(s)}\varepsilon^{3-s}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)} \\ &\quad\xrightarrow{\varepsilon\downarrow 0}\; -|\mathsf{G}_{s,\mu^s}\rangle\langle v|\circ\Big(\eta_s+\frac{\mu^{3-s}|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}\Big)^{-1}|\phi\rangle\langle\widetilde{\phi}|\circ|u\rangle\langle\mathsf{G}_{s,\mu^s}| \\ &\quad=\;-\frac{|\langle v,\phi\rangle_{L^2}|^2}{\eta_s+\dfrac{\mu^{3-s}|\langle v,\phi\rangle_{L^2}|^2}{2\pi s\sin\frac{3\pi}{s}}}\;|\mathsf{G}_{s,\mu^s}\rangle\langle\mathsf{G}_{s,\mu^s}| \end{aligned}$$

in the operator norm, having used $\langle\widetilde{\phi},u\rangle_{L^2}=\langle\phi,v\rangle_{L^2}$. Now, if $(-\Delta)^{\frac{s}{2}}+V$ is *not* zero-energy resonant, then $\langle v,\phi\rangle_{L^2}=0$, owing to Theorem 2.1.2(iv), and the conclusion is again

$$\big(h_\varepsilon^{(s/2)}+\mu^s\mathbb{1}\big)^{-1}\;\xrightarrow{\varepsilon\downarrow 0}\;\big((-\Delta)^{s/2}+\mu^s\mathbb{1}\big)^{-1}$$

in the operator norm. This proves part (i) of the present Theorem. If instead $(-\Delta)^{\frac{s}{2}}+V$ is zero-energy resonant, then using $\langle v,\phi\rangle_{L^2}\neq 0$ and plugging (*) back into (2.15) yields

$$\begin{aligned}\big(h_\varepsilon^{(s/2)}+\mu^s\mathbb{1}\big)^{-1}\;\xrightarrow{\varepsilon\downarrow 0}\;&\big((-\Delta)^{s/2}+\mu^s\mathbb{1}\big)^{-1} \\ &+\frac{1}{\dfrac{-\eta_s}{|\langle v,\phi\rangle_{L^2}|^2}-\dfrac{\mu^{3-s}}{2\pi s\sin\frac{3\pi}{s}}}\;|\mathsf{G}_{s,\mu^s}\rangle\langle\mathsf{G}_{s,\mu^s}|\end{aligned}$$

in the operator norm. Upon setting $\alpha:=-\eta_s|\langle v,\phi\rangle_{L^2}|^{-2}$ and $\lambda=\mu^s$, and comparing the resulting expression with (1.61), this means

$$\begin{aligned}\big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1}\;\xrightarrow{\varepsilon\downarrow 0}\;&\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}+\Big(\alpha-\tfrac{\lambda^{\frac{3}{s}-1}}{2\pi s\sin(\frac{3\pi}{s})}\Big)^{-1}|\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}| \\ =\;&(\mathsf{k}_\alpha^{(s/2)}+\lambda\mathbb{1})^{-1}\,,\end{aligned}$$

which proves part (ii) of the Theorem. □

## 2.4. Convergence of the 1D limit: resonant-driven case

The proof of the limit $h_\varepsilon^{(s/2)}\xrightarrow{\varepsilon\downarrow 0}\mathsf{k}_\alpha^{(s/2)}$ in dimension one when $s\in(\frac{1}{2},1)$ (Theorem 2.2.2) is technically analogous to that in three dimensions. Therefore, based on the detailed discussion of the preceding Section, we only present here the steps of the convergence scheme and a sketch of their proofs.

Prior to that, let us set up the key resolvent identity and useful scaling properties with a notation that we can use also in Section 2.5 when we will deal with the resonant-independent limit.

We then keep $s\in(\frac{1}{2},1)\cup(1,\frac{3}{2})$ generic for a moment and, in a unified form, we re-write (2.5) and (2.8) as

$$V_\varepsilon(x)\;=\;\frac{\eta(\varepsilon)}{\varepsilon^{\frac{s+\gamma}{2}}}\,V(\tfrac{x}{\varepsilon})\,. \tag{2.22}$$

Taking $\gamma=s$ in (2.22) yields (2.5) and taking $\gamma=2-s$ yields (2.8). Thus, setting

$$\begin{aligned} v(x)\;&:=\;|V(x)|^{\frac{1}{2}}\,, & u(x)\;&:=\;|V(x)|^{\frac{1}{2}}\,\mathrm{sign}(V(x))\,, \\ v_\varepsilon(x)\;&:=\;|V_\varepsilon(x)|^{\frac{1}{2}}\,, & u_\varepsilon(x)\;&:=\;|V_\varepsilon(x)|^{\frac{1}{2}}\,\mathrm{sign}(V_\varepsilon(x))\,, \end{aligned} \tag{2.23}$$

one has

$$v_\varepsilon(x)\;=\;\tfrac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{(s+\gamma)/4}}\,v(\tfrac{x}{\varepsilon})\,,\qquad u_\varepsilon(x)\;=\;\tfrac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{(s+\gamma)/2}}\,u(\tfrac{x}{\varepsilon})\,,\qquad v_\varepsilon u_\varepsilon\;=\;V_\varepsilon\,. \tag{2.24}$$

The 1D analogue $U_\varepsilon : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of the unitary scaling operator (2.12) acts as

$$(U_\varepsilon f)(x) \;:=\; \tfrac{1}{\varepsilon^{1/2}}\, f(\tfrac{x}{\varepsilon})\,, \tag{2.25}$$

which induces the scaling transformations

$$\begin{aligned} U_\varepsilon^* v_\varepsilon U_\varepsilon \;&=\; \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{\frac{s+\gamma}{4}}}\, v\,, \qquad U_\varepsilon^* u_\varepsilon U_\varepsilon \;=\; \frac{\sqrt{\eta(\varepsilon)}}{\varepsilon^{\frac{s+\gamma}{4}}}\, u\,, \\ U_\varepsilon^*\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}U_\varepsilon \;&=\; \varepsilon^s\big((-\Delta)^{s/2}+\lambda\varepsilon^s\mathbb{1}\big)^{-1}. \end{aligned} \tag{2.26}$$

Based on arguments that differ depending on whether $s \in (\frac{1}{2},1)$ or $s \in (1,\frac{3}{2})$ and which we shall prove in due time, the Konno-Kuroda-type resolvent identity

$$\begin{aligned} \big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1} \;&=\; \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1} - \\ &- \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1} v_\varepsilon \Big(\mathbb{1}+u_\varepsilon\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}v_\varepsilon\Big)^{-1} u_\varepsilon\,\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1} \end{aligned} \tag{2.27}$$

holds as an identity between bounded operators on $L^2(\mathbb{R})$ for every $\varepsilon > 0$ and every $-\lambda < 0$ in the resolvent set of $h_\varepsilon^{(s/2)}$. Inserting $U_\varepsilon U_\varepsilon^* = \mathbb{1}$ into (2.27) and applying (2.26) then yields

$$\big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1} = \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1} - A_\varepsilon^{(s)}\varepsilon^{\frac{2-s-\gamma}{2}}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)}\,, \tag{2.28}$$

having defined

$$\begin{aligned} A_\varepsilon^{(s)} \;&:=\; \varepsilon^{-\frac{2-s-\gamma}{2}}\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}(\eta(\varepsilon))^{-\frac{1}{2}}\, v_\varepsilon\, U_\varepsilon \\ B_\varepsilon^{(s)} \;&:=\; \eta(\varepsilon)\,\varepsilon^{\frac{s-\gamma}{2}}\, u\,\big((-\Delta)^{s/2}+\lambda\varepsilon^s\mathbb{1}\big)^{-1} v \\ C_\varepsilon^{(s)} \;&:=\; U_\varepsilon^*\, u_\varepsilon\,(\eta(\varepsilon))^{-\frac{1}{2}}\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}\varepsilon^{-\frac{2-s-\gamma}{2}}\,. \end{aligned} \tag{2.29}$$

We shall see in a moment (Lemma 2.4.3) that $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$ are Hilbert-Schmidt operators on $L^2(\mathbb{R})$.

The following scaling property too is going to be useful in both regimes $s \in (\frac{1}{2},1)$ and $s \in (1,\frac{3}{2})$.

Lemma 2.4.1. *For any $s,\gamma,\varepsilon > 0$ and any $x \in \mathbb{R}\backslash\{0\}$ one has*

$$\varepsilon^{\frac{s-\gamma}{2}}\,\mathsf{G}_{s,\lambda\varepsilon^s}(x) \;=\; \varepsilon^{\frac{2-s-\gamma}{2}}\,\mathsf{G}_{s,\lambda}(\varepsilon x)\,. \tag{2.30}$$

Proof. Owing to (1.39),

$$\begin{aligned} \varepsilon^{\frac{s-\gamma}{2}}\,\mathsf{G}_{s,\lambda\varepsilon^s}(x) \;&=\; \frac{1}{2\pi}\,\varepsilon^{\frac{s-\gamma}{2}}\int_{\mathbb{R}}\mathrm{d}p\, e^{\mathrm{i}px}\,\frac{1}{|p|^s+\lambda\varepsilon^s} \\ &=\; \frac{1}{2\pi}\,\varepsilon^{\frac{2-s-\gamma}{2}}\int_{\mathbb{R}}\mathrm{d}p\, e^{\mathrm{i}p(\varepsilon x)}\,\frac{1}{|p|^s+\lambda} \;=\; \varepsilon^{\frac{2-s-\gamma}{2}}\,\mathsf{G}_{s,\lambda}(\varepsilon x)\,, \end{aligned}$$

whence the thesis. □

We can now start the discussion for the proof of Theorem 2.2.2, thus working in the regime $s \in (\frac{1}{2},1)$.

First, we have the following properties.

Lemma 2.4.2. *Let $V : \mathbb{R} \to \mathbb{R}$ belong to $L^1(\mathbb{R}) \cap \mathcal{R}_{s,1}$ for some $s \in (\frac{1}{2},1)$. Then:*

- (i) *for every $\lambda \geqslant 0$, $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$;*
- (ii) *$|V|^{\frac{1}{2}} \ll (-\Delta)^{\frac{s}{4}}$ in the sense of infinitesimally bounded operators;*
- (iii) *the operator $(-\Delta)^{\frac{s}{2}}+V$ defined as a form sum is self-adjoint on $L^2(\mathbb{R})$, and $\sigma_{\mathrm{ess}}((-\Delta)^{\frac{s}{2}}+V) = [0,+\infty)$.*

Proof. The proof is completely analogous to that of Lemma 2.1.1 for the 3D case, and is based on the fact that the integral kernel of $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}}+\lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$, namely $|V(x)|^{\frac{1}{2}}\mathsf{G}_{s,\lambda}(x-y)|V(y)|^{\frac{1}{2}}$, belongs to $L^2(\mathbb{R}\times\mathbb{R},\mathrm{d}x\,\mathrm{d}y)$, as a direct consequence of the assumption $V\in L^1(\mathbb{R})\cap\mathcal{R}_{s,1}$. $\square$

Lemma 2.4.2 justifies the validity of the resolvent identity (2.27), and hence of the rescaled identity (2.28), owing again to the general argument of [**9**, Theorem B.1(b)].

Next, we monitor separately the following limits.

Lemma 2.4.3. *Let $V$ and $\eta$ satisfy Assumption $(\mathrm{I}_s^-)$ for some $s\in(\frac{1}{2},1)$. For every $\varepsilon>0$, the operators $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$ defined by* (2.22)-(2.24) *and* (2.29) *with $\gamma=s$ are Hilbert-Schmidt operators on $L^2(\mathbb{R})$ with limit*

$$\lim_{\varepsilon\downarrow 0}A_\varepsilon^{(s)} \;=\; |\mathsf{G}_{s,\lambda}\rangle\langle v| \tag{2.31}$$

$$\lim_{\varepsilon\downarrow 0}B_\varepsilon^{(s)} \;=\; B_0^{(s)} \;=\; u\,(-\Delta)^{-\frac{s}{2}}v \tag{2.32}$$

$$\lim_{\varepsilon\downarrow 0}C_\varepsilon^{(s)} \;=\; |u\rangle\langle\mathsf{G}_{s,\lambda}| \tag{2.33}$$

*in the Hilbert-Schmidt operator norm.*

Proof. Completely analogous to the proof of Lemma 2.3.3, the integral kernels being now (with $\gamma=s$)

$$\begin{aligned}A_\varepsilon^{(s)}(x,y) &= \mathsf{G}_{s,\lambda}(x-\varepsilon y)\,v(y)\\ B_\varepsilon^{(s)}(x,y) &= \eta(\varepsilon)\,u(x)\,\mathsf{G}_{s,\lambda\varepsilon^s}(x-y)\,v(y)\\ C_\varepsilon^{(s)}(x,y) &= u(x)\,\mathsf{G}_{s,\lambda}(\varepsilon x-y)\,.\end{aligned}$$

In particular, owing to (2.30),

$$B_\varepsilon^{(s)}(x,y) \;=\; \eta(\varepsilon)\,\varepsilon^{1-s}u(x)\,\mathsf{G}_{s,\lambda}(\varepsilon x-\varepsilon y)\,v(y)\,,$$

and using (1.49)-(1.50) one finds

$$B_\varepsilon^{(s)}(x,y)\;\xrightarrow{\varepsilon\downarrow 0}\;u(x)\,\frac{2^{1-s}\Gamma(\frac{1-s}{2})}{(2\pi)^{\frac{1}{2}}\Gamma(\frac{s}{2})}\,\frac{1}{|x-y|^{1-s}}\,v(y)\;=\;B_0^{(s)}(x,y)$$

pointwise almost everywhere. $\square$

Before plugging the limits found in Lemma (2.4.3) into (2.28), that now reads

$$\big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1} \;=\; \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}-A_\varepsilon^{(s)}\varepsilon^{1-s}\eta(\varepsilon)(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)}\,, \tag{2.34}$$

we see that, since $B_0^{(s)}$ is compact, $(\mathbb{1}+B_0^{(s)})^{-1}$ exists everywhere defined and bounded, in which case $\big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1}\to\big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1}$ as $\varepsilon\downarrow 0$, *unless* $B_0^{(s)}$ admits an eigenvalue $-1$. We then consider the following additional assumption.

**Assumption $(\mathrm{II}_s^-)$.** Assumption $(\mathrm{I}_s^-)$ holds. $B_0^{(s)}$ has eigenvalue $-1$, which is non-degenerate. $\phi\in L^2(\mathbb{R}^3)$ is a non-zero function such that $B_0^{(s)}\phi=-\phi$ and, in addition, $\langle\widetilde{\phi},\phi\rangle_{L^2}=-1$, where $\widetilde{\phi}:=(\mathrm{sign}V)\phi$.

Since $\langle\widetilde{\phi},\phi\rangle_{L^2}=-\langle(\mathrm{sign}V)\phi,(\mathrm{sign}V)v(-\Delta)^{-\frac{s}{2}}v\phi\rangle_{L^2}=-\|(-\Delta)^{-\frac{s}{4}}v\phi\|_{L^2}^2$, the normalisation $\langle\widetilde{\phi},\phi\rangle_{L^2}=-1$ is always possible.

When Assumption $(\mathrm{II}_s^-)$ holds, $(\mathbb{1}+B_\varepsilon^{(s)})^{-1}$ becomes singular in the limit $\varepsilon\downarrow 0$, with a singularity that now competes with the vanishing factor $\varepsilon^{1-s}$ of (2.34). To resolve this competing effect, we need first to expand $B_\varepsilon^{(s)}$ around $\varepsilon=0$ to a further order, than the limit (2.32). This expansion is valid irrespectively of Assumption $(\mathrm{II}_s^-)$.

LEMMA 2.4.4. *Let $s \in (\frac{1}{2}, 1)$ and $\lambda > 0$.*

(i) *For every $x \in \mathbb{R}\backslash\{0\}$*

$$\lim_{\lambda \downarrow 0} \frac{\mathsf{G}_{s,\lambda}(x) - \mathsf{G}_{s,0}(x)}{(s\sin(\frac{\pi}{s}))^{-1}\lambda^{\frac{1}{s}-1}} \;=\; 1\,. \tag{2.35}$$

(ii) *In the norm operator topology one has*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\lambda^{\frac{1}{s}-1}\varepsilon^{1-s}}\big(B_\varepsilon^{(s)} - B_0^{(s)}\big) \;=\; \frac{\eta_s}{\lambda^{\frac{1}{s}-1}}\,B_0^{(s)} + \frac{1}{s\sin(\frac{3\pi}{s})}\,|u\rangle\langle v|\,. \tag{2.36}$$

*Here $\eta_s \in \mathbb{R}$ is the constant that is part of Assumption $(\mathrm{I}_s^-)$.*

PROOF. Completely analogous to the proof of Lemma 2.3.4 for the 3D case. □

We can now monitor the competing effect in $\varepsilon^{1-s}(\mathbb{1} + B_\varepsilon^{(s)})^{-1}$ as $\varepsilon \downarrow 0$.

LEMMA 2.4.5. *Under the Assumptions $(\mathrm{I}_s^-)$ and $(\mathrm{II}_s^-)$ one has*

$$\lim_{\varepsilon \downarrow 0}\,\varepsilon^{1-s}\big(\mathbb{1} + B_\varepsilon^{(s)}\big)^{-1} \;=\; \Big(\eta_s + \frac{|\langle v, \phi\rangle_{L^2}|^2}{\lambda^{\frac{1}{s}-1}s\sin\frac{\pi}{s}}\Big)^{-1}\,|\phi\rangle\langle\widetilde{\phi}| \tag{2.37}$$

*in the operator norm topology.*

PROOF. Completely analogous to the proof of Lemma 2.3.5 for the 3D case. □

With these preliminaries at hand, we can prove Theorem 2.2.2.

PROOF OF THEOREM 2.2.2. The argument is the very same as the in the proof of Theorem 2.1.3 for the 3D case. Thus, the limit is the trivial one unless the potential in the approximating operators satisfy Assumptions $(\mathrm{I}_s^-)$ and $(\mathrm{II}_s^-)$, in which case, plugging the limits (2.31), (2.33), and (2.37) into (2.34), one has

$$\begin{aligned}\big(h_\varepsilon^{(s/2)} + \mu^s\mathbb{1}\big)^{-1} \;\xrightarrow{\varepsilon\downarrow 0}\; &\big((-\Delta)^{s/2} + \mu^s\mathbb{1}\big)^{-1} \\ &+ \frac{1}{\dfrac{-\eta_s}{|\langle v,\phi\rangle_{L^2}|^2} - \dfrac{1}{\lambda^{1-\frac{1}{s}}s\sin\frac{3\pi}{s}}}\,|\mathsf{G}_{s,\mu^s}\rangle\langle\mathsf{G}_{s,\mu^s}|\,.\end{aligned}$$

The comparison of the limit resolvent above with formulas (1.69) and (1.73) shows finally that the limit resolvent is precisely $(\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1}$ where the extension parameter satisfies $\alpha = -\eta_s|\int_{\mathbb{R}}\mathrm{d}x\,V(x)\psi(x)|^{-2}$, and this completes the proof. □

## 2.5. Convergence of the 1D limit: resonant-independent case

This Section contains the proof of Theorem 2.2.3. Thus, now $s \in (1, \frac{3}{2})$ and formulas (2.22)-(2.30) must be specialised with $\gamma = 2 - s$.

First, we observe that with $L^1$-potentials the following operator-theoretic properties hold.

LEMMA 2.5.1. *Let $V : \mathbb{R} \to \mathbb{R}$ belong to $L^1(\mathbb{R})$ and let $s \in (1, \frac{3}{2})$. Then:*

(i) *for every $\lambda \geqslant 0$, $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$;*

(ii) *$|V|^{\frac{1}{2}} \ll (-\Delta)^{\frac{s}{4}}$ in the sense of infinitesimally bounded operators;*

(iii) *the operator $(-\Delta)^{\frac{s}{2}} + V$ defined as a form sum is self-adjoint on $L^2(\mathbb{R})$, and $\sigma_{\mathrm{ess}}((-\Delta)^{\frac{s}{2}} + V) = [0, +\infty)$.*

PROOF. Since $s > 1$, (1.39) defines a function $\widehat{\mathsf{G}_{s,\lambda}} \in L^1(\mathbb{R})$, whence $\mathsf{G}_{s,\lambda} \in C_\infty(\mathbb{R})$ (continuous and vanishing at infinity). Therefore, the integral kernel of $|V|^{\frac{1}{2}}((-\Delta)^{\frac{s}{2}} + \lambda\mathbb{1})^{-1}|V|^{\frac{1}{2}}$, namely $|V(x)|^{\frac{1}{2}}\mathsf{G}_{s,\lambda}(x-y)|V(y)|^{\frac{1}{2}}$, belongs to $L^2(\mathbb{R} \times \mathbb{R}, \mathrm{d}x\,\mathrm{d}y)$, and this holds for any $\lambda \geqslant 0$. Based on this observation, the rest of the reasoning of the proof of Lemma 2.1.1 can be repeated verbatim. □

Following again the general argument of [**9**, Theorem B.1(b)], Lemma 2.5.1 justifies the validity of the resolvent identity (2.27), and hence of the rescaled identity (2.28), that now reads

$$\big(h_\varepsilon^{(s/2)}+\lambda\mathbb{1}\big)^{-1} = \big((-\Delta)^{s/2}+\lambda\mathbb{1}\big)^{-1} - \eta(\varepsilon)\,A_\varepsilon^{(s)}(\mathbb{1}+B_\varepsilon^{(s)})^{-1}C_\varepsilon^{(s)} \tag{2.38}$$

for every $\varepsilon>0$ and every $-\lambda<0$ in the resolvent set of $h_\varepsilon^{(s/2)}$.

LEMMA 2.5.2. *Let $V$ and $\eta$ satisfy Assumption $(\mathrm{I}_s^+)$ for some $s\in(1,\frac{3}{2})$. For every $\varepsilon>0$, the operators $A_\varepsilon^{(s)}$, $B_\varepsilon^{(s)}$, and $C_\varepsilon^{(s)}$ defined by (2.22)-(2.24) and (2.29) with $\gamma=2-s$ are Hilbert-Schmidt operators on $L^2(\mathbb{R})$ with limit*

$$\lim_{\varepsilon\downarrow 0} A_\varepsilon^{(s)} \;=\; |\mathsf{G}_{s,\lambda}\rangle\langle v| \tag{2.39}$$

$$\lim_{\varepsilon\downarrow 0} B_\varepsilon^{(s)} \;=\; B_0^{(s)} \;=\; \frac{\eta(0)}{\lambda^{1-\frac{1}{s}}s\,\sin\frac{\pi}{s}}\,|u\rangle\langle v| \tag{2.40}$$

$$\lim_{\varepsilon\downarrow 0} C_\varepsilon^{(s)} \;=\; |u\rangle\langle \mathsf{G}_{s,\lambda}| \tag{2.41}$$

*in the Hilbert-Schmidt operator norm.*

PROOF. The integral kernels are now

$$\begin{aligned} A_\varepsilon^{(s)}(x,y) &= \mathsf{G}_{s,\lambda}(x-\varepsilon y)\,v(y)\\ B_\varepsilon^{(s)}(x,y) &= \eta(\varepsilon)\,\varepsilon^{s-1}\,u(x)\,\mathsf{G}_{s,\lambda\varepsilon^s}(x-y)\,v(y)\\ C_\varepsilon^{(s)}(x,y) &= u(x)\,\mathsf{G}_{s,\lambda}(\varepsilon x-y)\,. \end{aligned}$$

For $A_\varepsilon^{(s)}$ and $C_\varepsilon^{(s)}$ we reason precisely as in the proof of Lemma 2.3.3. $B_\varepsilon^{(s)}$ is Hilbert-Schmidt as a consequence of Lemma 2.5.1. Re-writing

$$B_\varepsilon^{(s)}(x,y) \;=\; \eta(\varepsilon)\,u(x)\,\mathsf{G}_{s,\lambda}(\varepsilon x-\varepsilon y)\,v(y)$$

by means of (2.30), and observing that (1.39) implies

$$\mathsf{G}_{s,\lambda}(\varepsilon x-\varepsilon y) \xrightarrow{\varepsilon\downarrow 0} \mathsf{G}_{s,\lambda}(0) \;=\; \frac{1}{\lambda^{1-\frac{1}{s}}s\,\sin\frac{\pi}{s}}\,,$$

one deduces

$$B_\varepsilon^{(s)}(x,y) \xrightarrow{\varepsilon\downarrow 0} \frac{\eta(0)}{\lambda^{1-\frac{1}{s}}s\,\sin\frac{\pi}{s}}u(x)v(y)\,.$$

Then a dominated convergence argument, analogous to that used in the proof of Lemma 2.3.3, proves (2.40). □

It is now convenient to observe the following (see [**35**, Lemma 5.1] for an analogous argument).

LEMMA 2.5.3. *Assume that the data $s\in(1,\frac{3}{2})$, $\lambda>0$ with $-\lambda$ in the resolvent set of all the $h_\varepsilon^{(s/2)}$'s, and $V$ and $\eta$ matching Assumption $(\mathrm{I}_s^+)$,* do not *satisfy the exceptional relation*

$$1+\frac{\eta(0)}{\lambda^{1-\frac{1}{s}}s\,\sin\frac{\pi}{s}}\int_{\mathbb{R}}\mathrm{d}x V(x) \;=\; 0\,. \tag{2.42}$$

*Then the operator $\mathbb{1}+B_0^{(s)}$ is invertible with bounded inverse, everywhere defined on $L^2(\mathbb{R})$.*

Proof. Since $B_0^{(s)}$ is compact on $L^2(\mathbb{R})$, based on the Fredholm alternative we have to prove that the validity of (2.42) is equivalent to $B_0^{(s)}$ having eigenvalue $-1$. In fact, $B_0^{(s)}\phi = -\phi$ for some non-zero $\phi \in L^2(\mathbb{R})$ is the same as

$$\phi \;=\; \frac{\eta(0)}{\lambda^{1-\frac{1}{s}}\, s\, \sin\frac{\pi}{s}}\,\langle v,\phi\rangle_{L^2}\, u\,,$$

meaning that $\phi$ is not orthogonal to $v$ and $\phi$ is a multiple of $u$. When this is the case, $u$ itself must be an eigenfunction of $B_0^{(s)}$ with eigenvalue $-1$, and this is tantamount, owing to the identity above, as the validity of (2.42). □

For given $s$, $\eta$, and $V$, the exceptional value of $-\lambda$ satisfying (2.42) is going to correspond to the negative eigenvalue of $\mathsf{k}_\alpha^{(s/2)}$ described in Theorem 1.2.6(iv). As we are going to monitor the limit $h_\varepsilon^{(s/2)} \xrightarrow{\varepsilon\downarrow 0} \mathsf{k}_\alpha^{(s/2)}$ in the *resolvent sense*, not only must we discard the spectral points $-\lambda$ not belonging to the resolvent set of all the $h_\varepsilon^{(s/2)}$'s, but also the point $-\lambda$ given by (2.42). Thus, for our purposes the operator $\mathbb{1} + B_0^{(s)}$ is always invertible with everywhere defined bounded inverse.

In particular, (2.40) implies

$$(\mathbb{1} + B_\varepsilon^{(s)})^{-1} \;\xrightarrow{\varepsilon\downarrow 0}\; (\mathbb{1} + B_0^{(s)})^{-1} \tag{2.43}$$

in the operator norm.

Based on the preceding preparatory materials, we can now prove Theorem 2.2.3.

Proof of Theorem 2.2.3. Since (2.42) is excluded and therefore

$$(\mathbb{1} + B_0^{(s)})^{-1} u \;=\; \Big(1 + \frac{\eta(0)\int_{\mathbb{R}}\mathrm{d}x\, V(x)}{\lambda^{1-\frac{1}{s}}\, s\, \sin\frac{\pi}{s}}\Big)^{-1} u\,,$$

then plugging the limits (2.39), (2.41), and (2.43) into (2.38) yields

$$\big(h_\varepsilon^{(s/2)} + \lambda\mathbb{1}\big)^{-1} \;\xrightarrow{\varepsilon\downarrow 0}\; \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} - \frac{\eta(0)\int_{\mathbb{R}}\mathrm{d}x\, V(x)}{1 + \frac{\eta(0)\int_{\mathbb{R}}\mathrm{d}x\, V(x)}{\lambda^{1-\frac{1}{s}}\, s\, \sin\frac{\pi}{s}}}\, |\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}|$$

in the operator norm. Upon setting

$$\alpha \;:=\; -\Big(\eta(0)\int_{\mathbb{R}}\mathrm{d}x\, V(x)\Big)^{-1}$$

and comparing the resulting expression with (1.69) and (1.73), one finds

$$\begin{aligned}\big(h_\varepsilon^{(s/2)} + \lambda\mathbb{1}\big)^{-1} \;&\xrightarrow{\varepsilon\downarrow 0}\; \big((-\Delta)^{s/2} + \lambda\mathbb{1}\big)^{-1} + \frac{1}{\alpha - \frac{1}{\lambda^{1-\frac{1}{s}}\, s\, \sin\frac{\pi}{s}}}\, |\mathsf{G}_{s,\lambda}\rangle\langle\mathsf{G}_{s,\lambda}| \\ &=\; (\mathsf{k}_\alpha^{(s/2)} + \lambda\mathbb{1})^{-1}\,,\end{aligned}$$

which completes the proof. □

CHAPTER 3

# Time-dependent scattering theory

In this Chapter we discuss the time-dependent scattering theory of three-dimensional Schrödinger operators with point interactions. We begin by introducing some basic facts of scattering theory (see [**76, 97**] for a comprehensive discussion).

Consider a pair $(H, H_0)$ of self-adjoint operators on an Hilbert space $\mathcal{H}$. Assume moreover that the spectrum of $H_0$ is purely absolutely continuous (a typical choice is $H_0 = -\Delta$, the free negative Laplacian on $\mathcal{H} = L^2(\mathbb{R}^d)$). The *wave operators* $W^{\pm} := W^{\pm}(H, H_0)$ relative to the pair $(H, H_0)$ are defined by

$$W^{\pm} := \operatorname*{s\text{-}lim}_{t\to\pm\infty} e^{\mathrm{i}tH} e^{-itH_0},$$

whenever the strong limit exists.

Wave operators are of paramount importance for the study of the scattering governed by the interaction Hamiltonian $H$ in comparison with the free (reference) Hamiltonian $H_0$. Suppose, e.g., that $g \in \mathrm{ran} W^+$, viz. $g = W^+ f$ for some $f \in \mathcal{D}(W^+)$. It follows by the definition of wave operators that

$$\|e^{-\mathrm{i}tH} g - e^{-\mathrm{i}tH_0} f\|_{\mathcal{H}} \to 0 \qquad \text{as } t \to +\infty. \tag{3.1}$$

Hence, the perturbed unitary evolution $e^{-itH}g$ looks asymptotically free as $t \to +\infty$. Analogously, if $g \in \mathrm{ran} W^-$ then $e^{-itH}g$ looks asymptotically free as $t \to -\infty$. When $g \in \mathrm{ran} W^+ \cap \mathrm{ran} W^-$, we say that $g$ is a *scattering state*. A relevant situation occurs when every state in $L^2_{\mathrm{ac}}(H)$ (the absolutely continuous spectral subspace of $\mathcal{H}$ for $H$) is a scattering state. This motivates the following definition.

Definition 3.0.1. The wave operators $W^{\pm}$ relative to the pair $(H, H_0)$ are said to be *complete* if $\mathrm{ran}\, W^{\pm} = L^2_{\mathrm{ac}}(H)$.

Let us denote by $P_{\mathrm{ac}}(H)$ the orthogonal projection onto $L^2_{\mathrm{ac}}(H)$. Completeness of wave operators has a number of important consequences.

Proposition 3.0.2. *Assume that the wave operators $W^{\pm}$ relative to the pair $(H, H_0)$ are complete. Then*

(i) *the absolutely continuous part of $H$, namely the operator $HP_{\mathrm{ac}}(H)$, is unitarily equivalent to $H_0$.*
(ii) *$W^+$ and $W^-$ are unitary from $L^2(\mathbb{R}^3)$ onto $L^2_{\mathrm{ac}}(H)$.*
(iii) *$W^+$ and $W^-$* intertwine *$HP_{\mathrm{ac}}(H)$ and $H_0$, namely, for any Borel function $f$ on $\mathbb{R}$ one has the identity*

$$f(H)P_{\mathrm{ac}}(H) \;=\; W^{\pm} f(H_0)\,(W^{\pm})^* \,. \tag{3.2}$$

Through the intertwining formula (3.2), mapping properties of $f(H)P_{\mathrm{ac}}(H)$ can be deduced from those of $f(H_0)$, provided that the corresponding ones of $W^{\pm}$ are known. Thus, the $L^{p'} \to L^p$ boundedness of $f(H)P_{\mathrm{ac}}(H)$ follows from the $L^{p'} \to L^p$ boundedness of $f(H_0)$ and the $L^p \to L^p$ boundedness of $W^{\pm}$: more precisely, if $W^{\pm} \in \mathcal{B}(L^p(\mathbb{R}^d))$ for some $p \in [1, \infty]$, then $(W^{\pm})^* \in \mathcal{B}(L^{p'}(\mathbb{R}^d))$ and hence

$$\|f(H)P_{\mathrm{ac}}\|_{\mathcal{B}(L^{p'},L^p)} \;\leqslant\; C_p\, \|f(H_0)\|_{\mathcal{B}(L^{p'},L^p)}\,, \tag{3.3}$$

the constant $C_p$ being *independent* of $f$.

The literature on the $L^p$-boundedness of wave operators relative to actual Schrödinger operators of the form $-\Delta + V$, for sufficiently regular $V : \mathbb{R}^d \to \mathbb{R}$ vanishing at spatial infinity, is vast [**111, 112, 15, 108, 65, 30, 43, 66, 17, 114, 115, 18, 19**] and the problem is well known to depend crucially on the spectral properties of $-\Delta + V$ at the bottom of the absolutely continuous spectrum, that is, at energy zero.

For *singular perturbations* of the Schrödinger operators, the picture is much less developed and is essentially limited to the *one-dimensional* case [**38**]. In my recent work [**36**], in collaboration with G. Dell'Antonio, A. Michelangeli, and K. Yajima, we study $L^p$-bounds for the wave operators of the *three-dimensional* multi-centre point interaction Hamiltonian. For given $Y \subseteq \mathbb{R}^3$ and $\alpha \in \mathbb{R}^N$, let us set $H_{\alpha,Y} := -\Delta_{\alpha,Y}$, $H_0 = -\Delta$, and

$$W^{\pm}_{\alpha,Y} := W^{\pm}(H_{\alpha,Y}, H_0) = \underset{t\to\pm\infty}{s\text{-}\lim}\, e^{itH_{\alpha,Y}} e^{-itH_0}. \tag{3.4}$$

We set also

$$\begin{aligned} R_0(z) &:= (H_0 - z\mathbb{1})^{-1} \qquad z \in \mathbb{C} \setminus [0, +\infty)\,, \\ R_{\alpha,Y}(z) &:= (H_{\alpha,Y} - z\mathbb{1})^{-1} \qquad \zeta \in \mathbb{C} \setminus \sigma(H_{\alpha,Y})\,, \end{aligned} \tag{3.5}$$

that is, the resolvents of the operators $H_0$ and $H_{\alpha,Y}$.

Since the resolvent difference $R_{\alpha,Y}(\mu) - R_0(\mu)$ is of finite rank (Theorem 1.1.2(iii)), standard arguments from scattering theory [**97**] guarantee that the wave operators $W^{\pm}_{\alpha,Y}$ exist and are complete in $L^2(\mathbb{R}^3)$. We provide a manageable formula for (the integral kernel of) $W^{\pm}_{\alpha,Y}$, which we obtain by manipulating the resolvent difference $(H_{\alpha,Y} - z^2\mathbb{1})^{-1} - (H_0 - z^2\mathbb{1})^{-1}$: since this difference is an explicitly known finite rank operator for any dimensions $d = 1, 2, 3$, our derivation can be naturally exported also to lower dimensions.

Based on our representation of $W^{\pm}_{\alpha,Y}$, we then establish our main result:

THEOREM 3.0.3. *For any $Y \equiv \{y_1, \dots, y_N\} \subseteq \mathbb{R}^3$ and $\alpha \equiv (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$, the wave operators $W^{\pm}_{\alpha,Y}$ exist and are complete in $L^2(\mathbb{R}^3)$. Assume moreover that zero is not an eigenvalue for $H_{\alpha,Y}$, and that the matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \in \mathbb{R} \setminus \{0\}$. Then $W^{\pm}_{\alpha,Y}$ are* bounded *in $L^p(\mathbb{R}^3)$ for $1 < p < 3$, and* unbounded *for $p = 1$ and for $p \geqslant 3$.*

REMARK 3.0.4. We conjecture that the hypothesis of absence of poles $z \in \mathbb{R} \setminus \{0\}$ for $\Gamma_{\alpha,Y}(z)^{-1}$ is always satisfied, in analogy with the well-known picture for regular Schrödinger operators with short range potentials (see [**48**] and references therein). The conjecture can be proved by a direct computation in the case of few centres of interactions. Moreover, possible counterexamples could only occur for particular configurations of the centres and for a measure zero set of choices of $\alpha$.

REMARK 3.0.5. The fact that $L^p$-boundedness holds only for $p \in (1, 3)$ is consistent with the analogous result for actual Schrödinger operators. Indeed it is well known [**114, 115**] that the wave operators for three-dimensional Schrödinger operators $-\Delta + V$ admitting a zero-energy resonance are $L^p$-bounded if and only if $p \in (1, 3)$, and moreover, in complete analogy with the disussion of Chapter 2 in the single centre case, it can be proved [**9**, Theorem II.1.2.1] that $H_{\alpha,Y}$ is actually the strong resolvent limit in $L^2(\mathbb{R}^3)$, as $\varepsilon \downarrow 0$, of Schrödinger operators of the form

$$H^{(\varepsilon)} \;=\; -\Delta + \varepsilon^{-2} \sum_{j=1}^{N} \lambda_j(\varepsilon) V_j\Big(\frac{x - y_j}{\varepsilon}\Big) \tag{3.6}$$

for suitable real-analytic $\lambda_j(\varepsilon)$'s with $\lambda(0) = 1$ and real potentials $V_j$ of finite Rollnik norm such that $-\Delta + V_j$ has a zero-energy resonance for each $j \in \{1, \dots, N\}$. To fully substantiate such a parallelism between singular and regular Schrödinger operators, it would be of great interest to monitor the convergence, as bounded operators in $L^p(\mathbb{R}^3)$ for $p \in (1,3)$, of the wave operators for the pair $(H_\varepsilon, H_0)$ to the wave operator $W^\pm_{\alpha,Y}$. Along this line, in Section 3.5 we present the proof of this result in the special case $N=1$, $\alpha=0$.

Observe that, by virtue of Lemma 1.1.3, the complex conjugation $u \mapsto \mathcal{C}u := \overline{u}$ reverses the direction of time, i.e.,

$$\mathcal{C}^{-1} e^{-\mathrm{i}tH_{\alpha,Y}} \mathcal{C} \;=\; e^{\mathrm{i}tH_{\alpha,Y}}, \qquad \mathcal{C}^{-1} e^{-\mathrm{i}tH_0} \mathcal{C} \;=\; e^{\mathrm{i}tH_0}\,, \tag{3.7}$$

whence

$$W^-_{\alpha,Y} \;=\; \mathcal{C}^{-1}\, W^+_{\alpha,Y}\, \mathcal{C}\,. \tag{3.8}$$

Thus, once the $L^p$-boundedness is proved for $W^+_{\alpha,Y}$ and all $p \in (1,3)$, the same result follows for $W^-_{\alpha,Y}$ via (3.8). Analogously, it suffices to prove the $L^p$-unboundedness of $W^+_{\alpha,Y}$, for $p=1$ and $p \in [3,\infty)$, in order to have same result for $W^-_{\alpha,Y}$.

The first key ingredient of our analysis in an explicit representation of the (integral kernel of) wave operators $W^+_{\alpha,Y}$, based on the explicit resolvent difference $(H_{\alpha,Y} - z^2\mathbb{1})^{-1} - (H_0 - z^2\mathbb{1})^{-1}$. Then, as a second key ingredient, for the $L^p \to L^p$ estimate of $W^\pm_{\alpha,Y}$ we appeal to a large extent to some tools from harmonic analysis, the Calderón-Zygmund operators and the Muckenhaupt weighted inequalities.

We organise the material as follows. In Section 3.1 we produce the explicit stationary representation of the wave operators $W^\pm_{\alpha,Y}$ which the proof of Theorem 3.0.3 is based on. The $L^p$-boundedness part of Theorem 3.0.3 is proved in Section 3.2 for the single centre case, and in Section 3.3 for the multi-centre case. The $L^p$-unboundedness part is proved in Section 3.4. Last, in Section 3.5 we discuss the convergence of the wave operators relative to the family of Hamiltonians (3.6) to the wave operators $W^\pm_{\alpha,Y}$ (limit of shrinking potentials).

## 3.1. Stationary representation of wave operators

Following a standard procedure [**76**], in order to prove the $L^p$-boundedness of $W^+_{\alpha,Y}$ we want to represent $W^+_{\alpha,Y}$ by means of the boundary values attained by the resolvents of $H_{\alpha,Y}$ and $H_0$ on the reals.

To this aim, we introduce the operators $\Omega_{jk}$, $j,k \in \{1,\dots,N\}$, acting on $L^2(\mathbb{R}^3)$, defined by

$$\begin{aligned}(\Omega_{jk} f)(x) \;&:=\; \lim_{\delta\downarrow 0} \frac{1}{\pi\mathrm{i}} \int_0^{+\infty} \mathrm{d}\lambda\, \lambda\, e^{-\delta\lambda} \\ &\quad\times \bigg( \int_{\mathbb{R}^3} (\Gamma_{\alpha,Y}(-\lambda)^{-1})_{jk}\, \mathcal{G}_{-\lambda}(x) \big(\mathcal{G}_\lambda(y) - \mathcal{G}_{-\lambda}(y)\big)\, f(y)\mathrm{d}y \bigg)\,,\end{aligned} \tag{3.9}$$

and we also introduce the translation operators $T_{x_0} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, $x_0 \in \mathbb{R}^3$, defined by

$$(T_{x_0} f)(x) \;:=\; f(x - x_0)\,. \tag{3.10}$$

First of all, we show that the $\Omega_{ij}$'s are well-defined. It is convenient to re-write $\Omega_{jk}$ by using the spherical mean $M_u$ of a given function $u$, namely

$$M_u(r) \;:=\; \frac{1}{4\pi} \int_{\mathbb{S}^2} u(r\omega)\, \mathrm{d}\omega\,, \qquad r \in \mathbb{R}\,. \tag{3.11}$$

Observe that $\mathbb{R} \ni r \mapsto M_u(r)$ is even. It is also convenient to define the matrix-valued function $\lambda \mapsto F(\lambda) := (F_{jk}(\lambda))_{jk}$ by

$$F_{jk}(\lambda) \;:=\; \mathbf{1}_{(0,+\infty)}(\lambda)\, \lambda\, (\Gamma_{\alpha,Y}(-\lambda)^{-1})_{jk}\,, \quad j,k \in \{1,\dots,N\}\,, \tag{3.12}$$

where $\mathbf{1}_\Lambda$ denotes the characteristic function of the set $\Lambda$.

LEMMA 3.1.1. *Assume that zero is not an eigenvalue for $H_{\alpha,Y}$, and that the matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \in \mathbb{R} \setminus \{0\}$.*

(i) *The function $\lambda \mapsto F(\lambda)$ is* smooth *and* uniformly bounded *on $\mathbb{R}$, and*

$$\lim_{\lambda\to+\infty} F(\lambda) \;=\; -4\pi\mathrm{i}\mathbb{1}\,. \tag{3.13}$$

(ii) *The limit (3.9) exists in $L^2(\mathbb{R}^3)$ and $\Omega_{jk}$ may be written in the form*

$$(\Omega_{jk}u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\int_{\mathbb{R}} e^{-\mathrm{i}\lambda|x|}\,F_{jk}(\lambda)\widehat{(rM_u)}(-\lambda)\;\mathrm{d}\lambda\,. \tag{3.14}$$

If we introduce the distributional Fourier transform of $F_{jk}(\lambda)$ as

$$L_{jk}(\rho) \;:=\; \frac{1}{\sqrt{2\pi}}\lim_{\delta\downarrow 0}\int_0^{+\infty}\mathrm{d}\lambda\, e^{-\delta\lambda}e^{-\mathrm{i}\lambda\rho}F_{jk}(\lambda)\,, \tag{3.15}$$

it follows from (3.14) that

$$(\Omega_{jk}u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\,(L_{jk} * rM_u)(|x|)\,. \tag{3.16}$$

PROOF OF LEMMA 3.1.1. (i) Owing to our assumption and Theorem 1.1.4(ii), the only pole of $\Gamma_{\alpha,Y}(z)^{-1}$ on the real line can be $z=0$, in which case it is a pole of order one. It follows that $\lambda \mapsto \lambda\,\Gamma_{\alpha,Y}(-\lambda)^{-1}$ is smooth and bounded on compact sets of $(0,+\infty)$, and so is $\lambda \mapsto F(\lambda)$ on compact sets of $\mathbb{R}$. Concerning the behaviour as $\lambda \to +\infty$, we see from (1.28) that

$$\Gamma_{\alpha,Y}(-\lambda) \;=\; -(4\pi\mathrm{i})^{-1}\lambda\mathbb{1} + R(\lambda)$$

for some symmetric matrix $R(\lambda)$ that is uniformly bounded for $\lambda \in (0,\infty)$. Thus, as $\lambda \to +\infty$,

$$\frac{\Gamma_{\alpha,Y}(-\lambda)}{\lambda} \;=\; -(4\pi\mathrm{i})^{-1}\mathbb{1} + \frac{R(\lambda)}{\lambda} \;\to\; -(4\pi\mathrm{i})^{-1}\mathbb{1}\,,$$

which proves (3.13).

(ii) Let $u \in C_0^\infty(\mathbb{R}^3)$. Then, for $\lambda \in \mathbb{R}$,

$$\int_{\mathbb{R}^3}\mathcal{G}_\lambda(y)u(y)\,\mathrm{d}y \;=\; \int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\lambda|y|}}{4\pi|y|}u(y)\,\mathrm{d}y \;=\; \int_0^{+\infty} e^{\mathrm{i}\lambda r} rM_u(r)\,\mathrm{d}r\,.$$

Since $\mathbb{R} \ni r \mapsto M_u(r)$ is even, the identity above yields

$$\begin{aligned}\int_{\mathbb{R}^3}\big(\mathcal{G}_\lambda(y) - \mathcal{G}_{-\lambda}(y)\big)u(y)\,\mathrm{d}y \;&=\; \int_{\mathbb{R}} e^{i\lambda r}\, rM_u(r)\,\mathrm{d}r\\ &=\; \sqrt{2\pi}\,\widehat{(rM_u)}(-\lambda)\end{aligned} \tag{3.17}$$

and (3.9) may be rewritten as

$$(\Omega_{jk}u)(x) \;=\; \lim_{\delta\downarrow 0}\frac{1}{(2\pi)^{\frac{3}{2}}\mathrm{i}|x|}\int_0^{+\infty} e^{-\delta\lambda}\,F_{jk}(\lambda)e^{-i\lambda|x|}\,\widehat{(rM_u)}(-\lambda)\mathrm{d}\lambda\,. \tag{3.18}$$

Here $\widehat{(rM_u)}(-\lambda)$ is a square integrable function of $\lambda \in \mathbb{R}$ because Parseval's identity and Hölder's inequality yield

$$\|\widehat{(rM_u)}(-\lambda)\|_{L^2(\mathbb{R})} = \|rM_u\|_{L^2(\mathbb{R})} \leqslant (\sqrt{\pi})^{-1}\|u\|_{L^2(\mathbb{R}^3)}.$$

Since $F_{jk}(\lambda)$ is bounded, the Fourier inversion formula implies that the limit $\delta \downarrow 0$ in (3.18) exists in $L^2(\mathbb{R}^3_x)$ and (3.14) follows. □

The main result of this Section is the following representation formula for the wave operator.

PROPOSITION 3.1.2. *Assume that zero is not an eigenvalue for $H_{\alpha,Y}$, and that the matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \in \mathbb{R} \setminus \{0\}$. For any $u, v \in L^2(\mathbb{R}^3)$ we have*

$$\langle W^+_{\alpha,Y} u, v \rangle \;=\; \langle u, v \rangle + \sum_{j,k=1}^{N} \langle T_{y_j} \Omega_{jk} T^*_{y_k} u, v \rangle \,. \tag{3.19}$$

PROOF. It suffices to prove (3.19) for $u, v \in C_0^\infty(\mathbb{R}^3)$.

The limit (3.4) when $t \to +\infty$ equals its Abel limit, thus we re-write

$$\langle W^+_{\alpha,Y} u, v \rangle \;=\; \lim_{\varepsilon \downarrow 0} 2\varepsilon \int_0^{+\infty} \langle e^{-\mathrm{i}t(H_0 - \mathrm{i}\varepsilon \mathbb{1})} u, e^{-\mathrm{i}t(H_{\alpha,Y} - \mathrm{i}\varepsilon \mathbb{1})} v \rangle \, \mathrm{d}t \,. \tag{3.20}$$

Let now $\mu \in \mathbb{R}$. Exploiting the Fourier transform

$$(H_0 - (\mu + \mathrm{i}\varepsilon)\mathbb{1})^{-1} \;=\; \mathrm{i} \int_0^{+\infty} e^{\mathrm{i}\mu t} \, e^{-\mathrm{i}t(H_0 - \mathrm{i}\varepsilon \mathbb{1})} \, \mathrm{d}t \qquad (\varepsilon > 0)$$

(and the analogue for $H_{\alpha,Y}$), Parseval's formula in the r.h.s. of (3.20) yields

$$\langle W^+_{\alpha,Y} u, v \rangle \;=\; \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + \mathrm{i}\varepsilon) u, R_{\alpha,Y}(\lambda + \mathrm{i}\varepsilon) v \rangle \, \mathrm{d}\lambda \,. \tag{3.21}$$

Substituting $R_{\alpha,Y}(\lambda + \mathrm{i}\varepsilon)$ in the r.h.s. of (3.21) with the resolvent identity (1.32), one obtains

$$\begin{aligned} \langle W^+_{\alpha,Y} u, v \rangle \;&=\; \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + \mathrm{i}\varepsilon) u, R_0(\lambda + \mathrm{i}\varepsilon) v \rangle \, \mathrm{d}\lambda \\ &\quad + \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{j,k=1}^{N} \int_{\mathbb{R}} (\Gamma_{\alpha,Y}(\sqrt{\lambda + \mathrm{i}\varepsilon})^{-1})_{jk} \times \\ &\qquad \times \langle R_0(\lambda + \mathrm{i}\varepsilon)\, u, \mathcal{G}^{y_j}_{\sqrt{\lambda + \mathrm{i}\varepsilon}} \rangle \, \langle \overline{\mathcal{G}^{y_k}_{\sqrt{\lambda + \mathrm{i}\varepsilon}}}, v \, \rangle \, \mathrm{d}\lambda \,. \end{aligned} \tag{3.22}$$

The first summand in the r.h.s. of (3.22) gives

$$\begin{aligned} \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle R_0(\lambda + \mathrm{i}\varepsilon) u, R_0(\lambda + \mathrm{i}\varepsilon) v \rangle \, \mathrm{d}\lambda \;&=\; \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \langle u, R_0(\overline{\lambda + \mathrm{i}\varepsilon}) R_0(\lambda + \mathrm{i}\varepsilon) v \rangle \, \mathrm{d}\lambda \\ &=\; \frac{\varepsilon}{\pi} \int_{\mathbb{R}} \mathrm{d}\lambda \int_{\sigma(H_0)} \langle u, E^{(H_0)}(\mathrm{d}h) v \rangle \, \frac{1}{(h - \lambda)^2 + \varepsilon^2} \\ &=\; \int_{\sigma(H_0)} \langle u, E^{(H_0)}(\mathrm{d}h) v \rangle \, \frac{1}{\pi} \int_{\mathbb{R}} \mathrm{d}\lambda \, \frac{\varepsilon}{(h - \lambda)^2 + \varepsilon^2} \;=\; \langle u, v \rangle \,, \end{aligned}$$

thus (3.22) reads

$$\begin{aligned} \langle W^+_{\alpha,Y} u, v \rangle \;=\; \langle u, v \rangle + & \lim_{\varepsilon \downarrow 0} \frac{\varepsilon}{\pi} \sum_{j,k=1}^{N} \int_{\mathbb{R}} (\Gamma_{\alpha,Y}(\sqrt{\lambda + \mathrm{i}\varepsilon})^{-1})_{jk} \times \\ & \times \langle u, R_0(\overline{\lambda + \mathrm{i}\varepsilon})\, \mathcal{G}^{y_j}_{\sqrt{\lambda + \mathrm{i}\varepsilon}} \rangle \, \langle \overline{\mathcal{G}^{y_k}_{\sqrt{\lambda + \mathrm{i}\varepsilon}}}, v \, \rangle \, \mathrm{d}\lambda \,. \end{aligned} \tag{3.23}$$

We recall that $\sqrt{z}$ is chosen in the upper complex half plane and, for $z \in \mathbb{C} \setminus [0, \infty)$,

$$\mathcal{G}^{y_j}_{\sqrt{z}}(x) \;=\; \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}p(x - y_j)}}{p^2 - z} \, \mathrm{d}p \; \left( \equiv \lim_{L \to \infty} \frac{1}{(2\pi)^3} \int_{|p| < L} \frac{e^{\mathrm{i}p(x - y_j)}}{p^2 - z} \, \mathrm{d}p \right) . \tag{3.24}$$

Thus, for $z \equiv \lambda + \mathrm{i}\varepsilon$, both $\sqrt{\lambda+\mathrm{i}\varepsilon}$ and $\sqrt{\lambda-\mathrm{i}\varepsilon}$ belong to $\mathbb{C}^+$, and we compute

$$
\begin{aligned}
\frac{\varepsilon}{\pi}\, R_0(\lambda-\mathrm{i}\varepsilon)\, \mathcal{G}^{y_j}_{\sqrt{\lambda+\mathrm{i}\varepsilon}}(x) \;&=\; \frac{1}{(2\pi)^3}\,\frac{\varepsilon}{\pi}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}p(x-y_j)}}{(p^2-\lambda+\mathrm{i}\varepsilon)(p^2-\lambda-\mathrm{i}\varepsilon)}\,\mathrm{d}p \\
&=\; \frac{1}{(2\pi)^3}\,\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}^3} e^{\mathrm{i}p(x-y_j)}\Big(\frac{1}{(p^2-\lambda-\mathrm{i}\varepsilon)}-\frac{1}{(p^2-\lambda+\mathrm{i}\varepsilon)}\Big)\,\mathrm{d}p \\
&=\; \frac{1}{2\pi\mathrm{i}}\Big(\mathcal{G}^{y_j}_{\sqrt{\lambda+\mathrm{i}\varepsilon}}(x)-\mathcal{G}^{y_j}_{\sqrt{\lambda-\mathrm{i}\varepsilon}}(x)\Big)\,.
\end{aligned}
\tag{3.25}
$$

The second summand in the r.h.s. of (3.23) can be then written as

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\sum_{j,k=1}^{N}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\mathrm{d}\lambda\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y)}\,\Big(\mathcal{G}^{y_j}_{\sqrt{\lambda+\mathrm{i}\varepsilon}}(y)-\mathcal{G}^{y_j}_{\sqrt{\lambda-\mathrm{i}\varepsilon}}(y)\Big) \\
&\qquad\qquad\times\;(\Gamma_{\alpha,Y}(\sqrt{\lambda+\mathrm{i}\varepsilon})^{-1})_{jk}\,\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}^{y_k}_{\sqrt{\lambda+\mathrm{i}\varepsilon}}(x)\,v(x)\Big)\,.
\end{aligned}
\tag{3.26}
$$

Because $u$ and $v$ are smooth and with compact support, an integration by parts shows that both the d$x$-integral and the d$y$-integral in (3.26) above are bounded by $C\langle\lambda\rangle^{-\frac{1}{2}}$ *uniformly in* $\varepsilon$. Moreover, since we are assuming that zero is not an eigenvalue for $H_{\alpha,Y}$, it follows from Theorem 1.1.4(ii) that either the matrix $\Gamma_{\alpha,Y}(\sqrt{\lambda+\mathrm{i}\varepsilon})^{-1}$ has the singularity $(\sqrt{\lambda+\mathrm{i}\varepsilon})^{-1}$ near $\lambda=0$ (in the limit $\varepsilon\downarrow 0$), or it is bounded, with $\|\Gamma_{\alpha,Y}(\sqrt{\lambda+\mathrm{i}\varepsilon})^{-1}\|\leqslant C\langle\lambda\rangle^{-\frac{1}{2}}$. Therefore the $\lambda$-integrand is uniformly bounded by $C\lambda^{-\frac{1}{2}}\langle\lambda\rangle^{-1}$, dominated convergence is applicable in (3.26) above, the d$\lambda$-integration and the $\varepsilon\downarrow 0$-limit can be exchanged, and (3.26) becomes

$$
\begin{aligned}
&\sum_{j,k=1}^{N}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\mathrm{d}\lambda\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y)}\,\Big(\mathcal{G}^{y_j}_{\sqrt{\lambda+\mathrm{i}0}}(y)-\mathcal{G}^{y_j}_{\sqrt{\lambda-\mathrm{i}0}}(y)\Big) \\
&\qquad\qquad\times\;(\Gamma_{\alpha,Y}(\sqrt{\lambda+\mathrm{i}0})^{-1})_{jk}\,\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}^{y_k}_{\sqrt{\lambda+\mathrm{i}0}}(x)\,v(x)\Big)\,.
\end{aligned}
\tag{3.27}
$$

Owing to the difference $\mathcal{G}^{y_j}_{\sqrt{\lambda+\mathrm{i}0}}-\mathcal{G}^{y_j}_{\sqrt{\lambda-\mathrm{i}0}}$, we see that the $\lambda$-integration in (3.27) is only effective when $\lambda\geqslant 0$. Indeed, if $\lambda<0$, then $\sqrt{\lambda\pm\mathrm{i}0}=\mathrm{i}\sqrt{|\lambda|}$ and the integrand vanishes. We then consider (3.27) only with $\lambda\in[0,+\infty)$ and with the change of variable $\lambda\mapsto\lambda^2$ we obtain

$$
\begin{aligned}
&\text{second summand in the r.h.s. of (3.23)} \;= \\
&=\;\sum_{j,k=1}^{N}\frac{1}{\pi\mathrm{i}}\int_0^{+\infty}\mathrm{d}\lambda\,\lambda\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y)}\,\Big(\mathcal{G}^{y_j}_{\lambda}(y)-\mathcal{G}^{y_j}_{-\lambda}(y)\Big) \\
&\qquad\qquad\times\;(\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk}\,\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}^{y_k}_{\lambda}(x)\,v(x)\Big) \\
&=\;\lim_{\delta\downarrow 0}\sum_{j,k=1}^{N}\frac{1}{\pi\mathrm{i}}\int_0^{+\infty}\mathrm{d}\lambda\,\lambda\,e^{-\delta\lambda}\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y+y_k)}\,\big(\mathcal{G}_{\lambda}(y)-\mathcal{G}_{-\lambda}(y)\big) \\
&\qquad\qquad\times\;(\Gamma_{\alpha,Y}(\lambda)^{-1})_{jk}\,\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}^{y_j}_{\lambda}(x)\,v(x)\Big) \\
&=\;\lim_{\delta\downarrow 0}\int_{\mathbb{R}^3}\mathrm{d}x\,v(x)\sum_{j,k=1}^{N}\int_0^{+\infty}\mathrm{d}\lambda\,\lambda\,e^{-\delta\lambda}\int_{\mathbb{R}^3}\mathrm{d}y \\
&\qquad\times\;\overline{\Big(\frac{1}{\pi\mathrm{i}}\,(\Gamma_{\alpha,Y}(-\lambda)^{-1})_{jk}\,\mathcal{G}^{y_j}_{-\lambda}(x)\,\big(\mathcal{G}_{\lambda}(y)-\mathcal{G}_{-\lambda}(y)\big)u(y+y_k)\Big)}.
\end{aligned}
\tag{3.28}
$$

In the first step of (3.28) above we used the fact that $\sqrt{\lambda^2\pm i0}=\pm\lambda$ for $\lambda>0$. In the second step, the insertion of the exponential cut-off $e^{-\delta\lambda}$ is justified by the fact that the $\lambda$-integrand is uniformly bounded by $C\langle\lambda\rangle^{-\frac{5}{2}}$, as discussed above; we also exchanged $j \leftrightarrow k$, using the fact that $\Gamma_{\alpha,Y}(\lambda)^{-1}$ is symmetric, and made the change of variable $y \mapsto y+y_k$, using (1.27). In the third step we used the properties $\overline{\mathcal{G}_\lambda(x)} = \mathcal{G}_{-\lambda}(x)$ and $\overline{\Gamma_{\alpha,Y}(\lambda)^{-1}} = \Gamma_{\alpha,Y}(-\lambda)^{-1}$ that follow, respectively, from (1.27) and (1.28). The identity (3.19) then follows immediately from (3.28). $\square$

Summarising so far, we produced the representation (3.9)-(3.19) of the kernel of the wave operator $W^+_{\alpha,Y}$. Because of the obvious $L^p$-boundedness of $T_{x_0}$, in order to prove Theorem 3.0.3 it suffices to study the $L^p$-boundedness or unboundedness of each $\Omega_{jk}$, that is, to consider the quantities

$$\|\Omega_{jk}u\|^p_{L^p(\mathbb{R}^3)} \;=\; \frac{4\pi}{(2\pi)^{3p/2}}\int_0^{+\infty} |(L_{jk} * \rho M_u)(\rho)|^p\,\rho^{2-p}\,\mathrm{d}\rho\,, \tag{3.29}$$

whose expression follows from (3.16).

