Scuola Internazionale Superiore di Studi Avanzati

Ph.D. Course in Mathematical Analysis, Modelling and Applications

Dimension reduction problems in the modelling of hydrogel thin films

Ph.D. Thesis

Candidate:
Danka Lučić

Supervisors:
Prof. Antonio DeSimone
Dr. Virginia Agostiniani

Academic year 2017/2018

Il presente lavoro costituisce la tesi presentata da Danka Lučić, sotto la direzione del Prof. Antonio DeSimone e della Dr. Virginia Agostiniani, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Philosophiæ* presso la SISSA, Curriculum in Analisi Matematica, Modelli e Applicazioni. Ai sensi dell'art. 1, comma 4, dello Statuto della SISSA pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, Anno Accademico 2017/2018

Acknowledgements

Foremost, I would like to thank my advisors Prof. Antonio DeSimone and Dr. Virginia Agostiniani for having chosen an interesting research topic and for their guidance during my PhD study and research.

I am also grateful to Prof. Gianni Dal Maso and Prof. Marta Lewicka for having shared with me many interesting and valuable ideas, to Dr. Alessandro Lucantonio for many useful discussions and to the referees of my thesis Prof. Raz Kupferman and Prof. Corrado Maurini for many helpful comments and suggestions.

My sincere gratitude goes to Prof. Nicola Gigli for several useful discussions and his valuable advice.

These four years in Trieste will be unforgettable because they brought to me Poliane Teixeira, Maja Berović, Elio Marconi, Paolo Bonicatto, Giulia Vescovo, Ornela Mulita, Emanuele Tasso, Maicol Caponi, Flavia Santarcangelo and Jesus Núñez-Zimbrón - friends with whom I shared hundreds of beautiful and funny moments.

I am very grateful to Rosanna Cotterle. She was always there - to encourage me, to prepare a perfect coffee and *panino con cotto tagliato a mano*. I will miss her a lot.

I wish to thank Remigio and Morena for their warm hospitality and unconditional help, and obviously to Leonardo - it was the most delicious barbecue ever!

On the other side, every coming back home to Serbia couldn't finish with so many colorful memories without my friends Marina Stojanac, Milana Paldrmić, Marko Paldrmić and Gorica Vidaković - I thank them a lot.

A special thank goes to my grandmother Smiljana and my grandfather Velimir for all their support since I made my first steps and to my uncle Dragan - this thesis wouldn't be so nice without your drawings.

Finally, I am in debt with my parents Branko and Snežana and my sister Milica for their sincere advice, encouragements and for being close to me always.

- "Perchè lo ami?"
- "Non lo so."

Someone told me that it was the right answer.

Enrico, thank you for believing in me.

Trieste,
22 September 2018

Contents

List of Figures

"There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world."

Nikolai Ivanovich Lobachevsky

Introduction

The study of shape programming and morphing of surfaces is receiving considerable attention, especially in the field of *soft active* (or *smart*) materials, i.e. materials that deform in response to non-mechanical stimuli. Of particular interest is the problem of exploiting material heterogeneities to induce complex shape changes – for instance, to produce curved configurations from an initially flat state. In many natural systems [AHMD12, AD14], such as plants [AEKS11, DVR97], shape control is usually accomplished by growth, remodelling or swelling, in response to simple external stimuli (e.g. to a uniform change in the ambient temperature or humidity). In order to mimic such behaviours, thin sheets made of synthetic soft active materials appear as suitable candidates. In particular, in these systems curvature arises from heterogeneous in-plane [dHSSB+12, KES07, KHB+12, WMG+13, PSNH15] or through-the-thickness strains [AD15, AD17, PSNH16, SUT+10, SYU+11, SZT+12], which are induced by heterogeneous material properties, including variable anisotropy. In this thesis our attention will be especially focused on thin sheets made of *hydrogel*, namely an active material in which spontaneous deformations are induced by swelling due to the absorption of a liquid.

The behaviour of soft active materials can be described in the framework of nonlinear elasticity by exploiting models for *hyperelastic* materials whose response to the external stimuli is seen through a non-negative energy density function. More specifically, one considers a 3D hyperelastic sheet of small thickness h, occupying a domain $\Omega_h = \omega \times (-h/2, h/2) \subseteq \mathbb{R}^3$, where the midplate $\omega \subseteq \mathbb{R}^2$ is a bounded Lipschitz domain. In the sequel we shall use the notation $z = (z', z_3)$ for an arbitrary point in Ω_h. The total energy of the above described system associated with a deformation $v : \Omega_h \to \mathbb{R}^3$, in absence of external forces, is given by the quantity

$$\overline{\mathscr{E}}_h(v) = \int_{\Omega_h} \overline{W}_h\big(z, \nabla v(z)\big)\,\mathrm{d}z,$$

where $\overline{W}_h$ is the mentioned non-negative energy density function, defined on $\Omega_h \times \mathbb{R}^{3\times3}$ and satisfying the standard frame-indifference, growth and regularity conditions in non-linear elasticity. Non-trivial complex shapes arise as equilibrium configurations, or equivalently as *minimizers* of the above elastic energy functional. The investigation of the latter relies upon the understanding of *the relation between the three-dimensional theory and the lower-dimensional ones.* This task has become one of the fundamental problems in nonlinear elasticity and opened many interesting questions that lie on the interface between Mechanics, Geometry and Analysis. Several problems arise with regard to the above proposed functional. First, one has to determine the scaling of the infimum energy in terms of the small thickness of the sheet. Second, to derive the "limiting" lower dimensional theories under the identified scaling. Another problem is the study (from a geometric perspective) of the obtained dimensionally reduced models.

The lower dimensional theories (e.g. plate, shell, rod or ribbons) became a topic of interest in both the mathematical and the physical literature – [BBN15, ASK14, BLS16, SE10, LTD16, Mos15] and the references hereafter are some of the many examples of plate models, which will be the main focus of this thesis.

There are various approaches to the derivation of the plate theories in the context of nonlinear elasticity. The one that is most commonly used is the following: to make a certain ansatz for the expected form of the three dimensional deformation (based on physical intuition) and to perform a formal expansion with respect to the small thickness parameter. Clearly, different ansatzes may give a large variety of plate theories that on one hand are useful for some applications but on the other hand may lead to contradicting predictions (for a nice discussion see [DFMS17]).

A variational approach is instead ansatz-free and allows for the rigorous derivation of lower dimensional theories. It is known as the *dimension reduction* technique, based on the theory of Γ-convergence. The relevance of employing such a rigorously derived theory is twofold: one avoids inconsistencies, which are often present in ad-hoc formally-deduced models by computing a mathematically rigorous limit; the rigorous procedure guarantees that the minimizers of the 2D model faithfully reproduce the true behaviour of the sufficiently thin (yet, three-dimensional) sheets.

Up to now, using this approach, a wide class of plate models has been rigorously derived from three-dimensional elasticity. The nonlinear membrane theory has been derived in the limit of the vanishing sheet's thickness h by H. Le Dret and A. Raoult in [LDR95], corresponding to the asymptotic behaviour $\overline{\mathscr{E}}_h/h \sim 1$. The rigidity estimate proved by G. Friesecke, R.D. James and S. Müller in [FJM02] – providing a precise quantitative version of the idea that three-dimensional deformations with low energy should be very close to rigid motions – constituted a key ingredient in the rigorous derivation of the bending plate theory (under the scaling $\overline{\mathscr{E}}_h/h \sim h^2$) provided in the same paper. The same authors rigorously derived in [FJM06] von Kármán plate model (corresponding to the scaling order $\overline{\mathscr{E}}_h/h \sim h^4$), and introduced some further models corresponding to the intermediate scaling regimes. The above mentioned models are obtained in the setting of homogeneous thin structures, characterized by a z and h independent energy density function – namely, $\overline{W}_h(z, F) = W(F)$, with W minimized at the set of three-dimensional rotations SO(3). In this case, the energy scaling is driven by the magnitude of the applied forces.

Recently, a growing interest in the study of thin sheets that may assume non-trivial configurations in absence of external forces or boundary conditions motivated developments of the plate (and other lower dimensional) theories in the context of *pre-stretched* elastic bodies (also called "pre-stressed" or "pre-strained") – in other words, elastic bodies for which there is no natural notion of reference configuration (i.e. a configuration which is stress-free). Within this setting, the elastic properties of a thin sheet Ω_h are modeled by a density function $\overline{W}_h$ and by a positive definite, symmetric tensor field $U_h : \Omega_h \to \mathbb{R}^{3\times3}$ (referred to as *pre-stretch*) satisfying

$$\overline{W}_h(z, F) = W\big(FU_h(z)\big), \tag{0.1}$$

where W is a homogeneous density function as above. Thus, the state of deformation that minimizes the energy density is precisely its inverse $A_h = U_h^{-1}$, which is referred to as *spontaneous stretch.* The field A_h typically represents an active growth, a plasticity phenomenon or an inelastic one.

The representation of the energy densities in (0.1) is in the spirit of the *homogeneous* constitutive relation

$$\overline{W}(x,F) = \mathcal{W}(FU_x),$$

which dates back to early works of W. Noll and C.-C. Wang [Nol68, Wan66, Wan68]. There, an elastic body is modeled as a three-dimensional manifold M with the associated energy density function $\mathcal{W} : \mathbb{R}^{3\times 3} \to \mathbb{R}$ called *archetypal* and a field $U_x : TM \to \mathbb{R}^3$ called *implant map* – namely, such map represents how an orthonormal frame is implanted in the tangent space of M in each point. It clearly induces a reference metric on M.

An alternative way (discussed bellow) to the study of pre-stretched materials, due to E. Efrati and E. Sharon [KES07, ESK09, SRS07], is that to endow the body manifold M with an *incompatible* Riemannian metric, which will determine a (not necessarily unique) implant map on M (following the above terminology).

In the context of pre-stretched materials, the techniques of [FJM02] have been extended on one side by B. Schmidt in [Sch07a, Sch07b] to the case of heterogeneous multilayers – namely, elastic materials characterized by a pre-stretch depending only on the transversal variable and being h-close to the identity. The corresponding Kirchhoff plate model has been used in the study of bending behaviour of thin sheets made of nematic elastomers by V. Agostiniani and A. DeSimone in [AD15], [AD17] and [DeS18]. An important contribution to the modelling of nematic elastomers has been previously given by A. DeSimone and coauthors in [DD02, DT09, CD11, CDD02b, CDD02a, DeS99].

On the other side, models with only lateral variations of the pre-stretches, namely $U_h(z) = U(z')$ (and consequently $A_h(z) = A(z')$), were considered in [LP11, BLS16] and [LRR17] under the Kirchhoff and the von Kármán scaling regime, respectively. These theories, often referrd to as the *non-Euclidean plate theories*, were inspired by experimental results in [KES07], where it has been postulated that the complex changes detected in the experiments are due to the attempt of the body to realize the configuration with the prescribed pull-back metric $G = A^2$. When such a metric is *incompatible* (i.e. there is no orientation preserving deformation realizing it), it is clear that the infimum of the functional $\overline{\mathscr{E}}_h$ is strictly positive, pointing out the existence of the non-zero stress at free equilibria. Similarly, in [LMP10, LOP$^+$15] pre-stretches U_h with a particular asymptotic behaviour are considered. The corresponding plate models are derived under the higher (then Kirchhoff) scaling regime and the discussion on the energy scaling driven by the asymptotic behaviour of the difference $U_h - \mathrm{I}_3$ is provided.

Regarding the problem of different energy scalings in the case of pre-stretched materials or, equivalently, those characterized by the prescription of a general Riemannian metric G_h on Ω_h, it is expected that the hierarchy of plate models should be (naturally) driven by the scaling (in the sheet's thickness h) of the components of the corresponding Riemann curvature tensor, see [Lew11].

Moreover, general and strong results have been proved in the abstract setting of Riemannian manifolds in [KS14, KM14, MS18], covering many of (Euclidean and non-Euclidean) plate, shell and rod theories that have been previously derived, as well as providing a variety of new ones. In [KS14] Γ-convergence statements were proved for any dimension and codimension in the $O(h^2)$ scaling regime (generalizing the results of [LP11] and [BLS16]), while in a work parallel to the non-oscillatory case (NO) presented in the Chapter 5 [MS18], the authors analyze scaling orders $o(h^2)$, $O(h^4)$ and $o(h^4)$, extending the condition (5.27), Lemma 5.4.1 for the case (NO) and condition (5.86) to arbitrary manifolds.

Other results concerning the energy scaling for materials with residual strain are derived in [BK14] by imposing suitable boundary data (see also [SRS07]).

Coming back to our motivation from the very beginning of this Introduction, a natural question arises: which shapes can be taken by a thin elastic sheet with a prescribed non-Euclidean (incompatible) metric?

For instance, in order to generate bending deformations of an initially flat thin sheet and in this way produce curved shapes, one may employ the bilayer (or multilayer) models mentioned above. The corresponding 2D bending model is in this case described via a curvature functional measuring the difference between curvature tensor associated with the limiting deformation and a constant "target" (or "preferred") curvature arising in the presence of a transversal variation of the pre-stretch. It turns out that the corresponding configuration of minimal energy is a "piece" of cylindrical surface [Sch07b]. The proof of such result heavily relies upon the fine properties of $W^{2,2}$-isometric immersions, proved by M.R. Pakzad in the case of convex domains [Pak04] and later generalized by P. Hornung to more general domains [Hor11b].

Then another question that is much less explored in the literature is: how to control the curvature (and thus obtain the desired target shape) by tuning the internal growth (in view of the prescribed metric), the material heterogeneity or the external stimuli? Some recent results on the optimal control and the design problems related to plates and shells can be found, for instance, in [JM15, HMVV16, HMVV17].

Without any attempt of being complete, since the literature is vast and is growing at a very fast pace, in the above discussion we pointed out some problems (of different nature) in non-linear elasticity related to the thin structures and to their two-dimensional counterparts that will be subject of this thesis. The aim of this thesis, whose brief presentation is provided below, is to provide some further developments in the derivation and analysis of the dimensionally reduced theories inspired by the modelling of thin sheets made of hydrogel.

Organization of the thesis

We shall now present an overview of the content of this thesis. In Chapter 1 and Chapter 2 we provide an introduction to the necessary background in mathematics and elasticity to ensure good understanding of the forthcoming arguments:

• **Chapter 1: Notation and mathematical preliminaries.** We set up a general notation that will be used throughout. We introduce the basic notions and some important results from Riemannian geometry and Mathematical Analysis. In particular, we will be concerned with: the notion of Γ-convergence; existence and regularity of an isometric immersion of a planar domain ω equipped with an arbitrary metric g into Euclidean space $\mathbb{R}^3$; fine properties of $W^{2,2}$-isometric immersions of the flat metric $g = I_2$;

• **Chapter 2: From three-dimensionall elasticity to nonlinear plate theory.** This chapter is decomposed in the two main parts: in the first part we present the 3D model of a thin hyperelastic sheet, concentrating our attention on the properties of energy density functions and on the notion of a *material characterized by a spontaneous stretch*. Such material is described via energy density function $\overline{W}_h$ and a symmetric, positive definite tensor field $A_h : \Omega_h \to \mathbb{R}^{3\times 3}$ satisfying

$$\operatorname{argmin}_{\mathbb{R}^{3\times 3}} \overline{W}_h(z, \cdot) = SO(3)A_h(z), \qquad \text{for a.e. } z \in \Omega_h.$$

More precisely, we consider a more general spontaneous stretch distribution, not necessarily coming from the inverse of a pre-stretch. Moreover, we introduce all the physical quantities and their rescaled versions involved in the definition of the elastic energy functional $\overline{\mathscr{E}}_h$ and in the variational problem we will be concerned with.

The second part of the chapter is devoted to a presentation of the different energy scaling regimes and some of the key ideas in the dimension reduction. We concentrate on the Kirchhoff (bending) regime recalling the main ideas from [FJM02] – in particular, the rigidity estimate and a truncation argument. We shall see that these ideas and their counterparts in the non-Euclidean setting (developed in [LP11]) will be carried over Chapter 3 and Chapter 5, respectively.

We conclude this chapter by Appendix 2.A, dedicated to the properties of the quadratic form $\mathcal{Q}_3$ (coming from the second differential of the homogeneous energy density function W) and its linearized version $\mathcal{Q}_2$.

The remaining three chapters contain the results that were obtained in the articles [ALL17], [ADLL] and [LL18], respectively. Let us single out some of the major topics that we will be concerned with. Namely, we will

- draw the attention to the materials characterized by a *spontaneous stretch*, motivated by the properties of the Flory-Rehner-type model for thin sheets made of hydrogel;
- derive *new plate models* in the Kichhoff and von Kármán energy scaling regimes for more general forms of the spontaneous stretch, generalizing the results of [Sch07b, LP11, BLS16, LRR17]; we discuss as well the *scaling of the elastic energy* as a function of the thickness h;
- analyze the *energy minimizers* of the 2D Kirchhoff model characterized by a piecewise constant target curvature tensor (recall that the constant case has been treated in [Sch07b]) and employ the obtained results to study the *folding structures*, also addressing an *inverse (design)* problem;
- prove some *coercivity inequalities* for the 2D limiting models obtained in the Kirchhoff and von Kármán regimes;

A brief presentation of these three chapters follows. We provide a more detailed introduction at the beginning of each of them.

• **Chapter 3: Dimension reduction for materials with a spontaneous stretch distribution.** In this chapter we present the results obtained in [ALL17] in collaboration with V. Agostiniani and A. Lucantonio. We rigorously derive (in Theorem 3.2.2) a Kirchhoff plate theory from a three-dimensional model that describes the finite elasticity of an elastically heterogeneous, thin sheet. The heterogeneity in the elastic properties of the material results in a spontaneous stretch A_h that depends on both the thickness and the plane variables z'. At the same time, the spontaneous stretch is h-close to the identity (recall, h is the parameter quantifying the thickness), namely it is of the form

$$A_h(z) = \mathrm{I}_3 + hB\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \Omega_h$$

where B is a bounded *spontaneous strain* field defined on the rescaled domain $\Omega := \omega \times (-1/2, 1/2) \subseteq \mathbb{R}^3$. The 2D Kirchhoff limiting model is constrained to the set of isometric immersions of the mid-plane of the plate into $\mathbb{R}^3$, with a corresponding energy that penalizes deviations of the curvature tensor associated with a deformation from a z'-dependent target curvature tensor. Moreover, we provide a discussion on the 2D energy minimizers in the case where the target curvature tensor is piecewise constant with the eye on the further applications in Chapter 4.

In addition to the presented results from [ALL17], in Section 3.4 we discuss the additional condition (3.20) imposed on the spontaneous strain B in Theorem 3.2.2. We present two different ideas that might be used to remove it.

Also, this chapter contains an appendix (Appendix 3.A) in which we provide, for the convenience of the reader, complete proofs of the results obtained in [Sch07b] regarding energy minimizers in the case of constant target curvature.

• **Chapter 4: Application to heterogeneous thin gel sheets.** The results presented in this chapter are obtained in [ADLL] in collaboration with V. Agostiniani, A. DeSimone and A. Lucantonio. We discuss self-folding of a thin sheet by using patterned hydrogel bilayers, which act as hinges connecting flat faces. Folding is actuated by heterogeneous swelling due to different cross-linking densities of the polymer network in two layers.

For our analysis we use a dimensionally reduced plate model, obtained by applying the theory (and especially the study of pointwise minimizers) presented in Chapter 3 and further adapted, at the beginning of this chapter, to this particular case. It provides us with an explicit connection between material properties and the curvatures induced at the hinges – this connection offers a recipe for the fabrication of the bilayers, by providing the values of the cross-linking density of each layer that need to be imprinted during polymerization to produce a desired folded shape upon swelling.

• **Chapter 5: Dimension reduction for thin sheets with transversally varying pre-stretch.** This chapter is devoted to the results obtained in [LL18] in collaboration with M. Lewicka, where we study the Γ-limits in the vanishing sheet's thickness h of the non-Euclidean (incompatibile) elastic energy functionals in the description of pre-streched thin sheets. More precisely, the sheets are characterized by the incompatibility smooth tensor fields G_h defined on Ω_h (representing a Riemannian metric) and by the energy density function having the pre-stretch form (0.1) with the corresponding spontaneous stretch $A_h := U_h^{-1} = G_h^{1/2}$. Moreover, we consider tensor fields G_h of the form

$$G_h(z) = \bar{\mathcal{G}}(z') + h\mathcal{G}^1\left(z', \frac{z_3}{h}\right) + \frac{h^2}{2}\mathcal{G}^2\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \Omega_h,$$

where $\bar{\mathcal{G}}$ is a positive definite symmetric matrix field, while $\mathcal{G}^1$ and $\mathcal{G}^2$ are symmetric matrix fields and $\mathcal{G}^1$ satisfies $\int_{-1/2}^{1/2} \mathcal{G}^1(\cdot, t)\,\mathrm{d}t = 0$. The above metric tensor G_h is referred to as the "oscillatory" case. In such a case the dependence on the transversal variable at the first order is not necessarily linear; the linear case can be seen as a subcase of the former one and is considered within the "non-oscillatory" setting. In particular, G_h as above provides a generalization of the tensor field $G_h = A_h^2$ considered in Chapter 3 in the sense that the leading order metric I_3 present therein is now a general z'-dependent metric tensor $\bar{\mathcal{G}}$.

We derive the corresponding 2D models in the Kirchhoff and the von Kármán scaling regime and exhibit connections between the "oscillatory" and the "non-oscillatory" case. It turns out that (under the appropriate compatibility assumptions on $\mathcal{G}^1$ and $\mathcal{G}^2$) the vanishing of the additional (purely metric-related) terms in the 2D models corresponding to the oscillatory case leads to the 2D models in the "non-oscillatory" case corresponding to the appropriate effective metric.

We also study the scaling of the elastic energy per unit volume (i.e. $\overline{\mathscr{E}}_h/h$) as a power of h and discuss the scaling regimes up to power h^6 (see Theorem 5.2.7, Theorem 5.4.14 and Theorem 5.7.1). In Section 5.3 and Section 5.5 we prove some coercivity inequalities for the obtained Γ-limits in the "non-oscillatory" case at h^2- and h^4- scaling orders, respectively, while disproving the full coercivity of the classical von Kármán energy functional at scaling h^4 (see Example 5.5.3).

1

Notation and mathematical preliminaries

Contents

1.1 General notation

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\overline{\mathbb{R}}$ the set of natural, integer, real and extended real numbers, respectively. For fixed $n \in \mathbb{N}$ we will denote by

- $\mathbb{R}^n$ the Euclidean n-dimensional vector space,
- $\mathbb{R}^{n\times n}$ the vector space of real $n \times n$ matrices,
- $\mathrm{I}_n \in \mathbb{R}^{n\times n}$ the identity matrix,
- $\mathrm{Sym}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal = M\}$ the vector space of symmetric matrices, where by $M^\intercal \in \mathbb{R}^{n\times n}$ we denote the transpose of the matrix $M \in \mathbb{R}^{n\times n}$,
- $\mathrm{Skew}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal = -M\}$ the set of skew-symmetric matrices,
- $\mathrm{O}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal M = \mathrm{I}_n\}$ the set of all orthogonal transformations of $\mathbb{R}^n$,
- $\mathrm{SO}(n) := \{M \in \mathbb{R}^{n\times n} : M^\intercal M = \mathrm{I}_n, \det(M) = 1\}$ the set of all rotations of $\mathbb{R}^n$,
- $\mathrm{Trs}(n) := \{T_\mathsf{v} := \cdot + \mathsf{v} : \mathsf{v} \in \mathbb{R}^n\}$ the set of all translations in $\mathbb{R}^n$. Sometimes, to distinguish between translations in $\mathbb{R}^2$ and $\mathbb{R}^3$, we denote by τ_v the elements of $\mathrm{Trs}(2)$,

- $M_{\rm sym} := \frac{M+M^\intercal}{2}$ the symmetric part of the matrix $M \in \mathbb{R}^{n\times n}$,
- $\operatorname{tr} M$ the trace of the matrix M and $\operatorname{tr}^2 M := (\operatorname{tr} M)^2$,
- $|M| := \sqrt{\sum_{i,j=1}^n |m_{ij}|^2} = \sqrt{\operatorname{tr}(M^\intercal M)}$, Frobenius norm of $M = [m_{ij}]_{i,j=1}^n \in \mathbb{R}^{n\times n}$,
- $M : N = \operatorname{tr}(M^\intercal N)$ the scalar product between $M, N \in \mathbb{R}^{n\times n}$,
- $\mathsf{v} \cdot \mathsf{w} = \mathsf{v}^\intercal \mathsf{w}$ the scalar product between $\mathsf{v}, \mathsf{w} \in \mathbb{R}^n$, sometimes also denoted by $\langle \mathsf{v}, \mathsf{w} \rangle$,
- $\mathsf{v} \otimes \mathsf{w} = \mathsf{v}\mathsf{w}^\intercal$ the tensor product between $\mathsf{v}, \mathsf{w} \in \mathbb{R}^n$,
- $\mathsf{v} \wedge \mathsf{w}$ the exterior product between $\mathsf{v}, \mathsf{w} \in \mathbb{R}^n$,
- $\mathcal{L}^n$ the n-dimensional Lebesgue measure; also the notation $|\cdot|$ will be equivalently used,
- $\mathcal{H}^n$ the n-dimensional Hausdorff measure.

Functional spaces. Let X and Y be finite dimensional Banach spaces and let $\mathcal{U} \subseteq X$ be an open set. We will use the following notation

- $C(\mathcal{U}, Y)$ the space of continuous functions from $\mathcal{U}$ to Y,
- $C^m(\mathcal{U}, Y)$ the space of m-differentiable functions,
- $C^\infty(\mathcal{U}, Y)$ the space of smooth functions,
- $C_c^\infty(\mathcal{U}, Y)$ the space of smooth functions with compact support in $\mathcal{U}$,
- $\mathrm{L}^p(\mathcal{U}, Y)$ the space of Lebesgue (p-integrable) functions, $1 \le p < \infty$,
- $\mathrm{L}^\infty(\mathcal{U}, Y)$ the space of essentially bounded functions,
- $\mathrm{L}^p_{\rm loc}(\mathcal{U}, Y)$ the space of locally p-integrable functions, $1 \le p < \infty$,
- $\mathrm{W}^{m,p}(\mathcal{U}, Y)$ the space of Sobolev functions, $1 \le p \le \infty$,
- $\mathrm{W}^{m,p}_{\rm loc}(\mathcal{U}, Y)$ the space of locally Sobolev functions, $1 \le p \le \infty$.

We refer the reader to the book [Bre10] for some properties of Sobolev functions that will be used throughout.

By $\mathcal{N}(\mathcal{U})$ we denote the family of all open neighbourhoods of a set $\mathcal{U} \subseteq X$. The closure of a set $\mathcal{U} \subseteq X$ is denoted by $\overline{\mathcal{U}}$. Given a compact subset $\mathcal{K}$ of X, we say that $\Phi \in C^m(\mathcal{K})$ if there exists $\mathcal{U} \subseteq X$ open, containing $\mathcal{K}$, such that $\Phi \in C^m(\mathcal{U})$. An open ball of radius $r > 0$ centered in $x \in X$ is denoted by $B_r(x)$.

Furthermore, we give the following definitions:

- given $M \in \mathbb{R}^{2\times 2}$, the 3×3 matrix with principal minor equal M and all other entries equal to 0 is denoted by M^*,
- the matrix $M_{2\times 2}$ denotes the 2×2 principal minor of a given matrix $M \in \mathbb{R}^{3\times 3}$,
- $\mathscr{L}(\mathbb{R}^{3\times 3})$ is the space of all linear functions from $\mathbb{R}^{3\times 3}$ to $\mathbb{R}$,
- $\mathscr{L}_2(\mathbb{R}^{3\times 3})$ is the space of all bilinear functions from $\mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ to $\mathbb{R}$.

We denote by $\{\mathsf{e}_1, \mathsf{e}_2\}$ the standard basis of $\mathbb{R}^2$ and by $\{\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3\}$ the standard basis of $\mathbb{R}^3$. An open connected subset of $\mathbb{R}^n$, $n = 2, 3$, will be called *domain*. Sometimes, for the sake of brevity, an open subset of $\mathbb{R}^n$ with Lischitz boundary will be called a *Lipschitz subset* of $\mathbb{R}^n$.

1.2 The notion of Γ-convergence

In this section we shall concentrate on the remarkable notion of Γ-convergence, introduced by E. De Georgi and T. Franzoni [DF75] in 1975. We recall the basic definition and state some of the important properties (for instance, the compactness result and the convergence of minimizers). For their proofs and more details about this topic we refer the reader to [DM12] and [Bra02].

Let (X, d) be a metric space. We say that a function $\mathscr{F} : \mathrm{X} \to \overline{\mathbb{R}}$ is

- *lower semicontinuous* provided $\mathscr{F}(x) \leq \liminf_k \mathscr{F}(x_k)$, for every sequence $(x_k)_k$ converging to x as $k \to +\infty$,
- *coercive* if the set $\{x \in \mathrm{X} : \mathscr{F}(x) \leq t\}$ is precompact for every $t \in \mathbb{R}$.

We recall that

Theorem 1.2.1. *Let $\mathscr{F} : \mathrm{X} \to \overline{\mathbb{R}}$ be lower semicontinuous and coercive. Then $\mathscr{F}$ has a minimum in* X*.*

Through the rest of this section $(\mathscr{F}_k)_k$ will be a sequence of functions $\mathscr{F}_k$ from X to $\overline{\mathbb{R}}$.

Definition 1.2.2 (The Γ-limit)**.** *Fix $x \in \mathrm{X}$. We define the* Γ-lower limit *of a sequence $(\mathscr{F}_k)_k$ at the point x as*

$$\big(\Gamma\text{-}\liminf_{k\to+\infty} \mathscr{F}_k\big)(x) := \inf\big\{\liminf_{k\to+\infty} \mathscr{F}_k(x_k) : x_k \to x\big\},$$

and the Γ-upper limit *of a sequence $(\mathscr{F}_k)_k$ at the point x as*

$$\big(\Gamma\text{-}\limsup_{k\to+\infty} \mathscr{F}_k\big)(x) := \inf\big\{\limsup_{k\to+\infty} \mathscr{F}_k(x_k) : x_k \to x\big\}.$$

If there exists a function $\mathscr{F} : \mathrm{X} \to \overline{\mathbb{R}}$ such that $\Gamma\text{-}\liminf_k \mathscr{F}_k = \Gamma\text{-}\limsup_k \mathscr{F}_k = \mathscr{F}$, we say that $(\mathscr{F}_k)_k$ Γ-converges to $\mathscr{F}$ in X*, as $k \to +\infty$. We write $\mathscr{F} = \Gamma\text{-}\lim_k \mathscr{F}_k$.*

Equivalently, the above definition is saying that $(\mathscr{F}_k)_k$ Γ-converges to $\mathscr{F}$ in X if

(*i*) for every $x \in \mathrm{X}$ and every sequence $(x_k)_k \subseteq \mathrm{X}$ such that $x_k \xrightarrow{k} x$ it holds

$$\liminf_{k\to+\infty} \mathscr{F}_k(x_k) \geq \mathscr{F}(x),$$

(*ii*) for every $x \in \mathrm{X}$ there exists a sequence $(x_k)_k \subseteq \mathrm{X}$ such that $x_k \xrightarrow{k} x$ and

$$\limsup_{k\to+\infty} \mathscr{F}_k(x_k) \leq \mathscr{F}(x).$$

Such a sequence is usually referred to as a *recovery sequence.*

Since we will often deal with the family of functions $\{\mathscr{F}_h\}_h$ depending on the continuous parameter $h > 0$, we will say that $\{\mathscr{F}_h\}_h$ Γ-converges to $\mathscr{F}$ as $h \to 0$ if for every sequence $h_k \to 0$ as $k \to +\infty$ it holds that $(\mathscr{F}_{h_k})_k$ Γ-converges to $\mathscr{F}$.

We now point out some properties of Γ-convergence and Γ-limits that will be important later on. First of all, by using diagonal argument one can show that a Γ-limit is a lower semicontinuous function. Second, by its very definition, it can be easily verified that a Γ-limit is *stable under continuous perturbations*, i.e. given a continuous function $\mathscr{G} : \mathrm{X} \to \overline{\mathbb{R}}$ and letting $\mathscr{F} = \Gamma\text{-}\lim_k \mathscr{F}_k$, we have that $\mathscr{F} + \mathscr{G} = \Gamma\text{-}\lim_k \mathscr{F}_k + \mathscr{G}$.

Lemma 1.2.3 (Γ-limsup on a dense subset). *Let $D \subseteq \mathrm{X}$ be dense in X and let $\mathscr{F} : \mathrm{X} \to \overline{\mathbb{R}}$ be a continuous function. Assume that $\big(\Gamma\text{-}\limsup_k \mathscr{F}_k\big)(x) \le \mathscr{F}(x)$ for every $x \in D$. Then $\big(\Gamma\text{-}\limsup_k \mathscr{F}_k\big)(x) \le \mathscr{F}(x)$ for every $x \in \mathrm{X}$.*

Proposition 1.2.4 (Compactness of Γ-convergence). *Assume that (X, d) is a separable metric space. Then there exists a subsequence $(\mathscr{F}_{k_j})_j$ of $(\mathscr{F}_k)_k$ such that $\Gamma\text{-}\lim_j \mathscr{F}_{k_j}$ exists at each point x of X.*

Proposition 1.2.5 (Convergence of minimizers). *Assume that $(\mathscr{F}_k)_k$ Γ-converges to $\mathscr{F}$ as $k \to +\infty$ and that there exists a compact set $K \subseteq \mathrm{X}$ such that $\inf_{\mathrm{X}} \mathscr{F}_k = \inf_K \mathscr{F}_k$ for every $k \in \mathbb{N}$. Then the minimum of $\mathscr{F}$ is attained and it holds that*

$$\min_{\mathrm{X}} \mathscr{F} = \lim_{k\to+\infty} \inf_{\mathrm{X}} \mathscr{F}_k.$$

If $(x_k)_k \subseteq \mathrm{X}$ is a precompact sequence such that $\lim_k \mathscr{F}_k(x_k) = \lim_k \inf_{\mathrm{X}} \mathscr{F}_k$, then every converging subsequence of $(x_k)_k$ converges to a minimizer of $\mathscr{F}$.

1.3 Some results from Riemannian geometry

In this section we recall some definitions and results in Riemannian geometry that will be used throughout. We refer the reader to [HH06], [Cia06] and [Spi75] and the references appearing hereafter for more details.

Let Ω be a simply-connected domain in $\mathbb{R}^3$. Let $G : \Omega \to \mathbb{R}^{3\times3}$ be a smooth matrix field, with values in the set of positive definite symmetric matrices, denoted by Psym(3). Thus, we identify the map G with a Riemannian metric defined in Ω. We will use the following notation

$$G = [G_{ij}] \quad \text{and} \quad G^{-1} = [G^{ij}].$$

The Christoffel symbols $\{\Gamma^i_{kl}\}$ corresponding to the metric G are given by

$$\Gamma^i_{kl} = \frac{1}{2} G^{im}\big(\partial_l G_{mk} + \partial_k G_{ml} - \partial_m G_{kl}\big), \tag{1.1}$$

while the components of the Riemann curvature tensor Riem_G associated to G and its covariant version (fourth-order tensor field) read as

$$R^s_{klm} = \partial_l \Gamma^s_{km} - \partial_m \Gamma^s_{kl} + \Gamma^s_{lj}\Gamma^j_{km} - \Gamma^s_{mj}\Gamma^j_{kl} \quad \text{and} \quad R_{iklm} = G_{is} R^s_{klm}, \tag{1.2}$$

respectively. Note that the second equality can be written in the following form:

$$R_{iklm} = \frac{1}{2}\Big(\partial_{kl} G_{im} + \partial_{im} G_{kl} - \partial_{km} G_{il} - \partial_{il} G_{km}\Big) + \sum_{n,p=1}^{3} G_{np}\big(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\big). \tag{1.3}$$

Above, the Einstein summation convention for the sum over repeated indices is adopted.

A manifold (Ω, G) is said to be *flat* if all the components of the Riemann curvature tensor vanish, i.e. $\mathrm{Riem}_G \equiv 0$. Observe that $\mathrm{Riem}_G \equiv 0$ if and only if $R_{iklm} \equiv 0$ for all $i, k, l, m \in \{1, 2, 3\}$. It is a well-known result in Riemannian geometry that the latter condition is equivalent to the existence of an *orientation-preserving isometric immersion* of (Ω, G) into the Euclidean space $\mathbb{R}^3$, namely of a sufficiently smooth map $u : \Omega \to \mathbb{R}^3$ such that

$$(\nabla u)^{\mathsf{T}} \nabla u = G \quad \text{and} \quad \det \nabla u > 0.$$

We remark that such a result is true in any space dimension $n \geq 2$. We refer to its version with lower regularity requirement on G given in [Mar04], where $G \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^{3\times 3})$, and the corresponding immersion u belongs to the space $\mathrm{W}^{2,\infty}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$. In this case, clearly, the flatness condition $\mathrm{Riem}_G = 0$ is asked to be satisfied in the distributional sense.

A natural question arises in the case when (Ω, G) is not flat: is there any $n > 3$ and any immersion $u : \Omega \to \mathbb{R}^n$ such that $(\nabla u)^{\mathsf{T}} \nabla u = G$? A positive answer to this question is given by the results of Nash [Nas54] and Kuiper [Kui55], which says that any d-dimensional Riemannian manifold can be isometrically embedded in a Euclidean space of dimension $n \geq d+1$, with an immersion u of class C^1.

Further, we recall a result regarding the existence of an isometric immersion with a prescribed second fundamental form, restricting our attention to the following case: let $\omega \subseteq \mathbb{R}^2$ be a simply-connected domain, let $g = [g_{ij}] : \omega \to \mathrm{Psym}(2)$ and $\Pi = [h_{ij}] : \omega \to \mathrm{Sym}(2)$ be given C^2-matrix fields. The existence of a C^3-immersion $y : \omega \to \mathbb{R}^3$ satisfying

$$(\nabla y)^{\mathsf{T}} \nabla y = g \quad \text{and} \quad \Pi_y = \Pi$$

is guaranteed if the coefficients of g and Π satisfy the following Gauss-Codazzi-Mainardi systems of equations:

$$\partial_k h_{ij} - \partial_j h_{ik} = \Gamma^l_{ik} h_{ij} - \Gamma^l_{ij} h_{lk} \quad \text{and} \quad R_{iljk} = h_{ij} h_{lk} - h_{lj} h_{ik}, \tag{1.4}$$

where Γ^l_{ij} and R_{iljk} are related to $[g_{ij}]$ via the formulas (1.1), (1.3) and Π_y is the pull-back of the second fundamental form of the surface $y(\omega)$. The above compatibility result holds even when the regularity of g and Π is reduced to the spaces $\mathrm{W}^{1,\infty}_{\mathrm{loc}}\big(\omega, \mathrm{Sym}(2)\big)$ and $\mathrm{L}^{\infty}_{\mathrm{loc}}(\omega, \mathrm{Sym}(2))$, respectively, as proved in [Mar03]. The associated immersion y then belongs to the space $\mathrm{W}^{2,\infty}_{\mathrm{loc}}(\omega, \mathbb{R}^3)$. We also recall that

$$\Pi_y = (\nabla y)^{\mathsf{T}} \nabla \nu, \quad \text{where } \nu := \frac{\partial_1 y \wedge \partial_2 y}{|\partial_1 y \wedge \partial_2 y|}. \tag{1.5}$$

From the fact that $\partial_i y \cdot \nu = 0$ for $i = 1, 2$, we equivalently have

$$(\Pi_y)_{ij} = -\partial_{ij} y \cdot \nu, \qquad \text{for every } i, j = 1, 2. \tag{1.6}$$

We conclude this section by stating two compatibility results that will be useful in the sequel. A more general version of the first result below, for $A \in \mathrm{L}^p(\omega, \mathrm{Sym}(2))$ with $p \geq 2$ (actually, in any space dimension $n \geq 2$), can be found in [MSG15].

Theorem 1.3.1 (St. Venant compatibility condition [CC05])**.** *Let $\omega \subseteq \mathbb{R}^2$ be a simply-connected bounded Lipschitz domain and let $A \in \mathrm{L}^2(\omega, \mathrm{Sym}(2))$. Then*

$$\mathrm{curl}\big(\mathrm{curl}\, A\big) = 0 \ \textit{ in the distributional sense} \quad \Longleftrightarrow \quad A = \nabla_{\mathrm{sym}} w, \ w \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^2). \tag{1.7}$$

Moreover, w is unique up to rigid displacements.

Theorem 1.3.2 ([CC05])**.** *Let $\omega \subseteq \mathbb{R}^2$ be a simply-connected bounded Lipschitz domain and let $A \in \mathrm{L}^2(\omega, \mathrm{Sym}(2))$. Then*

$$\mathrm{curl}\ A = 0 \ \textit{ in the distributional sense} \iff A = -\nabla^2 v \ \textit{ a.e. in } \omega, \ \textit{ for } v \in \mathrm{W}^{2,2}(\omega). \tag{1.8}$$

The function v is unique up to addition of an affine function.

Here, the symbol ∇_{sym} stands for the symmetrized gradient, i.e. $\nabla_{\mathrm{sym}} = \frac{\nabla^{\mathsf{T}} + \nabla}{2}$. The symbol $\mathrm{curl}\, A$ is used to denote the differential operator which associates to A the vector-valued map $(\mathrm{curl}\, a^1, \mathrm{curl}\, a^2)^{\mathsf{T}}$, where $a^1, a^2 : \omega \to \mathbb{R}^2$ are the rows of A, in symbols $A = \big(a^1 \,|\, a^2\big)^{\mathsf{T}}$. Also, we use the convention $\mathrm{curl}\, w = \partial_{x_1} w^2 - \partial_{x_2} w^1$, where $x' = (x_1, x_2) \in \omega$ and w^1, w^2 are the components of the vector-valued map $w : \omega \to \mathbb{R}^2$.

1.4 The space of $\mathrm{W}^{2,2}$-isometric immersions

Given a bounded Lipschitz domain $\omega \subseteq \mathbb{R}^2$, consider the class of the isometric immersions (or briefly, *isometries*) of ω endowed with a flat metric $[g_{ij}] = [\delta_{ij}]$ into $\mathbb{R}^3$

$$\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega,\mathbb{R}^3) := \big\{ y \in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3) : (\nabla y)^\top \nabla y = \mathrm{I}_2 \text{ a.e. in } \omega \big\}. \tag{1.9}$$

For the sake of brevity, we equivalently use the symbol $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$.

Remark 1.4.1. It is easy to verify that for any $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ it holds that

$$\partial_i y \cdot \partial_j y = \delta_{ij} \text{ and } \partial_{ij} y \cdot \partial_k y = 0, \qquad \text{for } i,j,k = 1,2. \tag{1.10}$$

These identities can be used to verify that for any $w \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^2)$ and $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ it holds

$$(\nabla y)^\top \nabla\big((\nabla y) w\big) = \nabla w, \qquad \text{a.e. in } \omega. \tag{1.11}$$

■

As a consequence of the above remark and Lemma 1.3.1 we have the following result.

Lemma 1.4.2. *Let $\omega \subseteq \mathbb{R}^2$ be a simply connected bounded Lipschitz domain and let $A \in \mathrm{L}^2\big(\omega, \mathrm{Sym}(2)\big)$. The following two conditions are equivalent:*

(*i*) $\mathrm{curl}(\mathrm{curl}\, A) = 0$ *in the distributional sense;*

(*ii*) *for every $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ there exists $w_y \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)$ such that $A = \big((\nabla y)^\top \nabla w_y\big)_{\mathrm{sym}}$ a.e. in ω.*

Proof. $(i) \Rightarrow (ii)$. Note first that (i) implies that there exists $w \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^2)$ such that $A = \nabla_{\mathrm{sym}} w$ a.e. in ω, by (1.7). Pick $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ and let $w_y := (\nabla y) w \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)$. By Remark 1.4.1 we have

$$\big((\nabla y)^\top \nabla w_y\big)_{\mathrm{sym}} = \nabla_{\mathrm{sym}}\, w = A.$$

$(ii) \Rightarrow (i)$. Let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ be given by $y(x') := (x_1, x_2, 0)$ for every $x' \in \omega$. By (ii), there exists $w_y \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)$ such that $A = \big((\nabla y)^\top \nabla w_y\big)_{\mathrm{sym}}$. Given that $\nabla y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, it holds $A = \nabla_{\mathrm{sym}}\, w_y^{\tan}$, where $w_y^{\tan} \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^2)$ satisfies $w_y = (w_y^{\tan}, w_y^3)^\top$. Therefore, $\mathrm{curl}(\mathrm{curl}\, A) = 0$, in view of (1.7). □

It is a well-known result in differential geometry that every smooth $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ satisfies $\det \Pi_y = 0$ in ω (see (1.4)). By an approximation argument – see [Pak04, Lemma 2.5] or [MP05, Proposition 3] – one can deduce that the same property holds for any arbitrary $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, a.e. in ω, namely:

Lemma 1.4.3. *Let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. Then $\Pi_y \in \mathrm{L}^2\big(\omega, \mathrm{Sym}(2)\big)$ and the following equations are satisfied in the sense of distributions*

$$\partial_2(\Pi_y)_{11} = \partial_1(\Pi_y)_{12}, \quad \partial_2(\Pi_y)_{21} = \partial_1(\Pi_y)_{22}, \quad \det \Pi_y = 0.$$

1.4.1 Fine properties of isometric immersions

We here recall some important results regarding the class $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. We will mainly use the notation and the terminology from [MP05, Pak04, Hor11a, Hor11b]. We refer the reader to the former references for more details about this topic. Following [Hor11a] and [Hor11b], let us first introduce the *directed distance function* $\nu^\omega : \omega \times (\mathbb{R}^2 \setminus \{0\}) \to [0, +\infty)$, defined by

$$\nu^\omega(x', \mathsf{v}) := \inf\{a > 0 : x' + a\mathsf{v} \notin \omega\}, \qquad \text{for every } (x', \mathsf{v}) \in \omega \times (\mathbb{R}^2 \setminus \{0\}).$$

Note that the open line segment with endpoints $x' \pm \nu^\omega(x', \pm\mathsf{v})\mathsf{v}$ is the maximal subinterval of the line $x' + \mathbb{R}\mathsf{v}$ contained in ω and containing the point x'. Further, given $y \in C^1(\omega, \mathbb{R}^3)$, we define the set

$$C_{\nabla y} := \{x' \in \omega : \nabla y \text{ is constant in a neighbourhood of } x'\}.$$

We say that ∇y is *countably developable* on ω if it is *developable* on $\omega \setminus C_{\nabla y}$ in the following sense: there exists a (unique) vector field $q_{\nabla y} : \omega \setminus C_{\nabla y} \to \mathbb{R}^2$ of unit length such that for all $x' \in \omega \setminus C_{\nabla y}$ the gradient ∇y is constant on the open line segment

$$[x'] := \big(x' - \nu^\omega(x', -q_{\nabla y}(x'))\, q_{\nabla y}(x'),\, x' + \nu^\omega(x', q_{\nabla y}(x'))\, q_{\nabla y}(x')\big) \tag{1.12}$$

and we have that

$$[x'] \cap [z'] \neq \emptyset \Longrightarrow [x'] = [z'] \quad \text{for every } x', z' \in \omega \setminus C_{\nabla y}.$$

Denote by $\hat{C}_{\nabla y}$ the union of all connected components U of $C_{\nabla y}$ whose relative boundary $\omega \cap \partial U$ consists of at least three connected components. As emphasised in [Hor11a], the map $q_{\nabla y}$ can be extended to $\omega \setminus \hat{C}_{\nabla y}$, but this extension, in general, is not unique. We further have the following result:

Theorem 1.4.4. *If $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ then $y \in C^1(\omega, \mathbb{R}^3)$ and ∇y is countably developable.*

The proof of the first part of this result, about the regularity of $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, can be found in [MP05, Proposition 5]. The second part has been proved in [Hor11b]. Moreover, from [Hor11b, Theorem 4] it follows that the domain ω can be decomposed, up to a null set, in finitely many subdomains (touching each other on a finite union of line segments) on which y is either a plane, or a cylinder, or a cone or tangent developable. More specifically, the affine regions (i.e. regions on which y is a plane) are subdomains $U \subseteq \hat{C}_{\nabla y}$ on which ∇y is constant, while the regions on which y is developable are subdomains of ω of the form $[\Gamma(0, T)]$, where Γ is the line of curvature for y (see the following definitions, in particular (1.14)). The same result in the case of convex domain ω has been previously proved in [Pak04].

Definition 1.4.5 (Line of curvature)**.** *Let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. Then any (arclength parametrized) curve $\Gamma \in \mathrm{W}^{2,\infty}\big([0, T], \omega \setminus \hat{C}_{\nabla y}\big)$ satisfying (for some extension of $q_{\nabla y}$ to $\omega \setminus \hat{C}_{\nabla y}$)*

$$\begin{aligned} \Gamma'(t) = -q^{\perp}_{\nabla y}\big(\Gamma(t)\big), &\qquad \text{for all} \quad t \in [0, T], \text{ and} \\ \Gamma'(t) \cdot \Gamma'(t') > 0, &\qquad \text{for all} \quad t, t' \in [0, T] \end{aligned}$$

is called a line of curvature *of y.*

To any curve $\Gamma \in \mathrm{W}^{2,\infty}([0,T],\mathbb{R}^2)$ one can associate its normal and its curvature, respectively given by

$$N = (\Gamma')^{\perp} \quad \text{and} \quad \kappa_{\mathsf{t}} = \Gamma'' \cdot N. \tag{1.13}$$

Let the functions $s_\Gamma^\pm : [0,T] \to \mathbb{R}$ be such that $\pm s_\Gamma^\pm$ are positive, lower semicontinuous and bounded and define

- the bounded domain $M_{s_\Gamma^\pm} := \big\{(s,t) : t \in (0,T), s \in \big(s_\Gamma^-(t), s_\Gamma^+(t)\big)\big\}$;
- the map $\Phi_\Gamma : M_{s_\Gamma^\pm} \to \mathbb{R}^2$ by $\Phi_\Gamma(s,t) := \Gamma(t) + sN(t)$, for every $(s,t) \in M_{s_\Gamma^\pm}$;
- the maps $\beta_\Gamma^\pm : [0,T] \to \mathbb{R}^2$ by $\beta_\Gamma^\pm(t) := \Gamma(t) + s_\Gamma^\pm(t)N(t)$, for every $t \in [0,T]$.

Denote further

$$[\Gamma(t)] := \big(\beta_\Gamma^-(t), \beta_\Gamma^+(t)\big) \quad \text{and} \quad [\Gamma(0,T)] := \Phi_\Gamma(M_{s_\Gamma^\pm}) = \bigcup_{t\in(0,T)} [\Gamma(t)]. \tag{1.14}$$

We say that Γ is *uniformly admissible* if

$$\begin{aligned} &[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset, && \text{for every } t_1, t_2 \in [0,T], t_1 \neq t_2, \\ &1 - s_\Gamma^\pm(t)\kappa_{\mathsf{t}}(t) \geq c > 0, && \text{for a.e. } t \in [0,T]. \end{aligned} \tag{1.15}$$

If Γ satisfies only the first condition in (1.15), then we say that Γ is *admissible*. It can be shown that:

- if Γ is a line of curvature for some $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ then Γ is admissible,
- if Γ has values in ω then one can define $s_\Gamma^\pm(t) := \pm\nu^\omega\big(\Gamma(t), \pm N(t)\big)$ for all $t \in [0,T]$.