For a more compact notation, it is convenient to introduce the matrix functions

$$L(\rho) \;:=\; \big(L_{jk}(\rho)\big)_{jk}\,, \quad \Omega(\rho) \;:=\; \big(\Omega_{jk}(\rho)\big)_{jk}\,, \tag{3.30}$$

in terms of which

$$(\Omega u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\int_0^{+\infty} e^{-\mathrm{i}\lambda|x|}\,F(\lambda)\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda \tag{3.31}$$

and

$$(\Omega u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\,(L * rM_u)(|x|)\,. \tag{3.32}$$

The additional formulas (3.31)/(3.32) have the virtue of reducing the problem to the estimate of singular integral operators in one dimension and will play an important role in our next arguments – although in certain steps we need to go back to the more complicated, but more flexible expression (3.9).

## 3.2. $L^p$-bounds for the single centre case

In this Section and in the two following ones we present the proof of Theorem 3.0.3. In fact, only the statements concerning the boundedness and the unboundedness of $W^\pm_{\alpha,Y}$ need be proved, because the existence of $W^\pm_{\alpha,Y}$ in $L^2(\mathbb{R}^3)$ and their completeness follow at once from the Birman-Kato-Pearson Theorem [**97**], due to the fact (Theorem 1.1.2(i), identity (1.32)) that the resolvent difference $R_{\alpha,Y}(z) - R_0(z)$ is a rank-$N$ operator.

We first introduce some fundamental result from harmonic analysis, in particular in the theory of Calderón-Zygmund singular operators. For the definition of Calderón-Zygmund operators we refer to [**54**, Definitions 7.4.1, 7.4.2] and to [**55**, Definitions 4.1.2 and 4.1.8], whereas for the definition of $A_p$ Muckenhaupt weights we refer to [**54**, Definitions 7.1.3]. We shall use interchangeably the same symbol for a Calderón-Zygmund operator and for its integral kernel.

The following properties are known.

Theorem 3.2.1.

(i) *The convolution operator on $\mathbb{R}$ with a function $L(x)$ is a Calderón-Zygmund operator if $\widehat{L}(\xi)$ is bounded and, for a constant $C>0$, one has*

$$|L(x)| \;\leqslant\; C\,|x|^{-1} \qquad \textit{and} \qquad \Big|\frac{\mathrm{d}L}{\mathrm{d}x}(x)\Big| \;\leqslant\; C\,|x|^{-2} \qquad \textit{for } x\neq 0\,.$$

(ii) *If $L$ is a Calderón-Zygmund operator and $w$ is an $A_p$-weight for some $p\in(1,\infty)$, then $L$ is bounded in $L^p(\mathbb{R},w(x)dx)$ in the sense that*

$$\int_{\mathbb{R}} |(Lu)(x)|^p\,w(x)\,\mathrm{d}x \;\leqslant\; \int_{\mathbb{R}} |u(x)|^p\,w(x)\,\mathrm{d}x \qquad \forall u\in C^\infty_0(\mathbb{R})\,. \tag{3.33}$$

(iii) *If $w$ is an $A_p$-weight for some $p \in (1,\infty)$ and*

$$(\mathcal{M}(u))(x) \;:=\; \sup_{r>0} \frac{1}{2r} \int_{|x-y|<r} |u(y)|\,\mathrm{d}y \tag{3.34}$$

*is the Hardy-Littlewood maximal function of some $u \in C_0^\infty(\mathbb{R})$, then*

$$\int_{\mathbb{R}} |(\mathcal{M}(u))(x)|^p\, w(x)\,\mathrm{d}x \;\leqslant\; \int_{\mathbb{R}} |u(x)|^p\, w(x)\,\mathrm{d}x\,. \tag{3.35}$$

*If, for some function $L(x)$ one has $|L(x)| \leqslant A(x)$ in $\mathbb{R}$ for some $A \in L^1(\mathbb{R})$ which is bounded, non-negative, even, and non-increasing on $(0,+\infty)$, then $|(L*u)(x)| \leqslant C(\mathcal{M}(u))(x)$, hence the convolution operator on $\mathbb{R}$ with the function $L(x)$ is bounded in $L^p(\mathbb{R}, w(x)dx)$.*

(iv) *The function $|x|^a$ is an $A_p$-weight on $\mathbb{R}$ if and only if $a \in (-1, p-1)$.*

Concerning part (i) we refer to [**55**, Remark 4.1.1]. Part (ii) is a corollary of [**54**, Theorem 7.4.6]. The first and second statement of part (iii) are respectively [**105**, Theorem 1, Section V.3] and the Proposition in page 57 of [**105**, Section II.2.1]. For part (iv) we refer to [**54**, Example 7.1.7].

We start with the proof of the boundedness part of Theorem 3.0.3 in the special case of $N = 1$ centre. This case is simpler, for the oscillating terms $\mathcal{G}_\lambda^{y_j,y_k}$ are now absent, nevertheless it retains most of the essential ideas needed in the proof of the general case, which is the object of the following Section 3.3.

We shall control the two regimes $p \in (1, \frac{3}{2})$ and $p \in (\frac{3}{2}, 3)$ separately. Then the overall $L^p$-boundedness for $p \in (1,3)$ follows by interpolation.

**3.2.1. $L^p$-boundedness of $W_{\alpha,Y}^+$ for $N = 1$ and $p \in (\frac{3}{2}, 3)$.** In this regime the proof is based on Theorem 3.2.1 and on the following fact.

Lemma 3.2.2. *Suppose that $[0,+\infty) \ni \lambda \mapsto W(\lambda)$ is a smooth and bounded function such that $\lambda \mapsto W'(\lambda)$ and $\lambda \mapsto \lambda W''(\lambda)$ are both integrable. Let $Z(\rho)$, $\rho \in \mathbb{R}$, be the Fourier transform of $W(\lambda)$, in the sense of distributions, defined by*

$$Z(\rho) \;=\; \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}\, W(\lambda)\,.$$

*Then, the convolution operator with $Z(\rho)$ is a Calderón-Zygmund operator on $\mathbb{R}$. In particular, the operator $u \mapsto L*u$, where $L$ is defined in* (3.15) *for the case $N = 1$, is of Calderón-Zygmund type.*

Proof. The operator of convolution with $Z$ is bounded in $L^2(\mathbb{R})$ because $Z$ is the Fourier transform of a bounded function $W$. Integration by parts, using $e^{-\mathrm{i}\lambda\rho} = \mathrm{i}\rho^{-1}\partial_\lambda e^{-\mathrm{i}\lambda\rho}$, yields

$$Z(\rho) \;=\; \frac{\mathrm{i}}{\rho\sqrt{2\pi}} W(0) - \frac{\mathrm{i}}{\rho\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}\, W'(\lambda) \;\leqslant_{|\cdot|}\; \frac{C}{|\rho|}\,, \quad \rho \neq 0\,,$$

and differentiating further in $\rho$ yields

$$\begin{aligned} Z'(\rho) \;=\; & -\frac{\mathrm{i}\,W(0)}{\rho^2\sqrt{2\pi}} + \frac{\mathrm{i}}{\rho^2\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}\, W'(\lambda) \\ & - \frac{1}{\rho\sqrt{2\pi}} \int_0^{+\infty} \mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}\, \lambda\, W'(\lambda)\,. \end{aligned}$$

The first two summands in the r.h.s. above are obviously bounded in absolute value by $C|\rho|^{-2}$ for $\rho \neq 0$; so too is the third summand, as follows from integration by parts:

$$\begin{aligned} \frac{-1}{\rho\sqrt{2\pi}}\int_0^{+\infty}\mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}\,\lambda\, W'(\lambda) \;&=\; \frac{\mathrm{i}}{\rho^2\sqrt{2\pi}}\int_0^{+\infty}\mathrm{d}\lambda\, e^{-\mathrm{i}\lambda\rho}(W'(\lambda)+\lambda W''(\lambda)) \\ &\leqslant_{|\cdot|}\; \frac{C}{\rho^2}\,, \quad \rho\neq 0\,. \end{aligned}$$

Thus, we conclude from Theorem 3.2.1(i) that $u\mapsto Z*u$ is a Calderón-Zygmund operator on $\mathbb{R}$. Concerning the second statement of the thesis, we see that in the case $N=1$ (3.12) reads

$$F(\lambda) \;=\; \lambda\Big(\alpha+\frac{\mathrm{i}\lambda}{4\pi}\Big)^{-1}. \tag{3.36}$$

$F$ is therefore bounded and smooth on $[0,+\infty)$ and both $F'(\lambda)$ and $\lambda\,F''(\lambda)$ are integrable, whence the conclusion for the operator of convolution by $L$ defined in (3.15). □

The proof of the $L^p$-boundedness of $W^+_{\alpha,Y}$ for $N=1$ and $p\in(\frac{3}{2},3)$ then becomes particularly simple. First, we recall from (3.29) that

$$\|\Omega u\|^p_{L^p(\mathbb{R}^3)} \;=\; \frac{4\pi}{(2\pi)^{3p/2}}\int_0^{+\infty}|(L*\rho M_u)(\rho)|^p\,\rho^{2-p}\,\mathrm{d}\rho\,,$$

where $\rho^{2-p}$ is an $A_p$-weight for $p\in(\frac{3}{2},3)$ (Theorem 3.2.1(iv)) and the convolution with $L$ is a Calderón-Zygmund operator on $\mathbb{R}$ (Lemma 3.2.2). Then it follows from Theorem 3.2.1(ii) that

$$\begin{aligned} \|\Omega u\|^p_{L^p(\mathbb{R}^3)} \;&\leqslant\; \int_0^{+\infty}|(\rho M_u)(\rho)|^p\,\rho^{2-p}\,\mathrm{d}\rho \;=\; \int_0^{+\infty}|M_u(\rho)|^p\,\rho^2\,\mathrm{d}\rho \\ &\leqslant\; C_p\,\|u\|^p_{L^p(\mathbb{R}^3)} \end{aligned} \tag{3.37}$$

for some constant $C_p>0$, whence the conclusion.

**3.2.2. $L^p$-boundedness of $W^+_{\alpha,Y}$ for $N=1$ and $p\in(1,\frac{3}{2})$.** In the regime $p\in(1,\frac{3}{2})$ the general harmonic analysis treatment provided by Theorem 3.2.1 only allows us to find an $L^p$-bound to *part* of the function (see (3.14) above)

$$(\Omega u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\int_0^{+\infty}e^{-\mathrm{i}\lambda|x|}\,F(\lambda)\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\,,$$

whereas for the remaining part we need to produce further analysis.

Integrating by parts the above expression of $\Omega u$, using $e^{-\mathrm{i}\lambda\rho}=\mathrm{i}\rho^{-1}\partial_\lambda e^{-\mathrm{i}\lambda\rho}$, yields

$$\Omega u \;=\; \Omega_1 u+\Omega_2 u\,, \tag{3.38}$$

where

$$\begin{aligned} (\Omega_1 u)(x) \;&:=\; \frac{-\mathrm{i}}{(2\pi)^{\frac{3}{2}}|x|^2}\int_0^{+\infty}e^{-\mathrm{i}\lambda|x|}\,F(\lambda)\widehat{(r^2M_u)}(-\lambda)\,\mathrm{d}\lambda\,, \\ (\Omega_2 u)(x) \;&:=\; \frac{-1}{(2\pi)^{\frac{3}{2}}|x|^2}\int_0^{+\infty}e^{-\mathrm{i}\lambda|x|}\,F'(\lambda)\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\,. \end{aligned} \tag{3.39}$$

Now, concerning $\Omega_1 u$, we re-write

$$(\Omega_1 u)(x) \;=\; \frac{-\mathrm{i}}{(2\pi)^{\frac{3}{2}}|x|^2}(L*r^2M_u)(|x|) \tag{3.40}$$

with $L$ given by (3.15). Owing to Lemma 3.2.2, $u \mapsto L * u$ is a Calderón-Zygmund operator, and owing to Theorem 3.2.1(iv), $|x|^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p \in (1, \frac{3}{2})$. Therefore,

$$\begin{aligned} \|\Omega_1 u\|^p_{L^p(\mathbb{R}^3)} \;&=\; \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(L * \rho^2 M_u)(\rho)|^p \,\rho^{2-2p}\,\mathrm{d}\rho \\ &\leqslant\; \int_0^{+\infty} |\rho^2 M_u(\rho)|^p\,\rho^{2-2p}\,\mathrm{d}\rho \;=\; \int_0^{+\infty} |M_u(\rho)|^p\,\rho^2\,\mathrm{d}\rho \\ &\leqslant\; C_p\,\|u\|^p_{L^p(\mathbb{R}^3)} \end{aligned} \tag{3.41}$$

for some constant $C_p > 0$, where in the second step we applied Theorem 3.2.1(ii). This proves the $L^p$-boundedness of $\Omega_1$.

Concerning $\Omega_2 u$, instead, we re-write

$$(\Omega_2 u)(x) \;=\; \frac{-1}{(2\pi)^{\frac{3}{2}}\rho^2}(\mathcal{L} * r M_u)(\rho)\,, \tag{3.42}$$

where $\mathcal{L}$ is the Fourier transform of the function $\mathbf{1}_{(0,\infty)}F'(\lambda)$, and

$$F'(\lambda) \;=\; \alpha\Big(\alpha + \frac{\mathrm{i}\,\lambda}{4\pi}\Big)^{-2}. \tag{3.43}$$

Thus, in the non-trivial case $\alpha \neq 0$ $F'$ is smooth and bounded, and correspondingly both $F''$ and $\lambda F'''$ are integrable. This implies, through Lemma 3.2.2, that $u \mapsto \mathcal{L}*u$ is a Calderón-Zygmund operator on $\mathbb{R}$. Since $|x|^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p \in (1, \frac{3}{2})$ (Theorem 3.2.1(iv)), then Theorem 3.2.1(ii) yields

$$\begin{aligned} \|\Omega_2 u\|^p_{L^p(\mathbb{R}^3)} \;&=\; \frac{4\pi}{(2\pi)^{3p/2}} \int_0^{+\infty} |(\mathcal{L} * \rho\, M_u)(\rho)|^p\,\rho^{2-2p}\,\mathrm{d}\rho \\ &\leqslant\; \int_0^{+\infty} |\rho\, M_u(\rho)|^p\,\rho^{2-2p}\,\mathrm{d}\rho \;\leqslant\; C\int_{\mathbb{R}^3} \frac{|u(x)|^p}{|x|^p}\,\mathrm{d}x \end{aligned} \tag{3.44}$$

for some constant $C > 0$. This shows that

$$\|\Omega_2 \mathbf{1}_{\{|x|\geqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)} \;\leqslant\; C\,\|\mathbf{1}_{\{|x|\geqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)}\,. \tag{3.45}$$

For $L^p$-functions supported on $|x| \leqslant 1$ a further argument is needed. In other words, so far from (3.41) and (3.45) we have

$$\begin{aligned} &\|\Omega u\|^p_{L^p(\mathbb{R}^3)} \\ &\quad\leqslant\; 2\|\Omega_1 u\|^p_{L^p(\mathbb{R}^3)} + 2\|\Omega_2 \mathbf{1}_{\{|x|\geqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)} + 2\|\Omega_2 \mathbf{1}_{\{|x|\leqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)} \\ &\quad\leqslant\; C_p\|u\|^p_{L^p(\mathbb{R}^3)} + C\|\mathbf{1}_{\{|x|\geqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)} + 2\|\Omega_2 \mathbf{1}_{\{|x|\leqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)}\,, \end{aligned} \tag{3.46}$$

and we are left with producing the estimate

$$\|\Omega_2 \mathbf{1}_{\{|x|\leqslant 1\}} u\|_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\|\mathbf{1}_{\{|x|\leqslant 1\}} u\|^p_{L^p(\mathbb{R}^3)}\,. \tag{3.47}$$

To this aim, let us establish first the following result.

LEMMA 3.2.3. *Suppose that* $[0,+\infty) \ni y \mapsto Y(y)$ *is a bounded* $C^1$*-function such that* $\lambda \mapsto \lambda^\theta Y(\lambda)$ *and* $\lambda \mapsto (1+\lambda)^\theta Y'(\lambda)$ *are both integrable for all* $\theta \in (0,1)$*, and let*

$$T(x,y) \;:=\; \frac{1}{|x|^2}\int_0^{+\infty} \Big(\frac{e^{-\mathrm{i}\lambda(|x|-|y|)} - e^{-\mathrm{i}\lambda(|x|+|y|)}}{4\pi|y|}\Big) Y(\lambda)\,\mathrm{d}\lambda \tag{3.48}$$

*for* $x,y \in \mathbb{R}^3$*. Then, for any* $R > 0$ *and* $p \in (1, \frac{3}{2})$*, the integral operator* $T$ *on* $\mathbb{R}^3$ *with the integral kernel* $T(x,y)$ *is* $L^p(\Lambda_R) \to L^p(\mathbb{R}^3)$ *bounded, with* $\Lambda_R := \{x \in \mathbb{R}^3 \,|\, |x| \leqslant R\}$*.*

PROOF. We only consider the case $R=1$, the proof for generic $R$ is similar. Let us deal with the region $|x|\leqslant 10$ first. Since $|e^{-\mathrm{i}\lambda(|x|-|y|)}-e^{-\mathrm{i}\lambda(|x|+|y|)}|\leqslant 2(\lambda|y|)^{1-\theta}$ for any $\theta\in(0,1)$, and since $\lambda^{1-\theta}Y\in L^1(0,+\infty)$, then

$$|T(x,y)| \;\leqslant\; \frac{1}{2\pi|x|^2|y|^\theta}\int_0^{+\infty}|Y(\lambda)|\,\lambda^{1-\theta}\,\mathrm{d}\lambda \;\leqslant\; \frac{C_\theta}{|x|^2|y|^\theta}\,,\qquad |x|\leqslant 10\,,$$

for some constant $C_\theta>0$. For fixed $p$ in $(1,\frac{3}{2})$, we take $\theta\in(0,1)$ such that $p'\theta<3$, where $p'=\frac{p}{p-1}$ as usual. With this choice, $|y|^{-\theta}\in L^{p'}(\Lambda_1)$ and $|x|^{-2}\in L^p(\Lambda_{10})$, with $\Lambda_R=\{x\in\mathbb{R}^3\,|\,|x|\leqslant R\}$ as in the statement of the Lemma. For each $f\in L^p(\Lambda_1)$, Hölder's inequality and the above bound for $|T(x,y)|$ then imply

$$\|Tf\|_{L^p(\Lambda_{10})}\;\leqslant\; C_\theta\,\||x|^{-2}\|_{L^p(\Lambda_{10})}\cdot\|f\|_{L^p(\Lambda_1)}\cdot\||y|^{-\theta}\|_{L^{p'}(\Lambda_1)}\;=\;\kappa_p^-\,\|f\|_{L^p(\Lambda_1)}$$

for some constant $\kappa_p^->0$. Next, let us consider the region $|x|\geqslant 10$. Integration by parts gives

$$\begin{aligned}
T(x,y)\;&=\;\frac{1}{4\pi|x|^2|y|}\int_0^{+\infty}\partial_\lambda\Big(\frac{e^{-\mathrm{i}\lambda(|x|-|y|)}}{-\mathrm{i}(|x|-|y|)}-\frac{e^{-\mathrm{i}\lambda(|x|+|y|)}}{-\mathrm{i}(|x|+|y|)}\Big)\,Y(\lambda)\,\mathrm{d}\lambda\\
\text{(I)}\qquad&=\;\frac{1}{4\pi\,\mathrm{i}\,|x|^2|y|}\Big(\frac{1}{|x|-|y|}-\frac{1}{|x|+|y|}\Big)\,Y(0)\\
\text{(II)}\qquad&\quad+\;\frac{1}{4\pi\,\mathrm{i}\,|x|^2|y|}\int_0^{+\infty}\Big(\frac{e^{-\mathrm{i}\lambda(|x|-|y|)}}{|x|-|y|}-\frac{e^{-\mathrm{i}\lambda(|x|+|y|)}}{|x|+|y|}\Big)\,Y'(\lambda)\,\mathrm{d}\lambda\,.
\end{aligned}$$

Since $|x|\pm|y|\geqslant\frac{9}{10}|x|\geqslant 9$ whenever $|x|\geqslant 10$ and $|y|\leqslant 1$, and since $Y$ is bounded, then clearly

$$|(\mathrm{I})|\;\leqslant\;\frac{C}{|x|^4}\;\leqslant\;\frac{C}{|x|^3\,|y|^\theta}$$

for some constant $C>0$ and any $\theta\in(0,1)$. As for the summand (II), since

$$\begin{aligned}
\frac{e^{-\mathrm{i}\lambda(|x|-|y|)}}{|x|-|y|}-\frac{e^{-\mathrm{i}\lambda(|x|+|y|)}}{|x|+|y|}\;&\leqslant_{|\,\cdot\,|}\;\frac{2\,|y|}{|x|^2-|y|^2}+\frac{(2\lambda|y|)^{1-\theta}}{|x|+|y|}\\
&\leqslant\;C\Big(\frac{|y|}{|x|^2}+\frac{(\lambda|y|)^{1-\theta}}{|x|}\Big)
\end{aligned}$$

for some constant $C>0$ and any $\theta\in(0,1)$, and since $(1+\lambda)^{1-\theta}\,Y'\in L^1(0,+\infty)$, then

$$|(\mathrm{II})|\;\leqslant\;\frac{C}{|x|^2|y|}\Big(\frac{|y|}{|x|^2}+\frac{|y|^{1-\theta}}{|x|}\Big)\;=\;C\Big(\frac{1}{|x|^4}+\frac{1}{|x|^3|y|^\theta}\Big)\;\leqslant\;\frac{2C}{|x|^3\,|y|^\theta}$$

and hence also

$$|T(x,y)|\;\leqslant\;\frac{C}{|x|^3\,|y|^\theta}\,,\qquad |x|\geqslant 10\,,$$

for some constant $C>0$ and any $\theta\in(0,1)$. For fixed $p\in(1,\frac{3}{2})$, we take $\theta\in(0,1)$ such that $p'\theta<3$ and $f\in L^p(\Lambda_1)$: with this choice, Hölder's inequality yields

$$\begin{aligned}
\|Tf\|_{L^p(\mathbb{R}^3\setminus\Lambda_{10})}\;&\leqslant\;C_\theta\,\||x|^{-3}\|_{L^p(\mathbb{R}^3\setminus\Lambda_{10})}\cdot\|f\|_{L^p(\Lambda_1)}\cdot\||y|^{-\theta}\|_{L^{p'}(\Lambda_1)}\\
&=\;\kappa_p^+\,\|f\|_{L^p(\Lambda_1)}
\end{aligned}$$

for some constant $\kappa_p^+>0$. Combining the above bounds yields the boundedness of $T$ as a map from $L^p(\Lambda_1)$ to $L^p(\mathbb{R}^3)$. $\square$

Let us now complete the proof of the $L^p$-boundedness of $W^+_{\alpha,Y}$ for $N=1$ and $p\in(1,\frac{3}{2})$. We only need to show (3.47). Upon re-writing the second equation in (3.39) by means of (3.17), that is,

$$(3.49)\quad(\Omega_2u)(x)\;=\;\frac{-1}{(2\pi)^2|x|^2}\int_0^{+\infty}e^{-\mathrm{i}\lambda|x|}F'(\lambda)\Big(\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\lambda|y|}-e^{-\mathrm{i}\lambda|y|}}{4\pi|y|}\,u(y)\,\mathrm{d}y\Big)\mathrm{d}\lambda\,,$$

it is immediate to recognise that

$$\Omega_2 u \;=\; -(2\pi)^{-2}\, Tu\,, \tag{3.50}$$

where $T$ is the integral operator given by (3.48) with $Y \equiv F'$, and $F'$ does satisfy the assumptions of Lemma 3.2.3. From this, we conclude (3.47) at once.

## 3.3. $L^p$-bounds for the general multi-centre case

The additional complication in the case $N \geqslant 2$ is due to the presence, in the function $F$ defined in (3.12) and (3.30), of the terms $\mathcal{G}_\lambda^{y_j y_k}$ (definitions (1.27)-(1.28)), which are *oscillatory* in $\lambda$.

Let us start the discussion by re-writing

$$F(\lambda) \;=\; \lambda\,\Gamma_{\alpha,Y}(-\lambda)^{-1} \;=\; \lambda\Big(\mathcal{A}+\frac{\mathrm{i}\lambda}{4\pi}\mathbb{1}-\widetilde{G}(-\lambda)\Big)^{-1}, \quad \lambda>0\,, \tag{3.51}$$

with

$$\mathcal{A} \;:=\; \mathrm{diag}(\alpha_1,\dots,\alpha_N)\,, \tag{3.52}$$

$$\widetilde{G}(\lambda) \;:=\; (\mathcal{G}_\lambda^{y_j y_k})_{j,k=1,\dots,N}\,. \tag{3.53}$$

We decompose $F(\lambda)$ into a small-$\lambda$ and a large-$\lambda$ contribution by means of two cut-off functions $\omega_<$ and $\omega_>$ such that

$$\begin{aligned} &\omega_< \;\in\; C_0^\infty(\mathbb{R})\,, \qquad \omega_>(\lambda) \;:=\; 1-\omega_<(\lambda)\,,\\ &\omega_<(\lambda) \;=\; \begin{cases} 1 & \text{if } |\lambda|\leqslant\gamma\\ 0 & \text{if } |\lambda|\geqslant 2\gamma\,,\end{cases}\end{aligned} \tag{3.54}$$

where $\gamma>0$ is a sufficiently large number so that,

$$\|\mathcal{A}-\widetilde{G}(-\lambda)\| \;<\; |\lambda|(16\pi)^{-1}\,, \qquad |\lambda|\;\geqslant\;\gamma \tag{3.55}$$

($\|E\|$ being the operator norm of the matrix $E$ as an operator on $\mathbb{C}^N$), and the r.h.s. of (3.51) is invertible. Explicitly,

$$F \;=\; F^< + F^>\,, \qquad F^< \;:=\; \omega_< F\,, \qquad F^> \;:=\; \omega_> F\,. \tag{3.56}$$

From (3.51) and (3.55) we expand

$$\begin{aligned} F^>(\lambda) \;=\; &-4\pi\mathrm{i}\,\omega_>(\lambda)\Big\{\mathbb{1}-\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)+\Big(\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)\Big)^2\Big\}\\ &-4\pi\mathrm{i}\,\omega_>(\lambda)\Big(\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)\Big)^3\Big(\mathbb{1}-\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)\Big)^{-1}.\end{aligned} \tag{3.57}$$

We collect all terms that do not contain $\widetilde{G}(-\lambda)$ or for which the oscillation of $\widetilde{G}(-\lambda)$ is harmless into the quantity

$$\begin{aligned} F^{(0)}(\lambda) \;:=\; &F^<(\lambda)-4\pi\mathrm{i}\,\omega_>(\lambda)\Big\{\mathbb{1}-\frac{4\pi\mathrm{i}}{\lambda}\mathcal{A}-\frac{16\pi^2}{\lambda^2}\mathcal{A}^2\Big\}\\ &-4\pi\mathrm{i}\,\omega_>(\lambda)\Big(\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)\Big)^3\Big(\mathbb{1}-\frac{4\pi\mathrm{i}}{\lambda}\big(\mathcal{A}-\widetilde{G}(-\lambda)\big)\Big)^{-1},\end{aligned} \tag{3.58}$$

whereas

$$\begin{aligned} &F^{(1)}(\lambda) \;:=\; 4\pi\mathrm{i}\,\omega_>(\lambda)\\ &\quad\times\Big\{-\frac{4\pi\mathrm{i}}{\lambda}\widetilde{G}(-\lambda)-\frac{16\pi^2}{\lambda^2}\big(\mathcal{A}\,\widetilde{G}(-\lambda)+\widetilde{G}(-\lambda)\,\mathcal{A}\big)+\frac{16\pi^2}{\lambda^2}\widetilde{G}(-\lambda)^2\Big\}\end{aligned} \tag{3.59}$$

contains the oscillations explicitly.

Thus,

$$F \;=\; F^{(0)}+F^{(1)} \qquad \text{and} \qquad \Omega \;=\; \Omega^{(0)}+\Omega^{(1)}\,, \tag{3.60}$$

where, by means of (3.31),

$$
\begin{aligned}
(\Omega^{(\ell)}u)(x) \;&:=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}\int_0^{+\infty} e^{-\mathrm{i}\lambda|x|}\,F^{(\ell)}(\lambda)\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\,, \\
&\quad \ell\in\{0,1\}\,.
\end{aligned}
\tag{3.61}
$$

**3.3.1. $L^p$-boundedness of $\Omega^{(0)}$.** The $L^p$-boundedness of the map $u\mapsto\Omega^{(0)}u$ can be established via a straightforward adaptation of the arguments of Section 3.2, of course understanding that this is done for each component $\Omega^{(0)}_{jk}$, and this is possible precisely thanks to the lack of relevant oscillations in $\Omega^{(0)}$.

This means that first we write, in analogy to (3.32),

$$
(\Omega^{(0)}u)(x) \;=\; \frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}(\widehat{F^{(0)}}*rM_u)(|x|)\,,
\tag{3.62}
$$

and it is easy to check that $F^{(0)}$ satisfies the properties of the function $W$ in Lemma 3.2.2, from which, reasoning as in (3.37),

$$
\|\Omega^{(0)}u\|_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\,\|u\|_{L^p(\mathbb{R}^3)}\,,\qquad p\in(\tfrac{3}{2},3)\,,
\tag{3.63}
$$

for some constant $C_p>0$.

Then, in analogy to (3.38), (3.39), (3.40), (3.42), and (3.49), we split

$$
\Omega^{(0)}u \;=\; \Omega^{(0)}_1u+\Omega^{(0)}_2u
\tag{3.64}
$$

with

$$
\begin{aligned}
(\Omega^{(0)}_1u)(x) \;&:=\; \frac{-\mathrm{i}}{(2\pi)^{\frac{3}{2}}|x|^2}\int_0^{+\infty} e^{-\mathrm{i}\lambda|x|}\,F^{(0)}(\lambda)\widehat{(r^2M_u)}(-\lambda)\,\mathrm{d}\lambda\,, \\
&=\; \frac{-\mathrm{i}}{(2\pi)^{\frac{3}{2}}|x|^2}(\widehat{F^{(0)}}*r^2M_u)(|x|)
\end{aligned}
\tag{3.65}
$$

and

$$
\begin{aligned}
(\Omega^{(0)}_2u)(x) \;&:=\; \frac{-1}{(2\pi)^{\frac{3}{2}}|x|^2}\int_0^{+\infty} e^{-\mathrm{i}\lambda|x|}\,F^{(0)\prime}(\lambda)\widehat{(rM_u)}(-\lambda)\;\mathrm{d}\lambda \\
&=\; \frac{-1}{(2\pi)^{\frac{3}{2}}\rho^2}(\mathcal{L}^{(0)}*rM_u)(\rho) \\
&=\; \frac{-1}{(2\pi)^2|x|^2}\int_0^{+\infty} e^{-\mathrm{i}\lambda|x|}F^{(0)\prime}(\lambda)\Big(\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\lambda|y|}-e^{-\mathrm{i}\lambda|y|}}{4\pi|y|}\,u(y)\,\mathrm{d}y\Big)\mathrm{d}\lambda\,,
\end{aligned}
\tag{3.66}
$$

where $\mathcal{L}^{(0)}$ is the Fourier transform of the function $\mathbf{1}_{(0,\infty)}F^{(0)\prime}$.

Since, as observed already, $F^{(0)}$ behaves like $W$ in Lemma 3.2.2, we have, reasoning as in (3.41),

$$
\|\Omega^{(0)}_1u\|_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\,\|u\|_{L^p(\mathbb{R}^3)}\,,\qquad p\in(1,\tfrac{3}{2})\,,
\tag{3.67}
$$

and since $\mathbf{1}_{(0,\infty)}F^{(0)\prime}$ too satisfies the properties of the function $W$ in Lemma 3.2.2, we have, using the *second* line in the r.h.s. of (3.66) and reasoning as in (3.44)-(3.45),

$$
\|\Omega^{(0)}_2\mathbf{1}_{\{|x|\geqslant1\}}u\|_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\,\|\mathbf{1}_{\{|x|\geqslant1\}}u\|_{L^p(\mathbb{R}^3)}\,,\qquad p\in(1,\tfrac{3}{2})\,,
\tag{3.68}
$$

for some constant $C_p>0$. Last, since $\mathbf{1}_{(0,\infty)}F^{(0)\prime}$ satisfies the properties of the function $Y$ in Lemma 3.2.3, we have, using the *third* line in the r.h.s. of (3.66) and reasoning as in (3.49)-(3.50) and (3.47),

$$
\|\Omega^{(0)}_2\mathbf{1}_{\{|x|\leqslant1\}}u\|_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\,\|\mathbf{1}_{\{|x|\leqslant1\}}u\|_{L^p(\mathbb{R}^3)}\,,\qquad p\in(1,\tfrac{3}{2})\,.
\tag{3.69}
$$

Combining together the bounds (3.63), (3.67), (3.68), and (3.69), plus interpolation so as to cover also the case $p=\frac{3}{2}$, yields finally

$$\|\Omega^{(0)}u\|_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|u\|_{L^p(\mathbb{R}^3)}\,,\qquad p\in(1,3)\,,\tag{3.70}$$

for some constant $C_p>0$.

**3.3.2. $L^p$-boundedness of $\Omega^{(1)}$.** The proof of the $L^p$-boundedness of the map $u\mapsto\Omega^{(1)}u$ is somewhat more involved, however the basic idea of the proof is similar to that for $\Omega^{(0)}$. First we re-write (3.61) in analogy to (3.31) and (3.62) as

$$(\Omega^{(1)}u)(x)\;=\;\frac{1}{\mathrm{i}(2\pi)^{\frac{3}{2}}|x|}(\widehat{F^{(1)}}*rM_u)(|x|)\,.\tag{3.71}$$

Owing to (3.59), the matrix elements of $F^{(1)}(\lambda)$ entering (3.61) and (3.71) above are of the form

$$\frac{\omega_>(\lambda)}{\lambda}\,\frac{e^{-\mathrm{i}\lambda|y_j-y_k|}}{|y_j-y_k|}\,,\qquad\frac{\omega_>(\lambda)}{\lambda^2}\,\frac{e^{-\mathrm{i}\lambda|y_j-y_k|}}{|y_j-y_k|}\,,\qquad\frac{\omega_>(\lambda)}{\lambda^2}\,\frac{e^{-\mathrm{i}\lambda|y_j-y_k|}}{|y_j-y_k|}\,\frac{e^{-\mathrm{i}\lambda|y_r-y_s|}}{|y_r-y_s|}$$

(observe that the $\lambda$-dependence of the matrix elements of $\widetilde{G}$ in (3.59) is $\widetilde{G}(-\lambda)$). This means that denoting by $a>0$ any of the numbers $|y_j-y_k|$ or $|y_j-y_k|+|y_r-y_s|$ and by $X(\lambda)$ the function $\lambda^{-1}\omega_>(\lambda)$ or $\lambda^{-2}\omega_>(\lambda)$, formulas (3.61) and (3.71) imply that $\Omega^{(1)}u$ is a linear combination of terms of the form

$$\begin{aligned}(\Xi\,u)(x)\;&:=\;\frac{1}{\mathrm{i}\,|x|}\int_0^{+\infty}e^{-\mathrm{i}\lambda(|x|+a)}X(\lambda)\,\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\\&=\;\frac{1}{\mathrm{i}\,|x|}(\widehat{X}*rM_u)(|x|+a)\,,\end{aligned}\tag{3.72}$$

and we need to prove the $L^p$-boundedness of the map $u\mapsto\Xi\,u$. In fact, we shall establish it for each of the two terms of the bound

$$\|\Xi\,u\|_{L^p(\mathbb{R}^3)}\;\leqslant\;\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi\,u\|_{L^p(\mathbb{R}^3)}+\|\mathbf{1}_{\{|x|\leqslant R\}}\,\Xi\,u\|_{L^p(\mathbb{R}^3)}\tag{3.73}$$

for a suitable $R>0$.

Let us cast the discussion of such two terms into the following two Lemmas. The combination of (3.73) above with (3.74) and (3.75) below will then complete the proof of the $L^p$-boundedness of $\Omega^{(1)}$.

LEMMA 3.3.1. *For any $p\in(1,3)$ and $R>a$ there exists a constant $C_p>0$ such that*

$$\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi\,u\|_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|u\|_{L^p(\mathbb{R}^3)}\tag{3.74}$$

*for all $u\in L^p(\mathbb{R}^3)$, where $\Xi\,u$ is defined in* (3.72).

PROOF. We consider first the case $p\in(\frac{3}{2},3)$. From (3.72) and from the fact that $\rho\geqslant R+a$ implies $\frac{1}{2}\rho\leqslant\rho-a\leqslant\rho$,

$$\begin{aligned}\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi\,u\|^p_{L^p(\mathbb{R}^3)}\;&=\;4\pi\int_R^{+\infty}\rho^{2-p}\big|(\widehat{X}*rM_u)(\rho+a)\big|^p\,\mathrm{d}\rho\\&=\;4\pi\int_{R+a}^{+\infty}(\rho-a)^{2-p}\big|(\widehat{X}*rM_u)(\rho)\big|^p\,\mathrm{d}\rho\\&\leqslant\;C_p\int_0^{+\infty}\big|(\widehat{X}*rM_u)(\rho)\big|^p\,\rho^{2-p}\mathrm{d}\rho\,.\end{aligned}$$

Now, $\rho^{2-p}$ is an $A_p$-weight on $\mathbb{R}$ because $p\in(\frac{3}{2},3)$ (Theorem 3.2.1(iv)) and the convolution with $\widehat{X}$ is a Calderón-Zygmund operator on $\mathbb{R}$ because the function $X$

obviously satisfies the properties of the function $W$ in Lemma 3.2.2. Then it follows from Theorem 3.2.1(ii) that

$$\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi\,u\|^p_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\int_0^{+\infty}|(\rho M_u)(\rho)|^p\,\rho^{2-p}\,\mathrm{d}\rho \;\;\leqslant\; C'_p\|u\|^p_{L^p(\mathbb{R}^3)}$$

for suitable $C'_p>0$. The Lemma is then proved in the case $p\in(\frac{3}{2},3)$.

Next we consider the case $p\in(1,\frac{3}{2})$. Integration by parts in (3.72), using $e^{-\mathrm{i}\lambda(\rho+a)}=\mathrm{i}(\rho+a)^{-1}\partial_\lambda e^{-\mathrm{i}\lambda(\rho+a)}$, yields

$$\Xi\,u \;=\; \Xi_1 u+\Xi_2 u$$

with

$$\begin{aligned}(\Xi_1 u)(x) \;&:=\; \frac{-\mathrm{i}}{|x|(|x|+a)}\int_0^{+\infty}e^{-\mathrm{i}\lambda(|x|+a)}X(\lambda)\,\widehat{(r^2M_u)}(-\lambda)\,\mathrm{d}\lambda\\ &=\; \frac{-\mathrm{i}}{|x|(|x|+a)}(\widehat{X}*r^2M_u)(|x|+a)\end{aligned}$$

and

$$\begin{aligned}(\Xi_2 u)(x) \;&:=\; \frac{-1}{|x|(|x|+a)}\int_0^{+\infty}e^{-\mathrm{i}\lambda(|x|+a)}X'(\lambda)\,\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\\ &=\; \frac{-1}{|x|(|x|+a)}(\widehat{X'}*rM_u)(|x|+a)\,.\end{aligned}$$

Up to a change of variable, the quantity $\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_1 u\|_{L^p(\mathbb{R}^3)}$ is estimated precisely as the quantity $\|\Omega_1 u\|_{L^p(\mathbb{R}^3)}$ in Section 3.2.2 – see (3.41) above. Indeed,

$$\begin{aligned}\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_1 u\|^p_{L^p(\mathbb{R}^3)} \;&=\; \int_R^{+\infty}\frac{4\pi\rho^2}{\rho^p(\rho+a)^p}\,|(\widehat{X}*r^2M_u)(\rho+a)|^p\,\mathrm{d}\rho\\ &=\; \int_{R+a}^{+\infty}4\pi(\rho-a)^{2-p}\rho^{-p}\,|(\widehat{X}*r^2M_u)(\rho)|^p\,\mathrm{d}\rho\\ &\leqslant\; C\int_0^{+\infty}|(\widehat{X}*r^2M_u)(\rho)|^p\,\rho^{2-2p}\,\mathrm{d}\rho\\ &\leqslant\; C\int_0^{+\infty}|M_u(\rho)|^p\,\rho^2\,\mathrm{d}\rho \;\leqslant\; C_p\|u\|^p_{L^p(\mathbb{R}^3)}\end{aligned}$$

for some constants $C,C_p>0$, having used $\frac{1}{2}\rho\leqslant\rho-a\leqslant\rho$ in the third step and Theorem 3.2.1(ii) in the fourth step. This was possible because $\rho^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p\in(1,\frac{3}{2})$ (Theorem 3.2.1(iv)) and because $f\mapsto\widehat{X}*f$ is a Calderón-Zygmund operator on $\mathbb{R}$ (the function $X$ does satisfy the assumptions on the function $W$ in Lemma 3.2.2).

It remains to estimate the quantity $\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2 u\|_{L^p(\mathbb{R}^3)}$ in the regime $p\in(1,\frac{3}{2})$ and we proceed by splitting

$$\begin{aligned}&\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2 u\|^p_{L^p(\mathbb{R}^3)} \;=\\ &\qquad=\;\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2\mathbf{1}_{\{|x|\geqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}+\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2\mathbf{1}_{\{|x|\leqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}\,.\end{aligned}$$

For estimating $\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2\mathbf{1}_{\{|x|\geqslant R\}}u\|_{L^p(\mathbb{R}^3)}$ we observe that

$$\begin{aligned}\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2 u\|^p_{L^p(\mathbb{R}^3)} \;&=\; \int_R^{+\infty}\frac{4\pi\rho^2}{\rho^p(\rho+a)^p}\,|(\widehat{X'}*rM_u)(\rho+a)|^p\,\mathrm{d}\rho\\ &=\; \int_{R+a}^{+\infty}4\pi(\rho-a)^{2-p}\rho^{-p}\,|(\widehat{X'}*rM_u)(\rho)|^p\,\mathrm{d}\rho\\ (*)\qquad &\leqslant\; C\int_0^{+\infty}|(\widehat{X'}*rM_u)(\rho)|^p\,\rho^{2-2p}\,\mathrm{d}\rho\end{aligned}$$

for some constant $C>0$, where we used again $\frac{1}{2}\rho<\rho-a\leqslant\rho$. Then we can proceed exactly as in (3.44)-(3.45), because $\rho^{2-2p}$ is an $A_p$-weight on $\mathbb{R}$ for $p\in(1,\frac{3}{2})$ and $f\mapsto\widehat{X'}*f$ is a Calderón-Zygmund operator on $\mathbb{R}$; the conclusion is the same as in (3.45), that is,

$$\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2\mathbf{1}_{\{|x|\geqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|\mathbf{1}_{\{|x|\geqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}$$

for some constant $C_p>0$. We also observe from (*) that

$$\begin{aligned}&\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2 u\|^p_{L^p(\mathbb{R}^3)}\\&\qquad\leqslant\;\int_{\mathbb{R}^3}\mathrm{d}x\,\Big|\,\frac{1}{|x|^2}\int_0^{+\infty}e^{-\mathrm{i}\lambda|x|}\,X'(\lambda)\,\widehat{(rM_u)}(-\lambda)\,\mathrm{d}\lambda\,\Big|^p\;=\;\|\widetilde{\Xi}_2u\|^p_{L^p(\mathbb{R}^3)}\,,\end{aligned}$$

where $\widetilde{\Xi}_2u$ has precisely the same structure as $\Omega_2u$ in (3.39) with the function $X'$ here in place of the function $F'$ therein. Therefore, as argued in (3.49)-(3.50), since $X'$ satisfies the assumptions on the function $Y$ in Lemma 3.2.3, the conclusion is the same as in (3.47), that is,

$$\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2\mathbf{1}_{\{|x|\leqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|\mathbf{1}_{\{|x|\leqslant R\}}u\|^p_{L^p(\mathbb{R}^3)}$$

for some constant $C_p>0$. Therefore,

$$\|\mathbf{1}_{\{|x|\geqslant R\}}\,\Xi_2u\|^p_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|u\|^p_{L^p(\mathbb{R}^3)}$$

and Lemma is then proved in the case $p\in(1,\frac{3}{2})$.

Last, by interpolation the Lemma is also proved in the case $p=\frac{3}{2}$. $\square$

LEMMA 3.3.2. *For any $p\in(1,3)$ and $R>100a$ there exists a constant $C_p>0$ such that*

$$\|\mathbf{1}_{\{|x|\leqslant R\}}\,\Xi\,u\|_{L^p(\mathbb{R}^3)}\;\leqslant\;C_p\,\|u\|_{L^p(\mathbb{R}^3)}\tag{3.75}$$

*for all $u\in L^p(\mathbb{R}^3)$, where $\Xi\,u$ is defined in* (3.72).

PROOF. By means of (3.17) we see that the map $u\mapsto\Xi\,u$ defined in (3.72) is an integral operator with kernel $\frac{\mathrm{i}}{4\pi}\,K_\Xi(x,y)$ given by

$$K_\Xi(x,y)\;:=\;\frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{e^{-\mathrm{i}\lambda(|x|+a)}(e^{-\mathrm{i}\lambda|y|}-e^{\mathrm{i}\lambda|y|})}{|x|\,|y|}\,X(\lambda)\,\mathrm{d}\lambda\,.\tag{3.76}$$

Since $X(\lambda)=\lambda^{-1}\omega_>(\lambda)$ or $\lambda^{-2}\omega_>(\lambda)$, obviously $\rho\mapsto\widehat{X}(\rho)$ is smooth for $\rho\neq0$ and with rapid decrease as $\rho\to+\infty$. Moreover, since $X\in L^q(\mathbb{R})$ for any $q>1$, $\widehat{X}\in L^p(\mathbb{R})$ for any $p\in[2,\infty)$, owing to the Hausdorff-Young inequality. Thus, $\widehat{X}\in L^p(\mathbb{R})$ for any $p\in[1,\infty)$.

We shall prove the Lemma by splitting

$$\begin{aligned}&\|\mathbf{1}_{\{|x|\leqslant R\}}\,\Xi\,u\|^p_{L^p(\mathbb{R}^3)}\\&\quad=\;\|\mathbf{1}_{\{|x|\leqslant R\}}\,\Xi\,\mathbf{1}_{\{|x|\geqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)}+\|\mathbf{1}_{\{|x|\leqslant R\}}\,\Xi\,\mathbf{1}_{\{|x|\leqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)}\end{aligned}\tag{3.77}$$

and estimating separately the two summands in the r.h.s. above.

When $R>100a$, $|x|\leqslant R$, and $|y|\geqslant10R$, one has $|\widehat{X}(|x|\pm|y|+a)|\leqslant C_n\langle y\rangle^{-n}$ for any $n\in\mathbb{N}$ and suitable constants $C_n>0$, which follows from the rapid decrease of $\widehat{X}$. Then the identity

$$K_\Xi(x,y)\;=\;\frac{\widehat{X}(|x|+a+|y|)-\widehat{X}(|x|+a-|y|)}{|x||y|}\tag{3.78}$$

shows that in this regime $|K_\Xi(x,y)| \leqslant 2\,C_n|x|^{-1}|y|^{-1}\langle y\rangle^{-n}$. Therefore, for any $p\in(1,3)$ and corresponding $n$ large enough,

$$\|K_\Xi(x,\cdot)\mathbf{1}_{\{|\cdot|\geqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \;\leqslant\; \frac{2C_n}{|x|}\Big(\int_{|y|\geqslant 10R}\frac{1}{|y|^{p'}\langle y\rangle^{np'}}\Big)^{1/p'} \;\leqslant\; \frac{C_p}{|x|}$$

for some constant $C_p>0$. The latter bound and Hölder's inequality then yield, for any $p\in(1,3)$,

$$\begin{aligned} &\|\mathbf{1}_{\{|\cdot|\leqslant R\}}\,\Xi\,\mathbf{1}_{\{|\cdot|\geqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)} \\ &\quad\leqslant\; \int_{\mathbb{R}^3}\mathrm{d}x\,\mathbf{1}_{\{|x|\leqslant R\}}(x)\,\Big|\int_{\mathbb{R}^3}\mathrm{d}y\,K_\Xi(x,y)\mathbf{1}_{\{|y|\geqslant 10R\}}(y)\,u(y)\Big|^p \\ &\quad\leqslant\; C_p\,\Big\|\frac{\mathbf{1}_{\{|x|\leqslant R\}}}{|x|}\Big\|^p_{L^p(\mathbb{R}^3)}\,\|\mathbf{1}_{\{|\cdot|\geqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)} \\ &\quad=\; C_p'\,\|\mathbf{1}_{\{|\cdot|\geqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)} \end{aligned} \tag{3.79}$$

for some constant $C_p'>0$.

This provides the first partial estimate for the proof of (3.75): the proof is completed when we show in addition that

$$\|\mathbf{1}_{\{|\cdot|\leqslant R\}}\,\Xi\,\mathbf{1}_{\{|\cdot|\leqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)} \;\leqslant\; C_p\,\|\mathbf{1}_{\{|\cdot|\leqslant 10R\}}u\|^p_{L^p(\mathbb{R}^3)} \tag{3.80}$$

for any $p\in(1,3)$ and suitable constant $C_p>0$. We shall establish (3.80) above in three separate regimes: $p\in(2,3)$, $p\in(\frac{3}{2},2)$, and $p\in(1,\frac{3}{2})$. By interpolation, also the cases $p=\frac{3}{2}$ and $p=2$ will then be covered.