Now let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ and $\Gamma \in \mathrm{W}^{2,\infty}([0,T], \omega \setminus \hat{C}_{\nabla y})$ be its line of curvature. Denote by

$$\gamma(t) := y\big(\Gamma(t)\big), \quad v(t) := \partial_{N(t)} y\big(\Gamma(t)\big), \qquad \text{for all } t \in [0,T].$$

Then set $n := \gamma' \wedge v$ and $\kappa_{\mathsf{n}} := \gamma'' \cdot n$. In [Hor11a, Proposition 1] it has been shown that $\kappa_{\mathsf{t}} \in \mathrm{L}^\infty(0,T)$, $\kappa_{\mathsf{n}} \in \mathrm{L}^2(0,T)$, $\gamma \in \mathrm{W}^{2,2}\big((0,T),\mathbb{R}^3\big)$ and $R := \big(\gamma'|v|n\big)^{\mathsf{T}} \in \mathrm{W}^{1,2}\big((0,T),\mathbb{R}^{3\times3}\big)$, $R(t) \in \mathrm{SO}(3)$ for a.e. $t \in (0,T)$, solves the ODE

$$r' = \begin{pmatrix} 0 & \kappa_{\mathsf{t}} & \kappa_{\mathsf{n}} \\ -\kappa_{\mathsf{t}} & 0 & 0 \\ -\kappa_{\mathsf{n}} & 0 & 0 \end{pmatrix} r. \tag{1.16}$$

Moreover,

$$y\big(\Phi_\Gamma(s,t)\big) = \gamma(t) + sv(t) \quad \text{and} \quad \nabla y\big(\Phi_\Gamma(s,t)\big) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)$$

for a.e. $(s,t) \in \overline{M}_{s_\Gamma^\pm}$. Finally, we have that (up to a sign)

$$\Pi_y\big(\Phi_\Gamma(s,t)\big) = \frac{\kappa_{\mathsf{n}}(t)}{1 - s\kappa_{\mathsf{t}}(t)}\big(\Gamma'(t) \otimes \Gamma'(t)\big) \quad \text{for a.e. } (s,t) \in M_{s_\Gamma^\pm}, \tag{1.17}$$

with Π_y given by (1.5). Another important result given in [Hor11a, Proposition 2] is telling us how to construct an isometric immersion:

Proposition 1.4.6. *Let* $\Gamma \in \mathrm{W}^{2,\infty}\big((0,T),\mathbb{R}^2\big)$, $\kappa_{\mathsf{n}} \in \mathrm{L}^2(0,T)$ *and let* $s_\Gamma^\pm \in \mathrm{L}^\infty(0,T)$ *be such that* $\pm s_\Gamma^\pm$ *are lower semicontinuous and uniformly bounded from below by a positive constant. Denote by* $\tilde{R} := \big(\sigma|\tilde{v}|\tilde{n}\big)^\mathsf{T} \in \mathrm{W}^{1,2}\big((0,T),\mathbb{R}^{3\times3}\big)$, $\tilde{R}(t) \in \mathrm{SO}(3)$ *for a.e.* $t \in (0,T)$, *the solution of* (1.16) *with initial value* $\tilde{R}(0) = \mathrm{I}_3$. *Set* $\tilde{\gamma}(t) := \int_0^t \sigma(s)\,ds$ *for every* $t \in (0,T)$ *and define* $(\Gamma,\kappa_{\mathsf{n}}) : [\Gamma(0,T)] \to \mathbb{R}^3$ *by*

$$(\Gamma,\kappa_{\mathsf{n}})\big(\Gamma(t) + sN(t)\big) := \tilde{\gamma}(t) + s\tilde{v}(t) \quad \textit{for every } (s,t) \in M_{s_\Gamma^\pm}.$$

If Γ *is uniformly admissible, i.e. if* (1.15) *holds, then* $(\Gamma,\kappa_{\mathsf{n}}) \in \mathrm{W}^{2,2}_{\mathrm{iso}}\big([\Gamma(0,T)]\big)$.

Finally, the above mentioned fine properties of isometric immersions have been used in order to show the following density result.

Theorem 1.4.7 (Approximation by smooth functions [Hor11a])**.** *Assume that* $\omega \subseteq \mathbb{R}^2$ *is a bounded Lipschitz domain which satisfies*

$$\text{(1.18)} \qquad \begin{aligned} &\textit{there exists a closed subset } \Sigma \subset \partial\omega \textit{ with } \mathcal{H}^1(\Sigma) = 0 \textit{ such that} \\ &\textit{the outer unit normal exists and is continuous on } \partial\omega \setminus \Sigma. \end{aligned}$$

Then $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \cap C^\infty(\overline{\omega},\mathbb{R}^3)$ *is* $\mathrm{W}^{2,2}$*-strongly dense in* $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$.

1.4.2 Cylinders

This subsection regards the sub-class of $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ consisting of cylinders. Given $r \in (0,+\infty]$, we define the map $C_r : \mathbb{R}^2 \to \mathbb{R}^3$ as

$$C_r(x') := \begin{cases} \Big(r\big(\cos(x_1/r) - 1\big), r\sin(x_1/r), x_2\Big)^\mathsf{T}, & r \in (0,+\infty), \\ \big(0, x_1, x_2\big)^\mathsf{T}, & r = +\infty, \end{cases}$$

for every $x' = (x_1,x_2) \in \mathbb{R}^2$. Then we define the family of maps

$$\text{(1.19)} \quad \mathrm{Cyl} := \big\{T_{\mathsf{v}} \circ R \circ C_r \circ \varrho : \mathbb{R}^2 \to \mathbb{R}^3 \,\big|\, r \in (0,+\infty],\ T_{\mathsf{v}} \in \mathrm{Trs}(3),\ R \in \mathrm{SO}(3), \varrho \in \mathrm{O}(2)\big\}$$

and we call its elements *cylinders*. Note that the above defined family of cylinders includes also *planes* - the elements of Cyl with $r = +\infty$.

Remark 1.4.8. Observe that any cylinder $y = T_{\mathsf{v}} \circ R \circ C_r \circ \varrho$ maps lines parallel to $\varrho^\mathsf{T}\mathsf{e}_2$ to the lines of zero curvature - rulings. More in general, we have

$$\text{(1.20)} \qquad \nabla y(x') \;=\; R\,\nabla C_r\big(\varrho(x')\big)\,\varrho = R \begin{pmatrix} -\sin\Big(\frac{x'\cdot\varrho^\mathsf{T}\mathsf{e}_1}{r}\Big) & 0 \\ \cos\Big(\frac{x'\cdot\varrho^\mathsf{T}\mathsf{e}_1}{r}\Big) & 0 \\ 0 & 1 \end{pmatrix} \varrho, \qquad \text{for all } x' \in \mathbb{R}^2,$$

so that

$$\nabla y(a\,\varrho^\mathsf{T}\mathsf{e}_2) = R \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \varrho, \qquad \text{for every } a \in \mathbb{R}.$$

■

By direct computations one can see that a map $y = T_{\mathsf{v}} \circ R \circ C_r \circ \varrho \in \mathrm{Cyl}$ is an isometry whose second fundamental form is given by

$$\mathrm{II}_y(x') = (\det\varrho)\,\varrho^{\mathsf{T}} \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & 0 \end{pmatrix} \varrho, \qquad \text{for every } x' \in \mathbb{R}^2. \tag{1.21}$$

By Theorem 1.4.3 we have that at a.e. $x' \in \omega$ the second fundamental form $\mathrm{II}_y(x')$ associated to some isometry $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ belongs to the set

$$\mathcal{S}_0 := \big\{ S \in \mathrm{Sym}(2) : \det S = 0 \big\}. \tag{1.22}$$

Moreover, the set $\mathcal{S}_0$ can be characterised as follows

$$\mathcal{S}_0 = \mathbb{R}\,\{\mathsf{n}\otimes\mathsf{n} : \mathsf{n} \in \mathbb{R}^2, |\mathsf{n}| = 1\} = \mathbb{R}\left\{ \varrho^{\mathsf{T}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \varrho \,:\, \varrho \in \mathrm{SO}(2) \right\}. \tag{1.23}$$

The second equality in (1.23) is trivial. To see that the first equality holds, let $0 \neq S \in \mathcal{S}_0$. Since $\det S = 0$ (hence $\mathrm{rank} S = 1$), there exist $\mathsf{v}, \mathsf{w} \in \mathbb{R}^2 \setminus \{0\}$ such that

$$S = \mathsf{v}\otimes\mathsf{w} = \begin{pmatrix} \mathsf{v}_1\mathsf{w}_1 & \mathsf{v}_1\mathsf{w}_2 \\ \mathsf{v}_2\mathsf{w}_1 & \mathsf{v}_2\mathsf{w}_2 \end{pmatrix}.$$

Suppose wlog that $\mathsf{v}_1 \neq 0$. By the symmetry of S we have that

$$\mathsf{v}_1\mathsf{w}_2 = \mathsf{v}_2\mathsf{w}_1 \quad \Longrightarrow \quad \mathsf{w} = \left(\mathsf{w}_1, \frac{\mathsf{v}_2}{\mathsf{v}_1}\mathsf{w}_1\right).$$

By wiriting $\mathsf{w} = \left(\frac{\mathsf{w}_1}{\mathsf{v}_1}\mathsf{v}_1, \frac{\mathsf{v}_2}{\mathsf{v}_1}\mathsf{w}_1\right)$, one can see that $\mathsf{w} = c\,\mathsf{v}$ with $c := \mathsf{w}_1/\mathsf{v}_1$. Therefore $S = c\,\mathsf{v}\otimes\mathsf{v}$. For $0 \in \mathcal{S}_0$ we have that $0 = 0 \cdot \mathsf{n}\otimes\mathsf{n}$ for any $\mathsf{n} \in \mathbb{R}^2$, $|\mathsf{n}| = 1$.

Lemma 1.4.9. *The set $\mathcal{S}_0$ defined by (3.31) is the set of (constant) second fundamental forms of cylinders.*

Proof. $(\subseteq)$: Let $y : \mathbb{R}^2 \to \mathbb{R}^3$ be such that for every $x' \in \mathbb{R}^2$ it holds

$$\big(\nabla y(x')\big)^{\mathsf{T}} \nabla y(x') = \mathrm{I}_2 \quad \text{and} \quad \mathrm{II}_y(x') = c\,\mathsf{n}\otimes\mathsf{n}, \quad \text{for } c \in \mathbb{R}\setminus\{0\},\, \mathsf{n} \in \mathbb{R}^2, |\mathsf{n}| = 1. \tag{1.24}$$

Our aim is to show that $y \in \mathrm{Cyl}$. Choose $\varrho \in \mathrm{O}(2)$ such that $\varrho\mathsf{e}_1 = \mathsf{n}$ and $\varrho\mathsf{e}_2 = \mathrm{sgn}(c)\mathsf{n}^{\perp}$. Then define $\mathsf{v} := \nabla y(0)\varrho\mathsf{e}_1$ and $\mathsf{w} := \nabla y(0)\varrho\mathsf{e}_2$ and choose $R \in \mathrm{SO}(3)$ such that

$$R(\mathsf{v}\wedge\mathsf{w}) = \mathsf{f}_1,\; R\mathsf{v} = \mathsf{f}_2 \;\text{ and }\; R\mathsf{w} = \mathsf{f}_3.$$

Let $T_{-y(0)} \in \mathrm{Trs}(3)$. Finally, define $\underline{y} := R \circ T_{-y(0)} \circ y \circ \varrho$ and let us show that $\underline{y} = C_r$, with $r = \frac{1}{|c|}$. By simple computations we get that

$$\nabla\underline{y}(x') = R\,\nabla y\big(\varrho(x')\big)\,\varrho \;\text{ and }\; \underline{\nu}(x') = (\det\varrho)\,\big(R\circ\nu\big)\big(\varrho(x')\big) = \mathrm{sgn}(c)\,\big(R\circ\nu\big)\big(\varrho(x')\big), \tag{1.25}$$

and therefore

$$\mathrm{II}_{\underline{y}}(x') = \big(\nabla\underline{y}(x')\big)^{\mathsf{T}}\nabla\underline{\nu}(x') = \mathrm{sgn}(c)\varrho^{\mathsf{T}}\mathrm{II}_y\big(\varrho(x')\big)\varrho = |c|\mathsf{e}_1\otimes\mathsf{e}_1$$

for every $x' \in \mathbb{R}^2$. Note that $\nabla\underline{y}$ is constant along e_2 direction, in other words that $\partial_2\nabla\underline{y} \equiv 0$. Indeed, this can be seen by differentiating with respect to x_1 and x_2 the identities

$$\partial_i\underline{y}\cdot\partial_j\underline{y} \equiv \delta_{ij} \quad \text{and} \quad \partial_i\underline{y}\cdot\underline{\nu} \equiv 0, \qquad i,j = 1,2 \tag{1.26}$$

and by using that, from (1.24), it holds

$$\partial_i \underline{y} \cdot \partial_j \underline{\nu} \equiv |c| \text{ if } i = j = 1 \text{ and } \partial_i \underline{y} \cdot \partial_j \underline{\nu} \equiv 0 \text{ otherwise.} \tag{1.27}$$

Further, note that

$$\begin{aligned} \underline{y}(0) =& (R \circ T_{-y(0)} \circ y \circ \varrho)(0) = R\Big(y\big(\varrho(0)\big) - y(0)\Big) = (R \circ y)(0) - (R \circ y)(0) = 0 \\ \partial_1 \underline{y}(0) =& R\,\nabla y(0)\,\varrho \mathsf{e}_1 = R\mathsf{v} = \mathsf{f}_2 \\ \partial_2 \underline{y}(0) =& R\,\nabla y(0)\,\varrho \mathsf{e}_2 = R\mathsf{w} = \mathsf{f}_3. \end{aligned}$$

Hence

$$\begin{aligned} \underline{y}(x_1, x_2) =& \underline{y}(0, x_2) + \int_0^{x_1} \partial_1 \underline{y}(t, x_2)\,dt = \underline{y}(0, x_2) + \int_0^{x_1} \partial_1 \underline{y}(t, 0)\,dt \\ =& \underline{y}(0,0) + \int_0^{x_2} \nabla \underline{y}(0,0)\mathsf{e}_2\,dt + \eta(x_1) = x_2 \mathsf{f}_3 + \eta(x_1) \end{aligned}$$

where $\eta(x_1) := \int_0^{x_1} \partial_1 \underline{y}(t,0)\,dt \in \mathbb{R}^3$ for every $x_1 \in \mathbb{R}$. Summarizing, we have that

$$\underline{y}(x') = \eta(x_1) + x_2 \mathsf{f}_3 \text{ and } \underline{\nu}(x') = \eta'(x_1) \wedge \mathsf{f}_3, \qquad \text{for every } x' = (x_1, x_2) \in \mathbb{R}^2.$$

Since $\eta''(x_1) = \partial_{11}\underline{y}(x_1, 0)$, by using (1.26) and (1.27) we get that $\eta''(x_1) = -|c|\underline{\nu}(x_1, 0)$. Finally, it holds

$$\eta''(x_1) = -|c|\,\eta'(x_1) \wedge \mathsf{f}_3 \quad \text{for every } x_1 \in \mathbb{R}. \tag{1.28}$$

Note that $\eta'(x_1) \in \operatorname{span}\{\mathsf{f}_1, \mathsf{f}_2\}$. Hence $\eta'(x_1) = \eta_1'(x_1)\mathsf{f}_1 + \eta_2'(x_1)\mathsf{f}_2$ and consequently we have that $\eta'(x_1) \wedge \mathsf{f}_3 = -\eta_1'(x_1)\mathsf{f}_2 + \eta_2'(x_1)\mathsf{f}_1$. Now (1.28) becomes

$$\begin{cases} \eta_1''(x_1) = -\,|c|\eta_2'(x_1), \\ \eta_2''(x_1) = |c|\eta_1'(x_1), \end{cases} \qquad \text{for every } x_1 \in \mathbb{R}.$$

It is easy to see that

$$\eta(x_1) := \left(\frac{1}{|c|}\big(\cos(|c|x_1) - 1\big), \frac{1}{|c|}\sin(|c|x_1), 0\right) \quad \text{for every } x_1 \in \mathbb{R}$$

solves the above system of ODE with initial data $\eta_1'(0) = 0$ and $\eta_2'(0) = 1$. By setting $r := 1/|c|$ we finally get that

$$\underline{y}(x') = \left(r\left(\cos\left(\frac{x_1}{r}\right) - 1\right), r\cos\left(\frac{x_1}{r}\right), x_2\right) = C_r(x') \qquad \text{for all } x' \in \mathbb{R}^2.$$

Therefore

$$y = T_{-y(0)}^{-1} \circ R^{\mathsf{T}} \circ \underline{y} \circ \varrho^{\mathsf{T}} = T_{-y(0)}^{-1} \circ R^{\mathsf{T}} \circ C_r \circ \varrho^{\mathsf{T}} \in \mathrm{Cyl}.$$

In the case when $c = 0$, we have that $\Pi_y = 0 = 0 \cdot \mathsf{n} \otimes \mathsf{n}$ for any $\mathsf{n} \in \mathbb{R}^2$, $|\mathsf{n}| = 1$. One can apply the same reasoning to $\underline{y}$ defined as above and deduce that $\underline{y}(x') = \eta(x_1) + x_2\mathsf{f}_2$ and $\eta''(x_1) = 0$ for every $x_1 \in \mathbb{R}$, given that $c = 0$. The solution of this differential equation with initial data $\eta_1'(0) = 0$, $\eta_2'(0) = 1$ and $\eta_1(0) = \eta_2(0) = 0$ is $\eta(x_1) = (0, x_1)$ for every $x_1 \in \mathbb{R}$. Hence $\underline{y}(x') = (0, x_1, x_2) = C_{+\infty}(x')$ for every $x' \in \mathbb{R}^2$ and in turn $y = T_{-y(0)}^{-1} \circ R^{\mathsf{T}} \circ C_{+\infty} \circ \varrho^{\mathsf{T}} \in \mathrm{Cyl}$ proving this inclusion.

($\supseteq$) : Suppose that $y \in \mathrm{Cyl}$, i.e. $y = T_{\mathsf{v}} \circ R \circ C_r \circ \varrho$ for some $T_{\mathsf{v}} \in \mathrm{Trs}(3), R \in \mathrm{SO}(3)$, $\varrho \in \mathrm{O}(2)$ and $r \in (0, +\infty]$. By straightforward computations, as above, we have that

$$\nabla y(x') = R\nabla C_r\big(\varrho(x)\big)\varrho \text{ and } \nabla \nu(x') = (\det \varrho)\, R\nabla \nu_r\big(\varrho(x')\big)\varrho,$$

where we denote by $\nu_r := \partial_1 C_r \wedge \partial_2 C_r$. Therefore

$$\Pi_y(x') = (\det\varrho)\,\varrho^\mathsf{T}\big(\nabla C_r\big(\varrho(x')\big)\big)^\mathsf{T} R^\mathsf{T} R \nabla\nu_r\big(\varrho(x')\big)\varrho = (\det\varrho)\,\varrho^\mathsf{T}\Pi_{C_r}\big(\varrho(x')\big)\varrho.$$

Suppose that $r \in (0,+\infty)$. Since

$$\nabla C_r(x') = \begin{pmatrix} -\sin(x_1/r) & 0 \\ \cos(x_1/r) & 0 \\ 0 & 1 \end{pmatrix} \quad\text{and}\quad \nabla\nu_r(x') = \begin{pmatrix} -\frac{1}{r}\sin(x_1/r) & 0 \\ \frac{1}{r}\cos\big(x_1/r\big) & 0 \\ 0 & 1 \end{pmatrix}$$

we have that $\Pi_{C_r}(x') = \operatorname{diag}\big(\frac{1}{r},0\big) = \frac{1}{r}\mathsf{e}_1\otimes\mathsf{e}_1$. Finally, by defining $\mathsf{n} := \varrho^\mathsf{T}\mathsf{e}_1$ and $c := (\det\varrho)/r$, we have $\Pi_y(x') = c\,\mathsf{n}\otimes\mathsf{n} \in \mathcal{S}_0$. In the case $r = +\infty$, it is clear that $\nabla\nu_{+\infty} = 0$. Hence $\Pi_y \equiv 0 \in \mathcal{S}_0$, proving the thesis. □

The following lemma (proved in [ALL17]) will be the main ingredient for the proof of Theorem 3.3.5. It gives a "recipe" on how two cylinders can be patched together. We refer to Remark 1.4.12 below for the notation and the properties of roto-translations used in this section.

Lemma 1.4.10. *Let $\omega \subseteq \mathbb{R}^2$ be a bounded Lipschitz domain. Let $\gamma : [0,1] \to \overline{\omega}$ be a continuous injective curve such that $[\gamma] \cap \partial\omega = \{\gamma(0),\gamma(1)\}$ and such that two connected components ω_1 and ω_2 of $\omega\setminus[\gamma]$ are Lipschitz. Let $y_1, y_2 \in \mathrm{Cyl}$, say $y_1 = T_{\mathsf{v}_1}\circ R_1\circ C_{r_1}\circ\varrho_1$ and $y_2 = T_{\mathsf{v}_2}\circ R_2\circ C_{r_2}\circ\varrho_2$, with $r_1, r_2 \in (0,+\infty)$ such that $\det\varrho_1 = -\det\varrho_2$ whenever $r_1 = r_2$. The map defined as*

$$y := y_1\chi_{\omega_1} + y_2\chi_{\omega_2}, \qquad \textit{a.e. in } \omega,$$

belongs to $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ if and only if the following conditions hold:

(i) *$[\gamma]$ is a line segment spanned by some $\mathsf{e} \in \mathbb{R}^2\setminus\{0\}$;*

(ii) *$\varrho_1^\mathsf{T}\mathsf{e}_2$ and $\varrho_2^\mathsf{T}\mathsf{e}_2$ are parallel to e. This in particular implies that $\varrho_1\varrho_2^\mathsf{T} = \operatorname{diag}(\sigma_1,\sigma_2)$, for some $\sigma_1,\sigma_2 \in \{\pm1\}$;*

(iii) *Setting $\mathsf{w}_k := \varrho_k\big(\gamma(0)-(0,0)\big)$ and $\theta_k := (\mathsf{w}_k\cdot\mathsf{e}_1)/r_k$, for $k = 1,2$, we have*

$$(1.29)\quad \big(R_1\hat{R}_{\theta_1}\big)^\mathsf{T}\big(R_2\hat{R}_{\theta_2}\big) = \operatorname{diag}\big(\sigma_1\sigma_2,\sigma_1,\sigma_2\big) \ \textit{and}\ \mathsf{v}_1 + R_1C_{r_1}(\mathsf{w}_1) = \mathsf{v}_2 + R_2C_{r_2}(\mathsf{w}_2).$$

Proof. NECESSITY. Here, we show that if the deformation $y := y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$ is in $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, then it complies with conditions (i), (ii) and (iii). First of all, we recall from Teorem 1.4.4 that the very condition $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ implies $y \in C^1(\omega,\mathbb{R}^3)$. At the same time, from the specific expression of y we have that $\nabla y = \nabla y_k$ in ω_k for $k = 1,2$, where

$$\nabla y_k = R_k \begin{pmatrix} -\sin\left(\frac{x'\cdot\varrho_k^\mathsf{T}\mathsf{e}_1}{r_k}\right) & 0 \\ \cos\left(\frac{x'\cdot\varrho_k^\mathsf{T}\mathsf{e}_1}{r_k}\right) & 0 \\ 0 & 1 \end{pmatrix}\varrho_k. \tag{1.30}$$

This expression says in particular that ∇y is bounded and in turn that $y \in C^1(\overline{\omega},\mathbb{R}^3)$. Let us first prove the necessity of the conditions (i), (ii) and (iii) in the case when $\gamma(0) = (0,0)$. The continuity of y and ∇y at the point $(0,0)$ gives, respectively, that $\mathsf{v}_1 = \mathsf{v}_2$ (obtained by imposing $y_1(0,0) = y_2(0,0)$), and

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\varrho_1\varrho_2^\mathsf{T} = R_1^\mathsf{T}R_2\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \ \Leftrightarrow\ R_1^\mathsf{T}R_2 = \left(\begin{array}{c|cc} \det(\varrho_1\varrho_2^\mathsf{T}) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\varrho_1\varrho_2^\mathsf{T}$}} \\ 0 & & \end{array}\right) \tag{1.31}$$

(obtained from $\nabla y_1(0,0) = \nabla y_2(0,0)$ and from expression (1.30)), which proves (iii). The continuity of ∇y gives also that $\nabla y_1(\gamma(t)) = \nabla y_2(\gamma(t))$ for each $t \in [0,1]$, that is

$$\begin{pmatrix} -\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ \cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ 0 & 1 \end{pmatrix} \varrho_1\varrho_2^\mathsf{T} = R_1^\mathsf{T} R_2 \begin{pmatrix} -\sin\left(\frac{\gamma(t)\cdot\varrho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ \cos\left(\frac{\gamma(t)\cdot\varrho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ 0 & 1 \end{pmatrix}.$$

In turn, using the second condition in (1.31) and the notation $\varrho_1\varrho_2^\mathsf{T} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, we have

(1.32)
$$\begin{pmatrix} -m_1\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & -m_2\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) \\ m_1\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & m_2\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} -\det(\varrho_1\varrho_2^\mathsf{T})\sin\left(\frac{\gamma(t)\cdot\varrho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ m_1\cos\left(\frac{\gamma(t)\cdot\varrho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & m_2 \\ m_3\cos\left(\frac{\gamma(t)\cdot\varrho_2^\mathsf{T}\mathsf{e}_1}{r_2}\right) & m_4 \end{pmatrix}.$$

By the equality between the elements of the first row in the above expression one deduces that $\varrho_1^\mathsf{T}\mathsf{e}_2$ and $\varrho_2^\mathsf{T}\mathsf{e}_2$ must be parallel. This proves one part of the statement in (ii) and implies, in particular, that $\varrho_1\varrho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$ with $m_1, m_4 \in \{\pm 1\}$. In order to conclude the proof of (ii) and in the same time prove (i), we need to show that $[\gamma]$ is a line segment parallel to $\varrho_1^\mathsf{T}\mathsf{e}_2$ (and to $\varrho_2^\mathsf{T}\mathsf{e}_2$). Observe that $\varrho_1\varrho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$ implies $\varrho_2^\mathsf{T}\mathsf{e}_1 = m_1\varrho_1^\mathsf{T}\mathsf{e}_1$, so that the equation (1.32) simplifies to

$$\begin{pmatrix} -m_1\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ m_1\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) & 0 \\ 0 & m_4 \end{pmatrix} = \begin{pmatrix} -m_4\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ m_1\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right) & 0 \\ 0 & m_4 \end{pmatrix}, \tag{1.33}$$

for every $t \in [0,1]$. By differentiating the above equality restricted to the first elements of the first and second rows one gets

$$\begin{aligned} m_1\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1} &= m_4\cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right)\frac{\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2} \\ \sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1} &= \sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2}\right)\frac{\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_2} \end{aligned} \tag{1.34}$$

It turns out that (1.33) and (1.34) can be satisfied only if

$$\dot\gamma(t)\cdot\left(\varrho_1^\mathsf{T}\mathsf{e}_1\right) = 0, \qquad \text{for every } t \in [0,1], \tag{1.35}$$

which implies that $[\gamma]$ is a line segment parallel to $\varrho_1^\mathsf{T}\mathsf{e}_2$ (thus accordingly also to $\varrho_2^\mathsf{T}\mathsf{e}_2$). To prove previous assertion, we distinguish two cases:

- if $r_1 \neq r_2$, call $s := m_1/m_4$ and fix $t \in [0,1]$. Condition (1.33) grants that $\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1/r_1$ and $s\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1/r_2$ have the same sine and cosine. Then, since sine and cosine cannot simultaneously vanish, (1.34) yields $\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1/r_1 = s\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1/r_2$, whence necessarily $\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1 = 0$.

- if $r_1 = r_2$, by hypotheses we have that $\det\varrho_1 = -\det\varrho_2$. Since $\varrho_1\varrho_2^\mathsf{T} = \mathrm{diag}(m_1, m_4)$, we conclude that $m_1 m_4 = -1$, or equivalently $m_1 = -m_4$. Now the first condition in (1.34) gives

$$\frac{d}{dt}\sin\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right) = \cos\left(\frac{\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1}\right)\frac{\dot\gamma(t)\cdot\varrho_1^\mathsf{T}\mathsf{e}_1}{r_1} = 0,$$

so that the map $t \mapsto \gamma(t) \cdot \varrho_1^\top \mathsf{e}_1/r_1$ is constant and accordingly that $\dot\gamma(t) \cdot \varrho_1^\top \mathsf{e}_1 = 0$ for every $t \in [0,1]$.

This concludes the proof of the necessary condition of the lemma in the case where $\gamma(0) = (0,0)$.

Considering now the case $\mathsf{v} := \gamma(0) - (0,0) \neq 0$, define $\hat\omega := \omega - \mathsf{v}$ and $\hat y_k := y_k \circ \tau_{\mathsf{v}}$, $k = 1,2$ (recall form Section 1.1 that $\tau_{\mathsf{v}} := \cdot + \mathsf{v} \in \mathrm{Trs}(2)$). By Remark 1.4.12, one can easily verify that

$$\hat y_k \;=\; T_{\mathsf{u}_k} \circ R_k \circ \hat R_{\theta_k} \circ C_{r_k} \circ \varrho_k, \tag{1.36}$$

where $\theta_k := (\mathsf{w}_k \cdot \mathsf{e}_1)/r_k$ and $\mathsf{u}_k := \mathsf{v}_k + R_k \circ C_{r_k}(\mathsf{w}_k)$, with $\mathsf{w}_k := \varrho_k\big(\gamma(0) - (0,0)\big)$, for $k = 1,2$. Observe that the domain $\hat\omega$ is partitioned into $\hat\omega_1$ and $\hat\omega_2$ by the subdivision curve $[\gamma] - \mathsf{v}$ which satisfies the condition $\gamma(0) - \mathsf{v} = (0,0)$. It is now clear that $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ implies $\hat y := y \circ \tau_{\mathsf{v}} = \hat y_1 \chi_{\hat\omega_1} + \hat y_2 \chi_{\hat\omega_2} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\hat\omega)$, which further implies that $[\gamma] - \mathsf{v}$ (and hence $[\gamma]$) is a line segment parallel to $\varrho_1^\top \mathsf{e}_2$ and $\varrho_2^\top \mathsf{e}_2$, implying $\varrho_1 \varrho_2^\top = \mathrm{diag}(\sigma_1, \sigma_2)$, for some $\sigma_1, \sigma_2 \in \{\pm 1\}$, and that

$$\mathsf{v}_1 + R_1 \circ C_{r_1}(\mathsf{w}_1) = \mathsf{v}_2 + R_2 \circ C_{r_2}(\mathsf{w}_2) \quad \text{and} \quad \big(R_1 \hat R_{\theta_1}\big)^\top \big(R_2 \hat R_{\theta_2}\big) = \mathrm{diag}\big(\sigma_1\sigma_2, \sigma_1, \sigma_2\big),$$

which are precisely conditions (i), (ii) and (iii).

Sufficiency. Let $y_1, y_2 \in \mathrm{Cyl}$ satisfy conditions (ii) and (iii). Let $\mathsf{v} := \gamma(0) - (0,0)$ and let $\varrho \in \mathrm{SO}(2)$ be a rotation which brings the line segment $[\gamma] - \mathsf{v}$ to the vertical position. Let $\underline{y}_k := y_k \circ \tau_{\mathsf{v}} \circ \varrho^\top$. By denoting $\mathsf{u} := \mathsf{v}_1 + R_1 \circ C_{r_1}(\mathsf{w}_1)$ and $R := R_1 \circ \hat R_{\theta_1}$ we have by (iii) that $\underline{y} := \underline{y}_1 \chi_{\omega_1} + \underline{y}_2 \chi_{\omega_2}$ is of the form

$$\underline{y}(x_1, x_2) = \begin{cases} T_{\mathsf{u}} R\Big(r_1\big(\cos(x_1/r_1) - 1\big), \sigma_1^1 r_1 \sin(x_1/r_1), \sigma_2^1 x_2\Big)^\top, & x_1 \le 0, \\ T_{\mathsf{u}} R\Big(\sigma_1\sigma_2\, r_2\big(\cos(x_1/r_2) - 1\big), \sigma_1^1 r_2 \sin(x_1/r_2), \sigma_2^1 x_2\Big)^\top, & x_1 > 0, \end{cases} \tag{1.37}$$

where $\sigma_k^1 \in \{\pm 1\}$ are such that $\varrho_1 \varrho^\top = \mathrm{diag}(\sigma_1^1, \sigma_2^1)$ (which follows form the fact that $\varrho_1^\top \mathsf{e}_2 \parallel [\gamma]$). By construction, $\underline{y} \in C^1(\underline{\omega}, \mathbb{R}^3)$ with $\underline{\omega} = \varrho(\omega - \mathsf{v})$. Simple computations give $\partial_1 \underline{y}, \partial_2 \underline{y} \in \mathrm{W}^{1,2}(\underline{\omega}, \mathbb{R}^3)$, which implies that $\underline{y} \in \mathrm{W}^{2,2}(\underline{\omega}, \mathbb{R}^3)$. Note also that $\nabla \underline{y}(x')^\top \nabla \underline{y}(x') = \mathrm{I}_3$ for a.e. $x' \in \underline{\omega}$. Therefore $\underline{y} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\underline{\omega})$, thus accordingly $y := \underline{y} \circ \varrho \circ \tau_{-\mathsf{v}} \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. □

Remark 1.4.11. Observe that the condition “$\det\varrho_1 = -\det\varrho_2$ whenever $r_1 = r_2$” permits to exclude the trivial case where we patch together pieces of cylinders y_1 and y_2 having the same curvatures (i.e. $\det\varrho_1/r_1 = \det\varrho_2/r_2$, according to formula (1.21)). Clearly, this case does not force any condition on $[\gamma]$.

Moreover, an argument similar to that in the proof of Lemma 1.4.10 allows to prove necessary and sufficient conditions for having $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ of the form $y = y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$ with, say, y_2 affine (using our terminology, a cylinder with $r_2 = +\infty$). In this case, condition (i) remains the same and condition (ii) reduces to $\varrho_1^\top \mathsf{e}_2 \parallel [\gamma]$ (while $\varrho_2 \in \mathrm{O}(2)$ can be arbitrarily chosen). Moreover, for a chosen $\varrho_2 \in \mathrm{O}(2)$, condition (iii) becomes

$$(R_1 \hat R_{\theta_1})^\top R_2 \hat R_{\theta_2} = \left(\begin{array}{c|cc} \det(\varrho_1\varrho_2^\top) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\varrho_1\varrho_2^\top$}} \\ 0 & & \end{array}\right) \quad \text{and} \quad \mathsf{v}_1 + R_1 C_{r_1}(\mathsf{w}_1) = \mathsf{v}_2 + R_2 C_{r_2}(\mathsf{w}_2)$$

with $\mathsf{w}_k := \varrho_k\big(\gamma(0) - (0,0)\big)$ and $\theta_k := \mathsf{w}_k \cdot \mathsf{e}_1/r_k$. ■

Remark 1.4.12 (Properties of "roto-translations")**.** The following two properties, regarding the composition of cylinders, translations and rotations, can be easily proved.

(*i*) Fix $R \in SO(3)$ and $T_{\mathsf{v}} \in \mathrm{Trs}(3)$. Then $R \circ T_{\mathsf{v}} = T_{R\mathsf{v}} \circ R$.

(*ii*) Let $\tau_{\mathsf{w}} \in \mathrm{Trs}(2)$ and $\hat{R}_\theta \in SO(3)$ be defined by

$$\hat{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $C_r \circ \tau_{\mathsf{w}} = T_{C_r(\mathsf{w})} \circ \hat{R}_{(\mathsf{w}\cdot\mathsf{e}_1)/r} \circ C_r$, for every positive real number r.

In particular, property (*ii*) justifies the choice of the representation used for the elements in Cyl and it is useful for the proof of Lemma 1.4.10. ■

2

From three-dimensional elasticity to nonlinear plate theory

Contents

2.1 The 3D model for a thin hyperelastic sheet

2.1.1 The 3D energy density function

In the general framework in which we will work in, a thin elastic sheet of small thickness $0 < h \ll 1$ is modeled via the three-dimensional *reference domain* (representing the initial, reference configuration)

$$\Omega_h := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right) \subseteq \mathbb{R}^3, \tag{2.1}$$

where the mid-plane of the sheet, $\omega \subseteq \mathbb{R}^2$, is a bounded Lipschitz domain. An arbitrary point in Ω_h will be denoted by $z = (z', z_3) = (z_1, z_2, z_3)$.

In particular, we will consider thin sheets made of a *hyperelastic* material – an elastic material whose constitutive equations postulate the existence of an *energy density function* $\overline{W}_h$, which is a non-negative scalar function defined on a product space $\Omega_h \times \mathbb{R}^{3\times3}$. In other words, this function depends on the point in the domain and on the deformation gradient, as well as on an accurate choice of the material parameters. In Mechanics, $\overline{W}_h$ is also called stored-energy function or strain-energy function (see [Gur82]).

In order to faithfully model the system, $\overline{W}_h$ has to verify some natural physical conditions, that we now present. Let us consider, for the time being, a homogeneous hyperelastic material characterized by $\overline{W}_h = W$, where $W : \mathbb{R}^{3\times3} \to [0, +\infty]$ is a homogeneous energy density – a continuous function satisfying the following properties:

W1 Frame indifference: $W(F) = W(RF)$, for every $F \in \mathbb{R}^{3\times3}$ and $R \in \mathrm{SO}(3)$;

W2 Normalization: $W(\mathrm{I}_3) = \min_{\mathbb{R}^{3\times3}} W = 0$;

W3 Quadratic growth: there exists $C > 0$ such that

$$W(F) \geq C\,\mathrm{dist}^2\big(F, \mathrm{SO}(3)\big) \text{ holds for every } F \in \mathbb{R}^{3\times3}; \tag{2.2}$$

W4 Regularity near to the energy wells: there exists $\mathcal{U} \in \mathcal{N}\big(\mathrm{SO}(3)\big)$ such that $W \in C^2(\mathcal{U})$.

The property W1 means that the energy is invariant under the change of the observer who uses the another orthogonal frame; the normalization condition W2 is saying that, in the absence of the external loads, the equilibrium configuration is achieved by the deformation $v(z) = z$; the quadratic growth condition W3 expresses non-degeneracy of W near its well and grants that W is coercive. In particular, it implies that

$$W(F) \to +\infty, \qquad |F| \to +\infty.$$

Moreover, observe that W is allowed to take the value $+\infty$. This gives possibility to consider only orientation preserving deformations, i.e. those whose gradient F satisfies $\det F > 0$, by imposing the constraint

$$W(F) = +\infty, \qquad \det F \leq 0.$$

For this reason, the growth conditions from above are not allowed.

Let us mention also the class of *isotropic* energy density functions, namely those satisfying the condition

$$W(QFR) = W(F), \qquad \text{for every } F \in \mathbb{R}^{3\times3} \text{ and every } Q, R \in \mathrm{SO}(3), \tag{2.3}$$

in addition to W1 – W4. Energy densities with this property are often present in the models for hyperelastic materials – for instance, this is the case of the model for thin sheets made of polymer gel (see Chapter 4). We shell see in Appendix 2.A below that an isotropic density W provide us with an explicit form of the 2D energy density $\mathcal{Q}_2$ in (2.7) that appears in the description of the 2D models and as such can be better used in the analysis of the 2D energy minimizers that we provide in Chapter 3.

A spontaneous stretch distribution. The heterogeneous materials which we are going to discuss in Chapter 3 and Chapter 5, will be those characterised by a *spontaneous stretch distribution* $A_h : \Omega_h \to \mathrm{Sym}(3)$, which is generally a bounded, invertible tensor field, representing an active stretch, growth, plasticity or other inelastic phenomena.

The stretch A_h being *spontaneous* for the material is modeled by introducing an energy density $\overline{W}_h$, dependent both on the domain variable and the deformation gradient, whose minimum state is precisely $A_h(z)$ at each point $z \in \Omega_h$, modulo superposed rigid body rotations. Namely,

$$\overline{W}_h(z, \cdot) \quad \text{is minimized precisely at} \quad \mathrm{SO}(3)A_h(z). \tag{2.4}$$

The most interesting scenarios occur when the *Cauchy-Green* distribution A_h^2 associated with the spontaneous stretch distribution A_h is not kinematically compatible, i.e. there is no orientation-preserving deformation $v : \Omega_h \to \mathbb{R}^3$ such that $(\nabla v)^\intercal \nabla v = A_h^2$ in Ω_h. We also recall that, since $A_h^2(z)$ is a positive definite symmetric matrix, the distribution A_h^2 can be interpreted as a metric on Ω_h and that, in this framework, the kinematic compatibility of A_h^2 is equivalent to the condition that the Riemann curvature tensor associated with A_h^2 vanishes identically in Ω_h (see Section 1.3).

A typical example of $\overline{W}_h$ with this property is that associated to a *pre-stretched* material, which fulfills

$$\overline{W}_h(z, F) = W\big(FA_h^{-1}(z)\big), \tag{2.5}$$

for some homogeneous energy density W satisfying W1 – W4. The inverse tensor field A_h^{-1} is referred to as a *pre-stretch*. Based on the seminal results of [FJM02, FJM06] in the "Euclidean" case, the models for pre-stretched materials has been recently considered, on the one side, by B. Schmidt in [Sch07a] and [Sch07b] regarding heterogeneous multilayers characterized by the pre-stretch (and thus also the spontaneous stretch) varying only in the vertical (thickness) direction and being at each point h-close to the identity, namely $A_h^{-1}(z) = A_h^{-1}(z_3) = \mathrm{I}_3 + hB(z_3/h)$, with the *strain* distribution $B \in \mathrm{L}^\infty\big((-1/2, 1/2), \mathrm{Sym}(3)\big)$.

On the other side, the relevant case where the pre-stretch is only z'-dependent has been addressed by M. Lewicka and coauthors in [LP11], [BLS16] and [LRR17] and has given rise to the fortunate route of the mathematical treatment of the "non-Euclidean plate theories", introduced from a physical and mechanical view point by the pioneering work of E. Sharon and coauthors in [ESK09] and [KES07]. Further generalizations of the non-Eucidean plate models have been provided in [KS14] and [MS18] in the framework of Riemannian geometry, where z-dependent pre-stretches are considered as well.

We point out that, however, not always an energy density $\overline{W}_h$ associated with the material characterized by a spontaneous stretch A_h can be expressed in the pre-stretch form (2.5) – this is the case of the model energy densities of Flory-Rehner type (4.6) associated to a heterogeneous thin sheet made of polymer gels presented in Chapter 4. This feature depends on the different structure of such an energy density with respect to the models based on the representation (2.5), which originates from physical considerations. More precisely, in the case of pre-stretched materials $\overline{W}_h$ has the physical meaning of purely elastic energy, while in models for polymer gels $\overline{W}_h$ is the sum of two energy contributions (elastic and mixing energies) that concurrently define the energy minimum, but none of them is separately minimized at $\mathrm{SO}(3)A_h$.

This has been one of the motivations for the derivation of the plate model in Chapter 3 precisely for the materials with a spontaneous stretch, not necessarily satisfying (2.5) but rather the assumption (*iii*) in Definition 3.1.1.

Let us anticipate some definitions that we will use throughout the paper that are directly related to the homogeneous densities W.

The 2D energy densities. Given any homogeneous energy density W satisfying the properties W1 – W4, by using a standard notation we define the following quadratic form:

$$\mathcal{Q}_3(F) := D^2W(\mathrm{I}_3)[F]^2, \qquad \text{for every } F \in \mathbb{R}^{3\times 3}. \tag{2.6}$$

Moreover, we set

$$\mathcal{Q}_2(G) := \min_{\mathsf{c}\in\mathbb{R}^3} \mathcal{Q}_3\big(G^* + \mathsf{c}\otimes \mathsf{f}_3\big), \qquad \text{for every } G \in \mathbb{R}^{2\times 2}, \tag{2.7}$$

referring to Subsection 1.1 for the notation G^*. From the properties of W, one can deduce that $\mathcal{Q}_2$ is indeed a quadratic form and that $\mathcal{Q}_k$, for $k = 2, 3$, has the following properties:

- $\mathcal{Q}_k$ is positive semi-definite on $\mathbb{R}^{k\times k}$ and positive definite when restricted to $\mathrm{Sym}(k)$,
- $\mathcal{Q}_k|_{\mathrm{Skew}(k)} = 0$,
- $\mathcal{Q}_k$ is strictly convex on $\mathrm{Sym}(k)$.

The proof of some of the listed properties can be found for instance in [BLS16, FJM02]. For completeness, we provide detailed proofs of all of them in Appendix 2.A.

We further define the map $\ell : \mathrm{Sym}(2) \to \mathbb{R}^3$ by

$$\ell(G) \;:=\; \operatorname*{argmin}_{\mathsf{c}\in\mathbb{R}^3} \mathcal{Q}_3\big(G^* + (\mathsf{c}\otimes \mathsf{f}_3)_{\mathrm{sym}}\big). \tag{2.8}$$

By writing down the first order necessary condition for the minimum problem defining $\ell(G)$, one can easily deduce that the map ℓ is linear.

Moreover, we fix $r > 0$ such that $B_{2r}(\mathrm{I}_3) \subseteq \mathcal{U}$ and define

$$\rho^0(F) := W(\mathrm{I}_3 + F) - \frac{1}{2} D^2 W(\mathrm{I}_3)[F]^2 \quad \text{and} \quad \rho(s) := \sup_{|F|\le s} |\rho^0(F)| \tag{2.9}$$

for every $F \in B_r(0)$ and every $0 < s < r$. As a direct consequence of the regularity of W, we have that

$$\rho(s)/s^2 \to 0, \qquad \text{as } s \to 0. \tag{2.10}$$

2.1.2 The variational problem

With the aim to determine the equilibrium configurations of a thin elastic sheet, in the case of hyperelastic materials one can equivalently study (see for instance [Ped97]) the variational problem consisting in finding the minimizers of the *total energy functional* given by

$$\overline{\mathscr{F}}_h(v) \;:=\; \int_{\Omega_h} \overline{W}_h\big(z, \nabla v(z)\big)\,\mathrm{d}z - \int_{\Omega_h} f(z)\cdot v(z)\,\mathrm{d}z, \tag{2.11}$$

for each deformation $v : \Omega_h \to \mathbb{R}^3$, where $f : \Omega_h \to \mathbb{R}^3$ represents a body force acting on the sheet. For this purpose, very often a two-dimensional approximate model is used.

Dimension reduction, based on the theory of Γ-convergence, is a natural mathematical tool for the rigorous derivation of the lower dimensional models. The value of applying the arguments of Γ-convergence can be seen through Theorem 1.2.5, which ensures that the minimizers of the dimensionally reduced model faithfully describe the true behaviour of a sufficiently thin 3D sheet.

Hereafter we concentrate on the case in which the applied body forces are null. The total energy of the system in the absence of external loads is described through the *free-energy functional* $\overline{\mathscr{E}}_h : \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3) \to [0, +\infty]$ defined by

$$\overline{\mathscr{E}}_h(v) := \int_{\Omega_h} \overline{W}_h\big(z, \nabla v(z)\big)\,\mathrm{d}z, \qquad \text{for every } v \in \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3), \tag{2.12}$$

where $\overline{W}_h : \Omega_h \times \mathbb{R}^{3\times 3} \to [0,+\infty]$ is a (jointly) Borel energy density function associated with the elastic material. Additional properties will be specified separately in each of the models we will be concerned with. As in the homogeneous case, we will deal with the energy densities that have quadratic growth, so that the choice of $\mathrm{W}^{1,2}(\Omega_h,\mathbb{R}^3)$ as the admissible set of deformations in the definition of the free-energy functional is natural.

The rescaled 3D model. In order to perform a rigorous derivation of the lower dimensional limiting theories in terms of Γ-convergence, it will be convenient to work with maps defined on the fixed, *rescaled domain*

$$\Omega := \Omega_1 = \omega \times \left(-\frac{1}{2},\frac{1}{2}\right). \tag{2.13}$$

We denote by $x = (x', x_3) \in \Omega$ an arbitrary point in Ω. Given any $y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ and a thickness parameter $0 < h \ll 1$, we denote by

$$\nabla' y := \big(\partial_1 y \big|\, \partial_2 y\big) \quad \text{and} \quad \nabla_h y := \left(\nabla' y \,\middle|\, \frac{1}{h}\partial_3 y\right). \tag{2.14}$$

To any deformation $v \in \mathrm{W}^{1,2}(\Omega_h,\mathbb{R}^3)$ we associate, in a bijective way, the deformation $y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ by setting

$$y(x) := v(x', hx_3), \qquad \text{for a.e. } x \in \Omega. \tag{2.15}$$

Moreover, the following relation holds

$$\nabla_h y(x) = \nabla v(x', hx_3), \qquad \text{for a.e. } x \in \Omega. \tag{2.16}$$

For every $0 < h \ll 1$, we establish the notion of the rescaled energy density function and the rescaled free-energy functional. Namely, a *rescaled energy density* function is a (jointly) Borel function $W_h : \Omega \times \mathbb{R}^{3\times 3} \to [0,+\infty]$ defined by

$$W_h(x,F) := \overline{W}_h\big((x', hx_3), F\big), \qquad \text{for every } x \in \Omega \text{ and every } F \in \mathbb{R}^{3\times 3}. \tag{2.17}$$

A *rescaled free-energy functional* $\mathscr{E}_h : \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3) \to [0,+\infty]$ is defined by

$$\mathscr{E}_h(y) := \int_\Omega W_h\big(x, \nabla_h y(x)\big)\,\mathrm{d}x, \qquad \text{for every } y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3). \tag{2.18}$$

It is straightforward to check that

$$\mathscr{I}_h(v) := \frac{1}{h}\overline{\mathscr{E}}_h(v) = \frac{1}{h}\int_{\Omega_h} \overline{W}_h(z, \nabla v(z))\,\mathrm{d}z = \int_\Omega W_h\big(x, \nabla_h y(x)\big)\,\mathrm{d}x = \mathscr{E}_h(y), \tag{2.19}$$

for any $v \in \mathrm{W}^{1,2}(\Omega_h,\mathbb{R}^3)$ and $y = v(\cdot, h\cdot) \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$.

2.2 Plate models

2.2.1 Hierarchy of the plate models with respect to the energy scaling

Starting from the 3D models for the elastic sheets of small thickness $0 < h \ll 1$, there are several 2D limiting theories, obtained in the limit of vanishing thickness h. Different dimensionally reduced models correspond to the different scaling of the energy per unit volume $\overline{\mathscr{E}}_h/h = \mathscr{I}_h$, as a function of the thickness h (see also the discussion in the Introduction).

Heuristically [FRS93], the scaling $\overline{\mathscr{E}}_h/h \sim 1$ leads to the membrane theory, while $\overline{\mathscr{E}}_h/h \sim h^2$ corresponds to a bending deformation (which leaves the midplane ω unstretched) leading to the nonlinear plate theory that has been proposed by Kirchhoff in 1850. Scaling $\overline{\mathscr{E}}_h/h \sim h^4$ corresponds to von Kármán theory of plates. The above mentioned theories has been rigorously derived in the setting of homogeneous materials in [LDR95], [FJM02] and [FJM06], respectively. A detailed hierachy of those models, together with the derivation of the theories corresponding to some intermediate scaling regimes can be found in [FJM06].

We recall that in a homogeneous case the scaling of the energy is driven by the scaling of the applied forces. An expectation is [Lew11] that a hierarchy of limiting theories for pre-stretched materials could be possibly derived and lead by the embeddability properties of the target metrics, seen though the scalings of the different components of their Riemann curvature tensors. A discussion on this topic can be found in [LOP$^+$15] and [LMP10], regarding a special case of the metric tensor (or equivalently, the spontaneous stretch A_h) being a bifurcation from the identity, i.e. of the form $A_h = \mathrm{I}_3 + h^\gamma S(z') + h^{\gamma/2} z_3 D(z')$, $z \in \Omega_h$, with $\gamma \in (0, 2)$. It turns out that the energy scaling depends on the asymptotic behaviour of $\mathrm{I}_3 - A_h$, as a function of h.

We will tackle this topic in Chapter 5, where we work in a more general non-Euclidean setting, with zero order (with respect to h) metric non neccesarily flat. This will also give further generalizations of the results regarding the relation between energy scaling and curvature that have been addressed in [LP11], [BLS16] and [LRR17] in the case of thin sheets characterized by an h- and z_3-independent pre-stretch (arising in the prescription of a Riemannian metric $G = G(z')$ on Ω_h).

In [BLS16, LP11] it has been shown that $\inf \overline{\mathscr{E}_h}/h = O(h^2)$ if and only if $(\omega, G_{2\times 2})$ can be isometrically immersed into $\mathbb{R}^3$; in [BLS16, LRR17] it has been proved that $\inf \overline{\mathscr{E}}_h/h = o(h^2)$ if and only if the Riemann curvatures R_{1212}, R_{1213} and R_{1223} associated with the metric G identically vanish – if this is the case, then: $\inf \overline{\mathscr{E}}_h/h = O(h^4)$ and further, $\inf \overline{\mathscr{E}}_h/h = o(h^4)$ if and only if $\mathrm{Riem}_G \equiv 0$. Generalizations of these results have been provided in [KS14] and [MS18] within the setting of Riemannian manifolds of any dimension and co-dimension, putting the emphasis on their geometric interpretation – we shall see that the analysis in [MS18] comprises the results about the relation between h^2- and h^4-energy scaling and curvature in the case (NO) presented in Chapter 5.

2.2.2 Key tools in dimension reduction

A rigorous derivation of the 2D model corresponding to the Kirchhoff energy scaling, i.e. $\overline{\mathscr{E}}_h/h \sim h^2$ (or equivalently $\mathscr{E}_h \sim h^2$, in terms of rescaled quantities) has been achieved in the seminal paper [FJM02] of G. Friesecke, R. D. James and S. Müller for homogeneous materials, i.e. those materials modeled via energy density $\overline{W}_h = W$, with W satisfying W1 – W4.

The limiting 2D functional is constrained to the set of $\mathrm{W}^{2,2}$-isometric immersions of ω into $\mathbb{R}^3$ (denoted by $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$), and coincides with the one predicted by Kirchhoff, which is given by

$$\mathscr{E}_0^{\mathrm{hom}}(y) = \frac{1}{24} \int_\omega \mathcal{Q}_2\big(\Pi_y(z')\big)\, \mathrm{d}z', \qquad \text{for every } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \tag{2.20}$$

where Π_y denotes the pullback of the second fundamental form associated with the surface $y(\omega)$ and the quadratic form $\mathcal{Q}_2$ is given by (2.7).

The 2D model (2.20) is obtained through the compactness and the Γ-convergence results involving the sequence of functionals $\{\frac{1}{h^2}\mathscr{E}_h\}_h$, latter consisting in the Γ-lim inf and Γ-lim sup inequalities (see 1.2.2). Hereafter, we shall point out some crucial steps in the rigorous derivation from [FJM02] which will, together with their counterparts in non-Euclidean setting [LP11], lead the way in the derivation of the plate models for materials with a spontaneous stretch in Chapter 3 and Chapter 5.

The proof of the Γ-lim sup inequality – or more precisely, the construction of the recovery sequence – is based on a heuristic idea: namely, the fact that any smooth isometric immersion y can be approximated in $\mathrm{W}^{1,2}$-topology by a sequence of smooth deformations of the form

$$y^h(x) = y(x') + hx_3\nu(x') + \frac{h^2x_3^2}{2}d(x'), \text{ for all } x \in \Omega, \quad \text{where} \quad \nu = \frac{\partial_1 y \wedge \partial_2 y}{|\partial_1 y \wedge \partial_2 y|} \tag{2.21}$$

and satisfying $\frac{1}{h^2}\mathscr{E}_h(y^h) \cong \mathscr{E}_0^{\mathrm{hom}}(y) + o(h^2)$. With the help of a truncation argument proved in [LT97] (see Theorem 2.2.1 below) for the approximation of $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ by $\mathrm{W}^{2,\infty}(\omega,\mathbb{R}^3)$-maps or, alternatively, using the density of smooth functions in $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ (see Subsection 1.4.1), this heuristic idea can be made into a rigorous proof of the Γ-lim sup inequality. We remark that the presence of the smooth vector field $d : \omega \to \mathbb{R}^3$ above is important for the passage from $\mathcal{Q}_3 = D^2W(\mathrm{I}_3)$ to the associated quadratic form $\mathcal{Q}_2$ and this is where the true recovery sequence differ from the Kirchhoff-Love original ansatz containing only the first two terms in (2.21). It is worth comparing the dependence on the variable x_3 of the last term (containing d) in (2.21) with the corresponding one in the recovery sequences (3.24) and (5.21) in Chapter 3 and Chapter 5, respectively – it turns out that the dependence on x_3 is dictated by the dependence on x_3 of the present spontaneous strain.

Regarding the opposite direction, the rigidity estimate stated in Theorem 2.2.2 below was crucial in proving that every sequence $\{y^h\}_h$ of bounded bending energy, i.e. such that $\mathscr{E}_h(y^h)/h^2$ is bounded uniformly in h, is asymptotically of the form (2.21). More precisely, Theorem 2.2.2 gave the possibility to associate to any sequence $\{y^h\}_h$ converging in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ to some y and being of bounded bending energy a sequence of piecewise constant maps $R_h : \omega \to \mathrm{SO}(3)$ with the property that

$$\|\nabla_h y^h - R_h\|_{\mathrm{L}^2(\Omega)} \leq Ch^2. \tag{2.22}$$

This has been further used to show that the limiting map y is an element of $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ and that $\{\nabla_h y^h\}_h$ (and, in turn, also $\{R_h\}_h$) converges in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times3})$ to $R = (\nabla y|\nu) : \omega \to \mathrm{SO}(3)$, with ν as in (2.21). Once such a compactness result is proved, the Γ-lim inf inequality follows by observing from (2.22) that the sequence of maps

$$S_h := \frac{R_h^{\mathsf{T}}\nabla_h y^h - \mathrm{I}_3}{h} \tag{2.23}$$

is bounded in $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$, thus converges to some $S \in \mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$. It has been further shown that the corresponding limiting strain $S \in \mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$ satisfies

$$S_{2\times2}(x) = S_{2\times2}(x',0) + x_3\Pi_y(x'), \qquad \text{for a.e. } x \in \Omega. \tag{2.24}$$

We remark that the map $S_{2\times2}(\cdot,0)$ can be better characterized when a higher than Kirchhoff energy scaling order is present (see [FJM06]). In the case of Kirchhoff scaling and homogeneous materials as in [FJM02] one is allowed to simply neglect it in the derivation of the 2D model, thus its properties have been never discussed. It turns out that in the setting of heterogeneous materials (characterized by a spontaneous stretch) that we consider in Chapter 3 it will be important to detect some more properties of $S_{2\times 2}(\cdot, 0)$ – for instance, to understand if its symmetric part could be a symmetrized gradient of some map or equivalently, a linear strain with respect to the leading order midplate metric I_2. We will come back to this issue in Chapter 3, Section 3.4.

We conclude this section recalling some of the important results mentioned above, which form a basis for our further developments in the derivation of the dimensionally reduced models. The following theorem can be found in [FJM06] as a special case of the truncation result proved in [LT97].

Theorem 2.2.1 (Approximation by $\mathrm{W}^{m,\infty}$-maps)**.** *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and let $1 < p < \infty$, $m \in \mathbb{N}$ and $\lambda > 0$. Let $v \in W^{m,p}(\mathcal{U}, \mathbb{R}^n)$ and define $|v|_m := \sum_{|\alpha|\le m} |\nabla^\alpha v|$. Then there exists $v^\lambda \in \mathrm{W}^{m,\infty}$ such that*

$$\begin{aligned} \|v^\lambda\|_{\mathrm{W}^{m,\infty}(\mathcal{U})} &\le C(p,m,\mathcal{U})\lambda \\ \mathcal{L}^n\big(\{x \in \mathcal{U} :\, v^\lambda(x) \neq v(x)\}\big) &\le \frac{C(p,m)}{\lambda^p} \int_{|v|_m \ge \lambda/2} |v|_m^p(x)\,\mathrm{d}x \\ \|v^\lambda\|_{\mathrm{W}^{m,p}(\mathcal{U})} &\le C(p,m,\mathcal{U}) \|v\|_{\mathrm{W}^{m,p}(\mathcal{U})}. \end{aligned}$$

Observe that, in particular, it holds

$$\lim_{\lambda\to\infty} \lambda^p \mathcal{L}^n\big(\{x \in \mathcal{U} :\, v^\lambda(x) \neq v(x)\}\big) = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \|v^\lambda - v\|_{\mathrm{W}^{m,p}(\mathcal{U})} = 0.$$

The rigidity theorem of [FJM02] (Theorem 2.2.2 below), crucial for the rigorous derivation of the dimentionally reduced theories for thin elastic sheets, represents a quantitative version of a classical result in geometry and mechanics, known as Liouville's theorem, which says that a map whose gradient coincides with a constant rotation almost everywhere is a rigid motion.

Theorem 2.2.2 (Geometric rigidity [FJM02])**.** *Let $n \ge 2$ and let $\mathcal{U} \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. There exists a constant $C(\mathcal{U}) > 0$ with the following property: for every $v \in \mathrm{W}^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there is an associated rotation $R \in \mathrm{SO}(n)$ satisfying*

$$\|\nabla v - R\|_{\mathrm{L}^2(\mathcal{U})} \;\le\; C(\mathcal{U})\, \|\mathrm{dist}^2(\nabla v, \mathrm{SO}(n))\|_{\mathrm{L}^2(\mathcal{U})}. \tag{2.25}$$

Moreover, the constant $C(\mathcal{U})$ can be chosen uniformly or a family of the domains which are biLipschitz equivalent with controlled Lipschitz constants. $C(\mathcal{U})$ is invariant under translation and dilation.

In the non-Euclidean setting it is possible to estimate the deviation of the deformation from an affine map only at the expense of an extra term that is proportional to the gradient of a given metric g:

Theorem 2.2.3 (Geometric rigidity: non-Eulidean version [LP11])**.** *Let $n \ge 2$ and let $\mathcal{U} \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain with a given smooth Riemannian metric g. For every $y \in \mathrm{W}^{1,2}(\mathcal{U}, \mathbb{R}^n)$ there exists $Q \in \mathbb{R}^{n\times n}$ such that*

$$\int_{\mathcal{U}} |\nabla y(x) - Q|^2\,\mathrm{d}x \le C\left(\int_{\mathcal{U}} \mathrm{dist}^2\big(\nabla y(x), \mathrm{SO}(n)\sqrt{g}(x)\big)\,\mathrm{d}x + \|\nabla g\|^2_{\mathrm{L}^\infty(\mathcal{U})}(\mathrm{diam}\,\mathcal{U})^2\mathcal{L}^n(\mathcal{U})\right),$$

where the constant C depends only on $\|g\|_{\mathrm{L}^\infty(\mathcal{U})}$, $\|g^{-1}\|_{\mathrm{L}^\infty(\mathcal{U})}$ and $\mathcal{U}$ and is uniform for a family of domains which are biLipschitz equivalent with controlled Lischitz constants.

Moreover, a generalization of the Liouville's rigidity theorem from Euclidean to Riemannian setting has been recently proved in [KMS18] by R. Kupferman and coauthors.

2.A Appendix

In this section we discuss the properties of the quadratic forms $\mathcal{Q}_3$ and $\mathcal{Q}_2$ associated to a homogeneous density W satisfying W1 – W4 and examine more in details the case of W isotropic.

Lemma 2.A.1. *The maps $\mathcal{Q}_k, k = 3, 2$ defined by (2.6) and (2.7) respectively, satisfy:*

(i) $\mathcal{Q}_k$ is positive semi-definite on $\mathbb{R}^{k\times k}$ and positive definite when restricted to $\mathrm{Sym}(k)$*,*

(ii) $\mathcal{Q}_k|_{\mathrm{Skew}(k)} = 0$,

(iii) $\mathcal{Q}_k$ is strictly convex on $\mathrm{Sym}(k)$*.*

Before proving the above lemma, we recall from [DM12] the following proposition that provides a useful characterization of quadratic forms.

Proposition 2.A.2. *Let $n \in \mathbb{N}$ and let $\mathcal{Q} : \mathbb{R}^{n\times n} \to [0, +\infty]$ be an arbitrary function. If*

(a) $\mathcal{Q}(0) = 0$,

(b) $\mathcal{Q}(tM) \le t^2\mathcal{Q}(M)$ for every $M \in \mathbb{R}^{n\times n}$ and every $t > 0$,

(c) $\mathcal{Q}(M + N) + \mathcal{Q}(M - N) \le 2\mathcal{Q}(M) + 2\mathcal{Q}(N)$ for every $M, N \in \mathbb{R}^{n\times n}$

then $\mathcal{Q}$ is a quadratic form. Conversely, if $\mathcal{Q}$ is a quadratic form then (a), (b) and (c) are satisfied and, in addition,

(b′) $\mathcal{Q}(tM) = t^2\mathcal{Q}(M)$ for every $M \in \mathbb{R}^{n\times n}$ and every $t > 0$,

(c′) $\mathcal{Q}(M + N) + \mathcal{Q}(M - N) = 2\mathcal{Q}(M) + 2\mathcal{Q}(N)$ for every $M, N \in \mathbb{R}^{n\times n}$.

Proof of Lemma 2.A.1. First of all observe that, by its very definition, $\mathcal{Q}_3$ is a quadratic form on $\mathbb{R}^{3\times3}$ according to Proposition 2.A.2. Let us prove that $\mathcal{Q}_3$ satisfies properties *(i)*, *(ii)* and *(iii)*.

(i) Since W attains its minimum at identity, we have that

$$\mathcal{Q}_3(F) = D^2W(\mathrm{I}_3)[F, F] \ge 0 \quad \text{for every } F \in \mathbb{R}^{3\times3}.$$

Hence $\mathcal{Q}_3$ is positive semidefinite. Now let $\overline{F} \in \mathrm{Sym}(3)$ and $r > 0$ be as in (2.9). We have that

$$W(\mathrm{I}_3 + t\overline{F}) = D^2W(\mathrm{I}_3)[t\overline{F}, t\overline{F}] + \rho^0(tF)$$

for every $t > 0$ such that $t\overline{F} \in B_r(0)$. Recall that

$$\mathrm{dist}\big(F, SO(3)\big) \ge |\sqrt{F^{\intercal}F} - \mathrm{I}_3| \quad \text{for every } F \in \mathbb{R}^{3\times3}. \tag{2.26}$$

By super quadratic growth of W, i.e. $W(\mathrm{I}_3 + F) \ge C\mathrm{dist}^2(\mathrm{I}_3 + F, \mathrm{SO}(3))$ for every $F \in B_r(0)$ and $C > 0$, we get that

$$\begin{aligned} D^2W(\mathrm{I}_3)[t\overline{F}, t\overline{F}] + \rho^0(t\overline{F}) =& W(\mathrm{I}_3 + t\overline{F}) \ge C\mathrm{dist}^2(\mathrm{I}_3 + t\overline{F}, \mathrm{SO}(3)) \\ \overset{(2.26)}{\ge}& C\Big|(\mathrm{I}_3 + t\overline{F}) - \mathrm{I}_3\Big|^2 = C|t\overline{F}|^2. \end{aligned}$$

Therefore, by dividing the above inequality by t^2 and by letting $t \to 0$ we obtain

$$\mathcal{Q}_3(\overline{F}) = D^2W(\mathrm{I}_3)[\overline{F}, \overline{F}] \geq C|\overline{F}|^2,$$

proving that $\mathcal{Q}_3$ is positive definite on Sym(3), by arbitrariness of $\overline{F}$.

(ii) Let $F \in \mathbb{R}^{3\times3}$ and $A \in \mathrm{Skew}(3)$. Then $e^{tA} \in \mathrm{SO}(3)$ for every $t \in \mathbb{R}$. By frame indifference of W, we have that

$$\begin{aligned} W(\mathrm{I}_3 + tF) =& W\big(e^{tA}(\mathrm{I}_3 + tF)\big) \\ =& D^2W(\mathrm{I}_3)\big[t(A+F) + o(t), t(A+F) + o(t)\big] + \rho^0\big(t(A+F) + o(t)\big) \\ =& t^2\mathcal{Q}_3\big((A+F) + o(1)\big) + \rho^0\big(t(A+F) + o(t)\big) \end{aligned}$$

and contemporary

$$W(\mathrm{I}_3 + tF) = D^2W(\mathrm{I}_3)[tF, tF] + \rho^0(tF) = t^2\mathcal{Q}_3(F) + \rho^0(tF),$$

for every $t \in \mathbb{R}$ such that $tF \in B_r(0)$. Therefore

$$\begin{aligned} \big|\mathcal{Q}_3(F) - \mathcal{Q}_3\big(A + F + o(1)\big)\big| =& \big|\rho^0(tF)/t^2 - \rho^0\big(t(A+F) + o(t)\big)/t^2\big| \\ \leq& \rho(|tF|)/t^2 + \rho\big(|t(A+F) + o(t)|\big)/t^2. \end{aligned}$$

By letting $t \to 0$ we obtain that $\mathcal{Q}_3(F) = \mathcal{Q}_3(F + A)$. Moreover, since the previous equality holds for every $F \in \mathbb{R}^{3\times3}$, by taking $F = 0$ we get that $\mathcal{Q}_3(A) = 0$ for every $A \in \mathrm{Skew}(3)$.

(iii) Fix $\overline{F} \in \mathrm{Sym}(3)$. By directly computing the second differential of $\mathcal{Q}_3$, one has

$$D^2\mathcal{Q}_3(\overline{F})[F, F] = 2\mathcal{Q}_3(F) \geq 2C|F|^2 \quad \text{for every } F \in \mathrm{Sym}(3).$$

Hence $\mathcal{Q}_3$ is strictly convex on Sym(3), since its second differential is positive definite.

Now observe that $\mathcal{Q}_2$ defined by (2.7) satisfies (a), (b) and (c) of Proposition 2.A.2:

(a) $\mathcal{Q}_2(0) = 0$ is satisfied by the definition of $\mathcal{Q}_2$.

(b) Fix $G \in \mathbb{R}^{2\times2}$ and $t > 0$. Then, recalling the definition of the map ℓ form (2.8), we have

$$\mathcal{Q}_2(tG) = \mathcal{Q}_3(tG^* + t\ell(G) \otimes \mathsf{f}_3) = t^2\mathcal{Q}_2(G).$$

(c) Note that, by the linearity of the map ℓ, it holds

$$\mathcal{Q}_2(G_1 \pm G_2) = \mathcal{Q}_3\big(G_1 \pm G_2 + (\ell(G_1) \pm \ell(G_2)) \otimes \mathsf{f}_3\big)$$

and thus,

$$\begin{aligned} \mathcal{Q}_2(G_1 + G_2) + \mathcal{Q}_2(G_1 - G_2) = 2&\mathcal{Q}_3(G_1 + \ell(G_1) \otimes \mathsf{f}_3) \\ &+ 2\mathcal{Q}_3(G_2 + \ell(G_2) \otimes \mathsf{f}_3) = 2\mathcal{Q}_2(G_1) + 2\mathcal{Q}_2(G_2). \end{aligned}$$

Hence $\mathcal{Q}_2$ is a quadratic form. Then it is straightforward to check that also $\mathcal{Q}_2$ satisfies *(i)*, *(ii)* and *(iii)*, as claimed. □

Isotropic energy density function. Suppose that the energy density W is isotropic, i.e. satisfies the condition (2.3). Then (see [Gur82]) there exist constants $\mathbf{G} > 0$ and $\lambda \in \mathbb{R}$ such that

$$\mathcal{Q}_3(F) = D^2W(\mathrm{I}_3)[F,F] = 2\mathbf{G}|F_{\mathrm{sym}}|^2 + \lambda \mathrm{tr}^2(F)$$

for every $F \in \mathbb{R}^{3\times 3}$. Then we have

$$\begin{aligned}\mathcal{Q}_2(G) &= \min_{\mathsf{c}\in\mathbb{R}^3} \mathcal{Q}_3(G^* + \mathsf{c}\otimes \mathsf{f}_3) = \min_{\mathsf{b}\in\mathbb{R}^2, a\in\mathbb{R}} \left\{2\mathbf{G}\left(|G_{\mathrm{sym}}|^2 + |\mathsf{b}|^2/2 + a^2\right) + \lambda\left((\mathrm{tr}G + a)^2\right)\right\} \\ &= 2\mathbf{G}|G_{\mathrm{sym}}|^2 + \lambda \mathrm{tr}^2 G + \min_{a\in\mathbb{R}} \left\{(\mu+\lambda)a^2 + 2(\mathrm{tr}G)\lambda\, a\right\} \\ &= 2\mathbf{G}|G_{\mathrm{sym}}|^2 + \frac{\lambda \mathbf{G}}{\mathbf{G}+\lambda/2}\mathrm{tr}^2 G = 2\mathbf{G}|G_{\mathrm{sym}}|^2 + \frac{2\mathbf{G}\lambda}{2\mathbf{G}+\lambda}\mathrm{tr}^2 G,\end{aligned}$$

for every $G \in \mathbb{R}^{2\times 2}$. By denoting

$$\boldsymbol{\Lambda} := \frac{2\mathbf{G}\lambda}{2\mathbf{G}+\lambda}, \tag{2.27}$$

we finally have

$$\mathcal{Q}_2(G) = 2\mathbf{G}|G_{\mathrm{sym}}|^2 + \boldsymbol{\Lambda}\mathrm{tr}^2 G, \qquad \text{for all } G \in \mathbb{R}^{2\times 2}. \tag{2.28}$$

The positiveness of $\mathbf{G}$ is implied by the positive definiteness of $\mathcal{Q}_3$ – this can be easily seen by plugging $F = \mathrm{diag}(1,-1,0)$ into the expression for $\mathcal{Q}_3$. For the same reason, by computing $\mathcal{Q}_3(\mathrm{I}_3)$ we get that $2\mathbf{G} + 3\lambda > 0$. The latter further implies that $\boldsymbol{\Lambda} > -1/2$, which provides an explicit verification that also $\mathcal{Q}_2$ is positive definite. The constants $\mathbf{G}$ and $\boldsymbol{\Lambda}$ are called non-dimensional Lamé constants.

3

Dimension reduction for materials with a spontaneous stretch distribution

Contents

In this chapter we provide a dimensionally reduced model describing the bending behavior of heterogeneous thin elastic sheets with in-plane modulation of the through-the-thickness variation of the spontaneous strain.

More precisely, we are interested in thin elastic sheets occupying the reference domain $\Omega_h = \omega \times (-h/2, h/2)$ with $0 < h \ll 1$, of a material characterized by a spontaneous stretch distribution A_h (cfr. Section 2.1.1, formula (2.4)) of the form

$$A_h(z) = \mathrm{I}_3 + h\, B\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \Omega_h. \tag{3.1}$$

Here, B is a given strain field defined on the rescaled domain $\Omega = \Omega_1$, with values in the space $\mathrm{Sym}(3)$ of the 3×3 symmetric matrices.

Apart from the energy-well structure (2.4), we make standard assumptions on the family of densities $\overline{W}_h$ modeling the 3D system such as frame indifference and quadratic growth, and we suppose that the rescaled family $\{W_h\}_h$ converges uniformly, as $h \to 0$, to a limiting homogeneous density function W. We refer the reader to Section 3.1 for details on the assumptions on the 3D model. Thin gel sheets provide an interesting example of material which can be described through a family of densities of Flory-Rehner type (given in (4.6)) fulfilling precisely the mentioned assumptions, as we shall see in Chapter 4.

In Section 3.2 the same arguments as in [Sch07b] (which are in turn a slight variant of those employed in the seminal work [FJM02], presented in Subsection 2.2.2) has been used to find the corresponding limiting Kirchhoff plate model, under the assumption that

$$\text{(3.2)} \qquad \operatorname{curl}\big(\operatorname{curl}\mathcal{B}^0_{2\times2}\big) = 0, \quad \text{with} \quad \mathcal{B}^0_{2\times2}(z') := \int_{-1/2}^{1/2} B_{2\times2}(z',t)\,\mathrm{d}t \qquad \text{for a.e. } z' \in \omega.$$

We recall from Lemma 1.7 that the condition (3.2) guarantees that $\mathcal{B}^0_{2\times2}$ is a symmetrized gradient, and in turn allows for the construction of a “standard” ansatz for the recovery sequence. When instead condition (3.2) is violated, usual arguments such as local modifications or perturbation arguments seem insufficient to prove the same Γ-limit, while a heuristic argument make us believe that the general Γ-limit has to include a nonlocal term, which can be interpreted as a “first order stretching term”. A more detailed discussion is provided in Section 3.4.

The Kirchhoff-like model resulting from the dimension reduction is constrained to the set of isometric immersions of the mid-plane of the plate into $\mathbb{R}^3$, with a corresponding energy that penalizes deviations of the curvature tensor associated with a deformation from a x'-dependent *target curvature tensor* $\mathcal{B}^1_{2\times2}$ below. Namely, it is governed by the energy functional

$$\text{(3.3)} \quad \mathscr{E}_0(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\left(\Pi_y(z') - \mathcal{B}^1_{2\times2}(z')\right)\mathrm{d}z' + \text{ad.t.}, \;\; \text{with} \;\; \mathcal{B}^1_{2\times2} := 12\int_{-1/2}^{1/2} tB_{2\times2}(\cdot,t)\,\mathrm{d}t,$$

on each $W^{2,2}$-isometry y, where ad.t. stays for “additional terms” not depending on y. In the above formula, $\mathcal{Q}_2$ is the 2D density related to the limiting, homogeneous density W via (2.7), while Π_y is the pull-back of the second fundamental form of the surface $y(\omega)$. Let us observe that this plate theory stands between those of [Sch07b], on one hand, and of [LP11] and [BLS16], on the other hand, and represents (to the best of our knowledge) the first attempt to considering Kirchhoff plate theories (within the framework of Friesecke-James-Müller theory) originated in 3D energies characterized by pre-stretches or, more in general, spontaneous stretches which are heterogeneous in plane as well as along the thickness. Both in-plane and through-the-thickness variation of pre-stretch has been considered also in [KS14], within the framework of Riemannian geometry. Pre-stretches of the form (3.1) have been very recently treated in [CRS17] and [KO17] to derive corresponding *rod* models with misfit. Moreover, similar pre-stretches have been considered in [LOP+15] to obtain 2D models in the case of scaling orders higher than the Kirchhoff one.

It is worth mentioning that beam theories derived from 2D energies of the form (3.3), in the limit as $\varepsilon \to 0$ when $\omega = (-\ell/2, \ell/2) \times (\varepsilon/2 \times \varepsilon/2)$, can be found in [ADK16] for the case $\mathcal{B}^1_{2\times2}$ constant and in [FHMP16] in the case $\mathcal{B}^1_{2\times2} = \mathcal{B}^1_{2\times2}(z_1)$. To use a common terminology, these 1D theories may describe *narrow ribbons* of soft active materials.

Apart from the derivation of the 2D model, we will give some insight on the minimizers of the derived 2D model (3.3) and, keeping eye on the further application to the modeling of

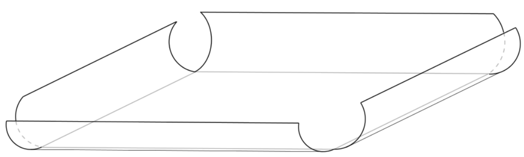

Figure 3.1: Example of a 2D minimum energy configuration.

foldable structures in Chapter 4, we focus on the case where the spontaneous strain B is an odd function of the thickness variable (which trivially fulfills condition (3.2)), being at the same time a piecewise constant function of the planar variable. This case leads in turn to a piecewise constant target curvature tensor.

In Section 3.3, we recall that in the case where $\mathcal{B}^1_{2\times 2}$ is constant, then a minimizer of the 2D energy $\mathscr{E}_0$ actually minimizes the integrand function pointwise – indeed, in this case both 2D target metric and curvature satisfy Gaus-Codazzi-Mainardi compatibility equations (1.4) – and the corresponding deformed configuration is a piece of cylindrical surface (see Lemma 3.3.2 and the discussion preceding it). In the case of a piecewise constant $\mathcal{B}^1_{2\times 2}$, some conditions (specified in Theorem 3.3.5) under which cylindrical surfaces can be patched together resulting into an isometry must be fulfilled for the pointwise minimizer to exist. When these conditions hold, an example of minimum energy configuration – a patchwork of cylindrical surfaces – is sketched in Figure 3.1.

3.1 Setting

Throughout this chapter $\omega \subseteq \mathbb{R}^2$ will be a simply-connected, bounded Lipschitz domain satisfying the condition

$$\text{there exists a closed subset } \Sigma \subset \partial\omega \text{ with } \mathcal{H}^1(\Sigma) = 0 \text{ such that the outer unit normal exists and is continuous on } \partial\omega \setminus \Sigma. \tag{3.4}$$

The requirement that ω is a simply-connected domain has to do with the "compatibility" condition of Theorem 1.3.1, which is imposed on the 2×2 part of the tensor-valued map $\mathcal{B}^0$ defined by (3.5) and (3.9). The condition (3.4) is a standard requirement on the domain in order to have the density results for the space of $\mathrm{W}^{2,2}$-isometric immersions of ω into $\mathbb{R}^3$ (see Theorem 1.4.7).

We are interested in a thin sheet $\Omega_h := \omega \times (-h/2, h/2)$, with $0 < h \ll 1$, of a material characterized by a spontaneous stretch given at each point of Ω_h in the form $A_h(z) = \mathrm{I}_3 + hB\big(z', \frac{z_3}{h}\big)$, for a suitable *spontaneous strain* $B \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3))$.

More in general, we consider a family $\mathscr{B} = \{B_h\}_{h\geq 0}$ of spontaneous strains such that

$$B_h \to B_0 =: B \qquad \text{in } \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3)), \text{ as } h \to 0, \tag{3.5}$$

the corresponding family $\{A_h\}_{h\geq 0}$ of spontaneous stretches defined as

$$A_h(x', hx_3) := \mathrm{I}_3 + hB_h(x) \qquad \text{for a.e. } x \in \Omega \text{ and for every } h \geq 0, \tag{3.6}$$

and the associated family $\{W_h\}_{h>0}$ of (rescaled; see Section 2.1) energy densities $W_h : \Omega \times \mathbb{R}^{3\times 3} \to [0, +\infty]$, which we suppose to be Borel functions satisfying the following properties:

(*i*) for a.e. $x \in \Omega$, the map $W_h(x, \cdot)$ is frame indifferent, i.e.

$$W_h(x, F) = W_h(x, RF) \qquad \text{for every } F \in \mathbb{R}^{3\times 3} \text{ and every } R \in \mathrm{SO}(3);$$

(*ii*) for a.e. $x \in \Omega$, $W_h(x, \cdot)$ is minimized precisely at $\mathrm{SO}(3)A_h(x', hx_3)$;

(*iii*) there exists an open neighbourhood $\mathcal{U}$ of $\mathrm{SO}(3)$ and $W \in C^2(\overline{\mathcal{U}})$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \big\| W_h(x, \cdot) - W \big\|_{C^2(\overline{\mathcal{U}})} \to 0, \qquad \text{as } h \to 0; \tag{3.7}$$

(iv) there exists a constant $C > 0$, independent of h, such that for a.e. $x \in \Omega$ it holds that

$$W_h(x, F) \geq C\mathrm{dist}^2\big(F, \mathrm{SO}(3)A_h(x', hx_3)\big), \qquad \text{for every } F \in \mathbb{R}^{3\times 3}. \tag{3.8}$$

Definition 3.1.1 (Admissible family of free-energy densities)**.** *Given $\mathscr{B} = \{B_h\}_{h\geq 0}$ satisfying (3.5) and the associated family $\{A_h\}_{h\geq 0}$ defined in (3.6), we call $\mathscr{B}$-*admissible *a family $\{W_h\}_{h>0}$ of Borel functions from $\Omega \times \mathbb{R}^{3\times 3}$ to $[0, +\infty]$ fulfilling (i) - (iv).*

Let $\{W_h\}_{h>0}$ be a given $\mathscr{B}$-admissible family of free-energy densities, with associated limiting density function W. Observe that the limiting density W inherits properties (i), (ii) and (iv) from convergence (3.7). Thus we can associate to W the quadratic forms $\mathcal{Q}_k$, $k = 2, 3$ via (2.6) and (2.7).

We also introduce the following notation:

$$\mathcal{B}^0(x') := \int_{-1/2}^{1/2} B(x', t)dt \qquad \text{and} \qquad \mathcal{B}^1(x') := 12\int_{-1/2}^{1/2} tB(x', t)dt, \tag{3.9}$$

for a.e. $x' \in \omega$. Above defined maps will play the crucial role in our derivation and analysis of the 2D model. Note also from hypothesis (3.5) that $\mathcal{B}^0, \mathcal{B}^1 \in \mathrm{L}^\infty(\omega, \mathrm{Sym}(2))$. Our limiting 2D model will be related to the 2D density function $\overline{\mathcal{Q}}_2 : \omega \times \mathbb{R}^{2\times 2} \to [0, +\infty)$ defined as

$$\overline{\mathcal{Q}}_2(x', G) := \min_{D\in\mathbb{R}^{2\times 2}} \int_{-1/2}^{1/2} \mathcal{Q}_2\big(D + tG - B_{2\times 2}(x', t)\big)\, \mathrm{d}t,$$

for a.e. $x' \in \omega$ and every $G \in \mathbb{R}^{2\times 2}$, where $B_{2\times 2}$ is related to the 3D model through (3.5), using the notation introduced in Subsection 1.1. Since $\mathcal{Q}_2$ does not depend on the skew-symmetric part of its argument, we can think of $\overline{\mathcal{Q}}_2$ to be defined only on $\omega \times \mathrm{Sym}(2)$ as

$$\overline{\mathcal{Q}}_2(x', G) = \min_{D\in\mathrm{Sym}(2)} \int_{-1/2}^{1/2} \mathcal{Q}_2\big(D + tG - B_{2\times 2}(x', t)\big)\, \mathrm{d}t. \tag{3.10}$$

This minimum problem can be solved explicitly, as stated by the following lemma.

Lemma 3.1.2. *For a.e. $x' \in \omega$ and every $G \in \mathrm{Sym}(2)$, the minimizer in (3.10) is unique and coincides with $\mathcal{B}^0_{2\times 2}(x')$. In other words, we have that*

$$\overline{\mathcal{Q}}_2(x', G) = \int_{-1/2}^{1/2} \mathcal{Q}_2\Big(\mathcal{B}^0_{2\times 2}(x') + tG - B_{2\times 2}(x', t)\Big)\, \mathrm{d}t \tag{3.11}$$

for a.e. $x' \in \omega$ and every $G \in \mathrm{Sym}(2)$.

Proof. By using the bilinear form $\mathcal{L}_2$ associated with $\mathcal{Q}_2$ it is easy to see that for a.e. $x' \in \omega$ and every $G \in \mathrm{Sym}(2)$ it holds

$$\begin{aligned}
&\min_{D\in\mathrm{Sym}(2)} \int_{-1/2}^{1/2} \mathcal{Q}_2(D + tG - B_{2\times 2}(x', t))\, \mathrm{d}t \\
&= \min_{D\in\mathrm{Sym}(2)} \int_{-1/2}^{1/2} \Big(\mathcal{Q}_2(D) + \mathcal{Q}_2\big(tG - B_{2\times 2}(x', t)\big) + 2\mathcal{L}_2\big(D, tG - B_{2\times 2}(x', t)\big)\Big)\, \mathrm{d}t \\
&= \min_{D\in\mathrm{Sym}(2)} \int_{-1/2}^{1/2} \Big(\mathcal{Q}_2(D) + \mathcal{Q}_2\big(tG - B_{2\times 2}(x', t)\big) + 2t\,\mathcal{L}_2\big(D, G\big) + 2\mathcal{L}_2\big(D, -B_{2\times 2}(x', t)\big)\Big)\, \mathrm{d}t \\
&= \int_{-1/2}^{1/2} \mathcal{Q}_2\big(tG - B_{2\times 2}(x', t)\big)\, \mathrm{d}t + \min_{D\in\mathrm{Sym}(2)} \Big(\mathcal{Q}_2(D) + 2\mathcal{L}_2\Big(D, -\int_{-1/2}^{1/2} B_{2\times 2}(x', t)\, \mathrm{d}t\Big)\Big) \\
&= \int_{-1/2}^{1/2} \mathcal{Q}_2\big(tG - B_{2\times 2}(x', t)\big)\, \mathrm{d}t - \mathcal{Q}_2\big(\mathcal{B}^0_{2\times 2}(x')\big) + \min_{D\in\mathrm{Sym}(2)} \mathcal{Q}_2\big(D - \mathcal{B}^0_{2\times 2}(x')\big).
\end{aligned}$$

From this equality, the thesis trivially follows.

□

Note that the minimizer in (3.10), which is in principle dependent on G from its definition, turns out to be independent of G in the end. This is not the case when, e.g., the limiting density function W depends explicitly on x_3, not just through its spontaneous stretch, see [Sch07b]. Finally, observe for future reference that from (3.11) one can rewrite $\overline{\mathcal{Q}}_2$ in the more explicit form

$$\overline{\mathcal{Q}}_2(x',G) = \frac{1}{12}\mathcal{Q}_2\big(G-\mathcal{B}^1_{2\times 2}(x')\big) + \int_{-1/2}^{1/2}\mathcal{Q}_2\big(B_{2\times 2}(x',t)-\mathcal{B}^0_{2\times 2}(x')\big)\,\mathrm{d}t - \frac{1}{12}\mathcal{Q}_2\big(\mathcal{B}^1_{2\times 2}(x')\big), \tag{3.12}$$

for a.e. $x'\in\omega$ and every $G\in\mathrm{Sym}(2)$. We will better use this expression for $\overline{\mathcal{Q}}_2$ more than (3.11) when we look for pointwise minimizers in Section 3.3. In fact, the only relevant part of $\overline{\mathcal{Q}}_2$ for our minimization purposes is the first summand on the right hand side of (3.12).

Before passing to the rigorous derivation of the 2D model, we provide a technical lemma consisting in two estimates for the family $\{W_h\}_{h>0}$ of energy densities and for its uniform limit W defined in a neighbourhood $\mathcal{U}$ of SO(3). They are elementary consequences of properties (ii) and (iii) of Definition 3.1.1. These estimates will be used in the proof of the Γ-$\liminf$ and the Γ-$\limsup$.

Lemma 3.1.3. *Let $r>0$ be related to the limiting energy density W by (2.9). For every $\varepsilon>0$ there exists $h_\varepsilon>0$ and $C_\varepsilon>0$ such that for a.e. $x\in\Omega$, every $F\in B_r(0)$ and every $h\in(0,h_\varepsilon]$ it holds that*

$$\Big|W_h\Big(x, A_h(x',hx_3)+F\Big) - W\Big(\mathrm{I}_3+F\Big)\Big| \le \varepsilon|F|^2, \tag{3.13}$$

$$\Big|W_h\Big(x, A_h(x',hx_3)+F\Big)\Big| \le C_\varepsilon|F|^2. \tag{3.14}$$

Proof. Fix $\varepsilon>0$ and choose $h_\varepsilon>0$ such that for a.e. $x\in\Omega$, every $h\in[0,h_\varepsilon]$ and every $F\in B_r(0)$ we have that $A_h(x',hx_3)+F\in B_{2r}(\mathrm{I}_3)$. Define $H_h: B_r(0)\to[0,+\infty)$ by $H_h(F):=W_h\big(x,A_h(x',hx_3)+F\big)$ for every $h\in(0,h_\varepsilon]$ and $H_0(F):=W\big(\mathrm{I}_3+F\big)$, $F\in B_r(0)$. Fix $F\in B_r(0)$ and $h\in[0,h_\varepsilon]$. We have the following estimate:

$$\begin{aligned}\big|H_h(F)-H_0(F)\big| &\le \sup_{t\in[0,1]}\big|DH_h(tF)F-DH_0(tF)F\big|\\ &\le |F|\sup_{t\in[0,1]}\big\|DH_h(tF)-DH_0(tF)\big\|_{\mathscr{L}(\mathbb{R}^{3\times 3})}.\end{aligned}$$

Similarly,

$$\begin{aligned}\big\|DH_h(tF)-DH_0(tF)\big\|_{\mathscr{L}(\mathbb{R}^{3\times 3})} &\le \sup_{s\in[0,1]}\sup_{|M|\le 1}\big|D^2H_h(stF)[tF,M]-D^2H_0(stF)[tF,M]\big|\\ &\le \sup_{B_r(0)}\big\|D^2H_h-D^2H_0\big\|_{\mathscr{L}_2(\mathbb{R}^{3\times 3})}|t||F|.\end{aligned}$$

By putting together the above estimates we have (after possibly shrinking h_ε) that (3.13) holds. By using (2.9) and the estimate (3.13) we obtain

$$\Big|W_h\Big(x,A_h(x',hx_3)+F\Big)\Big| \le \Big|W\big(\mathrm{I}_3+F\big)\Big| + \varepsilon|F|^2 \le \Big|D^2W\big(\mathrm{I}_3\big)[F]^2\Big| + \rho(|F|)+\varepsilon|F|^2,$$

for a.e. $x \in \Omega$, every $F \in B_r(0)$ and every $h \in (0, h_\varepsilon]$. It is now clear, by regularity of W and (2.10) that the right hand side above divided by $|F|^2$ is bounded by a constant independent of F (after possibly shrinking r), proving (3.14). □

3.2 Rigorous derivation of the Kirchhoff-like plate model

Given a $\mathscr{B}$-admissible family $\{W_h\}_{h>0}$ of energy densities in the sense of Definition 3.1.1, for every $h > 0$ we recall from (2.18) that the rescaled free energy functional $\mathscr{E}_h : \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3) \to [0, +\infty]$ is given by

$$\mathscr{E}_h(y) = \int_\Omega W_h\big(x, \nabla_h y(x)\big)\,\mathrm{d}x, \qquad \text{for every } y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3). \tag{3.15}$$

The following compactness result says in particular that if the rescaled energy $\mathscr{E}_h/h^2$ is bounded on y^h, uniformly in h, then the sequence $\{y^h\}_h$ converges to a deformation y which belongs to the class of isometries $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$.

Theorem 3.2.1 (Compactness)**.** *Let $\{y^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ be a sequence which satisfies*

$$\limsup_{h\to 0} \frac{1}{h^2}\mathscr{E}_h(y^h) < +\infty. \tag{3.16}$$

Then $\{\nabla_h y^h\}_h$ is precompact in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$, that is: there exists a (not relabeled) subsequence such that $\nabla_h y^h \to (\nabla' y|\nu)$ in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$, where $\nu(x) := \partial_1 y(x) \wedge \partial_2 y(x)$. Moreover, the limit $(\nabla' y|\nu)$ has the following properties:

(i) $(\nabla' y|\nu)(x) \in \mathrm{SO}(3)$ *for a.e.* $x \in \Omega$,

(ii) $(\nabla' y|\nu) \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^{3\times 3})$ *and*

(iii) $(\nabla' y|\nu)$ *is independent of* x_3.

In other words, the limiting deformation y (identified up to additive constants) belongs to the class $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ defined as in (1.9).

To prove this compactness result, we can use the same argument as in the proof of the corresponding result in [FJM02] where the spontaneous stretch is I_3 in place of our $\mathrm{I}_3 + hB$. Note that the same argument holds in the case of spontaneous stretch of the form $\mathrm{I}_3 + h^\alpha B$ with $\alpha \geq 1$.

Proof of Theorem 3.2.1. We will show that the sequence $\{\nabla_h y^h\}_h \subseteq \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ satisfies

$$\limsup_{h\to 0} \frac{1}{h^2} \int_\Omega \mathrm{dist}^2\Big(\nabla_h y^h(x), \mathrm{SO}(3)\Big)\,\mathrm{d}x < +\infty. \tag{3.17}$$

The thesis then directly follows by applying Theorem 4.1 from [FJM02]. Fix $h > 0$ and $F \in \mathbb{R}^{3\times 3}$. For a.e. $x \in \Omega$ there exists $R_{h,F}(x) \in \mathrm{SO}(3)$ such that

$$\mathrm{dist}\Big(F, \mathrm{SO}(3)\big(\mathrm{I}_3 + hB_h(x)\big)\Big) = \big|F - R_{h,F}(x)(\mathrm{I}_3 + hB_h(x))\big|.$$

We have the following estimate:

$$\begin{aligned} \mathrm{dist}^2(F, \mathrm{SO}(3)) &\leq 2\Big|F - R_{h,F}(x)\big(\mathrm{I}_3 + hB_h(x)\big)\Big|^2 + 2\Big|hR_{h,F}(x)B_h(x)\Big|^2 \\ &\overset{(3.8)}{\leq} \frac{2}{C} W_h(x, F) + 6h^2\big|B_h(x)\big|^2 \end{aligned} \tag{3.18}$$

for a.e. $x \in \Omega$. By (3.16) and (3.18) we have that (3.17) holds true. □

Before stating the following convergence theorem, let us anticipate that our limiting 2D model will be described by the energy functional $\mathscr{E}_0 : \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3) \to [0,+\infty]$ defined as

$$\mathscr{E}_0(y) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\omega \overline{\mathcal{Q}}_2\big(x', \Pi_y(x')\big)\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.19}$$

where $\overline{\mathcal{Q}}_2$ is defined through (3.5) and (3.11) and Π_y is the pull-back of the second fundamental form of the surface $y(\omega)$ given by (1.5).

Theorem 3.2.2 (Γ-limit)**.** *The following convergence results hold true:*

(i) Γ- $\liminf$: *for every sequence $\{y^h\}_h$ and every y such that $y^h \rightharpoonup y$ weakly in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ it holds*

$$\mathscr{E}_0(y) \le \liminf_{h\to 0} \frac{1}{h^2}\mathscr{E}_h(y^h),$$

(ii) Γ- $\limsup$: *under the hypothesis*

$$\operatorname{curl}\big(\operatorname{curl}\mathcal{B}^0_{2\times 2}\big) = 0 \quad \textit{in the distributional sense,} \tag{3.20}$$

with $\mathcal{B}^0$ defined by (3.5) and (3.9), we have that for every $y \in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ there exists a sequence $\{y^h\}_h$ such that $y^h \to y$ in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$, fulfilling

$$\mathscr{E}_0(y) = \lim_{h\to 0} \frac{1}{h^2}\mathscr{E}_h(y^h).$$

The convergence results of the previous theorem amount to saying that the sequence of energy functionals $\frac{1}{h^2}\mathscr{E}_h$ Γ-converge to $\mathscr{E}_0$, as $h \to 0$, in the strong and weak topology of $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$. We postpone the proof of the theorem after the following example.

Example 3.2.3. Note that when $\mathcal{B}^0_{2\times 2}$ is constant, condition (3.20) is trivially satisfied. In particular, recalling definition (3.9), condition (3.20) is trivially satisfied whenever the map $x \mapsto B_{2\times 2}$ is constant in x'. At the same time, the same condition is satisfied with $\mathcal{B}^0_{2\times 2} \equiv 0$ by every map $x \mapsto B_{2\times 2}(x)$ which is nothing but odd in x_3. We also note that it is possible to realize $\mathcal{B}^0_{2\times 2} \equiv C \neq 0$ through a map $x \mapsto B_{2\times 2}(x)$ which is not constant in x'. To construct such an example, we fix $\overline{B}_0 \in \mathrm{Sym}(2) \setminus \{0\}$ and define $B \in \mathrm{L}^\infty\big(\Omega, \mathrm{Sym}(3)\big)$ by setting

$$B_{2\times 2}(x) := \sum_{i=1}^n \overline{B}_i(x_3)\chi_{\omega_i}(x'), \qquad \text{for a.e. } x \in \Omega,$$

where $\{\omega_i\}_{i=1}^n$ is a partition of ω and $\{\overline{B}_i\}_{i=1}^n \subseteq \mathrm{L}^\infty\big((-1/2,1/2),\mathrm{Sym}(2)\big)$ is a family of tensor valued maps satisfying

$$\int_{-1/2}^{1/2} \overline{B}_i(x_3)\,\mathrm{d}x_3 = \overline{B}_0, \qquad \text{for every } i = 1,\dots,n,$$

while all the remaining entries of the matrix $B(x)$ are set to be zero for a.e. $x \in \Omega$. This gives rise to $B_{2\times 2}$ which is piecewise constant in x' (but, in general, not constant in the same variable), and in turn to

$$\mathcal{B}^0_{2\times 2}(x') = \sum_{i=1}^n \chi_{\omega_i}(x') \int_{-1/2}^{1/2} \overline{B}_i(x_3)\,\mathrm{d}x_3 = \overline{B}_0.$$

Note also that the above defined map $B_{2\times 2}$ can give rise to a non-constant tensor valued map $x' \mapsto \mathcal{B}^1_{2\times 2}(x')$, with $\mathcal{B}^1$ given by (3.9), which is interpreted in Section 3.3 (in each point x') as the target curvature tensor which appears in the 2D limiting model. Indeed, in the case of $n = 2$, by choosing $\overline{B}_1(x_3) := (x_3 + 1)\mathrm{I}_2$ and $\overline{B}_2(x_3) := (x_3^3 + 1)\mathrm{I}_2$ for all $x_3 \in (-1/2, 1/2)$, we obtain a simple example of $B_{2\times 2}$ for which $\mathcal{B}^0_{2\times 2}$ is constant, while the tensor-valued map $x' \mapsto \mathcal{B}^1_{2\times 2}(x')$ is piecewise constant. △

The proof of the Γ- lim inf is a straightforward adaptation to the case of a family of energy densities $\{W_h\}_{h>0}$ with wells $\mathrm{SO}(3)\big(\mathrm{I}_3 + hB_h\big)$, of the corresponding result in [FJM02] pertaining the case of a homogeneous W (minimized at SO(3)). For the construction of the recovery sequence in the proof of the Γ- lim sup one has instead to add an additional term with respect to the classical construction (see the third summand on the right-hand side of (3.24)). Such additional term gives rise, in the limit as $h \to 0$, to a symmetrized gradient (see formula (3.27)), in a position where the map $\mathcal{B}^0_{2\times 2}$ should appear in order to match the Γ-limit (cfr. (3.10) and (3.19)). For this purpose, condition (3.20) guarantees that the map $\mathcal{B}^0_{2\times 2}$ is a symmetrized gradient, thanks to Theorem 1.3.1. Throughout the following proof $\overline{C}$ is a generic positive constant, varying form line to line and independent of all other quantities.