From (3.78) we estimate

$$\begin{aligned} &\|K_\Xi(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \\ &\qquad\leqslant\; \frac{(4\pi)^{\frac{1}{p'}}}{|x|}\sum_{\pm}\Big(\int_0^{10R}\mathrm{d}\rho\,\rho^{2-p'}\,|\widehat{X}(|x|+a\pm\rho)|^{p'}\Big)^{1/p'}. \end{aligned} \tag{3.81}$$

When $p\in(2,3)$, and hence $p'\in(\frac{3}{2},2)$, we have $\rho^{2-p'}\leqslant(10R)^{2-p'}$ for every $\rho\in[0,10R]$, and (3.81) then yields

$$\|K_\Xi(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \;\leqslant\; C\,\frac{\|\widehat{X}\|_{L^{p'}(\mathbb{R})}}{|x|}\,. \tag{3.82}$$

When instead $p\in(\frac{3}{2},2)$, and hence $p'\in(2,3)$, the r.h.s. of (3.81) is estimated with Hölder's inequality, with weights $q=\frac{p'-1}{2(p'-2)}$ and $q'=\frac{p'-1}{3-p'}$, as

$$\begin{aligned} &\|K_\Xi(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \\ &\qquad\leqslant\; \frac{C}{|x|}\Big(\int_0^{10R}\frac{\mathrm{d}\rho}{\rho^{\frac{p'-1}{2}}}\Big)^{\frac{2(2-p)}{p'}}\,\|\widehat{X}\|_{L^{\frac{p'(p'-1)}{3-p'}}(\mathbb{R})}\,. \end{aligned} \tag{3.83}$$

In order to obtain analogous estimates to (3.82)-(3.83) in the remaining regime $p\in(1,\frac{3}{2})$, it is convenient to integrate by parts in (3.76), using $e^{-\mathrm{i}\lambda(|x|+a)} = \mathrm{i}(|x|+a)^{-1}\partial_\lambda e^{-\mathrm{i}\lambda(|x|+a)}$, so as to split

$$K_\Xi(x,y) \;=\; K_\Xi^{(1)}(x,y)+K_\Xi^{(2)}(x,y) \tag{3.84}$$

with

$$\begin{aligned} &K_\Xi^{(1)}(x,y) \\ &\quad:=\; \frac{-1}{\sqrt{2\pi}\,|x|(|x|+a)}\int_0^{+\infty}(e^{-\mathrm{i}\lambda(|x|+a+|y|)}+e^{\mathrm{i}\lambda(|x|+a-|y|)})\,X(\lambda)\,\mathrm{d}\lambda \\ &\quad=\; \frac{-1}{|x|(|x|+a)}\big(\widehat{X}(|x|+a+|y|)+\widehat{X}(|x|+a-|y|)\big) \end{aligned} \tag{3.85}$$

and

$$\begin{aligned} &K_{\Xi}^{(2)}(x,y) \\ &\quad := \frac{-\mathrm{i}}{\sqrt{2\pi}\,|x|(|x|+a)|y|}\int_0^{+\infty} e^{-\mathrm{i}\lambda(|x|+a)}(e^{-\mathrm{i}\lambda|y|}-e^{\mathrm{i}\lambda|y|})\,X'(\lambda)\,\mathrm{d}\lambda\,. \end{aligned}\tag{3.86}$$

Using (3.85) we get

$$\begin{aligned} &\|K_{\Xi}^{(1)}(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)} \\ &\qquad \leqslant \frac{(4\pi)^{\frac{1}{p'}}}{|x|(|x|+a)}\sum_{\pm}\Big(\int_0^{10R}\mathrm{d}\rho\,\rho^2\,|\widehat{X}(|x|+a\pm\rho)|^{p'}\Big)^{1/p'} \\ &\qquad \leqslant \frac{2\,(4\pi)^{\frac{1}{p'}}\,(10R)^2}{|x|(|x|+a)}\,\|\widehat{X}\|_{L^{p'}(\mathbb{R})}\,. \end{aligned}\tag{3.87}$$

As for $K_{\Xi}^{(2)}$, we exploit (3.86) using the bound $|X'(\lambda)|\leqslant C\langle\lambda\rangle^{-2}$ for some $C>0$, which follows from the fact that $X(\lambda)=\lambda^{-1}\omega_{>}(\lambda)$ or $\lambda^{-2}\omega_{>}(\lambda)$, and the bound $|e^{-\mathrm{i}\lambda|y|}-e^{\mathrm{i}\lambda|y|}|\leqslant 2(\lambda|y|)^{1-\theta}$ $\forall\theta\in(0,1)$. Thus,

$$|K_{\Xi}^{(2)}(x,y)|\;\leqslant\;\frac{1}{|x|(|x|+a)|y|}\int_0^{+\infty}2(\lambda|y|)^{1-\theta}\,|X'(\lambda)|\,\mathrm{d}\lambda\;\leqslant\;\frac{C}{|x|(|x|+a)|y|^{\theta}}\,,$$

whence

$$\|K_{\Xi}^{(2)}(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)}\;\leqslant\;\frac{C'}{|x|(|x|+a)}\,\Big\|\frac{\mathbf{1}_{\{|y|\leqslant 10R\}}}{|y|^{\theta}}\Big\|_{L^{p'}(\mathbb{R}^3)}\,,\tag{3.88}$$

for suitable constants $C,C'>0$, where the $L^{p'}$-norm in the r.h.s. is finite whenever $\theta p'<3$.

The estimates (3.82), (3.83), (3.84), (3.87), and (3.88) together then imply that, for some constant $C_p>0$,

$$\|K_{\Xi}(x,\cdot)\mathbf{1}_{\{|\cdot|\leqslant 10R\}}\|_{L^{p'}(\mathbb{R}^3)}\;\leqslant\;\frac{C_p}{|x|}\,,\qquad p\in(1,\tfrac{3}{2})\cup(\tfrac{3}{2},2)\cup(2,3)\,.\tag{3.89}$$

Then (3.89) and Hölder's inequality yield

$$\begin{aligned} &\|\mathbf{1}_{\{|\cdot|\leqslant R\}}\,\Xi\,\mathbf{1}_{\{|\cdot|\leqslant 10R\}}u\|_{L^p(\mathbb{R}^3)}^p \\ &\qquad \leqslant \int_{\mathbb{R}^3}\mathrm{d}x\,\mathbf{1}_{\{|x|\leqslant R\}}(x)\,\Big|\int_{\mathbb{R}^3}\mathrm{d}y\,K_{\Xi}(x,y)\mathbf{1}_{\{|y|\leqslant 10R\}}(y)\,u(y)\Big|^p \\ &\qquad \leqslant C_p\,\Big\|\frac{\mathbf{1}_{\{|x|\leqslant R\}}}{|x|}\Big\|_{L^p(\mathbb{R}^3)}^p\,\|\mathbf{1}_{\{|\cdot|\leqslant 10R\}}u\|_{L^p(\mathbb{R}^3)}^p \\ &\qquad = C_p'\,\|\mathbf{1}_{\{|\cdot|\leqslant 10R\}}u\|_{L^p(\mathbb{R}^3)}^p \end{aligned}\tag{3.90}$$

for some constant $C_p'>0$.

We have thus obtained precisely the desired estimate (3.80). This completes the proof because, as commented already, (3.79) and (3.80) together give (3.75). □

## 3.4. Unboundedness in $L^1(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$, $p\geqslant 3$

In this Section we complete the proof of Theorem 3.0.3 as far as the unboundedness part is concerned, hence showing that the wave operators $W_{\alpha,Y}^{\pm}$ are unbounded in $L^p(\mathbb{R}^3)$ whenever $p\in\{1\}\cup[3,+\infty]$. As commented already, it is enough to prove this property for $W_{\alpha,Y}^{+}$.

**3.4.1. Unboundedness of $W^+_{\alpha,Y}$ in $L^p(\mathbb{R}^3)$ for $p \in [3,+\infty]$.** Because of the $L^p$-boundedness of $W^+_{\alpha,Y}$ for $p \in (1,3)$, it is clear that we only need to prove that $W^+_{\alpha,Y}$ is unbounded in $L^3(\mathbb{R}^3)$, for any $L^p$-boundedness for $p > 3$ would then contradict, by interpolation, the unboundedness when $p = 3$.

Let us assume for contradiction that $W^+_{\alpha,Y}$ is bounded in $L^3(\mathbb{R}^3)$, which by duality implies also that $(W^+_{\alpha,Y})^*$ is bounded in $L^{3/2}(\mathbb{R}^3)$.

Choose $c > 0$ sufficiently large so as to make the matrix $\Gamma_{\alpha,Y}(\mathrm{i}c)$ non-singular. Correspondingly, $R_0(-c^2)$ maps continuously $L^{3/2}(\mathbb{R}^3)$ into $W^{2,3/2}(\mathbb{R}^3)$ and hence also $L^{3/2}(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ for any $q \in [\frac{3}{2},\infty)$, owing to a Sobolev embedding. Thus, the $L^{3/2}$-boundedness of $(W^+_{\alpha,Y})^*$, the $L^{3/2} \to L^3$-boundedness of $R_0(-c^2)$, and the $L^3$-boundedness of $W^+_{\alpha,Y}$ imply, by means of the intertwining property (3.2), that also the operator

$$R_{\alpha,Y}(-c^2)P_{\mathrm{ac}}(H_{\alpha,Y}) \;=\; W^+_{\alpha,Y}R_0(-c^2)(W^+_{\alpha,Y})^*$$

is continuous from $L^{3/2}(\mathbb{R}^3)$ to $L^3(\mathbb{R}^3)$. As a consequence, we read out from the the resolvent identity (1.32) that for any $u \in L^2_{\mathrm{ac}}(H_{\alpha,Y}) \cap L^{3/2}(\mathbb{R}^3)$ the function

$$\begin{aligned}(*)\qquad & R_{\alpha,Y}(-c^2)u - R_0(-c^2)u \\ & \qquad = \sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(\mathrm{i}c)^{-1})_{jk}\, \mathcal{G}^{y_j}_{\mathrm{i}c}(x) \int_{\mathbb{R}^3} \mathcal{G}^{y_k}_{\mathrm{i}c}(y)\, u(y)\,\mathrm{d}y\end{aligned}$$

must belong to $L^3(\mathbb{R}^3)$.

Let us make now a choice of $u$ for which the r.h.s. of (*) above *fails* instead to belong to $L^3(\mathbb{R}^3)$. Since $u \in L^2_{\mathrm{ac}}(H_{\alpha,Y})$, then $u$ is orthogonal to all the eigenfunctions of $H_{\alpha,Y}$, that is, owing to Theorem 1.1.4(i), $u$ is orthogonal to an (at most) $N$-dimensional subspace spanned by suitable linear combinations of $\mathcal{G}^{y_1}_{\mathrm{i}\lambda_k}, \dots, \mathcal{G}^{y_N}_{\mathrm{i}\lambda_k}$ for $k \in \{1,\dots,N\}$, where $-\lambda_1^2, \dots, -\lambda_N^2$ are the eigenvalues of $H_{\alpha,Y}$. Because of our choice of $c$, in such an orthogonal complement there is surely $u$ which is *not* orthogonal to the $\overline{\mathcal{G}^{y_k}_{\mathrm{i}c}}$'s, namely,

$$\int_{\mathbb{R}^3} \mathcal{G}^{y_k}_{\mathrm{i}c}(y)\, u(y)\,\mathrm{d}y \;\neq\; 0 \qquad \forall k \in \{1\dots,N\}\,.$$

(In fact, such a $u$ can be also found in $C^\infty_0(\mathbb{R}^3) \cap L^2_{\mathrm{ac}}(H_{\alpha,Y})$: indeed, the point spectral subspace of $H_{\alpha,Y}$ is at most $N$-dimensional, whereas the set of $u$'s that satisfy the non-vanishing condition above is open in the topology of the space of test functions.) For such $u$, because of the invertibility of the matrix $\Gamma_{\alpha,Y}(\mathrm{i}c)$, the expression

$$\sum_{j,k=1}^{N} (\Gamma_{\alpha,Y}(\mathrm{i}c)^{-1})_{jk}\, \mathcal{G}^{y_j}_{\mathrm{i}c}(x) \int_{\mathbb{R}^3} \mathcal{G}^{y_k}_{\mathrm{i}c}(y)\, u(y)\,\mathrm{d}y$$

is a linear combination of the $\mathcal{G}^{y_j}_{\mathrm{i}c}$'s with at least one non-zero coefficient, say, the one for $j = j_0$. Therefore, in a sufficiently small neighbourhood of $y_{j_0}$ (so small as not to contain any other of the $y_j$'s of $Y$, for $j \neq j_0$) the latter function must be of the form $c_{j_0}|x-y_{j_0}|^{-1} + R(x)$ for some constant $c_{j_0} \neq 0$ and some bounded (in fact, smooth) function $R(x)$. This would mean that in the considered neighbourhood of $y_{j_0}$ $R_{\alpha,Y}(-c^2)u - R_0(-c^2)u$ is *not* a $L^3$-function, a contradiction.

**3.4.2. Unboundedness of $W^+_{\alpha,Y}$ in $L^1(\mathbb{R}^3)$.** For this case the following preliminary observation is going to be useful.

REMARK 3.4.1. Let $g \in C^\infty_0(\mathbb{R})$. Then

$$\frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\mathrm{i}\lambda\rho}\, \widehat{g}(-\lambda)\,\mathrm{d}\lambda \;=\; g(\rho) - \mathrm{i}(\mathcal{H}g)(\rho)\,, \tag{3.91}$$

where $g \mapsto \mathcal{H}g$ denotes the Hilbert transform, defined as

$$(\mathcal{H}g)(\rho) \;:=\; \frac{1}{\pi}\,\mathrm{P.V.}\!\int_{-\infty}^{+\infty} \frac{g(\tau)}{\rho-\tau}\,\mathrm{d}\tau. \tag{3.92}$$

Indeed, following from the fact [**54**, Eq. (5.1.13)] that the Hilbert transform is the Fourier multiplier

$$\widehat{(\mathcal{H}g)}(\lambda) \;=\; -\mathrm{i}\,\mathrm{sgn}(\lambda)\,\widehat{g}(\lambda)\,,$$

one has

$$\begin{aligned} g(\rho) - \mathrm{i}(\mathcal{H}g)(\rho) \;&=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\mathrm{i}\rho\lambda}(1-\mathrm{sgn}(\lambda))\,\widehat{g}(\lambda)\,\mathrm{d}\lambda \\ &=\; \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{\mathrm{i}\rho\lambda}\,\widehat{g}(\lambda)\,\mathrm{d}\lambda \;=\; \frac{2}{\sqrt{2\pi}}\int_{0}^{+\infty} e^{-\mathrm{i}\lambda\rho}\,\widehat{g}(-\lambda)\,\mathrm{d}\lambda\,. \end{aligned}$$

Let us now prove the fact that the wave operator $W_{\alpha,Y}^{+}$ is unbounded in $L^1(\mathbb{R}^3)$. We may assume without loss of generality to take the set $Y=\{y_1,\dots,y_N\}$ of interaction centres so that $y_1=0$.

Let $u\in C_0^\infty(\mathbb{R}^3)$ be rotationally invariant, and we write $u(x)=f(|x|)$ for some $f:[0,+\infty)\to\mathbb{C}$ which is smooth and compactly supported. We extend $f$ to an even function on the whole $\mathbb{R}$. By construction, $f(r)=M_u(r)$, the spherical mean of $u$.

Our starting point is the stationary representation (3.19) for $W_{\alpha,Y}^{+}u$, that is,

$$W_{\alpha,Y}^{+}u \;=\; u+\sum_{j,k=1}^{N} T_{y_j}\Omega_{jk}T_{-y_k}u\,, \tag{3.93}$$

and for each $j,k\in\{1,\dots,N\}$ we set $K_{jk}u := T_{y_j}\Omega_{jk}T_{-y_k}u$. Explicitly,

$$\begin{aligned} &(K_{jk}u)(x) \\ &\quad=\; \frac{1}{\mathrm{i}\pi}\int_{\mathbb{R}^3}\mathrm{d}y\,u(y)\int_0^{+\infty}\mathrm{d}\lambda\,F_{jk}(\lambda)\,\frac{e^{-\mathrm{i}\lambda|x-y_j|}}{4\pi|x-y_j|}\,\frac{e^{\mathrm{i}\lambda|y-y_k|}-e^{-\mathrm{i}\lambda|y-y_k|}}{4\pi|y-y_k|}\,, \end{aligned} \tag{3.94}$$

where we used (3.9) and (3.12).

We now proceed by re-scaling $u$ and $f$ as

$$u_\varepsilon(x) \;:=\; \varepsilon^{-3}u(\varepsilon^{-1}x)\,,\qquad f_\varepsilon(r) \;:=\; \varepsilon^{-3}f(\varepsilon^{-1}r)\,,\qquad \varepsilon>0\,, \tag{3.95}$$

which makes the norms

$$4\pi\|r^2 f_\varepsilon\|_{L^1(0,+\infty)} \;=\; \|u_\varepsilon\|_{L^1(\mathbb{R}^3)} \;=\; \|u\|_{L^1(\mathbb{R}^3)} \;=\; 4\pi\|r^2 f\|_{L^1(0,+\infty)} \tag{3.96}$$

$\varepsilon$-independent. This re-scaling is devised so as to make all interaction centres but $y_1$ ineffective, because $u_\varepsilon$ is only bumped around the origin, and then to reduce the question to the unboundedness of the wave operator relative to a single-centre point interaction Hamiltonian, for which the answer will then come by direct inspection.

From (3.94) and (3.96),

$$\begin{aligned} &(K_{jk}u_\varepsilon)(x) \\ &\quad=\; \frac{1}{\mathrm{i}\pi\varepsilon^2}\int_{\mathbb{R}^3}\mathrm{d}y\,u(y)\int_0^{+\infty}\mathrm{d}\lambda\,F_{jk}(\tfrac{\lambda}{\varepsilon})\,\frac{e^{-\mathrm{i}\frac{\lambda}{\varepsilon}|x-y_j|}}{4\pi|x-y_j|}\,\frac{e^{\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}-e^{-\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\,, \end{aligned} \tag{3.97}$$

having made the changes of variables $y\to\varepsilon y$ and $\lambda\to\varepsilon^{-1}\lambda$ in the integrations. If we now define, for arbitrary $v\in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} &(K_{jk}^{(\varepsilon)}v)(x) \\ &\quad=\; \frac{1}{\mathrm{i}\pi}\int_{\mathbb{R}^3}\mathrm{d}y\,v(y)\int_0^{+\infty}\mathrm{d}\lambda\,F_{jk}(\tfrac{\lambda}{\varepsilon})\,\frac{e^{-\mathrm{i}\lambda|x-\frac{y_j}{\varepsilon}|}}{4\pi|x-\frac{y_j}{\varepsilon}|}\,\frac{e^{\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}-e^{-\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\,, \end{aligned} \tag{3.98}$$

then for the considered $u$ and its re-scaled $u_\varepsilon$ we have

$$\Big\| \sum_{j,k=1}^{N} K_{jk} u_\varepsilon \Big\|_{L^1(\mathbb{R}^3)} \;=\; \Big\| \sum_{j,k=1}^{N} K_{jk}^{(\varepsilon)} u \Big\|_{L^1(\mathbb{R}^3)}\,, \tag{3.99}$$

which follows by making the change of variable $x \mapsto \varepsilon x$ in the integration on the l.h.s.

We now want to study the contribution of each term $K_{jk}^{(\varepsilon)} u$ as $\varepsilon \downarrow 0$. We shall establish the following limits

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} (K_{11}^{(\varepsilon)} u)(x) \;&=\; -\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{e^{-\mathrm{i}\lambda|x|}}{|x|}\,\widehat{(rf)}(-\lambda)\,\mathrm{d}\lambda\,, \\ \lim_{\varepsilon\downarrow 0} (K_{jk}^{(\varepsilon)} u)(x) \;&=\; 0\,, \qquad\quad (j,k)\neq(1,1)\,, \end{aligned} \tag{3.100}$$

pointwise for a.e. $x \in \mathbb{R}^3$.

To this aim, we first find the bound

$$\int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|} - e^{-\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\,u(y)\,\mathrm{d}y \;\leqslant_{|\,\cdot\,|}\; C_u\,\langle\lambda\rangle^{-2} \int_{\operatorname{supp} u} \frac{\mathrm{d}y}{|y-\varepsilon^{-1}y_k|} \tag{3.101}$$

for some constant $C_u > 0$ depending on $u$, but not on $\varepsilon$. (3.101) is obvious for small $\lambda$'s, since $u$ is compactly supported, whereas for large $\lambda$'s we apply the distributional identity

$$(-\Delta_y - \lambda^2)\Big(\frac{e^{\pm\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\Big) \;=\; \delta(y-\tfrac{y_k}{\varepsilon})\,,$$

and integrating by parts we find

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|} - e^{-\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\,u(y)\,\mathrm{d}y \\ &\quad=\; \lambda^{-2} \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|} - e^{-\mathrm{i}\lambda|y-\frac{y_k}{\varepsilon}|}}{4\pi|y-\frac{y_k}{\varepsilon}|}\,(-\Delta)u(y)\mathrm{d}y \\ &\quad\leqslant_{|\,\cdot\,|}\; C_u\,\langle\lambda\rangle^{-2} \int_{\operatorname{supp} u} \frac{\mathrm{d}y}{|y-\varepsilon^{-1}y_k|}\,, \end{aligned}$$

thus, (3.101) is proved.

Next, in order to prove the first of the limits (3.100) by taking $\varepsilon \downarrow 0$ in (3.98), we use the asymptotics (3.13), namely,

$$\lim_{\varepsilon\downarrow 0} F_{11}(\varepsilon^{-1}\lambda) \;=\; -4\pi\mathrm{i}\,,$$

and we also recognise that the asymptotics as $\varepsilon \downarrow 0$ of the $y$-integration of (3.98) is precisely the quantity

$$\int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\lambda|y|} - e^{-\mathrm{i}\lambda|y|}}{4\pi|y|}\,u(y)\,\mathrm{d}y \;=\; \sqrt{2\pi}\,\widehat{(rM_u)}(-\lambda) \;=\; \sqrt{2\pi}\,\widehat{(rf)}(-\lambda)$$

discussed in (3.17). The limit $\varepsilon \downarrow 0$ can be exchanged with the integrations in $\lambda$ and in $y$ by dominated convergence, because $F_{11}(\frac{\lambda}{\varepsilon})$ is uniformly bounded (see Lemma 3.1.1(i)) and (3.101) provides a majorant that is integrable in $\lambda$. Thus,

$$\begin{aligned} \lim_{\varepsilon\downarrow 0} (K_{11}^{(\varepsilon)} u)(x) \;&=\; \frac{1}{\mathrm{i}\pi}(-4\pi\mathrm{i}) \int_0^{+\infty} \frac{e^{-\mathrm{i}\lambda|x|}}{4\pi|x|}\,\sqrt{2\pi}\,\widehat{(rf)}(-\lambda)\,\mathrm{d}\lambda \\ &=\; -\sqrt{\frac{2}{\pi}} \int_0^{+\infty} \frac{e^{-\mathrm{i}\lambda|x|}}{|x|}\,\widehat{(rf)}(-\lambda)\,\mathrm{d}\lambda\,, \quad x\neq 0\,, \end{aligned}$$

and the first limit of (3.100) is proved.

Concerning now (3.100) when $(j,k) \neq (1,1)$, from our estimate (3.101) we deduce

$$
\begin{aligned}
|(K_{jk}^{(\varepsilon)}u)(x)| \;&\leqslant\; \frac{C_u}{|x-\frac{y_j}{\varepsilon}|}\Big(\int_0^{+\infty}|F_{jk}(\tfrac{\lambda}{\varepsilon})|\,\langle\lambda\rangle^{-2}\,\mathrm{d}\lambda\Big)\Big(\int_{\operatorname{supp}u}\frac{\mathrm{d}y}{|y-\frac{y_k}{\varepsilon}|}\Big) \\
&\leqslant\; C_u'\,\|F_{jk}\|_{L^\infty(0,\infty)}\,\frac{1}{|x-\frac{y_j}{\varepsilon}|}\int_{\operatorname{supp}u}\frac{\mathrm{d}y}{|y-\frac{y_k}{\varepsilon}|}
\end{aligned}
\tag{3.102}
$$

for some new constant $C_u' > 0$. Since at least one among $y_j$ and $y_k$ does not coincide with the origin, and since $u$ is compactly supported, we conclude at once that

$$
\lim_{\varepsilon\downarrow 0}(K_{jk}^{(\varepsilon)}u)(x) \;=\; 0\,,\quad x\neq 0 \;\text{ if } j=0.
$$

The proof of (3.100) is thus completed, and in turn (3.100) implies

$$
\lim_{\varepsilon\downarrow 0}\sum_{j,k=1}^{N}(K_{jk}^{(\varepsilon)}u)(x) \;=\; -\frac{2}{\sqrt{2\pi}}\int_0^{+\infty}\frac{e^{-\mathrm{i}\lambda|x|}}{|x|}\,(\widehat{rf})(-\lambda)\,\mathrm{d}\lambda
\tag{3.103}
$$

pointwise for a.e. $x\in\mathbb{R}^3$.

This latter fact allows us to take the limit $\varepsilon\downarrow 0$ in the r.h.s. of (3.99), provided that the $L^1$-norm is taken on compacts of $\mathbb{R}^3$. Indeed, for fixed $R>0$ and any sufficiently small $\varepsilon>0$ such that $|x-\frac{y_j}{\varepsilon}|\geqslant|x|$ for any $x\in\{x||x|\leqslant R\}\cup\operatorname{supp}u$ and $j=1,\dots,N$, the estimate (3.102) implies ($\mathbf{1}_R\equiv$ the characteristic function of the ball $|x|\leqslant R$)

$$
\mathbf{1}_R(x)\sum_{j,k=1}^{N}|(K_{jk}^{(\varepsilon)}u)(x)| \;\leqslant\; N^2\,\frac{C_{u,R}}{|x|}\int_{\operatorname{supp}u}\frac{\mathrm{d}y}{|y|} \;\leqslant\; N^2\,\frac{C_{u,R}'}{|x|}
$$

for suitable constants $C_{u,R},C_{u,R}'>0$, which gives a majorant in $L^1(\mathbb{R}^3)$. Then, by (3.103) and dominated convergence,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\int_{|x|\leqslant R}\Big|\sum_{j,k=1}^{N}(K_{jk}^{(\varepsilon)}u)(x)\Big|\,\mathrm{d}x \\
&\qquad =\; \frac{2}{\sqrt{2\pi}}\int_{|x|\leqslant R}\mathrm{d}x\,\Big|\int_0^{+\infty}\mathrm{d}\lambda\,\frac{e^{-\mathrm{i}\lambda|x|}}{|x|}\,(\widehat{rf})(-\lambda)\,\Big| \\
&\qquad =\; \sqrt{32\pi}\int_0^R\mathrm{d}\rho\,\Big|\int_0^\infty\rho\,e^{-\mathrm{i}\lambda\rho}\,(\widehat{rf})(-\lambda)\,\mathrm{d}\lambda\,\Big|\,.
\end{aligned}
\tag{3.104}
$$

An integration by parts and formula (3.91) in Remark 3.4.1 yield

$$
\begin{aligned}
\sqrt{32\pi}\int_0^\infty\rho\,e^{-\mathrm{i}\lambda\rho}\,(\widehat{rf})(-\lambda)\,\mathrm{d}\lambda \;&=\; \sqrt{32\pi}\int_0^{+\infty}e^{-\mathrm{i}\lambda\rho}\,(\widehat{r^2f})(-\lambda)\,\mathrm{d}\lambda \\
&=\; 4\pi\,\big((r^2f)(\rho)-\mathrm{i}(\mathcal{H}r^2f)(\rho)\big).
\end{aligned}
\tag{3.105}
$$

In the integration by parts the boundary term does not appear because $r\mapsto rf(r)$ is an odd function and $(\widehat{rf})(0)=0$. The conclusion from (3.104) and (3.105) is therefore

$$
\lim_{\varepsilon\downarrow 0}\Big\|\,\mathbf{1}_R\sum_{j,k=1}^{N}K_{j,k}^{(\varepsilon)}u\,\Big\|_{L^1(\mathbb{R}^3)} \;=\; 4\pi\int_0^R\big|(\mathbb{1}-\mathrm{i}\mathcal{H})(r^2f)(\rho)\big|\,\mathrm{d}\rho\,.
\tag{3.106}
$$

The proof of the $L^1$-unboundedness of $W_{\alpha,Y}^+$ is completed as follows. Suppose for contradiction that $W_{\alpha,Y}^+$ is instead $L^1$-bounded. Then, for arbitrary $R>0$,

$$
\begin{aligned}
4\pi\int_0^R \big|(\mathbb{1}-\mathrm{i}\mathcal{H})(r^2 f)(\rho)\big|\,\mathrm{d}\rho \;&=\; \lim_{\varepsilon\downarrow 0}\Big\|\mathbf{1}_R\sum_{j,k=1}^{N}K_{j,k}^{(\varepsilon)}u\Big\|_{L^1(\mathbb{R}^3)}\\
&\leqslant\; \liminf_{\varepsilon\downarrow 0}\Big\|\sum_{j,k=1}^{N}K_{j,k}^{(\varepsilon)}u\Big\|_{L^1(\mathbb{R}^3)}\\
&=\; \liminf_{\varepsilon\downarrow 0}\Big\|\sum_{j,k=1}^{N}K_{j,k}u_\varepsilon\Big\|_{L^1(\mathbb{R}^3)}\\
&=\; \liminf_{\varepsilon\downarrow 0}\|(W_{\alpha,Y}^{+}-\mathbb{1})u_\varepsilon\|_{L^1(\mathbb{R}^3)}\\
&\leqslant\; (1+\|W_{\alpha,Y}^{+}\|_{\mathcal{B}(L^1(\mathbb{R}^3))})\|u_\varepsilon\|_{L^1(\mathbb{R}^3)}\\
&\leqslant\; (1+\|W_{\alpha,Y}^{+}\|_{\mathcal{B}(L^1(\mathbb{R}^3))})\|r^2 f\|_{L^1(0,\infty)}\,,
\end{aligned}
$$

where we applied (3.106) in the first step, (3.99) in the third step, (3.93) in the fourth step, the assumption of $L^1$-boundedness in the fifth step, and the scale invariance (3.96) in the last step. Moreover, due to the arbitrariness of $R$, the estimate above also implies

$$
(*)\qquad 4\pi\|(\mathbb{1}-\mathrm{i}\mathcal{H})(r^2 f)\|_{L^1(0,\infty)} \;\leqslant\; (1+\|W_{\alpha,Y}\|_{\mathcal{B}(L^1(\mathbb{R}^3))})\|r^2 f\|_{L^1(0,\infty)}\,.
$$

However, the inequality (*) can be surely violated. Indeed it is well-known that the Hilbert transform on $\mathbb{R}$ maps *even* functions into *odd* functions, but fails to map even (and compactly supported) $L^1$-functions into $L^1$-functions, as one may see with (a suitable mollification, so as to make it $C_0^\infty$ and even, of) the function $f_0(r) = (r^2+1)^{-1}$, the Hilbert transform of which is $(\mathcal{H}f_0)(r) = r(r^2+1)^{-1}$. Therefore (*) is a contradiction. The conclusion is that $W_{\alpha,Y}^{+}$ is necessarily *unbounded* on $L^1(\mathbb{R}^3)$.

## 3.5. $L^p$-convergence of wave operators

In this concluding Section we establish a result of $L^p$-convergence of wave operators in the limit when a regular Schrödinger Hamiltonian converges to a singular point interaction Hamiltonian. This is part of the general picture outlined in Remark 3.0.5 concerning the connection between two completely analogous results, on the one hand our main result (Theorem 3.0.3) of $L^p$-boundedness for $p\in(1,3)$ and $L^p$-unboundedness for $p\in\{1\}\cup[3,\infty)$ of the wave operators relative to the point interaction Hamiltonian $H_{\alpha,Y}$, and on the other hand the analogous results available in the previous literature, precisely in the same regimes of $p$, for wave operators relative to Schrödinger Hamiltonians of the form $-\Delta+V$.

For concreteness we restrict our attention to the case $N=1$ and $\alpha=0$, thus taking without loss of generality $Y=\{0\}$.

Let us conside a real measurable potential $V$, such that $|V(x)|\leqslant\langle x\rangle^{-\delta}$ for some $\delta>7$. Under these hypothesis, $V$ satisfies part (i) of the Assumption ($\mathrm{I}_2$) introduced in Chapter 2. Owing to Lemma 2.1.1(iii), the Schrödinger operator $H := -\Delta+V$ defined as a form sum is self-adjoint on $L^2(\mathbb{R}^3)$. It is well known [**76**] that the wave operators

$$
W^{\pm} \;:=\; s\text{-}\lim_{t\to\pm\infty} e^{\mathrm{i}tH}e^{-\mathrm{i}tH_0} \tag{3.107}
$$

relative to the pair $(H,H_0)$ exist and are complete in $L^2(\mathbb{R}^3)$; $W^{\pm}$ extend to bounded operators on $L^p(\mathbb{R}^3)$ in the following regimes: for all $p\in[1,+\infty]$ if zero is neither a resonance nor eigenvalue of $H$ [**18**], and only for $p\in(1,3)$ if zero is a resonance [**114**].

With reference to Assumption ($\mathrm{I}_2$), we take $\eta_2 \equiv 1$ as a distortion factor, and accordingly to (2.2) we consider the re-scaled operator

$$H^{(\varepsilon)} := h_\varepsilon^{(1)} = -\Delta + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0\,. \tag{3.108}$$

In analogy to (3.107), we consider also the wave operators relative to the pair $(H^{(\varepsilon)}, H_0)$, defined as

$$W_\varepsilon^{\pm} := \underset{t\to\pm\infty}{s\text{-lim}}\; e^{\mathrm{i}tH^{(\varepsilon)}} e^{-\mathrm{i}tH_0}\,. \tag{3.109}$$

Theorem 2.1.3 shows that $H^{(\varepsilon)} \to H_{\alpha,Y}|_{\alpha=0,Y=\{0\}}$ as $\varepsilon \downarrow 0$ in the norm resolvent sense of operators on $L^2(\mathbb{R}^3)$, and this in turn motivates us to investigate the relation between $W_\varepsilon^{\pm}$ and $W_{\alpha,Y}^{\pm}$ when $\varepsilon \downarrow 0$, as bounded operators on $L^p(\mathbb{R}^3)$ for $p \in (1,3)$. Our result is the following.

Proposition 3.5.1. *Suppose that $V$ is a real measurable potential such that $|V(x)| \leqslant C\langle x\rangle^{-\delta}$ for some $\delta > 7$. Then, for any $p \in (1,3)$ the wave operators $W_\varepsilon^{\pm}$ extend to bounded operators on $L^p(\mathbb{R}^3)$. If zero is a resonance but not an eigenvalue for the self-adjoint operator $H = -\Delta + V$ on $L^2(\mathbb{R}^3)$, then in the weak topology of $L^p(\mathbb{R}^3)$ with $p \in (1,3)$, and hence also in the strong topology of $L^2(\mathbb{R}^3)$,*

$$\lim_{\varepsilon\downarrow 0} W_\varepsilon^{\pm} u = W_{\alpha,Y}^{\pm} u\,, \qquad u \in L^p(\mathbb{R}^3)\,. \tag{3.110}$$

Proof. The statement on the $L^p$-boundedness of $W_\varepsilon^{\pm}$ follows directly from [**114**]. Concerning the limit (3.110), we shall prove it for $W_\varepsilon^{+}$, the argument for $W_\varepsilon^{-}$ being completely analogous.

Let us consider the scaling operator $U_\varepsilon : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ defined by (2.12). $U_\varepsilon$ induces the unitary equivalence

$$H^{(\varepsilon)} = U_\varepsilon(\varepsilon^{-2}H)\, U_\varepsilon^{*}. \tag{3.111}$$

As a consequence, $W_\varepsilon^{+}$ and $W^{+}$ are unitarily equivalent too as operators on $L^2(\mathbb{R}^3)$, for

$$\begin{aligned} W_\varepsilon^{+} &= \underset{t\to+\infty}{s\text{-lim}}\; e^{itH^{(\varepsilon)}} e^{-itH_0} \\ &= U_\varepsilon\, \underset{t\to+\infty}{s\text{-lim}}\; e^{it\varepsilon^{-2}H} e^{-it\varepsilon^{-2}H_0}\, U_\varepsilon^{*} = U_\varepsilon W^{+} U_\varepsilon^{*}\,. \end{aligned} \tag{3.112}$$

Moreover, for any $\varepsilon > 0$ and $p \in [1,+\infty]$ the operator $U_\varepsilon$ is a bounded bijection on $L^p(\mathbb{R}^3)$ with norm

$$\|U_\varepsilon\|_{\mathcal{B}(L^p(\mathbb{R}^3))} = \varepsilon^{3(\frac{1}{p}-\frac{1}{2})} \tag{3.113}$$

and inverse

$$(U_\varepsilon)^{-1} = U_{\varepsilon^{-1}}\,. \tag{3.114}$$

Combining (3.112), (3.113), and (3.114), it follows that for any $p \in (1,3)$

$$\|W_\varepsilon^{+}\|_{\mathcal{B}(L^p(\mathbb{R}^3))} = \|W^{+}\|_{\mathcal{B}(L^p(\mathbb{R}^3))} < +\infty. \tag{3.115}$$

For the proof of (3.110) it suffices to show that, when $\alpha = 0$ and $Y = \{0\}$,

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^3} \overline{(W_\varepsilon^{+}u)(x)}\, v(x)\, \mathrm{d}x = \int_{\mathbb{R}^3} \overline{(W_{\alpha,Y}u)(x)}\, v(x) \tag{3.116}$$

for any $u$ and $v$ in

$$\mathcal{D} := \{u \in \mathcal{S}(\mathbb{R}^3)\,|\, \widehat{u} \in C_0^\infty(\mathbb{R}^3)\} \tag{3.117}$$

which is dense in $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$. Indeed by means of a straightforward density argument, applicable because of the uniform norm-boundedness (3.115), the result (3.116) can then be lifted to any $u \in L^p(\mathbb{R}^d)$ and $v \in L^{p'}(\mathbb{R}^d)$, whence the conclusion. Moreover, with the choice (3.117) we can equivalently re-write (3.116) in Hilbert scalar product notation as

$$\lim_{\varepsilon\downarrow 0}\langle W_\varepsilon^+ u, v\rangle \;=\; \langle W_{\alpha,Y}u,v\rangle\,. \tag{3.118}$$

Aimed at establishing (3.118), let us fix $u,v\in\mathcal{D}$. Then there is $R>0$ such that $\widehat{u}(\xi)=0$ for $|\xi|>R$, and also

$$(\widehat{U_\varepsilon^* u})(\xi) \;=\; \frac{1}{\varepsilon^{3/2}}\,\widehat{u}\Big(\frac{\xi}{\varepsilon}\Big)\,,\qquad (\widehat{U_\varepsilon^* u})(\xi)\;=\;0\quad\text{for } |\xi|>R\varepsilon\,. \tag{3.119}$$

We shall make crucial use of the well-known fact from the stationary scattering theory [**76**] that

$$W^+ \;=\; \mathbb{1}-\frac{1}{\mathrm{i}\pi}\int_0^{+\infty}G_0(-\lambda)\,V\,(\mathbb{1}+G_0(-\lambda)V)^{-1}\,(G_0(\lambda)-G_0(\lambda))\,\lambda\,\mathrm{d}\lambda\,, \tag{3.120}$$

where

$$G_0(\pm\lambda)\;:=\;\lim_{\eta\downarrow 0}\,(H_0-(\lambda^2\pm\mathrm{i}\eta)\mathbb{1})^{-1}\;=\;\lim_{\eta\downarrow 0}\,R_0(\lambda^2\pm\mathrm{i}\eta)\,,\qquad \lambda\geqslant 0\,. \tag{3.121}$$

Then (3.112) and (3.120), together with $G_0(\pm\lambda)^*=G_0(\mp\lambda)$, yield

$$\begin{aligned}&\langle W_\varepsilon^+ u,v\rangle-\langle u,v\rangle\\ &\quad=\;\frac{1}{\mathrm{i}\pi}\int_0^{+\infty}\big\langle(\mathbb{1}+G_0(-\lambda)V)^{-1}\,(G_0(\lambda)-G_0(-\lambda))\,U_\varepsilon^* u\;,V\,G_0(\lambda)\,U_\varepsilon^* v\big\rangle\,\lambda\,\mathrm{d}\lambda\,.\end{aligned} \tag{3.122}$$

In fact, the $\lambda$-integration in (3.122) is only effective for $\lambda<R\varepsilon$. To see this, we compute the Fourier transform

$$\begin{aligned}&\big((G_0(\lambda)-G_0(-\lambda))\,U_\varepsilon^* u\big)\widehat{\;}(\xi)\\ &\quad=\;\lim_{\eta\downarrow 0}\,\big((\xi^2-\lambda^2-\mathrm{i}\eta)^{-1}-(\xi^2-\lambda^2+\mathrm{i}\eta)\big)^{-1}\,(\widehat{U_\varepsilon^* u})(\xi)\\ &\quad=\;\lim_{\eta\downarrow 0}\,\frac{2\,\mathrm{i}\,\eta}{(\xi^2-\lambda^2)^2+\eta^2}\,(\widehat{U_\varepsilon^* u})(\xi)\end{aligned} \tag{3.123}$$

and we argue that the function in (3.123) surely vanishes when $|\xi|>R\varepsilon$, owing to (3.119), and when in addition $\lambda>R\varepsilon$ such function also vanishes when $|\xi|\leqslant R\varepsilon$, because in this case $(\xi^2-\lambda^2)^2>0$ and the above limit in $\eta$ is zero. Thus,

$$(G_0(\lambda)-G_0(-\lambda))\,U_\varepsilon^* u\;\equiv\;0\qquad\text{when } \lambda>R\varepsilon\,. \tag{3.124}$$

By exploiting the scaling in $\varepsilon$ in (3.122) we obtain

$$\begin{aligned}&\langle W_\varepsilon^+ u,v\rangle-\langle u,v\rangle\\ &\quad=\;\frac{\varepsilon^2}{\mathrm{i}\pi}\int_0^{+\infty}\big\langle(\mathbb{1}+G_0(-\varepsilon\lambda)V)^{-1}(G_0(\varepsilon\lambda)-G_0(-\varepsilon\lambda))\,U_\varepsilon^* u\;,V G_0(\varepsilon\lambda)\,U_\varepsilon^* v\big\rangle\lambda\,\mathrm{d}\lambda\,,\end{aligned} \tag{3.125}$$

where it has to be remembered that, owing to (3.124), the integration actually only takes place when $\lambda\in[0,R]$.

Next, in order to compute the limit $\varepsilon\downarrow 0$ in (3.125), we consider separately the behaviour of the operators

$$\varepsilon^{\frac{1}{2}}G_0(\pm\varepsilon\lambda)\,U_\varepsilon^*\qquad\text{and}\qquad \varepsilon(\mathbb{1}+G_0(-\varepsilon\lambda)V)^{-1}\,.$$

Indeed, we shall see that they do converge strongly in a suitable Banach space. A weak-type Hölder's inequality implies that

$$\big(\varepsilon^{\frac{1}{2}}G_0(\pm\varepsilon\lambda)\,U_\varepsilon^* u\big)(x)\;=\;\int_{\mathbb{R}^3}\frac{e^{\pm\mathrm{i}\lambda|\varepsilon x-y|}}{4\pi|\varepsilon x-y|}\,u(y)\,\mathrm{d}y$$

is bounded by a constant (by $(4\pi)^{-1}\||x|^{-1}\|_{L^{3,\infty}}\|u\|_{L^{\frac{3}{2},1}}$ in terms of Lorentz norms); therefore, uniformly for $\lambda\in[0,R]$ and $x$ in compact sets,

$$\lim_{\varepsilon\downarrow 0}\big(\varepsilon^{\frac{1}{2}}G_0(\pm\varepsilon\lambda)\,U_\varepsilon^* u\big)(x)\;=\;\int_{\mathbb{R}^3}\frac{e^{\pm \mathrm{i}\lambda|y|}}{4\pi|y|}\,u(y)\,\mathrm{d}y\;=\;\langle\overline{\mathcal{G}_{\pm\lambda}},u\rangle\,.$$

As a consequence, we deduce that

$$\lim_{\varepsilon\downarrow 0}\big\|\varepsilon^{\frac{1}{2}}G_0(\pm\varepsilon\lambda)\,U_\varepsilon^* u-\langle\overline{\mathcal{G}_{\pm\lambda}},u\rangle\mathbf{1}\big\|_{L^2_{-\beta}(\mathbb{R}^3)}\;=\;0 \tag{3.126}$$

for $\beta>\frac{3}{2}$, where $L^2_{-\beta}(\mathbb{R}^3)\equiv L^2(\mathbb{R}^3,\langle x\rangle^{-2\beta}\mathrm{d}x)$ and $\mathbf{1}$ denotes the function $\mathbf{1}(x)=1$ $\forall x\in\mathbb{R}^3$. Moreover, owing to the spectral and decay assumptions on $V$, it is a standard fact [**113**, Theorem 4.8] that

$$\lim_{\varepsilon\downarrow 0}\Big\|\,\varepsilon(\mathbb{1}+G_0(-\varepsilon\lambda)V)^{-1}-\frac{4\pi\mathrm{i}}{\lambda a^2}\,|\varphi\rangle\langle V\varphi|\,\Big\|_{\mathcal{B}(L^2_{-\beta}(\mathbb{R}^3))}\;=\;0 \tag{3.127}$$

for $\lambda>0$ and $\beta\in(\frac{3}{2},\delta-\frac{1}{2})$, where $\varphi$ is a zero-energy resonance for $H=-\Delta+V$, uniquely identified by the conditions $\int_{\mathbb{R}^3}V|\varphi|^2\mathrm{d}x=-1$ and $\int_{\mathbb{R}^3}V\varphi\,\mathrm{d}x>0$, and where $a:=\int_{\mathbb{R}^3}V\varphi\,\mathrm{d}x$.

If we now and henceforth restrict $\beta$ to the regime $\beta\in(\frac{3}{2},\frac{\delta}{2})$, then (3.126) and (3.127) are still valid, and in addition the multiplication by $V$ is a $L^2_{-\beta}(\mathbb{R}^3)\to L^2_{\beta}(\mathbb{R}^3)$ continuous map. Thus, the $L^2$-scalar product appearing in the r.h.s. of (3.125) can be also regarded as a $L^2_{-\beta}$-$L^2_\beta$ duality product. Using this fact, and by means of (3.126) and (3.127), which are applicable because the $\lambda$-integration in (3.125) is actually only effective for $\lambda\in[0,R]$, we find

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}(\text{r.h.s. of (3.125)})\\
&=\;\frac{1}{\mathrm{i}\pi}\int_0^{+\infty}\Big\langle\,\frac{4\pi\mathrm{i}}{\lambda a^2}\,|\varphi\rangle\langle V\varphi|\big(\langle\overline{\mathcal{G}_\lambda},u\rangle-\langle\overline{\mathcal{G}_{-\lambda}},u\rangle\big)\mathbf{1}\,,V\langle\overline{\mathcal{G}_\lambda},v\rangle\,\mathbf{1}\Big\rangle_{L^2_{-\beta},L^2_\beta}\lambda\,\mathrm{d}\lambda\\
&=\;-4\int_0^{+\infty}\mathrm{d}\lambda\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y)}\,\big(\mathcal{G}_{-\lambda}(y)-\mathcal{G}_\lambda(y)\big)\Big)\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}_\lambda(x)\,v(x)\Big)\,.
\end{aligned}$$

Summarising, we have found

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\langle W_\varepsilon^+u,v\rangle\;&=\;\langle u,v\rangle\\
&\quad+4\int_0^{+\infty}\mathrm{d}\lambda\,\Big(\int_{\mathbb{R}^3}\mathrm{d}y\,\overline{u(y)}\,\big(\mathcal{G}_\lambda(y)-\mathcal{G}_{-\lambda}(y)\big)\Big)\Big(\int_{\mathbb{R}^3}\mathrm{d}x\,\mathcal{G}_\lambda(x)\,v(x)\Big)\,.
\end{aligned} \tag{3.128}$$

Since the r.h.s. above is precisely the quantity $\langle W_{\alpha,Y}u,v\rangle$ that we obtained in (3.9) in the special case $N=1$, $\alpha=0$, the limit $\langle W_\varepsilon^+u,v\rangle\to\langle W_{\alpha,Y}u,v\rangle$ of (3.118) is then established and, as already argued, this completes the proof. $\square$

CHAPTER 4

# Global smoothing properties

In this Chapter we study the smoothing properties of the dynamics generated by singular perturbations of the three-dimensional Laplacian.

As already discussed in the Introduction of this thesis, one main motivation is the investigation of perturbative non-linear problems of the form

$$\mathrm{i}\partial_t u \;=\; -\Delta_{\alpha,Y} u + \mathcal{N}(u)\,, \quad t \in \mathbb{R}\,,\, x \in \mathbb{R}^3\,, \tag{4.1}$$

which naturally arise in the context of many-body quantum systems subject to fixed impurities, where the non-linear term $\mathcal{N}(u)$ describes the interactions between the particles, in a suitable scaling regime. In fact, the smoothing properties of the unitary group $\{e^{\mathrm{i}t\Delta_{\alpha,Y}}\}_{t\in\mathbb{R}}$ are a fundamental tool in order to prove local well-posedness of (4.1), by means of a fixed point argument.

We begin our discussion by recalling some basic facts on the free Schrödinger evolution. The unitary propagator $e^{\mathrm{i}t\Delta}$ has an explicit integration kernel:

$$\big(e^{\mathrm{i}t\Delta} f\big)(x) = (4\pi \mathrm{i}t)^{-3/2} \int_{\mathbb{R}^3} e^{-\mathrm{i}\frac{|x-y|^2}{4t}} f(y)\mathrm{d}y, \qquad t \neq 0. \tag{4.2}$$

Owing to (4.2) and Young inequality one gets

$$\|e^{\mathrm{i}t\Delta} f\|_{L^\infty(\mathbb{R}^3)} \;\lesssim\; |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^1(\mathbb{R}^3)}, \qquad p \in [2,+\infty]\,, \quad t \neq 0\,. \tag{4.3}$$

Interpolating (4.3) with the trivial $L^2$-bound, one deduces the well-known *dispersive* (or $L^{p'} - L^p$) estimates:

$$\|e^{\mathrm{i}t\Delta} f\|_p \;\lesssim\; |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{p'}, \qquad p \in [2,+\infty]\,, \quad t \neq 0\,. \tag{4.4}$$

Furthermore, the free propagator $e^{\mathrm{i}t\Delta}$ satisfies a class of space-time estimates, known as *Strichartz estimates*:

$$\begin{aligned} &\|e^{\mathrm{i}t\Delta} f\|_{L^q(\mathbb{R}_t, L^p(\mathbb{R}^d_x))} \;\leqslant\; C\|f\|_{L^2(\mathbb{R}^3)}, \\ &\frac{2}{q} + \frac{3}{p} = \frac{3}{2}, \qquad p \in [2,6]. \end{aligned} \tag{4.5}$$

In the non-endpoint case $p \neq 6$, (4.5) follows by dispersive estimates (4.4) by means of a duality argument and fractional integration [**49, 110**]. The proof in the endpoint case $p = 6$ is more involved, and it is due to Keel and Tao [**70**].

The literature on dispersive and Strichartz estimates for actual Schrödinger operators of the form $-\Delta + V$, for sufficiently regular $V : \mathbb{R}^3 \to \mathbb{R}$ vanishing at spatial infinity, is vast [**67, 111, 96, 52, 41, 94, 64, 51, 87, 113**] and the problem is well known to depend on the spectral properties of $-\Delta + V$ at the bottom of the absolutely continuous spectrum, that is, at zero energy.

For singular perturbations of the Laplacian, the picture is much less developed. In this direction, the first result was achieved by D'Ancona, Pierfelice, and Teta [**33**], who proved weighted $L^1 - L^\infty$ estimates, the weights being suitably chosen in order to compensate the local singularities due to the point interactions.

In my work [**61**], in collaboration with F. Iandoli, we proved in the single centre case non-weighted dispersive and Strichartz estimates, that is, the analogue of (4.4)-(4.5), in the regime $p \in [2, 3)$. In my work [**36**], in collaboration with G. Dell'Antonio, A. Michelangeli, and K. Yajima, the result was extended to the general multi-centre case, as a consequence of the $L^p$-boundedness of the singular wave operators (Theorem 3.0.3).

The Chapter is organised as follows. In Section 1 we discuss weighted $L^1 - L^\infty$ estimates and their consequences. In Section 2 we prove non-weighted dispersive estimates, in the regime $p \in (1, 3)$. As a by-product, we deduce Strichartz estimates in the same range of $p$'s.

## 4.1. Weighted dispersive estimates

In this Section we study the weighted dispersive estimates for the unitary flow generated by singular perturbations of the three-dimensional Laplacian. We start our discussion with the case of a single point interaction, which can be assumed to be centred at the origin. Hence we consider the family of self-adjoint operators $\{-\Delta_\alpha\}_{\alpha\in\mathbb{R}}$ identified in Theorem 1.1.1. An important feature of the single-centre case is that the unitary propagator $e^{\mathrm{i}t\Delta_\alpha}$ has an explicit expression. In particular, we have the following characterisation [**5, 100**], valid for any $f \in L^2(\mathbb{R}^3)$ and $t \neq 0$.

(i) For $\alpha > 0$,

$$\begin{aligned}(e^{\mathrm{i}t\Delta_\alpha} f)(x) &= (e^{\mathrm{i}t\Delta} f)(x) + \frac{1}{(4\pi\mathrm{i}t)^{3/2}|x|} \times \\ &\times \int_{\mathbb{R}^3} \Big(\frac{f(y)}{|y|} \int_0^{+\infty} e^{-4\pi\alpha s}(s+|x|+|y|)e^{-\mathrm{i}\frac{(s+|x|+|y|)^2}{4t}} ds\Big)dy.\end{aligned} \tag{4.6}$$

(ii) For $\alpha < 0$,

$$\begin{aligned}(e^{\mathrm{i}t\Delta_\alpha} f)(x) &= (e^{\mathrm{i}t\Delta} f)(x) - e^{\mathrm{i}t(4\pi\alpha)^2}\langle\psi_\alpha, f\rangle\psi_\alpha(x) + \frac{1}{(4\pi\mathrm{i}t)^{3/2}|x|} \times \\ &\times \int_{\mathbb{R}^3} \Big(\frac{f(y)}{|y|} \int_0^{+\infty} e^{4\pi\alpha s}(s-|x|-|y|)e^{-\mathrm{i}\frac{(s-|x|-|y|)^2}{4t}} ds\Big)dy,\end{aligned} \tag{4.7}$$

where

$$\psi_\alpha(x) := \pi\sqrt{32|\alpha|}\,\mathsf{G}_{|E_\alpha|} = \sqrt{2|\alpha|}\,\frac{e^{4\pi\alpha|x|}}{|x|}$$

is the normalised eigenfunction associated to the eigenvalue $E_\alpha = -(4\pi\alpha)^2$.

(iii) For $\alpha = 0$,

$$(e^{\mathrm{i}t\Delta_0} f)(x) = (e^{-\mathrm{i}t\Delta} f)(x) + \frac{1}{4\pi(\pi\mathrm{i}t)^{1/2}|x|}\int_{\mathbb{R}^3} e^{-\mathrm{i}\frac{(|x|+|y|)^2}{4t}}\frac{f(y)}{|y|}dy. \tag{4.8}$$

Owing to the explicit formulas above, the following result can be easily proved (the original proof can be found in [**33**]).

THEOREM 4.1.1. *Let $w(x) := 1 + |x|^{-1}$.*

(i) *For every $\alpha \neq 0$, the following estimate holds:*

$$\|w^{-1}e^{\mathrm{i}t\Delta_\alpha}P_{ac}(-\Delta_\alpha)f\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-\frac{3}{2}}\|wf\|_{L^1(\mathbb{R}^3)}, \qquad t \neq 0. \tag{4.9}$$

(ii) *In the zero-energy resonant case $\alpha = 0$, the following estimate holds:*

$$\|w^{-1}e^{\mathrm{i}t\Delta_0}f\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-\frac{1}{2}}\|wf\|_{L^1(\mathbb{R}^3)}, \qquad t \neq 0. \tag{4.10}$$

REMARK 4.1.2. It is worth observing that the presence of a weight in (4.9)-(4.10) is unavoidable, because of the structure of the domain of $-\Delta_\alpha$ identified in (1.6). Indeed, even if one take $f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$, the evolution $e^{\mathrm{i}t\Delta_\alpha}f$ instantaneously developes a local singularity of order $|x|^{-1}$, which prevents a non-weighted estimate like (4.3). Furthermore, it emerges from Theorem 4.1.1 that the occurence of a zero-energy resonance results in a slower time-decay for the dispersive estimates, a typical and well-known phenomenon for regular Schrödinger operators of the form $-\Delta + V$ [**64, 94, 87**].

Proof of Theorem 4.1.1. Let use start with the case $\alpha > 0$. Combining the $L^1 - L^\infty$-estimate (4.3) for the free propagator $e^{\mathrm{i}t\Delta}$ with the trivial bounds

$$|e^{-\mathrm{i}\frac{(s+|x|+|y|)^2}{4t}}| \leqslant 1,$$

$$\int_0^{+\infty} e^{-4\pi\alpha s}(s+|x|+|y|)ds \lesssim 1+|x|+|y|,$$

one deduces that for $t \neq 0$ and for almost every $x \in \mathbb{R}^3$

$$|(e^{\mathrm{i}t\Delta_\alpha} f)(x)| \lesssim t^{-3/2} \int_{\mathbb{R}^3} \Big(1 + \frac{1+|x|+|y|}{|x||y|}\Big)|f(y)|dy. \tag{4.11}$$

Since

$$1 + \frac{1+|x|+|y|}{|x||y|} = w(x)w(y),$$

then (4.11) yields

$$\|w^{-1}e^{\mathrm{i}t\Delta_\alpha} f\|_{L^\infty(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}}\|wf\|_{L^1(\mathbb{R}^3)}, \qquad t \neq 0,$$

which is the desired estimate, the absolutely continuous subspace for $-\Delta_\alpha$ being the whole $L^2(\mathbb{R}^3)$.