Proof of Theorem 3.2.2. (i) Γ- lim inf: Let $y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ and $\{y^h\}_h$ be such that $y^h \rightharpoonup y$ weakly in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$. Assume that $\liminf_{h\to 0} \mathscr{E}_h(y^h)/h^2 < +\infty$, otherwise the proof is trivial. Then, as shown in [FJM02] and up to a (not relabeled) subsequence, there exists a family of piecewise constant maps $R_h : \mathsf{Q}_h \to \mathrm{SO}(3)$ such that

$$\int_{\mathsf{Q}_h\times(-1/2,1/2)} \left|\nabla_h y^h(x) - R_h(x')\right|^2 \mathrm{d}x \le \overline{C}h^2, \tag{3.21}$$

and $R_h \to (\nabla y|\nu)$ in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ as $h \to 0$, where $\mathsf{Q}_h := \bigcup_{\mathsf{Q}_{a,3h}\subseteq\omega} \mathsf{Q}_{a,h}$ and $\mathsf{Q}_{a,h} := a + (-h/2, h/2)^2$ for every $h > 0$ and $a \in h\mathbb{Z}^2$. Moreover, the sequence $S_h : \Omega \to \mathbb{R}^{3\times 3}$ defined by

$$S_h(x', x_3) := \begin{cases} \dfrac{R_h^{\mathsf{T}}(x')\nabla_h y^h(x', x_3) - \mathrm{I}_3}{h} & \text{for } x \in \mathsf{Q}_h \times (-1/2, 1/2), \\ 0 & \text{elswhere in } \Omega, \end{cases} \tag{3.22}$$

converges weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$, as $h \to 0$, to some $S \in \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ such that

$$S_{2\times 2}(x) = S_{2\times 2}(x', 0) + x_3\Pi_y(x'), \qquad \text{for a.e. } x \in \Omega. \tag{3.23}$$

Letting χ_h be the characteristic function of the set $\mathsf{Q}_h \cap \big\{|S_h(x)| \le 1/\sqrt{h}\big\}$ we also have that $\chi_h S_h \rightharpoonup S$ in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ as $h \to 0$. Now, by using also the convergence in (3.5) we have

$$S_h - B_h \rightharpoonup S - B \text{ in } \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3}) \quad \text{and} \quad \|h(S_h - B_h)\|_{\mathrm{L}^\infty\left(\mathsf{Q}_h\cap\{|S_h(x)|\le 1/\sqrt{h}\}\right)} \to 0.$$

By using frame indifference of W_h, (2.9) and the estimate (3.13) from Lemma 3.1.3, we have that for a fixed $\varepsilon > 0$, there exists $\bar{h} > 0$ so that for all $h \in (0, \bar{h})$ the following estimates hold:

$$
\begin{aligned}
&\frac{1}{h^2}\int_\Omega W_h(x,\nabla_h y^h(x))\,\mathrm{d}x\\
\geq&\frac{1}{h^2}\int_\Omega \chi_h W^h(x,R_h^{\mathsf{T}}(x')\nabla_h y^h(x))\,\mathrm{d}x\\
=&\frac{1}{h^2}\int_\Omega \chi_h W^h\Big(x,\big(\mathrm{I}_3+hB_h(x)\big)+h(S_h-B_h)(x)\Big)\,\mathrm{d}x\\
\geq&\frac{1}{h^2}\int_\Omega \chi_h\frac{1}{2}D^2W(\mathrm{I}_3)\big[h(S_h-B_h)(x)\big]^2-\chi_h\varepsilon|h(S_h-B_h)(x)|^2+\chi_h\rho^0\big(h(S_h-B_h)(x)\big)\,\mathrm{d}x\\
\geq&\int_\Omega \chi_h\frac{1}{2}\mathcal{Q}_3\big((S_h-B_h)(x)\big)-\chi_h\varepsilon|(S_h-B_h)(x)|^2-\chi_h\rho\big(\big|h(S_h-B_h)(x)\big|\big)\,\mathrm{d}x,
\end{aligned}
$$

where ρ^0 and ρ are defined in (2.9). Since $\mathcal{Q}_3$ is lower semicontinuous in the weak topology of $\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})$ and since (2.10) holds, passing to lim inf as $h\to0$ in the above inequality we obtain

$$
\liminf_{h\to0}\frac{1}{h^2}\int_\Omega W_h(x,\nabla_h y^h(x))\,\mathrm{d}x\geq\int_\Omega\frac{1}{2}\mathcal{Q}_3\big(S(x)-B(x)\big)\,\mathrm{d}x-\overline{C}\varepsilon,
$$

where $\overline{C}>0$ is such that $||S_h-B_h||_{\mathrm{L}^2(\Omega)}\leq\overline{C}$. Finally, by letting $\varepsilon\to0$ and by using the fact that $\mathcal{Q}_3(F)\geq\mathcal{Q}_2(F_{2\times2})$ for every $F\in\mathbb{R}^{3\times3}$ we get that

$$
\begin{aligned}
\liminf_{h\to0}\frac{1}{h^2}\int_\Omega W_h(x,\nabla_h y^h(x))\,\mathrm{d}x\geq&\frac{1}{2}\int_\Omega\mathcal{Q}_2\big(S_{2\times2}(x',0)+x_3\Pi_y(x')-B_{2\times2}(x',x_3)\big)\,\mathrm{d}x\\
\geq&\frac{1}{2}\int_\omega\overline{\mathcal{Q}}_2\big(x',\Pi_y(x')\big)\,\mathrm{d}x',
\end{aligned}
$$

which proves Γ- lim inf inequality.

(ii) Γ- lim sup: Let us prove Γ- lim sup inequality for a given $y\in\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)\cap C^\infty(\overline{\omega},\mathbb{R}^3)$. Once we have proved it, Γ-lim sup inequality will follow for any $y\in\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ by the density result of Theorem 1.4.7 and the continuity of the limiting functional $\mathscr{E}_0$ with respect to $\mathrm{W}^{2,2}$ convergence. Suppose that $\mathscr{E}_0(y)<+\infty$ (otherwise the proof is trivial). Let $d\in C^\infty_c(\Omega,\mathbb{R}^3)$ and define $\tilde{d}:\Omega\to\mathbb{R}^3$ by

$$
\tilde{d}(x',x_3):=\int_0^{x_3}d(x',t)\,\mathrm{d}t,\qquad\text{for every }(x',x_3)\in\omega\times(-1/2,1/2)=\Omega.
$$

Let $\tilde{w}\in C^\infty_c(\mathbb{R}^2,\mathbb{R}^2)$. We consider the family of functions y^h of the form

$$
y^h(x):=y(x')+h\big[x_3\nu(x')+\nabla y(x')\tilde{w}(x')\big]+h^2\tilde{d}(x',x_3),\tag{3.24}
$$

for every $x\in\Omega$ and every $h>0$, whose (h-rescaled) gradient $\nabla_h y^h$ reads as

$$
\nabla_h y^h(x)=(\nabla y(x')|\nu(x'))+h\big(\nabla'\big[x_3\nu(x')+\nabla y(x')\tilde{w}(x')\big]\big|d(x)\big)+h^2\big(\nabla'\tilde{d}(x)|0\big),
$$

for every $x\in\Omega$ and every $h>0$. One can easily verify that $\{y^h\}_h\subseteq\mathrm{W}^{2,\infty}(\Omega,\mathbb{R}^3)$ and that it converges in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ to y, as $h\to0$. Denote by $R(x'):=\big(\nabla y(x')\big|\nu(x')\big)$ for every $x'\in\omega$. Set

$$
P_h(x):=R^{\mathsf{T}}(x')\Big(\big(\nabla'\big[x_3\nu(x')+\nabla y(x')\tilde{w}(x')\big]\big|d(x)\big)+h\big(\nabla'\tilde{d}(x)|0\big)\Big)-B_h(x),\quad\text{for a.e. }x\in\Omega,
$$

and note that P_h converges in $\mathrm{L}^\infty(\Omega,\mathbb{R}^{3\times3})$ to the function

$$
\Omega\ni x\mapsto R^{\mathsf{T}}(x')\big(\nabla'\big[x_3\nu(x')+\nabla y(x')\tilde{w}(x')\big]\big|d(x)\big)-B(x)\in\mathbb{R}^{3\times3}.
$$

With this notation, we have that $R^{\mathsf{T}}(x')\nabla_h y^h(x) = A_h(x', hx_3) + hP_h(x)$ for a.e. $x \in \Omega$ with A_h given by (3.6). By the frame indifference of $W_h(x,\cdot)$, boundedness of P_h and B_h in L^∞-norm and the estimate (3.14) from Lemma 3.1.3, there exists $\overline{C}, \bar{h} > 0$ such that

$$\frac{1}{h^2}W_h\Big(x, \nabla_h y^h(x)\Big) = \frac{1}{h^2}W_h\Big(x, R^{\mathsf{T}}(x')\nabla_h y^h(x)\Big) = \frac{1}{h^2}W_h\Big(x, A_h(x', hx_3) + hP_h(x)\Big) \le \overline{C},$$

for a.e. $x \in \Omega$ and every $0 < h \le \bar{h}$. Moreover, by using the estimate (3.13) from Lemma 3.1.3 and the regularity of W, we get that

$$\frac{1}{h^2}W_h\Big(x, \nabla_h y^h(x)\Big) \to \frac{1}{2}\mathcal{Q}_3\Big(R^{\mathsf{T}}(x')\big(\nabla'\big[x_3\nu(x') + \nabla y(x')\tilde{w}(x')\big]\big|d(x)\big) - B(x)\Big)$$

pointwise almost everywhere in Ω, as $h \to 0$. Then, by dominated convergence theorem we have

$$\frac{1}{h^2}\int_\Omega W_h\Big(x, \nabla_h y^h(x)\Big)\,\mathrm{d}x \to \frac{1}{2}\int_\Omega \mathcal{Q}_3\Big(R^{\mathsf{T}}(x')\big(\nabla'\big[x_3\nu(x') + \nabla y(x')\tilde{w}(x')\big]\big|d(x)\big) - B(x)\Big)\,\mathrm{d}x$$

as $h \to 0$. To proceed, for a.e. $x \in \Omega$ we denote by $\overline{F}(x)$ the 2×2 part of the argument of $\mathcal{Q}_3$ in the above integral. Define $\bar{d} : \Omega \to \mathbb{R}^3$ as

$$\bar{d}(x) := R(x')\left(\ell\big(\overline{F}_{\mathrm{sym}}(x)\big) + \begin{pmatrix} 2B_{13} \\ 2B_{23} \\ B_{33}\end{pmatrix}(x) - \begin{pmatrix} \Big(\nabla\big(\nabla y(x')\tilde{w}(x')\big)\Big)^{\mathsf{T}}\nu(x') \\ 0 \end{pmatrix}\right),$$

for a.e. $x \in \Omega$, where $\ell(\overline{F}_{\mathrm{sym}})$ is given by (2.8) and $B = (B_{ij})_{i,j=1}^3$. Given that $\overline{F}_{\mathrm{sym}} \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(2))$, $B \in \mathrm{L}^\infty(\Omega, \mathrm{Sym}(3))$ and y and $\tilde{w}$ are smooth vector fields, $\bar{d}$ belongs to $\mathrm{L}^2(\Omega, \mathbb{R}^3)$. By choosing d to be equal to $\bar{d}$, one can readily check that

$$\begin{aligned}(3.25)\quad &\Big(R^{\mathsf{T}}(x')\big(\nabla'\big[x_3\nu(x') + \nabla y(x')\tilde{w}(x')\big]\big|\bar{d}(x)\big) - B(x)\Big)_{\mathrm{sym}} \\ &= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\overline{F}_{\mathrm{sym}}(x)} & \begin{matrix}0\\0\end{matrix} \\ \hline 0 & 0 & 0\end{array}\right) + \Big(\ell\big(\overline{F}_{\mathrm{sym}}(x)\big)\otimes \mathsf{f}_3\Big)_{\mathrm{sym}}.\end{aligned}$$

Recall from Remark 1.4.1 that $(\nabla y)^{\mathsf{T}}\nabla\big((\nabla y)\tilde{w}\big) = \nabla\tilde{w}$. Now, by direct computation we obtain

$$(3.26)\qquad \overline{F}_{\mathrm{sym}}(x) = x_3\Pi_y(x') + \nabla_{\mathrm{sym}}\tilde{w}(x') - B_{2\times 2}(x), \qquad \text{for a.e. } x \in \Omega.$$

Finally, by definition of $\mathcal{Q}_2$, (3.25) and (3.26) it holds that

$$\begin{aligned}&\int_\Omega \mathcal{Q}_3\Big(R^{\mathsf{T}}(x')\big(\nabla'\big[x_3\nu(x') + \nabla y(x')\tilde{w}(x')\big]\big|\bar{d}(x)\big) - B(x)\Big)\,\mathrm{d}x \\ &= \int_\Omega \mathcal{Q}_2\big(x_3\Pi_y(x') + \nabla_{\mathrm{sym}}\tilde{w}(x') - B_{2\times 2}(x)\big)\,\mathrm{d}x.\end{aligned}$$

Therefore, the density of $C_c^\infty(\Omega, \mathbb{R}^3)$ in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ and a diagonal argument give us that

$$(3.27)\qquad \limsup_{h\to 0}\frac{1}{h^2}\int_\Omega W_h(x, \nabla_h y^h(x))\,\mathrm{d}x = \frac{1}{2}\int_\omega\int_{-1/2}^{1/2}\mathcal{Q}_2\big(x_3\Pi_y(x') + \nabla_{\mathrm{sym}}\tilde{w}(x') - B_{2\times 2}(x)\big)\,\mathrm{d}x_3\,\mathrm{d}x'.$$

The compatibility assumption (3.20) on $\mathcal{B}^0_{2\times 2}$ and Theorem 1.3.1 guarantee the existence of the map $w \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^2)$ such that $\mathcal{B}^0_{2\times 2}(x') = \nabla_{\mathrm{sym}}w(x')$ for a.e. $x' \in \omega$. Thus, by using the density of $C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ (when restricted to ω) in $\mathrm{W}^{1,2}(\omega, \mathbb{R}^2)$ and a diagonal argument one more time, we prove Γ-lim sup inequality for a given $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \cap C^\infty(\overline{\omega}, \mathbb{R}^3)$. ☐

Remark 3.2.4. By standard arguments of Γ-convergence it can be shown that the above analysis holds also in the case when the appropriate body forces are present. More precisely, the above results can be applied to the sequence of functionals $\{\mathscr{F}_h\}_{h>0}$ defined by

$$\mathscr{F}_h(y) := \mathscr{E}_h(y) - \int_\Omega f_h(x) \cdot y(x)\,\mathrm{d}x, \qquad \text{for every } y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3),$$

where $\{f_h\}_{h\geq 0} \subseteq \mathrm{L}^2(\Omega, \mathbb{R}^3)$ is the family of body forces such that

$$\frac{f_h}{h^2} \rightharpoonup f_0 \quad \text{weakly in } \mathrm{L}^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad \int_\Omega f_h(x)\,\mathrm{d}x = 0 \text{ for every } h \geq 0.$$

The sequence $\{\mathscr{F}_h\}$ Γ−converges, as $h \to 0$, to

$$\mathscr{F}_0(y) := \begin{cases} \mathscr{E}_0(y) - \displaystyle\int_\omega f(x') \cdot y(x')\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

where $f(x') := \int_{-1/2}^{1/2} f_0(x', t)\,\mathrm{d}t$ for a.e. $x' \in \omega$. ■

3.3 2D energy minimizers

In this section, we discuss the minimizers of the derived 2D model in some special cases. Recall that the 2D limiting energy functional $\mathscr{E}_0$ is given by

$$\mathscr{E}_0(y) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\omega \overline{\mathcal{Q}}_2\big(x', \Pi_y(x')\big)\,\mathrm{d}x', & \text{for } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega), \\ +\infty, & \text{otherwise,} \end{cases}$$

where $\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ is the set of $\mathrm{W}^{2,2}$-isometric immersions of ω into $\mathbb{R}^3$, defined by (1.9). From formula (3.12), we have that

$$\mathscr{E}_0(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\Pi_y(x') - \mathcal{B}^1_{2\times 2}(x')\Big)\,\mathrm{d}x' + \text{ad.t.} \tag{3.28}$$

for every $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$, where ad.t. stays for “additional term” (not depending on y) and is given by

$$\text{ad.t.} := \frac{1}{2}\int_\omega \int_{-1/2}^{1/2} \mathcal{Q}_2\big(B_{2\times 2}(x', t) - \mathcal{B}^0_{2\times 2}(x')\big)\,\mathrm{d}t - \frac{1}{12}\mathcal{Q}_2\big(\mathcal{B}^1_{2\times 2}(x')\big)\mathrm{d}x'. \tag{3.29}$$

Observe that this term is irrelevant in the present discussion on the energy minimizers and will be discussed more in details in Chapter 5. We shall see that this term is non negative and vanishes if and only if the map $B_{2\times 2}$ is affine in x_3. More precisely, we have (up to a multiplicative constant) that

$$\text{ad.t.} = \operatorname{dist}^2(B_{2\times 2}, \mathbb{E}_1),$$

with $\mathbb{E}_1$ given in (5.12) – (5.13) and $\operatorname{dist}^2(B_{2\times 2}, \mathbb{E}_1)$ determined in Lemma 5.2.2 with $\mathcal{Q}_2(x', \cdot)$ being independent on x'.

Recall that Π_y is the pull-back of the second fundamental form associated with $y(\omega)$ (see (1.5)), hence it gives information on the curvature *realized* by the deformation y. On the other hand, when reading the expression for $\mathscr{E}_0$, it is natural to think of $\mathcal{B}^1_{2\times 2}$ as the *target* *curvature* tensor, which encodes the spontaneous curvature of the system. For the later use, we recall that

$$\mathcal{B}^1_{2\times2}(x') \;=\; 12\int_{-1/2}^{1/2} tB_{2\times2}(x',t)\,\mathrm{d}t, \qquad \text{for a.e. } x'\in\omega. \tag{3.30}$$

While, for a.e. x', the tensor $\mathcal{B}^1_{2\times2}(x')$ is a given $2{\times}2$ symmetric matrix with possibly nonzero determinat, we know by Lemma 1.4.3 that $\det \Pi_y = 0$ a.e. in ω.

Our aim is to determine explicitly some classes of minimizers. More precisely, recalling the definition of the set $\mathcal{S}_0$ from Section 1.4.2, namely

$$\mathcal{S}_0 \;=\; \big\{S\in \mathrm{Sym}(2) : \det S = 0\big\} \tag{3.31}$$

and having in mind the inequality

$$\min_{\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)} \mathscr{E}_0 \;\geq\; \frac{1}{24}\int_\omega \min_{S\in\mathcal{S}_0} \mathcal{Q}_2\big(S-\mathcal{B}^1_{2\times2}(x')\big)\,\mathrm{d}x' + \text{ad.t.},$$

we will focus our attention on *pointwise minimizers* of $\mathscr{E}_0$. Namely, on those $y\in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ such that

$$\mathscr{E}_0(y) \;=\; \frac{1}{24}\int_\omega \min_{S\in\mathcal{S}_0} \mathcal{Q}_2\big(S-\mathcal{B}^1_{2\times2}(x')\big)\,\mathrm{d}x' + \text{ad.t.} \;=\; \min_{\mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)} \mathscr{E}_0. \tag{3.32}$$

To go on, let us consider the set

$$\mathcal{N}(x') := \operatorname*{argmin}_{S\in\mathcal{S}_0} \mathcal{Q}_2\big(S-\mathcal{B}^1_{2\times2}(x')\big), \tag{3.33}$$

for a.e. $x'\in\omega$. Note that $\mathcal{N}(x')\neq\emptyset$ for a.e. $x'\in\omega$, because $\mathcal{Q}_2$ is a positive definite quadratic form (when restricted to $\mathrm{Sym}(2)$) and $\mathcal{S}_0$ is a closed subset of $\mathrm{Sym}(2)$. To accomplish our program, we would like to have some explicit representation of the elements of $\mathcal{N}(x')$, for a.e. $x'\in\omega$, also in view of the application which motivates our analysis (see Chapter 4.). Therefore, we restrict our attention to case of W *isotropic* (see Appendix 2.A), so that

$$\mathcal{Q}_2(G) \;=\; \min_{\mathsf{c}\in\mathbb{R}^3} \mathcal{Q}_3\big(G^* + \mathsf{c}\otimes \mathsf{f}_3\big) \;=\; 2\mathbf{G}\left(|G_{\mathrm{sym}}|^2 + \boldsymbol{\Lambda}\,\mathrm{tr}^2 G\right), \qquad \text{for every } G\in\mathbb{R}^{2\times2}, \tag{3.34}$$

referring to (2.28) and the subsequent discussion on the properties of $\mathbf{G}$ and $\boldsymbol{\Lambda}$. In particular, $\boldsymbol{\Lambda} > -1/2$ and this fact guarantees that the quantities appearing in the statement of Lemma 3.3.1 below are well defined.

Note that in the case when $\mathcal{B}^1_{2\times2}$ is constant in ω, pointwise minimizers of $\mathscr{E}_0$ always exist. More precisely, as noticed in [Sch07a] and [Sch07b] (see Lemma 3.3.2 below), any minimizer y of $\mathscr{E}_0$ with $\mathcal{B}^1_{2\times2}$ constant is characterised by the property $\Pi_y(x') \equiv \text{const.} \in \mathcal{N}$ for a.e. $x'\in\omega$, where

$$\mathcal{N} \;:=\; \operatorname*{argmin}_{S\in\mathcal{S}_0} \mathcal{Q}_2\big(S-\mathcal{B}^1_{2\times2}\big). \tag{3.35}$$

Clearly, in the case of nonconstant $\mathcal{B}^1_{2\times2}$, this is not always true. Now, while the analysis of the minimizers of $\mathscr{E}_0$, with an arbitrary nonconstant $\mathcal{B}^1_{2\times2}$, is behind the scope of the present paper, it is natural in our context to try to understand under which conditions the existence of *pointwise minimizers* of $\mathscr{E}_0$ is guaranteed. In Subsection 3.3.1 we answer this question in the case when $\mathcal{B}^1_{2\times2}$ is piecewise constant. To do this, we need a structure result for the set $\mathcal{N}$ in the case of constant $\mathcal{B}^1_{2\times2}$. This is the content of the following lemma.

Lemma 3.3.1. *Let a and b be two real numbers and let $\mathbf{\Lambda}$ be given by (3.34). The following implications hold:*

(*i*) *If* $\mathcal{B}^1_{2\times 2} \equiv \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ *then* $\mathcal{N} = \left\{ \varrho^\mathsf{T} \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \varrho \,:\, \varrho \in \mathrm{SO}(2) \right\}$ *with* $\mathfrak{r} = a\frac{1+2\mathbf{\Lambda}}{1+\mathbf{\Lambda}}$.

(*ii*) *If* $\mathcal{B}^1_{2\times 2} \equiv \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -\mathfrak{r} \end{pmatrix} \right\}$ *with* $\mathfrak{r} = \frac{a}{1+\mathbf{\Lambda}}$.

(*iii*) *If* $\mathcal{B}^1_{2\times 2} \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $|a| > |b|$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \right\}$ *with* $\mathfrak{r} = a + \frac{b\mathbf{\Lambda}}{1+\mathbf{\Lambda}}$.

(*iv*) *If* $\mathcal{B}^1_{2\times 2} \equiv \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $|b| > |a|$ *then* $\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{r} \end{pmatrix} \right\}$ *with* $\mathfrak{r} = b + \frac{a\mathbf{\Lambda}}{1+\mathbf{\Lambda}}$.

Before giving the proof of the above statement, let us make a couple of comments. First, note that the lemma, though restricted to the case of $\mathcal{B}^1_{2\times 2}$ diagonal, covers all the interesting cases, from the simple observation that, with abuse of notation, $\mathcal{N}_{\mathcal{B}^1_{2\times 2}} = \bar{\varrho}\,\mathcal{N}_{\overline{D}}\,\bar{\varrho}^\mathsf{T}$, where $\bar{\rho} \in \mathrm{O}(2)$ is such that $\bar{\varrho}^\mathsf{T}\,\mathcal{B}^1_{2\times 2}\,\bar{\varrho}$ coincides with the diagonal matrix $\overline{D}$. Moreover, interpreting the elements of $\mathcal{N}$ as second fundamental forms of *cylinders* (see the discussion below), the parameter $\mathfrak{r}$, when different from zero, corresponds to the nonzero principal curvature. In this case, observe also that, with abuse of notation, the set $\mathcal{N}_{(ii)}$ is never a subset of $\mathcal{N}_{(i)}$ and that, as for the (two) elements of $\mathcal{N}_{(ii)}$, the elements of $\mathcal{N}_{(i)}$ are pairwise linearly independent. This can be easily read off from the simple fact that

$$\mathcal{N}_{(i)} = \mathfrak{r}\left\{ \mathsf{n}\otimes\mathsf{n} :\, \mathsf{n} \in \mathbb{R}^2 \text{ with } |\mathsf{n}| = 1 \right\}.$$

Finally, the set of the directions corresponding to $\pm\mathfrak{r}$ in the cases (*i*), (*ii*), (*iii*), and (*iv*) is given by $\{\varrho\,\mathsf{e}_1 : \varrho \in \mathrm{SO}(2)\}, \{\mathsf{e}_1, \mathsf{e}_2\}, \{\mathsf{e}_1\}, \{\mathsf{e}_2\}$, respectively. This fact can be interpreted saying that, in order to reduce the energy, while in case (*i*) rolling up along all the possible directions is equally favorable, in the remaining cases the system rolls up along the direction corresponding to the greater (in modulus) eigenvalue of the target curvature tensor $\mathcal{B}^1_{2\times 2}$.

Proof of Lemma 3.3.1. Let $a, b \in \mathbb{R}$ and let $\mathcal{B}^1_{2\times 2} \equiv \mathrm{diag}(a, b)$. By representing any $S \in \mathrm{Sym}(2)$ by $\left(\begin{smallmatrix} \xi & \zeta \\ \zeta & \upsilon \end{smallmatrix}\right)$, $\zeta, \xi, \upsilon \in \mathbb{R}$ and recalling that $\mathcal{Q}_2$ is of the form (3.34), the minimization problem to be solved is:

$$\min_{S\in\mathcal{S}_0} \left\{ \left|S - \mathcal{B}^1_{2\times 2}\right|^2 + \mathbf{\Lambda}\,\mathrm{tr}^2\big(S - \mathcal{B}^1_{2\times 2}\big) \right\} = \min_{\substack{(\xi,\upsilon)\in\mathbb{R}^2,\,\zeta\in\mathbb{R} \\ \xi\upsilon = \zeta^2}} \left\{ \left| \begin{pmatrix} \xi - a & \zeta \\ \zeta & \upsilon - b \end{pmatrix} \right|^2 + \mathbf{\Lambda}\,\mathrm{tr}^2 \begin{pmatrix} \xi - a & \zeta \\ \zeta & \upsilon - b \end{pmatrix} \right\} \tag{3.36}$$

Denote $P := \left\{ (\xi, \upsilon) \in \mathbb{R}^2 \,\middle|\, \xi\upsilon \geq 0 \right\}$ and define for every $(\xi, \upsilon) \in P$ the function

$$f(\xi, \upsilon) := (1 + \mathbf{\Lambda})(\xi + \upsilon)^2 - 2\big(a(1 + \mathbf{\Lambda}) + b\mathbf{\Lambda}\big)\xi - 2\big(b(1 + \mathbf{\Lambda}) + a\mathbf{\Lambda}\big)\upsilon + a^2 + b^2 + \mathbf{\Lambda}(a + b)^2,$$

so that the minimization problem becomes $\min_{(\xi,\upsilon)\in P} f(\xi, \upsilon)$. Let us consider first the case when $a \neq b$. Observe that in this case f attains its minimum on $\partial P = \left\{ (\xi, \upsilon) \in \mathbb{R}^2 \,|\, \xi\upsilon = 0 \right\}$.

Indeed, supposing by contradiction that the minimum is attained at a point $(\bar{\xi}, \bar{\upsilon}) \in \mathrm{int}(P) = \{(\xi, \upsilon) \in \mathbb{R}^2 : \xi\upsilon > 0\}$ would give

$$\partial_\xi f(\bar{\xi}, \bar{\upsilon}) = 2(1 + \mathbf{\Lambda})(\bar{\xi} + \bar{\upsilon}) - 2\big(a(1 + \mathbf{\Lambda}) + \beta b\big) = 0,$$
$$\partial_\upsilon f(\bar{\xi}, \bar{\upsilon}) = 2(1 + \mathbf{\Lambda})(\bar{\xi} + \bar{\upsilon}) - 2\big(b(1 + \mathbf{\Lambda}) + \mathbf{\Lambda} a\big) = 0,$$

and in turn $a = b$, leading to a contradiction. Now (ii), (iii) and (iv) follow by straightforward computations. To prove (i), we first note that the set of stationary points of f in $\mathrm{int}(P)$ is given by

$$\left\{\left(\eta_\zeta^\pm, \eta_\zeta^\mp\right) \in \mathbb{R}^2 : \zeta \in \left[-\frac{|\mathfrak{r}|}{2}, \frac{|\mathfrak{r}|}{2}\right] \setminus \{0\}\right\},$$

where

$$\mathfrak{r} = \frac{a(1 + 2\beta)}{(1 + \beta)} \quad \text{and} \quad \eta_\zeta^\pm := \frac{\mathfrak{r}}{2} \pm \frac{\sqrt{\mathfrak{r}^2 - 4\zeta^2}}{2}, \qquad \text{for every } \zeta \in \left[-\frac{|\mathfrak{r}|}{2}, \frac{|\mathfrak{r}|}{2}\right] \setminus \{0\}.$$

Moreover, the value of f at these stationary points is $f(\eta_\zeta^+, \eta_\zeta^-) = f(\eta_\zeta^-, \eta_\zeta^+) = a\mathfrak{r}$. At the same time, we have that $\operatorname*{argmin}_{(\xi,\upsilon)\in\partial P} f(\xi, \upsilon) = \{(\mathfrak{r}, 0), (0, \mathfrak{r})\}$, and that $f(\mathfrak{r}, 0) = f(0, \mathfrak{r}) = a\mathfrak{r}$. Hence,

$$\operatorname*{argmin}_{(\xi,\upsilon)\in P} f(\xi, \upsilon) = \left\{\left(\eta_\zeta^\pm, \eta_\zeta^\mp\right) \in \mathbb{R}^2 : \zeta \in \left[-\frac{|\mathfrak{r}|}{2}, \frac{|\mathfrak{r}|}{2}\right]\right\}.$$

In turn, the elements of $\mathcal{N}$ are all the matrices of the form

$$S_\zeta^\pm := \begin{pmatrix} \eta_\zeta^\pm & \zeta \\ \zeta & \eta_\zeta^\mp \end{pmatrix} \quad \text{with } |\zeta| \le \frac{|\mathfrak{r}|}{2}.$$

The proof of the lemma, point (i), can be finished by observing that the following identity holds

$$\left\{\varrho^\mathsf{T} \begin{pmatrix} \mathfrak{r} & 0 \\ 0 & 0 \end{pmatrix} \varrho \,:\, \varrho \in \mathrm{SO}(2)\right\} = \left\{S_\zeta^\pm \,:\, |\zeta| \le \frac{|\mathfrak{r}|}{2}\right\}.$$

□

Now, let us go back to the set $\mathcal{S}_0$ defined in (3.31). The fact that the set $\mathcal{S}_0$ coincides with the set of (constant) second fundamental forms of cylinders, proved in Lemma 3.31, can be used to show, in the case where the target curvature tensor $\mathcal{B}^1_{2\times 2}$ is constant, that

(3.37) $\qquad y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ is a minimizer of $\mathscr{E}_0$ if and only if y is a pointwise minimizer.

This is the first step of the proof of Lemma 3.3.2 below. The second part of the proof consists then in showing that

(3.38) $$\Pi_y(x') \in \mathcal{N} \quad \text{for a.e. } x' \in \omega \quad \Longrightarrow \quad \Pi_y \equiv \text{const.}.$$

This property is at the core of our investigations in the following subsection and can be proved using some fine properties of isometric immersions (see Section 1.4.1 and [Hor11a], [Hor11b] and [Pak04] for more details). The proof of the following lemma can be found in [Sch07b, Proposition 4.2]. For completeness, in Appendix 3.A we provide a detailed proof of this result.

Lemma 3.3.2. *Let $\mathcal{B}^1_{2\times 2}$ be constant (cfr. (3.28)–(3.30)) and let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ be a minimizer of $\mathscr{E}_0$. Then $y = u|_\omega$ for some $u \in \mathrm{Cyl}$. In particular, y has constant second fundamental form.*

3.3.1 The case of piecewise constant target curvature tensor

In this subsection, we consider the case where the target curvature is a piecewise constant tensor valued map $x' \mapsto \mathcal{B}^1_{2\times 2}(x')$. More precisely, given $n \in \mathbb{N}$, $n \geq 2$, we say that the map $\mathcal{B}^1_{2\times 2} \in \mathrm{L}^\infty\big(\omega, \mathrm{Sym}(2)\big)$ is *piecewise constant* if it is of the form

$$\mathcal{B}^1_{2\times 2} = \sum_{k=1}^{n} \overline{B}_k \, \chi_{\omega_k} \quad \text{a.e. in } \omega, \qquad \text{with } \overline{B}_k = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix}, \quad a_k, b_k \in \mathbb{R}, \tag{3.39}$$

where $\{\omega_k\}_{k=1}^n$ is a partition of ω made of Lipschitz subdomains ω_k according to Definition 3.3.3. Clearly, it is convenient distinguishing between two different neighboring subdomain only when the corresponding spontaneous curvature are different from each other. Namely, we suppose that $\overline{B}_k \neq \overline{B}_j$ for every $k \neq j$ such that $\partial\omega_j \cap \partial\omega_k \neq \emptyset$. With such target curvature, our 2D energy functional takes the form

$$\mathscr{E}_0(y) = \frac{1}{24} \sum_{k=1}^{n} \int_{\omega_k} \mathcal{Q}_2\big(\Pi_y(x') - \overline{B}_k\big) \, \mathrm{d}x' + \text{ad.t.}, \qquad \text{for every } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega).$$

We want to determine the conditions the map $x' \mapsto \mathcal{B}^1_{2\times 2}(x')$ has to satisfy in order to guarantee the existence of pointwise minimizers of $\mathscr{E}_0$, i.e. to guarantee that there exists $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ such that $\Pi_y(x') \in \mathcal{N}(x')$ for a.e. $x' \in \omega$, where $\mathcal{N}(x')$ is defined by (3.33). In view of (3.39), we equivalently look for the necessary and sufficient conditions such that

$$\text{exists } y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \text{ such that } \Pi_y(x') \in \mathcal{N}_k \text{ for a.e. } x' \in \omega_k, \text{ for all } k = 1, \dots, n, \tag{3.40}$$

where

$$\mathcal{N}_k := \mathrm{argmin}_{S \in \mathcal{S}_0} \, \mathcal{Q}_2\big(S - \overline{B}_k\big), \qquad \text{for every } k = 1, \dots, n. \tag{3.41}$$

Note from (3.38) that a deformation satisfying (3.40) is, roughly speaking, a "patchwork" of cylinders. Therefore, conditions on $\mathcal{B}^1_{2\times 2}$ guaranteeing (3.40) translates into conditions under which cylinders can be patched together resulting into an isometry. This is the content of the main result of the present section, namely of Theorem 3.3.5 below. In order to state and prove it, we need the following definition.

Definition 3.3.3 (Lipschitz n-subdivision)**.** *Fix $n \in \mathbb{N}$, $n \geq 2$. A family $\{\omega_k\}_{k=1}^n$ of open, bounded and connected subsets of $\mathbb{R}^2$ is said to be a* Lipschitz n-subdivision *of ω provided it can be obtained* via *the following procedure:*

- *Call $\omega'_1 := \omega$.*
- *Suppose that for every $k = 1, \dots, n-1$ there exists a continuous injective curve $\gamma_k : [0,1] \to \overline{\omega}'_k$ such that $\partial\omega'_k \cap [\gamma_k] = \big\{\gamma_k(0), \gamma_k(1)\big\}$ (note that $\gamma_k(0) \neq \gamma_k(1)$) and the two connected components of $\omega'_k \setminus [\gamma_k]$ are Lipschitz. Then call ω'_{k+1} one of such connected components.*
- *Once the domains $\omega'_1, \dots, \omega'_n$ are defined, let $\omega_k := \omega'_k \setminus \overline{\omega}'_{k+1}$ for every $k = 1, \dots, n-1$ and let $\omega_n := \omega'_n$.*

In particular, the subdomains $\omega_1, \dots, \omega_n$ of ω are Lipschitz domains such that

$$\omega = \bigcup_{k=1}^{n} \omega_k \cup \bigcup_{k=1}^{n-1} \gamma_k\big((0,1)\big).$$

Remark 3.3.4. Since each ω_k is a Lipschitz domain, one has that its boundary $\partial\omega_k$ has null $\mathcal{L}^2$-measure. In particular, we deduce that $\mathcal{L}^2\big(\omega \setminus \bigcup_{k=1}^n \omega_k\big) = 0$. ■

Given a piecewise constant $\mathcal{B}^1_{2\times 2}$ and referring to Lemma 3.3.1 (see also the discussion after its statement), we set

$$\mathfrak{r}_k := \begin{cases} \dfrac{a_k(1+2\mathbf{\Lambda})}{1+\mathbf{\Lambda}}, & \text{if} \quad b_k = a_k, \\ \dfrac{a_k}{1+\mathbf{\Lambda}}, & \text{if} \quad b_k = -a_k, \\ a_k + \dfrac{b_k\,\mathbf{\Lambda}}{1+\mathbf{\Lambda}}, & \text{if} \quad |a_k| > |b_k|, \\ b_k + \dfrac{a_k\mathbf{\Lambda}}{1+\mathbf{\Lambda}}, & \text{if} \quad |b_k| > |a_k|, \end{cases} \qquad \text{for every } k = 1, \dots, n. \tag{3.42}$$

Recall that $\{0, \pm\mathfrak{r}_k\}$ are the eigenvalues (principal curvatures) of the (constant) curvature tensors ranging in $\mathcal{N}_k$.

Theorem 3.3.5. *Let $\mathcal{B}^1_{2\times 2}$ be of the form (3.39). Assume that $\mathfrak{r}_k \neq \mathfrak{r}_j$ for all $1 \le k < j \le n$ such that $\mathcal{H}^1\big(\partial\omega_k \cap \partial\omega_j\big) > 0$. Then there exists a pointwise minimizer $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ of $\mathscr{E}^0$ if and only if the following conditions are satisfied:*

(a) $[\gamma_k]$ is a line segment with $\gamma_k(0), \gamma_k(1) \in \partial\omega$, for every $k = 1, \dots, n-1$;

(b) $\gamma_k\big((0,1)\big) \cap \gamma_j\big((0,1)\big) = \emptyset$ for all $k \neq j = 1, \dots, n-1$;

(c) every non flat *region ω_k, i.e. ω_k with corresponding $\mathfrak{r}_k \neq 0$, satisfies: $\partial\omega_k \cap \omega$ consists of connected components which are orthogonal to some eigenvector (principal curvature direction) of the matrices of $\mathcal{N}_k$ corresponding to $\mathfrak{r}_k$.*

Proof. The sufficiency part of the statement follows by straightforward computations, as in the proof of Lemma 1.4.10. In order to prove necessity, we focus on the case $n = 2$, when ω is subdivided into two Lipschitz subdomains ω_1 and ω_2 by a curve $\gamma := \gamma_1$ as in Definition 3.3.3, since the general case can be achieved by an induction argument as a consequence of our definition of Lipschitz subdivision of the domain ω.

Let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ be a pointwise minimizer of $\mathscr{E}_0$. Note that on both subdomains ω_1 and ω_2 the target curvature tensor $\mathcal{B}^1_{2\times 2}$ is constant. Then by the definition of pointwise minimizers, by Lemma 3.3.2 and Lemma 3.3.1 we deduce that $y = y_1\chi_{\omega_1} + y_2\chi_{\omega_2}$, with $y_k = T_{\mathsf{v}_k} \circ R_k \circ C_{1/|\mathfrak{r}_k|} \circ \varrho_k \in \mathrm{Cyl}$, $k = 1, 2$, where $\mathfrak{r}_k$ is given by (3.42) and ϱ_k is such that $\Pi_{y_k} \equiv (\det \varrho_k)\, \varrho_k^{\mathsf{T}} \operatorname{diag}\big(|\mathfrak{r}_k|, 0\big) \varrho_k \in \mathcal{N}_k$. Since $\mathfrak{r}_1 \neq \mathfrak{r}_2$, by Lemma 1.4.10 and Remark 1.4.11 we obtain that $[\gamma]$ must be a line segment and that $\varrho_k^{\mathsf{T}}\mathsf{e}_2$ must be parallel to $[\gamma]$ (or equivalently that the eigenvector $\varrho_k^{\mathsf{T}}\mathsf{e}_1$ of Π_{y_k} is orthogonal to $[\gamma]$) whenever $\mathfrak{r}_k \neq 0$, $k = 1, 2$, which is precisely the statement of (a) and (c) in the case in which $n = 2$. □

Remark 3.3.6. Let k and j be such that $\mathcal{H}^1(\partial\omega_k \cap \partial\omega_j) > 0$. Observe that when $\mathfrak{r}_k = \mathfrak{r}_j$ (this may happen, though $\overline{B}_k \neq \overline{B}_j$), this condition does not impose that $\partial\omega_k \cap \partial\omega_j$ is a line segment. Indeed, when $\mathfrak{r}_k = \mathfrak{r}_j$, a pointwise minimizer y, when restricted to ω_k and ω_j, will be given by some cylinders y_k and y_j with $r_k = 1/|\mathfrak{r}_k|$ and $r_j = 1/|\mathfrak{r}_j|$, respectively, which have the same curvatures $\det \varrho_k |\mathfrak{r}_k| = \mathfrak{r}_k = \mathfrak{r}_j = \det \varrho_j |\mathfrak{r}_j|$. This fact, as observed in Remark 1.4.11, does not impose any further conditions on $\partial\omega_k \cap \partial\omega_j$. ■

Note that, if the target curvature does not induce any flat region, the presence of a pointwise minimizer forces the subdivision lines $[\gamma_k]$ to be all parallel (see Figure 3.2, (a) and (b)). When instead a flat region is present in the subdivision, this can give rise to a pointwise minimizer, even if the $[\gamma_k]$ are not mutually parallel (see Figure 3.2, (c) and (d)). Finally, observe that in this case a subdomain of type (iii) and (iv) can coexist (tough they cannot be neighbors).

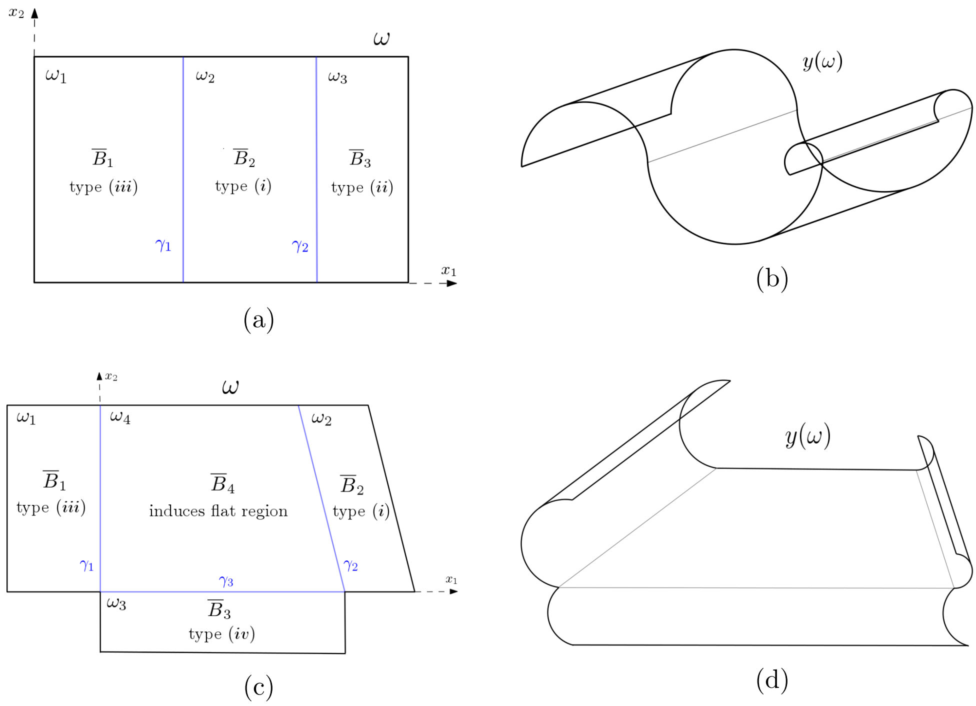

Figure 3.2: Examples of reference domains with given target curvature $\mathcal{B}^1_{2\times 2} = \sum_{k=1}^{3} \overline{B}_k \chi_{\omega_k}$, which guarantees the existence of a pointwise minimizer y in the case when there are no flat regions induced (figure (a)) and in the case when a flat regions are present (figure (c)). Corresponding examples of $y(\omega)$ are illustrated in pictures (b) and (d), respectively.

Point (c) above implies that for every k and j such that ω_k and ω_j are neighbor (i.e. share a piece of boundary, in symbols $\mathcal{H}^1(\partial\omega_k \cap \partial\omega_j) > 0$) it cannot be that $\overline{B}_k$ is of type (iii) (see Lemma 3.3.1) and $\overline{B}_j$ is of type (iv) at the same time. This is because, if not so, from point (c) above it would follow that the line segment $[\gamma] = \partial\omega_k \cap \partial\omega_j$ is simultaneously parallel to e_2 and to e_1, which is absurd. Hence, a reference domain endowed with target curvature as in Figure 3.3 does not admit a pointwise minimizer.

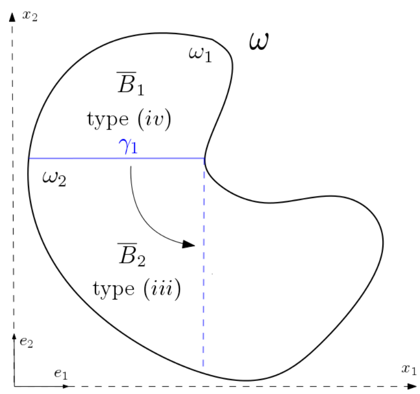

Figure 3.3: An example of reference domain with given target curvature $\mathcal{B}^1_{2\times 2} = \overline{B}_1\chi_{\omega_1} + \overline{B}_2\chi_{\omega_2}$ which does not allow for a pointwise minimizer y. This is because $\overline{B}_1$ of type (iv) forces $[\gamma_1]$ to be parallel to e_1, while $\overline{B}_2$ of type (iii) forces $[\gamma_1]$ to be parallel to e_2.

3.4 A discussion on the "curl curl" condition

In this section we discuss the case in which the spontaneous strain $\mathcal{B}^0_{2\times 2}$ does not satisfy the condition (3.20) (that we will refer to as the "curl curl" condition), i.e. it is not a symmetrized gradient of some map. We present two different conjectures: the first one favorizes the existence of an additional term in the corresponding 2D model, while the second one leads to the "old" limiting 2D model derived in Theorem 3.2.2.

Idea 1: An additional "stretching" term. Let us suppose here, for simplicity, that the energy densities $\overline{W}_h$ can be written in the pre-stretch form, namely

$$\overline{W}_h(z, F) = W\big(FA_h^{-1}(z)\big), \qquad \text{for a.e. } z \in \Omega_h,$$

with the spontaneous stretch A_h given by

$$A_h(z) = \mathrm{I}_3 + hB\left(z', \frac{z_3}{h}\right), \qquad \text{with } B \in \mathrm{L}^\infty\big(\Omega, \mathrm{Sym}(3)\big).$$

Observe also that $A_h^{-1}(x', hx_3) = \mathrm{I}_3 - hB(x) + o(h)$ for a.e. $x \in \Omega$. In order to study the limiting behaviour of the bending energy functionals

$$\frac{1}{h^3}\overline{\mathscr{E}}_h(v^h) = \frac{1}{h^3}\int_{\Omega_h} W\big(\nabla v^h(z)A_h^{-1}(z)\big)\,\mathrm{d}z, \qquad v^h : \Omega_h \to \mathbb{R}^3$$

and in order to give an ansatz for the limiting model, we shall suppose that v^h is sufficiently smooth to justify the following passages: first, Taylor's expansion at each fixed z' (and h) with respect to z_3:

$$v^h(z', z_3) = w^h(z') + z_3 b^h(z') + \frac{z_3^2}{2}d^h(z') + o(z_3^2), \qquad \text{for every } z \in \Omega_h. \tag{3.43}$$

Then, Taylor's expansion at each fixed z' with respect to h:

$$\begin{aligned} w^h(z') &= w(z') + h\widetilde{w}(z') + o(h), & \nabla w^h(z') &= \nabla w(z') + h\nabla\widetilde{w}(z') + o(h), \\ b^h(z') &= b(z') + h\tilde{b}(z') + o(h), & \nabla b^h(z') &= \nabla b(z') + h\nabla\tilde{b}(z') + o(h). \end{aligned} \tag{3.44}$$

Setting $S := \nabla v^h A_h^{-1}$, we have that

$$S^{\intercal} S = \left(\begin{array}{c|c} (\nabla w)^{\intercal}\nabla w & (\nabla w)^{\intercal} b \\ \hline \big((\nabla w)^{\intercal} b\big)^{\intercal} & b^{\intercal} b \end{array}\right) + h\mathbf{S} \tag{3.45}$$

with

$$\begin{aligned}\mathbf{S} := &\left(\begin{array}{c|c} 2\big((\nabla w)^{\intercal}\nabla \widetilde{w}\big)_{\mathrm{sym}} & (\nabla w)^{\intercal}\tilde{b} + (\nabla \tilde{w})^{\intercal} b \\ \hline \big((\nabla w)^{\intercal}\tilde{b} + (\nabla \tilde{w})^{\intercal} b\big)^{\intercal} & b^{\intercal}\tilde{b} + \tilde{b}^{\intercal} b \end{array}\right) + O(h) \\ &+ 2\left(\begin{array}{c|c} \frac{z_3}{h}\big((\nabla w)^{\intercal}\nabla b\big)_{\mathrm{sym}} & \frac{z_3}{h}(\nabla b)^{\intercal} b \\ \hline \frac{z_3}{h}\big((\nabla b)^{\intercal} b\big)^{\intercal} & 0 \end{array}\right) - 2\left(B\left(\begin{array}{c|c} (\nabla w)^{\intercal}\nabla w & (\nabla w)^{\intercal} b \\ \hline \big((\nabla w)^{\intercal} b\big)^{\intercal} & b^{\intercal} b \end{array}\right)\right)_{\mathrm{sym}}.\end{aligned}$$

By the frame indifference of W, using the above notation we have

$$\begin{aligned}\overline{W}_h\big(z, \nabla v^h(z)\big) &= W\big(\nabla v^h(z) A_h^{-1}(z)\big) = W(S) = W\big(\sqrt{S^{\intercal} S}\big) \\ &= \frac{1}{8}\mathcal{Q}_3\left((\nabla w|b)^{\intercal}(\nabla w|b) - \mathrm{I}_3 + h\mathbf{S}\right) + o(|S^{\intercal} S - \mathrm{I}_3|^2)\end{aligned}$$

In turn, the bending energy $\frac{1}{h^3}\overline{\mathscr{E}}^h(v^h)$ associated with v^h, can be expressed as follows:

$$\begin{aligned}\frac{\overline{\mathscr{E}}_h(v^h)}{h^3} &= \frac{1}{8h^3}\int_{\Omega_h} \mathcal{Q}_3\left((\nabla w|b)^{\intercal}(\nabla w|b) - \mathrm{I}_3 + h\mathbf{S}\right) + o(|S^{\intercal} S - \mathrm{I}_3|^2)\,\mathrm{d}z \\ &= \frac{1}{8}\int_{\Omega} \mathcal{Q}_3\left(\frac{(\nabla w|b)^{\intercal}(\nabla w|b) - \mathrm{I}_3}{h} + \mathbf{S}\right) + o(|S^{\intercal} S - \mathrm{I}_3|^2)\,\mathrm{d}x.\end{aligned}$$

Now it is clear that $\frac{1}{h^3}\overline{\mathscr{E}}_h(v^h) \leq C$ for some constant $C > 0$, leads to

$$(\nabla w|b)^{\intercal}(\nabla w|b) = \mathrm{I}_3 \quad \rightsquigarrow \quad \begin{cases} w \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega) \\ b = \partial_1 w \times \partial_2 w \end{cases} \tag{3.46}$$

Finally, by using that $\mathcal{Q}_3(F) \geq \mathcal{Q}_2(F_{2\times 2})$ for all $F \in \mathrm{Sym}(3)$, we have that

$$\begin{aligned}\frac{\overline{\mathscr{E}}_h(v^h)}{h^3} &= \frac{1}{8}\int_{\Omega} \mathcal{Q}_3(\mathbf{S}) + o(|S^{\intercal} S - \mathrm{I}_3|^2)\,\mathrm{d}x \\ &\geq \frac{1}{8}\int_{\Omega} \mathcal{Q}_2(\mathbf{S}_{2\times 2}) + o(|S^{\intercal} S - \mathrm{I}_3|^2)\,\mathrm{d}x \\ &\gtrapprox \frac{1}{2}\int_{\Omega} \mathcal{Q}_2\Big(\underbrace{\big((\nabla w)^{\intercal}\nabla \widetilde{w}\big)_{\mathrm{sym}}(x') + x_3\,\Pi_w(x') - B_{2\times 2}(x)}_{\frac{1}{2}\mathbf{S}_{2\times 2}}\Big)\,\mathrm{d}x.\end{aligned} \tag{3.47}$$

By adding and subtracting $\mathcal{B}^0_{2\times 2}$ in the argument of $\mathcal{Q}_2$ and using the fact that, by definition of $\mathcal{B}^0_{2\times 2}$ in (3.9), it holds

$$\int_{-1/2}^{1/2} \big(B_{2\times 2}(x', t) - \mathcal{B}^0_{2\times 2}(x')\big)\,\mathrm{d}t = 0,$$

for a.e. $x' \in \omega$, we have that

$$\begin{aligned}\frac{\overline{\mathscr{E}}_h(v^h)}{h^3} \gtrapprox\ & \frac{1}{2}\int_{\Omega} \mathcal{Q}_2\Big(\big((\nabla w)^{\intercal}\nabla \widetilde{w}\big)_{\mathrm{sym}} - \mathcal{B}^0_{2\times 2}\Big) + \mathcal{Q}_2\Big(x_3\Pi_w - (B_{2\times 2} - \mathcal{B}^0_{2\times 2})\Big)\,\mathrm{d}x \\ &+ \int_{\omega} \mathcal{L}_2\Big(\big((\nabla w)^{\intercal}\nabla \widetilde{w}\big)_{\mathrm{sym}} - \mathcal{B}^0_{2\times 2}, \int_{-1/2}^{1/2}\Big(x_3\Pi_w - (B_{2\times 2} - \mathcal{B}^0_{2\times 2})\Big)\,\mathrm{d}x_3\Big)\,\mathrm{d}x' \\ =\ & \frac{1}{2}\int_{\Omega} \mathcal{Q}_2\Big(\big((\nabla w)^{\intercal}\nabla \widetilde{w}\big)_{\mathrm{sym}} - \mathcal{B}^0_{2\times 2}\Big) + \mathcal{Q}_2\Big(x_3\Pi_w - (B_{2\times 2} - \mathcal{B}^0_{2\times 2})\Big)\,\mathrm{d}x.\end{aligned}$$

Now, as in Lemma 3.1.2, one can easily see that

$$\frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x_3\Pi_w - (B_{2\times 2} - \mathcal{B}^0_{2\times 2})\Big)\,\mathrm{d}x_3\,\mathrm{d}x' = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\Pi_w - \mathcal{B}^1_{2\times 2}\Big) + \text{ad.t.},$$

with $\mathcal{B}^1$ given by the second formula in (3.9) and

$$\text{ad.t.} = \frac{1}{2}\int_\omega \bigg(\int_{-1/2}^{1/2} \mathcal{Q}_2\big(B_{2\times 2}(x',t)\big)\,\mathrm{d}t - \mathcal{Q}_2\big(\mathcal{B}^0_{2\times 2}(x',t)\big) - \frac{1}{12}\mathcal{Q}_2\big(\mathcal{B}^1_{2\times 2}(x')\big)\bigg)\,\mathrm{d}x'.$$

Hence, we are left with

$$\frac{\overline{\mathscr{E}}_h(v^h)}{h^3} \gtrapprox \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(\big((\nabla w)^\intercal \nabla \widetilde{w}\big)_{\mathrm{sym}} - \mathcal{B}^0_{2\times 2}\Big)\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(\Pi_w - \mathcal{B}^1_{2\times 2}\Big)\,\mathrm{d}x' + \text{ad.t.}$$

The above inequality suggests the following ansatz for the limiting energy in place of $\mathscr{E}_0$ in Theorem 3.2.2, in the case the condition (3.20) is violated:

$$\text{(3.48)}\quad \mathscr{E}_0^{\text{new}}(y) := \frac{1}{2}\inf_{\widetilde{w}\in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)} \int_\omega \mathcal{Q}_2\Big(\big((\nabla y)^\intercal \nabla \widetilde{w}\big)_{\mathrm{sym}} - \mathcal{B}^0_{2\times 2}\Big)\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\big(\Pi_y - \mathcal{B}^1_{2\times 2}\big)\,\mathrm{d}x' + \text{ad.t.}$$

for $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$. Indeed, when $\mathcal{B}^0_{2\times 2}$ satisfies the compatibility condition (3.20), Lemma 1.4.2 grants that the first term in $\mathscr{E}_0^{\text{new}}(y)$ is minimized at zero and thus $\mathscr{E}_0^{\text{new}}(y)$ reduces to our "old" quantity $\mathscr{E}_0(y)$.