Let us consider now the case $\alpha < 0$. Owing to (4.7), one can easily distinguish the absolutely continuous part of the evolution:

$$\begin{aligned}(e^{\mathrm{i}t\Delta_\alpha} P_{ac}(-\Delta_\alpha) f)(x) &= (e^{\mathrm{i}t\Delta} f)(x) + \frac{1}{(4\pi \mathrm{i}t)^{3/2}|x|}\times \\ &\times \int_{\mathbb{R}^3} \Big(\frac{f(y)}{|y|}\int_0^{+\infty} e^{4\pi\alpha s}(s-|x|-|y|)e^{-\mathrm{i}\frac{(s-|x|-|y|)^2}{4t}}ds\Big)dy,\end{aligned} \tag{4.12}$$

Proceeding as above, estimate (4.9) immediately follows from (4.12).

Last, when $\alpha = 0$, we can proceed again in the same way, and owing to (4.8) we deduce (4.10). □

In the general multi-centre case, we do not have an explicit formula for the propagator, and one needs to resort to spectral calculus and the resolvent identity (1.32). In this perspective, the following result was proved in [**33**].

Theorem 4.1.3. *Let $\alpha \in \mathbb{R}^N$ and $Y \equiv \{y_1, \dots, y_N\} \subset \mathbb{R}^3$, and assume that the matrix $\Gamma_{\alpha,Y}(z)$ defined by (1.28) is invertible for $z \in \mathbb{R}$, with locally bounded inverse. Set*

$$w(x) = \sum_{j=1}^N \left(1 + \frac{1}{|x-y_j|}\right). \tag{4.13}$$

*The following dispersive estimate holds:*

$$\|w^{-1}e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y})f\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-\frac{3}{2}}\|wf\|_{L^1(\mathbb{R}^3)}, \qquad t \neq 0. \tag{4.14}$$

The assumption on $\Gamma_{\alpha,Y}$ implies in particular that zero is netiher an eigenvalue nor a resonance for $-\Delta_{\alpha,Y}$ (see the discussion in Chapter 1). Unlike the single centre case, no results are available when there is a zero-energy obstruction.

Furthermore, interpolating (4.14) with the trivial $L^2$-bound, one deduces that under the same assumption on $\Gamma_{\alpha,Y}$ as in Theorem 4.1.3 the following weighted dispersive inequalities holds true, for $p \in [2, +\infty]$:

$$\|w^{-\left(1-\frac{2}{p}\right)} e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y}) f\|_{L^p(\mathbb{R}^3)} \lesssim \\ |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|w^{\frac{2}{p'}-1} f\|_{L^{p'}(\mathbb{R}^3)}, \quad t \neq 0. \tag{4.15}$$

### 4.2. Non-weighted dispersive and Strichartz estimates

Observe that, when $p < 3$, the function $w$ given by (4.13) belongs to $L^p_{loc}(\mathbb{R}^3)$. Hence one could expect to prove, in the regime $p \in [2,3)$, an non-weighted version of (4.15). Indeed, we have the following result [**36**]:

THEOREM 4.2.1. *Assume that zero in not an eigenvalue for $-\Delta_{\alpha,Y}$, and that the matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \in \mathbb{R} \setminus \{0\}$. Then, for $p \in [2,3)$, one has the dispersive estimates*

$$\|e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y}) f\|_{L^p(\mathbb{R}^3)} \lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^3)}, \qquad t \neq 0 \tag{4.16}$$

PROOF. As commented in Chapter 3, the wave operators $W^{\pm}_{\alpha,Y}$ relative to the pair $(-\Delta_{\alpha,Y}, -\Delta)$ exist and are complete in $L^2(\mathbb{R}^3)$. In particular, the intertwining property (3.2) yields

$$e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y}) = W^{\pm}_{\alpha,Y} e^{\mathrm{i}t\Delta} (W^{\pm}_{\alpha,Y})^*. \tag{4.17}$$

Moreover, for a given $p \in [2,3)$, Theorem 3.0.3 guarantees that $W^{\pm}_{\alpha,Y}$ are bounded in $L^p(\mathbb{R}^3)$, whence by duality $(W^{\pm}_{\alpha,Y})^*$ are bounded in $L^{p'}(\mathbb{R}^3)$. It follows that

$$\begin{aligned} \|e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y}) f\|_{L^p(\mathbb{R}^3)} &= \|W^{\pm}_{\alpha,Y} e^{\mathrm{i}t\Delta} (W^{\pm}_{\alpha,Y})^* f\|_{L^p(\mathbb{R}^3)} \\ &\lesssim \|e^{\mathrm{i}t\Delta} (W^{\pm}_{\alpha,Y})^* f\|_{L^p(\mathbb{R}^3)} \\ &\lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|(W^{\pm}_{\alpha,Y})^* f\|_{L^{p'}(\mathbb{R}^3)} \\ &\lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^3)}, \end{aligned}$$

where in the third step we used the dispersive estimates (4.4) for the free Laplacian. The proof is complete. □

REMARK 4.2.2. Unlike the weighted $L^1-L^\infty$ estimates discussed in the previous Section, Theorem 4.2.1 covers the zero-energy resonant case also in the multi-centre setting. Moreover, in the considered regime $p \in [2,3)$, a zero-energy resonance does not produce a slower time decay for the $L^{p'} - L^p$ estimates.

Interpolating (4.16) with (4.14) we get that for $p \in [3,+\infty)$

$$\|w^{-\left(1-\frac{3-\varepsilon}{p}\right)} e^{\mathrm{i}t\Delta_{\alpha,Y}} P_{ac}(-\Delta_{\alpha,Y}) f\|_{L^p(\mathbb{R}^3)} \lesssim \\ |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|w^{-\left(1-\frac{3-\varepsilon}{p}\right)} f\|_{L^{p'}(\mathbb{R}^3)}, \quad \varepsilon > 0,\, t \neq 0. \tag{4.18}$$

In the regime $p \in [3,+\infty)$, estimate (4.18) *improves* the weights appearing in (4.15).

As a consequence of the dispersive estimates (4.16), one can deduce a class of Strichartz estimates for $-\Delta_{\alpha,Y}$ [**36**]. We shall call a pair of exponents $(q,p)$ *admissible for* $-\Delta_{\alpha,Y}$ if

$$p \in [2,3) \qquad \text{and} \qquad 0 \leqslant \frac{2}{q} = 3\Big(\frac{1}{2}-\frac{1}{p}\Big) < \frac{1}{2}\,, \tag{4.19}$$

that is, $q = \frac{4p}{3(p-2)} \in (4,+\infty]$.

THEOREM 4.2.3 (Strichartz estimates for $-\Delta_{\alpha,Y}$). *Assume that zero in not an eigenvalue for $-\Delta_{\alpha,Y}$, and that the matrix $\Gamma_{\alpha,Y}(z)$ is invertible for $z \in \mathbb{R}\setminus\{0\}$. Let $(q,p)$ and $(s,r)$ be two admissible pairs for $-\Delta_{\alpha,Y}$. Then*

$$\|e^{\mathrm{i}t\Delta_{\alpha,Y}}P_{\mathrm{ac}}(-\Delta_{\alpha,Y})f\|_{L^q(\mathbb{R}_t,L^p(\mathbb{R}^3_x))} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \tag{4.20}$$

*and*

$$\Big\|\int_0^t e^{\mathrm{i}(t-\tau)\Delta_{\alpha,Y}}P_{\mathrm{ac}}(-\Delta_{\alpha,Y})\,F(\tau)\,\mathrm{d}\tau\Big\|_{L^q(\mathbb{R}_t,L^p(\mathbb{R}^3_x))} \lesssim \|F\|_{L^{s'}(\mathbb{R}_t,L^{r'}(\mathbb{R}^3_x))}\,. \tag{4.21}$$

**4.2.1. Single centre case.** In the single centre case there is a more direct proof of Theorem 4.2.1, found in my work with F. Iandoli [**61**], which in a sense is elementary, since, unlike [**36**], it does not use any deep result from scattering theory. Our argument is based on the explicit characterisation of the unitary propagator $e^{\mathrm{i}t\Delta_\alpha}$ given by (4.6), (4.7), and (4.8). For our purposes, it is convient to re-write such formulas in the following equivalent form:

$$e^{\mathrm{i}t\Delta_\alpha}f=\begin{cases} e^{\mathrm{i}t\Delta}f+M_tf & \text{if } \alpha=0\\ e^{\mathrm{i}t\Delta_0}f+M_{\alpha,t}f & \text{if } \alpha>0\,,\\ e^{\mathrm{i}t\Delta_0}f+\widetilde{M}_{\alpha,t}f & \text{if } \alpha<0\end{cases}\qquad f\in L^2(\mathbb{R}^3),\ t\neq 0, \tag{4.22}$$

where

$$(M_tf)(x):=\frac{1}{4\pi(\pi\mathrm{i}t)^{1/2}|x|}\int_{\mathbb{R}^3}\frac{f(y)}{|y|}e^{-\mathrm{i}\frac{(|x|+|y|)^2}{4t}}dy, \tag{4.23}$$

$$(M_{\alpha,t}f)(x):=\frac{\alpha}{(\pi\mathrm{i}t)^{1/2}|x|}\int_{\mathbb{R}^3}\frac{f(y)}{|y|}\int_0^{+\infty}e^{-4\pi\alpha s}e^{-\mathrm{i}\frac{(|x|+|y|+s)^2}{4t}}dsdy, \tag{4.24}$$

$$\begin{split}(\widetilde{M}_{\alpha,t}f)(x):=-e^{\mathrm{i}t(4\pi\alpha)^2}\langle\psi_\alpha,f\rangle\psi_\alpha(x)-\frac{\alpha}{(\pi\mathrm{i}t)^{1/2}|x|}\times\\ \times\int_{\mathbb{R}^3}\frac{f(y)}{|y|}\int_0^{+\infty}e^{4\pi\alpha s}e^{-\mathrm{i}\frac{(|x|+|y|-s)^2}{4t}}dsdy,\end{split} \tag{4.25}$$

Unlike the proof of weighted $L^1-L^\infty$ estimates (4.9)-(4.10), here we need to deal with the oscillating terms in (4.23), (4.24), and (4.25). The relevant tool is the following result from harmonic analysis, due to Pitt [**92**]:

THEOREM 4.2.4 (Pitt's theorem). *Let $1<\gamma\leqslant\eta<\infty$ and $0<b<\frac{1}{\gamma'}$ be such that $\beta:=\frac{1}{\gamma'}-\frac{1}{\eta}-b<0$, and define $v(x)=|x|^{b\gamma}$ for all $x\in\mathbb{R}$. There is a constant $C>0$ such that*

$$\left(\int_{\mathbb{R}}|\widehat{f}(\xi)|^\eta|\xi|^{\beta\eta}d\xi\right)^{1/\eta}\leqslant C\left(\int_{\mathbb{R}}|f(x)|^\gamma|x|^{b\gamma}dx\right)^{1/\gamma}, \tag{4.26}$$

*for all $f\in L^\gamma(\mathbb{R},v(x)dx)$.*

Theorem 4.2.4 is a one-dimensional extension of the well known Hausdorff-Young inequality in the context of weighted Lebesgue spaces.

ALTERNATIVE PROOF OF THEOREM 4.2.1, SINGLE CENTRE CASE. By means of a standard density argument, it is enough to prove the thesis for $f\in C^\infty_0(\mathbb{R}^3)$. Let $R:=R(f)$ be such that $f(x)=0$ for $|x|\geqslant R$. In view of the discussion in Chapter 1.1, we know that, with respect to the canonical angular decomposition of $L^2(\mathbb{R}^3)$ given by (1.17), the singular Laplacian $-\Delta_\alpha$ differs from the free Laplacian only on the sector of zero angular moment. Hence we may further assume $f$ to be radial, viz. $f(x) = \widetilde{f}(|x|)$ for some $\widetilde{f} : [0, +\infty) \to \mathbb{C}$, with $\operatorname{supp} \tilde{f} \subseteq [0, R]$. We need to show that the following estimates hold, uniformly with respect to $f$.

$$\|M_t f\|_{L^p} \lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}}, \tag{4.27}$$

$$\|M_{\alpha,t} f\|_{L^p} \lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}}, \tag{4.28}$$

$$\|\widetilde{M}_{\alpha,t} P_{ac}(-\Delta_\alpha) f\|_{L^p} \lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}}. \tag{4.29}$$

The latter inequalities, combined with the dispersive estimates (4.4) for the free Laplacian, are clearly sufficient to prove (4.16) in the single centre case.

Let us start by proving (4.27). Using spherical coordinates in both variables $x$ and $y$ we get

$$\|M_t f\|_{L^p} \lesssim |t|^{-1/2} \left[ \int_0^{+\infty} r^{2-p} \left| \int_0^R \exp\left( -i\frac{\rho r}{2t} - i\frac{\rho^2}{4t} \right) \rho \widetilde{f}(\rho) d\rho \right|^q dr \right]^{1/p}. \tag{4.30}$$

Setting

$$h(\rho) := \begin{cases} e^{-i\rho^2/4t} \rho \widetilde{f}(\rho) & 0 \leqslant \rho \leqslant R \\ 0 & \rho \in \mathbb{R} \setminus [0, R], \end{cases} \tag{4.31}$$

the latter expression becomes

$$|t|^{-1/2} \left[ \int_0^{+\infty} r^{2-p} \left| \widehat{h}\left( \frac{r}{2t} \right) \right|^p dr \right]^{1/p}, \tag{4.32}$$

which is equal to

$$|t|^{-3(\frac{1}{2}-\frac{1}{p})} \left[ \int_0^{+\infty} r^{2-p} |\widehat{h}(r)|^p dr \right]^{1/p}. \tag{4.33}$$

Since $p < 3$, we may use Theorem 4.2.4 in the case $\eta = p$, $\gamma = p'$, $\beta = \frac{2-p}{p}$, and $b = \frac{2-p'}{p'}$, obtaining

$$|t|^{-3(\frac{1}{2}-\frac{1}{p})} \left[ \int_0^{+\infty} r^{2-p} |\widehat{h}(r)|^p dr \right]^{1/p} \lesssim |t|^{-3(\frac{1}{2}-\frac{1}{p})} \left[ \int_0^{+\infty} |h(r)|^{p'} r^{2-p'} dr \right]^{1/p'}, \tag{4.34}$$

which is the desired estimate, for

$$\left[ \int_0^{+\infty} |h(r)|^{p'} r^{2-p'} dr \right]^{1/p'} \approx \|f\|_{L^{p'}}. \tag{4.35}$$

Let us prove now (4.28). Since $p < 3$, the function $1/|y|$ belongs to $L^p_{loc}(\mathbb{R}^3)$, hence we can exchange the order of integration in (4.24) and use Minkowski inequality:

$$\begin{aligned} \|M_{\alpha,t} f\|_{L^p} &\lesssim |t|^{-1/2} \int_0^{+\infty} \left\| \int_{|y| \leqslant R} \frac{1}{|x|} e^{-4\pi\alpha s - i\frac{(|x|+|y|+s)^2}{4t}} \frac{f(y)}{|y|} dy \right\|_{L^p} ds \\ &= |t|^{-1/2} \int_0^{+\infty} e^{-4\pi\alpha s} \left\| \int_{|y| \leqslant R} \frac{1}{|x|} e^{-i\frac{|y|^2}{4t} - i\frac{|x||y|}{2t} - i\frac{s|y|}{2t}} \frac{f(y)}{|y|} dy \right\|_{L^p} ds. \end{aligned} \tag{4.36}$$

An integration in spherical coordinates yields

$$\|M_{\alpha,t} f\|_{L^p} \lesssim |t|^{-1/2} \int_0^{+\infty} e^{-4\pi\alpha s} \left( \int_0^{+\infty} r^{2-p} \left| \int_0^R e^{-i\frac{r\rho}{2t}} h_s(\rho)(\rho) d\rho \right|^p dr \right) ds, \tag{4.37}$$

where

$$h_s(\rho) := \begin{cases} e^{-i\frac{\rho^2}{4t}-i\frac{s\rho}{2t}}\rho\widetilde{f}(\rho) & 0 \leqslant \rho \leqslant R \\ 0 & \rho \in \mathbb{R}\setminus[0,R] \end{cases} . \tag{4.38}$$

The quantity (4.37) is nothing but

$$|t|^{-1/2}\int_0^{+\infty} e^{-4\pi\alpha s}\left(\int_0^{+\infty} r^{2-p}\left|\widehat{h_s}\left(\frac{r}{2t}\right)\right|^p dr\right)^{1/p} ds, \tag{4.39}$$

which, arguing as before, is bounded by $|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}}$. This concludes the proof of (4.28).

The proof of (4.29) is analogous: indeed, after projecting $f$ onto the the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ for $-\Delta_\alpha$, the first summand in the right hand side of (4.25) disappears, hence the remaining part can be treated exactly in the same way as done in the proof of (4.28). □

CHAPTER 5

# Singular-perturbed Sobolev spaces

In this Chapter we study the fractional powers of the non-negative, three-dimensional 'singular perturbed' Laplacian $-\Delta_\alpha$, for $\alpha \geqslant 0$. Since $-\Delta_\alpha$ is semi-bounded from below for any $\alpha \in \mathbb{R}$, then up to a non-essential shift our discussion could be also exported to negative $\alpha$'s.

We focus on the operators $(-\Delta_\alpha)^{s/2}$, $s \in \mathbb{R}$ thus denoting the number of '*singular fractional derivatives*', aiming at covering the regime of main relevance, that is, $s \in (0,2)$ (the power $s=0$ corresponds to the identity operator, the power $s=2$ corresponds to the actual $-\Delta_\alpha$).

Among the motivations for the interest on $(-\Delta_\alpha)^{s/2}$, central is surely the observation that its domain provides a 'singular-perturbed' version of the classical Sobolev space $H^s(\mathbb{R}^3)$, *adapted* to the self-adjoint operator $-\Delta_\alpha$ – we shall denote it with $H^s_\alpha(\mathbb{R}^3)$ in our results. As already discussed in the Introduction of this thesis, the knowledge of such singular Sobolev spaces, of their induced singular Sobolev norms, and of the mutual control between classical and singular Sobolev norms, constitutes a crucial tool, combined with dispersive properties of $-\Delta_\alpha$, for the study of the well-posedness of semi-linear 'singular' Schrödinger equations of the form

$$\mathrm{i}\partial_t u \;=\; -\Delta_\alpha u + \mathcal{N}(u) \tag{5.1}$$

with non-linearities of relevance such as $\mathcal{N}(u) = |u|^\gamma u$ or $\mathcal{N}(u) = |x|^{-\gamma} * |u|^2$, $\gamma > 0$. These are non-linear PDE's that model, in a suitable regime, the presence of a localised impurity. The natural energy space for equation (5.1) is $H^1_\alpha(\mathbb{R}^3)$, and one would like to address also a higher or lower regularity theory, whence the importance of the understanding of the spaces $H^s_\alpha(\mathbb{R}^3)$.

The material presented in this Chapter is based on my recent work [**46**], in collaboration with V. Georgiev and A. Michelangeli, where we provide a thorough characterisation of the *singular Sobolev spaces* $H^s_\alpha(\mathbb{R}^3)$, in the regime $s \in (0,2)$.

In the first of our main results, Theorem 5.1.1, we determine the precise structure of $H^s_\alpha(\mathbb{R}^3)$, identifying regular and singular part of a generic $g \in H^s_\alpha(\mathbb{R}^3)$ in all the regimes of $s$ for which such decomposition is meaningful. In our second main result, Theorem 5.1.3, we present a mutual control between classical and singular Sobolev norms, and in our third main result, Theorem 5.1.4, we find an explicit formula for the computation of $(-\Delta_\alpha)^{s/2}\, u$.

These results and related remarks are stated in Section 5.1. In particular, there arise three natural regimes of increasing regularity, $s \in (0, \frac{1}{2})$, $s \in (\frac{1}{2}, \frac{3}{2})$, and $s \in (\frac{3}{2}, 2)$: the first is so low that no canonical decomposition between regular and singular part is possible; the second is large enough to produce indeed a decomposition, however with no constraint between regular and singular component; the third is so high as to induce a constraint between the two components, which is completely analogous to what was already known for the space $H^2_\alpha(\mathbb{R}^3)$, i.e., the domain of $-\Delta_\alpha$. The transition cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ are discussed separately in Section 5.1 and then in Propositions 5.6.1 and 5.6.2.

A similar analysis was been done for Schrödinger operators with inverse square potentials [**71**]. Despite their singular behavior, it turns out that the corresponding adapted Sobolev spaces agree with the classical Sobolev space $H^s(\mathbb{R}^3)$, for every fractional power $s \in (0,2)$.

The material of this Chapter is organised as follows. In Sections 5.2 through 5.5 we develop an amount of preparatory material for the proof of our main results, which is then the object of our concluding Section 5.6. In particular, in Section 5.2 we establish a spectral-theorem-based canonical decomposition of the domain of $(-\Delta_\alpha)^{s/2}$ and in Section 5.3 we study the regularity of each term of such a decomposition. This leads us to identify convenient subspaces of the fractional space $H^s_\alpha(\mathbb{R}^3)$ in Section 5.4, an information that we find convenient for the sake of clarity to re-cast in an operator-theoretic language in terms of suitable fractional maps, Section 5.5.

### 5.1. The fractional singular Laplacian $(-\Delta_\alpha)^{s/2}$: main results

For $\alpha \geqslant 0$, the singular perturbed Laplacian $-\Delta_\alpha$, characterised by Theorem 1.1.1, is a non-negative self-adjoint operator on $L^2(\mathbb{R}^3)$ and the spectral theorem provides an unambiguous definition of its fractional powers $(-\Delta_\alpha)^{s/2}$. Special cases are $s=0$, yielding the identity operator on $L^2(\mathbb{R}^3)$, and $s=2$, yielding the operator $-\Delta_\alpha$ itself, whereas $s=1$ (the square root) corresponds to an operator whose domain is the form domain of $-\Delta_\alpha$.

For general $s \in (0,2)$ we are able to provide the following amount of information.

Our first result concerns the '*fractional domains*', namely the domains of the fractional powers of $-\Delta_\alpha$. We find that for small $s$ the fractional domain is the Sobolev space of order $s$, whereas when $s > \frac{1}{2}$ for each element of $\mathcal{D}((-\Delta_\alpha)^{s/2})$ we retrieve a notion of a *regular part* in $H^s(\mathbb{R}^3)$ and a *singular part* proportional to the Green's function $\mathsf{G}_\lambda$ defined in (1.4), thus carrying a local $|x|^{-1}$ singularity. This is in complete analogy to what happens with the operator domain $\mathcal{D}(-\Delta_\alpha)$ and the form domain $\mathcal{D}[-\Delta_\alpha]$ – see (1.6) and (1.8). In particular, when $s > \frac{3}{2}$ the singular part is also continuous, and its evaluation at $x=0$ provides the proportionality constant in front of the singular part, the very same kind of boundary condition displayed by the elements of $\mathcal{D}(-\Delta_\alpha)$.

THEOREM 5.1.1. *Let $\alpha \geqslant 0$, $\lambda > 0$, and $s \in (0,2)$. The following holds.*

(i) *If $s \in (0, \frac{1}{2})$, then*

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;=\; H^s(\mathbb{R}^3)\,. \tag{5.2}$$

(ii) *If $s \in (\frac{1}{2}, \frac{3}{2})$, then*

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;=\; H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\}\,, \tag{5.3}$$

*where $\mathsf{G}_\lambda$ is the function* (1.4).

(iii) *If $s \in (\frac{3}{2}, 2)$, then*

$$\begin{aligned} &\mathcal{D}((-\Delta_\alpha)^{s/2}) \;= \\ &\;=\; \Big\{ g \in L^2(\mathbb{R}^3) \,\Big|\, g = F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, \mathsf{G}_\lambda \text{ with } F_\lambda \in H^s(\mathbb{R}^3) \Big\}. \end{aligned} \tag{5.4}$$

Separating the three regimes above, two different transitions occur. When $s$ decreases from larger values, the first transition arises at $s = \frac{3}{2}$, namely the level of $H^s$-regularity at which continuity is lost. Correspondingly, the elements in $\mathcal{D}((-\Delta_\alpha)^{3/4})$ still decompose into a regular $H^{\frac{3}{2}}$-part plus a multiple of $\mathsf{G}_\lambda$ (singular part), and the decomposition is still of the form $F_\lambda + c_{F_\lambda}\mathsf{G}_\lambda$, except that now $F_\lambda$

cannot be arbitrary in $H^{\frac{3}{2}}(\mathbb{R}^3)$: indeed, $F_\lambda$ has additional properties, among which the fact that its Fourier transform is integrable (a fact that is false for generic $H^{\frac{3}{2}}$-functions), and for such $F_\lambda$'s the constant $c_{F_\lambda}$ has a form that is completely analogous to the constant in (5.4), that is,

$$c_{F_\lambda} \;=\; \frac{1}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F_\lambda}(p)$$

(see (5.86) below). Then, for $s<\frac{3}{2}$, the link between the two components disappears completely.

Decreasing $s$ further, the next transition occurs at $s=\frac{1}{2}$, namely the level of $H^s$-regularity below which the Green's function itself belongs to $H^s(\mathbb{R}^3)$ and it does not necessarily carry the leading singularity any longer. At the transition $s=\frac{1}{2}$, the elements in $\mathcal{D}((-\Delta_\alpha)^{1/4})$ still exhibit a decomposition into a regular $H^{\frac{1}{2}}$-part plus a more singular $H^{\frac{1}{2}^-}$-part, except that $H^{\frac{1}{2}^-}$-singularity is not explicitly expressed in terms of the Green's function $G_\lambda$. Then, for $s<\frac{1}{2}$, only $H^s$-functions form the fractional domain.

We shall discuss these transition points in Propositions 5.6.1 and 5.6.2.

REMARK 5.1.2. It is worth comparing $(-\Delta_\alpha)^{s/2}$ (the fractional power of the singular perturbation of the Laplacian) with $\mathsf{k}_\alpha^{(s/2)}$ (the singular perturbation of the fractional Laplacian, identified in Theorem 1.2.3), in the regime $s\in(\frac{3}{2},2)$. The elements of the domains of both operators split into a regular $H^s$-part plus a singular term, with a local boundary condition constraining the two components; however, in the former case the local singularity is $|x|^{-1}$ for all considered powers, whereas in the latter it is the singularity of the function $\mathsf{G}_{s,\lambda}$ defined by (1.39), namely $|x|^{-(3-s)}$.

Our next result concerns the 'singular' Sobolev norm induced by each fractional power $(-\Delta_\alpha)^{s/2}$ on its domain, in comparison with the corresponding ordinary Sobolev norm of the same order. Recall that $(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}\geqslant\lambda^{s/2}\mathbb{1}$ and hence $g\mapsto\|(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}g\|_2$ defines a norm on $\mathcal{D}((-\Delta_\alpha)^{s/2})$, with respect to which the fractional domain is complete.

THEOREM 5.1.3. *Let $\alpha\geqslant 0$, $\lambda>0$, and $s\in(0,2)$. Denote by $H^s_\alpha(\mathbb{R}^3)$, the 'singular Sobolev space' of fractional order $s$, the Hilbert space $\mathcal{D}((-\Delta_\alpha)^{s/2})$ equipped with the 'fractional singular Sobolev norm'*

$$\|g\|_{H^s_\alpha}\;:=\;\|(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}g\|_2\,,\qquad g\in\mathcal{D}(-\Delta_\alpha)^{s/2}\,. \tag{5.5}$$

*The following holds.*

(i) *If $s\in(0,\frac{1}{2})$, then*

$$\|g\|_{H^s_\alpha}\;\approx\;\|g\|_{H^s}\qquad\forall g\in\mathcal{D}(-\Delta_\alpha)^{s/2}=H^s(\mathbb{R}^3) \tag{5.6}$$

*in the sense of equivalence of norms. The constant in* (5.6) *is bounded, and bounded away from zero, uniformly in $\alpha$.*

(ii) *If $s\in(\frac{1}{2},\frac{3}{2})$ and $g=F+c\,\mathsf{G}_\lambda$ is a generic element in $H^s_\alpha(\mathbb{R}^3)$ according to the decomposition* (5.3)*, then*

$$\|F+c\,\mathsf{G}_\lambda\|_{H^s_\alpha}\;\approx\;\|F\|_{H^s}+(1+\alpha)|c|\,. \tag{5.7}$$

(iii) *If $s\in(\frac{3}{2},2)$ and $g=F_\lambda+\frac{F_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda$ is a generic element in $H^s_\alpha(\mathbb{R}^3)$ according to the decomposition* (5.4)*, then*

$$\Big\|F_\lambda+\tfrac{F_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda\Big\|_{H^s_\alpha}\;\approx\;\|F_\lambda\|_{H^s}\,. \tag{5.8}$$

*The constant in* (5.8) *is bounded, and bounded away from zero, uniformly in* $\alpha$.

It is worth remarking that in the limit $\alpha \to +\infty$ (recall that $\Delta_{\alpha=\infty}$ is the self-adjoint Laplacian on $L^2(\mathbb{R}^3)$) the equivalence of norms (5.7) tends to be lost, consistently with the fact that the function $\mathsf{G}_\lambda$ does not belong to $H^s(\mathbb{R}^3)$. Instead, the norm equivalences (5.6) and (5.8) remain valid in the limit $\alpha \to +\infty$, which is also consistent with the structure of the space $H^s_\alpha(\mathbb{R}^3)$ in those two cases.

Last, we examine the action of $-\Delta_\alpha$ on generic functions of its domain and in particular, when applicable, on the function $\mathsf{G}_\lambda$. We prove a computationally useful expression of $(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\varphi$ in terms of the classical fractional derivative $(-\Delta + \lambda\mathbb{1})^{s/2}\varphi$.

Theorem 5.1.4. *Let* $\alpha \geqslant 0$, $\lambda > 0$, *and* $s \in (0,2)$.

(i) *For each* $\varphi \in L^2(\mathbb{R}^3)$ *one has the distributional identity*

$$
\begin{aligned}
(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}\varphi \; &= \\
&= \; (-\Delta + \lambda\mathbb{1})^{s/2}\varphi - 4\sin\tfrac{s\pi}{2}\int_0^{+\infty}\mathrm{d}t\;\frac{t^{s/2}\,\kappa_\varphi(t)}{4\pi\alpha+\sqrt{\lambda+t}}\;\frac{e^{-\sqrt{\lambda+t}\,|x|}}{4\pi|x|}\,,
\end{aligned}
\tag{5.9}
$$

*where*

$$
\kappa_\varphi(t) \;:=\; \int_{\mathbb{R}^3}\mathrm{d}y\;\frac{e^{-\sqrt{\lambda+t}\,|y|}}{4\pi|y|}\,\varphi(y)\,. \tag{5.10}
$$

*When* $\varphi \in \mathcal{D}((-\Delta_\alpha)^{s/2}) \cap H^s(\mathbb{R}^3)$ (5.9) *is understood as an identity between* $L^2$*-functions, whereas when* $\varphi \in \mathcal{D}((-\Delta_\alpha)^{s/2})\backslash H^s(\mathbb{R}^3)$ *the r.h.s. in the* $L^2$*-identity* (5.9) *is understood as the difference of two distributional contributions.*

(ii) *The function* $\mathsf{G}_\lambda$ *defined in* (1.4) *belongs to* $\mathcal{D}((-\Delta_\alpha)^{s/2})$ *if and only if* $s \in (0,\frac{3}{2})$, *in which case*

$$
(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\mathsf{G}_\lambda \in H^{\sigma-}(\mathbb{R}^3)\,, \quad \sigma := \min\{\tfrac{3}{2}-s,\tfrac{1}{2}\}\,, \quad s \in (0,\tfrac{3}{2})\,. \tag{5.11}
$$

*Explicitly,*

$$
(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\mathsf{G}_\lambda \;=\; J_\lambda\,, \tag{5.12}
$$

*where* $J_\lambda$ *is the* $L^2$*-function given by*

$$
\widehat{J_\lambda}(p) \;:=\; \frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\;\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t}\,, \qquad p \in \mathbb{R}^3\,, \tag{5.13}
$$

*and*

$$
\phi(t) \;:=\; \frac{4\pi\alpha+\sqrt{\lambda}}{4\pi\alpha+\sqrt{\lambda+t}}\,, \qquad t \geqslant 0\,. \tag{5.14}
$$

Let us stress that the last Theorem applies to all the considered regimes of $s$, unlike the separation into various regimes made in the previous main Theorems. This way, formula (5.9) has the virtue to provide the explicit additional (distributional, in general) correction in the action of $(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}$ besides the 'classical' contribution $(-\Delta + \lambda\mathbb{1})^{s/2}$. Underlying (5.9), and in fact equivalent to it, we shall discuss in Section 5.2 another key formula for the action of $(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}$, where, in complete analogy to (1.7) we express such an action on $\varphi$ as the classical action $(-\Delta + \lambda\mathbb{1})^{s/2}$ on a suitable regular component of $\varphi$.

As mentioned already, the proofs of Theorems 5.1.1, 5.1.3, and 5.1.4 are deferred to Section 5.6, after developing the preparatory material in Sections 5.2-5.5; the only exception is the integral formula (5.9), that for its technical relevance in our discussion will be proved in advance, at the end of Section 5.2.

## 5.2. Canonical decomposition of the domain of $(-\Delta_\alpha)^{s/2}$

In this Section we present an intermediate technical lemma that is crucial for our analysis and gives a canonical decomposition of the domain of $(-\Delta_\alpha)^{s/2}$ for powers $s \in (0,2)$.

Based on the same argument, we then prove the integral formula (5.9) and hence part (i) of Theorem 5.1.4.

PROPOSITION 5.2.1. *Fix $\alpha \geqslant 0$ and $\lambda > 0$. Let $s \in (0,2)$ and $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$. Then*

$$g \;=\; f_g + h_g \tag{5.15}$$

*where $f_g \in H^s(\mathbb{R}^3)$ is given by*

$$f_g \;:=\; (-\Delta + \lambda\mathbb{1})^{-s/2}(-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g \tag{5.16}$$

*and $h_g \in L^2(\mathbb{R}^3)$ is given by*

$$h_g(x) \;:=\; 4\sin\tfrac{s\pi}{2}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-s/2}\,c_g(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{e^{-\sqrt{\lambda+t}\,|x|}}{4\pi|x|}\,, \tag{5.17}$$

*having set*

$$c_g(t) \;:=\; \int_{\mathbb{R}^3}\mathrm{d}y\,\frac{e^{-\sqrt{\lambda+t}\,|y|}}{4\pi|y|}\,((-\Delta_\alpha+\lambda\mathbb{1})^{s/2}g)(y)\,. \tag{5.18}$$

*When $g$ runs in $\mathcal{D}((-\Delta_\alpha)^{s/2})$ then the corresponding component $f_g$ in the decomposition* (5.15) *spans the whole $H^s(\mathbb{R}^3)$. In terms of this decomposition,*

$$(-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g \;=\; (-\Delta + \lambda\mathbb{1})^{s/2} f_g\,. \tag{5.19}$$

PROOF. (5.19) follows from (5.16), so the proof consists of showing that (5.16) and (5.17) give (5.15). Our argument is based on the identity

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;=\; \mathcal{D}((-\Delta_\alpha+\lambda\mathbb{1})^{s/2}) \;=\; (-\Delta_\alpha+\lambda\mathbb{1})^{-s/2}L^2(\mathbb{R}^3)\,, \tag{5.20}$$

which follows from the spectral theorem, owing to $-\Delta_\alpha \geqslant \mathbb{O}$, and on the integral identity

$$x^{s/2} \;=\; \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-1}\,\frac{x}{t+x}\,,\qquad x\geqslant 0\,,\quad s\in(0,2)\,. \tag{5.21}$$

By the functional calculus of $-\Delta_\alpha$, (5.21) gives

$$\begin{aligned}
(-\Delta_\alpha&+\lambda\mathbb{1})^{-s/2}\\
&= \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-1}\,(-\Delta_\alpha+\lambda\mathbb{1})^{-1}(t+(-\Delta_\alpha+\lambda\mathbb{1})^{-1})^{-1}\\
&= \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-2}\,(-\Delta_\alpha+(\lambda+t^{-1})\mathbb{1})^{-1}
\end{aligned}$$

and by means of the resolvent formula (1.10) and of (5.21) again one finds

$$\begin{aligned}
(-\Delta_\alpha+\lambda\mathbb{1})^{-s/2} \;&=\; \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-2}\,(-\Delta+(\lambda+t^{-1})\mathbb{1})^{-1}\\
&\qquad + \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-2}\Big(\alpha+\frac{\sqrt{\lambda+t^{-1}}}{4\pi}\Big)^{-1}|\mathsf{G}_{\lambda+t^{-1}}\rangle\langle\overline{\mathsf{G}_{\lambda+t^{-1}}}|\\
&= (-\Delta+\lambda\mathbb{1})^{-s/2}+\\
&\qquad + \frac{\sin s\frac{\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{s/2-2}\Big(\alpha+\frac{\sqrt{\lambda+t^{-1}}}{4\pi}\Big)^{-1}|\mathsf{G}_{\lambda+t^{-1}}\rangle\langle\overline{\mathsf{G}_{\lambda+t^{-1}}}|\,.
\end{aligned} \tag{5.22}$$

Let now $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$: applying the operator identity (5.22) to the $L^2$-function $(-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g$ gives $g$ itself in the l.h.s. and two summands on the r.h.s., the first of which is precisely $f_g$ defined in (5.16), whereas the second is

$$\begin{aligned}
&\frac{\sin s\frac{\pi}{2}}{\pi} \int_0^{+\infty} \mathrm{d}t\, t^{s/2-2} \Big(\alpha + \frac{\sqrt{\lambda + t^{-1}}}{4\pi}\Big)^{-1} \mathsf{G}_{\lambda+t^{-1}} \langle \overline{\mathsf{G}_{\lambda+t^{-1}}}\,, (-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g \,\rangle \\
&= \; 4 \sin \tfrac{s\pi}{2} \int_0^{+\infty} \mathrm{d}t \; \frac{t^{-s/2}}{4\pi\alpha + \sqrt{\lambda + t}} \; \mathsf{G}_{\lambda+t} \langle \overline{\mathsf{G}_{\lambda+t}}\,, (-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g \,\rangle \; = \; h_g
\end{aligned}$$

defined in (5.17)-(5.18). This proves that $h_g = g - f_g \in L^2(\mathbb{R}^3)$ and yields (5.15).

Not only is $f_g \in H^s(\mathbb{R}^3)$ for given $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$, but also, conversely, given an arbitrary $f \in H^s(\mathbb{R}^3)$ the function $(-\Delta_\alpha + \lambda\mathbb{1})^{-s/2}(-\Delta + \lambda\mathbb{1})^{s/2} f$ clearly belongs to $\mathcal{D}((-\Delta_\alpha)^{s/2})$ and its component $f_g$ is precisely $f$. Thus, $f_g$ does span $H^s(\mathbb{R}^3)$ when $g$ runs in $\mathcal{D}((-\Delta_\alpha)^{s/2})$. $\square$

REMARK 5.2.2. Formula (5.22) shows that, unlike what happens for singular perturbations of the fractional Laplacian, the resolvent of $(-\Delta_\alpha)^{s/2}$ is not a finite-rank perturbations of the resolvent of the free fractional Laplacian $(-\Delta)^{s/2}$.

PROOF OF THEOREM 5.1.4(I). We follow the same line of reasoning that has led to Proposition (5.2.1). By (5.21) and the functional calculus of $-\Delta_\alpha$,

$$(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\varphi \;=\; \frac{\sin s\frac{\pi}{2}}{\pi} \int_0^{+\infty} \mathrm{d}t\, t^{s/2-1} \, (-\Delta_\alpha + \lambda\mathbb{1})(-\Delta_\alpha + (\lambda + t)\mathbb{1})^{-1})\varphi \,.$$

Taking the difference between the identity above for generic $\alpha$ and for $\alpha = \infty$ (namely for the operator $-\Delta$ instead of $-\Delta_\alpha$), together with the resolvent formula (1.10), yields

$$\begin{aligned}
&(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\varphi - (-\Delta + \lambda\mathbb{1})^{s/2}\varphi \; = \\
&= \; -\frac{\sin s\frac{\pi}{2}}{\pi} \int_0^{+\infty} \mathrm{d}t\, t^{s/2} \Big( (-\Delta_\alpha + (\lambda + t)\mathbb{1})^{-1})\varphi - (-\Delta + (\lambda + t)\mathbb{1})^{-1})\varphi \Big) \\
&= \; -\frac{\sin s\frac{\pi}{2}}{\pi} \int_0^{+\infty} \mathrm{d}t\, t^{s/2} \Big(\alpha + \frac{\sqrt{\lambda + t}}{4\pi}\Big)^{-1} \frac{e^{-\sqrt{\lambda+t}\,|x|}}{4\pi|x|} \int_{\mathbb{R}^3} \mathrm{d}y \, \frac{e^{-\sqrt{\lambda+t}\,|y|}}{4\pi|y|} \, \varphi(y) \,,
\end{aligned}$$

which leads to (5.9), by means of the definition (5.10). $\square$

## 5.3. Regularity properties

In this Section we discuss the regularity and asymptotic properties of functions of the form $h_g$ that emerge in the the canonical decomposition of Proposition 5.2.1.

Preliminary, we state a Schur-test bound (see [**46**, Corollary A.3]) that we will use systematically for the estimate of the norm of a number of integral operators.

LEMMA 5.3.1. *For given constants $\beta \in (-\frac{1}{2}, \frac{1}{2})$ and $\gamma, \delta > 0$, and a measurable function $f$ on $\mathbb{R}^+$, let*

$$Q_{\beta,\gamma,\delta}(u,v) \;:=\; \frac{u^{(\frac{1}{2}-\beta)\gamma - \frac{1}{2}} \, v^{(\frac{1}{2}+\beta)\delta - \frac{1}{2}}}{u^\gamma + v^\delta} \,, \qquad u, v > 0 \,, \tag{5.23}$$

*and*

$$(Q_{\beta,\gamma,\delta} f)(u) \;:=\; \int_0^{+\infty} \mathrm{d}v \, Q_{\beta,\gamma,\delta}(u,v) \, f(v) \,. \tag{5.24}$$

*Then $Q_{\beta,\gamma,\delta}$ defines a bounded linear map on $L^2(\mathbb{R}^+)$ with norm*

$$\|Q_{\beta,\gamma,\delta}\|_{L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)} \;\leqslant\; \frac{1}{\sqrt{\gamma\delta}} \, \frac{\pi}{\cos \beta\pi} \,. \tag{5.25}$$

For given $\alpha \geqslant 0$, $\lambda > 0$, $s \in (0,2)$, and $f \in H^s(\mathbb{R}^3)$, we define

$$c_f(t) \;:=\; \int_{\mathbb{R}^3} \mathrm{d}y\, \frac{e^{-\sqrt{\lambda+t}\,|y|}}{4\pi|y|}\,((-\Delta+\lambda\mathbb{1})^{s/2}f)(y) \tag{5.26}$$

and

$$h_f(x) \;:=\; 4\sin\tfrac{s\pi}{2}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-s/2}\,c(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{e^{-\sqrt{\lambda+t}\,|x|}}{4\pi|x|}\,. \tag{5.27}$$

Equivalently, in Fourier transform,

$$c_f(t) \;:=\; \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{(p^2+\lambda)^{s/2}}{p^2+\lambda+t}\,\widehat{f}(p) \tag{5.28}$$

and

$$\widehat{h_f}(p) \;:=\; \frac{4\sin\frac{s\pi}{2}}{(2\pi)^{3/2}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-s/2}\,c(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,. \tag{5.29}$$

It is also convenient to introduce the function $w_f$ whose Fourier transform is

$$\widehat{w_f}(p) \;:=\; -\frac{1}{p^2+\lambda}\,\frac{4\sin\frac{s\pi}{2}}{(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{1-\frac{s}{2}}\,c(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,. \tag{5.30}$$

Formally,

$$h_f \;=\; q_f\,\mathsf{G}_\lambda + w_f \tag{5.31}$$

where $\mathsf{G}_\lambda$ is the function (1.4) and

$$q_f \;:=\; 4\sin\tfrac{s\pi}{2}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,. \tag{5.32}$$

LEMMA 5.3.2. *For given $\alpha \geqslant 0$, $\lambda > 0$, $s \in (0,2)$, and $f \in H^s(\mathbb{R}^3)$, the function $c(t)$ defined in* (5.26) *is continuous in $t \in [0,+\infty)$ and satisfies the bounds*

$$|c_f(t)| \;\lesssim\; \|f\|_{H^s}(1+t)^{-\frac{1}{4}} \tag{5.33}$$

*and*

$$\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{1}{2}}|c_f(t)|^2 \;\leqslant\; \tfrac{1}{2}\,\|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2^2 \;\approx\; \|f\|_{H^s}^2\,. \tag{5.34}$$

PROOF. The continuity of $t \mapsto c_f(t)$ is immediately checked by re-writing (5.26) as $c_f(t) = \langle \mathsf{G}_{\lambda+t}, (-\Delta+\lambda\mathbb{1})^{s/2}f\rangle$. From

$$\|\mathsf{G}_{\lambda+t}\|_2 \;=\; (8\pi\sqrt{\lambda+t})^{-\frac{1}{2}} \;\leqslant\; (8\pi\sqrt{\lambda})^{-\frac{1}{2}}\,,$$

a Schwarz inequality yields

$$|c_f(t)| \;\leqslant\; \|\mathsf{G}_{\lambda+t}\|_2\,\|(-\Delta+\lambda\mathbb{1})^{s/2}f\|_2 \;\lesssim\; \|f\|_{H^s}$$

and

$$|c_f(t)| \;\lesssim\; t^{-1/4}\|f\|_{H^s}\,,$$

whence (5.33). Next, we consider the function

$$\eta_\omega(\varrho) \;:=\; \varrho(\varrho^2+\lambda)^{\frac{s}{2}}\widehat{f}(\varrho,\omega)\,, \qquad \varrho\in\mathbb{R}^+\,,\ \omega\in\mathbb{S}^2\,,$$

where we wrote $\widehat{f}(p) = \widehat{f}(\rho,\omega)$ in polar coordinates $p \equiv (\varrho,\omega)$, $\varrho := |p|$, $\omega \in \mathbb{S}^2$. Clearly,

$$\begin{aligned}\int_{\mathbb{S}^2}\mathrm{d}\omega\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;&=\; \int_{\mathbb{S}^2}\mathrm{d}\omega\int_0^{+\infty}\mathrm{d}\varrho\,\varrho^2|(\varrho^2+\lambda)^{\frac{s}{2}}\widehat{f}(\varrho,\omega)|^2\\ &=\; \|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2^2 \;\approx\; \|f\|^2_{H^s(\mathbb{R}^3)}\,,\end{aligned}$$

and we estimate

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{1}{2}}|c_f(t)|^2 \;&=\; 2\int_0^{+\infty}\mathrm{d}t\,|c_f(t^2)|^2 \;=\; \frac{1}{4\pi^3}\int_0^{+\infty}\mathrm{d}t\,\Big|\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{(p^2+\lambda)^{\frac{s}{2}}\widehat{f}(p)}{p^2+\lambda+t^2}\Big|^2\\
&=\;\frac{1}{4\pi^3}\int_0^{+\infty}\mathrm{d}t\,\Big|\int_{\mathbb{S}^2}\mathrm{d}\omega\int_0^{+\infty}\mathrm{d}\varrho\,\frac{\varrho\,\eta_\omega(\varrho)}{\varrho^2+\lambda+t^2}\Big|^2\\
&\leqslant\;\frac{1}{\pi^2}\int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,|(Q\eta_\omega)(t)|^2 \;=\; \frac{1}{\pi^2}\int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\,,\end{aligned}$$

where $\eta\mapsto Q\eta$ is the integral operator on functions on $\mathbb{R}^+$ defined by

$$(Q\eta)(t)\;:=\;\int_0^{+\infty}Q(t,\varrho)\,\eta(\varrho)\,\mathrm{d}\varrho\,,\qquad Q(t,\varrho)\;:=\;\frac{\varrho}{\varrho^2+t^2}\,.$$

We observe that $Q$ has precisely the form of the operator $Q_{\beta,\gamma,\delta}$ defined in (5.23)-(5.24) of Lemma 5.3.1 with $\beta=\frac{1}{4}$, $\gamma=\delta=2$. Then the Schur bound (5.25) yields

$$\|Q\eta\|_2\;\leqslant\;\frac{\pi}{\sqrt{2}}\|\eta\|_2\qquad\forall\eta\in L^2(\mathbb{R}^+,\mathrm{d}\varrho)\,.$$

Therefore,

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{1}{2}}|c_f(t)|^2\;&\leqslant\;\frac{1}{\pi^2}\int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\leqslant\;\frac{1}{2}\int_{\mathbb{S}^2}\mathrm{d}\omega\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\\
&=\;\tfrac{1}{2}\,\|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2^2\;\approx\;\|f\|^2_{H^s}\,,\end{aligned}$$

which gives (5.34). $\square$

Let us now exploit the above information on the behaviour of $c_f(t)$ in order to obtain information about the regularity of the functions $h$ and $w$ defined, respectively, in (5.27) and (5.30). To this aim, we shall make often use of the identity

$$\int_0^{+\infty}\mathrm{d}t\,\frac{t^{a-1}}{R+t}\;=\;\frac{\pi}{\sin a\pi}\,\frac{1}{R^{1-a}}\,,\qquad a\in(0,1)\,,\quad R>0\,,\tag{5.35}$$

whence also the useful limit

$$\lim_{R\to+\infty}\Big(\frac{\pi}{\sin a\pi}\,\frac{1}{R^{1-a}}\Big)^{-1}\int_1^{+\infty}\mathrm{d}t\,\frac{t^{a-1}}{R+t}\;=\;1\,,\qquad a\in(0,1)\,.\tag{5.36}$$

We start with the function $h$ in the regime of small $s$.

Proposition 5.3.3. *For given $\alpha\geqslant0$, $\lambda>0$, $s\in(0,\frac{1}{2}]$, and $f\in H^s(\mathbb{R}^3)$, let $h_f$ be the function defined in* (5.26)-(5.27).

(i) *If $s\in(0,\frac{1}{2})$, then $h_f\in H^s(\mathbb{R}^3)$ with*

$$\|h_f\|_{H^s}\;\lesssim\;\|f\|_{H^s}\,,\qquad s\in(0,\tfrac{1}{2})\,.\tag{5.37}$$

(ii) *If $s=\frac{1}{2}$, then $h_f\in H^{\frac{1}{2}^-}(\mathbb{R}^3)$, but in general $h_f\notin H^{1/2}(\mathbb{R}^3)$.*

Proof. (i) Using (5.29) and setting $\mu_f(t):=t^{-\frac{1}{4}}c_f(t)$, we observe that

$$\begin{aligned}\|h_f\|^2_{H^s}\;&\approx\;\int_{\mathbb{R}^3}\mathrm{d}p\,|(p^2+\lambda)^{\frac{s}{2}}\widehat{h_f}(p)|^2\\
&\approx\;\int_{\mathbb{R}^3}\mathrm{d}p\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{(p^2+\lambda)^{\frac{s}{2}}}{p^2+\lambda+t}\Big|^2\\
&\lesssim\int_0^{+\infty}\mathrm{d}\varrho\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{1}{t^{\frac{1}{4}+\frac{s}{2}}}\,\frac{\varrho(\varrho^2+\lambda)^{\frac{s}{2}}}{\varrho^2+\lambda+t}\,\mu_f(t)\Big|^2\\
&\lesssim\int_0^1\mathrm{d}\varrho\,\varrho^2(\varrho^2+\lambda)^s\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{\mu_f(t)}{t^{\frac{1}{4}+\frac{s}{2}}(\lambda+t)}\Big|^2+\int_1^{+\infty}\mathrm{d}\varrho\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{\varrho^{1+s}}{t^{\frac{1}{4}+\frac{s}{2}}\,(\varrho^2+t)}\,\mu_f(t)\Big|^2\\
&\lesssim\;\|\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}+\|Q\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\,,\end{aligned}$$

the last step following by a Schwarz inequalities and by setting

$$(Q\mu_f)(\varrho) \;:=\; \int_0^{+\infty} \mathrm{d}t\, Q(\varrho,t)\,\mu_f(t)\,, \qquad Q(\varrho,t) \;:=\; \frac{\varrho^{1+s}}{t^{\frac{1}{4}+\frac{s}{2}}\,(\varrho^2+t)}\,.$$

In fact, this defines an integral operator $Q$ on functions on $\mathbb{R}^+$ which has precisely the form of the operator $Q_{\beta,\gamma,\delta}$ defined in (5.23)-(5.24) of Lemma 5.3.1 with $\beta = -\frac{1}{4}-\frac{s}{2}$, $\gamma=2$, $\delta=1$. Then the Schur bound (5.25) yields

$$\|Q\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\leqslant\; \frac{\pi}{\sqrt{2}\,\cos(\frac{\pi}{4}+\frac{s\pi}{2})}\,\|\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\,.$$

This, together with the bound (5.34), gives

$$\|h_f\|^2_{H^s} \;\lesssim\; \|\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)} + \|Q\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\lesssim\; \|\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)} \;\lesssim\; \|f\|^2_{H^s(\mathbb{R}^3)}\,,$$

which completes the proof of (5.37) and of part (i).

(ii) When $s=\frac{1}{2}$, (5.29) reads

$$\widehat{h_f}(p) \;=\; \frac{1}{\pi^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{1}{4}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,.$$

We consider the non-empty case of a non-zero $f\in H^{1/2}(\mathbb{R}^3)$ with positive Fourier transform and hence with non-zero $c_f(t)\geqslant 0$, due to (5.28). Owing to (5.33) and dominated convergence,

$$\frac{1}{\pi^{\frac{3}{2}}}\int_0^{1}\mathrm{d}t\,\frac{t^{-\frac{1}{4}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t} \;\approx\; C_1\frac{1}{p^2+\lambda} \qquad \text{as } |p|\to+\infty$$

with constant

$$C_1 \;:=\; \int_0^1\mathrm{d}t\,\frac{\pi^{-\frac{3}{2}}\,t^{-\frac{1}{4}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}} \;\in\; (0,+\infty)\,,$$

namely a contribution to $h_f$ that is a $H^{\frac{1}{2}^-}$-function not belonging to $H^{\frac{1}{2}}(\mathbb{R}^3)$. The remaining contribution to $h_f$ is given by the integration over $t\in[1,+\infty)$, and it is again a positive function of $p$, which therefore cannot compensate the singularity of the first contribution, i.e., it cannot make $h_f$ more regular than $H^{\frac{1}{2}^-}(\mathbb{R}^3)$. $\square$

Next we show that for given $f\in H^s(\mathbb{R}^3)$ with $s\in(\frac{1}{2},2)$ the corresponding $h_f$ is a $H^{\frac{1}{2}^-}$-function given by the sum of the $H^{\frac{1}{2}^-}$-function $q_f\mathsf{G}_\lambda$, that carries the leading singularity of $h_f$, and the more regular function $w_f\in H^s(\mathbb{R}^3)$. This is seen first discussing $q_f$ and then $w_f$.