In other words, the above heuristic argument make us believe that the Γ-limit in the general case (without additional assumptions on B) has to be "higher", i.e. to include one more term, which can be interpreted as a "first order stretching term". The proof of the Γ-lim sup inequality is standard in this case, since, as we have seen in Section 3.2, one can always produce terms of the form "$(\nabla y)^\intercal \nabla \widetilde{w}$" in the recovery sequence. The difficulty in making this argument rigorous lies in proving the Γ-lim inf inequality.

An idea would be to prove, in addition to the compactness result of Theorem 3.2.1, that the sequence of deformations $\{y^h\}_h$ with bounded bending energy also satisfies the following:

$$\text{(3.49)}\qquad \|\nabla_h y^h - (\nabla y|\nu)\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{3\times 3})} \le Ch^2.$$

Assume for the moment that (3.49) holds in Theorem 3.2.1. Let us now repeat the key steps in the proof of Γ-lim inf inequality in Theorem 3.2.2. Let $\{S_h\}_h$ be as in (3.22) and observe that

$$\int_{-1/2}^{1/2} S_h(\cdot,t)\,\mathrm{d}t = \int_{-1/2}^{1/2} \frac{R_h^\intercal \nabla_h y^h(\cdot,t) - \mathrm{I}_3}{h}\,\mathrm{d}t = \int_{-1/2}^{1/2} \frac{R_h^\intercal\big(\nabla_h y^h(\cdot,t) - (\nabla y|\nu)\big)}{h}\,\mathrm{d}t + \frac{R_h^\intercal(\nabla y|\nu) - \mathrm{I}_3}{h}.$$

Due to (3.21) and (3.49), the second summand on the right hand side in the above formula converges weakly in $\mathrm{L}^2(\omega,\mathbb{R}^{3\times 3})$. Moreover, its L^2-weak limit is a skew-symmetric matrix field. To verify the latter, denote $Q_h := R_h^\intercal(\nabla y|\nu) : \omega \to \mathrm{SO}(3)$ and observe that

$$\frac{(Q_h - \mathrm{I}_3)_{\mathrm{sym}}}{h} = h\frac{(Q_h - \mathrm{I}_3)}{h}\,\frac{(Q_h^\intercal - \mathrm{I}_3)}{h}.$$

Thus

$$\left\| \frac{(Q_h - \mathrm{I}_3)_{\mathrm{sym}}}{h} \right\|_{\mathrm{L}^1(\omega,\mathbb{R}^{3\times3})} \leq h \left\| \frac{Q_h - \mathrm{I}_3}{h} \right\|^2_{\mathrm{L}^2(\omega,\mathbb{R}^{3\times3})} \overset{(3.49)}{\leq} Ch \to 0, \qquad \text{as } h \to 0.$$

This grants that (up to passing to a further subsequence) the pointwise (and thus also the L^2-weak) limit of $\frac{(Q_h-\mathrm{I}_3)_{\mathrm{sym}}}{h}$ equals to 0. Therefore, the L^2-weak limit S of the sequence $\{S_h\}_h$ satisfy

$$S_{2\times2,\mathrm{sym}}(\cdot,0) = \mathrm{L}^2\text{-weak limit of} \int_{-1/2}^{1/2} \frac{1}{h}\Big((R_h^{\mathsf{T}})_{2\times3}\big(\nabla' y^h(\cdot,t) - \nabla y\big)\Big)_{\mathrm{sym}} \,\mathrm{d}t = \big((\nabla y)^{\mathsf{T}}\nabla\tilde{w}\big)_{\mathrm{sym}},$$

which is precisely the term that appears in the heuristic argument presented above. The last equality follows from (3.49), which guarantees in particular the existence of $\widetilde{w} \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)$ such that

$$\int_{-1/2}^{1/2} \frac{\nabla' y^h(\cdot,x_3) - \nabla y}{h}\,\mathrm{d}x_3 \rightharpoonup \nabla\widetilde{w}, \qquad \text{weakly in } \mathrm{L}^2(\omega,\mathbb{R}^{3\times2}). \tag{3.50}$$

Thus the validity of (3.49) would allow us to obtain the same lower bound as in (3.47), by performing the rigorous procedure as in the proof of Γ-lim inf inequality in Theorem 3.2.2.

One may observe that the validity of (3.49) has nothing to do with the presence of the strain field B – namely, if this convergence result holds, it should hold even as a part of the compatness result in [FJM02]. However, from the rigidity estimate in [FJM02] the only thing we know is that to any sequence $\{y^h\}_h$ of bounded bending energy one can associate a sequence of rotation-valued matrix fields R_h satisfying

$$\|\nabla_h y^h - R_h\|_{\mathrm{L}^2(\Omega,\mathbb{R}^{3\times3})} \leq Ch^2 \quad \text{and} \quad R_h \xrightarrow{\mathrm{L}^2} (\nabla y|\nu),$$

as observed in Section 2.2.2, and there is, in general, no reason for (3.49) to hold.

Idea 2: A “patchwork” of the recovery sequences. Since it is not clear if one can prove (3.49) or make the above heuristics rigorous in some other way, we here discuss the possibility to reach the “old” Γ-limit $\mathscr{E}_0$ even in the absence of the assumption (3.20) on the spontaneous strain B.

Let us also add at this point a trivial but important observation: the difficulties one encounter in removing the compatibility assumption (3.20) on the matrix field $\mathcal{B}^0_{2\times2}$ do not originate from the dependence of the spontaneous strain B on the thickness variable, since they persist even in the case where such a dependence is absent.

To start with, let us observe that the constant matrix field B trivially satisfy (3.20) and that in this case the limit functional $\mathscr{E}_0$, derived in Theorem 3.2.2, reads as

$$\mathscr{E}_0(y,\omega) := \mathscr{E}_0(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\big(\Pi_y(x')\big)\,\mathrm{d}x', \qquad y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega).$$

For the purposes of this argument we adopt a new notation for the energy functionals, indicating explicitly the planar domain of integration. We further suppose that B is only x'-dependent and is piecewise constant, namely

$$B = B_1\chi_{\omega_1} + B_2\chi_{\omega_2},$$

where B_1, B_2 are constant 3×3 matrices and $\{\omega_1,\omega_2\}$ form a Lipschitz partition of ω. We aim to show that

$$(\Gamma\text{-}\lim_h \frac{1}{h^2}\mathscr{E}_h)(y,\omega)=\mathscr{E}_0(y,\omega_1)+\mathscr{E}_0(y,\omega_2), \qquad \text{for every } y\in \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3),$$

where $\mathscr{E}_h$ is as in (3.15). Proving the additivity of the limiting functional and approximating any $B\in \mathrm{L}^\infty(\omega,\mathrm{Sym}(3))$ by a sequence of piecewise constant maps would lead us to the conclusion that the "old" Γ-limit is the "right one" also in the general case. In this case the proof of the Γ-lim inf inequality in Theorem 3.2.2 yields $(\Gamma\text{-}\liminf_h \frac{1}{h^2}\mathscr{E}_h)(y,\omega)\geq \mathscr{E}_0(y,\omega_1)+\mathscr{E}_0(y,\omega_2)$. Thus it remains to show that

$$(\Gamma\text{-}\limsup_h \frac{1}{h^2}\mathscr{E}_h)(y,\omega)\leq \mathscr{E}_0(y,\omega_1)+\mathscr{E}_0(y,\omega_2). \tag{3.51}$$

An idea for proving (3.51) is to construct a recovery sequence $\{y^h\}_h$ by "patching together" the recovery sequences that we know how to construct on ω_1 and ω_2 where B is constant. Behind this idea lies *the fundamental estimate*, which plays a central role in proving that the $\overline{\Gamma}$-limit of a sequence of increasing functionals is a measure. We do not enter into details about this topic, refering the reader to [DM12] for a thorough account on the notion of $\overline{\Gamma}$ convergence.

In the present case, let us consider the following: let $\omega_1'\subseteq\mathbb{R}^2$ be an open set such that $\omega_1\Subset\omega_1'$ and let $\{y_i^h\}$, $i=1,2$ be the recovery sequence as in (3.24), namely

$$y_i^h(x)=y(x')+h\big[x_3\nu(x')+\nabla' y(x')\tilde{w}_i(x')\big]+h^2\tilde{d}(x), \qquad \text{for every } x\in(\omega_1'\cup\omega_2)\times(-1/2,1/2),$$

with $\tilde{w}_i(x')=(B_i)_{2\times 2}\,x'$. Denote $P:=\omega_1'\cap\omega_2$ and (re)define $B:=B_1\chi_{\omega_1'}+B_2\chi_{\omega\setminus\omega_1'}$. We would like to find a sequence of cut-off functions $\varphi^h\in C_c^\infty(\omega_1')$ between ω_1 and ω_1' (in the sense of [DM12, Definition 18.1]), so that the convex combinations

$$y^h=\varphi^h y_1^h+(1-\varphi^h)y_2^h, \qquad \text{a.e. in } (\omega_1'\cup\omega_2)\times(-1/2,1/2),$$

satisfy

$$\frac{1}{h^2}\mathscr{E}_h(y^h,\omega)\leq\frac{1}{h^2}\mathscr{E}_h(y_1^h,\omega_1')+\frac{1}{h^2}\mathscr{E}_h(y_2^h,\omega_2)+\frac{1}{h^2}\mathscr{E}_h\big(y^h,P\big), \tag{3.52}$$

with $\lim_{|P|\to 0}\limsup_{h\to 0}\mathscr{E}_h\big(y^h,P\big)=0$. Relaying on the proof of [DM12, Theorem 19.1] and on the strategy of choosing cut-off functions that has been used therein, we are able to arrive to the following bound:

$$\frac{1}{h^2}\mathscr{E}_h\big(y^h,P\big)\leq C_1\big(|P|\big)+C_2\big(|P|\big)\frac{\|y_1^h-y_2^h\|^2_{\mathrm{L}^2(P)}}{h^2},$$

where $C_1\to 0$ and $C_2\to\infty$, as $|P|\to 0$. The fact that $C_2(|P|)\to\infty$ as $|P|\to 0$ suggest that the second term in the above expression must be canceled out by firstly passing to the limit as $h\to 0$. This is where the argument fails, since the L^2-norm of the difference of the recovery sequences is precisely of order h. This guarantees only the boundedness of the last term in the latter expression, but not its convergence to zero as $h\to 0$.

Clearly, this attempt does not exclude completely the possibility of patching together the recovery sequences (and, more in general, the possibility that the family of functionals $\frac{1}{h^2}\mathscr{E}_h$ satisfy the fundamental estimate uniformly), since one can still try to argue in the same way but, for instance: with a different choice of the cut-off functions used in the construction of convex combinations of the known recovery sequences; or by modifying the recovery sequences themselves.

3.A Appendix

For completeness and better understanding of the analysis of the pointwise minimizers of $\mathscr{E}_0$ in the case of a piecewise constant target curvature $\mathcal{B}^1_{2\times 2}$, we here provide a detailed proof (through the two lemmas below) of [Sch07b, Proposition 4.2], where the minimizers of $\mathscr{E}_0$ have been determined in the case of constant $\mathcal{B}^1_{2\times 2}$.

Let $\mathscr{E}_0$ be given by (3.28) and suppose that $\mathcal{B}^1_{2\times 2} \equiv \overline{B} \in \mathrm{Sym}(2)$. Then the set $\mathcal{N}(x')$ in (3.33) at each point $x' \in \omega$ equals

$$\mathcal{N}_{\overline{B}} := \operatorname*{argmin}_{S\in\mathcal{S}_0} \mathcal{Q}_2\big(S - \overline{B}\big),$$

where the set $\mathcal{S}_0$ is given by (1.22).

Lemma 3.A.1. *Every two distinct elements of the set $\mathcal{N}_{\overline{B}}$ are linearly independent. In particular, if $\operatorname{card}\mathcal{N}_{\overline{B}} \geq 2$ then $0 \notin \mathcal{N}_{\overline{B}}$.*

Proof. Suppose that $S_1, S_2 \in \mathcal{N}$ are two distinct elements. Note that the function $\mathcal{Q}_2\big(\cdot - \overline{B}\big)$ is strictly convex on $\mathrm{Sym}(2)$, being composition of a linear function with the strictly convex function $\mathcal{Q}_2$. This implies that

$$tS_1 + (1-t)S_2 \notin \mathcal{S}_0 \quad \text{for every } t \in (0,1). \tag{3.53}$$

Indeed, if $tS_1 + (1-t)S_2 \in \mathcal{S}_0$ for some $t \in (0,1)$ then

$$\mathcal{Q}_2\big(tS_1 + (1-t)S_2 - \overline{B}\big) < t\mathcal{Q}_2\big(S_1 - \overline{B}\big) + (1-t)\mathcal{Q}_2\big(S_2 - \overline{B}\big) = \min_{\mathcal{S}_0} \mathcal{Q}_2\big(\cdot - \overline{B}\big),$$

which leads to a contradiction, proving (3.53). Finally, we claim that S_1, S_2 are linearly independent: if not, there exists $c \in \mathbb{R}$ such that $S_1 = c\,S_2$ (up to possibly interchanging S_1 and S_2). Hence, for any $t \in (0,1)$, we have that

$$0 \overset{(3.53)}{\neq} \det\big(tS_1 + (1-t)S_2\big) = \det\big((ct + 1 - t)S_2\big) = 0,$$

which is a contradiction. □

Lemma 3.A.2. *Let $y \in \mathcal{M} := \operatorname{argmin}_{W^{2,2}_{\mathrm{iso}}(\omega)} \mathscr{E}_0$. Then $y = u|_\omega$ for some $u \in \mathrm{Cyl}$. In particular, the second fundamental form of y is constant.*

Proof. To simplify notation, we call $\mathcal{Q} := \mathcal{Q}_2(\cdot - \overline{B})$. First, we will show that

$$y \in W^{2,2}_{\mathrm{iso}}(\omega) \text{ is a minimizer of } \mathscr{E}_0 \text{ if and only if } y \text{ is a pointwise minimizer.} \tag{3.54}$$

Let $y \in \mathcal{M}$ and fix $S_{\min} \in \mathcal{N}_{\overline{B}}$. By Theorem 1.4.3, we know that $\det\big(\Pi_y(x')\big) = 0$ for a.e. $x' \in \omega$. Therefore we have that

$$\mathcal{Q}\big(\Pi_y(x')\big) \geq \mathcal{Q}(S_{\min}) \quad \text{for a.e. } x' \in \omega. \tag{3.55}$$

Suppose that there exists a Borel set $P \subseteq \omega$, with $\mathcal{L}^2(P) > 0$, such that $\mathcal{Q}\big(\Pi_y(x')\big) > \mathcal{Q}(S_{\min})$ for a.e. $x' \in P$. As shown in Section 1.4.2, $\mathcal{S}_0$ is the set of (constant) second fundamental forms of cylinders, thus there exists $u \in W^{2,2}_{\mathrm{iso}}(\omega)$ such that $\Pi_u(x') = S_{\min}$ for a.e. $x' \in \omega$. By integrating the inequality (3.55) over ω we get

$$\begin{aligned}\int_\omega \mathcal{Q}\big(\Pi_y(x')\big)\,dx' &= \int_{\omega\setminus P} \mathcal{Q}\big(\Pi_y(x')\big)\,dx' + \int_P \mathcal{Q}\big(\Pi_y(x')\big)\,dx' \\ &> \int_{\omega\setminus P} \mathcal{Q}\big(\Pi_u(x')\big)\,dx' + \int_P \mathcal{Q}\big(\Pi_u(x')\big)\,dx' = \int_\omega \mathcal{Q}\big(\Pi_u(x')\big)\,dx',\end{aligned}$$

which contradicts the fact that $y \in \mathcal{M}$. Therefore $\Pi_y(x') \in \mathcal{N}_{\overline{B}}$ for a.e. $x' \in \omega$.

For the converse implication, let $y \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ and suppose that $\Pi_y(x') \in \mathcal{N}_{\overline{B}}$ for a.e. $x' \in \omega$. Then

$$\mathcal{Q}\big(\Pi_y(x')\big) \leq \mathcal{Q}\big(\Pi_u(x')\big) \quad \text{for a.e. } x' \in \omega, \tag{3.56}$$

since $\Pi_u(x') \in \mathcal{S}_0$ for every $u \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\omega)$ and a.e. $x' \in \omega$. By integrating (3.56) over ω, we get that $y \in \mathcal{M}$. Hence (3.54) is proved. In particular, some cylinders are contained in $\mathcal{M}$. It remains to show that $\mathcal{M}$ contains only cylinders.

Let $y \in \mathcal{M}$. This implies that $\Pi_y(x') \in \mathcal{N}_{\overline{B}}$ for a.e. $x' \in \omega$. If $\mathcal{N}_{\overline{B}}$ is a singleton, then Π_y is constant a.e. in ω, thus accordingly $y = u|_\omega$ for some $u \in \mathrm{Cyl}$. If $\operatorname{card} \mathcal{N}_{\overline{B}} \geq 2$ then every two elements of $\mathcal{N}_{\overline{B}}$ are linearly independent, by Lemma 3.A.1, and therefore $0 \notin \mathcal{N}_{\overline{B}}$. This implies that y cannot be affine on any region. Hence, following the notation from Section 1.4.1, let us consider a subdomain of ω parametrized by a line of curvature $\Gamma \in \mathrm{W}^{2,\infty}\big((0,T),\omega\big)$. For a.e. $t \in (0,T)$ and $s_0, s_1 \in \big(s^-_\Gamma(t), s^+_\Gamma(t)\big)$, $s_0 \neq s_1$, we have, by (1.17), that

$$\Pi_y\big(\Gamma(t) + s_1 N(t)\big) = \frac{\kappa_{\mathsf{n}}(t)}{1 - s_1\kappa_{\mathsf{t}}(t)}\Gamma'(t) \otimes \Gamma'(t) = \frac{1 - s_0\,\kappa_{\mathsf{t}}(t)}{1 - s_1\kappa_{\mathsf{t}}(t)}\Pi_y\big(\Gamma(t) + s_0 N(t)\big).$$

Since the elements of $\mathcal{N}_{\overline{B}}$ are linearly independent we deuce that $\kappa_{\mathsf{t}}(t) = 0$ for a.e. $t \in (0,T)$. Therefore Γ' is constant on $(0,T)$. It remains to show that also κ_{n} is constant a.e. in $(0,T)$. For a.e. $t_0, t_1 \in (0,T)$ and $s \in \big(s^-_\Gamma(t_0), s^+_\Gamma(t_0)\big) \cap \big(s^-_\Gamma(t_1), s^+_\Gamma(t_1)\big)$ it holds that

$$\Pi_y\big(\Gamma(t_0) + sN(t_0)\big) = \frac{\kappa_{\mathsf{n}}(t_0)}{1 - s\kappa_{\mathsf{t}}(t_0)}\Gamma'(t_0) \otimes \Gamma'(t_0) = \frac{\kappa_{\mathsf{n}}(t_0)}{\kappa_{\mathsf{n}}(t_1)}\Pi_y\big(\Gamma(t_1) + sN(t_1)\big).$$

Again by linear independence of the elements in $\mathcal{N}_{\overline{B}}$, we deduce that κ_{n} is constant a.e. in $(0,T)$. Hence we showed that Π_y is locally constant in ω. By connectedness of ω, we conclude that Π_y is constant a.e. in ω. □

4

Application to heterogeneous thin gel sheets

Contents

This chapter is devoted to the analysis of *hydrogel*-based *folding* sheets exploiting the two-dimensional (nonlinear) plate model presented in Section 4.2, obtained by applying the rigorously derived theory in Chapter 3 for heterogeneous thin elastic plates.

Self-folding is a widespread phenomenon that occurs in natural systems, such as in the opening and closing of flowers. The understanding of such mechanisms offers interesting challenges and opportunities, not only from the point of view of biology, but also from the point of view of applications, for instance, in the fabrication of actuation systems and origami-like structures [Ion11]. Self-folding results from spatially heterogeneous deformations that are (in most cases) either induced by a spatial variation of the external stimulus or by modulations of the material properties imprinted during the fabrication process. We refer to [GLZ13, MYI$^+$15, NEB$^+$, LSDG17, HAB$^+$10, YXP$^+$14] for some examples of the approaches to folding and the numerical models that might be used in the study of folding mechanisms.

Hydrogel (a network of cross-linked polymer chains swollen with a liquid solvent) is example of an active material where spontaneous deformations are induced by swelling due to the absorption of a liquid. In the form of bilayers, hidrogels can be employed to produce curved shapes [HZL95]. In turn, hydrogel bilayers can be used in creating folding mechanisms – for instance, acting as hinges connecting flat faces (hydrogel monolayers), as we shall see below.

For concreteness, in Section 4.1 we consider heterogeneous thin sheets Ω_h made of hydrogel, by imprinting a heterogeneous density $\overline{N}_h$ of polymer chains, which corresponds to a heterogeneous shear modulus of the polymer network. Moreover, $\overline{N}_h$ is a (small) perturbation of order h of the average value $\overline{N}$, that is,

$$\overline{N}_h(z) = \overline{N} + hb\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \Omega_h = \omega \times (-h/2, h/2), \tag{4.1}$$

for some bounded function $b : \omega \times (-1/2, 1/2) \to \mathbb{R}$. Referring to the classical Flory-Rehner model [Doi09] for isotropic polymer gels, we obtain as a consequence of the above assumption on $\overline{N}_h$ that the free energy density $\overline{W}_h$ associated with the system is minimized at

$$\big(\alpha + h\beta\, b(z', z_3/h) + o(h)\big)\mathrm{SO}(3), \tag{4.2}$$

where the (dimensionless) constants α and β are functions of the material parameters appearing in the expression of $\overline{W}_h$, including $\overline{N}$. Thus after performing an appropriate change of variable (in Section 4.2) in order to pass from the energy wells that are h-close to $\alpha\mathrm{I}_3$ to those h-close to I_3, we can apply the theory derived in Chapter 3 and obtain the corresponding 2D model given in (4.23).

In Section 4.3 we discuss self-folding of thin sheets by using hydrogel bilayers, which act as hinges connecting flat faces. Folding is actuated by the heterogeneous swelling due to different stiffness (i.e. cross-linking density) of the polymer network across two layers. More precisely, we will focus on the case where the cross-linking density is different in the top and the bottom layers (see formula (4.5)) – a particular example of the heterogeneity presented in (4.1). We show that this structure allows to endow a thin gel sheet with a controlled curvature localized at the hinges (see Figure 4.1), which can be realized, upon swelling, at low energy cost. Furthermore, such a curvature can be expressed as a function of the material parameters of the layers. Specifically, we demonstrate the feasibility of the proposed folding mechanism with two examples, corresponding to specific patterns of flat faces and hinges (see Figure 4.2).

In Subsection 4.3.2 we consider the pointwise energy minimizers of the plate theory, which describe the configurations of a plate with bilayer-like hinges and thus provide a theoretical justification for the effectiveness of the folding mechanism. This is based on the previous study of the energy minimizers in Section 3.3.

In Subsection 4.3.3, with reference to an appropriate approximate variant of our Flory-Rehner-type model, we provide the explicit relation between the target curvature and the material parameters (see (4.40)). Such relation is then used in the design of folding bilayer sheets that morph into a cubes or pyramids.

In the last section of this chapter we provide a detailed analysis of Flory-Rehner-type energy densities, justifying the fact that the theory developed in Chapter 3 is the right one to apply in this case. An interesting fact that we notice regarding hydrogels is that their physics results in the following feature of the model: the energy densities (4.6) cannot be written in the "pre-stretched" form (2.5) reflecting that (4.6) is the sum of two energy contributions (elastic and mixing energies) that concurrently define the energy minimum.

4.1 Heterogeneous thin gel sheets: the 3D model

In the present context, a hydorgel (or, more in general, a polymer gel) is a network of cross-linked polymer chains swollen with a liquid solvent. We denote by $\overline{N}$ the density of polymer chains in the reference volume and we set $\mathbb{R}^{3\times3}_1 := \{F \in \mathbb{R}^{3\times3} : \det F \geq 1\}$. The dimensionless free-energy density for *isotropic* and *homogeneous* polymer gels is of Flory-Rehner type (see [Doi09]) and is given by the function $W^{\mathsf{FR}} : \mathbb{R}^{3\times3}_1 \to \mathbb{R}$ defined as

$$(4.3)\quad W^{\mathsf{FR}}(F) := \underbrace{\frac{\mathrm{v}\overline{N}}{2}\left(|F|^2 - 3\right)}_{\text{elastic energy}} + \underbrace{W^{\chi}_{vol}(\det F) - \frac{\mu}{k_{\mathsf{B}}\mathrm{T}}(\det F - 1)}_{\text{due-to-mixing energy}}, \qquad \text{for every } F \in \mathbb{R}^{3\times3}_1,$$

where the mixing energy $W^{\chi}_{vol} : (1, +\infty) \to (-\infty, 0]$ is given by

$$(4.4)\qquad W^{\chi}_{vol}(t) := (1-t)\log\left(\frac{t}{t-1}\right) - \frac{\chi}{t} + \chi, \qquad \text{for every } t > 1.$$

The physical parameters that appear in the above formula are

- k_{B} - Boltzmann's constant;
- T - absolute temperature;
- v - the volume per solvent molecule;
- $\chi \in (0, 1/2]$ - dimensionless measure of the enthalpy of mixing;
- $\mu \leq 0$ - the chemical potential of the solvent molecules.

We remark that the ranges $\chi \in (0, 1/2]$ and $\mu \leq 0$ correspond, respectively, to having a good solvent and a gel in contact with an external fluid that is either a vapor ($\mu < 0$) or a pure liquid ($\mu = 0$) in equilibrium with its own vapor. In particular, $\chi \in (0, 1/2]$ implies that the function W^{χ}_{vol} it is strictly decreasing and it fulfills

$$W^{\chi}_{vol}(1^+) = 0 \quad \text{and} \quad \lim_{t\to+\infty} W^{\chi}_{vol}(t) = \chi - 1 < 0.$$

3D model for heterogeneous gel sheet. Our attention will be focused on a *heterogeneous* thin gel sheet occupying the reference configuration Ω_h. More precisely, we consider a sheet characterized by a z-dependent cross-linking density, which in turn determines a z-dependent density $\overline{N}^h$ of polymer chains. At the same time, we suppose that $\overline{N}_h$ is a perturbation of a constant value $\overline{N}$, namely

$$(4.5)\qquad \overline{N}_h(z) := \overline{N} + hb\left(z', \frac{z_3}{h}\right) \quad \text{and} \quad \fint_{-h/2}^{h/2} \overline{N}_h(z', z_3)\,\mathrm{d}z_3 = \overline{N},$$

for a.e. $z' \in \omega$ and every $h > 0$, with $b \in \mathrm{L}^\infty(\Omega)$. Observe that the condition (4.5) is equivalent to $\int_{-1/2}^{1/2} b(z', t)\,\mathrm{d}t = 0$ for a.e. $z' \in \omega$.

Remark 4.1.1. More in general, one can replace b in (4.5) by some $b_h \in \mathrm{L}^\infty(\Omega)$ such that the sequence $\{b_h\}_h \subseteq \mathrm{L}^\infty(\Omega)$ satisfies $b_h \to b$ in $\mathrm{L}^\infty(\Omega)$ as $h \to 0$. ■

The energy of such heterogeneous system is described, by using the model densities (4.3), via the family of free-energy density functions $\overline{W}^{\mathsf{FR}}_h : \Omega_h \times \mathbb{R}^{3\times3} \to \mathbb{R} \cup \{+\infty\}$ defined as

$$(4.6)\qquad \overline{W}^{\mathsf{FR}}_h(z, F) := \frac{\mathrm{v}}{2}\left(\overline{N} + hb\left(z', \frac{z_3}{h}\right)\right)\left(|F|^2 - 3\right) + W^{\chi}_{vol}(\det F) - \frac{\mu}{k_{\mathsf{B}}\mathrm{T}}(\det F - 1)$$

for a.e. $z \in \Omega_h$ and every $F \in \mathbb{R}^{3\times3}_1$ and set to be $+\infty$ elsewhere in $\mathbb{R}^{3\times3}$, for every $h > 0$. Functions $\overline{W}^{\mathsf{FR}}_h$ are clearly isotropic (and thus frame indifferent). In Section 4.4 we provide a detailed analysis of the energy densities of Flory-Rehner type and, in particular, we show that each $\overline{W}_h^{\mathsf{FR}}$ satisfies the following properties:

REGULARITY: $\overline{W}_h^{\mathsf{FR}}$ is a (jointly) Borel function. Moreover, the function $\overline{W}_h^{\mathsf{FR}}(z,\cdot)$ is of class C^∞ on the set $\{F \in \mathbb{R}^{3\times3} : \det F > 1\}$.

ENERGY WELLS: there exist two constants

$$\alpha > 1 \qquad \text{and} \qquad \beta \neq 0, \tag{4.7}$$

which depend only on k_{B}, on the (fixed) material parameters of the gel $(\mathrm{v}, \overline{N}, \chi)$ and the environmental conditions (μ, T), such that

$$\overline{W}_h^{\mathsf{FR}}(z,\cdot) \quad \text{is minimized precisely on the set} \quad \mathrm{SO}(3)A_h(z),$$

where

$$A_h(z) = \alpha\, \mathrm{I}_3 + h\, \beta b\left(z', \frac{z_3}{h}\right) \mathrm{I}_3 + o(h), \qquad \text{for a.e. } z \in \Omega_h. \tag{4.8}$$

UNIFORM CONVERGENCE: There exists $\mathcal{U} \in \mathcal{N}\big(\alpha\mathrm{SO}(3)\big)$ such that the rescaled energy densities $W_h^{\mathsf{FR}}(x,F) := \overline{W}_h^{\mathsf{FR}}\big((x', hx_3), F\big)$, defined on $\Omega \times \mathbb{R}^{3\times3}$, satisfy

$$\operatorname*{ess\,sup}_{x\in\Omega} \|W_h^{\mathsf{FR}}(x,\cdot) - W^{\mathsf{FR}}\|_{C^2(\overline{\mathcal{U}})} \to 0, \qquad \text{as } h \to 0. \tag{4.9}$$

QUADRATIC GROWTH: There exists a constant $C > 0$ (independent of h) such that

$$\overline{W}_h^{\mathsf{FR}}(z,F) \geq C \operatorname{dist}^2\big(F, \mathrm{SO}(3)A_h(z)\big), \qquad \text{for a.e. } z \in \Omega_h, \text{ every } F \in \mathbb{R}^{3\times3}. \tag{4.10}$$

Due to (4.8), hereafter we may suppose (without loss of generality) that $\overline{W}_h^{\mathsf{FR}}$ are non-negative. We recall that the free-energy functional $\overline{\mathscr{E}}_h^{\mathrm{gel}} : \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3) \to [0, +\infty]$ associated to the above described system is given by

$$\overline{\mathscr{E}}_h^{\mathrm{gel}}(v) := \int_{\Omega_h} \overline{W}_h^{\mathsf{FR}}\big(z, \nabla v(z)\big)\, \mathrm{d}z, \qquad \text{for every } v \in \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3). \tag{4.11}$$

4.2 The corresponding Kirchhoff-like plate model

In this section we will show that the properties of $\overline{W}_h^{\mathsf{FR}}$ listed above and a suitable change of variables will allow us to apply the general theory developed in Chapter 3 to the case of the heterogeneous thin gel sheets and to derive the corresponding plate model in the Kirchhoff regime. The corresponding rescaled version of the functional $\overline{\mathscr{E}}_h^{\mathrm{gel}}$ is denoted by $\mathscr{E}_h^{\mathrm{gel}}$ and is given by

$$\mathscr{E}_h^{\mathrm{gel}}(y) := \int_\Omega W_h^{\mathsf{FR}}\big(x, \nabla_h y(x)\big)\, \mathrm{d}x, \qquad y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3),$$

following the rescaling procedure presented in Section 2.1.2.

The α-rescaled model. Let α be given by (4.7) and (4.8). We define, for each $0 < h \ll 1$ the energy density functions $\overline{W}_{\alpha h} : \Omega_{\alpha h}^{\alpha} \times \mathbb{R}^{3\times3} \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\overline{W}_{\alpha h}(\eta, F) := \overline{W}_h^{\mathsf{FR}}\left(\frac{\eta}{\alpha}, \alpha F\right), \qquad \text{for a.e. } \eta \in \Omega_{\alpha h} \text{ and every } F \in \mathbb{R}^{3\times3}, \tag{4.12}$$

where $\Omega^{\alpha}_{\alpha h} := \alpha\omega \times (-\alpha h/2, \alpha h/2)$. By considering as a small thickness parameter αh instead of h, it can be readily checked that the family $\{W_{\alpha h}\}_{\alpha h>0}$ of the rescaled densities

$$W_{\alpha h}\big((\eta', x_3), F\big) := \overline{W}_{\alpha h}\big((\eta', \alpha h x_3), F\big), \qquad (\eta', x_3) \in \Omega^{\alpha} := \alpha\omega \times (-1/2, 1/2)$$

is a $\mathscr{B}$-admissible family (according to Definition 3.1.1) with

$$\mathscr{B} = \{B_{\alpha h}\}_{\alpha h \geq 0} \quad \text{and} \quad B_{\alpha h} = B := \frac{\beta}{\alpha^2}\, b\left(\frac{\cdot}{\alpha}\right) \mathrm{I}_3. \tag{4.13}$$

Observe that the corresponding limiting density function W satisfy

$$W \;=\; W^{\mathsf{FR}}(\alpha\,\cdot). \tag{4.14}$$

Then, given $y \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ we set $u := y(\frac{\cdot}{\alpha}, \cdot) : \Omega^{\alpha} \to \mathbb{R}^3$ and note that

$$\begin{aligned} \frac{1}{h^2}\mathscr{E}^{\mathrm{gel}}_h(y) =& \frac{1}{h^2}\int_{\Omega} W^{\mathsf{FR}}_h\big(x, \nabla_h y(x)\big)\,\mathrm{d}x = \frac{1}{h^2}\int_{\Omega} W_{\alpha h}\left((\alpha x', x_3), \frac{1}{\alpha}\nabla_h y(x)\right)\mathrm{d}x \\ =& \frac{1}{(\alpha h)^2}\int_{\alpha\omega}\int_{-1/2}^{1/2} W_{\alpha h}\big((\eta', x_3), \nabla_{\alpha h} u(\eta', x_3)\big)\,\mathrm{d}\eta'\,\mathrm{d}x_3 =: \frac{1}{(\alpha h)^2}\mathscr{E}_{\alpha h}(u). \end{aligned} \tag{4.15}$$

We can apply Theorem 3.2.2 to the sequence of functionals $\frac{1}{(\alpha h)^2}\mathscr{E}_{\alpha h}$, getting that it Γ-converges in the weak and the strong topology of $\mathrm{W}^{1,2}\big(\Omega^{\alpha}, \mathbb{R}^3\big)$, as $\alpha h \to 0$, to the functional $\mathscr{E}_0$ given by

$$\mathscr{E}_0(u) := \frac{1}{24}\int_{\alpha\omega} \overline{\mathcal{Q}}_2\big(\eta', \Pi_u(\eta')\big)\mathrm{d}\eta', \quad u \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\alpha\omega),$$

and $+\infty$ elsewhere in $\mathrm{W}^{1,2}(\Omega^{\alpha}, \mathbb{R}^3)$. More explicitly, by the definition of $\mathcal{Q}_2$ and $\overline{\mathcal{Q}}_2$ associated to the limiting density W via (2.7) and (3.10), respectively, we have that given $u \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\alpha\omega)$ it holds

$$\begin{aligned} \mathscr{E}_0(u) = & \frac{1}{24}\int_{\alpha\omega} \mathcal{Q}_2\Big(\Pi_u(\eta') - \mathcal{B}^1_{2\times 2}(\eta'/\alpha)\Big)\,\mathrm{d}\eta' \\ & + \frac{1}{2}\int_{\alpha\omega}\int_{-1/2}^{1/2} \mathcal{Q}_2\left(B_{2\times 2}(\eta'/\alpha, t)\right)\,\mathrm{d}t\,\mathrm{d}\eta' - \frac{1}{24}\int_{\alpha\omega} \mathcal{Q}_2\left(\mathcal{B}^1_{2\times 2}(\eta'/\alpha)\right)\mathrm{d}\eta', \end{aligned} \tag{4.16}$$

with $\mathcal{B}^1$ being associated to B defined in (4.13) via the second formula in (3.9), namely

$$\mathcal{B}^1 = 12\frac{\beta}{\alpha^2}\int_{-1/2}^{1/2} t\, b(\cdot, t)\,\mathrm{d}t\,\mathrm{I}_3, \qquad \text{a.e. in } \omega. \tag{4.17}$$

The corresponding 2D model. Let us define

$$\mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega, \mathbb{R}^3) \;:=\; \big\{y \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^3) : (\nabla y)^{\mathsf{T}}\nabla y = \alpha^2\,\mathrm{I}_2 \;\text{ a.e. in } \omega\big\}. \tag{4.18}$$

We will refer to any $y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega, \mathbb{R}^3)$ as to an α-isometry. For the sake of brevity, we will use the notation $\mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega)$ instead of $\mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega, \mathbb{R}^3)$. Observe that given $y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega)$, the map

$$u(\eta') := y\left(\frac{\eta'}{\alpha}\right), \quad \eta' \in \alpha\omega, \quad \text{belongs to } \mathrm{W}^{2,2}_{\mathrm{iso}}(\alpha\omega). \tag{4.19}$$

The second fundamental forms associated with y and u are related via

$$\Pi_y(x') = \frac{1}{\alpha^2}\Pi_u(\alpha x'), \qquad \text{for a.e. } x' \in \omega. \tag{4.20}$$

Let the quadratic form $\mathcal{Q}_2^{\mathsf{FR}}$ be defined by

$$\mathcal{Q}_2^{\mathsf{FR}}(G) := \min_{\mathsf{c}\in\mathbb{R}^3} D^2W^{\mathsf{FR}}(\alpha\mathrm{I}_3)[G^* + \mathsf{c}\otimes\mathsf{f}_3]^2, \qquad \text{for every } G \in \mathbb{R}^{2\times 2}.$$

Since W^{FR} is isotropic and coincides with W_1 defined in (4.52), we have by (4.60) and (4.61) with $\theta = 1$ and by (2.28) that $\mathcal{Q}_2^{\mathsf{FR}}$ reads as

$$\mathcal{Q}_2^{\mathsf{FR}}(G) = 2\mathbf{G}|G_{\mathrm{sym}}|^2 + \mathbf{\Lambda}(\alpha)\,\mathrm{tr}^2 G, \qquad \text{for every } G \in \mathbb{R}^{2\times 2}. \tag{4.21}$$

In the latter formula, $\mathbf{G}$ and $\mathbf{\Lambda}(\alpha)$ are non-dimensional, positive Lamé constants, which are given by

$$\mathbf{G} := \mathrm{v}\overline{N} \text{ and } \mathbf{\Lambda}(\alpha) := \frac{2\mathbf{G}\lambda(\alpha)}{2\mathbf{G}+\lambda(\alpha)}, \quad \text{with} \quad \lambda(\alpha) = -\mathrm{v}\overline{N} - \frac{1}{\alpha^2} + \frac{\alpha}{\alpha^3-1} - \frac{2\chi}{\alpha^5} > 0. \tag{4.22}$$

The positivity of the constant $\lambda(\alpha)$, which is not self-evident, follows from (4.62). Observe that

$$D^2W(\mathrm{I}_3) = \alpha^2 D^2W^{\mathsf{FR}}(\alpha\mathrm{I}_3) \quad \text{and consequently} \quad \mathcal{Q}_2 = \alpha^2\mathcal{Q}_2^{\mathsf{FR}},$$

where W is given by (4.14). Let the functional $\mathscr{E}_0^{\mathrm{gel}} : \mathrm{W}^{1,2}(\Omega,\mathbb{R}^3) \to [0,+\infty]$ be defined by

$$\mathscr{E}_0^{\mathrm{gel}}(y) := \frac{1}{24}\int_\omega \mathcal{Q}_2^{\mathsf{FR}}\big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big)\,\mathrm{d}x' + \frac{1}{2}\int_\Omega \mathcal{Q}_2^{\mathsf{FR}}\big(\beta\, b(x)\mathrm{I}_2\big)\,\mathrm{d}x - \frac{1}{24}\int_\omega \mathcal{Q}_2^{\mathsf{FR}}\big(\mathcal{B}_{\mathrm{gel}}(x')\big)\,\mathrm{d}x', \tag{4.23}$$

for every $y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega,\mathbb{R}^3)$ and $+\infty$ elsewhere in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$. In the expression (4.23), $\mathcal{B}_{\mathrm{gel}}$ (the target curvature tensor) is given by

$$\mathcal{B}_{\mathrm{gel}} := 12\beta\int_{-1/2}^{1/2} t\, b(\cdot,t)\,\mathrm{d}t\,\mathrm{I}_2, \qquad \text{a.e. in } \omega \qquad \text{and we have} \qquad \mathcal{B}^1_{2\times 2} = \frac{1}{\alpha^2}\mathcal{B}_{\mathrm{gel}}. \tag{4.24}$$

Due to (4.15), Γ-convergence of the α-rescaled functionals $\frac{1}{(\alpha h)^2}\mathscr{E}_{\alpha h}$ and the above listed relations between α-rescaled and original quantities it follows that the sequence of functionals $\frac{1}{h^2}\mathscr{E}_h^{\mathrm{gel}}$ Γ-converges to $\mathscr{E}_0^{\mathrm{gel}}$, in the strong and the weak topology of $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$, as $h \to 0$.

Let us define a fourth-order tensor $\mathbb{C}$ by

$$\mathbb{C} := 2\mathbf{G}\,\mathbb{I}_2 + \mathbf{\Lambda}(\alpha)\mathrm{I}_2\otimes\mathrm{I}_2, \tag{4.25}$$

where $\mathbb{I}_2$ stands for the identity tensor of rank 4. From now on, we will use the following (equivalent) expression for the 2D energy functional $\mathscr{E}_0^{\mathrm{gel}}$, which is more appropriate from the point of view of applications. By the very definition of the tensor $\mathbb{C}$ it is clear that $\mathscr{E}_0^{\mathrm{gel}}$ can be written in the following equivalent form:

$$\mathscr{E}_0^{\mathrm{gel}}(y) = \frac{1}{24}\int_\omega \mathbb{C}\big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big) : \big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big)\,\mathrm{d}x' + \mathrm{ad.t.}, \quad y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega), \tag{4.26}$$

where ad.t. stands for “additional term” independent on y and given by

$$\mathrm{ad.t.} = \frac{1}{2}\int_\Omega \mathbb{C}\big(\beta\, b(x)\mathrm{I}_2\big) : \big(\beta\, b(x)\mathrm{I}_2\big)\,\mathrm{d}x - \frac{1}{24}\int_\omega \mathbb{C}\big(\mathcal{B}_{\mathrm{gel}}(x')\big) : \big(\mathcal{B}_{\mathrm{gel}}(x')\big)\,\mathrm{d}x'.$$

This non-negative term tells that the 2D energy, in general, cannot be minimized at zero: this fact originates in the incompatibility of the spontaneous strain distribution and the presence of residual stresses in the 3D reference configuration – it will be discussed in more details in Chapter 5. Clearly, the presence of such constant terms will be irrelevant for the further study of the energy minimizing maps.

As a direct consequence of the above rigorous derivation via Γ-convergence of the 2D model, we have the following:

Theorem 4.2.1. *Denote* $m_h := \inf_{W^{1,2}(\Omega,\mathbb{R}^3)} \mathscr{E}_h^{\mathrm{gel}}$ *and suppose that* $\{y^h\}_h \subseteq W^{1,2}(\Omega,\mathbb{R}^3)$ *is a low-energy sequence, i.e.*

$$\lim_{h\to 0} \frac{\mathscr{E}_h^{\mathrm{gel}}(y^h)}{h^2} = \lim_{h\to 0} \frac{m_h}{h^2}.$$

Then, up to a (not relabeled) subsequence, $y^h \to y$ *in* $W^{1,2}(\Omega,\mathbb{R}^3)$ *as* $h\to 0$*, where*

$$y \in W^{2,2}_{\alpha,\mathrm{iso}}(\omega) \quad \textit{solves} \quad m_0 = \min_{W^{2,2}_{\alpha,\mathrm{iso}}(\omega)} \mathscr{E}_0^{\mathrm{gel}}.$$

Moreover, $\dfrac{m_h}{h^2} \to m_0$*, as* $h \to 0$.

In terms of the physical quantities and the finite thickness h_0 (small with respect to the in-plane characteristic size of the plate), the following asymptotic approximate identity for the low-energy values $\overline{\mathscr{E}}_{h_0}^{\mathrm{gel}}(v^{h_0}) = \inf \overline{\mathscr{E}}_{h_0}^{\mathrm{gel}} + o(1)$:

$$\overline{\mathscr{E}}_{h_0}^{\mathrm{gel}}(v^{h_0}) \cong \frac{h_0^3}{24}\int_\omega \mathbb{C}\big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big) : \big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big)\, \mathrm{d}x' + h_0^3\, \mathrm{ad.t.}. \tag{4.27}$$

Hence, the minimizers of the 2D bending model (4.26) provide reliable estimates for the “almost minimal” values of the 3D energy given by (4.6) and (4.11).

4.3 Foldable structures made of hydrogel bilayer

4.3.1 Self-folding using bilayer gels

Let us call $\Omega_{h_0} = \omega \times (-h_0/2, h_0/2)$ the reference configuration of our foldable structure, where the reference thickness h_0 is much smaller than the in-plane dimensions. The planar domain ω, which represents the mid-plane of the plate, is hereafter assumed to be a union $\omega = \bigcup_{i=1}^n \omega_i$ of polygons. We say that the interfaces which delimit the polygons form a *pattern* on ω.

A thin sheet $\Omega_h = \omega \times (-h/2, h/2)$ with ideal small thickness h is supposed to be a heterogeneous system where the heterogeneity, both lateral and vertical, is due to a patterned bilayer structure made of hydrogels. This structure is engineered through an z-dependent density of cross-links in the polymer network, which in turn determines an z-dependent density $\overline{N}_h(z)$ of polymer chains that is of the form

$$\overline{N}_h(z) = \begin{cases} \overline{N} - \dfrac{h}{h_0} M_i, & \text{for } z \in \omega_i \times (-h/2, 0], \\[2mm] \overline{N} + \dfrac{h}{h_0} M_i, & \text{for } z \in \omega_i \times (0, h/2), \end{cases} \tag{4.28}$$

where $\overline{N}$ is the average density along the thickness and each M_i, for $i = 1, \ldots, n$, is a non-negative constant. This $\overline{N}_h$ corresponds to a particular choice of piecewise constant function b in (4.5). The system is a composite of flat faces (where $M_i = 0$) connected by hinges (where

$M_i > 0$). Starting from an initially dry state, folding will be accomplished by putting the gel in contact with a solvent. Upon swelling, the difference in $\overline{N}_h$ in the two layers of the hinges will induce bending. The prototype of this actuation mechanism is presented in Figure 4.1, where the larger (resp. lower) number of dots in the hinge (red patch) corresponds to a higher (resp. lower) cross-linking density in each layer.