For given $\alpha\geqslant 0$, $\lambda>0$, $s\in(\frac{1}{2},2)$, we introduce the $L^2$-function $\Upsilon_\lambda$ whose Fourier transform is given by

$$\widehat{\Upsilon_\lambda}(p) \;:=\; \frac{4\sin\frac{s\pi}{2}}{(2\pi)^{3/2}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-s/2}}{(4\pi\alpha+\sqrt{\lambda+t})(p^2+\lambda+t)}\,. \tag{5.38}$$

LEMMA 5.3.4. *For given $\alpha\geqslant 0$, $\lambda>0$, $s\in(\frac{1}{2},2)$, and $f\in H^s(\mathbb{R}^3)$, the corresponding constant $q_f$ defined in* (5.32) *satisfies*

$$q_f \;=\; \langle\Upsilon_\lambda,(-\Delta+\lambda\mathbb{1})^{s/2}f\rangle\,. \tag{5.39}$$

*In particular,*

$$|q_f| \;\lesssim\; \frac{1}{1+\alpha}\,\|f\|_{H^s} \tag{5.40}$$

*and*

$$q_f=0 \qquad\Leftrightarrow\qquad (-\Delta+\lambda\mathbb{1})^{s/2}f \;\perp\; \Upsilon_\lambda \tag{5.41}$$

*in the sense of $L^2$-orthogonality.*

PROOF. Because of (5.28) and (5.32),

$$\begin{aligned} q_f &= \frac{4\sin\frac{s\pi}{2}}{(2\pi)^{3/2}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-s/2}}{4\pi\alpha+\sqrt{\lambda+t}}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{(p^2+\lambda)^{\frac{s}{2}}\widehat{f}(p)}{p^2+\lambda+t} \\ &= \int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{\Upsilon_\lambda}(p)\,(p^2+\lambda)^{\frac{s}{2}}\widehat{f}(p)\,, \end{aligned}$$

whence (5.39). □

PROPOSITION 5.3.5. *For given $\alpha\geqslant 0$, $\lambda>0$, $s\in(\frac{1}{2},2)$, and $f\in H^s(\mathbb{R}^3)$, the functions $h_f$ and $w_f$ and the constant $q_f$ defined, respectively, in* (5.27), (5.30), *and* (5.32), *satisfy the identity*

$$h_f \;=\; q_f\,\mathsf{G}_\lambda + w_f\,, \tag{5.42}$$

*where $\mathsf{G}_\lambda$ is the function* (1.4). *Moreover, $w_f$ belongs to $H^s(\mathbb{R}^3)$ and*

$$\|(p^2+\lambda)^{\frac{s}{2}}\widehat{w_f}\|_2 \;\leqslant\; \frac{\sqrt{2}\,\sin\frac{s\pi}{2}}{\sin(\frac{s\pi}{2}-\frac{\pi}{4})}\,\|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2\,, \tag{5.43}$$

*whence also*

$$\|w_f\|_{H^s} \;\lesssim\; \|f\|_{H^s}\,. \tag{5.44}$$

PROOF. The decomposition (5.42) is an immediate consequence of the finiteness of $q_f$, namely of the bound (5.40). Using (5.30) and setting $\mu_f(t):=t^{-\frac{1}{4}}c_f(t)$, we observe that

$$\begin{aligned} &\|(p^2+\lambda)^{\frac{s}{2}}\widehat{w_f}\|_2^2 \;= \\ &\quad= \frac{2\sin^2\frac{s\pi}{2}}{\pi^3}\int_{\mathbb{R}^3}\mathrm{d}p\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{t^{1-\frac{s}{2}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{(p^2+\lambda)^{1-\frac{s}{2}}\,(p^2+\lambda+t)}\Big|^2 \\ &\quad\leqslant \frac{8\sin^2\frac{s\pi}{2}}{\pi^2}\int_0^{+\infty}\mathrm{d}\varrho\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{\varrho\,t^{\frac{3}{4}-\frac{s}{2}}\,\mu_f(t)}{(\varrho^2+\lambda)^{1-\frac{s}{2}}\,(\varrho^2+\lambda+t)}\Big|^2 \\ &\quad\leqslant \frac{8\sin^2\frac{s\pi}{2}}{\pi^2}\,\|Q\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\,, \end{aligned}$$

where for convenience we wrote

$$(Q\mu_f)(\varrho) \;:=\; \int_0^{+\infty}\mathrm{d}t\,Q(\varrho,t)\,\mu_f(t)\,,\qquad Q(\varrho,t)\;:=\;\frac{\varrho^{s-1}\,t^{\frac{3}{4}-\frac{s}{2}}}{\varrho^2+t}\,.$$

In fact this defines an integral operator $Q$ on functions on $\mathbb{R}^+$ which has precisely the form of the operator $Q_{\beta,\gamma,\delta}$ defined in (5.23)-(5.24) of Lemma 5.3.1 with $\beta=\frac{3}{4}-\frac{s}{2}$, $\gamma=2$, $\delta=1$. Then the Schur bound (5.25) yields

$$\|Q\|_{L^2(\mathbb{R}^+,\mathrm{d}t)\to L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\leqslant\; \frac{\pi}{\sqrt{2}\,\sin(\frac{s\pi}{2}-\frac{\pi}{4})}\,.$$

Combining the estimates above with (5.34) then yields

$$\begin{aligned} \|(p^2+\lambda)^{\frac{s}{2}}\widehat{w_f}\|_2^2 \;&\leqslant\; \frac{8\sin^2\frac{s\pi}{2}}{\pi^2}\,\|Q\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\leqslant\; \frac{4\sin^2\frac{s\pi}{2}}{\sin^2(\frac{s\pi}{2}-\frac{\pi}{4})}\,\|\mu_f\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)} \\ &\leqslant\; \frac{2\sin^2\frac{s\pi}{2}}{\sin^2(\frac{s\pi}{2}-\frac{\pi}{4})}\,\|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2^2 \end{aligned}$$

which is precisely (5.43). □

For the last noticeable property we want to discuss in this Section, as well as for later purposes, it is useful to highlight a few features, whose proof is elementary and will be omitted, of the function $t\mapsto\phi(t)$, $t\geqslant 0$, introduced in (5.14).

LEMMA 5.3.6. *For given $\alpha \geqslant 0$ and $\lambda > 0$, (5.14) defines a function $\phi \in C^\infty([0,+\infty))$ with*

$$\phi(t) \;=\; \frac{4\pi\alpha+\sqrt{\lambda}}{4\pi\alpha+\sqrt{\lambda+t}} \;=\; 1-\frac{t}{(4\pi\alpha+\sqrt{\lambda+t}\,)(\sqrt{\lambda+t}+\sqrt{\lambda})}\,, \tag{5.45}$$

$$0 \;<\; \phi(t) \;\leqslant\; \phi(0) \;=\; 1\,, \tag{5.46}$$

*and*

$$\phi(t) \;\lesssim\; (1+t)^{-1/2}\,. \tag{5.47}$$

*$\phi$ is strictly monotone decreasing and decays as $t\to+\infty$ with asymptotics*

$$\phi(t) \;=\; \frac{4\pi\alpha+\sqrt{\lambda}}{\sqrt{t}}-\frac{4\pi\alpha(4\pi\alpha+\sqrt{\lambda})}{t}+O(t^{-\frac{3}{2}}) \qquad \textit{as } t\to+\infty\,. \tag{5.48}$$

We turn now to the discussion of a relevant connection between the constant $q_f$ defined in (5.32) and the function

$$F_f \;:=\; f+w_f\,. \tag{5.49}$$

In fact, owing to Proposition 5.3.5, when $f\in H^s(\mathbb{R}^3)$ so is $w_f$, and hence $F_f$ too. When $s>\frac{3}{2}$, a standard Sobolev lemma implies that $F_f$ is continuous. We shall now see that, in this regime of $s$, $F_f(0)$ is a multiple of $q_f$. Significantly, an analogous property survives when $s=\frac{3}{2}$ (see Proposition 5.4.5(ii) in the next Section).

LEMMA 5.3.7. *For given $\alpha\geqslant 0$, $\lambda>0$, $s\in(\frac{3}{2},2)$, and $f\in H^s(\mathbb{R}^3)$, let $w_f$ and $q_f$ be, respectively, the function and the constant defined in* (5.30) *and* (5.32)*, and let $F_f$ be the function* (5.49)*. Then $F_f$ is continuous and*

$$F(0) \;=\; (\alpha+\tfrac{\sqrt{\lambda}}{4\pi})\,q_f\,. \tag{5.50}$$

REMARK 5.3.8. It is worth noticing that (5.50) is consistent also when $s\to 2$. Indeed, when $s=2$ and $f\in H^2(\mathbb{R}^3)$, then $w_f\equiv 0$, owing to (5.30), whence $F_f(0)=f(0)$. On the r.h.s. of (5.50), we re-write $q_f$ given by (5.32) as

$$q_f \;=\; \frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}\,c_f(t)}{\alpha+\frac{\sqrt{\lambda+t}}{4\pi}}\,.$$

As $s\to 2$ the pre-factor in front of the integral vanishes asymptotically as $(1-\frac{s}{2})$, whereas the integral diverges: indeed when $s=2$ we see from (5.28) that $c_f(t)\to f(0)$ as $t\to 0$, therefore when $s\to 2$ the leading (i.e., divergent) part of the integral is given by the integration around $t=0$, i.e.,

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}\,c_f(t)}{\alpha+\frac{\sqrt{\lambda+t}}{4\pi}} \;&\approx\; (\alpha+\tfrac{\sqrt{\lambda}}{4\pi})^{-1}f(0)\int_0^1\mathrm{d}t\,t^{-s/2}\\ &=\; (\alpha+\tfrac{\sqrt{\lambda}}{4\pi})^{-1}(1-\tfrac{s}{2})^{-1}f(0) \qquad \text{as } s\to 2\,.\end{aligned}$$

Thus, $(\alpha+\frac{\sqrt{\lambda}}{4\pi})\,q_f\to F_f(0)$ as $s\to 2$.

PROOF OF LEMMA 5.3.7. We have already argued before stating the Lemma that $F_f$ is continuous.

Since $f\in H^s(\mathbb{R}^3)$ for $s>\frac{3}{2}$, then $\widehat{f}\in L^1(\mathbb{R}^3)$ and

$$\begin{aligned}f(0) \;&=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{f}(p) \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{f}(p)\,\frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}(p^2+\lambda)^{\frac{s}{2}}}{p^2+\lambda+t}\\ &=\; \frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{s}{2}}\,c_f(t)\,,\end{aligned}$$

having used (5.35) in the second identity and (5.28) in the third one.

Also $w_f \in H^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$, owing to Proposition 5.3.5, and hence $\widehat{w_f} \in L^1(\mathbb{R}^3)$; from this fact and from (5.30) one obtains

$$\begin{aligned}
w_f(0) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} \mathrm{d}p\, \widehat{w_f}(p) \\
&= -\frac{4\sin\frac{s\pi}{2}}{(2\pi)^3}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{1-\frac{s}{2}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\int_{\mathbb{R}^3}\frac{\mathrm{d}p}{(p^2+\lambda+t)(p^2+\lambda)} \\
&= -\frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{1-\frac{s}{2}}\,c_f(t)}{(4\pi\alpha+\sqrt{\lambda+t}\,)(\sqrt{\lambda+t}+\sqrt{\lambda})} \\
&= -\frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{s}{2}}\,c_f(t) + \frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,t^{-\frac{s}{2}}\,c_f(t)\,\phi(t)\,,
\end{aligned}$$

where we used (5.45) for $\phi(t)$.

Combining the last two equations, and using (5.45) and (5.32), one obtains

$$\begin{aligned}
F_f(0) &= f(0)+w_f(0) = (4\pi\alpha+\sqrt{\lambda}\,)\,\frac{\sin\frac{s\pi}{2}}{\pi}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{s}{2}}\,c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}} \\
&= (\alpha+\tfrac{\sqrt{\lambda}}{4\pi})\,q_f\,,
\end{aligned}$$

thus proving (5.50). □

## 5.4. Subspaces of $\mathcal{D}((-\Delta_\alpha)^{s/2})$

In this Section we show that in the regime $s \in (0, \frac{3}{2})$ the domain of the fractional operator $(-\Delta_\alpha)^{s/2}$ contains two noticeable subspaces: the one-dimensional span of the Green function $\mathsf{G}_\lambda$ defined in (1.4) and the Sobolev space $H^s(\mathbb{R}^3)$. We also show that in the remaining regime $s \in [\frac{3}{2}, 2)$ none of these spaces is entirely contained in $\mathcal{D}((-\Delta_\alpha)^{s/2})$ – however, there is a proper subspace of $H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\}$ which is part of $\mathcal{D}((-\Delta_\alpha)^{s/2})$.

As a consequence, recalling that $\mathsf{G}_\lambda \in H^{\frac{1}{2}-}(\mathbb{R}^3)$, we will conclude that

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \supset H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\}\,, \qquad s \in [\tfrac{1}{2}, \tfrac{3}{2})\,, \tag{5.51}$$

and

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \supset H^s(\mathbb{R}^3)\,, \qquad s \in (0, \tfrac{1}{2})\,. \tag{5.52}$$

The first two main results of this Section are formulated as follows.

Proposition 5.4.1. *For given $\alpha \geqslant 0$, $\lambda > 0$, and $s \in (0, 2)$, one has*

$$(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\mathsf{G}_\lambda = J_\lambda \tag{5.53}$$

*in the distributional sense, where $J_\lambda$ is the function defined by* (5.13)-(5.14)*. In particular,*

$$\mathsf{G}_\lambda \in \mathcal{D}((-\Delta_\alpha)^{s/2}) \qquad \Leftrightarrow \qquad s \in (0, \tfrac{3}{2})\,, \tag{5.54}$$

*in which case*

$$\|(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}\mathsf{G}_\lambda\|_2 \lesssim 1 + \alpha\,. \tag{5.55}$$

Proposition 5.4.2. *For given $\alpha \geqslant 0$,*

(i) *if $s \in (0, \frac{3}{2})$, then $H^s(\mathbb{R}^3)$ is a subspace of $\mathcal{D}((-\Delta_\alpha)^{s/2})$ and for every $\lambda > 0$ and $F \in H^s(\mathbb{R}^3)$ one has*

$$\|(-\Delta_\alpha + \lambda\mathbb{1})^{s/2}F\|_{L^2} \lesssim \|F\|_{H^s}\,; \tag{5.56}$$

(ii) *if $s \in [\frac{3}{2}, 2)$, then $H^s(\mathbb{R}^3)$ is not a subspace of $\mathcal{D}((-\Delta_\alpha)^{s/2})$.*

The third main result of this Section will be discussed later, see Proposition 5.4.5 below. In order to prove Proposition 5.4.1 we establish the following properties.

LEMMA 5.4.3. *For given $\alpha \geqslant 0$, $\lambda > 0$, and $s \in (0,2)$, the function $J_\lambda$ defined by (5.13)-(5.14) has real and bounded Fourier transform that satisfies*

$$\widehat{J_\lambda}(p) \;=\; \frac{\kappa_s}{(p^2+\lambda)}\,(1+o(1))\,, \qquad 0<s<1\,, \tag{5.57}$$

$$\widehat{J_\lambda}(p) \;=\; \kappa_1\,\frac{\ln(p^2+\lambda+1)}{(p^2+\lambda)}\,(1+o(1))\,, \qquad s=1\,, \tag{5.58}$$

$$\widehat{J_\lambda}(p) \;=\; \frac{\kappa_s}{(p^2+\lambda)^{\frac{3}{2}-\frac{s}{2}}}\,(1+o(1))\,, \qquad 1<s<2\,, \tag{5.59}$$

*as $|p|\to+\infty$, where $\kappa_s>0$ depends only on $s$ (as well as on $\alpha$ and $\lambda$). As a consequence, $J_\lambda$ belongs to $L^2(\mathbb{R}^3)$ if and only if $s\in(0,\frac{3}{2})$. When this is the case,*

$$\|J_\lambda\|_2 \;\lesssim\; 1+\alpha\,, \tag{5.60}$$

*and moreover*

$$J_\lambda \in H^{\sigma-}(\mathbb{R}^3)\,, \qquad \sigma := \min\{\tfrac{3}{2}-s,\tfrac{1}{2}\}\,, \qquad s\in(0,\tfrac{3}{2})\,. \tag{5.61}$$

PROOF. In the case $s\in(0,1)$, owing to (5.46)-(5.47),

$$\kappa_s \;:=\; \frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,t^{\frac{s}{2}-1}\,\phi(t) \;\lesssim\; \int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}}{(1+t)^{\frac{1}{2}}} \;<\; +\infty\,,$$

whence

$$(p^2+\lambda)\,\widehat{J_\lambda}(p) \;=\; \frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,t^{\frac{s}{2}-1}\,\phi(t)\,\frac{p^2+\lambda}{p^2+\lambda+t} \;\xrightarrow{|p|\to+\infty}\; \kappa_s$$

by dominated convergence, which proves (5.57).

In the case $s=1$,

$$\widehat{J_\lambda}(p) \;=\; \frac{1}{\pi(2\pi)^{\frac{3}{2}}}\Big(\int_0^1\mathrm{d}t\,\frac{t^{-\frac{1}{2}}\,\phi(t)}{p^2+\lambda+t}+\int_1^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{1}{2}}\,\phi(t)}{p^2+\lambda+t}\Big)\,.$$

As $|p|\to+\infty$,

$$\int_0^1\mathrm{d}t\,\frac{t^{-\frac{1}{2}}\,\phi(t)}{p^2+\lambda+t} \;\approx\; \frac{\text{const.}}{p^2+\lambda}$$

by (5.46) and dominated convergence, and

$$\int_1^{+\infty}\mathrm{d}t\,\frac{t^{-\frac{1}{2}}\,\phi(t)}{p^2+\lambda+t} \;\approx\; (4\pi\alpha+\sqrt{\lambda}\,)\int_1^{+\infty}\mathrm{d}t\,\frac{t^{-1}}{p^2+\lambda+t} \;=\; (4\pi\alpha+\sqrt{\lambda}\,)\,\frac{\ln(p^2+\lambda+1)}{p^2+\lambda}$$

by (5.48) and dominated convergence, which proves (5.58) with $\kappa_1 := \frac{4\pi\alpha+\sqrt{\lambda}}{\pi(2\pi)^{3/2}}$.

In the case $s\in(1,2)$,

$$\widehat{J_\lambda}(p) \;=\; \frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\Big(\int_0^1\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t}+\int_1^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t}\Big)\,.$$

As $|p|\to+\infty$,

$$\int_0^1\mathrm{d}t\,\frac{t^{-\frac{1}{2}}\,\phi(t)}{p^2+\lambda+t} \;\approx\; \frac{\text{const.}}{p^2+\lambda}$$

by (5.46) and dominated convergence, and

$$
\begin{aligned}
\frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_1^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t} \;&\approx\; \frac{(4\pi\alpha+\sqrt{\lambda}\,)\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_1^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-\frac{3}{2}}}{p^2+\lambda+t} \\
&\approx\; \frac{(4\pi\alpha+\sqrt{\lambda}\,)\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-\frac{3}{2}}}{p^2+\lambda+t} \\
&=\; -\frac{(4\pi\alpha+\sqrt{\lambda}\,)\tan\frac{s\pi}{2}}{(2\pi)^{\frac{3}{2}}}\,\frac{1}{(p^2+\lambda)^{\frac{3}{2}-\frac{s}{2}}}
\end{aligned}
$$

by (5.48), (5.36), and dominated convergence, which proves (5.59) with

$$\kappa_s \;:=\; -(2\pi)^{-\frac{3}{2}}(4\pi\alpha+\sqrt{\lambda}\,)\tan\frac{s\pi}{2} > 0\,.$$

It is clear from the above arguments that in all cases $\widehat{J_\lambda}(p)$ is positive and uniformly bounded. Immediate consequences of the asymptotics (5.57)-(5.58)-(5.59) are the fact that $J_\lambda \in L^2(\mathbb{R}^3)$ if and only if $s\in(0,\frac{3}{2})$ and the gain of regularity (5.61). Then the point-wise bound

$$|\widehat{J_\lambda}(p)| \;\lesssim\; (1+\alpha)\,\frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}}{(p^2+1+t)\sqrt{1+t}} \tag{5.62}$$

yields immediately (5.60). □

We can now prove Proposition 5.4.1.

Proof of Proposition 5.4.1. By formula (5.9) of Theorem 5.1.4(i), re-written in Fourier transform, we have

$$((-\Delta_\alpha+\lambda\mathbb{1})^{\frac{s}{2}}\mathsf{G}_\lambda)\widehat{\ }(p) \;=\; ((-\Delta+\lambda\mathbb{1})^{\frac{s}{2}}\mathsf{G}_\lambda)\widehat{\ }(p) \;+\; \widehat{\mathcal{I}_\lambda}(p)\,,$$

where for convenience we set

$$\widehat{\mathcal{I}_\lambda}(p) \;:=\; -\frac{4\sin\frac{s\pi}{2}}{(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}}\,\kappa_{\mathsf{G}_\lambda}(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,,$$

and $\kappa_{\mathsf{G}_\lambda}$, given by (5.10), is now computed as

$$\kappa_{\mathsf{G}_\lambda}(t) \;=\; \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{1}{(p^2+\lambda+t)(p^2+\lambda)} \;=\; \frac{1}{4\pi}\,\frac{1}{\sqrt{\lambda+t}+\sqrt{\lambda}}\,.$$

(Formula (5.9) is indeed usable here, because it has been already demonstrated, in the end of Section 5.2.) Thus,

$$\widehat{\mathcal{I}_\lambda}(p) \;=\; -\frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}}{p^2+\lambda+t} + \frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t}\,,$$

where $\phi(t)$ is the function already introduced in (5.14) and (5.45). Owing to (5.35),

$$\frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}}{p^2+\lambda+t} \;=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\,\frac{1}{(p^2+\lambda)^{1-\frac{s}{2}}} \;=\; ((-\Delta+\lambda\mathbb{1})^{\frac{s}{2}}\mathsf{G}_\lambda)\widehat{\ }(p)\,,$$

whereas, according to our definition (5.13),

$$\frac{\sin\frac{s\pi}{2}}{\pi(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}\,\phi(t)}{p^2+\lambda+t} \;=\; \widehat{J_\lambda}(p)\,.$$

Therefore, $\widehat{\mathcal{I}_\lambda}(p) = -((-\Delta+\lambda\mathbb{1})^{\frac{s}{2}}\mathsf{G}_\lambda)\widehat{\ }(p)+\widehat{J_\lambda}(p)$, whence

$$(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}\mathsf{G}_\lambda \;=\; J_\lambda\,,$$

that is, the identity (5.53). As proved in Lemma 5.4.3, $J_\lambda\in L^2(\mathbb{R}^3) \Leftrightarrow s\in(0,\frac{3}{2})$: thus, $\mathsf{G}_\lambda\in\mathcal{D}((-\Delta_\alpha)^{s/2}) \Leftrightarrow s\in(0,\frac{3}{2})$, and (5.54) follows. (5.55) is then an immediate consequence of (5.60). □

Let us now pass to the proof of Proposition 5.4.2. First, we establish the following property.

LEMMA 5.4.4. *For given* $\lambda>0$, $s\in(0,\frac{3}{2})$, *and* $F\in H^s(\mathbb{R}^3)$, *let* $\kappa_F(t)$ *be the function defined in* (5.10), *namely*

$$\kappa_F(t)\;=\;\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{\widehat{F}(p)}{p^2+\lambda+t}\,. \tag{5.63}$$

*Then*

$$\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}}\;\lesssim\;\|F\|^2_{H^s}\,. \tag{5.64}$$

PROOF. Passing to polar coordinates $p\equiv(\varrho,\omega)$, $\varrho:=|p|$, $\omega\in\mathbb{S}^2$, $\widehat{F}(p)=\widehat{F}(\rho,\omega)$, we see that the function $\eta_\omega(\varrho):=\varrho(\varrho^2+\lambda)^{\frac{s}{2}}\widehat{F}(\varrho,\omega)$ belongs to $L^2(\mathbb{R}^+,\mathrm{d}\varrho)$ with

$$\int_{\mathbb{S}^2}\mathrm{d}\omega\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\;=\;\int_{\mathbb{S}^2}\mathrm{d}\omega\int_0^{+\infty}\mathrm{d}\varrho\,\varrho^2|(\varrho^2+\lambda)^{\frac{s}{2}}\widehat{F}(\varrho,\omega)|^2\;\approx\;\|F\|^2_{H^s(\mathbb{R}^3)}\,.$$

Moreover,

$$(*)\quad\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}}\;\leqslant\;\int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^{\frac{1}{2}}\varrho^{1-s}\,\eta_\omega(\varrho)}{(t^2+\lambda)^{\frac{1}{4}-\frac{s}{2}}(\varrho^2+\lambda+t^2)}\Big|^2,$$

because

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}}\;&=\;2\int_0^{+\infty}\mathrm{d}t\,\frac{t\,|\kappa_F(t^2)|^2}{(t^2+\lambda)^{\frac{1}{2}-s}}\\&=\;\frac{1}{4\pi^3}\int_0^{+\infty}\mathrm{d}t\,\frac{t}{(t^2+\lambda)^{\frac{1}{2}-s}}\Big|\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{(p^2+\lambda)^{\frac{s}{2}}\widehat{F}(p)}{(p^2+\lambda+t^2)(p^2+\lambda)^{\frac{s}{2}}}\Big|^2\\&=\;\frac{1}{4\pi^3}\int_0^{+\infty}\mathrm{d}t\,\Big|\int_{\mathbb{S}^2}\mathrm{d}\omega\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^{\frac{1}{2}}\varrho^2\,(\varrho^2+\lambda)^{\frac{s}{2}}\widehat{F}(\varrho,\omega)}{(t^2+\lambda)^{\frac{1}{4}-\frac{s}{2}}(\varrho^2+\lambda+t^2)\,(\varrho^2+\lambda)^{\frac{s}{2}}}\Big|^2\\&\leqslant\;\frac{1}{\pi^2}\int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^{\frac{1}{2}}\varrho^{1-s}\,\eta_\omega(\varrho)}{(t^2+\lambda)^{\frac{1}{4}-\frac{s}{2}}(\varrho^2+\lambda+t^2)}\Big|^2.\end{aligned}$$

There are two possible cases: $s\in[0,\frac{1}{2})$ and $s\in[\frac{1}{2},\frac{3}{2})$. In the first case one has $\frac{1}{4}-\frac{s}{2}\in(0,\frac{1}{4}]$, and (*) yields

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}}\;&\leqslant\;\int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^s\varrho^{1-s}}{\varrho^2+\lambda+t^2}\,\eta_\omega(\varrho)\Big|^2\\&\leqslant\;\int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\big|(Q\eta_\omega)(t)\big|^2\;=\;\int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\,,\end{aligned}$$

where $Q$ is the integral operator on functions on $\mathbb{R}^+$ defined by the kernel

$$Q(\varrho,t)\;:=\;\frac{t^s\,\varrho^{1-s}}{\varrho^2+t^2}\,.$$

In fact, $Q$ has precisely the form of the operator $Q_{\beta,\gamma,\delta}$ defined in (5.23)-(5.24) of Lemma 5.3.1 with $\beta=\frac{1}{4}-\frac{s}{2}$, $\gamma=\delta=2$, where in this case $\beta\in(0,\frac{1}{4}]$ and hence it is admissible (the admissibility condition in Lemma 5.3.1 is $\beta\in(-\frac{1}{2},\frac{1}{2})$): then the Schur bound (5.25) yields

$$\|Q\eta_\omega\|_{L^2(\mathbb{R}^+,\mathrm{d}t)}\;\leqslant\;\frac{\pi}{\sqrt{2}\,\cos(\frac{\pi}{4}-\frac{s\pi}{2})}\,\|\eta_\omega\|_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\,.$$

Therefore,

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}} &\leqslant \int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\\ &\lesssim \int_{\mathbb{S}^2}\mathrm{d}\omega\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \approx \|F\|^2_{H^s(\mathbb{R}^3)}\,,\end{aligned}$$

which proves (5.64) in the case $s\in[0,\frac{1}{2})$. In the second case, namely $s\in[\frac{1}{2},\frac{3}{2})$, one has $\frac{s}{2}-\frac{1}{4}\in[0,\frac{1}{2})$, and (*) yields

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}} &\leqslant \int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^{\frac{1}{2}}(t^2+\lambda)^{\frac{s}{2}-\frac{1}{4}}\varrho^{1-s}\,\eta_\omega(\varrho)}{\varrho^2+\lambda+t^2}\Big|^2\\ &\leqslant \int_0^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{(t^2+\lambda)^{\frac{s}{2}}\varrho^{1-s}\,\eta_\omega(\varrho)}{\varrho^2+\lambda+t^2}\Big|^2\\ &\lesssim \int_0^1\mathrm{d}t\,(t^2+\lambda)^s\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{\varrho^{1-s}}{\varrho^2+\lambda}\,\eta_\omega(\varrho)\Big|^2\\ &\qquad+\int_1^{+\infty}\mathrm{d}t\int_{\mathbb{S}^2}\mathrm{d}\omega\,\Big|\int_0^{+\infty}\mathrm{d}\varrho\,\frac{t^s\varrho^{1-s}}{\varrho^2+t^2}\,\eta_\omega(\varrho)\Big|^2\\ &\lesssim \int_{\mathbb{S}^2}\mathrm{d}\omega\,\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}+\int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\,,\end{aligned}$$

the integral operator $Q$ being defined as in the first case. Here $Q$ is of the form $Q_{\beta,\gamma,\delta}$ of(5.23)-(5.24) with $\beta=\frac{1}{4}-\frac{s}{2}$, $\gamma=\delta=2$, where in this case $\beta\in(-\frac{1}{2},0]$ and hence $\beta$ is again admissible: the above inequality and the Schur bound (5.25) then yield

$$\begin{aligned}\int_0^{+\infty}\mathrm{d}t\,\frac{|\kappa_F(t)|^2}{(t+\lambda)^{\frac{1}{2}-s}} &\lesssim \int_{\mathbb{S}^2}\mathrm{d}\omega\,\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}+\int_{\mathbb{S}^2}\mathrm{d}\omega\,\|Q\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}t)}\\ &\lesssim \int_{\mathbb{S}^2}\mathrm{d}\omega\|\eta_\omega\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \approx \|F\|^2_{H^s(\mathbb{R}^3)}\,,\end{aligned}$$

which proves (5.64) also in the case $s\in[\frac{1}{2},\frac{3}{2})$. □

We can now prove Proposition 5.4.2. To this aim, it is convenient to introduce the function $I_F$ whose Fourier transform is given by

$$\widehat{I_F}(p) := -\frac{4\sin\frac{s\pi}{2}}{(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}}\,\kappa_F(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,,\tag{5.65}$$

where

$$\kappa_F(t) := \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\frac{\widehat{F}(p)}{p^2+\lambda+t}\,.\tag{5.66}$$

Proof of Proposition 5.4.2. (i) By formulas (5.9)-(5.10) of Theorem 5.1.4(i), re-written in Fourier transform, we have

$$((-\Delta_\alpha+\lambda\mathbb{1})^{\frac{s}{2}}F)\widehat{\ }(p) = ((-\Delta+\lambda\mathbb{1})^{\frac{s}{2}}F)\widehat{\ }(p) + \widehat{I_F}(p)\,,\tag{5.67}$$

where the function $I_F$ is given by (5.65)-(5.66). By assumption, $(-\Delta+\lambda\mathbb{1})^{\frac{s}{2}}F\in L^2(\mathbb{R}^3)$; therefore, the fact that $F\in\mathcal{D}((-\Delta_\alpha)^{s/2})$ with $\|(-\Delta_\alpha+\lambda\mathbb{1})^{\frac{s}{2}}F\|_2\lesssim\|F\|_{H^s}$ follows at once from (5.67) if one proves that $I_F\in L^2(\mathbb{R}^3)$ with $\|I_F\|_{L^2}\lesssim\|F\|_{H^s}$. To this aim, setting $\mu(t):=(t+\lambda)^{-\frac{1}{4}+\frac{s}{2}}\kappa_F(t)$, we observe that

$$\begin{aligned}\|I_F\|_2^2 &\lesssim \int_{\mathbb{R}^3}\mathrm{d}p\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}}\,\kappa_F(t)}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\Big|^2\\ &\lesssim \int_0^{+\infty}\mathrm{d}\varrho\,\Big|\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{s}{2}}}{(\lambda+t)^{\frac{1}{4}+\frac{s}{2}}}\,\frac{\varrho}{\varrho^2+\lambda+t}\,\mu(t)\Big|^2 \leqslant \|Q\mu\|^2_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)}\,,\end{aligned}$$

where for convenience we wrote

$$(Q\mu)(\varrho) \;:=\; \int_0^{+\infty} \mathrm{d}t\, Q(\varrho,t)\,\mu(t)\,, \qquad Q(\varrho,t) \;:=\; \frac{t^{-\frac{1}{4}}\,\varrho}{\varrho^2+t}\,.$$

In fact, this defines an integral operator $Q$ on functions on $\mathbb{R}^+$ which has precisely the form of the operator $Q_{\beta,\gamma,\delta}$ defined in (5.23)-(5.24) of Lemma 5.3.1 with $\beta = -\frac{1}{4}$, $\gamma = 2$, $\delta = 1$. Then the Schur bound (5.25) yields

$$\|Q\mu\|_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\leqslant\; \pi\,\|\mu\|_{L^2(\mathbb{R}^+,\mathrm{d}t)}\,.$$

Combining the estimates above with (5.64) yields

$$\|I_F\|_2 \;\lesssim\; \|Q\mu\|_{L^2(\mathbb{R}^+,\mathrm{d}\varrho)} \;\lesssim\; \|\mu\|_{L^2(\mathbb{R}^+,\mathrm{d}t)} \;\lesssim\; \|F\|_{H^s}\,,$$

which completes the proof of part (i).

As for part (ii), if for contradiction $H^s(\mathbb{R}^3)$ was a subspace of $\mathcal{D}((-\Delta_\alpha)^{s/2})$, then the canonical decomposition (5.15)/(5.42) $g = f_g + c_{f_g}\mathsf{G}_\lambda + w_{f_g}$ of a generic element $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$ for suitable functions $f_g, w_{f_g} \in H^s(\mathbb{R}^3)$ would imply that $c_{f_g}\mathsf{G}_\lambda = g - f_g - w_{f_g} \in \mathcal{D}((-\Delta_\alpha)^{s/2})$. For those $g$'s with non-zero coefficient $c_g$ this would yield the contradiction that $\mathsf{G}_\lambda$ too belongs to $\mathcal{D}((-\Delta_\alpha)^{s/2})$, which was proved to be false in Proposition 5.4.1. $\square$

We move now to the third main result of this Section. It is formulated for $s \in (\frac{1}{2},2)$, but it is relevant for us in the regime of large $s$, namely $s \in [\frac{3}{2},2)$ (it provides no new information for lower $s$). As seen previously, in the latter regime neither $H^s(\mathbb{R}^3)$ nor $\mathrm{span}\{\mathsf{G}_\lambda\}$ are contained in $\mathcal{D}((-\Delta_\alpha)^{s/2})$. Nevertheless, we can identify a suitable proper subspace of $H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\}$ which is still contained in $\mathcal{D}((-\Delta_\alpha)^{s/2})$, as we shall now show.

To this aim, given $\alpha \geqslant 0$, $\lambda > 0$, and $s \in (\frac{1}{2},2)$, we introduce the subspace $\mathcal{D}_0^{(s)} \subset H^s(\mathbb{R}^3)$ defined by

$$\mathcal{D}_0^{(s)} \;:=\; \left\{ F \in H^s(\mathbb{R}^3) \;\middle|\; \begin{array}{c} F^{(0)} := \dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F}(p) < +\infty \\ I_F + \frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda \in L^2(\mathbb{R}^3) \end{array} \right\}, \tag{5.68}$$

where $I_F$ is the function defined by (5.65)-(5.66) for given $F$, and $J_\lambda$ is the function defined by (5.13)-(5.14).

Proposition 5.4.5. *Let $\alpha \geqslant 0$ and $\lambda > 0$.*

(i) *For $s \in (\frac{1}{2},2)$ one has*

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;\supset\; \Big\{ F + \frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda \;\Big|\; F \in \mathcal{D}_0^{(s)} \Big\}\,, \tag{5.69}$$

*the space $\mathcal{D}_0^{(s)} \subset H^s(\mathbb{R}^3)$ being defined in (5.68). In particular, $\mathcal{D}_0^{(s)}$ contains the Schwarz class $\mathcal{S}(\mathbb{R}^3)$, and*

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;\supset\; \Big\{ F + \frac{F(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda \;\Big|\; F \in \mathcal{S}(\mathbb{R}^3) \Big\}\,. \tag{5.70}$$

(ii) *For $s = \frac{3}{2}$ one has*

$$\mathcal{D}((-\Delta_\alpha)^{3/4}) \;=\; \Big\{ F + \frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda \;\Big|\; F \in \mathcal{D}_0^{(3/2)} \Big\}\,. \tag{5.71}$$

REMARK 5.4.6. Formula (5.71) qualifies the fractional domain in the transition case $s=\frac{3}{2}$ and implies the following interesting corollary: the only linear combinations $F+q\,\mathsf{G}_\lambda$ that it is possible to find in $\mathcal{D}((-\Delta_\alpha)^{3/4})$ for some $H^{\frac{3}{2}}$-function $F$ must satisfy $\int_{\mathbb{R}^3}\widehat{F}(p)\,\mathrm{d}p<+\infty$; as such, $F$ cannot be a generic function in $H^{\frac{3}{2}}(\mathbb{R}^3)$. Such a *loss of genericity* of the $H^{\frac{3}{2}}$-regular component in $\mathcal{D}((-\Delta_\alpha)^{3/4})$ is the distinctive feature of the transition at $s=\frac{3}{2}$, since both below and above this threshold the regular part of an element in the fractional domain $\mathcal{D}((-\Delta_\alpha)^{s/2})$ is indeed a generic $H^s$-function.

PROOF OF PROPOSITION 5.4.5. (i) Let $F\in\mathcal{D}_0^{(s)}$. In particular, $F\in H^s(\mathbb{R}^3)$ and $F^{(0)}$ is finite.

In order to prove (5.69) one needs to show that $(-\Delta_\alpha+\lambda\mathbb{1})^{s/2}(F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda)$ is square integrable. In fact, owing to (5.53) and (5.67),

$$(5.72)\quad (-\Delta_\alpha+\lambda\mathbb{1})^{s/2}\Big(F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda\Big)\;=\;(-\Delta+\lambda\mathbb{1})^{s/2}F+I_F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda\,,$$

which indeed belongs to $L^2(\mathbb{R}^3)$ because so do $(-\Delta+\lambda\mathbb{1})^{s/2}F$ and $I_F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda$, as a consequence of the fact that $F$ belongs to the space $\mathcal{D}_0^{(s)}$.

Next, in order to prove (5.70) we combine (5.13)-(5.14) and (5.65)-(5.66) so as to get

$$(5.73)\quad \begin{aligned}\widehat{I_F}(p)+&\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\widehat{J_\lambda}(p)\\ &=\;\frac{4\sin\frac{s\pi}{2}}{(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\!\mathrm{d}t\,\frac{t^{\frac{s}{2}-1}(F^{(0)}-t\,\kappa_F(t))}{4\pi\alpha+\sqrt{\lambda+t}}\,\frac{1}{p^2+\lambda+t}\,.\end{aligned}$$

When $F\in\mathcal{S}(\mathbb{R}^3)$ the finiteness of $F^{(0)}=F(0)$ is obvious, and

$$(5.74)\qquad |F^{(0)}-t\,\kappa_F(t)|\;=\;\Big|\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F}(p)\,\frac{p^2+\lambda}{p^2+\lambda+t}\Big|\;\lesssim\;\frac{\mathrm{const}(F)}{1+t}\,,$$

whence

$$(5.75)\qquad \Big|\widehat{I_F}(p)+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\widehat{J_\lambda}(p)\Big|\;\lesssim\;\frac{\mathrm{const}(F)}{p^2+\lambda}\,.$$

This shows that $I_F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda\in L^2(\mathbb{R}^3)$ whenever $F\in\mathcal{S}(\mathbb{R}^3)$, thus concluding that $\mathcal{S}(\mathbb{R}^3)\subset\mathcal{D}_0^{(s)}$.

(ii) One has to prove the opposite inclusion than (5.69) in the special case $s=\frac{3}{2}$. Let $g\in\mathcal{D}((-\Delta_\alpha)^{3/4})$. Necessarily $g=F_{f_g}+q_{f_g}\,\mathsf{G}_\lambda$ for functions $f_g,w_{f_g},F_{f_g}\in H^{\frac{3}{2}}(\mathbb{R}^3)$ with $F_{f_g}=f_g+w_{f_g}$ and for a constant $q_{f_g}\in\mathbb{C}$, as prescribed by the canonical decomposition (5.15)/(5.42). Let us suppress the index '$g$' in the following.

Now, we claim that

$$\text{(i)}\qquad F_f^{(0)}\;=\;\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F_f}(p)<+\infty\qquad\text{and}\qquad q_f\;=\;\frac{F_f^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,.$$

From this claim we deduce that $F_f+\frac{F_f^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda=g\in\mathcal{D}((-\Delta_\alpha)^{3/4})$; as a consequence, (5.72) implies that $I_{F_f}+\frac{F_f^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda\in L^2(\mathbb{R}^3)$. This completes the proof, because the finiteness of $F_f^{(0)}$ and the square-integrability of $I_{F_f} + \frac{F_f^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\, J_\lambda$ amount to $F_f \in \mathcal{D}_0^{(3/2)}$, and $g$ has the form $F_f + \frac{F_f^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\, \mathsf{G}_\lambda$.

Let us therefore establish (i). To this aim, we mimic the proof of Lemma 5.3.7: in that case we had $s > \frac{3}{2}$, which made the manipulation of all the indefinite integrals harmless; now, instead, $s = \frac{3}{2}$ and a truncation scheme is needed. Moreover, thanks to the linearity, let us assume, non restrictively, that $\widehat{f}(p) > 0$, and hence also $c_f(t) > 0$ and $-\widehat{w_f}(p) > 0$, as follows from (5.28) and (5.30).

First of all,

$$\text{(ii)}\qquad \begin{aligned} F_f^{(0)} &= \lim_{R\to+\infty} \int_{|p|<R} \widehat{F_f}(p)\,\mathrm{d}p \\ &= \lim_{R\to+\infty} \Big( \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \widehat{f}(p)\,\mathrm{d}p + \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \widehat{w_f}(p)\,\mathrm{d}p \Big)\,. \end{aligned}$$

In general, each integral in the r.h.s. above is in divergent as $R \to +\infty$, and we want to show that a compensation among them cancels this possible divergence.

By inserting into the first integrand in the r.h.s. of (ii) the quantity

$$1 \;=\; \frac{1}{\pi\sqrt{2}} \int_0^{+\infty} \mathrm{d}t\, \frac{t^{-\frac{3}{4}}(p^2+\lambda)^{\frac{3}{4}}}{p^2+\lambda+t}$$

(see (5.35)), it is immediately checked that dominated convergence and exchange of the truncated integration over $t$ and $p$ apply, so one has

$$\text{(iii)}\qquad \begin{aligned} &\frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \widehat{f}(p)\,\mathrm{d}p \;= \\ &\quad= \frac{1}{\pi\sqrt{2}\,(2\pi)^{\frac{3}{2}}} \lim_{T\to+\infty} \int_{|p|<R} \mathrm{d}p\, \widehat{f}(p) \int_0^T \mathrm{d}t\, \frac{t^{-\frac{3}{4}}(p^2+\lambda)^{\frac{3}{4}}}{p^2+\lambda+t} \\ &\quad= \frac{1}{\pi\sqrt{2}} \lim_{T\to+\infty} \int_0^T \mathrm{d}t\, t^{-\frac{3}{4}} \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \mathrm{d}p\, \frac{(p^2+\lambda)^{\frac{3}{4}} \widehat{f}(p)}{p^2+\lambda+t} \\ &\quad= \frac{1}{\pi\sqrt{2}} \lim_{T\to+\infty} \int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_{R,f}(t)\,, \end{aligned}$$

where for convenience we denoted by

$$c_{R,f}(t) \;:=\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \mathrm{d}p\, \frac{(p^2+\lambda)^{\frac{3}{4}} \widehat{f}(p)}{p^2+\lambda+t}$$

the finite-momentum truncation of the function $c_f(t)$ defined in (5.28).

An analogous use of dominated convergence and exchange of integration, using (5.30), yields

$$\begin{aligned} &\frac{1}{(2\pi)^{\frac{3}{2}}} \int_{|p|<R} \widehat{w_f}(p)\,\mathrm{d}p \;= \\ &\quad= -\frac{2\sqrt{2}}{(2\pi)^3} \int_{|p|<R} \mathrm{d}p \int_0^{+\infty} \mathrm{d}t\, \frac{t^{\frac{1}{4}} c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\, \frac{1}{(p^2+\lambda)(p^2+\lambda+t)} \\ &\quad= -\frac{2\sqrt{2}}{(2\pi)^3} \lim_{T\to+\infty} \int_{|p|<R} \mathrm{d}p \int_0^T \mathrm{d}t\, \frac{t^{\frac{1}{4}} c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\, \frac{1}{(p^2+\lambda)(p^2+\lambda+t)} \\ &\quad= -\frac{2\sqrt{2}}{(2\pi)^3} \lim_{T\to+\infty} \int_0^T \mathrm{d}t\, \frac{t^{\frac{1}{4}} c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}} \int_{|p|<R} \frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}\,. \end{aligned}$$

It is convenient to re-arrange the r.h.s. above as

$$
\text{(iv)}\qquad
\begin{aligned}
\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{|p|<R}\widehat{w_f}(p)\,\mathrm{d}p \;=\; & \\
= \; \frac{1}{\pi\sqrt{2}}\lim_{T\to+\infty}\int_0^T & \mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\phi(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \\
& -\frac{1}{\pi\sqrt{2}}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \\
& -\frac{2\sqrt{2}}{(2\pi)^3}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\,\frac{t^{\frac{1}{4}}c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\;\times \\
& \qquad \times\Big(\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}-\frac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\frac{R}{\sqrt{\lambda+t}}\Big)\,,
\end{aligned}
$$

where we inserted the function $\phi(t)$ defined in (5.14)/(5.45).

Plugging (iii) and (iv) into (ii),

$$
\text{(v)}\qquad
\begin{aligned}
F_f^{(0)} \;=\; \lim_{R\to+\infty}\Big\{ & \frac{1}{\pi\sqrt{2}}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\phi(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \\
& +\frac{1}{\pi\sqrt{2}}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\,\Big(c_{R,f}(t)-c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)\Big) \\
& -\frac{2\sqrt{2}}{(2\pi)^3}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\,\frac{t^{\frac{1}{4}}c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\;\times \\
& \qquad \times\Big(\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}-\frac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\frac{R}{\sqrt{\lambda+t}}\Big)\Big\}\,.
\end{aligned}
$$

The first term in the r.h.s. of (v) can be thought of as an integration over $t\in\mathbb{R}$ of the function $t\mapsto \mathbf{1}_{\{t\in[0,T]\}}t^{-\frac{3}{4}}\,c_f(t)\,\phi(t)(\frac{2}{\pi}\arctan\frac{R}{\sqrt{\lambda+t}})$. Recalling that $c_f(t)\lesssim(1+t)^{-\frac{1}{4}}$ (see (5.33)), $\phi(t)\lesssim(1+t)^{-\frac{1}{2}}$ (see (5.47)), and $\frac{2}{\pi}\arctan\frac{R}{\sqrt{\lambda+t}}<1$, we see that dominated convergence applies twice and

$$
\text{(vi)}\qquad
\begin{aligned}
& \frac{1}{\pi\sqrt{2}}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\phi(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \\
& \qquad \xrightarrow{T\to+\infty}\;\frac{1}{\pi\sqrt{2}}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\phi(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \\
& \qquad \xrightarrow{R\to+\infty}\;\frac{1}{\pi\sqrt{2}}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\phi(t)\;= \\
& \qquad\qquad\qquad =\;\frac{1}{\pi\sqrt{2}}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\,\frac{4\pi\alpha+\sqrt{\lambda}}{4\pi\alpha+\sqrt{\lambda+t}} \\
& \qquad\qquad\qquad =\;(\alpha+\tfrac{\sqrt{\lambda}}{4\pi})\,q_f\,,
\end{aligned}
$$

having used (5.45) and (5.32) in the last two steps.

From (v) and (vi) we find

$$
\begin{aligned}
F_f^{(0)} \;=\; & q_f\,(\alpha+\tfrac{\sqrt{\lambda}}{4\pi})+ \\
& +\frac{1}{\pi\sqrt{2}}\lim_{R\to+\infty}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\,\Big(c_{R,f}(t)-c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)\Big) \\
& -\frac{2\sqrt{2}}{(2\pi)^3}\lim_{R\to+\infty}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\,\frac{t^{\frac{1}{4}}c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\;\times \\
& \qquad \times\Big(\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}-\frac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\frac{R}{\sqrt{\lambda+t}}\Big)\Big\}\,,
\end{aligned}
$$

which implies (i) as long as one proves that

$$\text{(vii)} \qquad \lim_{R\to+\infty}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\Big(c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)-c_{R,f}(t)\Big) \;=\; 0$$

and

$$\text{(viii)} \qquad \begin{aligned} &\lim_{R\to+\infty}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\,\frac{t^{\frac{1}{4}}c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\;\times \\ &\qquad\times\Big(\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}-\frac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\frac{R}{\sqrt{\lambda+t}}\Big)\Big\} \;=\; 0\,. \end{aligned}$$

Last, let us establish (vii) and (viii), thus completing the proof. One has

$$\begin{aligned} \int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_{R,f}(t) \;&=\; \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\int_{|p|<R}\mathrm{d}p\,\frac{(p^2+\lambda)^{\frac{3}{4}}\widehat{f}(p)}{p^2+\lambda+t} \\ \xrightarrow{T\to+\infty}\; \frac{1}{(2\pi)^{\frac{3}{2}}}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\int_{|p|<R}\mathrm{d}p\,\frac{(p^2+\lambda)^{\frac{3}{4}}\widehat{f}(p)}{p^2+\lambda+t} \;&=\; \int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\, c_{R,f}(t) \end{aligned}$$

by dominated convergence, thanks to the uniform-in-$T$ summable majorant function $t\mapsto \mathrm{const}(R)\cdot t^{-\frac{3}{4}}(\lambda+t)^{-1}$. One also has

$$\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big) \;\xrightarrow{T\to+\infty}\; \int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\, c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)$$

by dominated convergence, thanks to the bound $\arctan(\frac{R}{\sqrt{\lambda+t}})\leqslant\frac{R}{\sqrt{\lambda+t}}$ and hence to the uniform-in-$T$ summable majorant function $t\mapsto R\,t^{-\frac{3}{4}}(\lambda+t)^{-\frac{1}{2}}$. Thus,

$$\begin{aligned} &\lim_{R\to+\infty}\lim_{T\to+\infty}\int_0^T \mathrm{d}t\, t^{-\frac{3}{4}}\Big(c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)-c_{R,f}(t)\Big) \;= \\ &\qquad=\;\lim_{R\to+\infty}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\Big(c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)-c_{R,f}(t)\Big) \end{aligned}$$

Now, since $c_{R,f}(t)\nearrow c_f(t)$ and $\frac{2}{\pi}\arctan\frac{R}{\sqrt{\lambda+t}}\nearrow 1$ as $R\to+\infty$, the functions

$$t\mapsto t^{-\frac{3}{4}}\Big(c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)-c_{R,f}(t)\Big)$$

form a decreasing-in-$R$ net of summable functions, whose point-wise limit as $R\to+\infty$ is the null function. Therefore, by monotone convergence,

$$\lim_{R\to+\infty}\int_0^{+\infty}\mathrm{d}t\, t^{-\frac{3}{4}}\Big(c_f(t)\big(\tfrac{2}{\pi}\arctan\tfrac{R}{\sqrt{\lambda+t}}\big)-c_{R,f}(t)\Big) \;=\; 0$$

and (vii) is proved.

Concerning (viii), with analogous bounds as above one takes the limit $T\to+\infty$ based on dominated convergence. In order to take the limit $R\to+\infty$ in the resulting quantity

$$\int_0^{+\infty}\mathrm{d}t\,\frac{t^{\frac{1}{4}}c_f(t)}{4\pi\alpha+\sqrt{\lambda+t}}\Big(\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}-\frac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\frac{R}{\sqrt{\lambda+t}}\Big)$$

one observes that

$$\begin{aligned} &\Big(\tfrac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\tfrac{R}{\sqrt{\lambda+t}}-\textstyle\int_{|p|<R}\frac{\mathrm{d}p}{(p^2+\lambda)(p^2+\lambda+t)}\Big) \;= \\ &\qquad=\;\tfrac{4\pi}{\sqrt{\lambda+t}+\sqrt{\lambda}}\arctan\tfrac{R}{\sqrt{\lambda+t}}-\tfrac{1}{t}\Big(\sqrt{\lambda+t}\,\arctan\tfrac{R}{\sqrt{\lambda+t}}-\sqrt{\lambda}\,\arctan\tfrac{R}{\sqrt{\lambda}}\Big) \\ &\qquad=\;\frac{\sqrt{\lambda}}{t}\Big(\arctan\tfrac{R}{\sqrt{\lambda}}-\arctan\tfrac{R}{\sqrt{\lambda+t}}\Big)\;\leqslant\;\frac{\pi\sqrt{\lambda}}{t}\,, \end{aligned}$$

which shows that the integrand vanishes point-wise in $t$ as $R \to +\infty$ and is bounded by a uniformly-in-$R$ integrable function: then dominated convergence applies and (viii) is also proved. □

## 5.5. Fractional maps

In this Section we revisit part of the results of Sections 5.2-5.4 relative to the regime $s \in (\frac{1}{2}, 2)$ in terms of certain linear maps which it is very natural to introduce and which provide a more compact formulation.

For $s \in (\frac{1}{2}, 2)$, we define the linear maps

$$\mathcal{R}_s : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)\,, \qquad \mathcal{R}_s f \;:=\; f + w_f \tag{5.76}$$

and

$$\mathcal{Q}_s : H^s(\mathbb{R}^3) \to \mathbb{C}\,, \qquad \mathcal{Q}_s f \;:=\; q_f\,, \tag{5.77}$$

where $w_f$ is the function defined in (5.30) and $q_f$ is the constant defined in (5.32), for given $\alpha \geqslant 0$ and $\lambda > 0$. Owing to Lemma 5.3.4 and Proposition 5.3.5, both maps are bounded:

$$\|\mathcal{R}_s f\|_{H^s} \;\lesssim\; \|f\|_{H^s}\,, \qquad |\mathcal{Q}_s f| \;\lesssim\; \frac{1}{1+\alpha}\|f\|_{H^s}\,. \tag{5.78}$$

As a consequence of Propositions 5.2.1 and 5.3.5,

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;=\; \{\mathcal{R}_s f + (\mathcal{Q}_s f)\mathsf{G}_\lambda \,|\, f \in H^s(\mathbb{R}^3)\}\,, \tag{5.79}$$

that is, when $f$ spans $H^s(\mathbb{R}^3)$, $\mathcal{R}_s f$ spans all possible regular components and $(\mathcal{Q}_s f)\mathsf{G}_\lambda$ spans all possible singular components of the elements of $\mathcal{D}((-\Delta_\alpha)^{s/2})$.

It is also convenient to write

$$\mathcal{R}_s \;=\; \mathbb{1} - \mathcal{W}_s\,, \qquad \mathcal{W}_s f \;:=\; -w_f\,. \tag{5.80}$$

The linear map $\mathcal{W}_s : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$ is bounded, because of Proposition 5.3.5.

Proposition 5.5.1.

(i) *When $s \in (\frac{1}{2}, \frac{3}{2})$, the maps $\mathcal{R}_s$ and $\mathcal{Q}_s$ are surjective and not injective; moreover, there are functions in* $\ker \mathcal{R}_s$ *that do not belong to* $\ker \mathcal{Q}_s$ *and vice versa.*

(ii) *Explicitly, when $s \in (\frac{1}{2}, \frac{3}{2})$, the non-zero $H^s$-function*

$$f_\star \;:=\; (-\Delta + \lambda\mathbb{1})^{-s/2} J_\lambda\,, \tag{5.81}$$

*where $J_\lambda$ is the function defined in* (5.14)*, satisfies*

$$\mathcal{R}_s f_\star \;=\; 0\,, \qquad \text{and} \qquad \mathcal{Q}_s f_\star \;=\; 1\,. \tag{5.82}$$

(iii) *For any $s \in (\frac{1}{2}, 2)$,*

$$\ker \mathcal{Q}_s \;=\; (-\Delta + \lambda\mathbb{1})^{-s/2}\big(\{\Upsilon_\lambda\}^\perp\big) \tag{5.83}$$

*in the sense of $L^2$-orthogonality, where $\Upsilon_\lambda$ is the function defined in* (5.38).

(iv) *When $s = \frac{3}{2}$, $\mathcal{R}_{3/2}$ is injective and not surjective, whereas $\mathcal{Q}_s$ is surjective and not injective.*

(v) *When $s \in (\frac{3}{2}, 2)$, $\mathcal{R}_s$ is surjective and injective, hence a bijection in $H^s(\mathbb{R}^3)$, whereas $\mathcal{Q}_s$ is surjective and not injective.*

Proof. (i) From (5.79) and from the fact that $H^s(\mathbb{R}^3) \subset \mathcal{D}((-\Delta_\alpha)^{s/2})$ (Proposition 5.4.2(i)) it follows that $\mathcal{R}_s$ is surjective and $\mathcal{Q}_s$ is not injective, and that there exist $f$'s in $H^s(\mathbb{R}^3)$ for which $\mathcal{R}_s f \neq 0$ whereas $\mathcal{Q}_s f = 0$. From (5.79) again and from the fact that $\mathrm{span}\{\mathsf{G}_\lambda\} \subset \mathcal{D}((-\Delta_\alpha)^{s/2})$ it follows that $\mathcal{Q}_s$ is surjective and

$\mathcal{R}_s$ is not injective, and that there exist $f$'s in $H^s(\mathbb{R}^3)$ for which $\mathcal{Q}_s f \neq 0$ whereas $\mathcal{R}_s f = 0$.

(ii) Owing to (5.19),

$$(-\Delta_\alpha + \lambda\mathbb{1})^{-s/2} J_\lambda \;=\; (-\Delta_\alpha + \lambda\mathbb{1})^{-s/2}(-\Delta + \lambda\mathbb{1})^{s/2} f_\star \;\in\; \mathcal{D}((-\Delta_\alpha)^{s/2})\,,$$

whence also, owing to (5.53), as well as to (5.15), (5.16), and (5.19),

$$\mathsf{G}_\lambda \;=\; (-\Delta_\alpha + \lambda\mathbb{1})^{-s/2} J_\lambda \;=\; \mathcal{R}_s f_\star + (\mathcal{Q}_s f_\star)\mathsf{G}_\lambda\,,$$

from which (5.82) follows.

(iii) The identity (5.83) is precisely equation (5.41) proved in Lemma 5.3.4.

(iv)-(v) The surjectivity of $\mathcal{Q}_s$ is obvious, and its non-injectivity is proved in general in part (iii) above.

For the injectivity of $\mathcal{R}_s$ when $s \in [\frac{3}{2}, 2)$ we exploit the fact, encoded in (5.79), that if $f \in H^s(\mathbb{R}^3)$, then $g := \mathcal{R}_s f + (\mathcal{Q}_s f)\mathsf{G}_\lambda$ is an element of $\mathcal{D}((-\Delta_\alpha)^{s/2})$ and (5.19) implies that $f = (-\Delta + \lambda\mathbb{1})^{-s/2}(-\Delta_\alpha + \lambda\mathbb{1})^{s/2} g$. Therefore, if $\mathcal{R}_s f = 0$, then necessarily $\mathcal{Q}_s f = 0$ (for otherwise $\mathsf{G}_\lambda$ would belong to $\mathcal{D}((-\Delta_\alpha)^{s/2})$ for $s \geqslant \frac{3}{2}$, which is forbidden by Proposition 5.4.1), whence also $g = 0$ and then $f = 0$: $\mathcal{R}_s$ is injective.

The lack of surjectivity of $\mathcal{R}_{3/2}$ is a consequence of Proposition 5.4.5(ii), as is evident from comparing the expressions (5.71) and (5.79) for $\mathcal{D}((\Delta_\alpha)^{3/4})$, taking into account that $\mathcal{D}_0^{(3/2)} \subsetneq H^{\frac{3}{2}}(\mathbb{R}^3)$.

When $s \in (\frac{3}{2}, 2)$ one can prove the invertibility of $\mathcal{R}_s = \mathbb{1} - \mathcal{W}_s$ as a bijection on $H^s(\mathbb{R}^3)$ by means of the following argument. The bound (5.43) found in Proposition 5.3.5 in the present notation reads

$$\|(p^2+\lambda)^{\frac{s}{2}}\,\widehat{\mathcal{W}_s f}\|_2 \;\leqslant\; \frac{\sqrt{2}\,\sin\frac{s\pi}{2}}{\sin(\frac{s\pi}{2} - \frac{\pi}{4})}\,\|(p^2+\lambda)^{\frac{s}{2}}\widehat{f}\|_2 \qquad \forall f \in H^s(\mathbb{R}^3)\,.$$

Since

$$s \;\longmapsto\; \frac{\sqrt{2}\,\sin\frac{s\pi}{2}}{\sin(\frac{s\pi}{2} - \frac{\pi}{4})}$$

is continuous and strictly monotone decreasing, attaining the value 1 at $s = \frac{3}{2}$, then for $s \in (\frac{3}{2}, 2)$ the map $\mathcal{F}\mathcal{W}_s\mathcal{F}^{-1}$ (where $\mathcal{F} : L^2(\mathbb{R}^3, \mathrm{d}x) \to L^2(\mathbb{R}^3, \mathrm{d}p)$ is the Fourier transform, inherited also on $H^s(\mathbb{R}^3, \mathrm{d}x)$) is bounded on the space $L^2(\mathbb{R}^3, (p^2+\lambda)^s \mathrm{d}p)$ *with norm strictly smaller than 1.* As a consequence, $\mathcal{F}\mathcal{R}_s\mathcal{F}^{-1} = \mathbb{1} - \mathcal{F}\mathcal{W}_s\mathcal{F}^{-1}$ is a bijection on such space. Using an obvious isomorphism $L^2(\mathbb{R}^3, (p^2+\lambda)^s \mathrm{d}p) \stackrel{\cong}{\longmapsto} L^2(\mathbb{R}^3, (p^2+1)^s \mathrm{d}p) = \mathcal{F}H^s(\mathbb{R}^3, \mathrm{d}x)$, one then concludes that the map $\mathcal{R}_s$ is a bijection on $H^s(\mathbb{R}^3, \mathrm{d}x)$. □

## 5.6. Proofs of the main results and transition behaviours

Proof of Theorem 5.1.1.

(i) Case $s \in (0, \frac{1}{2})$. Let $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$. Owing to Proposition 5.2.1, $g = f_g + h_g$ with $f_g \in H^s(\mathbb{R}^3)$ given by (5.16) and $h_g$ given by (5.17). In Proposition 5.3.3 we established that $h_g \in H^s(\mathbb{R}^3)$ too, therefore $\mathcal{D}((-\Delta_\alpha)^{s/2}) \subset H^s(\mathbb{R}^3)$. Conversely, in Proposition 5.4.2(i) we established that $\mathcal{D}((-\Delta_\alpha)^{s/2}) \supset H^s(\mathbb{R}^3)$. The conclusion is the identity (5.2).

(ii) Case $s \in (\frac{1}{2}, \frac{3}{2})$. Again, owing to Proposition 5.2.1, a generic $g \in \mathcal{D}((-\Delta_\alpha)^{s/2})$ decomposes as $g = f_g + h_g$ with $f_g \in H^s(\mathbb{R}^3)$ given by (5.16) and $h_g$ given by (5.17). In Proposition 5.3.5 we established that $h_g = q_{f_g}\mathsf{G}_\lambda + w_{f_g}$ for some $q_{f_g} \in \mathbb{C}$ and some $w_{f_g} \in H^s(\mathbb{R}^3)$. Therefore, $\mathcal{D}((-\Delta_\alpha)^{s/2}) \subset H^s(\mathbb{R}^3) + \mathrm{span}\{\mathsf{G}_\lambda\}$. Conversely, in Propositions 5.4.1 and 5.4.2(i) we established the opposite inclusion (5.51). The conclusion is the identity (5.3).

(iii) Case $s \in (\frac{3}{2}, 2)$. Owing to Propositions 5.2.1 and 5.3.5, $\mathcal{D}((-\Delta_\alpha)^{s/2})$ consists exactly of elements of the form $f + q_f \mathsf{G}_\lambda + w_f$, obtained by letting $f$ span the whole $H^s(\mathbb{R}^3)$ and by taking $w_f$ and $q_f$ according to (5.30) and (5.32). (With the same argument as in the proof of part (ii), this allows one to deduce again $\mathcal{D}((-\Delta_\alpha)^{s/2}) \subset H^s(\mathbb{R}^3) \dotplus \mathrm{span}\{\mathsf{G}_\lambda\}$, however the latter is now a strict inclusion, as established in Propositions 5.4.1 and 5.4.2(ii).) It follows from Proposition 5.3.5 that $F_f := f + w_f \in H^s(\mathbb{R}^3)$ and it follows from Lemma 5.3.7 that $F_\lambda$ is a continuous function on $\mathbb{R}^3$ with $F_f(0) = (\alpha + \frac{\sqrt{\lambda}}{4\pi})q_f$. Thus,

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;\subset\; \Big\{F + \frac{F(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda \,\Big|\, F \in H^s(\mathbb{R}^3)\Big\}\,.$$

Conversely, we established in Proposition 5.4.5 that

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;\supset\; \Big\{F + \frac{F(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda \,\Big|\, F \in H^s(\mathbb{R}^3)\Big\}\,,$$

because in this regime of $s$ the space $\mathcal{D}_0^{(s)}$ used in Proposition 5.4.5 is the whole $H^s(\mathbb{R}^3)$ and $F^{(0)} = F(0)$. The conclusion is the identity (5.4). Alternatively, in the equivalent language of the fractional maps introduced in Section 5.5, one argues as follows: according to (5.79),

$$\mathcal{D}((-\Delta_\alpha)^{s/2}) \;=\; \{\mathcal{R}_s f + (\mathcal{Q}_s f)\mathsf{G}_\lambda \,|\, f \in H^s(\mathbb{R}^3)\}\,,$$

Lemma 5.3.7 reads

$$\mathcal{Q}_s f \;=\; \frac{(\mathcal{R}_s f)(0)}{\alpha + \frac{\sqrt{\lambda}}{2\pi}}\,,$$

and Proposition 5.5.1(v) establishes that $\mathcal{R}_s : H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)$ is a bijection, which all together gives precisely the representation (5.4) for $\mathcal{D}((-\Delta_\alpha)^{s/2})$. $\square$

Proof of Theorem 5.1.3.
(i) Case $s \in (0, \frac{1}{2})$. The bound $\|g\|_{H^s_\alpha} \gtrsim \|g\|_{H^s}$ was proved in (5.56) of Proposition (5.4.2)(i). As for the opposite bound, Proposition 5.2.1 implies that $g = f_g + h_g$ and $\|g\|_{H^s_\alpha} = \|(-\Delta + \lambda\mathbb{1})^{s/2} f_g\|_2 \approx \|f_g\|_{H^s}$, Proposition 5.3.3 implies that $\|h_g\|_{H^s} \lesssim \|f_g\|_{H^s}$, therefore $\|g\|_{H^s} \lesssim \|f_g\|_{H^s} = \|g\|_{H^s_\alpha}$.

(ii) Case $s \in (\frac{1}{2}, \frac{3}{2})$. By means of the decomposition of Propositions 5.2.1 and 5.3.5, as well as the surjectivity of the map $f \mapsto f + w_f$ on $H^s(\mathbb{R}^3)$ (Proposition 5.5.1(i)), one has $g = F_{f_g} + q_{f_g}\mathsf{G}_\lambda$ with $F_{f_g} = f_g + w_{f_g}$, and $\|g\|_{H^s_\alpha} = \|(-\Delta + \lambda\mathbb{1})^{s/2} f_g\|_2 \approx \|f_g\|_{H^s}$. Combining this norm equivalence with the bounds $\|F_{f_g}\|_{H^s} \lesssim \|f_g\|_{H^s}$ and $(1+\alpha)|q_{f_g}| \lesssim \|f_g\|_{H^s}$ (Lemma 5.3.4 and Proposition 5.3.5, i.e., eq. (5.78)) one has $\|F_f\|_{H^s} + (1+\alpha)|q_{f_g}| \lesssim \|F_f + q_{f_g}\mathsf{G}_\lambda\|_{H^s_\alpha}$. For the opposite inequality we write $\|F_{f_g} + q_{f_g}\mathsf{G}_\lambda\|_{H^s_\alpha} \leqslant \|F_{f_g}\|_{H^s_\alpha} + |q_{f_g}|\|\mathsf{G}_\lambda\|_{H^s_\alpha}$ and we use $\|F_{f_g}\|_{H^s_\alpha} \lesssim \|F_{f_g}\|_{H^s}$ (eq. (5.56) in Proposition 5.4.2) and $\|\mathsf{G}_\lambda\|_{H^s_\alpha} \lesssim (1+\alpha)$ (eq. (5.55) in Proposition 5.4.1), whence the conclusion.

(iii) Case $s \in (\frac{3}{2}, 2)$. Arguing as in part (ii), for $F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda$ one has $F_\lambda = f + w_f = R_s f$, $\frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}} = Q_s f$, and $\big\|F_\lambda + \frac{F_\lambda(0)}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda\big\|_{H^s_\alpha} \approx \|f\|_{H^s}$. Since in the regime $s \in (\frac{3}{2}, 2)$ the map $R_s$ is invertible on $H^s(\mathbb{R}^3)$ (Proposition 5.5.1(v)), and hence also with bounded inverse, then $\|f\|_{H^s} \approx \|R_s f\|_{H^s} = \|F_\lambda\|_{H^s}$, which completes the proof. $\square$

Proof of Theorem 5.1.4.
Part (i) was proved already in the end of Section 5.2. Part (ii) is entirely proved in Proposition 5.4.1 and Lemma 5.4.3. $\square$

The transition cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ are characterised as follows.

PROPOSITION 5.6.1 (Transition case $s=\frac{1}{2}$). *Let $\alpha\geqslant 0$, $\lambda>0$, and $s=\frac{1}{2}$. Then*

$$\mathcal{D}((-\Delta_\alpha)^{1/4}) \;=\; \{f+h_f\,|\,f\in H^{\frac{1}{2}}(\mathbb{R}^3)\}\,, \tag{5.84}$$

*where, for given $f$, $h_f$ is the $H^{\frac{1}{2}^-}$-function defined in* (5.26)-(5.27) *and discussed in Proposition 5.3.3(ii). Moreover,*

$$H^{\frac{1}{2}}(\mathbb{R}^3)\dotplus\mathrm{span}\{\mathsf{G}_\lambda\}\;\subsetneq\;\mathcal{D}((-\Delta_\alpha)^{1/4})\;\subset\;H^{\frac{1}{2}^-}(\mathbb{R}^3)\,. \tag{5.85}$$

PROOF. The first statement is an immediate consequence of the canonical decomposition (5.15) of Proposition 5.2.1 and of the definition (5.26)-(5.27). The inclusion $\mathcal{D}((-\Delta_\alpha)^{1/4})\subset H^{\frac{1}{2}^-}(\mathbb{R}^3)$ of (5.85) follows at once from the decomposition (5.15) and from Proposition 5.3.3(ii), whereas the inclusion $H^{1/2}(\mathbb{R}^3)\dotplus\mathrm{span}\{\mathsf{G}_\lambda\}\subset\mathcal{D}((-\Delta_\alpha)^{1/4})$ is precisely the inclusion (5.51) for $s=\frac{1}{2}$, which follows from Propositions 5.4.1 and 5.4.2(ii). Last, in order to see that the latter inclusion is strict, we observe that in course of the proof of Proposition 5.3.3(ii) certain non-zero functions $f\in H^{1/2}(\mathbb{R}^3)$ were considered for which $\widehat{h_f}(p)\approx\langle p\rangle^{-2}\ln\langle p\rangle$ as $|p|\to+\infty$, which is logarithmically more singular than $\mathsf{G}_\lambda$ and than an $H^{\frac{1}{2}}$-function. □

PROPOSITION 5.6.2 (Transition case $s=\frac{3}{2}$). *Let $\alpha\geqslant 0$, $\lambda>0$, and $s=\frac{3}{2}$. Then*

$$\mathcal{D}((-\Delta_\alpha)^{3/4}) \;=\; \Big\{F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,\mathsf{G}_\lambda\;\Big|\;F\in\mathcal{D}_0^{(3/2)}\Big\}\,, \tag{5.86}$$

*where*

$$\mathcal{D}_0^{(3/2)} \;=\; \left\{F\in H^{\frac{3}{2}}(\mathbb{R}^3)\;\middle|\;\begin{array}{c}F^{(0)}:=\dfrac{1}{(2\pi)^{\frac{3}{2}}}\displaystyle\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F}(p)<+\infty\\ I_F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda\in L^2(\mathbb{R}^3)\end{array}\right\}\subsetneq\;H^{\frac{3}{2}}(\mathbb{R}^3)\,, \tag{5.87}$$

*$I_F$ is the function defined by* (5.65)-(5.66) *for given $F$ and $s=\frac{3}{2}$, and $J_\lambda$ is the function defined by* (5.13)-(5.14) *for $s=\frac{3}{2}$.*

PROOF. An immediate consequence of Proposition 5.4.5. □

REMARK 5.6.3. Let us elaborate further on the two conditions

$$\int_{\mathbb{R}^3}\mathrm{d}p\,\widehat{F}(p)<+\infty \tag{*}$$

and

$$I_F+\frac{F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,J_\lambda\;\in\;L^2(\mathbb{R}^3) \tag{**}$$

that characterise $\mathcal{D}_0^{(3/2)}$ and hence the fractional domain $\mathcal{D}((-\Delta_\alpha)^{3/4})$. The constraint (**) is actually a cancellation condition, as was seen in the proof of Proposition 5.4.5, formulas (5.73)-(5.75), in the special case of $F\in\mathcal{S}(\mathbb{R}^3)$. In general, because of the asymptotics (Lemma 5.4.3)

$$\widehat{J_\lambda}(p)\;=\;\frac{\kappa}{(p^2+\lambda)^{\frac{3}{4}}}+R(p)\qquad\text{as }|p|\to+\infty$$

for some $L^2$-function $R$, one must have

$$\widehat{I_F}(p)=\;-\frac{\kappa F^{(0)}}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\cdot\frac{1}{(p^2+\lambda)^{\frac{3}{4}}}+\widetilde{R}(p)\qquad\text{as }|p|\to+\infty \tag{***}$$

for some $L^2$-function $\widetilde{R}$ in order for a cancellation to occur and then for (**) to hold. One can see that the inverse Fourier transform of $(p^2+\lambda)^{-\frac{3}{4}}$ belongs to the Besov space $B^0_{2,\infty}(\mathbb{R}^3) \supsetneqq L^2(\mathbb{R}^3) = B^0_{2,2}(\mathbb{R}^3)$. Therefore, if $\widehat{I_F}$ satisfies the asymptotic expansion (***), the cancellation property is equivalent to the vanishing of the $B^0_{2,\infty}(\mathbb{R}^3)$-term in

$$I_F + \frac{F^{(0)}}{\alpha + \frac{\sqrt{\lambda}}{4\pi}}\, J_\lambda .$$

Moreover, a closer inspection of the argument in formulas (5.73)-(5.75) shows that condition (**), equivalently (***), follows as soon as one assumes for $\widehat{F}$ a slightly better large-$p$ decay than the one imposed by (*), for example

$$\int_{\mathbb{R}^3} \widehat{F}(p)\langle p\rangle^{\varepsilon}\mathrm{d}p < +\infty$$

for arbitrary $\varepsilon > 0$.

CHAPTER 6

# Singular Hartree equation

In this Chapter we study the singular Hartree equation in three dimension

$$\mathrm{i}\partial_t u \;=\; -\Delta_\alpha u + (w * |u|^2)u\,, \tag{6.1}$$

in the complex-valued unknown $u \equiv u(x,t)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, for a given measurable function $w : \mathbb{R}^3 \to \mathbb{R}$.

In order to avoid non-essential additional discussions, we restrict ourselves once and for all to positive $\alpha$'s. In fact, $-\Delta_\alpha$ is semi-bounded from below for every $\alpha \in \mathbb{R}$, thus shifting it up by a suitable constant one ends up with studying a modification of (6.1) with a trivial linear term that does not affect the solution theory of the equation.

The singular Hartree equation (6.1) can be interpreted as a classical Hartree equation

$$\mathrm{i}\partial_t u \;=\; -\Delta u + Vu + (w * |u|^2)u, \tag{6.2}$$

in which the external potential $V$ models a 'delta'-like impurity at the origin.

Among the several contexts of relevance of (6.2), one is surely the quantum dynamics of large Bose gases, where particles are subject to an external potential $V$ and interact through a two-body potential $w$. When $V$ is locally sufficiently regular, (6.2) emerges as the effective evolution equation, rigorously in the limit of infinitely many particles, of a many-body initial state that is scarcely correlated, say, $\Psi(x_1,\dots,x_N) \sim u_0(x_1)\cdots u_0(x_n)$, whose evolution can be proved to retain the approximate form $\Psi(x_1,\dots,x_N;t) \sim u(x_1,t)\cdots u(x_n,t)$ for some one-body orbital $u \in L^2(\mathbb{R}^d)$ that solves the Hartree equation (6.2) with initial condition $u(x,0) = u_0(x)$. The precise meaning of the control of the many-body wave function is in the sense of one-body reduced density matrices. The limit $N \to +\infty$ is taken with a suitable re-scaling prescription of the many-body Hamiltonian, so as to make the limit non-trivial. In the *mean field* scaling, that models particles paired by an interaction of long range and weak magnitude, the interaction term in the Hamiltonian has the form $N^{-1}\sum_{j<k} w(x_j - x_k)$, and when applied to a wave function of the approximate form $u(x_1)\cdots u(x_n)$ it generates indeed the typical self-interaction term $(w*|u|^2)u$ of (6.2). This scenario is today controlled in a virtually complete class of cases, ranging from bounded to locally singular potentials $w$, and through a multitude of techniques to control the limit (see, e.g., [**20**, Chapter 2] and the references therein).

Irrespectively of the technique to derive the Hartree equation from the many-body linear Schrödinger equation (hierarchy of marginals, Fock space of fluctuations, counting of the condensate particles, and others), one fundamental requirement is that at least for the time interval in which the limit $N \to +\infty$ is monitored the Hartree equation itself is well-posed. In fact, for sufficiently regular $V$, the Cauchy problem for (6.2) has been extensively studied, and nowadays its local and global well-posedness, as well as its long-time behavior are well understood – for the vast literature on the subject, we refer to the monograph [**25**], as well as to the recent work [**80**].

For the singular Hartree equation the picture is much less developed. The material presented here is based on my recent work [**83**], in collaboration with A. Michelangeli and A. Olgiati, where we study the Cauchy problem associated to (6.1)

$$\begin{cases} \mathrm{i}\partial_t u &= -\Delta_\alpha u + (w * |u|^2)u \\ u(0) &= f \in H^s_\alpha(\mathbb{R}^3), \end{cases} \tag{6.3}$$

where $H^s_\alpha(\mathbb{R}^3)$ are the singular Sobolev spaces introduced in Chapter 5. We are going to discuss *local* solution theory both in a regime of low (i.e., $s \in [0, \frac{1}{2})$), intermediate (i.e., $s \in (\frac{1}{2}, \frac{3}{2})$), and high (i.e., $s \in (\frac{3}{2}, 2]$) regularity. Then, exploiting the conservation of the mass and the energy, we are going to obtain a *global* theory in the mass space ($s = 0$) and the energy space ($s = 1$). Our result is the first fundamental step towards a rigorous derivation of (6.1) from the many-body quantum dynamics.

We deal with strong $H^s_\alpha$-solutions of the problem (6.3), meaning, functions $u \in C(I, H^s_\alpha(\mathbb{R}^3))$ for some interval $I \subseteq \mathbb{R}$ with $I \ni 0$, which are fixed points for the *solution map*

$$\Phi(u)(t) := e^{\mathrm{i}t\Delta_\alpha} f - \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha} (w * |u(\tau)|^2)u(\tau)\,\mathrm{d}\tau\,. \tag{6.4}$$

Let us recall the notion of *local* and *global well-posedness* (see [**25**, Section 3.1]).

Definition 6.0.1. We say that the Cauchy problem (6.3) is locally well-posed in $H^s_\alpha(\mathbb{R}^3)$ if the following properties hold:

(i) For every $f \in H^s_\alpha(\mathbb{R}^3)$, there exists a unique strong $H^s_\alpha$-solution $u$ to the equation

$$u(t) = e^{\mathrm{i}t\Delta_\alpha} f - \mathrm{i} \int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha} (w * |u(\tau)|^2)u(\tau)\,\mathrm{d}\tau \tag{6.5}$$

defined on the maximal interval $(-T_*, T^*)$, where $T_*, T^* \in (0, +\infty]$ depend on $f$ only.

(ii) There is the blow-up alternative: if $T^* < +\infty$ (resp., if $T_* < +\infty$), then $\lim_{t\uparrow T^*} \|u(t)\|_{H^s_\alpha} = +\infty$ (resp., $\lim_{t\downarrow T_*} \|u(t)\|_{H^s_\alpha} = +\infty$).

(iii) There is continuous dependence on the initial data: if $f_n \xrightarrow{n\to+\infty} f$ in $H^s_\alpha(\mathbb{R}^3)$, and if $I \subset (-T_*, T^*)$ is a closed interval, then the maximal solution $u_n$ to (6.3) with initial datum $f_n$ is defined on $I$ for $n$ large enough, and satisfies $u_n \xrightarrow{n\to+\infty} u$ in $C(I, H^s_\alpha(\mathbb{R}^3))$.

If $T_* = T^* = +\infty$, we say that the solution is global. If (6.3) is locally well-posed and for every $f \in H^s_\alpha(\mathbb{R}^3)$ the solution is global, we say that (6.3) is globally well-posed in $H^s_\alpha(\mathbb{R}^3)$.

Let us emphasize an important feature of solutions to the integral equation (6.5). As already mentioned in Section 1.1, $-\Delta_\alpha$ diagonalises w.r.t. the canonical angular decomposition (1.17) of $L^2(\mathbb{R}^3)$. In particular, the subspace of $L^2(\mathbb{R}^3)$ of definite rotational symmetry are invariant under the propagator $e^{\mathrm{i}t\Delta_\alpha}$. If both $f$ and $w$ are spherically symmetric, then, the solution $u$ to (6.5) is radial too. This makes the above definitions of strong solutions and well-posedness meaningful also with respect to the spaces

$$H^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3) := H^s_\alpha(\mathbb{R}^3) \cap L^2_{\ell=0}(\mathbb{R}^3)$$

equipped with the $H^s_\alpha$-norm. Part of the solution theory we found is set in such spaces.

We can finally formulate our main results. Let us start with the local theory.

THEOREM 6.0.2 ($L^2$-theory – local well-posedness). *Let $\alpha \geqslant 0$. Let $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ for $\gamma \in [0, \frac{3}{2})$. Then the Cauchy problem* (6.3) *is locally well-posed in $L^2(\mathbb{R}^3)$.*

THEOREM 6.0.3 (Low regularity – local well-posedness). *Let $\alpha \geqslant 0$ and $s \in (0, \frac{1}{2})$. Let $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ for $\gamma \in [0, 2s]$. Then the Cauchy problem* (6.3) *is locally well-posed in $H^s_\alpha(\mathbb{R}^3)$, which in this regime coincides with $H^s(\mathbb{R}^3)$.*

THEOREM 6.0.4 (Intermediate regularity – local well-posedness). *Let $\alpha \geqslant 0$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $w \in W^{s,p}(\mathbb{R}^3)$ for $p \in (2, +\infty)$. Then the Cauchy problem* (6.3) *is locally well-posed in $H^s_\alpha(\mathbb{R}^3)$.*

THEOREM 6.0.5 (High regularity – local well-posedness). *Let $\alpha \geqslant 0$ and $s \in (\frac{3}{2}, 2]$. Let $w \in W^{s,p}(\mathbb{R}^3)$ for $p \in (2, +\infty)$ and spherically symmetric. Then the Cauchy problem* (6.3) *is locally well-posed in $H^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$.*

The transition cases $s = \frac{1}{2}$ and $s = \frac{3}{2}$ are not covered explicitly for the mere reason that the structure of the perturbed Sobolev spaces $H^{1/2}_\alpha(\mathbb{R}^3)$ and $H^{3/2}_\alpha(\mathbb{R}^3)$ is not as clean as that of $H^s_\alpha(\mathbb{R}^3)$ when $s \notin \{\frac{1}{2}, \frac{3}{2}\}$ – see the disussion after Theorem 5.1.1.

Let us remark that for $s > 0$ we have an actual 'continuity' in $s$ of the assumption on $w$ in the three Theorems 6.0.3, 6.0.4, and 6.0.5 above – in the low regularity case our proof does not require any control on derivatives of $w$ and therefore we find it more informative to formulate the assumption in terms of the Lorentz space corresponding to $W^{s,p}(\mathbb{R}^3)$.

Such a 'continuity' is due to the fact that under the hypotheses of Theorems 6.0.3, 6.0.4, and 6.0.5 we can work in a locally-Lipschitz regime of the non-linearity. When instead $s = 0$ we have a 'jump' in the form of an extra range of admissible potentials $w$, which is due to the fact that for the $L^2$-theory we are able to make use of the Strichartz estimates for the singular Laplacian.

Next, we investigate the global theory in the mass and in the energy spaces.

THEOREM 6.0.6 (Global solution theory in the mass space). *Let $\alpha \geqslant 0$, and let $w \in L^\infty(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3)$, or $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ for $\gamma \in (0, \frac{3}{2})$. Then the Cauchy problem* (6.3) *is globally well-posed in $L^2(\mathbb{R}^3)$.*

THEOREM 6.0.7 (Global solution theory in the energy space). *Let $\alpha \geqslant 0$, $w \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^3)$ for $p \in (2, +\infty)$, and $f \in H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$.*

(i) *There exists a constant $C_w > 0$, depending only on $\|w\|_{W^{1,p}}$, such that if $\|f\|_{L^2} \leqslant C_w$, then the unique strong solution in $H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ to* (6.3) *with initial data $f$ is global.*

(ii) *If $w \geqslant 0$, then the Cauchy problem* (6.3) *is globally well-posed in $H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$.*

As stated in the Theorems above, part of the local and of the global solution theory is set for spherically symmetric potentials $w$ and solutions $u$. In a sense, this is the natural solution theory for the singular Hartree equation, for sufficiently high regularity. In particular, the spherical symmetry needed for the high regularity theory is induced naturally by the special structure of the space $H^s_\alpha(\mathbb{R}^3)$ (as opposite to $H^s(\mathbb{R}^3)$, or also to $H^s_\alpha(\mathbb{R}^3)$ for small $s$), where a boundary ('contact') condition holds between regular and singular component of $H^s_\alpha$-functions.

Before concluding this general introduction, it is worth mentioning that the *one-dimensional* version of the non-linear Schrödinger equation with point-like pseudo-potentials is much more deeply investigated and better understood, as compared to the so far virtually unexplored scenario in three dimensions.

In the last dozen years a systematic analysis was carried out of singular non-linear Schrödinger equation in one dimension, mainly with local non-linearity, initially motivated by phenomenological models of short-range obstacles in non-linear transport [**109**]. This includes local and global well-posedness in operator domain and energy space and blow-up phenomena [**3, 1, 2**], weak $L^p$-solutions [**91**], scattering [**16**], solitons [**62, 63**], as well as more recent modifications of the non-linearity [**14**]. None of such works has a three-dimensional counterpart.

The material of this Chapter is organised as follows. In Section 1 we collect some preliminary results which will turn out to be useful for our analysis. The local well-posedness in the low, intermediate, and high regularity will be carried out, respectively, in Section 2, 3 and 4. Last, in Section 5, we discuss the global theory in the mass and in the energy space.

## 6.1. Preparatory materials

We start this Section by recalling some fundamental tools of fractional calculus. One is the following fractional Leibniz rule by Kato and Ponce, also in the generalised version by Gulisashvili and Kon.

THEOREM 6.1.1 (Generalised fractional Leibniz rule, [**69, 58**]). *Suppose that $r \in (1,+\infty)$ and $p_1, p_2, q_1, q_2 \in (1,+\infty]$ with $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j \in \{1,2\}$, and suppose that $s, \mu, \nu \in [0,+\infty)$. Let $d \in \mathbb{N}$, then*

$$\begin{aligned} \|\mathcal{D}^s(fg)\|_{L^p(\mathbb{R}^d)} \;\lesssim\; & \|\mathcal{D}^{s+\mu}f\|_{L^{p_1}(\mathbb{R}^d)}\|\mathcal{D}^{-\mu}g\|_{L^{q_1}(\mathbb{R}^d)} \\ & + \|\mathcal{D}^{-\nu}f\|_{L^{p_2}(\mathbb{R}^d)}\|\mathcal{D}^{s+\nu}g\|_{L^{q_2}(\mathbb{R}^d)} \,, \end{aligned} \tag{6.6}$$

*where $\mathcal{D}^s = (-\Delta)^{\frac{s}{2}}$, the Riesz potential. The same result holds when $\mathcal{D}^s$ is the Bessel potential $(\mathbb{1}-\Delta)^{\frac{s}{2}}$.*

REMARK 6.1.2. As a direct consequence of Mihlin multiplier theorem [**22**, Section 6.1], the estimate (6.6) holds as well for $\mathcal{D}^s = (-\Delta + \lambda\mathbb{1})^{\frac{s}{2}}$ for any $\lambda \geqslant 0$.

We also need a more versatile re-distribution of the derivatives among the two factors $f$ and $g$ in (6.6): the following recent result by Fujiwara, Georgiev, and Ozawa provides a very useful refinement of the fractional Leibniz rule and is based on a careful treatment of the correction term

$$[f,g]_s \;:=\; f\,\mathcal{D}^s g + g\,\mathcal{D}^s f\,. \tag{6.7}$$

THEOREM 6.1.3 (Higher order fractional Leibniz rule, [**44**]). *Suppose that $p, q, r \in (1,+\infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and let $d \in \mathbb{N}$.*

(i) *Let $s_1, s_2 \in [0,1]$ and set $s := s_1 + s_2$. Then*

$$\|\mathcal{D}^s(fg) - [f,g]_s\|_{L^p(\mathbb{R}^d)} \;\lesssim\; \|\mathcal{D}^{s_1}f\|_{L^p(\mathbb{R}^d)}\|\mathcal{D}^{s_2}g\|_{L^q(\mathbb{R}^d)}\,. \tag{6.8}$$

(ii) *Let $s_1 \in [0,2]$, $s_2 \in [0,1]$ be such that $s := s_1 + s_2 \geqslant 1$. Then*

$$\begin{aligned} &\|\mathcal{D}^s(fg) - [f,g]_s + s\,\mathcal{D}^{s-2}(\nabla f\cdot\nabla g) + s\mathcal{D}^{s-2}(g\Delta f) - sg\mathcal{D}^{s-2}\Delta f\|_{L^p(\mathbb{R}^d)} \\ &\qquad\lesssim\; \|\mathcal{D}^{s_1}f\|_{L^p(\mathbb{R}^d)}\|\mathcal{D}^{s_2}g\|_{L^q(\mathbb{R}^d)}\,. \end{aligned} \tag{6.9}$$

*Moreover, since*

$$\mathcal{D}^{s-2}(\nabla f\cdot\nabla g) + \mathcal{D}^{s-2}(g\Delta f) - g\mathcal{D}^{s-2}\Delta f = \mathcal{D}^{s-2}\nabla\cdot(g\nabla f) + g\mathcal{D}^s f$$

*we can rewrite (6.9) in the more compact form*

$$\begin{aligned} &\|\mathcal{D}^s(fg) - f\mathcal{D}^s g + (s-1)g\mathcal{D}^s f + s\,\mathcal{D}^{s-2}\nabla\cdot(g\nabla f)\|_{L^p(\mathbb{R}^d)} \\ &\qquad\lesssim\; \|\mathcal{D}^{s_1}f\|_{L^p(\mathbb{R}^d)}\|\mathcal{D}^{s_2}g\|_{L^q(\mathbb{R}^d)}\,. \end{aligned} \tag{6.10}$$

For the fractional derivative of $|x|^{-1}e^{-\lambda|x|}$ we need, additionally, a point-wise estimate.

LEMMA 6.1.4. *Let $\lambda > 0$, $s \in (0,2]$. We have the estimate*

$$\Big|\mathcal{D}^s\,\frac{e^{-\lambda|x|}}{|x|}\Big| \;\lesssim\; \frac{e^{-\lambda|x|}}{|x|}+\frac{e^{-\lambda|x|}}{|x|^{1+s}}\,,\qquad x\neq 0\,. \tag{6.11}$$

PROOF. Obvious when $s=2$, and a straightforward consequence of the identity

$$\mathcal{D}^s\mathsf{G}_\lambda \;=\; \mathcal{F}^{-1}(|p|^s\widehat{\mathsf{G}_\lambda}(p)) \;=\; -\mathcal{F}^{-1}\Big(\frac{1}{(2\pi)^{\frac{3}{2}}}\,\frac{\lambda}{|p|^2+\lambda}\Big)+\mathcal{F}^{-1}\Big(\frac{1}{(2\pi)^{\frac{3}{2}}}\,\frac{|p|^s+\lambda}{|p|^2+\lambda}\Big)$$

when $s \in (0,2)$, where $\mathsf{G}_\lambda$ is the function (1.4) for some $\lambda>0$. $\square$

Based on the preceding properties, we derive now two useful estimates that we are going to apply systematically in our discussion when $s>\frac{1}{2}$. Let us recall that in this case

$$\|g_1g_2\|_{W^{s,q}} \;\lesssim\; \|g_1\|_{W^{s,q}}\,\|g_2\|_{W^{s,q}} \qquad\qquad (s>\tfrac{1}{2},\ q\in(6,+\infty))\,, \tag{6.12}$$

as follows, for example, from the fractional Leibniz rule (6.6) and Sobolev's embedding $W^{s,q}(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$.

We start with the estimate for the regime $s\in(\frac{1}{2},\frac{3}{2})$, for which we recall Sobolev's embedding

$$\begin{aligned} W^{s,q}(\mathbb{R}^3) \;&\hookrightarrow\; C^{0,\vartheta(s,q)}(\mathbb{R}^3)\\ \vartheta(s,q)\;&:=\;\min\{s-\tfrac{3}{q},1\} \qquad\qquad (s\in(\tfrac{1}{2},\tfrac{3}{2}),\ q\in(6,+\infty)\,. \end{aligned} \tag{6.13}$$

PROPOSITION 6.1.5. *Let $s\in(\frac{1}{2},\frac{3}{2})$ and $h\in W^{s,q}(\mathbb{R}^3)$ for some $q\in(6,+\infty)$. Then*

$$\big\|\mathcal{D}^s\big((h-h(0))\,\mathsf{G}_\lambda\big)\big\|_{L^2} \;\lesssim\; \|h\|_{W^{s,q}(\mathbb{R}^3)}\,, \tag{6.14}$$

PROOF. It is not restrictive to fix $\lambda=1$. We set $\widetilde{G}(x):=|x|^{-1}e^{-\frac{1}{2}|x|}$.

By means of the commutator bound (6.8) we find

$$\begin{aligned} \big\|\mathcal{D}^s\big((h-h(0))\,\mathsf{G}_1\big)\big\|_{L^2} \;&\approx\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\,\widetilde{G}\big)\big\|_{L^2}\\ &\leqslant\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\,\widetilde{G}\big)-\big[e^{-\frac{1}{2}|x|}(h-h(0)),\widetilde{G}\big]_s\big\|_{L^2}\\ &\qquad+\big\|e^{-\frac{1}{2}|x|}(h-h(0))\,\mathcal{D}^s\widetilde{G}\big\|_{L^2}+\big\|\widetilde{G}\,\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2}\\ &\lesssim\; \big\|\mathcal{D}^{s_1}\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^{q_1}}\|\mathcal{D}^{s_2}\widetilde{G}\|_{L^{q_2}}\\ &\qquad+\big\|e^{-\frac{1}{2}|x|}(h-h(0))\,\mathcal{D}^s\widetilde{G}\big\|_{L^2}+\big\|\widetilde{G}\,\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2}\\ &\equiv\; \mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3 \end{aligned} \tag{i}$$

for every $s_1,s_2\in[0,1]$ with $s_1+s_2=s$ and every $q_1,q_2\in[2,+\infty]$ such that $q_1^{-1}+q_2^{-1}=2^{-1}$.

Let us estimate the term $\mathcal{R}_3$. Since, by (6.12) and Sobolev's embedding,

$$\begin{aligned} \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^q} \;&\lesssim\; \big\|(\mathbb{1}-\Delta)^{\frac{s}{2}}\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^q}\\ &\leqslant\; \big\|e^{-\frac{1}{2}|x|}h\big\|_{W^{s,q}}+|h(0)|\,\|e^{-\frac{1}{2}|x|}\|_{W^{s,q}}\\ &\lesssim\; \|h\|_{W^{s,q}}+\|h\|_{L^\infty}\;\lesssim\;\|h\|_{W^{s,q}}\,, \end{aligned} \tag{ii}$$

and since $\|\widetilde{G}\|_{\frac{2q}{q-2}}<+\infty$ because $\frac{2q}{q-2}<3$ for $q>6$, then Holder's inequality yields

$$\mathcal{R}_3 \;\leqslant\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^q}\,\big\|\widetilde{G}\big\|_{L^{\frac{2q}{q-2}}} \;\lesssim\; \|h\|_{W^{s,q}}\,. \tag{iii}$$

Next, let us estimate $\mathcal{R}_1$. When $s\in(\frac{1}{2},1]$, we choose $s_1=s$, $s_2=0$, $q_1=q$, and $q_2=\frac{2q}{q-2}$ and we proceed exactly as for $\mathcal{R}_3$. When instead $s\in(1,\frac{3}{2})$, we choose

$s_1 = 1$, $s_2 = s-1 \in (0, \frac{1}{2})$, $q_1 = (\frac{1}{q} - \frac{s-1}{3})^{-1}$, and $q_2 = (\frac{1}{2} - \frac{1}{q_1})^{-1}$. Then Sobolev's embedding $W^{s,q}(\mathbb{R}^3) \hookrightarrow W^{1,q_1}(\mathbb{R}^3)$ and estimate (ii) above imply

$$\Big\|\mathcal{D}^{s_1}\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\Big\|_{L^{q_1}} \lesssim \Big\|(\mathbb{1}-\Delta)^{\frac{s}{2}}\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\Big\|_{L^q} \lesssim \|h\|_{W^{s,q}},$$

whereas estimate (6.11) and the fact that $q_2 < sq_2 < 3$ for $s \in (1, \frac{3}{2})$ imply

$$\|\mathcal{D}^{s_2}\widetilde{G}\|_{L^{q_2}} \lesssim \big\|e^{-\frac{1}{2}|x|}\big(\tfrac{1}{|x|} + \tfrac{1}{|x|^s}\big)\big\|_{L^{q_2}} < +\infty .$$

Thus, in either case $s \in \big(\frac{1}{2}, 1\big]$ and $s \in \big(1, \frac{3}{2}\big)$,

$$\text{(iv)} \qquad \mathcal{R}_1 \lesssim \|h\|_{W^{s,q}} .$$

Last, let us estimate $\mathcal{R}_2$. Because of the embedding (6.13),

$$\Big\|\frac{e^{-\frac{1}{2}|x|}(h-h(0))}{|x|^{\vartheta(s,q)}}\Big\|_{L^\infty} \lesssim \|h\|_{W^{s,q}} .$$

Moreover, since $\vartheta(s,q) > s - \frac{1}{2}$ and hence $2(1-\vartheta(s,q)) < 2(1+s-\vartheta(s,q)) < 3$ for every $s \in \big(\frac{1}{2}, \frac{3}{2}\big)$, estimate (6.11) implies

$$\big\||x|^{\vartheta(s,q)}\mathcal{D}^s\widetilde{G}\big\|_{L^2} \lesssim \big\|e^{-\frac{1}{2}|x|}\big(|x|^{-(1-\vartheta(s,q))} + |x|^{-(1+s-\vartheta(s,q))}\big)\big\|_{L^2} < +\infty .$$

Thus,

$$\text{(v)} \qquad \mathcal{R}_2 \leqslant \Big\|\frac{e^{-\frac{1}{2}|x|}(h-h(0))}{|x|^{\vartheta(s,q)}}\Big\|_{L^\infty} \big\||x|^{\vartheta(s,q)}\mathcal{D}^s\widetilde{G}\big\|_{L^2} \lesssim \|h\|_{W^{s,q}} .$$

Plugging (iii), (iv), and (v) into (i) the thesis follows. □

We establish now an analogous estimate for the regime $s \in (\frac{3}{2}, 2]$, for which we recall Sobolev's embedding

$$\text{(6.15)} \qquad \begin{aligned} W^{s,q}(\mathbb{R}^3) &\hookrightarrow C^{1,\vartheta(s,q)}(\mathbb{R}^3) \\ \vartheta(s,q) &:= s - 1 - \tfrac{3}{q} \qquad (s \in (\tfrac{3}{2}, 2],\ q \in (6,+\infty)) . \end{aligned}$$

Proposition 6.1.6. *Let $s \in (\frac{3}{2}, 2]$ and $h \in W^{s,q}(\mathbb{R}^3)$ for some $q \in (6, +\infty)$. Assume further that $h$ is spherically symmetric and that $(\nabla h)(0) = 0$. Then*

$$\text{(6.16)} \qquad \big\|\mathcal{D}^s\big((h-h(0))\,\mathsf{G}_\lambda\big)\big\|_{L^2} \lesssim \|h\|_{W^{s,q}(\mathbb{R}^3)} ,$$

*where $\mathsf{G}_\lambda$ is the function (1.4) for some $\lambda > 0$.*

Prior to proving Proposition 6.1.6 let us highlight the following property.

Lemma 6.1.7. *Under the assumptions of Proposition 6.1.6,*

$$\text{(6.17)} \qquad |h(x) - h(0)| \lesssim \|h\|_{W^{s,q}}\, |x|^{1+\vartheta(s,q)} ,$$

*where $\vartheta(s,q) = s - 1 - \frac{3}{q}$, as fixed in (6.15).*

Proof. By assumption, $h(x) = \widetilde{h}(|x|)$ for some even function $\widetilde{h} : \mathbb{R} \to \mathbb{C}$. Owing to the embedding (6.15), $\widetilde{h} \in C^{1,\vartheta(s,q)}(\mathbb{R})$, whence $\widetilde{h}' \in C^{0,\vartheta(s,q)}(\mathbb{R})$. Moreover, $\widetilde{h}'(0) = 0$, because $(\nabla h)(0) = 0$. Therefore,

$$|\widetilde{h}'(\rho)| = |\widetilde{h}'(\rho) - \widetilde{h}'(0)| \lesssim \|h\|_{W^{s,q}}\, |\rho|^{\vartheta(s,q)} .$$

As a consequence,

$$|h(x) - h(0)| \leqslant \int_0^{|x|} |\widetilde{h}'(\rho)|\,\mathrm{d}\rho \lesssim \|h\|_{W^{s,q}} \int_0^{|x|} \rho^{\vartheta(s,q)}\,\mathrm{d}\vartheta \lesssim \|h\|_{W^{s,q}}\, |x|^{1+\vartheta(s,q)} ,$$

which completes the proof. □

Proof of Proposition 6.1.6. It is not restrictive to fix $\lambda = 1$. We set $\widetilde{G}(x) := |x|^{-1}e^{-\frac{1}{2}|x|}$. Let us split

$$
\begin{aligned}
\big\|\mathcal{D}^s\big((h-h(0))\,\mathsf{G}\big)\big\|_{L^2} \;&\approx\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\,\widetilde{G}\big)\big\|_{L^2} \\
&\leqslant\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\,\widetilde{G}\big) - e^{-\frac{1}{2}|x|}(h-h(0))\mathcal{D}^s\widetilde{G} \\
&\quad + (s-1)\widetilde{G}\,\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big) + s\,\mathcal{D}^{s-2}\nabla\cdot\big(\widetilde{G}\,\nabla\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2} \\
&\quad + \big\|e^{-\frac{1}{2}|x|}(h-h(0))\,\mathcal{D}^s\widetilde{G}\big\|_{L^2} + (s-1)\big\|\widetilde{G}\,\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2} \\
&\quad + s\,\big\|\mathcal{D}^{s-2}\nabla\cdot\big(\widetilde{G}\nabla\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2} \\
&\equiv\; \mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4\,.
\end{aligned}
\tag{i}
$$

We estimate the term $\mathcal{R}_1$ by means of the commutator bound (6.10) with $s_1 = s$ and $s_2 = 0$ and of (6.12), namely

$$
\mathcal{R}_1 \;\lesssim\; \Big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\Big\|_{L^q}\|\widetilde{G}\|_{L^{\frac{2q}{q-2}}} \;\lesssim\; \|h\|_{W^{s,q}}
\tag{ii}
$$

(since $\frac{2q}{q-2}\in[2,3)$ and $\|\widetilde{G}\|_{\frac{2q}{q-2}} < +\infty$).

For the estimate of $\mathcal{R}_2$, we observe that $s-\vartheta(s,q) < \frac{3}{2}$ and hence (6.11) implies

$$
\big\|\,|x|^{1+\vartheta(s,q)}\,\mathcal{D}^s\widetilde{G}\big\|_{L^2} \;\lesssim\; \big\|\,e^{-\frac{1}{2}|x|}\big(\,|x|^{\vartheta(s,q)} + |x|^{-(s-\vartheta(s,q))}\big)\big\|_{L^2} \;<\; +\infty\,;
$$

this and the bound (6.17) yield

$$
\mathcal{R}_2 \;\leqslant\; \Big\|\frac{h-h(0)}{|x|^{1+\vartheta(s,q)}}\Big\|_{L^\infty}\,\big\|\,|x|^{1+\vartheta(s,q)}\mathcal{D}^s\widetilde{G}\big\|_{L^2} \;\lesssim\; \|h\|_{W^{s,q}}\,.
\tag{iii}
$$

For $\mathcal{R}_3$, Hölder's inequality, the property (6.12), and Sobolev's embedding yield

$$
\begin{aligned}
\mathcal{R}_3 \;&\lesssim\; \big\|\mathcal{D}^s\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\|_{L^q}\,\|\widetilde{G}\|_{L^{\frac{2q}{q-2}}} \\
&\lesssim\; \|e^{-\frac{1}{2}|x|}(h-h(0))\|_{W^{s,q}} \;\lesssim\; \|h\|_{W^{s,q}} + \|h\|_{L^\infty} \\
&\lesssim\; \|h\|_{W^{s,q}}\,.
\end{aligned}
\tag{iv}
$$

For $\mathcal{R}_4$, one has

$$
\begin{aligned}
\mathcal{R}_4 \;&\lesssim\; \big\|\mathcal{D}^{s-1}\big(\widetilde{G}\,\nabla\big(e^{-\frac{1}{2}|x|}(h-h(0))\big)\big\|_{L^2} \\
&\lesssim\; \big\|\nabla\big(e^{-\frac{1}{2}|x|}(h-h(0)\big)\big\|_{W^{s-1,q}} \;\lesssim\; \|h\|_{W^{s,q}}\,,
\end{aligned}
\tag{v}
$$

where we used the estimate (6.14) in the second inequality (indeed, $s-1\in(\frac{1}{2},1)$), and the property (6.12) and Sobolev's embedding in the last inequality.

Plugging the bounds (ii)-(v) into (i) completes the proof. $\square$

## 6.2. $L^2$-theory and low regularity theory

In this Section we prove Theorems 6.0.2 and 6.0.3. Let us start with Theorem 6.0.3 and then discuss the adaptation for $s = 0$. The proof for $s\in(0,\frac{1}{2})$ is based on a fixed point argument in the complete metric space $(\mathcal{X}_{T,M}, d)$ defined by

$$
\begin{aligned}
\mathcal{X}_{T,M} \;&:=\; \big\{u\in L^\infty([-T,T],H^s_\alpha(\mathbb{R}^3))\,|\,\|u\|_{L^\infty([-T,T],H^s_\alpha(\mathbb{R}^3))}\leqslant M\big\} \\
d(u,v) \;&:=\; \|u-v\|_{L^\infty([-T,T],L^2(\mathbb{R}^3))}
\end{aligned}
\tag{6.18}
$$

for given $T, M > 0$. This is going to be the same space for the contraction argument in the intermediate regularity regime $s\in(\frac{1}{2},\frac{3}{2})$ (Section 6.3), whereas for the high regularity regime $s\in(\frac{3}{2},2)$ (Section 6.4) we are going to only use the spherically symmetric sector of the space (6.18).

Proof of Theorem 6.0.3.

Since by assumption $s \in (0, \frac{1}{2})$, the spaces $H^s(\mathbb{R}^3)$ and $H^s_\alpha(\mathbb{R}^3)$ coincide and their norms are equivalent (Theorem 5.1.1), so we can interchange them in the computations that follow.

From the expression (6.4) for the solution map $\Phi(u)$ one finds

$$\|\Phi(u)\|_{L^\infty H^s_\alpha} \leqslant \|f\|_{H^s_\alpha} + T\,\|(w * |u|^2)u\|_{L^\infty H^s_\alpha}$$

and applying the fractional Leibniz rule (6.6) (Theorem 6.1.1), Hölder's inequality, and Young's inequality one also finds

$$\begin{aligned}
\|(w * |u|^2)u\|_{L^\infty H^s_\alpha} &= \|(w * |u|^2)u\|_{L^\infty H^s} \\
&\lesssim \|w * |u|^2\|_{L^\infty L^\infty}\,\|\mathcal{D}^s u\|_{L^\infty L^2} \\
&\qquad + \|\mathcal{D}^s(w * |u|^2)\|_{L^\infty L^{\frac{6}{\gamma},\infty}}\,\|u\|_{L^\infty L^{\frac{6}{3-\gamma}}} \\
&\lesssim \|w\|_{L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)}\,\|u\|^2_{L^\infty L^{\frac{6}{3-\gamma}}}\,\|\mathcal{D}^s u\|_{L^\infty L^2}\,.
\end{aligned}$$

Sobolev's embedding $H_\alpha(\mathbb{R}^3) = H^s(\mathbb{R}^3) \hookrightarrow H^{\frac{\gamma}{2}}(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3-\gamma}}(\mathbb{R}^3)$ then yields

$$\text{(i)} \qquad \|\Phi(u)\|_{L^\infty H^s_\alpha(\mathbb{R}^3)} \leqslant \|f\|_{H^s_\alpha} + C_1\,T\,\|w\|_{L^{\frac{3}{\gamma},\infty}}\,\|u\|^3_{L^\infty H^s_\alpha}\,.$$

for some constant $C_1 > 0$.

On the other hand, again by Hölder's and Young's inequality,

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L^\infty L^2} &\leqslant T\,\|(w * |u|^2)u - (w * |v|^2)v\|_{L^\infty L^2} \\
&\lesssim T\Big(\|(w * |u|^2)(u-v)\|_{L^\infty L^2} + \big\|\big(w * (|u|^2 - |v|^2)v\big\|_{L^\infty L^2}\Big) \\
&\lesssim T\Big(\|w\|_{L^{\frac{3}{\gamma},\infty}}\,\|u\|^2_{L^\infty L^{\frac{6}{3-\gamma}}}\,\|u-v\|_{L^\infty L^2} \\
&\qquad + \|w\|_{L^{\frac{3}{\gamma},\infty}}\,\|u-v\|_{L^\infty L^2}\,\||u| + |v|\|_{L^\infty L^{\frac{6}{3-\gamma}}}\,\|v|\|_{L^\infty L^{\frac{6}{3-\gamma}}}\Big)\,,
\end{aligned}$$

whence, by the same embedding $H_\alpha(\mathbb{R}^3) \hookrightarrow L^{\frac{6}{3-\gamma}}(\mathbb{R}^3)$ as before,

$$\text{(ii)} \qquad d(\Phi(u), \Phi(v)) \leqslant C_2\,\|w\|_{L^{\frac{3}{\gamma},\infty}}\big(\|u\|^2_{L^\infty H^s_\alpha} + \|v\|^2_{L^\infty H^s_\alpha}\big)\,T\,d(u,v)$$

for some constant $C_2 > 0$.

Thus, choosing $T$ and $M$ such that

$$M = 2\,\|f\|_{H^s_\alpha}\,, \qquad T = \tfrac{1}{4}\big(\max\{C_1, C_2\}\,M^2\|w\|_{L^{\frac{3}{\gamma},\infty}}\big)^{-1}\,,$$

estimate (i) reads $\|\Phi(u)\|_{L^\infty H^s_\alpha} \leqslant M$ and shows that $\Phi$ maps the space $\mathcal{X}_{T,M}$ defined in (6.18) into itself, whereas estimate (ii) reads $d(\Phi(u), \Phi(v)) \leqslant \frac{1}{2} d(u,v)$ and shows that $\Phi$ is a contraction on $\mathcal{X}_{T,M}$. By Banach's fixed point theorem, there exists a unique fixed point $u \in \mathcal{X}_{T,M}$ of $\Phi$ and hence a unique solution $u \in \mathcal{X}_{T,M}$ to (6.5), which is therefore also continuous in time.

Furthermore, by a customary continuation argument we can extend such a solution over a maximal interval for which the blow-up alternative holds true. Also the continuous dependence on the initial data is a direct consequence of the fixed point argument. We omit the standard details, they are part of the well-established theory of semi-linear Schrödinger equations (see [**25**, Section 4.4]). $\square$

We move now to the proof of Theorem 6.0.2. Crucial for this case are the Strichartz estimates of Theorem 4.2.3. To this aim, we modify the contraction space (6.18) to the complete metric space $(\mathcal{Y}_{T,M}, d)$ defined by

$$\begin{aligned}
\mathcal{Y}_{T,M} &:= \left\{ \begin{array}{c} u \in L^\infty([-T,T], L^2(\mathbb{R}^3)) \cap L^{q(\gamma)}([-T,T], L^{p(\gamma)}(\mathbb{R}^3)) \\ \text{s. t. } \|u\|_{L^\infty([-T,T],L^2(\mathbb{R}^3))} + \|u\|_{L^{q(\gamma)}([-T,T],L^{p(\gamma)}(\mathbb{R}^3))} \leqslant M \end{array} \right\} \\
d(u,v) &:= \|u - v\|_{L^\infty([-T,T],L^2(\mathbb{R}^3))} + \|u - v\|_{L^{q(\gamma)}([-T,T],L^{p(\gamma)}(\mathbb{R}^3))}
\end{aligned} \tag{6.19}$$

for given $T, M > 0$, where

$$q(\gamma) := \frac{6}{\gamma}, \qquad p(\gamma) := \frac{18}{9-2\gamma} \tag{6.20}$$

are defined so as to form an admissible pair $(q(\gamma), p(\gamma))$ for $-\Delta_\alpha$, in the sense of (4.19). For the rest of the proof let us drop the explicit dependence on $\gamma$ in $(q, p)$.