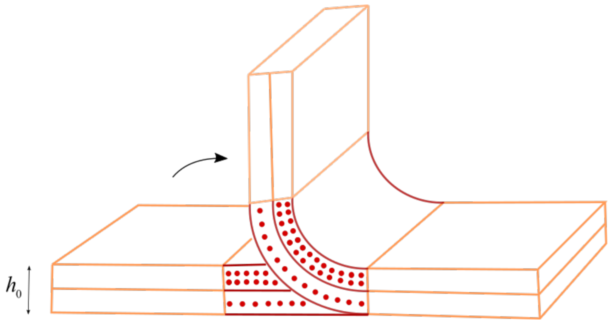

Figure 4.1: Sketch of the proposed folding mechanism. The hydrogel bilayer acts as a hinge that bends upon swelling. The number of red dots in each of the two layers is proportional to the amount of cross-links in the polymer network.

As a consequence of the "piecewise constant heterogeneity" in the density of polymer chains $\overline{N}_h$, for each $0 < h \ll 1$ the energy densities $\overline{W}_h^{\mathsf{FR}}$ given in (4.6) that models our system, satisfy (modulo rigid-body rotations)

$$\operatorname*{argmin}_{\mathbb{R}^{3\times 3}_1} \overline{W}_h^{\mathsf{FR}}(z,\cdot) = A_h(z) = \begin{cases} \alpha \mathrm{I}_3 - \dfrac{h}{h_0}\beta\, M_i \mathrm{I}_3 + o(h), & z \in \omega_i \times (-h/2, 0], \\ \alpha \mathrm{I}_3 + \dfrac{h}{h_0}\beta\, M_i \mathrm{I}_3 + o(h), & z \in \omega_i \times (0, h/2). \end{cases} \tag{4.29}$$

We recall that $\overline{W}_{h_0}^{\mathsf{FR}}$ represents the actual energy density corresponding to the finite (small) thickness system.

4.3.2 Minimal energy configurations: connection between shape parameters and gel properties

In this section, we study the minimizers of the limiting bending energy $\mathscr{E}_0^{\mathrm{gel}}$ given by (4.26). More precisely, we focus our attention on the class of *pointwise minimizers*, namely, on those deformations $y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega)$ whose second fundamental form Π_y minimizes pointwise the integrand in (4.26). Importantly, this analysis employs a simple structure from the fabrication viewpoint and provides the theoretical basis for successful and robust actuation using the proposed folding mechanism.

Recalling the relation between an isometry u on $\alpha\omega$ and an α-isometry y on ω, as well as the relation between their second fundamental forms given in formula (4.19), we have that

$$\begin{aligned} &\min_{y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega)} \int_\omega \mathbb{C}\big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big) : \big(\Pi_y(x') - \mathcal{B}_{\mathrm{gel}}(x')\big)\,\mathrm{d}x' \\ &= \min_{u \in \mathrm{W}^{2,2}_{\mathrm{iso}}(\alpha\omega)} \frac{1}{\alpha^2} \int_{\alpha\omega} \mathbb{C}\big(\alpha^2 \Pi_u(\eta') - \mathcal{B}_{\mathrm{gel}}(\eta'/\alpha)\big) : \big(\alpha^2 \Pi_u(\eta') - \mathcal{B}_{\mathrm{gel}}(\eta'/\alpha)\big)\,\mathrm{d}\eta'. \end{aligned} \tag{4.30}$$

Observe that the target curvature tensor $\mathcal{B}_{\rm gel}$ defined in (4.24) is in this case a piecewise constant map, which on each subdomain ω_i of ω reads as

$$\mathcal{B}_{\rm gel} = \overline{a}_i \, \mathrm{I}_2 \quad \text{with} \quad \overline{a}_i := \frac{3\beta}{h_0} M_i, \qquad i = 1, \dots n. \tag{4.31}$$

According to (4.30) and (4.31) and recalling that each isometry u is such that $\det \Pi_u = 0$ a.e. in $\alpha\omega$, our minimization problem reduces to finding those isometries u whose second fundamental form fulfills

$$\Pi_u(\eta') \in \mathcal{N}_i := \underset{\substack{S \in \mathrm{Sym}(2) \\ \det S = 0}}{\operatorname{argmin}} \, \mathbb{C}\big(\alpha^2 S - \overline{a}_i \, \mathrm{I}_2\big) : \big(\alpha^2 S - \overline{a}_i \, \mathrm{I}_2\big), \tag{4.32}$$

for a.e. η' in the i-th subdomain of $\alpha\omega$ and for every $i = 1, \dots, n$. A necessary and sufficient condition for (4.32) to hold is that Π_u is actually equal to a constant matrix $A_i \in \mathcal{N}_i$ on each i-th subdomain of $\alpha\omega$ (see Lemma 3.3.2).

The solution of the (finite dimensional) minimization problem in (4.32), as shown in Lemma 3.3.1 point (i), yields the following explicit representation for the set $\mathcal{N}_i$:

$$\mathcal{N}_i = \left\{ \frac{\kappa_i}{\alpha^2} \, \mathsf{n} \otimes \mathsf{n} : \; \mathsf{n} \in \mathbb{R}^2, \text{ with } |\mathsf{n}| = 1 \right\}, \quad \text{with} \quad \kappa_i := 2 \, \overline{a}_i \, \frac{\mathbf{G} + \mathbf{\Lambda}(\alpha)}{2\mathbf{G} + \mathbf{\Lambda}(\alpha)}, \tag{4.33}$$

for each $i = 1, \dots, n$. In other words, the second fundamental form Π_u of an energy minimizing isometry u, when restricted to the i-th subdomain of $\alpha\omega$ (that we refer to as the i-th patch) with $\kappa_i \neq 0$, corresponds to the second fundamental form of a *cylindrical surface* whose non-zero principal curvature equals κ_i/α^2, while the associated principal curvature direction might be (a-priori) given by any unit vector in $\mathbb{R}^2$.

However, the isometry constraint forces a precise choice of the principal curvature direction associated with the non-zero principal curvature κ_i/α^2 – it must be orthogonal to the interfaces between i-th patch and all the neighbouring ones. Equivalently, a cylindrical surface with $\kappa_i \neq 0$ can be glued to all the nighbouring cylindrical surface patches (or even plane patches) only if its rulings are parallel to the interface between them. It is thus clear that the existence of an isometry with the above described properties heavily depends on the compatibility between the pattern on ω and the target curvature, as proved in Theorem 3.3.5.

The theoretical results that we have just discussed provide the foundations for a successful self-folding strategy in patterned, bilayer thin sheets made of hydrogels (or other active materials). In particular, the planar domain ω is patterned in such a way that the heterogeneity in swelling of the gel due to variations in the cross-linking density induces piecewise constant target curvatures. Many interesting and feasible patterns on ω along with the induced target curvatures at the hinges satisfy the previously mentioned compatibility property, thus guaranteeing the existence of a pointwise minimizer of the 2D bending energy. To be concrete, we now consider the *pyramid*-type and the *cube*-type domains, sketched in the Figure 4.2 (A) and (B), respectively. Both of these two patterned domains consist of two different types of patches: *hinges* and *flat faces*. Each hinge, denoted by ω_{hinge}, has a nontrivial bilayer structure, characterized by

$$\overline{N}_h(z) = \begin{cases} \overline{N} - \dfrac{h}{h_0} M, & \text{for } z \in \omega_{\mathsf{hinge}} \times (-h/2, 0], \\[2ex] \overline{N} + \dfrac{h}{h_0} M, & \text{for } z \in \omega_{\mathsf{hinge}} \times (0, h/2), \end{cases} \tag{4.34}$$

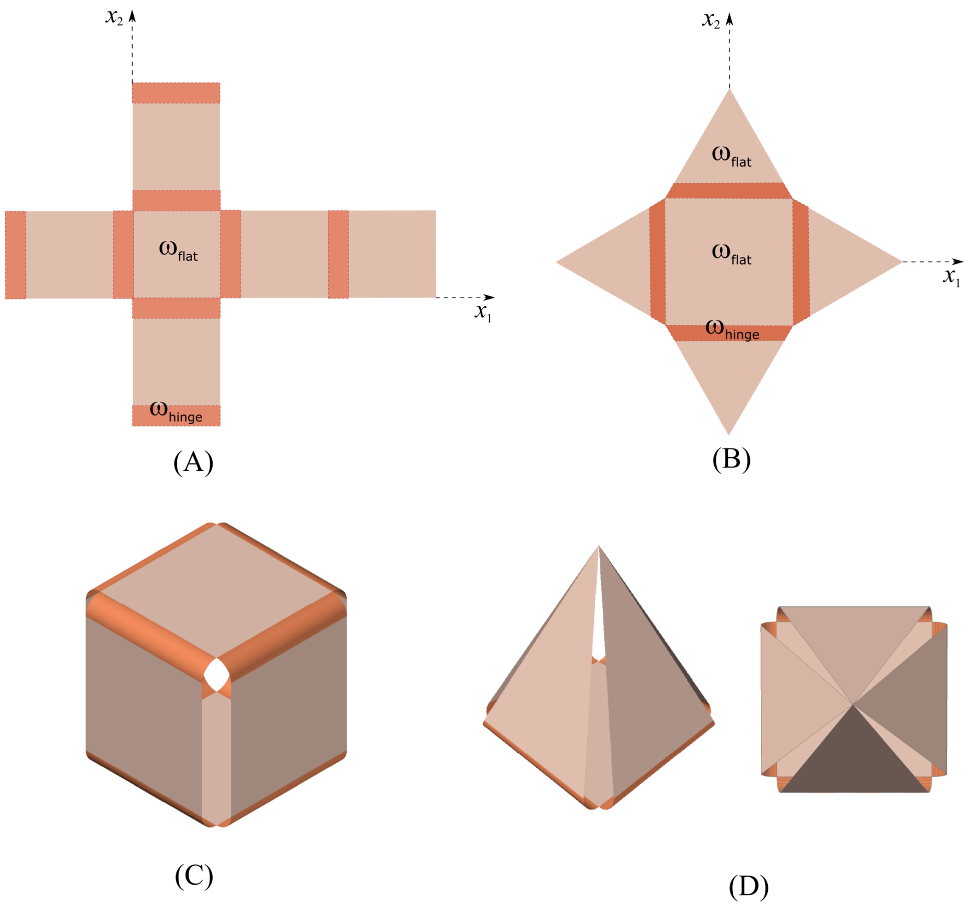

Figure 4.2: Cube-type domain (A) and pyramid-type domain (B) in the unfolded flat state. Folded cube shape (C) and pyramid shape (D) obtained as images of the corresponding patterned domains (A) and (B) under the pointwise minimizing deformation of the corresponding 2D bending energies.

with $M > 0$. On each flat face, denoted by ω_{flat}, the density of polymer chains N_h is constantly equal to $\overline{N}$, i.e. $M = 0$. Note that both in the pyramid and the cube case the interfaces between an ω_{hinge} and the (at most two) neighbouring ω_{flat} are mutually parallel (recall that the set of the nonzero principal curvature directions of a pointwise minimizer restricted to ω_{hinge} may be any unit vector of $\mathbb{R}^2$, see (4.33)). Hence the existence of a pointwise minimizer of $\mathscr{E}_0^{\mathrm{gel}}$ is guaranteed. More in details, such minimizer deforms the mid-plane ω in the following way: it dilates the domain ω by the factor α and maps each α-dilated hinge into a cylindrical surface with radius $r = \alpha^2/|\kappa|$, where

$$\kappa = \frac{6\beta M}{h_0} \frac{\mathrm{v}\overline{N} + \mathbf{\Lambda}(\alpha)}{2\mathrm{v}\overline{N} + \mathbf{\Lambda}(\alpha)}; \tag{4.35}$$

the cylinder's rulings are parallel to the interfaces which delimit the hinge from the flat faces. At the same time, each α-dilated flat face remains flat.

Looking at the expression (4.35) for the curvature κ that arises on hinges, it is clear that, maintaining the other physical constants fixed, an appropriate choice of the values $\overline{N}$ and M (recall that the $\overline{N}$- dependence is present in the term $\mathbf{\Lambda}(\alpha)$ as well) of the bilayer's structure at the hinges is needed to induce precise self-folding as in Figure 4.2 (C) and (D). We address the problem of designing the structure of each bilayer to produce the desired shape upon folding in the following section.

4.3.3 Bilayer design problem

In this section we address the "inverse" problem of finding the physical properties of a hydorgel that allow us to realize a given target shape upon self-folding. First, we introduce an approximate variant of the Flory-Rehner energies $\overline{W}_h^{\mathsf{FR}}$, for which the dependence of the target curvature $\mathcal{B}_{\rm gel}$ on the fixed physical parameters of the 3D system becomes explicit. Further, we restrict our attention to gels characterized by

- $\mu = 0$
- v, $\overline{N}$ and χ satisfy the relation $\overline{N} < \frac{1-2\chi}{2\mathrm{v}}$.

The first of these two conditions means that the gel is in contact with a pure liquid in equilibrium with its own vapor. The second condition is always satisfied whenever a hydrogel with a large swelling ratio is considered, as it occurs for the typical values $\overline{N}\mathrm{v} \sim 10^{-3}$ and $\chi \sim 10^{-1}$.

To present the approximate model we work with, let us go back to the general setting of a thin gel sheet occupying the domain $\Omega_h = \omega \times (-h/2, h/2)$ with $\omega = \bigcup_{i=1}^n \omega_i$ being a general patchwork of polygons. First of all, in the case $\mu = 0$ the Flory-Rehner energy density (4.6) reduces to

$$\overline{W}_h^{\mathsf{FR}}(z, F) = \frac{\mathrm{v}\overline{N}_h(z)}{2}\big(|F|^2 - 3\big) + W_{vol}^{\chi}(\det F), \qquad z \in \Omega_h \text{ and } F \in \mathbb{R}_1^{3\times 3}.$$

Then, by Taylor's expansion, $W_{vol}^{\chi}(t) = \frac{1-2\chi}{2t} + o\left(\frac{1}{t}\right) + \chi - 1$, for every $t \gg 1$. Thus, when deformation gradients with large determinants are considered, one may approximate $\overline{W}_h^{\mathsf{FR}}$ (discarding the constant term $\chi - 1$) by the function $\widehat{W}_h^{\mathsf{FR}}$

$$\widehat{W}_h^{\mathsf{FR}}(z, F) := \frac{\mathrm{v}\overline{N}_h(z)}{2}\big(|F|^2 - 3\big) + \frac{1-2\chi}{2\det F}, \qquad z \in \Omega_h \text{ and } F \in \mathbb{R}_1^{3\times 3}. \tag{4.36}$$

This approximate model is adequate from the point of view of applications. It has been considered for the first time in [Doi09], and afterward used, for instance, in [LNS14, DS11].

The functional form of the energy densities $\widehat{W}_h^{\mathsf{FR}}$ allow to determine the spontaneous stretch distribution (and therefore the set of minimizers) explicitly in terms of the fixed physical parameters of the model. Indeed, by Lemma 4.4.4 and Remark 4.4.5 in Section 4.4, for every $z \in \omega_i \times (-h/2, h/2)$, $i = 1, \dots, n$, it holds

$$\operatorname*{argmin}_{\mathbb{R}_1^{3\times 3}} \widehat{W}_h^{\mathsf{FR}}(z, \cdot) = \mathrm{SO}(3)\widehat{A}_h(z) \tag{4.37}$$

with $\widehat{A}_h(z)$, $z \in \Omega_h$, being (as in (4.29)) the scalar multiple of the identity matrix:

$$\widehat{A}_h(z) = \begin{cases} \alpha \mathrm{I}_3 - \dfrac{h}{h_0}\beta\, M_i \mathrm{I}_3 + o(h), & z \in \omega_i \times (-h/2, 0], \\[2ex] \alpha \mathrm{I}_3 + \dfrac{h}{h_0}\beta\, M_i \mathrm{I}_3 + o(h), & z \in \omega_i \times (0, h/2), \end{cases} \tag{4.38}$$

where the constants α and β are explicitly given by

$$\alpha = \left(\frac{1-2\chi}{2\mathrm{v}\overline{N}}\right)^{1/5} \qquad \text{and} \qquad \beta = -\frac{1}{5\overline{N}}\left(\frac{1-2\chi}{2\mathrm{v}\overline{N}}\right)^{1/5}. \tag{4.39}$$

In this case, the constant $\mathbf{\Lambda}(\alpha)$ appearing in (4.22) can be explicitly derived, as well. Thus, we obtain that the tensor $\mathbb{C}$ in (4.26), within this approximate 2D model (obtained via an analogous procedure presented in Section 4.2 for the general case, taking into account Remark 4.4.10 below), reads

$$\mathbb{C} = 2\mathrm{v}\overline{N}\Big(\mathbb{I}_2 + \frac{1}{3}\mathrm{I}_2 \otimes \mathrm{I}_2\Big).$$

Finally, as a consequence of (4.39), the target curvature $\mathcal{B}_{\mathrm{gel}}$ is given on each subdomain ω_i of ω by

$$\mathcal{B}_{\mathrm{gel}} = \bar{a}_i\,\mathrm{I}_2, \quad \text{with} \quad \bar{a}_i := -\frac{3}{5\overline{N}}\left(\frac{1-2\chi}{2\mathrm{v}\overline{N}}\right)^{1/5}\frac{M_i}{h_0}, \qquad i = 1,\dots,n. \tag{4.40}$$

Summarizing, the total energy (4.26) in this case is

$$\mathscr{E}_0^{\mathrm{gel}}(y) = \frac{\mathrm{v}\overline{N}}{12}\sum_{i=1}^{n}\int_{\omega_i} |\Pi_y(x') - \bar{a}_i\mathrm{I}_2|^2 + \frac{1}{3}\mathrm{tr}^2\big(\Pi_y(x') - \bar{a}_i\mathrm{I}_2\big)\,\mathrm{d}x' + \text{ad.t.}, \tag{4.41}$$

for every $y \in \mathrm{W}^{2,2}_{\alpha,\mathrm{iso}}(\omega)$. In particular, whenever a compatible pattern is considered, from the discussion at the end of Section 4.3.2 and from formula (4.33), it follows that the isometry u associated with a pointwise minimizer y of $\mathscr{E}_0^{\mathrm{gel}}$ via (4.19), maps each i-th subdomain of $\alpha\omega$ into a cylindrical surface of radius

$$r_i := \frac{4\,\alpha\,h_0}{3M_i} = \frac{4\,h_0}{3M_i}\left(\frac{1-2\chi}{2\mathrm{v}\overline{N}}\right)^{1/5} \tag{4.42}$$

and with rulings parallel to the interface with each neighbouring patch.

We are now ready to design a bilayer structure on each ω_{hinge} (i.e. to determine the values of $\overline{N}$ and M on such subdomain, see (4.34)) in order to induce precise self-folding of a pyramid and a cube (Figure 4.2 (C) and (D)), at low energy cost.

Pyramid The target shape that we refer to as a (precisely) *folded pyramid* is characterized by the following set of parameters (see Figure 4.3):

- ℓ – initial length of the pyramid basis,
- $\phi \in (0, \frac{\pi}{2})$ – vertex angle,
- H – height of the pyramid,
- $\alpha > 1$ – in-plane swelling factor.

By 'precisely folded pyramid' we mean that the four external vertices of the unfolded pyramid meet at a single point in the folded configuration. The above four parameters completely determine a (unique) 'precisely folded pyramid' shape, since from them one can derive

- r – radius of curvature of the deformed (α-dilated) hinges,
- ℓ_1 – initial hinge width,
- ℓ_2 – initial height of the pyramid side.

The following relations hold among these geometric paramters and the former set of parameters:

$$r = \frac{\sin\phi}{1+\sin\phi}\left(H - \frac{\alpha\,\ell}{2\text{tg}\,\phi}\right), \qquad \ell_1 = \frac{r}{\alpha}\left(\frac{\pi}{2}+\phi\right), \qquad \ell_2 = \frac{H - r(1+\sin\phi)}{\alpha\cos\phi}.$$

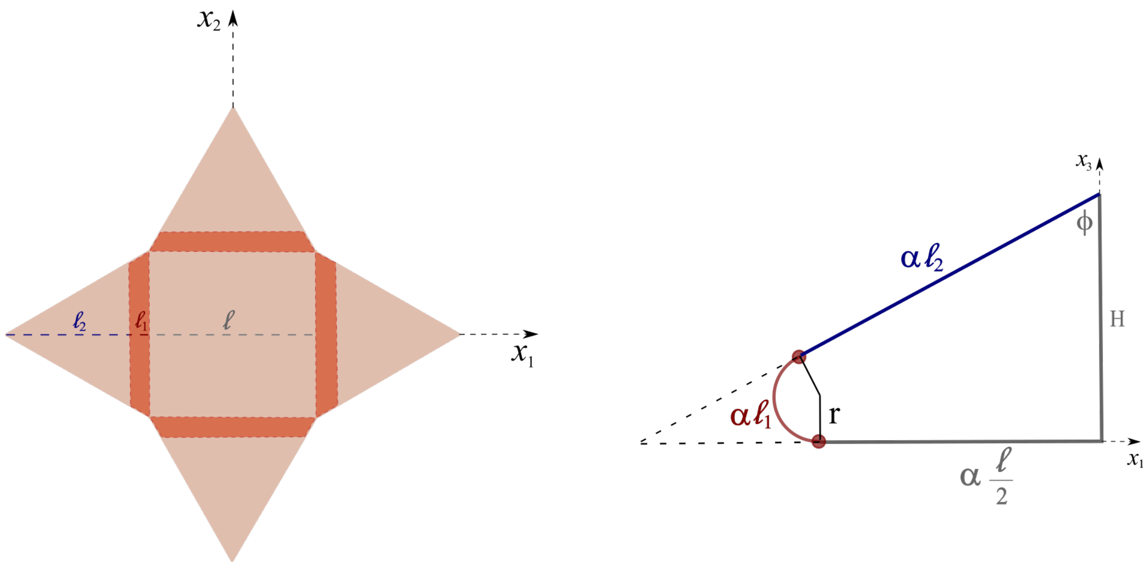

Figure 4.3: Unfolded pyramid (left) and geometry of the precisely folded pyramid (right).

By using the first formula in (4.39) and the expression (4.42), we deduce that the correct material properties ($\overline{N}$ and M) to be imprinted on the hinges in order to accomplish the desired self-folding are determined in terms of the swelling factor α and the curvature radius r by

$$\overline{N} = \frac{1-2\chi}{2\text{v}\alpha^5} \quad \text{and} \quad M = \frac{4\alpha h_0}{3r}. \tag{4.43}$$

Cube We consider as the target shape the (precisely) *folded cube* characterized by the following parameters (see Figure 4.4):

- ℓ_1 – initial hinge width,
- $\alpha > 1$ – swelling factor,
- $\phi = \pi/2$ – hinge closing angle.

The radius r of a cylindrical surface representing a deformed hinge, which is needed to obtain the desired folded cube from a flat, cube-type pattern, must satisfy the relation $r = \frac{2\alpha\ell_1}{\pi}$. As in the pyramid case, formulas (4.39) and (4.42) allows us to design the hinges in order to get a (precisely) folded cube. Namely, the material parameters that determine the cross-linking density in the bilayers that constitute the hinges are

$$\overline{N} = \frac{1-2\chi}{2\text{v}\alpha^5} \quad \text{and} \quad M = \frac{4\alpha h_0}{3r} = \frac{2\pi h_0}{3\ell_1}. \tag{4.44}$$

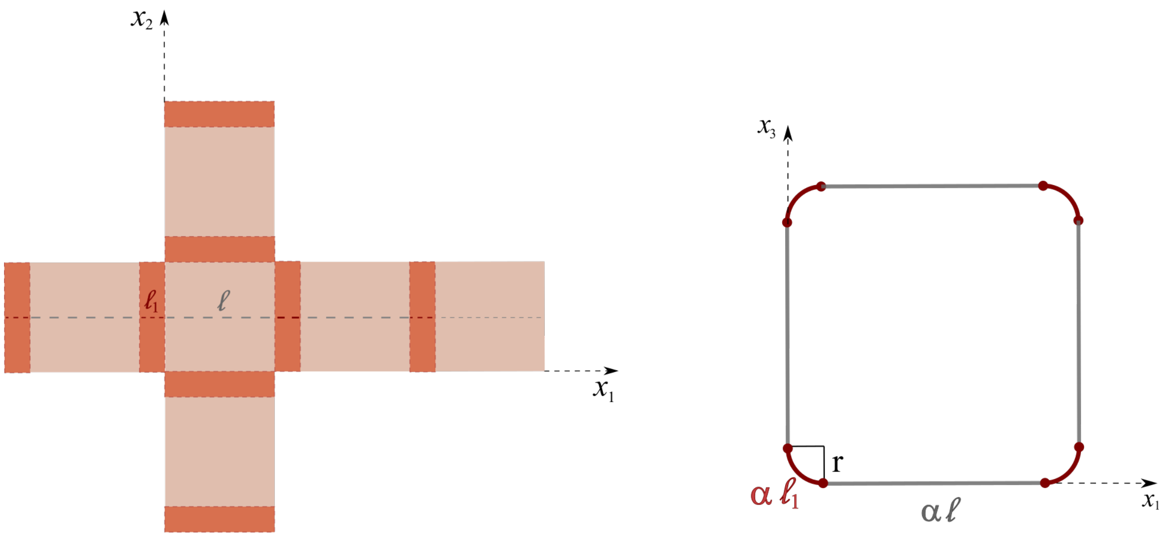

Figure 4.4: Unfolded cube on the left, geometry of the precisely folded cube on the right.

As it is clear from the above results, the final "size" of a folded cube or pyramid is controlled by the parameter $\overline{N}$ through its relation with the swelling factor α. Specifically, in order to produce larger folded cubes or pyramids, which correspond to a bigger value of α, a smaller value of $\overline{N}$ is needed. The curvature of the hinges $1/r$ is controlled by the parameter M, in such a way that, as one might expect, the value of r increases as the value of M decreases. This means that, on one side, flat plates (seen as cylindrical surfaces with radius $r = +\infty$) correspond to the isometric deformation of flat faces with no through-the-thickness variation of the cross linking density, i.e. $M = 0$. On the other side, shapes with sharp folds (i.e. with $r \to 0$) require large variations M in the cross-linking density along the bilayer that constitutes the hinge. In turn, since r and ℓ_1 are proportional, manufacturing a folded structure with a small hinge width ℓ_1 requires a large value of M.

4.4 Analysis of energy density functions of Flory-Rehner type

This section is dedicated to the analysis of the Flory-Rehenr energy density functions $\overline{W}_h^{\mathsf{FR}}$ introduced in (4.6). Let us recall that, given a small thickness parameter $0 < h \ll 1$,

$$\overline{W}_h^{\mathsf{FR}}(z, F) = \frac{\mathrm{v}\overline{N}_h(z)}{2}(|F|^2 - 3) + W_{vol}^{\chi}(\det F) - \frac{\mu}{k_{\mathsf{B}}\mathrm{T}}(\det F - 1),$$

for a.e. $z \in \Omega_h$ and $F \in \mathbb{R}^{3\times 3}_1$. For the forthcoming analysis we will use the following normalized representation of the density function $\overline{N}_h$:

$$\overline{N}_h(z) = \overline{N}\, f_h\left(z', \frac{z_3}{h}\right), \quad \text{where } f_h\left(z', \frac{z_3}{h}\right) := 1 + h\,\frac{b\left(z', \frac{z_3}{h}\right)}{\overline{N}}, \text{ for a.e. } z \in \Omega_h. \tag{4.45}$$

From the very definition of f_h, we have that $f_h \to 1$ in $\mathrm{L}^\infty(\Omega)$.

Now, set $\mathbb{R}^{3\times 3}_+ := \{F \in \mathbb{R}^{3\times 3} : \det F > 0\}$. By the standard algebraic inequality, one has for every $F \in \mathbb{R}^{3\times 3}_+$ that

$$|F|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \geq 3\big(\lambda_1^2\lambda_2^2\lambda_3^2\big)^{1/3} = 3\big(\det(F^\mathsf{T}F)\big)^{1/3} = 3(\det F)^{2/3}, \tag{4.46}$$

where $\lambda_1^2, \lambda_2^2, \lambda_3^2$ are the eigenvalues of the matrix $F^\mathsf{T}F$. Recall that the equality in (4.46) holds if and only if $\lambda_1^2 = \lambda_2^2 = \lambda_3^2$.

The inequality (4.46) allows us to switch the analysis of the energy densities defined in (4.6) to the analysis of the function $p : (0, +\infty) \times [1, +\infty) \to \mathbb{R}$ defined by

$$p(\theta, t) := \frac{3\mathrm{v}\overline{N}\theta}{2}\left(t^{2/3} - 1\right) + W^{\chi}_{vol}(t) - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}}(t-1) \tag{4.47}$$

for every $\theta \in (0, +\infty)$ and every $t \in [1, +\infty)$. Note that $p \in C^{\infty}\big((0, +\infty) \times (1, +\infty)\big)$ and that $p(\theta, \cdot) \in C\big([1, +\infty)\big)$ for every $\theta \in (0, +\infty)$.

In particular, observe that from (4.46) it follows that

$$\overline{W}^{\mathsf{FR}}_h\big((x', hx_3), F\big) \geq p\big(f_h(x), \det F\big), \qquad x \in \Omega.$$

This inequality, and the related rigidity characterizing the equality case, says in particular that $F_{\min}$ is a minimizer of $\overline{W}^{\mathsf{FR}}_h\big((x', hx_3), \cdot\big)$ if and only if $F_{\min} = t^{1/3}_{\min} R$, for some $R \in \mathrm{SO}(3)$, where $t_{\min}$ is a minimizer of $p\big(f_h(x), \cdot\big)$, $x \in \Omega$. This argument is detailed in the proof of Lemma 4.4.4 below.

In view of the previous discussion and since $f_h(x)$ is uniformly close to 1 for every h sufficiently small, in what follows we analyze the minimizers of $p(\theta, \cdot)$, for θ varying in some interval $\mathcal{I}$ containing 1. To do this, it is useful to recall the following sharp logarithmic estimate (which can be found in [Top06])

$$\log(1+t) \geq \frac{t^2 + 2t}{2(1+t)}, \qquad \text{for every } -1 < t \leq 0. \tag{4.48}$$

This can be checked just observing that the function defined as the left-hand side minus the right-hand side of (4.48) is null at zero and has a nonpositive derivative. We also recall the following 1-dimensional real analysis result, which will be useful as well.

Lemma 4.4.1. *Let $\mathcal{I} \subseteq \mathbb{R}$ be an open interval. Let $f \in C^1(\mathcal{I})$ and assume that*

$$f'(t) = 0,\, t \in \mathcal{I} \quad \Longrightarrow \quad \exists\, f''(t) > 0.$$

Then the function f has at most one stationary point, which is a global minimum.

Proof. First, we show the following:
CLAIM: For every $t_1, t_2 \in I$, $t_1 < t_2$, such that $f'(t_1) = f'(t_2) = 0$ there exists $t_0 \in (t_1, t_2)$ such that $f'(t_0) = 0$.

Indeed, since $f''(t_1)$ and $f''(t_2)$ are strictly positive, there exists t_1' in the right neighbourhood of t_1 such that $f'(t_1') > 0$ and t_2' in the left neighbourhood of t_2 such that $f'(t_2') < 0$. Hence there exist $t_0 \in (t_1', t_2') \subseteq (t_1, t_2)$ such that $f'(t_0) = 0$, by continuity of f'.

To prove Lemma, suppose by contradiction that there exist $t_1, t_2 \in \mathcal{I}$, $t_1 < t_2$ such that $f'(t_1) = f'(t_2) = 0$. By CLAIM we have that there exists $\tau_0 \in (t_1, t_2)$ such that $f'(\tau_0) = 0$. But there exists also $\tau_1 \in (t_1, \tau_0)$ such that $f'(\tau_1) = 0$, again by CLAIM. In this way we construct a decreasing sequence $\{\tau_k\}_{k \geq 0} \subseteq (t_1, t_2)$. Let $\tau = \lim_{k \to +\infty} \tau_k$. Consequently $f'(\tau) = 0$ and $f''(\tau) > 0$. On the other hand

$$f''(\tau) = \lim_{k \to +\infty} \frac{f'(\tau_k) - f'(\tau)}{\tau_k - \tau} = 0,$$

which is a contradiction. Therefore, if exists, stationary point t of f is unique and by hypothesis we have that $f''(t) > 0$, so it is a point of a global minimum. □

With the following lemma we show that $p(\theta, \cdot)$ has a unique global minimum in $[1, +\infty)$.

Lemma 4.4.2. *There exists an open neighbourhood* $\mathcal{I}$ *of* 1 *and a unique smooth function* $\varphi : \mathcal{I} \to (1, +\infty)$ *such that*

$$\operatorname*{argmin}_{t\in[1,+\infty)} p(\theta, t) = \varphi(\theta), \qquad \textit{for every } \theta \in \mathcal{I}.$$

Proof. Denote by $\overline{p} := p(1, \cdot)$. First, note that the derivative of $\overline{p}$ reads as

$$\overline{p}'(t) = \frac{\mathrm{v}N}{t^{1/3}} + \frac{d}{dt} W^{\chi}_{vol}(t) - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}} = \frac{\mathrm{v}N}{t^{1/3}} + \log\left(1 - \frac{1}{t}\right) + \frac{1}{t} + \frac{\chi}{t^2} - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}}, \qquad \text{for every } t > 1.$$

Let $h(t) := \log\left(\frac{t}{t-1}\right) - \frac{1}{t} - \frac{1}{2t^2}$ for every $t \in (1, +\infty)$. Note that $h(t) \geq 0$ for every $t \in (1, +\infty)$ and that $t^2 h(t) \to 0$ as $t \to +\infty$. It is straightforward to check that $\lim_{t\to 1} \overline{p}'(t) = -\infty$ and $\lim_{t\to+\infty} \overline{p}'(t) = -\frac{\mu}{k_{\mathrm{B}}\mathrm{T}} \geq 0$. Moreover, one has that

$$\overline{p}'(t) \geq 0 \quad \Leftrightarrow \quad t^2\overline{p}'(t) \geq 0 \quad \Leftrightarrow \quad \mathrm{v}N t^{5/3} - (1/2 - \chi) - t^2 h(t) - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}} t^2 \geq 0.$$

Since $\lim_{t\to+\infty} \mathrm{v}N t^{5/3} - (1/2-\chi) - t^2 h(t) - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}} t^2 = +\infty$, the existence of $t \in (1, +\infty)$ such that $\overline{p}'(t) = 0$ is guaranteed. The uniqueness of such $t \in (1, +\infty)$ follows from Lemma 4.4.1 above. More precisely, by using (4.48), one finds that $(t^2\overline{p}'(t))' > 0$ for every $t \in (1, +\infty)$. Since $(t^2\overline{p}'(t))' = 2t\overline{p}'(t) + t^2\overline{p}''(t)$, we get that $\overline{p}'(t) = 0$ implies $\overline{p}''(t) > 0$. Therefore there is a unique stationary point $t_{\min} \in (1, +\infty)$ and it is the point of the global minimum of $\overline{p}$. Recall that

$$0 = \overline{p}'(t_{\min}) = p_t(1, t_{\min}) \quad \text{and} \quad 0 < \overline{p}''(t_{\min}) = p_{tt}(1, t_{\min}).$$

By applying the Implicit Function Theorem we get that there exist an open neighbourhood $\mathcal{I} \subseteq (0, +\infty)$ of 1, an open neighbourhood $\mathcal{J} \subseteq (1, +\infty)$ of t_1 and a unique smooth function $\varphi : \mathcal{I} \to \mathcal{J}$, such that

$$\Big\{\big(\theta, \varphi(\theta)\big) : \theta \in \mathcal{I}\Big\} = \big\{(\theta, t) \in \mathcal{I} \times \mathcal{J} : p_t(\theta, t) = 0\big\}.$$

It easy to check (by using the same argument as in the case of $\overline{p}$ above) that $p_t\big(\theta, \varphi(\theta)\big) = 0$ implies $p_{tt}\big(\theta, \varphi(\theta)\big) > 0$ for every $\theta \in \mathcal{I}$. Hence we obtained a smooth map

$$(0, +\infty) \supseteq \mathcal{I} \ni \theta \mapsto \varphi(\theta) = \operatorname*{argmin}_{[1,+\infty)} p(\theta, \cdot) \in (1, +\infty). \tag{4.49}$$

□

Remark 4.4.3. We recall that the derivative of the function φ defined in (4.49) is given by

$$\varphi'(\theta) = -p_{t\theta}\big(\theta, \varphi(\theta)\big)\left(p_{tt}\big(\theta, \varphi(\theta)\big)\right)^{-1}, \qquad \text{for every } \theta \in \mathcal{I}. \tag{4.50}$$

In particular, by direct computations and by using the fact that $p_{tt}\big(\theta, \varphi(\theta)\big) > 0$ for every $\theta \in \mathcal{I}$, we have that $\varphi'(\theta) < 0$ for all $\theta \in \mathcal{I}$. ■

Hereafter, we let $\mathcal{I} \subseteq \mathbb{R}$ and the function φ be given by Lemma 4.4.2. For the sake of brevity, let us introduce the following notation. Given $\theta \in \mathcal{I}$ we denote

$$\alpha_\theta := \sqrt[3]{\varphi(\theta)} > 1 \qquad \text{and we set} \quad \alpha := \alpha_1. \tag{4.51}$$

For every $\theta \in \mathcal{I}$, we define the function $W_\theta : \mathbb{R}^{3\times3} \to \mathbb{R} \cup \{+\infty\}$ by

$$W_\theta(F) := \frac{\mathrm{v}N\theta}{2}\big(|F|^2 - 3\big) + W^{\chi}_{vol}(\det F) - \frac{\mu}{k_{\mathrm{B}}\mathrm{T}}(\det F - 1), \qquad \text{for every } F \in \mathbb{R}^{3\times3}_1, \tag{4.52}$$

and declare it to be equal to $+\infty$ elsewhere in $\mathbb{R}^{3\times3}$. Observe that $W_1 = W^{\mathsf{FR}}$ on $\mathbb{R}^{3\times3}_1$. Moreover, there exists $\bar{h} > 0$ such that

$$\overline{W}^{\mathsf{FR}}_h(z, F) = W_{f_h(z', \frac{z_3}{h})}(F), \qquad \text{for a.e. } z \in \Omega_h, \text{ every } F \in \mathbb{R}^{3\times3} \text{ and every } h \le \bar{h}. \tag{4.53}$$

We now show that the family of 3D energy densities $\{\overline{W}^{\mathsf{FR}}_h\}_h$ enjoys the properties announced in Section 4.1. The very same arguments used in the analysis of the original densities $\overline{W}^{\mathsf{FR}}_h$ below can be employed in the study of the properties of the approximate densities $\widehat{W}^{\mathsf{FR}}_h$ that we use in Subsection 4.3.3 (we shall indicate some important points in Remark 4.4.5 and Remark 4.4.10).

Regularity. For every $\theta \in \mathcal{I}$ the function W_θ is of class C^∞ on $\{F \in \mathbb{R}^{3\times3} : \det F > 1\}$ and continuous on $\mathbb{R}^{3\times3}$. Then, by (4.53), for a.e. $z \in \Omega_h$ the function $\overline{W}^{\mathsf{FR}}_h(z, \cdot)$ shares the same regularity properties. Furthermore, the function $\overline{W}^{\mathsf{FR}}_h(\cdot, F)$ is measurable for every $F \in \mathbb{R}^{3\times3}$, by its very definition. In particular, $\overline{W}^{\mathsf{FR}}_h$ is a Carathéodory function and thus jointly Borel (see, for instance, [AB99]).

Energy wells.

Lemma 4.4.4. *For every $\theta \in \mathcal{I}$ the function W_θ is minimized precisely on the set $\alpha_\theta\,\mathrm{SO}(3)$.*

Proof. Fix $\theta \in \mathcal{I}$ and denote $m_\theta := \min_{[1,+\infty)} p(\theta, \cdot) = p(\theta, \alpha_\theta^3)$. It follows by inequality in (4.46) that $W_\theta(F) - m_\theta \ge p(\theta, \det F) - m_\theta \ge 0$ for every $F \in \mathbb{R}^{3\times3}_1$. Suppose that $W_\theta(F) - m_\theta = 0$ for some $F \in \mathbb{R}^{3\times3}_1$. By the definition of p given by (4.47), F must satisfy $|F|^2 = 3(\det F)^{2/3}$. This implies that all eigenvalues of the positive symmetric matrix $F^\top F$ are equal to some $\lambda^2 \in \mathbb{R} \setminus \{0\}$, by (4.46), and accordingly $F \in \lambda\mathrm{SO}(3)$. On the other hand, it must be satisfied $\lambda^3 = \det F = \alpha_\theta^3$ (by uniqueness of the minimum point of the function $p(\theta, \cdot)$), implying that $\lambda = \alpha_\theta$. Conversely, let $F = \alpha_\theta R$ for some $R \in \mathrm{SO}(3)$. Then $|F|^2 = 3\alpha_\theta^2 = 3(\alpha_\theta^3)^{2/3} = 3(\det F)^{2/3}$ and thus $W_\theta(F) - m_\theta = 0$. □

The above lemma, translated in the language of Flory-Rehner energies, says that

$$\overline{W}^{\mathsf{FR}}_h(z, \cdot) \quad \text{attains its minimum precisely on} \quad \alpha_{f_h(z', \frac{z_3}{h})}\mathrm{SO}(3). \tag{4.54}$$

It only remains to understand the form of $\alpha_{f_h(z', \frac{z_3}{h})}$ for our particular choice of $f_h(z', \frac{z_3}{h})$ given in (4.45). By Taylor's expansion of the function φ around 1, we have, as claimed in (4.8), that

$$\alpha_{f_h(z', \frac{z_3}{h})} = \sqrt[3]{\varphi\left(1 + h\,\frac{b(z', \frac{z_3}{h})}{\overline{N}}\right)} = \alpha + h\,\beta b\Big(z', \frac{z_3}{h}\Big) + o(h), \qquad \text{with } \beta := \frac{\varphi'(1)}{3\overline{N}}. \tag{4.55}$$

This gives the energy well $A_h(x', hx_3)\mathrm{SO}(3)$ defined in (4.7)–(4.8). Let us stress the fact that a crucial property of A_h which allows us to use Theorem 3.2.2 in Chapter 3 is that such spontaneous stretch field has the structure $A_h(x', hx_3) = \alpha\mathrm{I}_3 + hC(x) + o(h)$, with $\int_{-1/2}^{1/2} C(x', x_3)\,\mathrm{d}x_3 = 0$. Note that this fact is connected with the structure of the chosen density $\overline{N}_h$ of polymer chains (see (4.5) and the subsequent sentence).

Remark 4.4.5. The above analysis of the energy wells can be applied also to the approximate version of Flory-Rehner energy densities given by the family of functions $\widehat{W}^{\mathsf{FR}}_h$ introduced in Section 4.3.3 (see (4.36)). In this case, the auxiliary functions p (given in (4.47)) and W_θ (given in (4.52)) might be replaced by the function $\hat{p}$ and $\widehat{W}_\theta$, respectively, where

$$\hat{p}(\theta, t) := \frac{3\mathrm{v}\overline{N}\theta}{2}\big(t^{2/3} - 1\big) + \frac{1 - 2\chi}{2t} \quad \text{and} \quad \widehat{W}_\theta(F) := \frac{\mathrm{v}\overline{N}\theta}{2}\big(|F|^2 - 1\big) + \frac{1 - 2\chi}{2\det F}, \tag{4.56}$$

for $t \in [1, +\infty)$, $\theta \in (0, +\infty)$ and $F \in \mathbb{R}^{3\times 3}_1$. The set of minimizers of $\hat{p}(\theta, \cdot)$ can be explicitly found, by directly computing the first and the second derivative of $\hat{p}(\theta, \cdot)$. Namely, we have that

$$\operatorname*{argmin}_{t\in[1,+\infty)} \hat{p}(\theta, t) = \left(\frac{1-2\chi}{2\mathrm{v}\overline{N}\theta}\right)^{3/5}, \qquad \text{for every } \theta \in (0, +\infty).$$

Then, by applying the same argument as in the proof of Lemma 4.4.4 to $\widehat{W}_\theta$ and $\hat{p}$ in place of W_θ and p, we can prove that

$$\operatorname*{argmin}_{\mathbb{R}^{3\times 3}_1} \widehat{W}_\theta = \left(\frac{1-2\chi}{2\mathrm{v}\overline{N}\theta}\right)^{1/5} \mathrm{SO}(3). \tag{4.57}$$

By observing also that $\widehat{W}^{\mathsf{FR}}_h(z, \cdot) = \widehat{W}_{f_h(z', \frac{z_3}{h})}$ and by using Taylor's expansion, (4.37) – (4.38) follows. ■

In order to prove uniform convergence and quadratic growth of W_θ, we should compute its first and second differentials. Observe that the convergence and growth properties of $\overline{W}^{\mathsf{FR}}_h$, announced in (4.9) and (4.10), respectively, will follow by using the identification $\overline{W}^{\mathsf{FR}}_h(z, F) = W_{f_h(z', \frac{z_3}{h})}(F)$ and the fact that $f_h \to 1$ in $\mathrm{L}^\infty(\Omega)$ as $h \to 0$.

By straightforward computations, for all $F \in \mathbb{R}^{3\times 3}$ with $\det F > 1$ and $M, N \in \mathbb{R}^{3\times 3}$, we have that

$$DW_\theta(F)[M] = \mathrm{v}\overline{N}\theta F : M + \left(1 - \det F \log\left(\frac{\det F}{\det F - 1}\right) + \frac{\chi}{\det F} - \frac{\mu}{kT}\det F\right) F^{-\mathsf{T}} : M, \tag{4.58}$$

and

$$\begin{aligned} D^2W_\theta(F)[M, N] = {}& \mathrm{v}\overline{N}\theta N : M \\ &+ \left(-1 + \det F \log\left(\frac{\det F}{\det F - 1}\right) - \frac{\chi}{\det F} + \frac{\mu}{kT}\det F\right) F^{-\mathsf{T}} N^{\mathsf{T}} F^{-\mathsf{T}} : M \\ &+ \left(-\det F \log\left(\frac{\det F}{\det F - 1}\right) + \frac{\det F}{\det F - 1} - \frac{\chi}{\det F} - \frac{\mu}{kT}\det F\right)(F^{-\mathsf{T}} : N)(F^{-\mathsf{T}} : M). \end{aligned} \tag{4.59}$$

Let us also notice that by plugging $F = \alpha_\theta \mathrm{I}_3$ into the expression (4.59), we get

$$D^2W_\theta(\alpha_\theta \mathrm{I}_3)[M]^2 = D^2W_\theta(\alpha_\theta \mathrm{I}_3)[M_{\mathrm{sym}}]^2 = 2\mathbf{G}_\theta |M_{\mathrm{sym}}|^2 + \lambda(\alpha_\theta)\,\mathrm{tr}^2 M, \tag{4.60}$$

for every $M \in \mathbb{R}^{3\times 3}$, where

$$\mathbf{G}_\theta := \mathrm{v}\overline{N}\theta > 0 \quad \text{and} \quad \lambda(\alpha_\theta) := -\mathrm{v}\overline{N}\theta - \frac{1}{\alpha_\theta^2} + \frac{\alpha_\theta}{\alpha_\theta^3 - 1} - \frac{2\chi}{\alpha_\theta^5}. \tag{4.61}$$

The estimate (4.48) grants that also $\lambda(\alpha_\theta) > 0$. Indeed, from the definition of α_θ in (4.51) it directly follows that

$$\mathrm{v}\overline{N}\theta + \alpha_\theta \log\left(1 - \frac{1}{\alpha_\theta^3}\right) + \frac{1}{\alpha_\theta^2} + \frac{\chi}{\alpha_\theta^5} - \frac{\alpha_\theta \mu}{k_{\mathsf{B}}\mathrm{T}} = 0.$$

Observe that the above equality is equivalent to $p_t(\theta, \alpha_\theta^3) = 0$. As a consequence, $\lambda(\alpha_\theta)$ can be equivalently written as

$$\lambda(\alpha_\theta) = \frac{\alpha_\theta}{\alpha_\theta^3 - 1} + \alpha_\theta \log\left(1 - \frac{1}{\alpha_\theta^3}\right) - \frac{\chi}{\alpha_\theta^5} - \frac{\alpha_\theta \mu}{k_{\mathsf{B}}\mathrm{T}}.$$

Then (4.48) yields

$$\lambda(\alpha_\theta) \geq \frac{(1-2\chi)\alpha_\theta^3 + 2\chi}{2\alpha_\theta^5(\alpha_\theta^3 - 1)} - \frac{\mu\alpha_\theta}{k_{\mathsf{B}}\mathrm{T}} > 0, \tag{4.62}$$

where the last strict inequality is due to the fact that $\alpha_\theta > 1$, $\chi \in (0, 1/2]$ and $\mu \leq 0$.

Uniform convergence.

Lemma 4.4.6. *Let $(\theta_k)_k \subseteq \mathcal{I}$, $\theta_k \to 1$ as $k \to +\infty$. The sequence $(W_{\theta_k})_k$ satisfies*

$$\|W_{\theta_k} - W^{\mathsf{FR}}\|_{C(\mathcal{K})} \to 0, \qquad \text{as } k \to \infty, \tag{4.63}$$

for every $\mathcal{K}$ compact subset of $\mathbb{R}^{3\times 3}_1$ and there exists $\mathcal{U} \in \mathcal{N}\big(\alpha \mathrm{SO}(3)\big)$ such that

$$||W_{\theta_k} - W^{\mathsf{FR}}||_{C^2(\overline{\mathcal{U}})} \to 0, \qquad \text{as } k \to \infty. \tag{4.64}$$

Proof. Pick $\mathcal{K} \subseteq \mathbb{R}^{3\times 3}_1$ and note that $\sup_{F\in\mathcal{K}} |W_{\theta_k}(F) - W^{\mathsf{FR}}(F)| \leq \frac{\mathrm{v}N}{2}|\theta_k - 1|\big|\overline{C} - 3\big|$, where $\overline{C} > 0$ is such that $|F|^2 \leq \overline{C}$ for every $F \in \mathcal{K}$. This grants the convergence in (4.63). To show (4.64), we first observe that, by continuity of the determinant, there exists $\mathcal{U} \in \mathcal{N}(\alpha\mathrm{SO}(3))$ such that $\det F > 1$ for every $F \in \mathcal{U}$. Up to shrinking $\mathcal{U}$, we have that W_{θ_k} and W^{FR} are of class C^2 in $\overline{\mathcal{U}}$. By (4.58) and (4.59), one can easily see that

$$||DW_{\theta_k} - DW^{\mathsf{FR}}||_{C(\overline{\mathcal{U}}, \mathscr{L}(\mathbb{R}^{3\times3}))} \leq \sup_{|M|\leq 1} \sup_{F\in\overline{\mathcal{U}}} \big\{\mathrm{v}N|\theta_k - 1||F||M|\big\} \leq \mathrm{v}N\,\overline{C}|\theta_k - 1|$$
$$||D^2W_{\theta_k} - D^2W^{\mathsf{FR}}||_{C(\overline{\mathcal{U}}, \mathscr{L}^2(\mathbb{R}^{3\times3}))} \leq \sup_{|M|\leq 1, |N|\leq 1} \mathrm{v}N|\theta_k - 1||M||N| \leq \mathrm{v}N|\theta_k - 1|.$$

with $\overline{C} > 0$ such that $|F| \leq \overline{C}$ for every $F \in \overline{\mathcal{U}}$. By the last two inequalities and by (4.63) with $\mathcal{K} = \overline{\mathcal{U}}$, we conclude the convergence result in (4.64). □

Taking into account (4.53), the above lemma implies that the uniform convergence in (4.9) holds.

Remark 4.4.7. In this remark we emphasize the fact that the (rescaled) energy densities $W^{\mathsf{FR}}_h(x, \cdot)$ defined by (4.6) are minimized on

$$\mathrm{SO}(3)A_h(x', hx_3), \qquad \text{for every } h > 0 \text{ and a.e. } x \in \Omega,$$

where A_h given by (4.7) – (4.8) and that they uniformly converge to W^{FR} given by (4.3), which is minimized at $\alpha\,\mathrm{SO}(3)$, $\alpha > 1$ as in (4.7). However, by directly confronting formulas (4.3) and (4.6), one can check that the densities W^{FR}_h cannot be rewritten in the pre-stretch form (see the discussion in Section 2.1.1)

$$W^{\mathsf{FR}}_h(x, F) = W\Big(FA_h^{-1}(x', hx_3)\Big),$$

where $W = W^{\mathsf{FR}}(\alpha\,\cdot)$. This is instead the case, for instance, in [AD15, AD17, DeS18] or [Sch07b]. ■

Quadratic growth.

Lemma 4.4.8. *Let $(\theta_k)_k \subseteq \mathcal{I}$, $\theta_k \to 1$ as $k \to +\infty$. There exists $C > 0$ and $\bar{k} \in \mathbb{N}$ such that*

$$W_{\theta_k}(F) \geq C\,\mathrm{dist}^2\big(F, \alpha_{\theta_k}\mathrm{SO}(3)\big), \quad \text{for every } F \in \mathbb{R}^{3\times3}_1 \text{ and every } k \geq \bar{k}, \tag{4.65}$$

$$W^{\mathsf{FR}}(F) \geq C\,\mathrm{dist}^2\big(F, \alpha\,\mathrm{SO}(3)\big), \quad \text{for every } F \in \mathbb{R}^{3\times3}_1. \tag{4.66}$$

Proof. We divide the proof of lemma into three cases. For the sake of brevity we denote $\alpha_k := \alpha_{\theta_k}$, $k \in \mathbb{N}$ and recall that $\alpha = \alpha_1$.

CASE 1. There exists $k_1 \in \mathbb{N}$, $r_1 > 0$ and $C_1 > 0$ such that for all $F \in \mathbb{R}^{3\times 3}_1$ it holds

$$\mathrm{dist}(F, \alpha\mathrm{SO}(3)) < r_1 \quad \Longrightarrow \quad W_{\theta_k}(F) \geq C_1 \mathrm{dist}^2(F, \alpha_k \mathrm{SO}(3)), \quad \text{for every } k \geq k_1.$$

PROOF OF CASE 1. Let $r_1 > 0$ be such that every F with $\mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) \leq r_1$ satisfies $\det F > 1$. Clearly, $\mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) < r_1$ implies that $\sqrt{F^\intercal F} \in B_{r_1}(\alpha \mathrm{I}_3)$. Given that $\alpha_k \to \alpha$ as $k \to +\infty$, there exists $k_1 \in \mathbb{N}$ such that $\alpha_k \mathrm{I}_3 \in B_{r_1}(\alpha \mathrm{I}_3)$ for every $k \geq k_1$. Then, by frame indifference of the function W_{θ_k}, its regularity and the fact that it vanishes at $\alpha_k \mathrm{I}_3$, we have that for every $k \geq k_1$ there exists $H_k \in B_{r_1}(\alpha \mathrm{I}_3)$ laying on the segment between $\sqrt{F^\intercal F}$ and $\alpha_k \mathrm{I}_3$ such that

$$W_{\theta_k}(F) = W_{\theta_k}(\sqrt{F^\intercal F}) = \frac{1}{2} D^2 W_{\theta_k}(\alpha_k \mathrm{I}_3)[\sqrt{F^\intercal F} - \alpha_k \mathrm{I}_3]^2 + \frac{D^3 W_{\theta_k}(H_k)}{3!}[\sqrt{F^\intercal F} - \alpha_k \mathrm{I}_3]^3$$

for every F with $\mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) < r_1$. Observe from (4.59) that all differentials of order greater or equal to 3 of W_{θ_k} are independent on k, for every $k \geq k_1$. Therefore, by denoting $\overline{C} := \sup_{\overline{B}_{r_1}(\alpha \mathrm{I}_3)} |D^3 W_{\theta_k}|$ and recalling (4.60) we have that

$$W_{\theta_k}(F) \geq \mathrm{v}N |\sqrt{F^\intercal F} - \alpha_k \mathrm{I}_3|^2 \left(1 - \frac{\overline{C}}{\mathrm{v}N} |\sqrt{F^\intercal F} - \alpha_k \mathrm{I}_3|\right)$$

for every $k \geq k_1$ and every F with $\mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) < r_1$. Pick $r_1 > 0$ so that $2r_1\overline{C}/\mathrm{v}N < 1/2$. Then for all F satisfying $\mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) < r_1$ we have that

$$W_{\theta_k}(F) \geq \frac{\mathrm{v}N}{2} |\sqrt{F^\intercal F} - \alpha_k \mathrm{I}_3|^2 = \frac{\mathrm{v}N}{2} \mathrm{dist}^2(F, \alpha_k \mathrm{SO}(3)),$$

proving the implication in CASE 1, with $C_1 = \frac{\mathrm{v}N}{2}$. Observe also that, up to shrinking r_1, by passing to the limit as $k \to +\infty$ in the above inequality, we also have that

$$W^{\mathsf{FR}}(F) \geq C_1 \mathrm{dist}^2\big(F, \alpha\mathrm{SO}(3)\big), \qquad \text{whenever } \mathrm{dist}\big(F, \alpha\mathrm{SO}(3)\big) < r_1.$$

CASE 2. There exists $k_2 \in \mathbb{N}$, $r_2 > 0$ and $C_2 > 0$ such that for every $F \in \mathbb{R}^{3\times 3}_1$ it holds that

$$\mathrm{dist}(F, \alpha\mathrm{SO}(3)) > r_2 \quad \Longrightarrow \quad W_{\theta_k}(F) \geq C_2 \, \mathrm{dist}^2(F, \alpha_k \mathrm{SO}(3)), \qquad \text{for every } k \geq k_2.$$

PROOF OF CASE 2. First observe that there exists $k_2 \in \mathbb{N}$ such that

$$|F|^2 \geq \frac{1}{2} \mathrm{dist}^2\big(F, \alpha_k \mathrm{SO}(3)\big) - 3(\alpha^2 + 1)$$

for every $k \geq k_2$ and every $F \in \mathbb{R}^{3\times 3}$. Hence, for every $k \geq k_2$ and every $F \in \mathbb{R}^{3\times 3}_1$ it holds

$$W_{\theta_k}(F) \geq \frac{\mathrm{v}N}{4} |F|^2 - C \geq \frac{\mathrm{v}N}{8} \mathrm{dist}^2\big(F, \alpha_k \mathrm{SO}(3)\big) - C,$$

with C being a positive constant depending only on the fixed physical parameters. Moreover, for every $F \in \mathbb{R}^{3\times 3}$ we have that

$$\mathrm{dist}^2\big(F, \alpha_k \mathrm{SO}(3)\big) \geq \frac{1}{2} \mathrm{dist}^2\big(F, \alpha\mathrm{SO}(3)\big) - 3|\alpha_k - \alpha|^2.$$

Thus, by choosing $r_2 > 0$ big enough, the inequality in CASE 2 holds with $C_2 = \frac{\mathrm{v}N}{16} - 1$.

Before proceeding to Case 3. we observe that, by using the same argument as above, one can show (up to increasing r_2) that

$$W^{\mathsf{FR}}(F) \geq \overline{C}_2 \operatorname{dist}^2(F, \alpha \mathrm{SO}(3)), \qquad \text{whenever } \operatorname{dist}\big(F, \alpha \mathrm{SO}(3)\big) > r_2,$$

for some $\overline{C}_2 > 0$. Since $W^{\mathsf{FR}} > 0$ on the set $\{F \in \mathbb{R}^{3\times 3}_1 : r_1 \leq \operatorname{dist}(F, \alpha\mathrm{SO}(3)) \leq r_2\}$, there exists $\overline{C}_3 > 0$ such that $W^{\mathsf{FR}}(F) \geq \overline{C}_3$ on $\{F \in \mathbb{R}^{3\times 3}_1 : r_1 \leq \operatorname{dist}(F, \alpha\mathrm{SO}(3)) \leq r_2\}$, by the continuity of W^{FR}. By setting $C_0 := \min\{C_1, \overline{C}_2, \overline{C}_3/r_1^2\}$, we prove that W^{FR} satisfies

$$W^{\mathsf{FR}}(F) \geq C_0 \operatorname{dist}^2\big(F, \alpha\mathrm{SO}(3)\big), \qquad \text{for every } F \in \mathbb{R}^{3\times 3}_1.$$

Case 3. There exists $k_3 \in \mathbb{N}$ and $C_3 > 0$ such that for every $F \in \mathbb{R}^{3\times 3}_1$

$$r_1 \leq \operatorname{dist}(F, \alpha\mathrm{SO}(3)) \leq r_2 \quad \Longrightarrow \quad W_{\theta_k}(F) \geq C_3, \qquad \text{for every } k \geq k_3.$$

Proof of Case 3. Note that there exists $k_3 \in \mathbb{N}$ such that for every $k \geq k_3$

$$W_{\theta_k}(F) \geq |W_{\theta_k}(F) - W^{\mathsf{FR}}(F)| + C_0 \operatorname{dist}^2\big(F, \alpha\mathrm{SO}(3)\big) \geq \frac{C_0 r_1^2}{2} =: C_3,$$

by the uniform convergence of W_{θ_k} to W^{FR} on the set $\{F \in \mathbb{R}^{3\times 3}_1 : r_1 \leq \operatorname{dist}\big(F, \alpha\mathrm{SO}(3)\big) \leq r_2\}$ and the quadratic growth of the W^{FR}. This concludes the proof of Case 3.

Now set $\bar{k} := \min\{k_1, k_2, k_3\}$ and $C := \min\{C_1, C_2, C_3/4r_1^2\}$. The above three cases yield

$$W_{\theta_k}(F) \geq C \operatorname{dist}^2\big(F, \alpha_k\mathrm{SO}(3)\big), \qquad \text{for every } k \geq \bar{k} \text{ and every } F \in \mathbb{R}^{3\times 3}_1,$$

concluding the proof of the lemma. □

We remark that the quadratic growth of W^{FR} in a neighbourhood of $\alpha\mathrm{SO}(3)$ is deduced by passing to the limit as $k \to +\infty$ in (4.65). In the following simple example we show that the converse is not necessarily true, i.e. the quadratic growth of the limiting function W^{FR} near to its well does not implies, in general, the same property of the sequence W_{θ_k}.

Example 4.4.9. Consider the sequence $(\psi_k)_k$ of functions $\psi_k : (-1, 1) \to \mathbb{R}$ defined by

$$\psi_k(t) := \left| t - \frac{1}{k} \right|^{2+\frac{1}{k}}, \qquad \text{for every } t \in (-1, 1).$$

Note that for every $k \in \mathbb{N}$ the function ψ_k is non negative, minimized precisely at $\frac{1}{k}$ and $\psi_k \to \psi_\infty$ in C^2-norm on $[-1, 1]$, as $k \to \infty$, with $\psi_\infty(t) := t^2$, for every $t \in [-1, 1]$. Clearly, the limiting function ψ_∞ is non negative, minmized at 0 and has quadratic growth. However, there is no $\bar{k} \in \mathbb{N}$, no constant $C > 0$ and no $\varepsilon > 0$ such that

$$\psi_k(t) \geq C \left(t - \frac{1}{k} \right)^2, \qquad \text{for every } t \in (-\varepsilon, \varepsilon) \subseteq (-1, 1) \text{ and every } k \geq \bar{k}.$$

△

Remark 4.4.10. Here, we make a couple of comments about the properties of the approximate energy densities $\widehat{W}^{\mathsf{FR}}_h$ given in (4.36). Recall also that $\widehat{W}^{\mathsf{FR}}_h(z, \cdot) = \widehat{W}_{f_h(z', \frac{z_3}{h})}$ for a.e. $z \in \Omega_h$ and h small enough, with $\widehat{W}_{f_h(z', \frac{z_3}{h})}$ given in (4.56) for $\theta = f_h(z', \frac{z_3}{h})$.

We start with the simple observation that the dependence on the thickness variable z_3 and the thickness parameter h in the approximate model energies $\widehat{W}^{\mathsf{FR}}_h$ remains unchanged with respect to the original $\overline{W}_h^{\mathsf{FR}}$. Thus, taking into account the energy well structure presented in Remark 4.4.5, it is straightforward to verify that $\widehat{W}_h^{\mathsf{FR}}$ shares the same regularity, uniform convergence and quadratic growth properties as $\overline{W}_h^{\mathsf{FR}}$. In particular, the 2D model (obtained by repeating the same procedure as in the general case provided in Section 4.2) will be governed by the functional $\mathscr{E}_0^{\mathrm{gel}}$ as in (4.23), with the corresponding quadratic form $\mathcal{Q}_2^{\mathsf{FR}}$ obtained via the standard minimization process involving the second differential of the (approximate) homogeneous density $\widehat{W}^{\mathsf{FR}} := \widehat{W}_1$ having the energy well $\alpha\mathrm{SO}(3)$ with $\alpha = (\frac{1-2\chi}{2\mathrm{v}\overline{N}})^{1/5}$ explicitly determined in (4.57) with $\theta = 1$. Namely, in this case $\mathcal{Q}_2^{\mathsf{FR}}$ in (4.23) is given by

$$\mathcal{Q}_2^{\mathsf{FR}}(G) = \min_{\mathsf{c}\in\mathbb{R}^3} D^2\widehat{W}^{\mathsf{FR}}(\alpha\mathrm{I}_3)[G^* + \mathsf{c}\otimes\mathsf{f}_3]^2, \qquad \text{for every } G \in \mathbb{R}^{2\times 2}.$$

By a straightforward computation one can find that

$$D^2\widehat{W}^{\mathsf{FR}}(\alpha\mathrm{I}_3)[F]^2 = 2\mathrm{v}\overline{N}|F_{\mathrm{sym}}|^2 + \mathrm{v}\overline{N}\,\mathrm{tr}^2 F, \qquad F \in \mathbb{R}^{3\times 3}$$

and thus (see Appendix 2.A) that

$$\mathcal{Q}_2^{\mathsf{FR}}(G) = 2\mathrm{v}\overline{N}\left(|G_{\mathrm{sym}}|^2 + \frac{1}{3}\mathrm{tr}^2 G\right), \qquad G \in \mathbb{R}^{2\times 2}.$$

■

5

Dimension reduction for thin sheets with transversally varying pre-stretch

Contents

In this chapter we will be concerned with the derivation of plate models under different energy scaling regimes, in the description of thin hyperelastic sheets Ω_h of small thickness h characterized by a spontaneous stretch A_h, which is precisely the inverse of a given pre-stretch (see Section 2.1.1). The pre-stretch is introduced via a smooth incompatibility tensor field G_h, representing a Riemannian metric on Ω_h. In particular, the leading order metric, which in Chapter 3 equals I_3 (viewing $A_h^2(z) = \mathrm{I}_3 + h2B_h(z', \frac{z_3}{h}) + O(h^2)$ as a metric on Ω_h), is here extended to an arbitrary metric on Ω_h depending on the planar variable only.

We start by giving (in Section 5.1) a precise description of the setting in which we will work throughout this chapter. Moreover, we provide a brief overview of the obtained results.

5.1 The set-up of the problem

We consider a family of thin hyperelastic sheets occupying the reference domains $\Omega_h := \omega \times (-h/2, h/2) \subseteq \mathbb{R}^3$, where $\omega \subseteq \mathbb{R}^2$ is a bounded, simply-connected, Lipschitz domain and $0 < h \ll 1$. We recall the referential rescaling of each Ω_h via: $\Omega_h \ni (z', z_3) \mapsto (z', z_3/h) \in \Omega$, referring to Section 2.1.2 for the relation between the physical and the rescaled quantities. In particular, recall that a point in the rescaled domain Ω is denoted by $x = (x', x_3)$.

Further, we suppose that the sheets are characterized by the incompatibility (Riemann metric) tensor fields $G_h \in C^\infty\big(\overline{\Omega}_h, \mathrm{Sym}(3)\big)$, satisfying $G_h(z) \in \mathrm{Psym}(3)$ for every $z \in \overline{\Omega}_h$ and the energy density function having the pre-stretch form

$$\overline{W}_h(z, F) := W\big(FG_h^{-1/2}(z)\big), \qquad z \in \Omega_h \text{ and } F \in \mathbb{R}^{3\times 3}.$$

The energy density W is the homogeneous density satisfying the properties W1 – W4 listed in Section 2.1.1. In view of (2.4), note that $A_h := G_h^{1/2}$ has the role of a spontaneous stretch. It explicit form is given in (5.4) below.

Moreover, we assume that G_h satisfy the following structure expansion assumption:

$$\text{(O)}\quad \left[\begin{array}{l} \qquad\qquad\qquad\textsc{Oscillatory case:} \\ G_h(z) = \mathcal{G}_h\left(z', \frac{z_3}{h}\right), \qquad \text{for all } z = (z', z_3) \in \Omega_h, \\ \mathcal{G}_h(x', t) = \bar{\mathcal{G}}(x') + h\mathcal{G}^1(x', x_3) + \frac{h^2}{2}\mathcal{G}^2(x', x_3) + o(h^2), \qquad \text{for all } (x', x_3) \in \overline{\Omega} \\ \text{where } \bar{\mathcal{G}} \in C^\infty\big(\overline{\omega}, \mathrm{Sym}(3)\big),\ \bar{\mathcal{G}}(x') \in \mathrm{Psym}(3) \text{ and } \mathcal{G}^1, \mathcal{G}^2 \in C^\infty\big(\overline{\Omega}, \mathrm{Sym}(3)\big) \text{ and} \\ (5.1) \qquad \displaystyle\int_{-1/2}^{1/2} \mathcal{G}^1(x', t)\,\mathrm{d}t = 0, \qquad \text{for all } x' \in \overline{\omega}. \end{array}\right.$$

The above requirement of $\bar{\mathcal{G}}$ being independent of the transversal variable $t \in (-1/2, 1/2)$ is essential for treating the energy order $\inf \mathscr{I}_h \leq Ch^2$ (see (5.3) below).

The zero mean requirement on $\mathcal{G}^1$ can be relaxed to requesting that $\int_{-1/2}^{1/2} \mathcal{G}^1(x', t)_{2\times 2}\,\mathrm{d}t$ be a linear strain with respect to the leading order midplate metric $\bar{\mathcal{G}}_{2\times 2}$, as it has been shown in Chapter 3 in the case $\bar{\mathcal{G}}_{2\times 2} = \mathrm{I}_2$. In the latter case the sufficient and necessary condition for this to happen is that $\mathrm{curl}\,\big(\mathrm{curl} \int_{-1/2}^{1/2} \mathcal{G}^1_{2\times 2}(x', t)\,\mathrm{d}t\big) = 0$ (see Lemma 1.4.2). As it can be seen from the discussion provided in Section 3.4, the question how to remove the "curl curl" condition above still remained open. We assume (5.1) here, in light of the special case (NO) below.

We refer to the family of thin sheets Ω_h pre-stretched by metrics in (O) as the "oscillatory" case:

$$G_h(z) = \bar{\mathcal{G}}(z') + h\mathcal{G}^1\left(z', \frac{z_3}{h}\right) + \frac{h^2}{2}\mathcal{G}^2\left(z', \frac{z_3}{h}\right) + o(h^2) \quad \text{for all } z = (z', z_3) \in \Omega_h, \tag{5.2}$$

and note that it includes a subcase of a single metric independent of h (to which we refer as the "non-oscillatory" case), upon taking:

$$\mathcal{G}^1(x', x_3) = x_3\bar{\mathcal{G}}^1(x'), \qquad \mathcal{G}^2(x', x_3) = x_3^2\bar{\mathcal{G}}^2(x').$$

Formula (5.2) becomes then Taylor's expansion, with $\bar{\mathcal{G}} = G_{|\omega\times\{0\}}$, $\bar{\mathcal{G}}^i = \big((\partial_3)^i G\big)_{|\omega\times\{0\}}$, $i = 1, 2$ in:

$$
\text{(NO)}\quad \left[\begin{array}{c}
\textsc{Non-oscillatory case:} \\[1em]
G_h = G_{|\overline{\Omega}_h} \quad \text{for some } \ G \in C^\infty(\overline{\Omega}, \mathrm{Sym}(3)), \ \text{ with values in } \mathrm{Psym}(3), \\[1em]
G_h(z) = \bar{\mathcal{G}}(z') + z_3\partial_3 G(z',0) + \dfrac{z_3^2}{2}\partial_{33}G(z',0) + o(z_3^2), \ \text{ for all } \ z = (z', z_3) \in \Omega_h.
\end{array}\right.
$$

Mechanically, the assumption (NO) describes thin sheets that have been cut out of a single specimen block Ω, pre-stretched according to a fixed (though arbitrary) tensor $A = G^{1/2}$. In practice, the pre-stretch in the energetical description (5.3) may be due to a variety of phenomena, such as: growth of leaves, differential swelling in gels or electrodes in electrochemical cells, tearing of plastic sheets or actuation of micro-mechanical devices, to name a few.

As we shall see, the general case (O) can be reduced to (NO) via the following effective metric:

$$
\text{(EF)}\quad \left[\begin{array}{l}
\qquad\qquad\qquad \textsc{Effective oscillatory case:} \\[1em]
\bar{G}_h(z) = \bar{G}(z) = \bar{\mathcal{G}}(z') + z_3\bar{\mathcal{G}}^1(z') + \dfrac{z_3^2}{2}\bar{\mathcal{G}}^2(z'), \qquad \text{for all } \ z = (z', z_3) \in \Omega_h, \\[1em]
\text{with: } \ \bar{\mathcal{G}}^1_{2\times 2} = 12\displaystyle\int_{-1/2}^{1/2} t\mathcal{G}^1_{2\times 2}(\cdot, t)\,\mathrm{d}t, \quad \bar{\mathcal{G}}^1\mathrm{f}_3 = -60\displaystyle\int_{-1/2}^{1/2}(2t^3 - \frac{1}{2}t)\mathcal{G}^1(\cdot, t)\mathrm{f}_3\,\mathrm{d}t \\[1em]
\text{and: } \ \bar{\mathcal{G}}^2_{2\times 2} = 30\displaystyle\int_{-1/2}^{1/2}(6t^2 - \frac{1}{2})\mathcal{G}^2_{2\times 2}(\cdot, t)\,\mathrm{d}t.
\end{array}\right.
$$

In this chapter we are interested in studying the limit behaviour of the nonnegative free-energy functionals (recall the relation (2.19)):

$$
\mathscr{I}_h(u) \; = \; \frac{1}{h}\overline{\mathscr{E}}_h(u) = \frac{1}{h}\int_{\Omega_h} \overline{W}_h\big(z, \nabla u(z)\big)\,\mathrm{d}z, \tag{5.3}
$$

defined on vector fields $u \in \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ that we interpret as deformations of Ω_h. We will be concerned with the regimes of curvatures of G_h in (O) which yield the incompatibility rate, quantified by $\inf \mathscr{I}_h$, of order higher than h^2 in the plate's thickness h.

5.1.1 A brief overview of the obtained results

With respect to the prior works in the context of pre-stretched materials, we propose the following new contributions.

The 2D model in the non-oscillatory case. We start by deriving (in Section 5.2), the Γ-limit of the rescaled energies $\frac{1}{h^2}\mathscr{I}_h$. In the setting of (NO), we obtain:

$$
\begin{aligned}
\mathscr{I}_2(y) &= \frac{1}{2}\big\|\mathrm{Tensor}_2\big\|^2_{\mathcal{Q}_2} = \frac{1}{2}\Big\|x_3\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}x_3\partial_3 G(x',0)\Big\|^2_{\mathcal{Q}_2} \\
&= \frac{1}{24}\Big\|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G(x',0)\Big\|^2_{\mathcal{Q}_2}.
\end{aligned}
$$

We now explain the notation above. Firstly, $\|\cdot\|_{\mathcal{Q}_2}$ is a weighted L^2 norm in (5.12) on the space $\mathbb{E}$ of $\mathrm{Sym}(2)$-valued tensor fields on Ω. The weights in (5.10) are determined by the elastic energy W together with the leading order metric coefficient $\bar{\mathcal{G}}$. The functional $\mathscr{I}_2$ is defined on the set of isometric immersions $\{y \in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3);\ (\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times 2}\}$; each such immersion generates the corresponding Cosserat vector $\vec{b}$, uniquely given by requesting: $\big[\partial_1 y,\ \partial_2 y,\ \vec{b}\big] \in \mathrm{SO}(3)\bar{\mathcal{G}}^{1/2}$ in ω.

The energy $\mathscr{I}_2$ measures then the bending quantity Tensor_2, which is linear in x_3, resulting in its reduction to the single nonlinear bending term, that coincides with the difference of the curvature form $\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}}$ from the preferred (target) curvature tensor $\frac{1}{2}\partial_3 G(\cdot,0)$ on the midplate ω. The same energy has been derived in [LP11, BLS16] under the assumption that G is independent of z_3 and in [KS14] for a general manifold (M^n,g) with any codimension submanifold $(N^k, g_{|N})$ replacing the midplate $\omega\times\{0\}$. The derivation of $\mathscr{I}_2$ can be seen as particular case of [KS14] (with $n=3$ and $k=2$), but also as a particular case of the result in case (O), as we shall see below.

In Subsection 5.2.2 we identify the necessary and sufficient conditions for $\min \mathscr{I}_2 = 0$, in terms of the vanishing of the Riemann curvatures $R_{1212}, R_{1213}, R_{1223}$ of G at $z_3 = 0$. In this case, it follows that $\inf \mathscr{I}_h \leq Ch^4$. For the discussed case (NO), the recent work [MS18] generalized the same statements for arbitrary dimension and codimension.

In Section 5.4 we then derive the Γ-limit of the rescaled energies $\frac{1}{h^4}\mathscr{I}_h$, which is given by:

$$\mathscr{I}_4(V,\mathbb{S}) = \frac{1}{2}\big\|\mathrm{Tensor}_4\big\|^2_{\mathcal{Q}_2},$$

defined on the spaces of: first order infinitesimal isometries

$$\mathscr{V}_{y_0} = \{V \in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3) : \big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0\}$$

and finite strains

$$\mathscr{S}_{y_0} = \big\{\mathbb{S} = \mathrm{L}^2\text{-}\lim_{n\to\infty}\big((\nabla y_0)^{\mathsf{T}}\nabla w_n\big)_{\mathrm{sym}} : w_n \in \mathrm{W}^{1,2}(\omega,\mathbb{R}^3)\big\}$$

on the deformed midplate $y_0(\omega) \subseteq \mathbb{R}^3$. Here, y_0 is the unique smooth isometric immersion of $\bar{\mathcal{G}}_{2\times 2}$ for which $\mathscr{I}_2(y_0) = 0$; recall that it generates the corresponding Cosserat's vector $\vec{b}_0$.

The expression in Tensor_4 is quite complicated but it has the structure of a quadratic polynomial in variable $x_3 \in (-1/2, 1/2)$. A key tool for identifying this expression, also in the general case (O), involves the subspaces $\{\mathbb{E}_n \subset \mathbb{E}\}_{n\geq 1}$ in (5.13), consisting of the tensorial polynomials in x_3 of order n. The bases of $\{\mathbb{E}_n\}$ are then naturally given in terms of the Legendre polynomials $\{p_n\}_{n\geq 0}$ on $\big(-\frac{1}{2},\frac{1}{2}\big)$. Since $\mathrm{Tensor}_4 \in \mathbb{E}_2$, we write the decomposition:

$$\mathrm{Tensor}_4 = p_0(x_3)\mathrm{Stretching}_4 + p_1(x_3)\mathrm{Bending}_4 + p_2(x_3)\mathrm{Curvature}_4,$$

which, as shown in Subsection 5.4.2, results in:

$$\begin{aligned}\mathscr{I}_4(V,\mathbb{S}) &= \frac{1}{2}\Big(\big\|\mathrm{Stretching}_4\big\|^2_{\mathcal{Q}_2} + \big\|\mathrm{Bending}_4\big\|^2_{\mathcal{Q}_2} + \big\|\mathrm{Curvature}_4\big\|^2_{\mathcal{Q}_2}\Big)\\ &= \frac{1}{2}\Big\|\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + \frac{1}{24}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{48}\partial_{33}G(x',0)_{2\times 2}\Big\|^2_{\mathcal{Q}_2}\\ &\quad + \frac{1}{24}\big\|\big[\langle\nabla_i\nabla_j V, \vec{b}_0\rangle\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2} + \frac{1}{1440}\big\|\big[R_{i3j3}\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}.\end{aligned}$$

Above, ∇_i denotes the covariant differentiation with respect to the metric $\bar{\mathcal{G}}$ and R_{i3j3} are the potentially non-zero curvatures of G on ω at $z_3 = 0$. When $y_0 = id_2$ (which occurs automatically when $\bar{\mathcal{G}} = \mathrm{I}_3$), then $\vec{b}_0 = \mathfrak{f}_3$ and the first two terms in $\mathscr{I}_4$ reduce to the stretching and the linear bending contents of the classical von Kármán energy. The third term is purely metric-related and measures the non-immersability of G relative to the present quartic scaling. These findings generalize the results of [BLS16] valid for x_3-independent G in (NO). We also point out that, following the same general principle in the h^2- scaling regime, one may readily decompose:

$$\text{Tensor}_2 = p_0(x_3)\text{Stretching}_2 + p_1(x_3)\text{Bending}_2;$$

since Tensor_2 is already a multiple of x_3, then $\text{Stretching}_2 = 0$ in the ultimate form of $\mathscr{I}_2$.

It is not hard to deduce (see Subsection 5.4.3) that the necessary and sufficient conditions for having $\min \mathscr{I}_4 = 0$ are precisely that $R_{ijkl} \equiv 0$ on $\omega \times \{0\}$, for all $i, j, k, l = 1 \ldots 3$. In that case, we show in Section 5.7 that $\inf \mathscr{I}_h \leq Ch^6$. We also identify the curvature term that will be present in the corresponding decomposition of Tensor_6. It turns out to be precisely $\big[\partial_3 R_{i3j3}(x', 0)\big]_{i,j=1,2} = \big[\nabla_3 R_{i3j3}(x', 0)\big]_{i,j=1,2}$. which in view of the second Bianchi identity carries the only potentially non-vanishing components of the covariant gradient $\nabla \text{Riem}(x', 0)$. This finding is consistent with results of Section 5.6, analyzing the conformal non-oscillatory metric $G = e^{2\phi(z_3)}\mathrm{I}_3$. Namely, we show that different orders of vanishing of ϕ at $z_3 = 0$ correspond to different even orders of scaling of $\mathscr{I}_h$ as $h \to 0$:

$$\phi^{(k)}(0) = 0 \quad \text{for } k = 1, \ldots, n-1 \quad \text{and} \quad \phi^{(n)}(0) \neq 0 \quad \Leftrightarrow \quad ch^{2n} \leq \inf \mathscr{I}_h \leq Ch^{2n}$$

with the lower bound: $\inf \mathscr{I}_h \geq c_n h^n \Big\| \big[\partial_3^{(n-2)} R_{i3j3}(x', 0)\big]_{i,j=1,2} \Big\|^2_{\mathcal{Q}_2}$.

The 2D model in the oscillatory case. We show that the analysis in the general case (O) may follow a similar procedure, where we first project the limiting quantity Tensor^O on an appropriate polynomial space and then decompose the projection along the respective Legendre basis. For the Γ-limit of $\frac{1}{h^2}\mathscr{I}_h$ in Section 5.2, we show that:

$$\text{Tensor}_2^O = x_3\big((\nabla y)^\mathsf{T}\nabla\vec{b}\big)_{\text{sym}} - \frac{1}{2}\mathcal{G}^1_{2\times 2} = p_0(x_3)\text{Stretching}_2^O + p_1(x_3)\text{Bending}_2^O + \text{Excess}_2,$$

with $\text{Excess}_2 = \text{Tensor}_2^O - \mathbb{P}_1(\text{Tensor}_2^O)$.

Consequently:

$$\begin{aligned}\mathscr{I}_2^O(y) &= \frac{1}{2}\Big(\big\|\text{Stretching}_2^O\big\|^2_{\mathcal{Q}_2} + \big\|\text{Bending}_2^O\big\|^2_{\mathcal{Q}_2} + \big\|\text{Excess}_2\big\|^2_{\mathcal{Q}_2}\Big)\\ &= \frac{1}{24}\Big\|\big((\nabla y)^\mathsf{T}\nabla\vec{b}\big)_{\text{sym}} - \frac{1}{2}\bar{\mathcal{G}}^1_{2\times 2}\Big\|^2_{\mathcal{Q}_2} + \frac{1}{8}\text{dist}^2_{\mathcal{Q}_2}\big(\mathcal{G}^1_{2\times 2}, \mathbb{E}_1\big),\end{aligned}$$

where again $\text{Stretching}_2^O = 0$ in view of the assumed $\int_{-1/2}^{1/2} \mathcal{G}_1 \,\mathrm{d}x_3 = 0$. For the same reason:

$$\text{Excess}_2 = -\frac{1}{2}\Big(\mathcal{G}^1_{2\times 2} - \mathbb{P}_1\big(\mathcal{G}^1_{2\times 2}\big)\Big) = -\frac{1}{2}\Big(\mathcal{G}^1_{2\times 2} - 12\int_{-1/2}^{1/2} x_3\mathcal{G}^1_{2\times 2}\,\mathrm{d}x_3\Big)$$

and also: $\mathbb{P}_1\big(\mathcal{G}^1_{2\times 2}\big) = x_3\bar{\mathcal{G}}^1_{2\times 2}$ with $\bar{\mathcal{G}}^1_{2\times 2}$ defined in (EF). The limiting oscillatory energy $\mathscr{I}_2^O$ consists thus of the bending term that coincides with $\mathscr{I}_2$ for the effective metric $\bar{G}$, plus the purely metric-related excess term. Observe that in the case when $\bar{\mathcal{G}} = \mathrm{I}_3$ the limiting functional $\mathscr{I}_2^O$ reduces to the functional $\mathscr{E}_0$ derived in Chapter 3 in the special case of the spontaneous strain B given in (3.5) satisfying $\int_{-1/2}^{1/2} B(\cdot, t)\,\mathrm{d}t = 0$.

It is easy to observe that: $\min \mathscr{I}_2^O = 0$ if and only if $\mathcal{G}^1_{2\times 2} = x_3\bar{\mathcal{G}}^1_{2\times 2}$ on $\omega \times \{0\}$ and $\text{Bending}_2^O = 0$. We show in Section 5.4 that this automatically implies: $\inf \mathscr{I}_h \leq Ch^4$. The

Γ-limit of $\frac{1}{h^4}\mathscr{I}_h$ is further derived in Subsection 5.4.1 and Subsection 5.4.2, by considering the decomposition:

$$\begin{aligned}&\mathrm{Tensor}_4^O = p_0(x_3)\mathrm{Stretching}_4^O + p_1(x_3)\mathrm{Bending}_4^O + p_2(x_3)\mathrm{Curvature}_4^O + \mathrm{Excess}_4,\\ &\text{with } \mathrm{Excess}_4 = \mathrm{Tensor}_4^O - \mathbb{P}_2(\mathrm{Tensor}_4^O).\end{aligned}$$

It follows that:

$$\begin{aligned}\mathscr{I}_4^O(V,\mathbb{S}) =& \frac{1}{2}\Big(\big\|\mathrm{Stretching}_4^O\big\|^2_{\mathcal{Q}_2} + \big\|\mathrm{Bending}_4^O\big\|^2_{\mathcal{Q}_2} + \big\|\mathrm{Curvature}_4^O\big\|^2_{\mathcal{Q}_2} + \big\|\mathrm{Excess}_4\big\|^2_{\mathcal{Q}_2}\Big)\\ =& \frac{1}{2}\big\|\mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + B_0\big\|^2_{\mathcal{Q}_2} + \frac{1}{24}\big\|\big[\langle\nabla_i\nabla_j V, \vec{b}_0\rangle\big]_{i,j=1,2} + B_1\big\|^2_{\mathcal{Q}_2}\\ &+ \frac{1}{1440}\big\|\big[R_{i3j3}\big]_{i,j=1,2}\big\|^2_{\mathcal{Q}_2}\\ &+ \frac{1}{2}\mathrm{dist}^2_{\mathcal{Q}_2}\Big(\frac{1}{4}\mathcal{G}^2_{2\times 2} - \int_0^{x_3}\big[\nabla_i\big((\mathcal{G}^1 e_3) - \frac{1}{2}\mathcal{G}^1_{33}e_3\big)\big]_{i,j=1,2,\mathrm{sym}}\,\mathrm{d}t, \mathbb{E}_2\Big),\end{aligned}$$

where $R_{1313}, R_{1323}, R_{2323}$ are the respective Riemann curvatures of the effective metric $\bar{G}$ in (EF) at $z_3 = 0$. The corrections B_0 and B_1 coincide with the same expressions written for $\bar{G}$ under two extra constraints (see Theorem 5.4.12), that can be seen as the h^4-order counterparts of the h^2-order condition $\int_{-1/2}^{1/2}\mathcal{G}^1\,\mathrm{d}t = 0$ that has been assumed throughout. In case these conditions are valid, the functional $\mathscr{I}_4^O$ is the sum of the effective stretching, bending and curvature in $\mathscr{I}_4$ for $\bar{G}$, plus the additional purely metric-related excess term.

Coercivity of $\mathscr{I}_2$ and $\mathscr{I}_4$. We additionally analyze the derived limiting functionals by identifying their kernels, when nonempty. In Section 5.3 we show that the kernel of $\mathscr{I}_2$ consists of the rigid motions of a single smooth deformation y_0 that solves:

$$(\nabla y_0)^{\mathsf{T}}\nabla y_0 = \bar{\mathcal{G}}_{2\times 2}, \qquad \big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(x',0)_{2\times 2}.$$

Further, $\mathscr{I}_2(y)$ bounds from above the squared distance of an arbitrary $W^{2,2}$ isometric immersion y of the midplate metric $\bar{\mathcal{G}}_{2\times 2}$, from the indicated kernel of $\mathscr{I}_2$.

In Section 5.5 we consider the case of $\mathscr{I}_4$. We first identify (see Theorem 5.5.1) the zero-energy displacement-strain couples $(V, \mathbb{S})$. In particular, we show that the minimizing displacements are exactly the linearised rigid motions of the referential y_0. We then prove that the bending term in $\mathscr{I}_4$, which is solely a function of V, boundes from above the squared distance of an arbitrary $W^{2,2}$ displacement obeying $\big((\nabla y_0)^{\mathsf{T}}\nabla V\big)_{\mathrm{sym}} = 0$, from the indicated minimizing set in V. On the other hand, the full coercivity result involving minimization in both V and $\mathbb{S}$ is false. In Remark 5.5.3 we exhibit an example in the setting of the classical von Kármán functional, where $\mathscr{I}_4(V_n, \mathbb{S}_n) \to 0$ as $n \to \infty$, but the distance of $(V_n, \mathbb{S}_n)$ from the kernel of $\mathscr{I}_4$ remains uniformly bounded away from 0. We note that this lack of coercivity is not prevented by the fact that the kernel is finite dimensional.

5.2 Rigorous derivation of Kirchhoff-like plate models $\mathscr{I}_2^O$ and $\mathscr{I}_2$

5.2.1 Compactness and Γ-limit under Ch^2 energy bound

Define the matrix fields $\bar{A} \in C^\infty(\overline{\omega}, \mathrm{Sym}(3))$ and $A_h, A_1, A_2 \in C^\infty(\overline{\Omega}, \mathrm{Sym}(3))$ such that $\bar{A}(z') \in \mathrm{Psym}(3)$ and, uniformly for all $(z', z_3) \in \Omega_h$, there holds:

(5.4) $$A_h(z', z_3) = G_h^{1/2}(z', z_3) = \bar{A}(z') + hA_1\left(z', \frac{z_3}{h}\right) + \frac{h^2}{2}A_2\left(z', \frac{z_3}{h}\right) + o(h^2).$$

Equivalently, $\bar{A}, A_1, A_2$ solve the following system of equations:

(5.5) $$\bar{A}^2 = \bar{\mathcal{G}}, \qquad 2(\bar{A}A_1)_{\mathrm{sym}} = \mathcal{G}^1, \qquad 2A_1^2 + 2(\bar{A}A_2)_{\mathrm{sym}} = \mathcal{G}^2 \qquad \text{in } \overline{\Omega}.$$

Observe that under the assumption (O), condition W3 on W easily implies:

$$\frac{1}{h}\int_{\Omega_h} \mathrm{dist}^2\big(\nabla u^h(z)\bar{A}^{-1}(z'), \mathrm{SO}(3)\big)\,\mathrm{d}z \le \frac{C}{h}\int_{\Omega_h} \mathrm{dist}^2\big(\nabla u^h(z)A_h^{-1}(z), \mathrm{SO}(3)\big) + h^2\,\mathrm{d}z$$
$$\le C\left(\mathscr{I}_h(u^h) + h^2\right).$$

Consequently, the results of [BLS16] automatically yield the following compactness properties of any sequence of deformations with the quadratic energy scaling:

Theorem 5.2.1. *Assume (O). Let $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ be a sequence of deformations satisfying:*

(5.6) $$\mathscr{I}_h(u^h) \le Ch^2.$$

Then the following properties hold for the rescalings $y^h \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ given by

$$y^h(x', x_3) = u^h(x', hx_3) - \fint_{\Omega_h} u^h(z)\,\mathrm{d}z :$$

(i) There exist $y \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3)$ and $\vec{b} \in \mathrm{W}^{1,2} \cap \mathrm{L}^\infty(\omega, \mathbb{R}^3)$ such that, up to a subsequence:

$$y^h \to y \ \text{ in } \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3) \quad \text{and} \quad \frac{1}{h}\partial_3 y^h \to \vec{b} \ \text{ in } \mathrm{L}^2(\Omega, \mathbb{R}^3), \ \text{ as } h \to 0.$$

(ii) The limit deformation y realizes the reduced midplate metric on ω:

(5.7) $$(\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times 2}.$$

In particular $\partial_1 y$, $\partial_2 y \in \mathrm{L}^\infty(\omega, \mathbb{R}^3)$ and the unit normal $\nu = \frac{\partial_1 y \wedge \partial_2 y}{|\partial_1 y \wedge \partial_2 y|}$ to the surface $y(\omega)$ satisfies: $\nu \in \mathrm{W}^{1,2} \cap \mathrm{L}^\infty(\omega, \mathbb{R}^3)$. The limit displacement $\vec{b}$ is the Cosserat field defined via:

(5.8) $$\vec{b} = (\nabla y)(\bar{\mathcal{G}}_{2\times 2})^{-1}\begin{bmatrix}\bar{\mathcal{G}}_{13}\\ \bar{\mathcal{G}}_{23}\end{bmatrix} + \frac{\sqrt{\det\bar{\mathcal{G}}}}{\sqrt{\det\bar{\mathcal{G}}_{2\times 2}}}\,\nu.$$

Recall that the results in [BLS16] also give:

(5.9) $$\liminf_{h\to 0}\frac{1}{h^2}\frac{1}{h}\int_{\Omega_h} W\big(\nabla u^h(z)\bar{\mathcal{G}}^{-1/2}(z')\big)\,\mathrm{d}z \ge \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x')\Big)\,\mathrm{d}x',$$

with the curvature integrand $(\nabla y)^{\mathsf{T}}\nabla\vec{b}$ quantified by the quadratic forms:

$$(5.10)\qquad \begin{aligned} &\mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3\big(\bar{A}^{-1}(x')\tilde{F}\bar{A}^{-1}(x')\big);\ \tilde{F}\in\mathbb{R}^{3\times 3} \text{ with } \tilde{F}_{2\times 2} = F_{2\times 2}\Big\},\\ &\mathcal{Q}_3(F) = D^2W(\mathrm{I}_3)[F]^2. \end{aligned}$$

The form $\mathcal{Q}_3$ is defined for all $F\in\mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x',\cdot)$ are defined on $F_{2\times 2}\in\mathbb{R}^{2\times 2}$. We remark that we will in this chapter, with abuse of notation, denote by $F_{2\times 2}$ also an arbitrary matrix in $\mathbb{R}^{2\times 2}$, not necessarily being a 2×2 minor of a given 3×3 matrix as in Section 1.1. Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2(x',\cdot)$ are nonnegative definite and depend only on the symmetric parts of their arguments, in view of the assumptions on the elastic energy density W, as it has been shown in Appendix 2.A. Recall also that, the minimization problem in (5.10) has a unique solution among symmetric matrices $\tilde{F}$ satisfying $\tilde{F}_{2\times 2} = F_{2\times 2}$, which is now for each $x'\in\omega$ described by the linear function $F_{2\times 2}\mapsto \ell(x', F_{2\times 2})\in\mathbb{R}^3$ in:

$$(5.11)\qquad \mathcal{Q}_2(x', F_{2\times 2}) = \min\Big\{\mathcal{Q}_3\big(\bar{A}^{-1}(x')(F^*_{2\times 2} + \mathsf{c}\otimes\mathsf{f}_3)_{\mathrm{sym}}\,\bar{A}^{-1}(x')\big):\ \mathsf{c}\in\mathbb{R}^3\Big\}.$$

Recall form Section 1.1 that for a given $F_{2\times 2}\in\mathbb{R}^{2\times 2}$, the 3×3 matrix with principal minor equal $F_{2\times 2}$ and all other entries equal to 0, is denoted by $F^*_{2\times 2}$.

The energy in the right hand side of (5.9) is a Kirchhoff-like fully nonlinear bending, which in case of $\bar{A}\mathsf{f}_3 = \mathsf{f}_3$ reduces to the classical bending content quantifying the second fundamental form $(\nabla y)^{\mathsf{T}}\nabla\vec{b} = (\nabla y)^{\mathsf{T}}\nabla\nu$ on the deformed surface $y(\omega)$. Observe that the vector field $\vec{b}$ in general contains also the shear direction, apart from the normal one. Namely, it determines the preferred direction of "stacking" copies of surfaces $y(\omega)$ on top of each other, in order to obtain the deformed three dimensional shell $u(\Omega_h)$ with minimal energy in (5.3) (see [LP16]).

We now proceed with the derivation of the new Kirchhoff-like plate model corresponding to a more general incompatibility metric G_h, as proposed in the following results.

We start with an observation about projections on polynomial subspaces of L^2. Consider the following Hilbert space, with its norm:

$$(5.12)\qquad \mathbb{E} := \Big(\mathrm{L}^2\big(\Omega,\mathrm{Sym}(2)\big), \|\cdot\|_{\mathcal{Q}_2}\Big),\qquad \|F\|_{\mathcal{Q}_2} = \Big(\int_\Omega \mathcal{Q}_2(x', F(x))\,\mathrm{d}x\Big)^{1/2},$$

associated to the scalar product:

$$\langle F_1, F_2\rangle_{\mathcal{Q}_2} = \int_\Omega \mathcal{L}_{2,x'}\big(F_1(x), F_2(x)\big)\,\mathrm{d}x,$$

where $\mathcal{L}_{2,x'}$ is a bilinear form associated to the quadratic form $\mathcal{Q}_2(x',\cdot)$.

We define $\mathbb{P}_1$ and $\mathbb{P}_2$, respectively, as the orthogonal projections onto the following subspaces of $\mathbb{E}$:

$$(5.13)\qquad \begin{aligned} &\mathbb{E}_1 = \Big\{x_3\mathcal{F}_1(x') + \mathcal{F}_0(x');\ \mathcal{F}_1,\mathcal{F}_0\in\mathrm{L}^2\big(\omega,\mathrm{Sym}(2)\big)\Big\},\\ &\mathbb{E}_2 = \Big\{x_3^2\mathcal{F}_2(x') + x_3\mathcal{F}_1(x') + \mathcal{F}_0(x');\ \mathcal{F}_2,\mathcal{F}_1,\mathcal{F}_0\in\mathrm{L}^2\big(\omega,\mathrm{Sym}(2)\big)\Big\}, \end{aligned}$$

obtained by projecting each $F(x',\cdot)$ on the appropriate polynomial subspaces of the space $\mathrm{L}^2\big((-1/2,1/2),\mathrm{Sym}(2)\big)$ whose orthonormal bases is given in terms of the Legendre polynomials $\{p_i\}_{i=0}^\infty$ in $\mathrm{L}^2(-1/2,1/2)$. The first three polynomials are:

$$p_0(t) = 1,\qquad p_1(t) = \sqrt{12}t,\qquad p_2(t) = \sqrt{5}\big(6t^2 - \frac{1}{2}\big).$$

Lemma 5.2.2. *For every $F \in \mathbb{E}$, we have:*

$$\mathbb{P}_1(F) = 12\Big(\int_{-1/2}^{1/2} tF(x',t)\,\mathrm{d}t\Big)x_3 + \int_{-1/2}^{1/2} F(x',t)\,\mathrm{d}t,$$
$$\mathbb{P}_2(F) = \Big(\int_{-1/2}^{1/2} (180t^2-15)F(x',t)\,\mathrm{d}t\Big)x_3^2 + 12\Big(\int_{-1/2}^{1/2} tF(x',t)\,\mathrm{d}t\Big)x_3 + \Big(\int_{-1/2}^{1/2} (-15t^2+\frac{9}{4})F(x',t)\,\mathrm{d}t\Big)$$

Moreover, the distances from spaces $\mathbb{E}_1$ and $\mathbb{E}_2$ are:

$$\mathrm{dist}^2(F,\mathbb{E}_1) = \int_\omega \Big[\int_{-1/2}^{1/2} \mathcal{Q}_2\Big(x',F(x',t)\Big)\,\mathrm{d}t - 12\mathcal{Q}_2\Big(x',\int_{-1/2}^{1/2} tF(x',t)\,\mathrm{d}t\Big) - \mathcal{Q}_2\Big(x',\int_{-1/2}^{1/2} F(x',t)\,\mathrm{d}t\Big)\Big]\,\mathrm{d}x',$$
$$\mathrm{dist}^2(F,\mathbb{E}_2) = \int_\omega \Big[\int_{-1/2}^{1/2} \mathcal{Q}_2\Big(x',F(x',t)\Big)\,\mathrm{d}t - 180\mathcal{Q}_2\Big(x',\int_{-1/2}^{1/2} (t^2-\frac{1}{12})F(x',t)\,\mathrm{d}t\Big) - 12\mathcal{Q}_2\Big(x',\int_{-1/2}^{1/2} tF(x',t)\,\mathrm{d}t\Big) - \mathcal{Q}_2\Big(x',\int_{-1/2}^{1/2} F(x',t)\,\mathrm{d}t\Big)\Big]\,\mathrm{d}x'.$$

Proof. Observe that:

$$\mathbb{P}_1(F) = p_1(x_3)\int_{-1/2}^{1/2} p_1(t)F(x',t)\,\mathrm{d}t + \int_{-1/2}^{1/2} p_0(t)F(x',t)\,\mathrm{d}t,$$

and similarly:

$$\mathbb{P}_2(F) = p_2(x_3)\int_{-1/2}^{1/2} p_2(t)F(x',t)\,\mathrm{d}t + p_1(x_3)\int_{-1/2}^{1/2} p_1(t)F(x',t)\,\mathrm{d}t + \int_{-1/2}^{1/2} p_0(t)F(x',t)\,\mathrm{d}t,$$

whereas:

$$\mathrm{dist}^2(F,\mathbb{E}_1) = \|F\|^2_{\mathcal{Q}_2} - \|\mathbb{P}_1(F)\|^2_{\mathcal{Q}_2} = \|F\|^2_{\mathcal{Q}_2} - \Big(\|\int_{-1/2}^{1/2} p_1F\,\mathrm{d}t\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_0F\,\mathrm{d}t\|^2_{\mathcal{Q}_2}\Big),$$
$$\mathrm{dist}^2(F,\mathbb{E}_2) = \|F\|^2_{\mathcal{Q}_2} - \|\mathbb{P}_2(F)\|^2_{\mathcal{Q}_2} = \|F\|^2_{\mathcal{Q}_2} - \Big(\|\int_{-1/2}^{1/2} p_2F\,\mathrm{d}t\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_1F\,\mathrm{d}t\|^2_{\mathcal{Q}_2} + \|\int_{-1/2}^{1/2} p_0F\,\mathrm{d}t\|^2_{\mathcal{Q}_2}\Big).$$

The Lemma results then by a straightforward calculation. □

Theorem 5.2.3. *In the setting of Theorem 5.2.1, $\liminf_{h\to 0} \frac{1}{h^2}\mathscr{I}_h(u^h)$ is bounded from below by:*

$$\mathscr{I}_2^O(y) = \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', x_3\nabla y(x')^\mathsf{T}\nabla\vec{b}(x') - \frac{1}{2}\mathcal{G}^1_{2\times 2}(x)\Big)\,\mathrm{d}x$$
$$= \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \big(\nabla y(x')^\mathsf{T}\nabla\vec{b}(x')\big)_{\mathrm{sym}} - \frac{1}{2}\bar{\mathcal{G}}^1_{2\times 2}(x')\Big)\,\mathrm{d}x' + \frac{1}{8}\mathrm{dist}^2\Big(\mathcal{G}^1_{2\times 2},\mathbb{E}_1\Big),$$

where $\bar{\mathcal{G}}^1$ is as in (EF). In the non-oscillatory case (NO) this formula becomes:

$$\mathscr{I}_2(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \big(\nabla y(x')^\mathsf{T}\nabla\vec{b}(x')\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0)\Big)\,\mathrm{d}x'.$$

The first term in $\mathscr{I}_2^O$ coincides with $\mathscr{I}_2$ for the effective metric $\bar{G}$ in (EF).