PROOF OF THEOREM 6.0.2. Clearly, when $\gamma = 0$ the very same argument used in the proof of Theorem 6.0.3 applies.

When $\gamma \in (0, \frac{3}{2})$ we exploit instead a contraction argument in the modified space (6.19).

One has

$$\begin{aligned}\|\Phi(u)\|_{L^\infty L^2} + \|\Phi(u)\|_{L^q L^p} \;&\leqslant\; \Big\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha} \big((w*|u|^2)u\big)(\tau)\,\mathrm{d}\tau \Big\|_{L^\infty L^2} \\ &+ \Big\| \int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha} \big((w*|u|^2)u\big)(\tau)\,\mathrm{d}\tau \Big\|_{L^q L^p} + \|f\|_{L^2} + \|e^{\mathrm{i}t\Delta_\alpha} f\|_{L^q L^p}\,,\end{aligned}$$

from which, by means of the Strichartz estimates (4.20)-(4.21), one deduces

$$\|\Phi(u)\|_{L^\infty L^2} + \|\Phi(u)\|_{L^q L^p} \;\leqslant\; C\Big(\|f\|_{L^2} + \|(w*|u|^2)u\|_{L^{q'} L^{p'}}\Big)$$

for some constant $C > 0$.

By Hölder's and Young's inequalities,

$$\begin{aligned}\|(w*|u|^2)u\|_{L^{q'} L^{p'}} \;&\leqslant\; \|w*|u|^2\|_{L^{q'} L^{\frac{9}{\gamma}}} \|u\|_{L^\infty L^2} \\ &\lesssim\; \|w\|_{L^{\frac{3}{\gamma},\infty}} \|u\|^2_{L^{2q'} L^r} \|u\|_{L^\infty L^2}\end{aligned}$$

and

$$\|u\|^2_{L^{2q'} L^r} \;\leqslant\; (2T)^{1-\frac{\gamma}{2}} \|u\|^2_{L^q L^r}\,,$$

whence

$$\text{(i)}\quad \|\Phi(u)\|_{L^\infty L^2} + \|\Phi(u)\|_{L^q L^p} \;\leqslant\; C_1\Big(\|f\|_{L^2} + T^{1-\frac{\gamma}{2}} \|w\|_{L^{\frac{3}{\gamma},\infty}} \|u\|^2_{L^q L^r} \|u\|_{L^\infty L^2}\Big)$$

for some constant $C_1 > 0$.

Following the very same scheme, one finds

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L^\infty L^2} + \|\Phi(u) - \Phi(v)\|_{L^q L^p} \;&\lesssim\; \|(w*|u|^2)u - (w*|v|^2)v\|_{L^{q'} L^{p'}} \\ &\leqslant\; \|(w*|u|^2)(u-v)\|_{L^{q'} L^{p'}} + \|w*(|u|^2-|v|^2)v\|_{L^{q'} L^{p'}}\,,\end{aligned}$$

and moreover

$$\|(w*|u|^2)(u-v)\|_{L^{q'} L^{p'}} \;\lesssim\; T^{1-\frac{\gamma}{2}} \|w\|_{L^{\frac{3}{\gamma},\infty}} \|u\|^2_{L^q L^r} \|u-v\|_{L^\infty L^2}$$

and

$$\|w*(|u|^2-|v|^2)v\|_{L^{q'} L^{p'}} \;\lesssim\; T^{1-\frac{\gamma}{2}} \|w\|_{L^{\frac{3}{\gamma},\infty}} \|u-v\|_{L^q L^p} \big(\|u\|_{L^q L^p} + \|v\|_{L^q L^p}\big) \|v\|_{L^\infty L^2}\,.$$

Thus,

$$\text{(ii)}\quad \begin{aligned} d(\Phi(u),\Phi(v)) \;&\leqslant C_2 \|w\|_{L^{\frac{3}{\gamma},\infty}} \big(\|u\|^2_{L^\infty L^2} + \|u\|^2_{L^q L^p} + \|v\|^2_{L^\infty L^2} + \|v\|^2_{L^q L^p}\big) \times \\ &\qquad \times\; T^{1-\frac{\gamma}{2}}\, d(u,v)\,.\end{aligned}$$

Therefore, choosing $T$ and $M$ such that

$$M \;=\; 2C_1\,\|f\|_{L^2}\,, \qquad T \;=\; \big(8\max\{C_1, C_2\} M^2 \|w\|_{L^{\frac{3}{\gamma},\infty}}\big)^{-1+\frac{\gamma}{2}}\,,$$

estimate (i) reads $\|\Phi(u)\|_{L^\infty L^2} + \|\Phi(u)\|_{L^q L^p} \leqslant M$ and shows that $\Phi$ maps $\mathcal{Y}_{T,M}$ into itself, whereas estimate (ii) reads $d(\Phi(u),\Phi(v)) \leqslant \frac{1}{2} d(u,v)$ and shows that $\Phi$ is a contraction on $\mathcal{Y}_{T,M}$.

The thesis then follows by Banach's fixed point theorem through the same arguments outlined in the end of the proof of Theorem 6.0.3. $\square$

For later purposes, let us conclude this Section with the following stability result.

PROPOSITION 6.2.1. *Let* $\alpha \geqslant 0$. *For given* $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$, $\gamma \in [0, \frac{3}{2})$, *and* $f \in L^2(\mathbb{R}^3)$, *let* $u$ *be the unique strong* $L^2$*-solution to the Cauchy problem* (6.3) *in the maximal interval* $(-T_*, T^*)$. *Consider moreover the sequences* $(w_n)_n$ *and* $(f_n)_n$ *of potentials and initial data such that* $w_n \xrightarrow{n\to+\infty} w$ *in* $L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$ *and* $f_n \xrightarrow{n\to+\infty} f$ *in* $L^2(\mathbb{R}^3)$. *Then there exists a time* $T := T(\|w\|_{L^{\frac{3}{\gamma},\infty}}, \|f\|_{L^2}) > 0$, *with* $[-T, T] \subset (-T_*, T^*)$, *such that, for sufficiently large* $n$, *the Cauchy problem* (6.3) *with potential* $w_n$ *and initial data* $f_n$ *admits a unique strong* $L^2$*-solution* $u_n$ *in the interval* $[-T, T]$. *Moreover,*

$$u_n \xrightarrow{n\to+\infty} u \qquad \text{in} \quad C([-T,T], L^2(\mathbb{R}^3))\,. \tag{6.21}$$

PROOF. As a consequence of Theorem 6.0.2, there exist an interval $[-T_n, T_n]$ for some $T_n := T_n(\|w_n\|_{L^{\frac{3}{\gamma},\infty}}, \|f_n\|_{L^2}) > 0$ and a unique $u_n \in C([-T_n, T_n], L^2(\mathbb{R}^3))$ such that

$$u_n(t) \;=\; e^{\mathrm{i}t\Delta_\alpha} f_n - \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}(w_n * |u_n(\tau)|^2)u_n(\tau)\,\mathrm{d}\tau\,. \tag{i}$$

Since $\|w_n\|_{L^{\frac{3}{\gamma},\infty}}$ and $\|f_n\|_{L^2}$ are asymptotically close, respectively, to $\|w\|_{L^{\frac{3}{\gamma},\infty}}$ and $\|f\|_{L^2}$, then there exists $T := T(\|w\|_{L^{\frac{3}{\gamma},\infty}}, \|f\|_{L^2})$ such that $T \leqslant T_n$ eventually in $n$, which means that $u_n$ is defined on $[-T, T]$. Let us set $\phi_n := u - u_n$.

By assumption $u$ solves (6.5), thus subtracting (i) from (6.5) yields

$$\begin{aligned}
\phi_n \;&=\; e^{\mathrm{i}t\Delta_\alpha}(f - f_n) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}\big((w * |u|^2)u - (w_n * |u_n|^2)u_n\big)(\tau)\,\mathrm{d}\tau \\
&=\; e^{\mathrm{i}t\Delta_\alpha}(f - f_n) - \mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}\Big\{\big((w - w_n) * |u|^2\big)u + (w_n * |u|^2)\phi_n \\
&\qquad\qquad + \big(w_n * (\overline{u}\phi_n + \overline{\phi_n}u_n)\big)u_n\Big\}(\tau)\,\mathrm{d}\tau\,.
\end{aligned} \tag{ii}$$

Let us first discuss the case $\gamma = 0$. From (ii) above, using Hölder's and Young's inequality in weak spaces, one has

$$\begin{aligned}
\|\phi_n\|_{L^\infty L^2} \;\lesssim\; &\|f - f_n\|_{L^2} + T\,\|w - w_n\|_{L^\infty}\,\|u\|^3_{L^\infty L^2} \\
&+ T\,\|w_n\|_{L^\infty}\big(\,\|u\|^2_{L^\infty L^2} + \|u_n\|^2_{L^\infty L^2}\,\big)\,\|\phi_n\|_{L^\infty L^2}\,.
\end{aligned}$$

Since $\|w_n\|_{L^\infty}$ and $\|u_n\|_{L^\infty L^2}$ are bounded uniformly in $n$, then the above inequality implies, decreasing further $T$ if needed,

$$\|\phi_n\|_{L^\infty L^2} \;\lesssim\; \|f - f_n\|_{L^2} + \|w - w_n\|_{L^{\frac{3}{\gamma},\infty}} \;\xrightarrow{n\to+\infty}\; 0\,,$$

which proves the proposition in the case $\gamma = 0$.

Let now $\gamma \in (0, \frac{3}{2})$. In this case, owing to Theorem 6.0.2, $u, u_n$ belong to $L^q([-T, T], L^p(\mathbb{R}^3))$, where $(q, p) = (\frac{6}{\gamma}, \frac{18}{9-2\gamma})$ is the admissible pair defined in the proof therein. We can then argue as in the proof of Theorem 6.0.2. Applying the Strichartz estimates (4.20)-(4.21) to the identity (ii) above, one gets

$$\begin{aligned}
\|\phi_n\|_{L^\infty L^2} + \|\phi_n\|_{L^q L^p} \;\lesssim\; &\|f - f_n\|_{L^2} + \big\|\big((w - w_n) * |u|^2\big)u\big\|_{L^{q'} L^{p'}} \\
&+ \big\|(w_n * |u|^2)\phi_n\big\|_{L^{q'} L^{p'}} + \big\|\big(|w_n| * (|u_n| + |u|)|\phi_n|\big)u_n\big\|_{L^{q'} L^{p'}}\,.
\end{aligned} \tag{iii}$$

By means of Hölder's and Young's inequality in weak spaces one finds

(iv)

$$
\begin{aligned}
\big\|\big((w-w_n)*|u|^2\big)u\big\|_{L^{q'}L^{p'}} &\lesssim T^{1-\frac{\gamma}{2}}\|w-w_n\|_{L^{\frac{3}{\gamma},\infty}}\|u_n\|^2_{L^qL^p}\,\|u\|_{L^\infty L^2}\\
\big\|(w_n*|u|^2)\phi_n\big\|_{L^{q'}L^{p'}} &\lesssim T^{1-\frac{\gamma}{2}}\|w_n\|_{L^{\frac{3}{\gamma},\infty}}\,\|u\|^2_{L^qL^p}\|\phi_n\|_{L^\infty L^2}\\
\big\|\big(|w_n|*(|u_n|+|u|)|\phi_n|\big)u_n\big\|_{L^{q'}L^{p'}} &\lesssim T^{1-\frac{\gamma}{2}}\|w_n\|_{L^{\frac{3}{\gamma},\infty}}\big(\|u_n\|_{L^qL^p}+\|u\|_{L^qL^p}\big)\times\\
&\qquad\times\|\phi_n\|_{L^qL^p}\|u\|_{L^\infty L^2}\,.
\end{aligned}
$$

Since $\|w_n\|_{L^{\frac{3}{\gamma},\infty}}$ and $\|u_n\|_{L^qL^p}$ are bounded uniformly in $n$, then inequalities (iii) and (iv) imply, decreasing further $T$ if needed,

$$
\|\phi_n\|_{L^\infty L^2}+\|\phi_n\|_{L^qL^p}\;\lesssim\;\|f-f_n\|_{L^2}+\|w-w_n\|_{L^{\frac{3}{\gamma},\infty}}\;\xrightarrow{n\to+\infty}\;0\,,
$$

which completes the proof. □

## 6.3. Intermediate regularity theory

In this Section we prove Theorem 6.0.4. The proof is based again on a contraction argument in the complete metric space $\mathcal{X}_{T,M}$, for suitable $T,M>0$, defined in (6.18), now with $s\in(\frac{1}{2},\frac{3}{2})$.

As a consequence, in the energy space ($s=1$) we shall deduce that the solution to the integral problem (6.5) is also a solution to the differential problem (6.3).

We conclude the Section with a stability result of the solution with respect to the initial datum $f$ and the potential $w$.

Let us start with two preparatory lemmas.

Lemma 6.3.1. *Let $\alpha\geqslant 0$ and $s\in(\frac{1}{2},\frac{3}{2})$. Let $w\in W^{s,p}(\mathbb{R}^3)$ for $p\in(2,+\infty)$. Then*

$$
\begin{aligned}
\|w*(\psi_1\psi_2)\|_{L^\infty(\mathbb{R}^3)} &\lesssim \|w*(\psi_1\psi_2)\|_{W^{s,3p}(\mathbb{R}^3)}\\
&\lesssim \|w\|_{W^{s,p}(\mathbb{R}^3)}\|\psi_1\|_{H^s_\alpha(\mathbb{R}^3)}\|\psi_2\|_{H^s_\alpha(\mathbb{R}^3)}
\end{aligned}
\tag{6.22}
$$

*for any $H^s_\alpha$-functions $\psi_1$, $\psi_2$, and $\psi_3$.*

Proof. The first inequality in (6.22) is due to Sobolev's embedding

$$
W^{s,3p}(\mathbb{R}^3)\;\hookrightarrow\;L^\infty(\mathbb{R}^3)\,.
$$

For the second inequality, let us observe preliminarily that

$$
H^s_\alpha(\mathbb{R}^3)\;\hookrightarrow\;L^{\frac{6p}{3p-2}}(\mathbb{R}^3)\,.
$$

Indeed, decomposing by means of (5.3) a generic $\psi\in H^s_\alpha(\mathbb{R}^3)$ as $\psi=\phi_\lambda+\kappa_\lambda\,\mathsf{G}_\lambda$ for some $\phi_\lambda\in H^s(\mathbb{R}^3)$ and some $\kappa_\lambda\in\mathbb{C}$, one has

$$
\|\psi\|_{L^{\frac{6p}{3p-2}}}\;\leqslant\;\|\phi_\lambda\|_{L^{\frac{6p}{3p-2}}}+|\kappa_\lambda|\,\|\mathsf{G}_\lambda\|_{L^{\frac{6p}{3p-2}}}\;\lesssim\;\|\phi_\lambda\|_{H^s}+|\kappa_\lambda|\;\approx\;\|\psi\|_{H^s_\alpha}\,,
$$

the second step following from Sobolev's embedding $H^s(\mathbb{R}^3)\hookrightarrow L^{\frac{6p}{3p-2}}(\mathbb{R}^3)$ and from $\mathsf{G}_\lambda\in L^{\frac{6p}{3p-2}}(\mathbb{R}^3)$, because $\frac{6p}{3p-2}\in[2,3)$ for $p\in(2,+\infty)$, the last step being the norm equivalence (5.7). Therefore Young's inequality yields

$$
\begin{aligned}
\|w*(\psi_1\psi_2)\|_{W^{s,3p}} &\approx \|(\mathbb{1}-\Delta)^{\frac{s}{2}}(w*(\psi_1\psi_2))\|_{L^{3p}} = \|((\mathbb{1}-\Delta)^{\frac{s}{2}}w)*(\psi_1\psi_2)\|_{L^{3p}}\\
&\lesssim \|(\mathbb{1}-\Delta)^{\frac{s}{2}}w\|_{L^p}\,\|\psi_1\|_{L^{\frac{6p}{3p-2}}}\,\|\psi_2\|_{L^{\frac{6p}{3p-2}}}\\
&\lesssim \|w\|_{W^{s,p}(\mathbb{R}^3)}\|\psi_1\|_{H^s_\alpha(\mathbb{R}^3)}\|\psi_2\|_{H^s_\alpha(\mathbb{R}^3)}\,,
\end{aligned}
$$

thus proving (6.22). □

LEMMA 6.3.2. *Let $\alpha \geqslant 0$ and $s \in (\frac{1}{2}, \frac{3}{2})$. Let $h \in W^{s,q}(\mathbb{R}^3)$ for $q \in (6, +\infty)$. Then $h\psi \in H^s_\alpha(\mathbb{R}^3)$ for each $\psi \in H^s_\alpha(\mathbb{R}^3)$ and*

$$\|h\,\psi\|_{H^s_\alpha(\mathbb{R}^3)} \lesssim \|h\|_{W^{s,q}(\mathbb{R}^3)}\,\|\psi\|_{H^s_\alpha(\mathbb{R}^3)}\,. \tag{6.23}$$

PROOF. Let us decompose $\psi \in H^s_\alpha(\mathbb{R}^3)$ as $\psi = \phi_\lambda + \kappa_\lambda \mathsf{G}_\lambda$ for some $\phi_\lambda \in H^s(\mathbb{R}^3)$ and $\kappa_\lambda \in \mathbb{C}$, according to (5.3). On the other hand, by the embedding (6.13) the function $h$ is continuous and $|h(0)| \leqslant \|h\|_{L^\infty(\mathbb{R}^3)} \lesssim \|h\|_{W^{s,q}(\mathbb{R}^3)}$. Thus,

$$h\,g \;=\; h\,\phi_\lambda + \kappa_\lambda\,(h - h(0))\,\mathsf{G}_\lambda + \kappa_\lambda\,h(0)\,\mathsf{G}_\lambda\,. \tag{i}$$

Applying the fractional Leibniz rule (6.6) and using Sobolev's embedding,

$$\begin{aligned} \|h\,\phi_\lambda\|_{H^s} &\approx \|(\mathbb{1}-\Delta)^{\frac{s}{2}}(h\,\phi_\lambda)\|_{L^2} \\ &\lesssim \|(\mathbb{1}-\Delta)^{\frac{s}{2}}h\|_{L^q}\,\|\phi_\lambda\|_{L^{\frac{2q}{q-2}}} + \|h\|_{L^\infty}\|(\mathbb{1}-\Delta)^{\frac{s}{2}}\phi_\lambda\|_{L^2} \\ &\lesssim \|h\|_{W^{s,q}}\|\phi_\lambda\|_{H^s}\,. \end{aligned} \tag{ii}$$

Moreover, since $\mathsf{G}_\lambda \in L^2(\mathbb{R}^3)$,

$$\|(h-h(0))\mathsf{G}_\lambda\|_{L^2} \lesssim \|h-h(0)\|_{L^\infty} \lesssim \|h\|_{W^{s,q}}\,;$$

this, together with the estimate (6.14), gives

$$\|\kappa_\lambda\,(h-h(0))\mathsf{G}_\lambda\|_{H^s} \lesssim |\kappa_\lambda|\,\|h\|_{W^{s,q}}\,. \tag{iii}$$

The bounds (ii) and (iii) imply that $h\psi$ is the sum of the function $h\phi_\lambda + \kappa_\lambda(h - h(0))\mathsf{G}_\lambda \in H^s(\mathbb{R}^3)$ and of the multiple $\kappa_\lambda h(0)\mathsf{G}_\lambda$ of $\mathsf{G}_\lambda$: as such, owing to (5.3), $h\psi$ belongs to $H^s_\alpha(\mathbb{R}^3)$ and its $H^s_\alpha$-norm is estimated, according to the norm equivalence (5.7), by

$$\begin{aligned} \|h\,\psi\|_{H^s_\alpha(\mathbb{R}^3)} &\approx \|h\,\phi_\lambda + \kappa_\lambda\,(h-h(0))\,\mathsf{G}_\lambda\|_{H^s_\alpha} + |\kappa_\lambda|\,|h(0)| \\ &\lesssim \|h\|_{W^{s,q}}(\|\phi_\lambda\|_{H^s} + |\kappa_\lambda|) + |\kappa_\lambda|\,\|h\|_{W^{s,q}} \\ &\approx \|h\|_{W^{s,q}}\,\|\psi\|_{H^s_\alpha}\,, \end{aligned}$$

which completes the proof. □

Combining Lemmas 6.3.1 and 6.3.2 one therefore has the trilinear estimate

$$\|(w*(u_1u_2))u_3\|_{H^s_\alpha(\mathbb{R}^3)} \lesssim \|w\|_{W^{s,p}(\mathbb{R}^3)}\prod_{j=1}^{3}\|u_j\|_{H^s_\alpha(\mathbb{R}^3)}\,. \tag{6.24}$$

Let us now prove Theorem 6.0.4.

PROOF OF THEOREM 6.0.4.
From the expression (6.4) for the solution map $\Phi(u)$ and from the bound (6.24) one finds

$$\begin{aligned} \|\Phi(u)\|_{L^\infty H^s_\alpha} &\leqslant \|f\|_{H^s_\alpha} + T\,\|(w*|u|^2)u\|_{L^\infty H^s_\alpha} \\ &\leqslant \|f\|_{H^s_\alpha} + C_1\,T\,\|w\|_{W^{s,p}}\,\|u\|^3_{L^\infty H^s_\alpha} \end{aligned} \tag{i}$$

for some constant $C_1 > 0$.
Moreover,

$$\begin{aligned} \|\Phi(u)-\Phi(v)\|_{L^\infty L^2} &\leqslant T\,\|(w*|u|^2)u - (w*|v|^2)v\|_{L^\infty L^2} \\ &\lesssim T\left(\|(w*|u|^2)(u-v)\|_{L^\infty L^2} + \left\|\big(w*(|u|^2-|v|^2)v\right\|_{L^\infty L^2}\right). \end{aligned} \tag{ii}$$

For the first summand in the r.h.s. above estimate (6.22) and Hölder's inequality yield

$$\begin{aligned} \|(w*|u|^2)(u-v)\|_{L^\infty L^2} &\leqslant \|w*|u|^2\|_{L^\infty L^\infty}\,\|u-v\|_{L^\infty L^2} \\ &\lesssim \|w\|_{W^{s,p}}\|u\|^2_{L^\infty H^s_\alpha}\,\|u-v\|_{L^\infty L^2}\,. \end{aligned} \tag{iii}$$

For the second summand, let us observe preliminarily that

$$\text{(iv)} \qquad H^s_\alpha(\mathbb{R}^3) \hookrightarrow L^{3,\infty}(\mathbb{R}^3)\,.$$

Indeed, decomposing by means of (5.3) a generic $\psi \in H^s_\alpha(\mathbb{R}^3)$ as $\psi = \phi_\lambda + \kappa_\lambda \, \mathsf{G}_\lambda$ for some $\phi_\lambda \in H^s(\mathbb{R}^3)$ and some $\kappa_\lambda \in \mathbb{C}$, one has

$$\|\psi\|_{L^{3,\infty}} \leqslant \|\phi_\lambda\|_{L^{3,\infty}} + |\kappa_\lambda| \, \|\mathsf{G}_\lambda\|_{L^{3,\infty}} \lesssim \|\phi_\lambda\|_{H^s} + |\kappa_\lambda| \approx \|\psi\|_{H^s_\alpha}\,,$$

the second step following from the Sobolev's embedding $H^s(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, the last step being the norm equivalence (5.7). Then (iv) above, Sobolev's embedding $W^{s,p}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, and an application of Holder's and Young's inequality in Lorentz spaces, yield

$$\text{(v)} \qquad \begin{aligned} \|(w*(|u|^2-|v|^2))v\|_{L^\infty L^2} &\leqslant \|w*(|u|^2-|v|^2)\|_{L^\infty L^{6,2}} \, \|v\|_{L^\infty L^{3,\infty}} \\ &\lesssim \|w\|_{L^3} \, \|u+v\|_{L^\infty L^{3,\infty}} \, \|u-v\|_{L^\infty L^2} \, \|v\|_{L^\infty L^{3,\infty}} \\ &\lesssim \|w\|_{W^{s,p}} \, \|u+v\|_{L^\infty H^s_\alpha} \, \|u-v\|_{L^\infty L^2} \, \|v\|_{L^\infty H^s_\alpha}\,. \end{aligned}$$

Thus, (ii), (iii), and (v) together give

$$\text{(vi)} \qquad d(\Phi(u),\Phi(v)) \leqslant C_2 \, T \, \|w\|_{W^{s,p}} \left( \|u\|^2_{L^\infty H^s_\alpha} + \|v\|^2_{L^\infty H^s_\alpha} \right) d(u,v)$$

for some constant $C_2 > 0$.

Now, setting $C := \max\{C_1, C_2\}$ and choosing $T$ and $M$ such that

$$M = 2\,\|f\|_{H^s_\alpha}\,, \qquad T = \tfrac{1}{4} \left( C\, M^2 \|w\|_{W^{s,p}} \right)^{-1},$$

estimate (i) reads $\|\Phi(u)\|_{L^\infty H^s_\alpha} \leqslant M$ and shows that $\Phi$ maps the space $\mathcal{X}_{T,M}$ defined in (6.18) into itself, whereas estimate (vi) reads $d(\Phi(u),\Phi(v)) \leqslant \frac{1}{2} d(u,v)$ and shows that $\Phi$ is a contraction on $\mathcal{X}_{T,M}$. By Banach's fixed point theorem, there exists a unique fixed point $u \in \mathcal{X}_{T,M}$ of $\Phi$ and hence a unique solution $u \in \mathcal{X}_{T,M}$ to (6.5), which is therefore also continuous in time.

Furthermore, by a standard continuation argument we can extend such a solution over a maximal interval for which the blow-up alternative holds true. Also the continuous dependence on the initial data is a direct consequence of the fixed point argument. $\square$

A straightforward consequence of Theorem 6.0.4 when $s = 1$ concerns the differential meaning of the local strong solution determined so far.

Corollary 6.3.3 (Integral and differential formulation). *Let $\alpha \geqslant 0$. For given $w \in W^{1,p}(\mathbb{R}^3)$, $p \in (2,+\infty)$, and $f \in H^1_\alpha(\mathbb{R}^3)$, let $u$ be the unique solution in the class $C([-T,T], H^1_\alpha(\mathbb{R}^3))$ to the integral equation (6.5) in the interval $[-T,T]$ for some $T > 0$, as given by Theorem 6.0.4. Then $u(0) = f$ and $u$ satisfies the differential equation (6.1) as an identity between $H^{-1}_\alpha$-functions, $H^{-1}_\alpha(\mathbb{R}^3)$ being the topological dual of $H^1_\alpha(\mathbb{R}^3)$.*

Proof. The bound (6.24) shows that the non-linearity defines a map $u \mapsto (w*|u|^2)u$ that is continuous from $H^1_\alpha(\mathbb{R}^3)$ into itself, and hence in particular it is continuous from $H^1_\alpha(\mathbb{R}^3)$ to $H^{-1}_\alpha(\mathbb{R}^3)$. Then the thesis follows by standard facts of the theory of linear semi-groups (see [**25**, Section 1.6]). $\square$

For later purposes, let us conclude this Section with the following stability result.

Proposition 6.3.4. *Let $\alpha \geqslant 0$ and $s \in (\frac{1}{2}, \frac{3}{2})$. For given $w \in W^{s,p}(\mathbb{R}^3)$, $p \in (2,+\infty)$, and $f \in H^s_\alpha(\mathbb{R}^3)$, let $u$ be the unique strong $H^s_\alpha$-solution to the Cauchy problem (6.3) in the maximal interval $(-T_*, T^*)$. Consider moreover the sequences $(w_n)_n$ and $(f_n)_n$ of potentials and initial data such that $w_n \xrightarrow{n\to+\infty} w$ in $W^{s,p}(\mathbb{R}^3)$ and $f_n \xrightarrow{n\to+\infty} f$ in $H^s_\alpha(\mathbb{R}^3)$. Then there exists a time $T := T(\|w\|_{W^{s,p}}, \|f\|_{H^s_\alpha}) >$*

*0, with* $[-T,T]\subset(T_*,T^*)$*, such that, for sufficiently large* $n$*, the Cauchy problem* (6.3) *with potential* $w_n$ *and initial data* $f_n$ *admits a unique strong* $H^s_\alpha$*-solution* $u_n$ *in the interval* $[-T,T]$*. Moreover,*

$$u_n \xrightarrow{n\to+\infty} u \qquad \text{in} \quad C([-T,T],H^s_\alpha(\mathbb{R}^3))\,. \tag{6.25}$$

Proof. As a consequence of Theorem 6.0.4, there exist an interval $[-T_n,T_n]$ for some $T_n:=T_n(\|w_n\|_{W^{s,p}},\|f_n\|_{H^s_\alpha})>0$ and a unique $u_n\in C([-T_n,T_n],H^1_\alpha(\mathbb{R}^3))$ such that

$$u_n(t) \;=\; e^{\mathrm{i}t\Delta_\alpha}f_n-\mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}(w_n*|u_n(\tau)|^2)u_n(\tau)\,\mathrm{d}\tau\,. \tag{*}$$

Since $\|w_n\|_{W^{s,p}}$ and $\|f_n\|_{H^s_\alpha}$ are asymptotically close, respectively, to $\|w\|_{W^{s,p}}$ and $\|f\|_{H^s_\alpha}$, then there exists $T:=T(\|w\|_{W^{s,p}},\|f\|_{H^s_\alpha})$ such that $T\leqslant T_n$ eventually in $n$, which means that $u_n$ is defined on $[-T,T]$. Let us set $\phi_n:=u-u_n$. By assumption $u$ solves (6.5), thus subtracting (*) from (6.5) yields

$$\begin{aligned}
\phi_n \;&=\; e^{\mathrm{i}t\Delta_\alpha}(f-f_n)-\mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}\big((w*|u|^2)u-(w_n*|u_n|^2)u_n\big)(\tau)\,\mathrm{d}\tau\\
&=\; e^{\mathrm{i}t\Delta_\alpha}(f-f_n)-\mathrm{i}\int_0^t e^{\mathrm{i}(t-\tau)\Delta_\alpha}\Big\{\big((w-w_n)*|u|^2\big)u+(w_n*|u|^2)\phi_n\\
&\qquad\qquad\qquad+\big(w_n*(\overline{u}\phi_n+\overline{\phi_n}u_n)\big)u_n\Big\}(\tau)\,\mathrm{d}\tau\,.
\end{aligned}$$

From the above identity, taking the $H^s_\alpha$-norm of $\phi_n$ boils down to repeatedly applying the estimate (6.24) to the summands in the integral on the r.h.s., thus yielding

$$\begin{aligned}
\|\phi_n\|_{L^\infty H^s_\alpha} \;&\lesssim\; \|f-f_n\|_{H^s_\alpha}+T\,\|w-w_n\|_{W^{s,p}}\,\|u\|^3_{L^\infty H^s_\alpha}\\
&\qquad+T\,\|w_n\|_{W^{s,p}}\big(\,\|u\|^2_{L^\infty H^s_\alpha}+\|u_n\|^2_{L^\infty H^s_\alpha}\,\big)\,\|\phi_n\|_{L^\infty H^s_\alpha}\,.
\end{aligned}$$

Since by assumption $\|w_n\|_{W^{s,p}}$ and $\|u_n\|^2_{L^\infty H^s_\alpha}$ are bounded uniformly in $n$, then the above inequality implies, decreasing further $T$ if needed,

$$\|\phi_n\|_{L^\infty H^s_\alpha}\;\lesssim\;\|f-f_n\|_{H^s_\alpha}+\|w-w_n\|_{W^{s,p}}\;\xrightarrow{n\to+\infty}\;0\,,$$

which completes the proof. $\square$

## 6.4. High regularity theory

In this Section we prove Theorem 6.0.5 for the regime $s\in(\frac{3}{2},2]$. The approach is again a contraction argument, that we now set in the spherically symmetric sector of the space $\mathcal{X}_{T,M}$ introduced in (6.18), namely in the complete metric space $(\mathcal{X}^{(0)}_{T,M},d)$ with

$$\begin{aligned}
\mathcal{X}^{(0)}_{T,M}\;&:=\;\big\{u\in L^\infty([-T,T],H^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3))\,|\,\|u\|_{L^\infty([-T,T],H^s_\alpha(\mathbb{R}^3))}\leqslant M\big\}\\
d(u,v)\;&:=\;\|u-v\|_{L^\infty([-T,T],L^2(\mathbb{R}^3))}
\end{aligned}\tag{6.26}$$

for suitable $T,M>0$.

A very much useful by-product of such a contraction argument will be the proof that when $s=2$ the solution to the integral problem (6.5) is also a solution to the differential problem (6.3), as we shall show in a moment.

Let us start with two preparatory lemmas.

Lemma 6.4.1. *Let* $\alpha\geqslant0$ *and* $s\in(\frac{3}{2},2]$*. Let* $w\in W^{s,p}(\mathbb{R}^3)$ *for* $p\in(2,+\infty)$ *and assume that* $w$ *is spherically symmetric. Then*

(i) *one has the estimate*

$$\begin{aligned}\|w*(\psi_1\psi_2)\|_{L^\infty(\mathbb{R}^3)} &\lesssim \|w*(\psi_1\psi_2)\|_{W^{s,3p}(\mathbb{R}^3)}\\ &\lesssim \|w\|_{W^{s,p}(\mathbb{R}^3)}\|\psi_1\|_{H^s_\alpha(\mathbb{R}^3)}\|\psi_2\|_{H^s_\alpha(\mathbb{R}^3)}\end{aligned} \tag{6.27}$$

*for any $H^s_\alpha$-functions $\psi_1$, $\psi_2$, and $\psi_3$;*

(ii) *if in addition $\psi_1$, $\psi_2$ are spherically symmetric, so too is $w*(\psi_1\psi_2)$ and*

$$\big(\nabla(w*(\psi_1\psi_2))\big)(0) \;=\; 0\,.$$

Proof. (i) The first inequality in (6.27) is due to Sobolev's embedding

$$W^{s,3p}(\mathbb{R}^3) \;\hookrightarrow\; L^\infty(\mathbb{R}^3)\,.$$

For the second inequality, let us observe preliminarily that

$$H^s_\alpha(\mathbb{R}^3) \;\hookrightarrow\; L^{\frac{6p}{3p-2}}(\mathbb{R}^3)\,.$$

Indeed, decomposing by means of (5.4) a generic $\psi\in H^s_\alpha(\mathbb{R}^3)$ as $\psi=\phi_\lambda+\frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda$ for some $\phi_\lambda\in H^s(\mathbb{R}^3)$, one has

$$\|\psi\|_{L^{\frac{6p}{3p-2}}} \;\lesssim\; \|\phi_\lambda\|_{L^{\frac{6p}{3p-2}}}+|\phi_\lambda(0)|\,\|\mathsf{G}_\lambda\|_{L^{\frac{6p}{3p-2}}} \;\lesssim\; \|\phi_\lambda\|_{H^s} \;\approx\; \|\psi\|_{H^s_\alpha}\,,$$

the second step following from Sobolev's embedding $H^s(\mathbb{R}^3)\hookrightarrow L^{\frac{6p}{3p-2}}(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$ and from $\mathsf{G}_\lambda\in L^{\frac{6p}{3p-2}}(\mathbb{R}^3)$, because $\frac{6p}{3p-2}\in[2,3)$ for $p\in(2,+\infty)$, the last step being the norm equivalence (5.8). Therefore Young's inequality yields

$$\begin{aligned}\|w*(\psi_1\psi_2)\|_{W^{s,3p}} &\approx \|(\mathbb{1}-\Delta)^{\frac{s}{2}}(w*(\psi_1\psi_2))\|_{L^{3p}}\\ &= \|((\mathbb{1}-\Delta)^{\frac{s}{2}}w)*(\psi_1\psi_2)\|_{L^{3p}}\\ &\lesssim \|(\mathbb{1}-\Delta)^{\frac{s}{2}}w\|_{L^p}\,\|\psi_1\|_{L^{\frac{6p}{3p-2}}}\,\|\psi_2\|_{L^{\frac{6p}{3p-2}}}\\ &\lesssim \|w\|_{W^{s,p}(\mathbb{R}^3)}\,\|\psi_1\|_{H^s_\alpha(\mathbb{R}^3)}\,\|\psi_2\|_{H^s_\alpha(\mathbb{R}^3)}\,,\end{aligned}$$

thus proving (6.27).

(ii) The spherical symmetry of $w*(\psi_1\psi_2)$ in this second case is obvious. From Sobolev's embedding $W^{s,3p}(\mathbb{R}^3)\hookrightarrow C^1(\mathbb{R}^3)$ we deduce that $\nabla(w*(\psi_1\psi_2))(x)$ is well defined for every $x\in\mathbb{R}^3$; moreover,

$$\nabla(w*(\psi_1\psi_2))(0) \;=\; \big((\nabla w)*(\psi_1\psi_2)\big)(0) \;=\; \int_{\mathbb{R}^3}(\nabla w)(-y)\psi_1(y)\psi_2(y)\,\mathrm{d}y \;=\; 0\,,$$

the above integral vanishing because the integrand is of the form $R(y)\frac{y}{|y|}$ for some spherically symmetric function $R$. $\square$

Lemma 6.4.2. *Let $\alpha\geqslant 0$ and $s\in(\frac{3}{2},2]$. Let $h\in W^{s,q}_{\mathrm{rad}}(\mathbb{R}^3)$ for some $q\in(6,+\infty)$ and assume that $(\nabla h)(0)=0$. Then $h\psi\in H^s_\alpha(\mathbb{R}^3)$ for each $\psi\in H^s_\alpha(\mathbb{R}^3)$ and*

$$\|h\psi\|_{H^s_\alpha(\mathbb{R}^3)} \;\lesssim\; \|h\|_{W^{s,q}(\mathbb{R}^3)}\,\|\psi\|_{H^s_\alpha(\mathbb{R}^3)}\,. \tag{6.28}$$

Proof. Let us decompose $\psi\in H^s_\alpha(\mathbb{R}^3)$ as $\psi=\phi_\lambda+\frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda$ for some $\phi_\lambda\in H^s(\mathbb{R}^3)$, according to (5.4). On the other hand, by the embedding (6.15) the function $h$ is continuous and $|h(0)|\leqslant\|h\|_{L^\infty(\mathbb{R}^3)}\lesssim\|h\|_{W^{s,q}(\mathbb{R}^3)}$. Thus,

$$h\,\psi \;=\; h\,\phi_\lambda+\frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,(h-h(0))\,\mathsf{G}_\lambda+\frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,h(0)\,\mathsf{G}_\lambda\,. \tag{i}$$

Applying the fractional Leibniz rule (6.6) and using Sobolev's embedding,

$$\begin{aligned}\|h\,\phi_\lambda\|_{H^s} &\approx \|(\mathbb{1}-\Delta)^{\frac{s}{2}}(h\,\phi_\lambda)\|_{L^2}\\ &\lesssim \|(\mathbb{1}-\Delta)^{\frac{s}{2}}h\|_{L^q}\,\|\phi_\lambda\|_{L^{\frac{2q}{q-2}}}+\|h\|_{L^\infty}\|(\mathbb{1}-\Delta)^{\frac{s}{2}}\phi_\lambda\|_{L^2}\\ &\lesssim \|h\|_{W^{s,q}}\|\phi_\lambda\|_{H^s}\,.\end{aligned} \tag{ii}$$

Moreover, since $\mathsf{G}_\lambda \in L^2(\mathbb{R}^3)$,

$$\Big\|\tfrac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}(h-h(0))\mathsf{G}_\lambda\Big\|_{L^2} \;\lesssim\; \|h-h(0)\|_{L^\infty}\,\|\phi_\lambda\|_{L^\infty} \;\lesssim\; \|h\|_{W^{s,q}}\,\|\phi_\lambda\|_{H^s}\,;$$

this, together with the estimate (6.16) (which requires indeed spherical symmetry), gives

$$\text{(iii)}\qquad \Big\|\tfrac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\,(h-h(0))\mathsf{G}_\lambda\Big\|_{H^s} \;\lesssim\; \|h\|_{W^{s,q}}\|\phi_\lambda\|_{H^s}\,.$$

The bounds (ii) and (iii) above imply that $F_\lambda := h\phi_\lambda + \frac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}(h-h(0))\mathsf{G}_\lambda$ belongs to $H^s(\mathbb{R}^3)$ with

$$\text{(iv)}\qquad \|F\|_{H^s} \;\lesssim\; \|h\|_{W^{s,q}}\|\phi_\lambda\|_{H^s}\,.$$

In particular, $F_\lambda$ is continuous. One has

$$F_\lambda(0) \;=\; h(0)\phi_\lambda(0) + \tfrac{\phi_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\lim_{|x|\to 0}\frac{h(x)-h(0)}{|x|} \;=\; h(0)\phi_\lambda(0)$$

because by assumption $(\nabla h)(0)=0$. In turn, (i) now reads $h\psi = F_\lambda + \frac{F_\lambda(0)}{\alpha+\frac{\sqrt{\lambda}}{4\pi}}\mathsf{G}_\lambda$, which means, in view of the domain decomposition (5.4), that $h\psi$ belongs to $H^s_\alpha(\mathbb{R}^3)$. Owing to (iv) above and to the norm equivalence (5.8), we conclude

$$\|h\psi\|_{H^s_\alpha} \;\approx\; \|F_\lambda\|_{H^s} \lesssim\; \|h\|_{W^{s,q}}\|\phi_\lambda\|_{H^s}\,,$$

which completes the proof. $\square$

Combining Lemmas 6.4.1 and 6.4.2 one therefore has the trilinear estimate

$$\text{(6.29)}\qquad \|(w*(u_1u_2))u_3\|_{H^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3)} \;\lesssim\; \|w\|_{W^{s,p}(\mathbb{R}^3)}\prod_{j=1}^{3}\|u_j\|_{H^s_{\alpha,\mathrm{rad}}(\mathbb{R}^3)}.$$

Let us now prove Theorem 6.0.5.

Proof of Theorem 6.0.5.

From the expression (6.4) for the solution map $\Phi(u)$ and from the bound (6.29) one finds

$$\text{(i)}\qquad \begin{aligned}\|\Phi(u)\|_{L^\infty H^s_{\alpha,\mathrm{rad}}} &\leqslant\; \|f\|_{H^s_{\alpha,\mathrm{rad}}} + T\,\|(w*|u|^2)u\|_{L^\infty H^s_{\alpha,\mathrm{rad}}}\\ &\leqslant\; \|f\|_{H^s_{\alpha,\mathrm{rad}}} + C_1\,T\,\|w\|_{W^{s,p}}\,\|u\|^3_{L^\infty H^s_{\alpha,\mathrm{rad}}}\end{aligned}$$

for some constant $C_1>0$.

Moreover,

$$\text{(ii)}\qquad \begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^\infty L^2} &\leqslant\; T\,\|(w*|u|^2)u-(w*|v|^2)v\|_{L^\infty L^2}\\ &\lesssim\; T\,\big(\|(w*|u|^2)(u-v)\|_{L^\infty L^2} + \big\|\big(w*(|u|^2-|v|^2)v\big\|_{L^\infty L^2}\big)\,.\end{aligned}$$

For the first summand in the r.h.s. above the bound (6.27) and Hölder's inequality yield

$$\text{(iii)}\qquad \begin{aligned}\|(w*|u|^2)(u-v)\|_{L^\infty L^2} &\leqslant\; \|w*|u|^2\|_{L^\infty L^\infty}\,\|u-v\|_{L^\infty L^2}\\ &\lesssim\; \|w\|_{W^{s,p}}\,\|u\|^2_{L^\infty H^s_\alpha}\,\|u-v\|_{L^\infty L^2}\,.\end{aligned}$$

For the second summand, let us observe preliminarily that the embedding

$$\text{(iv)}\qquad H^s_\alpha(\mathbb{R}^3) \;\hookrightarrow\; L^{3,\infty}(\mathbb{R}^3)$$

valid for $s \in (\frac{1}{2}, \frac{3}{2})$ and established in the proof of Theorem 6.0.4 holds true even more when $s \in (\frac{3}{2}, 2]$. Then (iv) above, Sobolev's embedding $W^{s,p}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, and an application of Holder's and Young's inequality in Lorentz spaces, yield

$$\text{(v)} \qquad \begin{aligned} \|(w*(|u|^2-|v|^2))v\|_{L^\infty L^2} &\leqslant \|w*(|u|^2-|v|^2)\|_{L^\infty L^{6,2}}\,\|v\|_{L^\infty L^{3,\infty}} \\ &\lesssim \|w\|_{L^3}\,\|u+v\|_{L^\infty L^{3,\infty}}\,\|u-v\|_{L^\infty L^2}\,\|v\|_{L^\infty L^{3,\infty}} \\ &\lesssim \|w\|_{W^{s,p}}\,\|u+v\|_{L^\infty H^s_\alpha}\,\|u-v\|_{L^\infty L^2}\,\|v\|_{L^\infty H^s_\alpha}\,. \end{aligned}$$

Combining (ii), (iii), and (v) we get

$$\text{(vi)} \qquad d(\Phi(u),\Phi(v)) \;\leqslant\; C_2\,T\,\|w\|_{W^{s,p}}\big(\|u\|^2_{L^\infty H^s_\alpha} + \|v\|^2_{L^\infty H^s_\alpha}\big)\,d(u,v)$$

for some constant $C_2 > 0$.

Thus, choosing $T$ and $M$ such that

$$M \;=\; 2\,\|f\|_{H^s_\alpha}\,, \qquad T \;=\; \tfrac{1}{4}\big(\max\{C_1,C_2\}\,M^2\|w\|_{W^{s,p}}\big)^{-1}\,,$$

estimate (i) reads $\|\Phi(u)\|_{L^\infty H^s_{\alpha,\mathrm{rad}}} \leqslant M$ and shows that $\Phi$ maps the space $\mathcal{X}^{(0)}_{T,M}$ into itself, whereas estimate (vi) reads $d(\Phi(u),\Phi(v)) \leqslant \frac{1}{2}d(u,v)$ and shows that $\Phi$ is a contraction on $\mathcal{X}^{(0)}_{T,M}$. By Banach's fixed point theorem, there exists a unique fixed point $u \in \mathcal{X}^{(0)}_{T,M}$ of $\Phi$ and hence a unique solution $u \in \mathcal{X}^{(0)}_{T,M}$ to (6.5), which is therefore also continuous in time.

Furthermore, by a standard continuation argument we can extend such a solution over a maximal interval for which the blow-up alternative holds true. Also the continuous dependence on the initial data is a direct consequence of the fixed point argument. □

A straightforward, yet crucial for us, consequence of Theorem 6.0.5 when $s = 2$ concerns the differential meaning of the local strong solution determined so far.

Corollary 6.4.3 (Integral and differential formulation). *Let $\alpha \geqslant 0$ and $w \in W^{2,p}(\mathbb{R}^3)$, $p \in (2,+\infty)$, a spherically symmetric potential. Assume moreover $f \in H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$. Let $u \in C([-T,T], H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3))$ the unique local to the Cauchy problem (6.3) in the interval $[-T,T]$, for some $T > 0$, i.e. $u$ satisfies the Duhamel formula (6.5). Then $u(0,\cdot) = f$ and $u$ satisfies the equation $\mathrm{i}\partial_t u = -\Delta_\alpha u + (w*|u|^2)u$ as an identity in between $L^2(\mathbb{R}^3)$-functions.*

Proof. The bound (6.29) shows that the non-linearity defines a map $u \mapsto (w*|u|^2)u$ that is continuous from $H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ into itself, and hence in particular it is continuous from $H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. Then the thesis follows by standard fact on the theory of linear semi-groups (see [**25**, Section 1.6]). □

## 6.5. Global solutions in the mass and in the energy space

In order to study the global solution theory of the Cauchy problem (6.3) when $s = 0$ (the mass space $L^2(\mathbb{R}^3)$) and $s = 1$ (the energy space $H^1_\alpha(\mathbb{R}^3)$), we introduce the following two quantities, that are formally conserved in time along the solutions.

Definition 6.5.1.

(i) Let $u \in L^2(\mathbb{R}^3)$. We define the *mass* of $u$ as

$$\mathcal{M}(u) \;:=\; \|u\|^2_{L^2}\,.$$

(ii) Let $\lambda > 0$ and let $u = \phi_\lambda + \kappa_\lambda \mathsf{G}_\lambda \in H^1_\alpha(\mathbb{R}^3)$, according to (5.3). We define the *energy* of $u$ as

$$\begin{aligned}\mathcal{E}(u) \;&:=\; \tfrac{1}{2}(-\Delta_\alpha)[u] + \tfrac{1}{4}\int_{\mathbb{R}^3}(w*|u|^2)|u|^2\,\mathrm{d}x \\ &=\; \tfrac{1}{2}\big(\lambda\|\phi_\lambda\|^2_{L^2} + \|\nabla\phi_\lambda\|^2_{L^2} + \big(\alpha + \tfrac{\sqrt{\lambda}}{4\pi}\big)|\kappa_\lambda|^2 - \lambda\|u\|^2_{L^2}\big) \\ &\qquad + \tfrac{1}{4}\int_{\mathbb{R}^3}(w*|u|^2)|u|^2\,\mathrm{d}x\,.\end{aligned}$$

REMARK 6.5.2. For given $u$, the value of $(-\Delta_\alpha)[u]$ (the quadratic form of $-\Delta_\alpha$) is independent of $\lambda$, and so too is the energy $\mathcal{E}(u)$.

We shall establish suitable conservation laws in order to prolong the local solution globally in time. The mass is conserved in $L^2(\mathbb{R}^3)$ in the following sense.

PROPOSITION 6.5.3 (Mass conservation in $L^2(\mathbb{R}^3)$).
*Let $\alpha \geqslant 0$, and let $w$ belong either to the class $L^\infty(\mathbb{R}^3)\cap W^{1,3}(\mathbb{R}^3)$ or to the class $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$, for $\gamma \in (0,\frac{3}{2})$. For a given $f \in L^2(\mathbb{R}^3)$, let $u$ be the unique local solution in $C((-T_*,T^*),L^2(\mathbb{R}^3))$ to the Cauchy problem (6.5) in the maximal interval $(-T_*,T^*)$, as given by Theorem 6.0.4. Then $\mathcal{M}(u(t))$ is constant for $t \in (-T_*,T^*)$.*

PROOF. Let us discuss first the case $w \in L^\infty(\mathbb{R}^3)\cap W^{1,3}(\mathbb{R}^3)$. Consider preliminarily an initial data $f \in H^1_\alpha(\mathbb{R}^3)$. Owing to Corollary 6.3.3, for each $t \in (-T_*,T^*)$ $u$ satisfies $\mathrm{i}\partial_t u = -\Delta_\alpha u + (w*|u|^2)u$ as an identity between $H^{-1}_\alpha$-functions, whence

$$\big\langle\, \mathrm{i}\partial_t u + \Delta_\alpha u - (w*|u|^2)u, u\big\rangle_{H^{-1}_\alpha, H^1_\alpha} \;=\; 0\,.$$

The imaginary part of the above identity gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^2_{L^2} \;=\; 0\,,$$

which implies that $\mathcal{M}(u(t))$ is constant on $(-T_*,T^*)$. For arbitrary $f \in L^2(\mathbb{R}^3)$ we use a density argument. Let $(f_n)_n$ be a sequence in $H^1_\alpha(\mathbb{R}^3)$ such that $f_n \xrightarrow{n\to\infty} f$ in $L^2(\mathbb{R}^3)$, and denote by $u_n$ the solution to the Cauchy problem (6.3) with initial datum $f_n$. Because of the continuous dependence on the initial data, we have that $u_n \to u$ in $C(I,L^2(\mathbb{R}^3))$, for every closed interval $I \subset (-T_*,T^*)$. Since $\mathcal{M}(u_n(t)) = \mathcal{M}(u_n(0)) = \mathcal{M}(f_n)$ for every $n$, we deduce that $\mathcal{M}(u(t)) = \mathcal{M}(f)$ for $t \in I$. Owing to the continuity of the map $t \mapsto \mathcal{M}(u(t))$, we conclude that $\mathcal{M}(u(t)) = \mathcal{M}(f)$ for $t \in (-T_*,T^*)$.

Let us discuss now the case $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$, $\gamma \in (0,\frac{3}{2})$. Consider preliminarily an initial data $f \in H^1_\alpha(\mathbb{R}^3)$ and a Schwartz potential $w$. Owing to Corollary 6.3.3, for each $t \in (-T_*,T^*)$ $u$ satisfies $\mathrm{i}\partial_t u = -\Delta_\alpha u + (w*|u|^2)u$ as an identity between $H^{-1}_\alpha$-functions, and reasoning as above we deduce that $\mathcal{M}(u(t))$ is constant on $(-T_*,T^*)$. For arbitrary $f \in L^2(\mathbb{R}^3)$ and $w \in L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$, $\gamma \in (0,\frac{3}{2})$, we use a density argument. Let $(f_n)_n$ be a sequence in $H^1_\alpha(\mathbb{R}^3)$ such that $f_n \xrightarrow{n\to\infty} f$ in $L^2(\mathbb{R}^3)$, $(w_n)_n$ be a sequence of Schwartz potentials such that $w_n \xrightarrow{n\to\infty} w$ in $L^{\frac{3}{\gamma},\infty}(\mathbb{R}^3)$, and denote by $u_n$ the $L^2$ strong solution to the Cauchy problem (6.3) with initial datum $f_n$ and potential $w_n$. The stability result given by Proposition 6.2.1 guarantees that $u_n \xrightarrow{n\to+\infty} u$ in $C([-T,T],L^2(\mathbb{R}^3))$ for some $T>0$, whence $\mathcal{M}(u_n(t)) \xrightarrow{n\to+\infty} \mathcal{M}(u(t))$ for $t \in [-T,T]$. Using the mass conservation for $u_n$ we deduce that $\mathcal{M}(u(t)) = \mathcal{M}(f)$ for $t \in [-T,T]$. Repeating the above argument with $f$ replaced by $u(t_0)$ for some $t_0 \in (-T_*,T^*)$ yields the property that $t \mapsto \mathcal{M}(u(t))$ is constant in a suitable interval around $t_0$ and hence, by the arbitrariness of $t_0$, it is locally constant on the whole $(-T_*,T^*)$. But $(-T_*,T^*) \ni t \mapsto \mathcal{M}(u(t))$ is also continuous, whence the conclusion. $\square$

We therefore conclude the following.

PROOF OF THEOREM 6.0.6. An immediate consequence of the conservation of the mass, i.e., conservation of the $L^2$-norm, and of the blow up alternative in $L^2$. □

Let us move now to the conservation of mass and energy in the energy space. We observe the following.

LEMMA 6.5.4. *Let $\alpha \geqslant 0$ and let $w \in W^{1,p}(\mathbb{R}^3)$ for some $p > 2$. If $v_n \xrightarrow{n\to+\infty} v$ in $H^1_\alpha(\mathbb{R}^3)$, then $\mathcal{E}(v_n) \xrightarrow{n\to+\infty} \mathcal{E}(v)$. As a consequence, if $u \in C([-T,T], H^1_\alpha(\mathbb{R}^3))$ for some $T > 0$, then $t \mapsto \mathcal{E}(u(t))$ is continuous on $[-T,T]$.*

PROOF. The limit $\mathcal{E}(v_n) \xrightarrow{n\to+\infty} \mathcal{E}(v)$ follows from the inequality

$$\begin{aligned} |\mathcal{E}(v) - \mathcal{E}(v_n)| \;&\lesssim\; |\,(-\Delta_\alpha)[v] - (-\Delta_\alpha)[v_n]\,| \\ &\qquad + \|(w * |v|^2)|v|^2 - (w * |v_n|^2)|v_n|^2\|_{L^1} \end{aligned}$$

combined with the estimates

$$|\,(-\Delta_\alpha)[v] - (-\Delta_\alpha)[v_n]\,| \;\lesssim\; \|v - v_n\|_{H^1_\alpha}\big(\|v\|_{H^1_\alpha} + \|v_n\|_{H^1_\alpha}\big)$$

and

$$\begin{aligned} &\|(w * |v|^2)|v|^2 - (w * |v_n|^2)|v_n|^2\|_{L^1} \\ &\quad\lesssim\; \|(w * |v|^2)(|v|^2 - |v_n|^2)\|_{L^1} + \|(w * (|v|^2 - |v_n|^2)|v_n|^2\|_{L^1} \\ &\quad\lesssim\; \|w * |v|^2\|_{L^\infty}\|v - v_n\|_2\big(\|v\|_2 + \|v_n\|_2\big) \\ &\qquad\qquad + \big\||w| * \big(|v - v_n|(|v| + |v_n|)\big)\big\|_{L^\infty}\|v_n\|^2_{L^2} \\ &\quad\lesssim\; \|w\|_{W^{1,p}}\|v - v_n\|_{H^1_\alpha}\big(\|v\|^2_{H^1_\alpha} + \|v_n\|^2_{H^1_\alpha}\big)\,, \end{aligned}$$

the last two steps above following from Hölder's and Young's inequality, and from the inequality (6.22). □

We then see that mass and energy are conserved in the spherically symmetric component of the energy space.

PROPOSITION 6.5.5 (Mass and energy conservation in $H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$). *Let $\alpha \geqslant 0$. For a given $w \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^3)$, $p \in (2,+\infty)$, and a given $f \in H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$, let $u$ be the unique local solution in $C((-T_*,T^*), H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3))$ to the Cauchy problem (6.5) in the maximal interval $(-T_*,T^*)$, as given by Theorem 6.0.4. Then $\mathcal{M}(u(t))$ and $\mathcal{E}(u(t))$ are constant for $t \in (-T_*,T^*)$.*

PROOF. We start proving the statement for the mass. Owing to Corollary 6.3.3, for each $t \in (-T_*,T^*)$ $u$ satisfies $\mathrm{i}\partial_t u = -\Delta_\alpha u + (w * |u|^2)u$ as an identity in $H^{-1}_\alpha(\mathbb{R}^3)$, whence

$$\big\langle\, \mathrm{i}\partial_t u + \Delta_\alpha u - (w * |u|^2)u, u \big\rangle_{H^{-1}_\alpha, H^1_\alpha} \;=\; 0\,.$$

The imaginary part of the above identity gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^2_{L^2} \;=\; 0\,,$$

which implies that $\mathcal{M}(u(t)$ is constant on $(-T_*,T^*)$.

Let us prove now that the energy is conserved, first in the special case $f \in H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ and $w \in W^{2,p}_{\mathrm{rad}}(\mathbb{R}^3)$, for $p \in (2,+\infty)$. Owing to Corollary 6.4.3, $u$ satisfies $\mathrm{i}\partial_t u = -\Delta_\alpha u + (w * |u|^2)u$ as an identity in $L^2(\mathbb{R}^3)$, whence

$$\big\langle\, \mathrm{i}\partial_t u + \Delta_\alpha u - (w * |u|^2)u, \partial_t u \,\big\rangle_{L^2} \;=\; 0\,.$$

The real part in the above identity gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{2}\langle -\Delta_\alpha u, u\rangle_{L^2} - \tfrac{1}{4}\textstyle\int_{\mathbb{R}^3}(w * |u|^2)|u|^2\mathrm{d}x\big) \;=\; 0\,,$$

which implies that $\mathcal{E}(u(t))$ is constant on $(-T_*, T^*)$.

For arbitrary $f \in H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ and $w \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^3)$ we use the stability result of Proposition 6.3.4. Let $(f_n)_n$ be a sequence in $H^2_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$ and $(w_n)_n$ be a sequence in $W^{2,p}_{\mathrm{rad}}(\mathbb{R}^3)$ such that $f_n \xrightarrow{n\to+\infty} f$ in $H^1_\alpha(\mathbb{R}^3)$ and $w_n \xrightarrow{n\to+\infty} w$ in $W^{1,p}(\mathbb{R}^3)$, and denote by $u_n$ the solution to the Cauchy problem (6.3) with initial datum $f_n$ and potential $w_n$. Then Proposition 6.3.4 guarantees that $u_n \xrightarrow{n\to+\infty} u$ in $C([-T,T], H^1_\alpha(\mathbb{R}^3))$ for some $T>0$, and Lemma 6.5.4 implies that $\mathcal{E}(u_n(t)) \xrightarrow{n\to+\infty} \mathcal{E}(u(t))$ for $t \in [-T,T]$. Using the energy conservation for $u_n$ we deduce that $\mathcal{E}(u(t)) = \mathcal{E}(f)$ for $t \in [-T,T]$. Repeating the above argument with $f$ replaced by $u(t_0)$ for some $t_0 \in (-T_*, T^*)$ yields the property that $t \mapsto \mathcal{E}(u(t))$ is constant in a suitable interval around $t_0$ and hence, by the arbitrariness of $t_0$, it is locally constant on the whole $(-T_*, T^*)$. But $(-T_*, T^*) \ni t \mapsto \mathcal{E}(u(t))$ is also continuous, whence the conclusion. $\square$

We are now ready to prove our result on the solution theory for the Cauchy problem (6.3).

Proof of Theorem 6.0.7.

Let $u \in C((-T_*, T^*), H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3))$ be the unique local strong solution to (6.3), on the maximal time interval $(-T_*, T^*)$, with given initial datum $f = \phi_\lambda + c\,\mathsf{G}_\lambda \in H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$, for some $\lambda > 0$, and given potential $w \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^3)$, for some $p \in (2,+\infty)$, as provided by Theorem 6.0.4. Then $(-T_*, T^*) \ni t \mapsto \mathcal{M}(u(t)) + \mathcal{E}(u(t))$ is the constant map, as follows from Propositions 6.5.5. Decomposing $u(t) = \phi_\lambda(t) + \kappa_\lambda(t)\mathsf{G}_\lambda$ for each $t \in (-T_*, T^*)$ and using (5.7) we find

$$\begin{aligned}
\|u(t)\|^2_{H^1_\alpha} &\approx \|\phi_\lambda(t)\|^2_{H^1} + |\kappa_\lambda(t)|^2 \\
&\lesssim (\lambda+1)\,\|u(t)\|_{L^2} \\
&\qquad + \big(\lambda\|\phi_\lambda(t)\|^2_{L^2} - \lambda\|u(t)\|^2_{L^2} + \|\nabla\phi_\lambda(t)\|^2_{L^2} + \big(\alpha + \tfrac{\sqrt{\lambda}}{4\pi}\big)|\kappa_\lambda(t)|^2\big) \\
(*)\qquad &\lesssim \mathcal{M}(u(t)) + \tfrac{1}{2}(-\Delta_\alpha)[u(t)]\,.
\end{aligned}$$

For part (i) of the statement, we observe that

$$\begin{aligned}
\sup_{t\in(-T_*,T^*)} \|u(t)\|^2_{H^1_\alpha} &\lesssim \sup_{t\in(-T_*,T^*)} \big(\mathcal{M}(u(t)) + \tfrac{1}{2}(-\Delta_\alpha)[u(t)]\big) \\
&\lesssim \sup_{t\in(-T_*,T^*)} \big(\mathcal{M}(u(t)) + \mathcal{E}(u(t)) + \|(w*|u(t)|^2)|u(t)|^2\|_{L^1_x}\big) \\
&\lesssim 1 + \sup_{t\in(-T_*,T^*)} \|w*|u|^2\|_{L^\infty_x}\,\|u\|^2_{L^2_x} \\
&\lesssim 1 + \sup_{t\in(-T_*,T^*)} \|w\|_{W^{s,p}}\|u\|^2_{H^1_\alpha}\,\|f\|^2_{L^2}\,,
\end{aligned}$$

having used (*), the estimate (6.22), and the mass and energy conservation. Therefore, if $\|f\|_{L^2}$ is sufficiently small (depending only on $\|w\|_{W^{s,p}}$), then

$$\sup_{t\in(-T_*,T^*)} \|u(t)\|^2_{H^1_\alpha} \lesssim 1\,,$$

and we conclude that solution is global, owing to the blow up alternative.

For part (ii) of the statement, the additional assumption $w \geqslant 0$ implies

$$\tfrac{1}{2}(-\Delta_\alpha)[u(t)] \leqslant \tfrac{1}{2}(-\Delta_\alpha)[u(t)] + \tfrac{1}{4}\int_{\mathbb{R}^3} (w*|u(t)|^2)|u(t)|^2\mathrm{d}x = \mathcal{E}(u(t))\,,$$

which, combined with (*) and the mass and energy conservation yields

$$\sup_{t\in(-T_*,T^*)} \|u(t)\|^2_{H^1_\alpha} \lesssim \sup_{t\in(-T_*,T^*)} \big(\mathcal{M}(u(t)) + \mathcal{E}(u(t))\big) \lesssim 1\,.$$

Therefore, the solution is global, by the blow up alternative. Since this is true for every initial datum $f \in H^1_{\alpha,\mathrm{rad}}(\mathbb{R}^3)$, we deduce global well-posedness for (6.3). $\square$

### 6.6. Comments on the spherically symmetric solution theory

As initially mentioned in the introduction of this Chapter, and then shown in the preceding discussion, part of the solution theory was established for spherically symmetric potentials and solutions (Theorems 6.0.5 and 6.0.7) and in this Section we collect our remarks on the emergence of such a feature.

This is indeed a natural phenomenon both for the local high regularity theory and for the global theory in the energy space, as we are now going to explain. Of course, the spherically symmetric solution theory is the most relevant in the study of the singular Hartree equation, since the linear part, namely the operator $-\Delta_\alpha$, differs from the ordinary $-\Delta$ precisely in the $L^2$-sector of rotationally symmetric functions.

For the local theory, one ineludible ingredient of the fixed point argument is the treatment of the non-linear part of the solution map (6.4) with a $\widetilde{H}^s_\alpha$-estimate that we close by means of the trilinear estimate (6.24)/(6.29).

This estimate is designed for functions of the form $hu$, where $h = w * |u|^2$, and it is crucially sensitive to the specific structure of the space $\widetilde{H}^s_\alpha(\mathbb{R}^3)$ for $s > \frac{1}{2}$ (Theorem 5.1.1(ii)-(iii)). In particular, in order to recognise that the regular part $hu$ is indeed a $H^s$-function, one must show that $(h-h(0))G_\lambda \in H^s(\mathbb{R}^3)$. Technically this is dealt with by means of the fractional Leibniz rule, suitably generalised so as to avoid the direct $L^p$-estimate of $s$ derivatives of each factor $h - h(0)$ and $G_\lambda$; already heuristically it is clear that this only works with a sufficient vanishing rate of $h - h(0)$ as $|x| \to 0$ in order to compensate the local singularity of $G_\lambda$.

For intermediate regularity (Proposition 6.1.5 and Lemmas 6.3.1-6.3.2) the vanishing rate $h(x) - h(0) \sim |x|^\theta$ that can be deduced from the embedding $w * |u^2| \in C^{0,\theta}(\mathbb{R}^3)$ is enough to close the argument and no spherical symmetry is required. For high regularity (Proposition 6.1.6 - Lemma 6.1.7, and Lemmas 6.4.1-6.4.2) the embedding $w * |u^2| \in C^{1,\theta}(\mathbb{R}^3)$ would only guarantee an insufficient vanishing rate $h(x) - h(0) \sim |x|$; since one needs $h(x) - h(0) \sim |x|^{1+\theta}$, this requires the additional condition $\nabla h(0) = 0$. For the latter condition to hold for $h = w * |u|^2$, as shown in the proof of Lemma 6.4.1(ii), the spherical symmetry of both $w$ and $u$ appears as the most natural and explicitly treatable assumption.

In fact, the condition $\nabla h(0) = 0$ is even more crucial and apparently unavoidable in one further point of the argument $hu \in \widetilde{H}^s_\alpha(\mathbb{R}^3)$, because unlike the intermediate regularity case, where it suffices to prove that the regular component of $hu$ is a $H^s$-function, in the high regularity case one must also prove that such regular component satisfies the correct *boundary condition* in connection with the singular component. As shown in the proof of Lemma 6.4.2, the correct boundary condition is *equivalent* to $|x|^{-1}(h(x) - h(0)) \to 0$ as $|x| \to 0$, for which $\nabla h(0) = 0$ is again necessary.

Concerning the global theory in the energy space, the emergence of a solution theory for spherically symmetric functions is due to one further mechanism. As usual, globalisation is based upon the mass and energy conservation. In the theory of semi-linear Schrödinger equations it is typical that the conservation laws are deduced from a suitably regularised problem (see, e.g., the proof of [**25**, Theorem 3.3.5]). In the present context (Proposition 6.5.5) we follow this scheme showing first the conservation laws at the level of $\widetilde{H}^2_\alpha$-regularity, and then controlling the stability of a density argument which is set for $\widetilde{H}^1_\alpha$-regularity. Clearly the first step appeals to the local $\widetilde{H}^2_\alpha$-theory, which is derived only for the spherically symmetric case, thus the stability argument can only work in the spherically symmetric sector of the energy space. It would be interesting to understand if this is just a technical issue, or if the conservation of energy fails in the non-radial setting.

CHAPTER 7

# Finite energy week solutions to magnetic NLS

In this last Chapter we move on to a different scenario that represents a second playground, besides the singular perturbations of pseudo-differential operators studied in the previous Chapters, for the general scheme for linear and non-linear Schrödinger equations presented in the Introduction.

Here the operator of interest is going to be the *magnetic Laplacian*. We shall then consider the initial value problem associated with the non-linear Schrödinger equation with magnetic potential

$$\mathrm{i}\partial_t u = -(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \tag{7.1}$$

in the complex-valued unknown $u \equiv u(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, where

$$\mathcal{N}(u) = |u|^{\gamma-1}u + (|\cdot|^{-\alpha} * |u|^2)u, \qquad \gamma \in (1,5]\,, \alpha \in (0,3) \tag{7.2}$$

is a defocusing non-linearity, both of local (pure power) and non-local (Hartree) type, and $A : \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{R}^3$ is the external time-dependent magnetic potential.

Formally, solutions of (7.1) conserve in time both the *mass* and the *energy*, defined respectively by

$$\begin{aligned}
(\mathcal{M}(u))(t) &:= \int_{\mathbb{R}^3} |u(t,x)|^2 \,\mathrm{d}x \\
(\mathcal{E}(u))(t) &:= \int_{\mathbb{R}^3} \Big( \tfrac{1}{2}|(\nabla - \mathrm{i}A(t))\,u|^2 + \tfrac{1}{\gamma+1}|u|^{\gamma+1} + \tfrac{1}{4}(|x|^{-\alpha} * |u|^2)|u|^2 \Big)\,\mathrm{d}x\,.
\end{aligned}$$

Let us highlight that we shall choose $A$ within a *considerably larger* class of rough potentials than what customarily considered in the literature so far. We will be in the condition to prove the existence of global-in-time, finite energy, weak solutions to (7.1), without attacking for the moment the general issue of global well-posedness and conservation of energy. To be concrete, let us state the conditions on the magnetic potential.

Assumption 1. *The magnetic potential $A$ belongs to one of the two classes $\mathcal{A}_1$ or $\mathcal{A}_2$ defined by*

$$\begin{aligned}
\mathcal{A}_1 &:= \widetilde{\mathcal{A}}_1 \cap \mathcal{R} \\
\mathcal{A}_2 &:= \widetilde{\mathcal{A}}_2 \cap \mathcal{R}\,,
\end{aligned}$$

*where*

$$\widetilde{\mathcal{A}}_1 := \left\{ A = A(t,x) \;\middle|\; \begin{array}{c} \operatorname{div} A = 0 \ \ \textit{for a.e. } t \in \mathbb{R}, \\ A = A_1 + A_2 \ \textit{such that, for } j \in \{1,2\}, \\ A_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3,\mathbb{R}^3)) \\ a_j \in (4,+\infty], \quad b_j \in (3,6), \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\}$$

*and*

$$\widetilde{\mathcal{A}}_2 := \left\{ A = A(t,x) \;\middle|\; \begin{array}{c} \operatorname{div} A = 0 \ \ \textit{for a.e. } t \in \mathbb{R}, \\ A = A_1 + A_2 \ \textit{such that, for } j \in \{1,2\}, \\ A_j \in L^{a_j}_{\mathrm{loc}}(\mathbb{R}, W^{1,\frac{3b_j}{3+b_j}}(\mathbb{R}^3,\mathbb{R}^3)) \\ a_j \in (2,+\infty], \quad b_j \in (3,+\infty], \quad \frac{2}{a_j} + \frac{3}{b_j} < 1 \end{array} \right\},$$

*and where*

$$\mathcal{R} := \left\{ A \in \widetilde{\mathcal{A}}_1 \text{ or } A \in \widetilde{\mathcal{A}}_2 \mid \partial_t A_j \in L^1_{\mathrm{loc}}(\mathbb{R}, L^{b_j}(\mathbb{R}^3,\mathbb{R}^3)),\ j=1,2 \right\}.$$

A few observations are in order. First and foremost, both classes $\mathcal{A}_1$ and $\mathcal{A}_2$ include magnetic potentials for which in general the validity of Strichartz estimates for the magnetic Laplacian is not known. In fact, for *non-smooth* magnetic potentials, Strichartz estimates are only available with a number of restrictions. When $A$ is time-independent, global-in-time magnetic Strichartz estimates were established by various authors under suitable spectral assumptions (absence of zero-energy resonances) on the magnetic Laplacian $A$ [**39, 40, 31**], or alternatively under suitable smallness of the so called non-trapping component of the magnetic field [**32**], up to the critical scaling $|A(x)| \sim |x|^{-1}$. Counterexamples at criticality are also known [**42**]. In the time-dependent case, magnetic Strichartz estimates are available only under suitable smallness condition of $A$ [**47, 104**].

A large part of our intermediate results, including in particular the local solution theory, are found with magnetic potentials in the larger classes $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{A}}_2$. The mild amount of regularity in time provided by the intersection with the class $\mathcal{R}$ is needed to infer suitable a priori bounds on the solution from the estimates on the total energy. Regularity in time of the external potential is not needed either, when equation (7.1) is studied in the *mass sub-critical* regime, i.e., when $\gamma \in (1, \frac{7}{3})$, $\alpha \in (0,2)$, and $\max\{b_1, b_2\} \in (3,6)$. In this case we are able to work with the more general condition $A \in \widetilde{\mathcal{A}}_1$. This is a customary fact in the context of Schrödinger equations with time-dependent potentials, as well known since [**110**] (compare Theorems [**110**, Theorem 1.1] and [**110**, Theorem 1.4] therein: $L^a$-integrability in time on the electric external potentials yields a $L^p$-theory in space, whereas additional $L^a$-integrability of the time derivative of the potential yields a $H^2$-theory in space). Our aim here of studying finite energy solutions to (7.1) thus requires some intermediate assumptions on the magnetic potential, determined by the class $\mathcal{R}$ above.

The additional requirement on $\nabla A$ present in the class $\mathcal{A}_2$ is taken to accommodate slower decay at infinity for $A$, way slower than the behaviour $|A(x)| \sim |x|^{-1}$ (and in fact even a $L^\infty$-behaviour) which, as mentioned before, is critical for the validity of magnetic Strichartz inequalities.

Last, it is worth remarking that the divergence-free condition, $\operatorname{div} A = 0$, is assumed merely for convenience: our whole analysis can be easily extended to the cases where $\operatorname{div} A$ belongs to suitable Lebesgue spaces and consider it as a given (electrostatic) scalar potential.

Before stating our main result, we collect some preliminary tools. We begin by defining the energy space for the magnetic Laplacian (here, with respect to our general setting, $A$ is meant to be a magnetic vector potential at a fixed time).

DEFINITION 7.0.1. Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$. We define the *magnetic Sobolev space*

$$H^1_A(\mathbb{R}^3) := \{ f \in L^2(\mathbb{R}^3) \mid (\nabla - \mathrm{i}\, A) f \in L^2(\mathbb{R}^d) \}$$

equipped with the norm

$$\|f\|^2_{H^1_A(\mathbb{R}^3)} := \|f\|^2_{L^2(\mathbb{R}^d)} + \|(\nabla - \mathrm{i}\, A) f\|^2_{L^2(\mathbb{R}^d)},$$

which makes $H^1_A(\mathbb{R}^3)$ a Banach space.

We recall [**78**, Theorem 7.21] that, when $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, any $f \in H^1_A(\mathbb{R}^3)$ satisfies the *diamagnetic inequality*

$$|(\nabla |f|)(x)| \leqslant |((\nabla - \mathrm{i}\, A) f)(x)| \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{7.3}$$

As a consequence of diamagnetic inequality, the following two useful Lemmas can be easily proved.

LEMMA 7.0.2. *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$. Then, for every $t \in \mathbb{R}$,*

$$\|2\,\mathrm{i}\,A(t)\cdot\nabla f + |A(t)|^2 f\|_{H^{-1}(\mathbb{R}^3)} \;\lesssim\; C_A(t)\|f\|_{H^1(\mathbb{R}^3)}\,, \tag{7.4}$$

*where*

$$C_A(t) \;:=\; 1 + \|A_1(t)\|^2_{L^{b_1}(\mathbb{R}^3)} + \|A_2(t)\|^2_{L^{b_2}(\mathbb{R}^3)}\,.$$

*In particular, for every $t \in \mathbb{R}$, $(\nabla - \mathrm{i}A(t))^2$ is a continuous map from $H^1(\mathbb{R}^3)$ to $H^{-1}(\mathbb{R}^3)$.*

LEMMA 7.0.3. *Let $A \in L^b(\mathbb{R}^3)$ with $b \in [3, +\infty]$. One has*

$$(1 + \|A\|_{L^b(\mathbb{R}^3)})^{-1}\|f\|_{H^1(\mathbb{R}^3)} \;\lesssim\; \|f\|_{H^1_A(\mathbb{R}^3)} \;\lesssim\; (1 + \|A\|_{L^b(\mathbb{R}^3)})\|f\|_{H^1(\mathbb{R}^3)}\,, \tag{7.5}$$

*whence $H^1_A(\mathbb{R}^3) \cong H^1(\mathbb{R}^3)$ as an isomorphism between Banach spaces*

REMARK 7.0.4. As an immediate consequence of Lemma 7.0.3, given a potential $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, for every $t > 0$ the magnetic Sobolev space $H^1_{A(t)}(\mathbb{R}^3)$ is equivalent to the ordinary Sobolev space $H^1(\mathbb{R}^3)$.

We can finally state our main result, proved in my work in collaboration with A. Michelangeli and P. Antonelli [**13**]. Clearly, there is no fundamental difference in studying solutions forward or backward in time, and as customary we shall only consider henceforth the problem for $t \geqslant 0$. Our entire discussion can be repeated for the case $t \leqslant 0$.

THEOREM 7.0.5 (Existence of global, finite energy weak solutions).
*Let the magnetic potential $A$ be such that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, and take $\gamma \in (1, 5]$, $\alpha \in (0, 3)$. Then, for every initial datum $f \in H^1(\mathbb{R}^3)$, the Cauchy problem*

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -(\nabla - \mathrm{i}\,A)^2 u + |u|^{\gamma-1}u + (|\cdot|^{-\alpha} * |u|^2)u \\ u(0,\cdot) = \; f \end{cases} \qquad t \in [0, +\infty), \quad x \in \mathbb{R}^3 \tag{7.6}$$

*admits a global weak $H^1$-solution*

$$u \;\in\; L^\infty_{\mathrm{loc}}([0, +\infty), H^1(\mathbb{R}^3)) \cap W^{1,\infty}_{\mathrm{loc}}([0, +\infty), H^{-1}(\mathbb{R}^3))\,,$$

*meaning that* (7.1) *is satisfied for a.e. $t \in [0, +\infty)$ as an identity in $H^{-1}$ and $u(0,\cdot) = f$. Moreover, the energy $\mathcal{E}(u)(t)$ is bounded on compact intervals.*

## 7.1. The heat-Schrödinger flow

As already mentioned, Strichartz estimates are in general not available for a magnetic potential $A$ in the class $\mathcal{A}_1$ or $\mathcal{A}_2$. To overcome this issue, we introduce a small dissipation term in equation (7.1)

$$i\partial_t u \;=\; -(1 - \mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \tag{7.7}$$

and we study the approximated problem. Similar *parabolic* (or *vanishing viscosity*) regularisation procedures are commonly used in PDEs, see for example the work by Guo, Nakamitsu, and Strauss [**59**] on the existence of global, finite energy weak solutions to the Maxwell-Schrödinger system.

Let us consider the Cauchy problem associated to (7.7)

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -(1 - \mathrm{i}\,\varepsilon)(\nabla - \mathrm{i}\,A)^2 u + \mathcal{N}(u) \\ u(0,\cdot) \;=\; f \end{cases} \qquad t \in [0, T]\,, \quad x \in \mathbb{R}^3\,. \tag{7.8}$$

We aim at establishing local well-posedness for (7.8), by considering $(1-\mathrm{i}\,\varepsilon)\Delta u$ as the main linear part and $-(1-\mathrm{i}\,\varepsilon)(2\,\mathrm{i}\,A\cdot\nabla u+|A|^2u)+\mathcal{N}(u)$ as a perturbation. The relevant tool is a family of space-time smoothing estimates for the heat-Schrödinger propagator $e^{(i+\varepsilon)t\Delta}$. We recall that a pair $(q,r)$ is said (Strichartz) admissible if

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2},\qquad r\in[2,6].$$

The pair $(2,6)$ is called *endpoint*. We have the following result [**13**, Proposition 2.12].

PROPOSITION 7.1.1 (Space-time estimates for the heat-Schrödinger flow). *Let $\varepsilon>0$ and let $(q,r)$ be an admissible pair.*

(i) *One has (homogeneous Strichartz estimate)*

$$\|e^{(\mathrm{i}+\varepsilon)t\Delta}f\|_{L^q([0,T],L^r(\mathbb{R}^3))}\ \lesssim\ \|f\|_{L^2(\mathbb{R}^3)}\,. \tag{7.9}$$

(ii) *Let $T>0$ and let the pair $(s,p)$ satisfy*

$$\frac{2}{s}+\frac{3}{p}=\frac{7}{2},\qquad \begin{cases}\frac12\leqslant\frac1p\leqslant1 & 2\leqslant r<3\\ \frac12\leqslant\frac1p<\frac1r+\frac23 & 3\leqslant r\leqslant6\,.\end{cases} \tag{7.10}$$

*Then (inhomogeneous retarded Strichartz estimate)*

$$\Big\|\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\Big\|_{L^q([0,T],L^r(\mathbb{R}^3))}\ \lesssim_\varepsilon\ \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}\,. \tag{7.11}$$

(iii) *Assume in addition that $(q,r)$ is non-endpoint. Let $T>0$ and let the pair $(s,p)$ satisfy*

$$\frac{2}{s}+\frac{3}{p}=\frac{5}{2},\qquad \frac12\leqslant\frac1p<\frac1r+\frac13\,. \tag{7.12}$$

*Then (inhomogeneous retarded smoothing-Strichartz estimate)*

$$\Big\|\nabla\!\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta}F(\tau)\,\mathrm{d}\tau\,\Big\|_{L^q([0,T],L^r(\mathbb{R}^3))}\ \lesssim_\varepsilon\ \|F\|_{L^s([0,T],L^p(\mathbb{R}^3))}\,. \tag{7.13}$$

REMARK 7.1.2. It is worth noticing that in (7.10) the range of admissible pairs $(s,p)$ is *larger* as compared to the case of the Schrödinger equation. See [**13**, Remark 2.13] for further observations on this point.

## 7.2. The viscous magnetic propagator

As an intermediate step towards the local well posedness of (7.8), in this Section we discuss the existence of the linear magnetic viscous propagator associated to the equation $\mathrm{i}\partial_t u=-(\nabla-\mathrm{i}A)^2$ and we prove that, under our assumptions on the magnetic potential, it satisifes a family of Strichartz-type estimates. We preliminary need a technical Lemma.

Let us first define, for $T>0$, the space

$$X^{(4,3)}[0,T]\ :=\ L^\infty([0,T],H^1(\mathbb{R}^3))\cap L^4([0,T],W^{1,3}(\mathbb{R}^3))$$

equipped with the Banach norm

$$\|\cdot\|_{X^{(4,3)}[0,T]}:=\|\cdot\|_{L^\infty([0,T],H^1(\mathbb{R}^3))}+\|\cdot\|_{L^4([0,T],W^{1,3}(\mathbb{R}^3))}\,.$$

Owing to the space-time estimates (7.11)-(7.13) for the heat-Schrödinger propagator, the following result can be proved [**13**, Lemmas 2.20 and 2.21]

LEMMA 7.2.1. *Let $A\in\widetilde{\mathcal{A}}_1$ or $A\in\widetilde{\mathcal{A}}_2$, and let $\varepsilon>0$. There exists a constant $\theta_A>0$ such that, for every $T\in(0,1]$,*

$$\Big\|\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta}\big(A(\sigma)\cdot\nabla+|A(\sigma)|^2\big)u(\sigma)\,\mathrm{d}\sigma\,\Big\|_{X^{(4,3)}[0,T]}\ \lesssim_{\varepsilon,A}\ T^{\theta_A}\|u\|_{X^{(4,3)}[0,T]}\,.$$

We can now prove a fundamental Theorem.

THEOREM 7.2.2. *Assume that* $A \in \widetilde{\mathcal{A}}_1$ *or* $A \in \widetilde{\mathcal{A}}_2$. *Let* $\tau \in \mathbb{R}$, $\varepsilon > 0$, $f \in H^1(\mathbb{R}^3)$, *and* $F \in L^s(\mathbb{R}, W^{1,p}(\mathbb{R}^3))$ *for some pair* $(s,p)$ *satisfying* (7.10) *with* $r = 3$. *Consider the inhomogeneous Cauchy problem*

$$\begin{cases} \mathrm{i}\,\partial_t u \;=\; -\,(1-\mathrm{i}\,\varepsilon)(\Delta u - 2\,\mathrm{i}\,A\cdot\nabla u - |A|^2 u) + F \\ u(\tau,\cdot) \;=\; f \end{cases} \tag{7.14}$$

*and the associated integral equation*

$$\begin{aligned} u(t,\cdot) \;&=\; e^{(\mathrm{i}+\varepsilon)(t-\tau)\Delta} f \\ &\quad -\mathrm{i}\int_\tau^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta}\big((1-\mathrm{i}\,\varepsilon)(2\,\mathrm{i}\,A\cdot\nabla u + |A|^2 u)(\sigma) + F(\sigma)\big)\,\mathrm{d}\sigma\,, \end{aligned} \tag{7.15}$$

*There exists a unique solution* $u \in C([\tau,+\infty), H^1(\mathbb{R}^3))$ *to* (7.15). *Moreover, for any* $T > \tau$ *and for any Strichartz pair* $(q,r)$, *with* $r \in [2,3]$,

$$\|u\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \;\lesssim_{\varepsilon,A,T}\; \|f\|_{H^1(\mathbb{R}^3)} + \|F\|_{L^s(\mathbb{R},W^{1,p}(\mathbb{R}^3))}. \tag{7.16}$$

PROOF. It is clearly non-restrictive to set the initial time $\tau = 0$. For given $T \in (0,1]$ and $M > 0$, we consider the ball of radius $M$ in $X^{(4,3)}[0,T]$, i.e.,

$$\mathcal{X}_{T,M} \;:=\; \{u \in X^{(4,3)}[0,T] \mid \|u\|_{X^{(4,3)}[0,T]} \leqslant M\}.$$

Moreover, we define the solution map $u \mapsto \Phi u$ where, for $t \in [0,T]$,

$$\begin{aligned} (\Phi u)(t) \;&:=\; e^{(\mathrm{i}+\varepsilon)t\Delta} f \\ &\quad -(\mathrm{i}+\varepsilon)\int_0^t e^{(\mathrm{i}+\varepsilon)(t-\sigma)\Delta}\big((2\,\mathrm{i}\,A(\sigma)\cdot\nabla + |A(\sigma)|^2)u(\sigma) + F(\sigma)\big)\,\mathrm{d}\sigma\,. \end{aligned} \tag{7.17}$$

Thus, finding a solution to the integral equation (7.15), with $\tau = 0$, is equivalent to finding a fixed point for the map $\Phi$. We shall then prove Theorem 7.2.2 by showing that, for suitable $T$ and $M$, the map $\Phi$ is a contraction on $\mathcal{X}_{T,M}$. To this aim, let us consider a generic $u \in \mathcal{X}_{T,M}$: owing to the Strichartz estimates (7.9) and (7.13) and to Lemma 7.2.1, there exist positive constants $C \equiv C_{\varepsilon,A}$ and $\theta \equiv \theta_A$ such that, for $T \in (0,1]$,

$$\|\Phi u\|_{X^{(4,3)}[0,T]} \;\leqslant\; C\left(\|f\|_{H^1(\mathbb{R}^3)} + \|F\|_{L^s([0,T],W^{1,p}(\mathbb{R}^3)} + T^\theta\|u\|_{X^{(4,3)}[0,T]}\right). \tag{7.18}$$

It is possible to restrict further $M$ and $T$ such that

$$M \;>\; 2C\left(\|f\|_{H^1(\mathbb{R}^3)} + \|F\|_{L^s([0,T],W^{1,p}(\mathbb{R}^3))}\right)$$

and $2CT^\theta < 1$, in which case (7.18) yields

$$\|\Phi u\|_{X^{(4,3)}[0,T]} \;\leqslant\; M(\tfrac{1}{2} + CT^\theta) \;<\; M\,.$$

This proves that $\Phi$ maps indeed $\mathcal{X}_{T,M}$ into itself. Next, for generic $u, v \in \mathcal{X}_{T,M}$, and with the above choice of $M$ and $T$, (7.18) also yields

$$\begin{aligned} \|\Phi u - \Phi v\|_{X^{(4,3)}[0,T]} \;&=\; \|\Phi(u-v)\|_{X^{(4,3)}[0,T]} \;\leqslant\; CT^\theta\|u-v\|_{X^{(4,3)}[0,T]} \\ &<\; \frac{1}{2}\|u-v\|_{X^{(4,3)}[0,T]}\,, \end{aligned}$$

which proves that $\Phi$ is indeed a contraction on $\mathcal{X}_{T,M}$. By Banach's fixed point theorem, we conclude that the integral equation $u = \Phi u$ has a unique solution in $\mathcal{X}_{T,M}$. Furthermore, $\Phi u \in C([0,T], H^1(\mathbb{R}^3))$. Hence, we have found a local solution $u \in C([0,T], H^1(\mathbb{R}^3))$ to the integral equation (7.15), which satisfies (7.16). Moreover, since the local existence time $T$ does not depend on the initial data, this solution can be extended globally in time, and (7.16) is satisfied for any $T > 0$. $\square$

Theorem 7.2.2 shows the existence of a unique solution $u$ to the integral equation (7.15). From the assumptions on the magnetic potential and the source term $F$ and by using standard arguments in the theory of evolution equations (see for example [**25**]) we may also infer that $u$ satisfies (7.14) for almost every $t \in \mathbb{R}$ in the sense of distributions. In the case when $F = 0$, the solution $u$ to (7.14) defines an evolution operator, namely for any $f \in H^1(\mathbb{R}^3)$ the *magnetic viscous evolution* is defined by $\mathcal{U}_{\varepsilon,A}(t,\tau)f := u(t)$ where $u$ is the solution to (7.14) with $F = 0$. As a consequence of Theorem 7.2.2 we have that $\mathcal{U}_{\varepsilon,A}(t,\tau)$ enjoys a class of Strichartz-type estimates [**13**, Proposition 3.2].

PROPOSITION 7.2.3. *The family $\{\mathcal{U}_{\varepsilon,A}(t,\tau)\}_{t,\tau}$ of operators on $H^1(\mathbb{R}^3)$ satisfies the following properties:*

(i) *$\mathcal{U}_{\varepsilon,A}(t,s)\mathcal{U}_{\varepsilon,A}(s,\tau) = \mathcal{U}_{\varepsilon,A}(t,\tau)$ for any $\tau < s < t$;*
(ii) *$\mathcal{U}_{\varepsilon,A}(t,t) = \mathbb{1}$;*
(iii) *the map $(t,\tau) \mapsto \mathcal{U}_{\varepsilon,A}(t,\tau)$ is strongly continuous in $H^1(\mathbb{R}^3)$;*
(iv) *for any admissible pair $(q,r)$ with $r \in [2,3]$, and for any $f \in H^1(\mathbb{R}^3)$ and $F \in L^s(\mathbb{R}, W^{1,p}(\mathbb{R}^3))$ for some pair $(s,p)$ satisfying (7.10) with $r = 3$ (in particular, $(s,p)$ can be the dual of an admissible pair), one has*

$$\|\mathcal{U}_{\varepsilon,A}(t,\tau)f\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,A,T} \|f\|_{H^1(\mathbb{R}^3)} \tag{7.19}$$

$$\Big\| \int_\tau^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\, F(\sigma)\,\mathrm{d}\sigma \Big\|_{L^q([\tau,T],W^{1,r}(\mathbb{R}^3))} \lesssim_{\varepsilon,A,T} \|F\|_{L^s([\tau,T],W^{1,p}(\mathbb{R}^3))}. \tag{7.20}$$

## 7.3. Solution theory for the regularised magnetic NLS

In this Section we study local and global solution theory for the approximated magnetic NLS (7.7). We can write the integral formulation for its associated Cauchy problem (7.8) in terms of the viscous magnetic propagator:

$$u(t) \;=\; \mathcal{U}_{\varepsilon,A}(t,\tau)f - \mathrm{i}\int_\tau^t \mathcal{U}_{\varepsilon,A}(t,\sigma)\,\mathcal{N}(u(\sigma))\,\mathrm{d}\sigma\,. \tag{7.21}$$

Owing to (7.21) and the Strichartz-type estimates (7.19)-(7.20) we can set up a fixed point argument and show the existence of solutions to (7.8). We first focus on the case of energy sub-critical non-linearities.

PROPOSITION 7.3.1 (Local well-posedness, energy sub-critical case). *Let $\varepsilon > 0$. Assume that $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$ and that the exponents in the non-linearity (7.2) are in the regime $\gamma \in (1,5)$ and $\alpha \in (0,3)$. Then for any $f \in H^1(\mathbb{R}^3)$ there exists a unique solution $u \in C([0,T_{max}), H^1(\mathbb{R}^3))$ to (7.21) on a maximal interval $[0,T_{max})$ such that the following blow-up alternative holds: if $T_{max} < +\infty$ then $\lim_{t\uparrow T_{max}} \|u(t)\|_{H^1} = +\infty$.*

PROOF. Since the linear magnetic viscous propagator $\mathcal{U}_{\varepsilon,A}(t,\tau)$ satisfies the Strichartz-type estimates (7.19)-(7.20), and since the non-linearities considered here are sub-critical perturbation of the linear flow, a customary contraction argument in the space

$$C([0,T],H^1(\mathbb{R}^3)) \cap L^{q(\gamma)}([0,T],W^{1,r(\gamma)}(\mathbb{R}^3)) \cap L^{q(\alpha)}([0,T],W^{1,r(\alpha)}(\mathbb{R}^3))\,, \tag{7.22}$$

where

$$(q(\gamma),r(\gamma)) \;:=\; \Big(\tfrac{4(\gamma+1)}{\gamma-1},\, \tfrac{3(\gamma+1)}{\gamma+2}\Big) \tag{7.23}$$

(see, e.g., [**80**, Theorems 2.1 and 3.1]) and

$$(q(\alpha),r(\alpha)) \;:=\; \begin{cases} (+\infty,2) & \alpha \in (0,2] \\ \Big(\tfrac{6}{\alpha-2},\, \tfrac{18}{13-2\alpha}\Big) & \alpha \in (2,3) \end{cases} \tag{7.24}$$

(see, e.g., [**79**, Section 5.2]), guarantees the existence of a unique local solution for sufficiently small $T$. We observe, in particular, that with the above choice one has $r(\gamma), r(\alpha) \in [2,3)$. Furthermore, by a customary continuation argument we can extend such a solution over a maximal interval for which the blow-up alternative holds true. ☐

In the presence of a energy-critical non-linearity ($\gamma = 5$) the above arguments cannot be applied. However, it is possible to exploit a similar idea as in [**26**] to infer a local well-posedness result when $\gamma = 5$.

Proposition 7.3.2 (Local existence and uniqueness, energy critical case). *Let $A \in \widetilde{\mathcal{A}}_1$ or $A \in \widetilde{\mathcal{A}}_2$ and let the exponents in the non-linearity (7.2) be in the regime $\gamma = 5$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$ and $f \in H^1(\mathbb{R}^3)$. There exists $\eta_0 > 0$ such that, if*

$$\|\nabla e^{\mathrm{i}t\Delta} f\|_{L^6([0,T],L^{\frac{18}{7}}(\mathbb{R}^3))} \leqslant \eta \tag{7.25}$$

*for some (small enough) $T > 0$ and some $\eta < \eta_0$, then there exists a unique solution $u \in \mathcal{C}([0,T], H^1(\mathbb{R}^3))$ to (7.21). Moreover, this solution can be extended on a maximal interval $[0, T_{max})$ such that the following blow-up alternative holds true: $T_{max} < \infty$ if and only if $\|u\|_{L^6([0,T_{max}),L^{18}(R^3))} = \infty$.*

Proof. A direct application of a well-known argument by Cazenave and Weissler [**26**] (we refer to [**72**, Section 3] for a more recent discussion). In particular, having established Strichartz estimates for $\mathcal{U}_{\varepsilon,A}(t,\tau)$ relative to the pair $(q,r) = (6, \frac{18}{7})$, we proceed exactly as in the proof of [**72**, Theorem 3.4 and Corollary 3.5], so as to find a unique solution $u$ to the integral equation (7.21) in the space

$$C([0,T], H^1(\mathbb{R}^3)) \cap L^6([0,T], W^{1,\frac{18}{7}}(\mathbb{R}^3)) \cap L^{q(\alpha)}([0,T], W^{1,r(\alpha)}(\mathbb{R}^3)) \tag{7.26}$$

with $(q(\alpha), r(\alpha))$ given by (7.24), together with the $L^6_t L^{18}_x$-blow-up alternative. ☐

Our next step is to establish some a priori estimates which will be needed in order to extend the local approximating solution to (7.21) over arbitrary time intervals. In particular, we show that the total mass and energy are uniformly bounded. Furthermore, by exploiting the dissipative regularisation, we will infer some a priori space-time bounds which will allow to extend globally the solution also in the energy-critical case.

We consider potentials $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, so to have the time regularity needed in order to study the energy functional. The following result can be proved [**13**, Proposition 5.2].

Proposition 7.3.3. *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, and that the exponents in the non-linearity (7.2) are in the whole regime $\gamma \in (1,5]$ and $\alpha \in (0,3)$. For fixed $\varepsilon > 0$, let $u_\varepsilon \in C([0,T), H^1(\mathbb{R}^3))$ be the local solution to the regularised equation (7.7) for some $T > 0$. Then the mass, the energy, and the $H^1$-norm of $u_\varepsilon$ are bounded in time over $[0,T)$, uniformly in $\varepsilon > 0$, that is,*

$$\sup_{t\in[0,T]} \mathcal{M}(u_\varepsilon) \lesssim 1 \tag{7.27}$$

$$\sup_{t\in[0,T]} \mathcal{E}(u_\varepsilon) \lesssim_{A,T} 1 \tag{7.28}$$

$$\|u_\varepsilon\|_{L^\infty([0,T),H^1(\mathbb{R}^3))} \lesssim_{A,T} 1\,, \tag{7.29}$$

*and moreover one has the a priori bounds*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3} \Big(&|(\nabla - \mathrm{i}A(t))u_\varepsilon|^2\big(|u_\varepsilon|^{\gamma-1} + (|x|^{-\alpha} * |u_\varepsilon|^2)\big) + (\gamma-1)|u_\varepsilon|^{\gamma-1}|\nabla|u_\varepsilon||^2 \\ &+ (|x|^{-\alpha} * \nabla|u_\varepsilon|^2)\nabla|u_\varepsilon|^2\Big)\,\mathrm{d}x\,\mathrm{d}t \;\lesssim_{A,T}\; \varepsilon^{-1}\,.\end{aligned} \tag{7.30}$$

REMARK 7.3.4. A virtue of Proposition 7.3.3 is to produce bounds (7.27)-(7.29) that are *uniform* in $\varepsilon$. The non-uniformity in $T$ of (7.28)-(7.29) is due to the fact that the magnetic potential is only $AC_{\mathrm{loc}}$ in time: for $AC$-potentials such bounds would be uniform in $T$ as well.

We can now exploit the a priori estimates for mass and energy so as to prove that the local solution to the regularised Cauchy problem (7.8) can be actually extended globally in time.

We discuss first the result in the energy sub-critical case.

THEOREM 7.3.5 (Global well-posedness, energy sub-critical case). *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, and that the exponents in the non-linearity* (7.2) *are in the regime $\gamma \in (1,5)$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$. Then the regularised non-linear magnetic Schrödinger equation* (7.7) *is globally well-posed in $H^1(\mathbb{R}^3)$. Moreover, the solution $u_\varepsilon$ to* (7.7) *with given initial datum $f \in H^1(\mathbb{R}^3)$ satisfies the bound*

$$\|u_\varepsilon\|_{L^\infty[0,T],H^1(\mathbb{R}^3)} \lesssim_T 1 \qquad \forall T \in (0,+\infty)\,, \tag{7.31}$$

*uniformly in $\varepsilon > 0$.*

PROOF. The local well-posedness is proved in Proposition 7.3.1. Because of (7.29), the $H^1$-norm of $u_\varepsilon$ is bounded on finite intervals of time. Therefore, by the blow-up alternative, the solution is necessarily global and in particular it satisfies the bound (7.31). □

We discuss now the analogous result in the energy-critical case.

THEOREM 7.3.6 (Global existence and uniqueness, energy critical case). *Assume that $A \in \mathcal{A}_1$ or $A \in \mathcal{A}_2$, and that the exponents in the non-linearity* (7.2) *are in the regime $\gamma = 5$ and $\alpha \in (0,3)$. Let $\varepsilon > 0$ and $f \in H^1(\mathbb{R}^3)$. The Cauchy problem* (7.8) *has a unique global strong $H^1$-solution $u_\varepsilon$. Moreover, $u$ satisfies the bound*

$$\|u_\varepsilon\|_{L^\infty[0,T],H^1(\mathbb{R}^3)} \lesssim_T 1 \qquad \forall T \in (0,+\infty)\,, \tag{7.32}$$

*uniformly in $\varepsilon > 0$.*

PROOF. The existence of a unique local solution $u_\varepsilon$ is proved in Proposition 7.3.2. The a priori bound (7.30) implies that

$$\int_0^T\!\!\int_{\mathbb{R}^3} \big(|u_\varepsilon|^2\,\nabla|u_\varepsilon|\big)^2\,\mathrm{d}x\,\mathrm{d}t \;\lesssim\; \varepsilon^{-1}\,,$$

which, together with Sobolev's embedding, yields

$$\begin{aligned} \|u_\varepsilon\|^6_{L^6([0,T],L^{18}(\mathbb{R}^3))} \;&=\; \|u_\varepsilon^3\|^2_{L^2([0,T],L^6(\mathbb{R}^3))} \;\lesssim\; \int_0^T\!\!\int_{\mathbb{R}^3} |\nabla|u_\varepsilon|^3|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\lesssim\; \int_0^T\!\!\int_{\mathbb{R}^3} |u_\varepsilon|^4\,|\nabla|u_\varepsilon||^2\,\mathrm{d}x\,\mathrm{d}t \;\lesssim\; \varepsilon^{-1} \,<\, +\infty\,. \end{aligned} \tag{7.33}$$

Owing to (7.33) and to the blow-up alternative proved in Proposition 7.3.2, we conclude that the solution $u$ can be extended globally and moreover, using again (7.29), it satisfies the bound (7.32). □

REMARK 7.3.7. As anticipated right after stating the assumptions on the magnetic potential, let us comment here about the fact that in the *mass sub-critical* regime ($\gamma \in (1,\frac{7}{3})$ and $\alpha \in (0,2)$) we can work with the larger class $\widetilde{\mathcal{A}}_1$ instead of $\mathcal{A}_1$ and still prove the extension of the local solution globally in time with finite $H^1$-norm on arbitrary finite time interval. This is due to the fact that, for a potential $u \in \widetilde{\mathcal{A}}_1$ and in the mass sub-critical regime, in order to extend the solution globally neither need we the estimate (7.29) as in the proof of Theorem 7.3.5, nor need we the estimate (7.30) as in the proof of Theorem 7.3.6. Indeed, we can first prove local well-posedness in $L^2(\mathbb{R}^3)$ for the regularised magnetic NLS (7.7), using a fixed point argument based on the space-time estimates for the heat-Schrödinger flow, in the very same spirit of the proof of Theorem 7.2.2. Then we can extend such a solution globally in time using only the mass a priori bound (7.27), for proving such a bound does not require any time-regularity assumption on the magnetic potential. Moreover, since the non-linearities are mass sub-critical and since we can prove convenient estimates on the commutator $[\nabla,(\nabla-\mathrm{i}\,A)^2]$ when $\max\{b_1,b_2\}\in(3,6)$, we can show that the global $L^2$-solution exhibits *persistence of $H^1$-regularity* in the sense that it stays in $H^1(\mathbb{R}^3)$ for every positive time provided that the initial datum belongs already to $H^1(\mathbb{R}^3)$. This way, we obtain existence and uniqueness of one global strong $H^1$-solution.

### 7.4. Removing the regularisations

In this Section we prove our main Theorem 7.0.5. The proof is based on a compactness argument, owing to the uniform bounds (7.31) and (7.32), so as to remove the $\varepsilon$-regularisation and leads to a *local* weak $H^1$-solution to (7.6). The crucial result is the following [**13**, Proposition 7.1].

PROPOSITION 7.4.1. *Assume that $A\in\mathcal{A}_1$ or $A\in\mathcal{A}_2$, and that the exponents in the non-linearity* (7.2) *are in the whole regime $\gamma\in(1,5]$ and $\alpha\in(0,3)$. Let $T>0$, and $f\in H^1(\mathbb{R}^3)$. For any sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n\downarrow 0$, let $u_n$ be the unique global strong $H^1$-solution to the Cauchy problem* (7.8) *with viscosity parameter $\varepsilon=\varepsilon_n$ and with initial datum $f$, as provided by Theorem 7.3.5 in the energy sub-critical case and by Theorem 7.3.6 in the energy critical case. Then, up to a subsequence, $u_n$ converges weakly-$*$ in $L^\infty([0,T],H^1(\mathbb{R}^3))$ to a local weak $H^1$-solution $u$ to the magnetic NLS* (7.1) *in the time interval $[0,T]$ and with initial datum $f$, meaning that*

$$u \;\in\; L^\infty([0,T],H^1(\mathbb{R}^3))\cap W^{1,\infty}([0,T],H^{-1}(\mathbb{R}^3))\,,$$

*satisfies* (7.1) *for a.e. $t\in[0,T]$ as an identity in $H^{-1}$, and $u(0,\cdot)=f$.*

PROOF. We only sketch here the arguments we presented in detail in [**13**, Section 7]. Owing to the *uniform-in-$\varepsilon$* bounds (7.31) and (7.32) and the Banach-Alaoglu theorem, we deduce that the sequence $(u_n)_n$ in the assumption of Proposition 7.4.1 admits, up to a subsequence, a weak-$*$ limit $u$ in $L^\infty([0,T],H^1(\mathbb{R}^3))$. Moreover, it can be proved [**13**, Corollary 7.5] that there exist indices $p_i$, $p_{ij}$, $p(\gamma)$, and $\widetilde{p}(\alpha)$ in $[\frac{6}{5},2]$, and there exists functions $X_i\in L^\infty([0,T],L^{p_i}(\mathbb{R}^3))$, $Y_{ij}\in L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3))$, $N_1\in L^\infty([0,T],L^{p(\gamma)}(\mathbb{R}^3))$, and $N_2\in L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3))$ such that, up to subsequences,

$$\begin{aligned}
A_i\cdot\nabla u_n \;&\to\; X_i \qquad \text{weakly-}*\text{ in } L^\infty([0,T],L^{p_i}(\mathbb{R}^3))\\
A_i\cdot A_j u_n \;&\to\; Y_{ij} \qquad \text{weakly-}*\text{ in } L^\infty([0,T],L^{p_{ij}}(\mathbb{R}^3))\\
|u_n|^{\gamma-1}u_n \;&\to\; N_1 \qquad \text{weakly-}*\text{ in } L^\infty([0,T],L^{p(\gamma)}(\mathbb{R}^3))\\
(|\cdot|^{-\alpha}*|u_n|^2)u_n \;&\to\; N_2 \qquad \text{weakly-}*\text{ in } L^\infty([0,T],L^{\widetilde{p}(\alpha)}(\mathbb{R}^3))\,.
\end{aligned}$$

As an application of the Aubin-Lions compactenss lemma (see, e.g., [**98**, Section 7.3]) one can deduce the following identities [**13**, Lemma 7.8]:

$$\begin{aligned}
A_i\cdot\nabla u \;&=\; X_i\\
A_i\cdot A_j u \;&=\; Y_{ij}\\
|u|^{\gamma-1}u \;&=\; N_1\\
\big(|\cdot|^{-\alpha}*|u|^2\big)u \;&=\; N_2\,.
\end{aligned}$$

We are now able to show that the function $u$ is actually a local weak $H^1$-solution to the magnetic NLS (7.1) with initial datum $f$ in the time interval $[0,T]$. Indeed, all the exponents $p_i$, $p_{ij}$, $p(\gamma)$ and $\widetilde{p}(\alpha)$ belong to the interval $[\frac{6}{5},2]$, and then by Sobolev's embedding the functions $X_i = A_i \cdot \nabla u$, $Y_{ij} = A_i \cdot A_j u$, $N_1 = |u|^{\gamma-1}u$, and $N_2 = (|\cdot|^{-\alpha} * u^2)u$ all belong to $H^{-1}(\mathbb{R}^3)$, and so too does $\Delta u$, obviously. Therefore (7.1) is satisfied by $u$ as an identity between $H^{-1}$-functions, which also implies $\partial_t u \in L^\infty([0,T],H^{-1}(\mathbb{R}^3))$. Thus, $u \in W^{1,\infty}([0,T],H^{-1}(\mathbb{R}^3))$. On the other hand $u_n \in C^1([0,T],H^{-1}(\mathbb{R}^3))$, whence

$$\int_0^T\!\!\int_{\mathbb{R}^3} \eta(t,x)\big(u_n(t,x)-u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t \;\to\; 0 \qquad \forall\eta \in L^1([0,T],H^{-1}(\mathbb{R}^3))\,.$$

For $\eta(t,x) = \delta(t-t_0,x)\varphi(x)$, where $t_0$ is arbitrary in $[0,T]$ and $\varphi$ is arbitrary in $L^2(\mathbb{R}^3)$, the limit above reads $u_n(t_0,\cdot) \to u(t_0,\cdot)$ weakly in $L^2(\mathbb{R}^3)$, whence $u(0,\cdot) = f(\cdot)$. □

It is already evident at this stage that had we assumed the magnetic potential to be an $AC$-function for all times, then the proof of the existence of a global weak solution with finite energy would be completed with the proof of Proposition 7.4.1 above, in full analogy with the scheme of the work [**59**] above mentioned.

Our potential being in general only $AC_{\mathrm{loc}}$ in time, we cannot appeal to bounds that are uniform in time (indeed, our (7.31) and (7.32) are $T$-dependent), and the following straightforward 'glueing' argument must be added in order to complete the proof of our main result.

Proof of Theorem 7.0.5. We set $T = 1$ and we choose an arbitrary sequence $(\varepsilon_n)_n$ of positive numbers with $\varepsilon_n \downarrow 0$. Let $u_n$ be the unique local strong $H^1$-solution to the regularised magnetic NLS (7.7) with viscosity parameter $\varepsilon = \varepsilon_n$ and with initial datum $f \in H^1(\mathbb{R}^3)$. By Proposition 7.4.1, there exists a subsequence $(\varepsilon_{n'})_{n'}$ of $(\varepsilon_n)_n$ such that $u_{n'} \to u_1$ weakly-$*$ in $L^\infty([0,1],H^1(\mathbb{R}^3))$, where $u_1$ is a local weak $H^1$-solution to the magnetic NLS (7.1) with $u_1(0) = f$. If we take instead $T = 2$ and repeat the argument, we find a subsequence $(\varepsilon_{n''})_{n''}$ of $(\varepsilon_{n'})_{n'}$ such that $u_{n''} \to u_2$ weakly-$*$ in $L^\infty([0,2],H^1(\mathbb{R}^3))$, where $u_2$ is a local weak $H^1$-solution to (7.1) with $u_2(0) = f$, now in the time interval $[0,2]$. Moreover, having refined the $u_{n'}$'s in order to obtain the $u_{n''}$'s, necessarily $u_2(t) = u_1(t)$ for $t \in [0,1]$. Iterating this process, we construct for any $N \in \mathbb{N}$ a function $u_N$ which is a local weak $H^1$-solution to (7.1) in the time interval $[0,N]$, with $u_N(0) = f$ and $u_N(t) = u_{N-1}(t)$ for $t \in [0,N-1]$. It remains to define

$$u(t,x) \;:=\; u_N(t,x) \qquad x \in \mathbb{R}^3\,, \quad t \in [0,+\infty) \quad N = [t]\,.$$

Since $u_N \in L^\infty([0,N],H^1(\mathbb{R}^3)) \cap W^{1,\infty}([0,N],H^{-1}(\mathbb{R}^3))$ for every $N \in N$, such $u$ turns out to be a global weak $H^1$-solution to (7.6) with finite energy for a.e. $t \in \mathbb{R}$, uniformly on compact time intervals. □

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