Proof. The argument follows the proof of [BLS16, Theorem 2.1] and thus we only indicate its new ingredients. Applying the compactness analysis for the z_3-independent metric $\bar{G}$, one obtains the sequence $\{R_h\}_h \subseteq \mathrm{L}^2(\omega, \mathbb{R}^{3\times 3})$ of approximating rotation-valued fields, satisfying:

$$\frac{1}{h}\int_{\Omega_h} |\nabla u^h(z)\bar{A}^{-1}(z') - R_h(z')|^2 \,\mathrm{d}z \leq Ch^2. \tag{5.14}$$

Recall $y^h = u^h(\cdot, h\cdot)$ and define now the family $\{S_h\}_h \subseteq \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ by:

$$S_h(x', x_3) = \frac{1}{h}\Big(R_h^{\mathsf{T}}(x')\nabla_h y^h(x) A_h^{-1}(x', hx_3) - \mathrm{I}_3\Big), \qquad x = (x', x_3) \in \Omega.$$

According to [BLS16], the same quantities, written for the metric $\bar{G}$ rather than G^h:

$$\bar{S}_h(x', x_3) = \frac{1}{h}\Big(R_h^{\mathsf{T}}(x')\nabla_h y^h(x) \bar{A}^{-1}(x') - \mathrm{I}_3\Big), \qquad x = (x', x_3) \in \Omega,$$

converge weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ to $\bar{S}$, such that:

$$\big(\bar{A}(x')\bar{S}(x', x_3)\bar{A}(x')\big)_{2\times 2} = \bar{s}(x') + x_3 \nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x'), \tag{5.15}$$

with some appropriate $\bar{s} \in \mathrm{L}^2(\omega, \mathbb{R}^{2\times 2})$. Observe that:

$$S_h(x', x_3) = \bar{S}_h(x', x_3) + R_h^{\mathsf{T}}(x')\nabla_h y^h(x') \frac{A_h^{-1}(x', hx_3) - \bar{A}^{-1}(x')}{h}$$

and that the term $R_h^{\mathsf{T}}(x')\nabla_h y^h(x)$ converges strongly in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ to $\bar{A}(x')$. On the other hand, the remaining factor converges uniformly on Ω as $h \to 0$, because:

$$\frac{1}{h}\big(A_h^{-1}(x', hx_3) - \bar{A}^{-1}(x')\big) = -\bar{A}^{-1}(x')A_1(x', x_3)\bar{A}^{-1}(x') + O(h) \tag{5.16}$$

Concluding, S_h converge weakly in $\mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ to S, satisfying by (5.15):

$$\big(\bar{A}(x')S(x', x_3)\bar{A}(x')\big)_{2\times 2} = \bar{s}(x') + x_3 \nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - \bar{A}(x')A_1(x', x_3). \tag{5.17}$$

Consequently, using the definition of S_h and frame invariance of W and Taylor expanding W at I_3 on the set $\{|S_h|^2 \leq 1/h\}$, we obtain:

$$\begin{aligned}
\liminf_{h\to 0} \frac{1}{h^2}\mathscr{I}_h(u^h) &= \liminf_{h\to 0} \frac{1}{h^2}\int_\Omega W\big(\mathrm{I}_3 + hS_h(x)\big)\,\mathrm{d}x \\
&\geq \liminf_{h\to 0} \frac{1}{2}\int_{\{|S_h|^2\leq 1/h\}} \mathcal{Q}_3(S_h(x)) + o(|S_h|^2)\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_\Omega \mathcal{Q}_3(S(x))\,\mathrm{d}x = \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', \big(\bar{A}(x')S(x)\bar{A}(x')\big)_{2\times 2}\Big)\,\mathrm{d}x \\
&\geq \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', \big(\bar{A}(x')S(x)\bar{A}(x')\big)_{2\times 2} - \bar{s}(x')\Big)\,\mathrm{d}x.
\end{aligned}$$

The last inequality follows since the effective integrand $\big(\bar{A}S\bar{A}\big)_{2\times 2}$ is the sum of an x_3-independent term $\bar{s}(x')$ and the remaining term with zero x_3-average. We call the right hand side functional above $\mathscr{I}_2^O$ and recall (5.17) and (5.5) to get:

$$\begin{aligned}
\mathscr{I}_2^O(y) &= \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', x_3\nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - \frac{1}{2}\mathcal{G}^1_{2\times 2}(x)\Big)\,\mathrm{d}x = \frac{1}{2}\big\|x_3(\nabla y)^{\mathsf{T}}\nabla\vec{b} - \frac{1}{2}\mathcal{G}^1_{2\times 2}\big\|^2_{\mathcal{Q}_2} \\
&= \frac{1}{2}\big\|\mathbb{P}_1\big(x_3(\nabla y)^{\mathsf{T}}\nabla\vec{b} - \frac{1}{2}\mathcal{G}^1_{2\times 2}\big)\big\|^2_{\mathcal{Q}_2} + \frac{1}{8}\big\|\mathcal{G}^1_{2\times 2} - \mathbb{P}_1(\mathcal{G}^1_{2\times 2})\big\|^2_{\mathcal{Q}_2} \\
&= \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - 6\int_{-1/2}^{1/2} t\mathcal{G}^1_{2\times 2}(x', t)\,\mathrm{d}t\Big)\,\mathrm{d}x' + \frac{1}{8}\mathrm{dist}^2\big(\mathcal{G}^1_{2\times 2}, \mathbb{E}_1\big),
\end{aligned}$$

where we have used the fact that $\mathcal{Q}_2(x', \cdot)$ is a function of its symmetrized argument and Lemma 5.2.2. The formula for $\mathscr{I}_2$ in case (NO) is immediate. □

Our next result is the upper bound, parallel to the lower bound in Theorem 5.2.3:

Theorem 5.2.4. *Assume (O). For every isometric immersion $y \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3)$ of the reduced midplate metric $\bar{\mathcal{G}}_{2\times 2}$ as in (5.7), there exists a sequence $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ such that the sequence $\{y^h = u^h(\cdot, h\cdot)\}_h$ converges in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ to y and:*

$$\lim_{h\to 0} \frac{1}{h^2}\mathscr{I}_h(u^h) = \mathscr{I}_2^O(y) \tag{5.18}$$

Automatically, $\frac{1}{h}\partial_3 y^h$ converges in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$ to $\vec{b} \in \mathrm{W}^{1,2} \cap L^\infty(\omega, \mathbb{R}^3)$ as in (5.8).

Proof. Given an admissible y, we define $\vec{b}$ by (5.8) and also define the matrix field:

$$Q = \big[\partial_1 y,\ \partial_2 y,\ \vec{b}\big] \in \mathrm{W}^{1,2} \cap L^\infty(\omega, \mathbb{R}^{3\times 3}). \tag{5.19}$$

It follows that $Q(x')\bar{A}^{-1}(x') \in \mathrm{SO}(3)$ on ω. The recovery sequence $\{y^h = u^h(\cdot, h\cdot)\}$ satisfying (5.18) is then constructed via a diagonal argument, applied to the explicit deformation fields below. Again, we only indicate the new ingredients with respect to the proof in [BLS16, Theorem 3.1].

We define the vector field $\vec{d} \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$ by:

$$\begin{aligned} \vec{d}(x', x_3) = Q(x')^{\mathsf{T},-1}\bigg(&\frac{x_3^2}{2}\Big(\ell(x', \nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x')) - \frac{1}{2}\begin{bmatrix} \nabla|\vec{b}|^2(x') \\ 0\end{bmatrix}\Big) \\ &- \frac{1}{2}\ell\Big(x', \int_0^{x_3} \mathcal{G}^1(x', t)_{2\times 2}\,\mathrm{d}t\Big) \\ &+ \int_0^{x_3} \mathcal{G}^1(x', t)\,\mathrm{d}t\,\mathsf{f}_3 - \frac{1}{2}\int_0^{x_3} \mathcal{G}^1(x', t)_{33}\,\mathrm{d}t\,\mathsf{f}_3\bigg). \end{aligned} \tag{5.20}$$

In view of definition (5.11), the formula in (5.20) is equivalent to the vector field $\partial_3\vec{d} \in \mathrm{L}^2(\Omega, \mathbb{R}^3)$ being, for each $(x', x_3) \in \Omega$, the unique solution to:

$$\begin{aligned} &\mathcal{Q}_2\Big(x', x_3\nabla y(x')^{\mathsf{T}}\nabla\vec{b}(x') - \frac{1}{2}\mathcal{G}^1_{2\times 2}(x', x_3)\Big) \\ &\quad = \mathcal{Q}_3\bigg(\bar{A}^{-1}(x')\Big(Q^{\mathsf{T}}(x')\Big[x_3\partial_1\vec{b}(x'),\ x_3\partial_2\vec{b}(x'),\ \partial_3\vec{d}(x', x_3)\Big] - \frac{1}{2}\mathcal{G}^1(x', x_3)\Big)\bar{A}^{-1}(x')\bigg). \end{aligned}$$

One then approximates $y, \vec{b}$ by sequences $\{\tilde{y}^h\}_h \subseteq W^{2,\infty}(\omega, \mathbb{R}^3)$, $\{\tilde{b}^h\}_h \subseteq W^{1,\infty}(\omega, \mathbb{R}^3)$ respectively, and request them to satisfy conditions exactly as in the proof of [BLS16, Theorem 3.1]. The warping field $\vec{d}$ is approximated by $d^h(x', x_3) = \int_0^{x_3} \bar{d}^h(x', t)\,\mathrm{d}t$, where:

$$\bar{d}^h \to \bar{d} = \partial_3\vec{d} \quad \text{strongly in } \mathrm{L}^2(\Omega, \mathbb{R}^3) \quad \text{and} \quad h\|\bar{d}^h\|_{\mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^3)} \to 0 \quad \text{as } h \to 0.$$

Finally, we define:

$$u^h(x', hx_3) = \tilde{y}^h(x') + hx_3\tilde{b}^h(x') + h^2 d^h(x', x_3), \qquad x = (x', x_3) \in \Omega, \tag{5.21}$$

so that, with the right approximation error, there holds:

$$\nabla u^h(x', hx_3) \approx Q(x') + h\big[x_3\partial_1\vec{b}(x'),\ x_3\partial_2\vec{b}(x'),\ \partial_3\vec{d}(x', x_3)\big].$$

Using Taylor's expansion of W, the definition (5.20) and the controlled blow-up rates of the approximating sequences, we conclude the construction. □

We conclude this section by noting the following easy direct consequence of Theorems 5.2.3 and 5.2.4:

Corollary 5.2.5. *If the set of* $\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)$ *isometric immersions of* $\bar{\mathcal{G}}_{2\times 2}$ *is nonempty, then the functional* $\mathscr{I}_2^O$ *attains its infimum and:*

$$\lim_{h\to 0}\frac{1}{h^2}\inf \mathscr{I}_h = \min\ \mathscr{I}_2^O.$$

The infima in the left hand side are taken over the set $\mathrm{W}^{1,2}(\Omega_h,\mathbb{R}^3)$, *whereas the minima in the right hand side are taken over* $\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)$ *isometric immersions* y *of* $\bar{\mathcal{G}}_{2\times 2}$.

5.2.2 Identification of the Ch^2 scaling regime

In this section, we identify the equivalent conditions for $\inf \mathscr{I}_h \sim h^2$ in terms of curvatures of the metric tensor $\bar{G}$ in case (NO). We begin by expressing the integrand tensor in the residual energy $\mathscr{I}_2$ in terms of the shape operator on the deformed midplate. Recall that we always use the Einstein summation convention over repeated indices running from 1 to 3.

Lemma 5.2.6. *In the the non-oscillatory setting (NO), let* $y\in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3)$ *be an isometric immersion of the metric* $\bar{\mathcal{G}}_{2\times 2}$, *so that (5.7) holds on* ω. *Define the Cosserat vector* $\vec{b}$ *according to (5.8). Then:*

$$\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0) = \frac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\Pi_y + \frac{1}{\bar{\mathcal{G}}^{33}}\begin{bmatrix}\Gamma^3_{11} & \Gamma^3_{12}\\ \Gamma^3_{12} & \Gamma^3_{22}\end{bmatrix}(x',0), \tag{5.22}$$

for all $x'\in\omega$. *Above,* $\bar{\mathcal{G}}^{33} = \langle \bar{\mathcal{G}}^{-1}\mathrm{f}_3,\mathrm{f}_3\rangle$, *whereas* $\Pi_y = (\nabla y)^{\mathsf{T}}\nabla\nu \in \mathrm{L}^2\big(\omega,\mathrm{Sym}(2)\big)$ *is the second fundamental form of the surface* $y(\omega)\subseteq\mathbb{R}^3$, *and* $\{\Gamma^i_{kl}\}_{i,k,l=1\dots 3}$ *are the Christofel symbols of* G, *as in (1.1).*

Proof. The proof is an extension of the arguments in [BLS16, Theorem 5.3], which we modify for the case of x_3-dependent metric G. Firstly, the fact that $Q^{\mathsf{T}}Q = \bar{\mathcal{G}}$ with Q defined in (5.19), yields:

$$\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} = \Big(\big[\partial_i\bar{\mathcal{G}}_{j3}\big]_{i,j=1,2}\Big)_{\mathrm{sym}} - \big[\langle\partial_{ij}y,\vec{b}\rangle\big]_{i,j=1,2}. \tag{5.23}$$

Also, $\partial_i\bar{\mathcal{G}} = 2\big((\partial_i Q)^{\mathsf{T}}Q\big)_{\mathrm{sym}}$ for $i=1,2$, results in:

$$\langle\partial_{ij}y,\partial_k y\rangle = \frac{1}{2}\big(\partial_i G_{kj} - \partial_j G_{kl} - \partial_k G_{ij}\big) \tag{5.24}$$

and:

$$(\nabla y)^{\mathsf{T}}\partial_{ij}y = \Gamma^m_{ij}(x',0)\begin{bmatrix}\bar{\mathcal{G}}_{m1}\\ \bar{\mathcal{G}}_{m2}\end{bmatrix} \qquad \text{for } \ i,j=1,2.$$

Consequently, we obtain the formula:

$$\begin{aligned}\big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}(\nabla y)^{\mathsf{T}}\partial_{ij}y &= \Big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23},\ \big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}\begin{bmatrix}\bar{\mathcal{G}}_{13}\\ \bar{\mathcal{G}}_{23}\end{bmatrix}\Big]\begin{bmatrix}\Gamma^1_{ij}\\ \Gamma^2_{ij}\\ \Gamma^3_{ij}\end{bmatrix}(x',0)\\ &= \bar{\mathcal{G}}_{m3}\Gamma^m_{ij}(x',0) - \frac{1}{\bar{\mathcal{G}}^{33}}\Gamma^3_{ij}(x',0).\end{aligned}$$

Computing the normal vector ν from (5.8) and noting that $\det \bar{\mathcal{G}}_{2\times 2}/\det \bar{\mathcal{G}} = \bar{\mathcal{G}}^{33}$, we get:

$$\begin{aligned}\Pi_{ij} &= -\langle \partial_{ij} y, \nu\rangle = -\sqrt{\bar{\mathcal{G}}^{33}}\Big(\langle \partial_{ij} y, \vec{b}\rangle - \big[\bar{\mathcal{G}}_{13},\ \bar{\mathcal{G}}_{23}\big]\,(\bar{\mathcal{G}}_{2\times 2})^{-1}(\nabla y)^{\mathsf{T}}\partial_{ij} y\Big)\\ &= \sqrt{\bar{\mathcal{G}}^{33}}\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}\,,ij} - \frac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\Gamma^3_{ij}(x',0) - \frac{\sqrt{\bar{\mathcal{G}}^{33}}}{2}\partial_3 G_{ij}(x',0), \qquad \text{for } \ i,j=1,2,\end{aligned}$$

which completes the proof of (5.22). □

The key result of this section is the following:

Theorem 5.2.7. *The energy scaling beyond the Kirchhoff regime:*

$$\lim_{h\to 0}\frac{1}{h^2}\inf \mathscr{I}_h = 0$$

is equivalent to the following conditions:

(i) in the oscillatory case (O)

(5.25) $\left[\begin{array}{l} \mathcal{G}^1_{2\times 2} \in \mathbb{E}_1 \textit{ or equivalently there holds:} \\ \qquad \mathcal{G}^1_{2\times 2}(x',x_3) = x_3\bar{\mathcal{G}}^1_{2\times 2}(x') \qquad \textit{for all } \ (x',x_3)\in\overline{\Omega}. \\ \textit{Moreover, condition (5.26) below must be satisfied with } G \textit{ replaced by} \\ \textit{the effective metric } \bar{G} \textit{ in (EF). This condition involves only } \bar{\mathcal{G}} \textit{ and } \bar{\mathcal{G}}^1_{2\times 2} \\ \textit{terms of } \bar{G}. \end{array}\right.$

(ii) in the non-oscillatory case (NO)

(5.26) $\left[\begin{array}{l} \textit{There exists } y_0 \in \mathrm{W}^{2,2}(\omega,\mathbb{R}^3) \textit{ satisfying (5.7) and such that:} \\ \qquad \Pi_{y_0}(x') = -\dfrac{1}{\sqrt{\bar{\mathcal{G}}^{33}}}\begin{bmatrix} \Gamma^3_{11} & \Gamma^3_{12} \\ \Gamma^3_{12} & \Gamma^3_{22}\end{bmatrix}(x',0) \qquad \textit{for all } x'\in\omega, \\ \textit{where } \Pi_{y_0} \textit{ is the second fundamental form of the surface } y_0(\omega) \textit{ and } \{\Gamma^i_{jk}\} \\ \textit{are the Christoffel symbols of the metric } G. \end{array}\right.$

The isometric immersion y_0 in (5.26) is automatically smooth (up to the boundary) and it is unique up to rigid motions. Further, condition (5.26) is equivalent to:

(5.27) $\left[\begin{array}{l} \textit{The following Riemann curvatures of the metric } G \textit{ vanish on } \omega\times\{0\}\textit{:} \\ \qquad R_{1212}(x',0) = R_{1213}(x',0) = R_{1223}(x',0) = 0 \qquad \textit{for all } x'\in\omega. \end{array}\right.$

The Riemann curvatures of a given metric G are given by (1.3).

Proof. By Corollary 5.2.5, it suffices to determine the equivalent conditions for $\min \mathscr{I}_2^O = 0$ and $\min \mathscr{I}_2 = 0$. In case (O), the linearity of $x_3 \mapsto \mathcal{G}^1_{2\times 2}(x',x_3)$ is immediate, while condition (5.26) follows in both cases (O) and (NO) by Lemma 5.2.6. Note that the Christoffel symbols $\{\Gamma^i_{jk}\}$ depend only on $\bar{\mathcal{G}}$ and $\partial_3 G_{2\times 2}(x',0)$ in the Taylor expansion of G. This completes the proof of (i) and (ii).

Regularity of y_0 is an easy consequence, via the bootstrap argument, of the continuity equation:

$$\partial_{ij} y_0 = \sum_{m=1}^{2} \gamma_{ij}^m \partial_m y_0 - (\Pi_{y_0})_{ij}\nu_0 \qquad \text{for } i,j=1,2, \tag{5.28}$$

where $\{\gamma_{ij}^m\}_{i,j,m=1\dots2}$ denote the Christoffel symbols of $\bar{\mathcal{G}}_{2\times2}$ on ω. Uniqueness of y_0 is a consequence of (5.26), due to uniqueness of isometric immersion with prescribed second fundamental form.

To show (iv), we argue as in the proof of [BLS16, Theorem 5.5]. The compatibility of $\bar{G}_{2\times2}$ and Π_{y_0} is equivalent to the satisfaction of the related Gauss-Codazzi-Mainardi equations (see (1.4)). By an explicit calculation, we see that the two Codazzi-Mainardi equations become:

$$\begin{aligned}
(\partial_2\Gamma_{11}^3 - \partial_1\Gamma_{12}^3) &- \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{11}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{12}^3\right) + \frac{1}{G^{33}}G^{m3}(\Gamma_{2m}^3\Gamma_{11}^3 - \Gamma_{1m}^3\Gamma_{12}^3) \\
&= \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{12}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{11}^m\right) + \frac{G^{32}}{G^{33}}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2), \\
(\partial_2\Gamma_{12}^3 - \partial_1\Gamma_{22}^3) &- \frac{1}{2}\left(\frac{\partial_2 G^{33}}{G^{33}}\Gamma_{12}^3 - \frac{\partial_1 G^{33}}{G^{33}}\Gamma_{22}^3\right) + \frac{1}{G^{33}}G^{m3}(\Gamma_{2m}^3\Gamma_{12}^3 - \Gamma_{1m}^3\Gamma_{22}^3) \\
&= \left(\sum_{m=1}^{2}\Gamma_{1m}^3\Gamma_{22}^m - \sum_{m=1}^{2}\Gamma_{2m}^3\Gamma_{12}^m\right) - \frac{G^{31}}{G^{33}}(\Gamma_{11}^3\Gamma_{22}^3 - (\Gamma_{12}^3)^2),
\end{aligned}$$

and are equivalent to $R_{121}^3 = R_{221}^3 = 0$ on $\omega \times \{0\}$. The Gauss equation is, in turn, equivalent to $R_{1212} = 0$ exactly as in [BLS16]. The simultaneous vanishing of $R_{121}^3, R_{221}^3, R_{1212}$ is equivalent with the vanishing of R_{1212}, R_{1213} and R_{1223}, which proves the claim in (5.27). □

5.3 Coercivity of the limiting energy $\mathscr{I}_2$

In this section we quantify the statement in Theorem 5.2.7 and prove that in case when either of the limiting energies $\mathscr{I}_2$ or $\mathscr{I}_2^O$ can be minimized to zero, the effective energy $\mathscr{I}_2(y)$ measures the distance of a given isometric immersion y from the kernel: $\ker \mathscr{I}_2 = \big\{Qy_0 + c;\ Q \in SO(3),\ c \in \mathbb{R}^3\big\}$.

Assume that the set of $W^{2,2}(\omega,\mathbb{R}^3)$ isometric immersions y of $\bar{\mathcal{G}}_{2\times2}$ is nonempty, which in view of Theorem 5.2.3 and Theorem 5.2.4 is equivalent to: $\inf \mathscr{I}_h \leq Ch^2$. For each such y, the continuity equation (5.28) combined with Lemma 5.2.6 gives the following formula, valid for all $i,j=1,2$:

$$\partial_{ij} y = \sum_{m=1,2}\gamma_{ij}^m\partial_m y - \sqrt{\bar{\mathcal{G}}^{33}}\Big(\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times2}(x',0)\Big)_{ij}\nu + \frac{\Gamma_{ij}^3}{\sqrt{\bar{\mathcal{G}}^{33}}}\nu \quad \text{on } \omega, \tag{5.29}$$

and resulting in:

$$|\nabla^2 y|^2 = |\Pi_y|^2 + \sum_{i,j=1,2}\bar{\mathcal{G}}_{2\times2} : [\gamma_{ij}^1,\ \gamma_{ij}^2]^{\otimes2} \quad \text{on } \omega.$$

By Lemma 5.2.6 and since $|\nabla y|^2 = \mathrm{tr}\big((\nabla y)^{\mathsf{T}}\nabla y\big) = \mathrm{tr}\,\bar{\mathcal{G}}_{2\times2}$, this yields the bound:

$$\|y - \fint_\omega y\|^2_{W^{2,2}(\omega,\mathbb{R}^3)} \leq C\big(\mathscr{I}_2(y) + 1\big), \tag{5.30}$$

where C is a constant independent of y. Clearly, when condition (5.27) does not hold, so that $\min \mathscr{I}_2 > 0$, the right hand side $C\big(\mathscr{I}_2(y)+1\big)$ above may be replaced by $C\mathscr{I}_2(y)$. On the other hand, in presence of (5.27), the bound (5.30) can be refined to the following coercivity result:

Theorem 5.3.1. *Assume the curvature condition (5.27) on a metric G as in (NO), and let y_0 be the unique (up to rigid motions in $\mathbb{R}^3$) isometric immersion of $\bar{\mathcal{G}}_{2\times 2}$ satisfying (5.26). Then, for all $y \in W^{2,2}(\omega,\mathbb{R}^3)$ such that $(\nabla y)^{\mathsf{T}}\nabla y = \bar{\mathcal{G}}_{2\times 2}$, there holds:*

$$\operatorname{dist}^2_{W^{2,2}(\omega,\mathbb{R}^3)}\Big(y,\ \big\{Ry_0 + \mathsf{c}:\ \ R\in SO(3),\ \mathsf{c}\in\mathbb{R}^3\big\}\Big) \leq C\mathscr{I}_2(y), \tag{5.31}$$

with a constant $C>0$ that depends on G,ω and W but it is independent of y.

Proof. Without loss of generality, we set $\fint_\omega y = \fint_\omega y_0 = 0$. For any $R\in SO(3)$, identity (5.29) implies:

$$\begin{aligned}
&\int_\omega \big|\nabla^2 y - \nabla^2(Ry_0)\big|^2\,\mathrm{d}x' \\
\leq & C\Big(\int_\omega \big|\nabla y - \nabla(Ry_0)\big|^2\,\mathrm{d}x' + \int_\omega \big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0)\big|^2\,\mathrm{d}x' \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \int_\omega |\nu - R\vec{\nu}_0|^2\,\mathrm{d}x'\Big) \\
\leq & C\Big(\int_\omega \big|\nabla y - \nabla(Ry_0)\big|^2\,\mathrm{d}x' + \int_\omega \big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0)\big|^2\,\mathrm{d}x'\Big),
\end{aligned}$$

where we used $\mathscr{I}_2(Ry_0)=0$ and the fact that $\int_\omega |\nu - R\vec{\nu}_0|^2\,\mathrm{d}x' \leq C\int_\omega \big|\nabla y - \nabla(Ry_0)\big|^2\,\mathrm{d}x'$ following, in particular, from $|\partial_1 y\times\partial_2 y| = |\partial_1(Ry_0)\times\partial_2(Ry_0)| = \sqrt{\det\bar{\mathcal{G}}_{2\times 2}}$. Also, the non-degeneracy of quadratic forms $\mathcal{Q}_2(x',\cdot)$ in (5.10), implies the uniform bound:

$$\int_\omega \big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0)\big|^2\,\mathrm{d}x' \leq C\mathscr{I}_2(y).$$

Taking $R\in SO(3)$ as in Lemma 5.3.2 below, (5.31) directly follows in view of (5.32). □

The next weak coercivity estimate has been the essential part of Theorem 5.3.1:

Lemma 5.3.2. *Let y and y_0 be as in Theorem 5.3.1. Then there exists $R\in SO(3)$ such that:*

$$\int_\omega |\nabla y - R\nabla y_0|^2\,\mathrm{d}x' \leq C\int_\omega \big|\big((\nabla y)^{\mathsf{T}}\nabla\vec{b}\big)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G_{2\times 2}(x',0)\big|^2\,\mathrm{d}x', \tag{5.32}$$

with a constant $C>0$ that depends on G,ω but it is independent of y.

Proof. Consider the natural extensions u and u_0 of y and y_0, namely:

$$u(z',z_3) = y(z') + z_3\vec{b}(z'),\quad u_0(z',z_3) = y_0(z') + z_3\vec{b}_0(z') \qquad \text{for all } (z',z_3)\in\Omega_h.$$

Clearly, $u\in W^{1,2}(\Omega_h,\mathbb{R}^3)$ and $u_0\in C^1(\overline{\Omega}_h,\mathbb{R}^3)$ satisfies $\det\nabla u_0>0$ for h sufficiently small. Write:

$$\omega = \bigcup_{k=1}^N \omega_k, \qquad \Omega_h = \bigcup_{k=1}^N \Omega_h^k$$

as the union of $N\geq 1$ open, bounded, connected domains with Lipschitz boundary, such that on each $\{\Omega_h^k = \omega_k\times(-\frac{h}{2},\frac{h}{2})\}_{k=1}^N$, the deformation $u_{0|\Omega_h^k}$ is a C^1 diffeomorphism onto its image $\mathcal{U}_h^k\subseteq\mathbb{R}^3$.

STEP 1. We first prove (5.32) under the assumption $N = 1$. Call $v = u \circ u_0^{-1} \in W^{1,2}(\mathcal{U}_h, \mathbb{R}^3)$ and apply the geometric rigidity estimate [FJM02] (see Theorem 2.2.2) for the existence of $R \in SO(3)$ satisfying:

$$\int_{\mathcal{U}_h} |\nabla v - R|^2 \,d\eta \leq C \int_{\mathcal{U}_h} \mathrm{dist}^2\big(\nabla v, SO(3)\big) \,d\eta, \tag{5.33}$$

with a constant C depending on a particular choice of h (and ultimately k, when $N > 1$), but independent of v. Since $\nabla v(u_0(z)) = \nabla u(z)\big(\nabla u_0(z)\big)^{-1}$ for all $z \in \Omega^h$, we get:

(5.34)
$$\begin{aligned}\int_{\mathcal{U}_h} |\nabla v - R|^2 \,d\eta &= \int_{\Omega_h} (\det \nabla u_0)\big|(\nabla u - R\nabla u_0)(\nabla u_0)^{-1}\big|^2 \,dz \\ &\geq C \int_{\Omega_h} |\nabla u - R\nabla u_0|^2 \,dz \\ &= C \int_{\Omega_h} \Big|\Big[\partial_1 y,\ \partial_2 y,\ \vec{b}\Big] - R\Big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\Big]\Big|^2 + z_3^2|\nabla\vec{b} - R\nabla\vec{b}_0|^2 \,dz \\ &\geq Ch \int_\omega |\nabla y - R\nabla y_0|^2 \,dz'.\end{aligned}$$

Likewise, the change of variables in the right hand side of (5.33) gives:

$$\int_{\mathcal{U}_h} \mathrm{dist}^2\big(\nabla v, SO(3)\big) \,d\eta \leq C \int_{\Omega_h} \mathrm{dist}^2\big((\nabla u)(\nabla u_0)^{-1}, SO(3)\big) \,dz. \tag{5.35}$$

Since $(\nabla u)^\mathsf{T}\nabla u(z', 0) = (\nabla u_0)^\mathsf{T}\nabla u_0(z', 0) = \bar{\mathcal{G}}(z')$, by polar decomposition it follows that: $\nabla u(z', 0) = Q(z') = \bar{R}\bar{\mathcal{G}}^{1/2}(z')$ and $\nabla u_0(z', 0) = Q_0(z') = \bar{R}_0\bar{\mathcal{G}}^{1/2}(z')$ for some $\bar{R}, \bar{R}_0 \in SO(3)$. The notation Q, Q_0 is consistent with that introduced in (5.19). Observe further:

$$\begin{aligned}\nabla u(z', z_3) = Q + z_3\Big[\partial_1\vec{b},\ \partial_2\vec{b},\ 0\Big] &= \bar{R}\bar{\mathcal{G}}^{1/2}\Big(\mathrm{I}_3 + z_3\bar{\mathcal{G}}^{-1}Q^\mathsf{T}\Big[\partial_1\vec{b},\ \partial_2\vec{b},\ 0\Big]\Big) \\ &= \bar{R}\bar{\mathcal{G}}^{1/2}\Big(\mathrm{I}_3 + z_3\bar{\mathcal{G}}^{-1}\Big((\nabla y)^\mathsf{T}\nabla\vec{b})^* + \mathrm{f}_3 \otimes (\nabla\vec{b}|0)^\mathsf{T}\vec{b}\Big)\Big),\end{aligned}$$

and similarly:

$$\nabla u_0(z', z_3) = \bar{R}_0\bar{\mathcal{G}}^{1/2}\Big(\mathrm{I}_3 + z_3\bar{\mathcal{G}}^{-1}\Big((\nabla y_0)^\mathsf{T}\nabla\vec{b}_0)^* + \mathrm{f}_3 \otimes (\nabla\vec{b}_0|0)^\mathsf{T}\vec{b}_0\Big)\Big).$$

Consequently, the integrand in the right hand side of (5.35) becomes:

$$\begin{aligned}(\nabla u)(\nabla u_0)^{-1} = \bar{R}\bar{\mathcal{G}}^{1/2}\bigg(\mathrm{I}_3 + z_3\bar{\mathcal{G}}^{-1}S\Big(\mathrm{I}_3 + z_3\bar{\mathcal{G}}^{-1}\big((\nabla y_0)^\mathsf{T}\nabla\vec{b}_0)^* \\ + \mathrm{f}_3 \otimes (\nabla\vec{b}_0|0)^\mathsf{T}\vec{b}_0\big)\Big)^{-1}\bigg)\bar{\mathcal{G}}^{-1/2}\bar{R}_0^\mathsf{T},\end{aligned} \tag{5.36}$$

where:

$$\begin{aligned}S = \Big((\nabla y)^\mathsf{T}\nabla\vec{b} - (\nabla y_0)^\mathsf{T}\nabla\vec{b}_0\Big)^* + \mathrm{f}_3 \otimes (\nabla\vec{b}|0)^\mathsf{T}\vec{b} - \mathrm{f}_3 \otimes (\nabla\vec{b}_0|0)^\mathsf{T}\vec{b}_0 \\ = \Big((\nabla y)^\mathsf{T}\nabla\vec{b} - (\nabla y_0)^\mathsf{T}\nabla\vec{b}_0\Big)^*_{\mathrm{sym}}.\end{aligned}$$

The last equality follows from the easy facts that, for $i, j = 1, 2$, we have:

$$\begin{aligned}&\langle\partial_i\vec{b}, \vec{b}\rangle = \langle\partial_i\vec{b}_0, \vec{b}_0\rangle = \frac{1}{2}\partial_i\bar{\mathcal{G}}_{33} \\ &\langle\partial_i y, \partial_j\vec{b}\rangle - \langle\partial_j y, \partial_i\vec{b}\rangle = \langle\partial_i y_0, \partial_j\vec{b}_0\rangle - \langle\partial_j y_0, \partial_i\vec{b}_0\rangle = \partial_j\bar{\mathcal{G}}_{i3} - \partial_i\bar{\mathcal{G}}_{j3}.\end{aligned}$$

Thus, (5.35) and (5.36) imply:

$$\begin{aligned}
\int_{\mathcal{U}_h} \operatorname{dist}^2\big(\nabla v, \mathrm{SO}(3)\big)\,\mathrm{d}\eta &\leq C\int_{\Omega_h} \big|(\nabla u)(\nabla u_0)^{-1} - \bar R \bar R_0^{\mathsf{T}}\big|^2\,\mathrm{d}z \\
&\leq C\int_{\Omega_h} \big|z_3 S(z', z_3)\big|^2\,\mathrm{d}z \\
&\leq C\int_{\omega} \Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \big((\nabla y_0)^{\mathsf{T}}\nabla\vec b_0\big)_{\mathrm{sym}}\Big|^2\,\mathrm{d}z' \\
&= Ch\int_{\omega} \Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac12\partial_3 G_{2\times2}(z',0)\Big|^2\,\mathrm{d}z'
\end{aligned} \tag{5.37}$$

with a constant C that depends on G, ω and h, but not on y. We conclude (5.32) in view of (5.33), (5.34) and (5.37).

STEP 2. To prove (5.32) in case $N > 1$, let $k, s : 1, \dots, N$ be such that $\omega_k \cap \omega_s \neq \emptyset$. Define:

$$F = \Big(\int_{\Omega_h^k\cap\Omega_h^s} \det\nabla u_0\,\mathrm{d}z\Big)^{-1}\int_{\Omega_h^k\cap\Omega_h^s} (\det\nabla u_0)(\nabla u)(\nabla u_0)^{-1}\,\mathrm{d}z \in \mathbb{R}^{3\times3}.$$

Denote by $R_k, R_s \in \mathrm{SO}(3)$ the corresponding rotations in (5.32) on ω_k, ω_s. For $i \in \{k, s\}$ we have:

$$\begin{aligned}
|F - R_i|^2 &= \Big|\Big(\int_{\Omega_h^k\cap\Omega_h^s} \det\nabla u_0\,\mathrm{d}z\Big)^{-1}\int_{\Omega_h^k\cap\Omega_h^s} (\det\nabla u_0)\big(\nabla u - R_i\nabla u_0\big)(\nabla u_0)^{-1}\,\mathrm{d}z\Big|^2 \\
&\leq C\int_{\Omega_h^k\cap\Omega_h^s} |\nabla u - R_i\nabla u_0|^2\,\mathrm{d}z \leq C\int_{\Omega_h^i} |\nabla u - R_i\nabla u_0|^2\,\mathrm{d}z \\
&\leq \int_{\omega_i} \Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac12\partial_3 G_{2\times2}(z',0)\Big|^2\,\mathrm{d}z',
\end{aligned}$$

where for the sake of the last bound we applied the intermediate estimate in (5.34) to the left hand side of (5.33), as discussed in the previous step. Consequently:

$$|R_k - R_s|^2 \leq C\int_{\omega} \Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac12\partial_3 G_{2\times2}(z',0)\Big|^2\,\mathrm{d}z',$$

and thus:

$$\begin{aligned}
\int_{\omega_k} |\nabla y - R_s\nabla y_0|^2\,\mathrm{d}z' &\leq 2\Big(\int_{\omega_k} |\nabla y - R_k\nabla y_0|^2\,\mathrm{d}z' + \int_{\omega_k} |R_k - R_s|^2|\nabla y_0|^2\,\mathrm{d}z'\Big) \\
&\leq C\int_{\omega} \Big|\big((\nabla y)^{\mathsf{T}}\nabla\vec b\big)_{\mathrm{sym}} - \frac12\partial_3 G_{2\times2}(z',0)\Big|^2\,\mathrm{d}z'.
\end{aligned}$$

This shows that one can take one and the same $R = R_1$ on each $\{\omega_k\}_{k=1}^N$, at the expense of possibly increasing the constant C by a controlled factor depending only on N. The proof of (5.32) is done. □

Remark 5.3.3. A similar reasoning as in the proof of Lemma 5.3.2, yields a quantitative version of the uniqueness of isometric immersion with a prescribed second fundamental form compatible to the metric by the Gauss-Codazzi-Mainardi equations. More precisely, given a smooth metric g in $\omega \subseteq \mathbb{R}^2$, for every two isometric immersions $y_1, y_2 \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3)$ of g, there holds:

$$\min_{R\in\mathrm{SO}(3)} \int_\omega |\nabla y_1 - R\nabla y_2|^2\,\mathrm{d}x' \leq C\int_\omega |\Pi_{y_1} - \Pi_{y_2}|^2\,\mathrm{d}x',$$

with a constant $C > 0$, depending on g and ω but independent of y_1 and y_2. ■

5.4 Rigorous derivation of von Kármán-like plate models $\mathscr{I}_4^O$ and $\mathscr{I}_4$

In this and the next sections we assume that:

$$\lim_{h\to 0}\frac{1}{h^2}\inf \mathscr{I}_h = 0. \tag{5.38}$$

Recall that by Theorem 5.2.7 this condition is equivalent to the existence of a (automatically smooth and unique up to rigid motions) vector field $y_0 : \overline{\omega} \to \mathbb{R}^3$ satisfying:

$$(\nabla y_0)^{\mathsf{T}}\nabla y_0 = \bar{\mathcal{G}}_{2\times 2} \qquad \text{and} \qquad \big((\nabla y_0)^{\mathsf{T}}\nabla \vec{b}_0\big)_{\mathrm{sym}} = \frac{1}{2}\bar{\mathcal{G}}^1_{2\times 2} \quad \text{on } \omega, \tag{5.39}$$

where in the oscillatory case (O) the symmetric x'-dependent matrix $\mathcal{G}^1$ is given in (EF) and there must be $\mathcal{G}^1_{2\times 2} = x_3\bar{\mathcal{G}}^1_{2\times 2}$, whereas in the non-oscillatory (NO) case $\bar{\mathcal{G}}^1(x')$ is simply $\partial_3 G(x',0)$. The (smooth) Cosserat field $\vec{b}_0 : \overline{\omega} \to \mathbb{R}^3$ in (5.8) is uniquely given by requesting that:

$$Q_0 := \Big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\Big] \qquad \text{satisfies:} \qquad Q_0^{\mathsf{T}}Q_0 = \bar{\mathcal{G}}, \quad \det Q_0 > 0 \quad \text{on } \omega,$$

with notation similar to (5.19). We now introduce the new vector field $\vec{d}_0 : \overline{\Omega} \to \mathbb{R}^3$ through:

$$Q_0^{\mathsf{T}}\Big[x_3\partial_1\vec{b}_0(x'),\ x_3\partial_2\vec{b}_0(x'),\ \partial_3\vec{d}_0(x',x_3)\Big] - \frac{1}{2}\mathcal{G}^1(x',x_3) \in \mathrm{Skew}(3), \tag{5.40}$$

justified by (5.39) and in agreement with the construction (5.20) of second order terms in the recovery sequence for the Kirchhoff limiting energies. Explicitly, we have:

$$\vec{d}_0(x',x_3) = Q_0^{-\mathsf{T}}(x')\Big(\int_0^{x_3}\mathcal{G}^1(x',t)\,\mathrm{d}t\,\mathsf{f}_3 - \frac{1}{2}\int_0^{x_3}\mathcal{G}^1(x',t)_{33}\,\mathrm{d}t\,\mathsf{f}_3 - \frac{x_3^2}{2}\begin{bmatrix}(\nabla\vec{b}_0)^{\mathsf{T}}\vec{b}_0(x')\\ 0\end{bmatrix}\Big).$$

In what follows, the smooth matrix field in (5.40) will be referred to as $P_0 : \overline{\Omega} \to \mathbb{R}^{3\times 3}$, namely:

$$P_0(x',x_3) = \Big[x_3\partial_1\vec{b}_0(x'),\ x_3\partial_2\vec{b}_0(x'),\ \partial_3\vec{d}_0(x',x_3)\Big]. \tag{5.41}$$

In the non-oscillatory case (NO), the above formulas become:

$$\begin{aligned}&\vec{d}_0 = \frac{x_3^2}{2}\tilde{d}_0(x'), \qquad P_0(x',x_3) = x_3\Big[\partial_1\vec{b}_0,\ \partial_2\vec{b}_0,\ \tilde{d}_0\Big](x'),\\ &\text{where:} \quad \tilde{d}_0(x') = Q_0^{\mathsf{T},-1}(x')\Big(\partial_3 G(x',0)\mathsf{f}_3 - \frac{1}{2}\partial_3 G(x',0)_{33}\mathsf{f}_3 - \begin{bmatrix}(\nabla\vec{b}_0)^{\mathsf{T}}\vec{b}_0(x')\\ 0\end{bmatrix}\Big).\end{aligned} \tag{5.42}$$

We also note that the assumption $\int_{-1/2}^{1/2}\mathcal{G}^1(x',t)\,\mathrm{d}t = 0$ implies:

$$\int_{-1/2}^{1/2} P_0(x',x_3)\,\mathrm{d}x_3 = 0 \qquad \text{for all } x' \in \overline{\omega}. \tag{5.43}$$

With the aid of $\vec{d}_0$ we now construct the sequence of deformations with low energy:

Lemma 5.4.1. *Assume (O). Then (5.38) implies:*

$$\inf \mathscr{I}_h \le Ch^4.$$

Proof. Define the sequence of smooth maps $u^h : \overline{\Omega}_h \to \mathbb{R}^3$ by:

$$u^h(z', z_3) = y_0(z') + z_3\vec{b}_0(x') + h^2\vec{d}_0\left(z', \frac{z_3}{h}\right), \qquad z = (z', z_3) \in \overline{\Omega}_h. \tag{5.44}$$

In order to compute $\nabla u^h A_h^{-1}$, recall the expansion of A_h^{-1}, so that:

$$\nabla u^h(z) A_h^{-1}(z) = Q_0(z')\bar{A}^{-1}(z')\big(\mathrm{I}_3 + hS_h(z) + O(h^2)\big), \tag{5.45}$$

where for every $z \in \Omega_h$:

$$S_h(z) = \bar{A}^{-1}(z')\Big(Q_0^{\mathsf{T}}(z')P_0\big(z', \frac{z_3}{h}\big) - \bar{A}(z')A_1\big(z', \frac{z_3}{h}\big)\Big)\bar{A}^{-1}(z').$$

By frame invariance of the energy density W and since $Q_0(z')\bar{A}^{-1}(z') \in \mathrm{SO}(3)$, we obtain:

$$\begin{aligned} W\big(\nabla u^h(z)A_h^{-1}(z)\big) &= W\big(\mathrm{I}_3 + hS_h(z) + O(h^2)\big) = W\big(\mathrm{I}_3 + h(S_h)_{\mathrm{sym}}(z) + O(h^2)\big) \\ &= W\big(\mathrm{I}_3 + O(h^2)\big) = O(h^4), \end{aligned}$$

where we also used the fact that $(S_h)_{\mathrm{sym}}(z) = 0$ following directly from the definition (5.40). This implies that $\mathscr{I}_h(u^h) = O(h^4)$ as well, proving the claim. ☐

Lemma 5.4.2. *Assume (O) and (5.38). For an open, Lipschitz subset $\mathcal{V} \subseteq \omega$, denote:*

$$\mathcal{V}_h = \mathcal{V} \times \left(-\frac{h}{2}, \frac{h}{2}\right), \qquad \mathscr{I}_h(u^h, \mathcal{V}_h) = \frac{1}{h}\int_{\mathcal{V}_h} W\big(\nabla u^h(z)A_h^{-1}(z)\big)\,\mathrm{d}z.$$

If y_0 is injective on $\mathcal{V}$, then for every $u^h \in \mathrm{W}^{1,2}(\mathcal{V}_h, \mathbb{R}^3)$ there exists $\bar{R}_h \in \mathrm{SO}(3)$ such that:

$$\frac{1}{h}\int_{\mathcal{V}_h}\Big|\nabla u^h(z) - \bar{R}_h\Big(Q_0(z') + hP_0\big(z', \frac{z_3}{h}\big)\Big)\Big|^2\,\mathrm{d}z \le C\left(\mathscr{I}_h(u^h, \mathcal{V}_h) + h^3|\mathcal{V}_h|\right), \tag{5.46}$$

with the smooth correction matrix field P_0 in (5.41). The constant C in (5.46) is uniform for all subdomains $\mathcal{V}_h \subseteq \Omega_h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants.

Proof. The proof, similar to [LRR17, Lemma 2.2], is a combination of the change of variable argument in Lemma 5.3.2 and the low energy deformation construction in Lemma 5.4.1. Observe first that:

$$Q_0(z') + hP_0\left(z', \frac{z_3}{h}\right) = \nabla Y^h(z', z_3) + O(h^2),$$

where by $Y^h : \bar{\Omega}_h \to \mathbb{R}^3$ we denote the smooth vector fields in (5.44). It is clear that for sufficiently small $h > 0$, each $Y^h_{|\mathcal{V}_h}$ is a smooth diffeomorphism onto its image $\mathcal{U}_h \subseteq \mathbb{R}^3$, satisfying uniformly: $\det \nabla Y^h > c > 0$. We now consider $v^h = u^h \circ (Y^h)^{-1} \in \mathrm{W}^{1,2}(\mathcal{U}_h, \mathbb{R}^3)$. By the rigidity estimate [FJM02]:

$$\int_{\mathcal{U}_h} |\nabla v^h - \bar{R}_h|^2\,\mathrm{d}\eta \le C\int_{\mathcal{U}_h} \mathrm{dist}^2\big(\nabla v^h, \mathrm{SO}(3)\big)\,\mathrm{d}\eta, \tag{5.47}$$

for some rotation $\bar{R}_h \in \mathrm{SO}(3)$. Noting that: $(\nabla v^h) \circ Y^h = (\nabla u^h)(\nabla Y^h)^{-1}$ in the set $\mathcal{V}_h$, the change of variable formula yields for the left hand side in (5.47):

$$\begin{aligned} \int_{\mathcal{U}_h} |\nabla v^h - \bar{R}_h|^2\,\mathrm{d}\eta &= \int_{\mathcal{V}_h} (\det \nabla Y^h)\big|(\nabla u^h)(\nabla Y^h)^{-1} - \bar{R}_h\big|^2\,\mathrm{d}z \\ &\ge c\int_{\mathcal{V}_h}\Big|\nabla u^h - \bar{R}_h\Big(Q_0(z') + hP_0\big(z', \frac{z_3}{h}\big) + O(h^2)\Big)\Big|^2\,\mathrm{d}z \\ &\ge c\int_{\mathcal{V}_h}\Big|\nabla u^h - \bar{R}_h\Big(Q_0(z') + hP_0\big(z', \frac{z_3}{h}\big)\Big)\Big|^2\,\mathrm{d}z - c\int_{\mathcal{V}_h} O(h^4)\,\mathrm{d}z. \end{aligned}$$

Similarly, the right hand side in (5.47) can be estimated by:

$$\begin{aligned}\int_{\mathcal{U}_h} \operatorname{dist}^2\big(\nabla v^h, \mathrm{SO}(3)\big)\,\mathrm{d}\eta &= \int_{\mathcal{V}_h} (\det \nabla Y^h)\operatorname{dist}^2\big((\nabla u^h)(\nabla Y^h)^{-1}, \mathrm{SO}(3)\big)\,\mathrm{d}z\\ &\leq C\int_{\mathcal{V}_h} \operatorname{dist}^2\big((\nabla u^h)A_h^{-1}A_h(\nabla Y^h)^{-1}, \mathrm{SO}(3)\big)\,\mathrm{d}z\\ &\leq C\int_{\mathcal{V}_h} \operatorname{dist}^2\Big((\nabla u^h)A_h^{-1}, \mathrm{SO}(3)(\nabla Y^h)A_h^{-1}\Big)\,\mathrm{d}z.\end{aligned}$$

Recall that from (5.45) we have: $(\nabla Y^h)A_h^{-1} \in \mathrm{SO}(3)\big(\mathrm{I}_3 + hS_h + O(h^2)\big) \subseteq \mathrm{SO}(3)\big(\mathrm{I}_3 + O(h^2)\big)$, since $S_h \in \mathrm{Skew}(3)$. Consequently, the above bound becomes:

$$\begin{aligned}\int_{\mathcal{U}_h} \operatorname{dist}^2\big(\nabla v^h, \mathrm{SO}(3)\big)\,\mathrm{d}\eta &\leq C\int_{\mathcal{V}_h} \operatorname{dist}^2\Big((\nabla u^h)A_h^{-1}, \mathrm{SO}(3)(\mathrm{I}_3 + O(h^2))\Big)\,\mathrm{d}z\\ &\leq C\int_{\mathcal{V}_h} \operatorname{dist}^2\Big((\nabla u^h)A_h^{-1}, \mathrm{SO}(3)\Big) + O(h^4)\,\mathrm{d}z.\end{aligned}$$

The estimate (5.46) follows now in view of (5.47) and by the lower bound on energy density W. □

The well-known approximation technique [FJM02] in a combination with the arguments in [LRR17, Corollary 2.3], yield the following approximation result that can be seen as a higher order counterpart of (5.14):

Corollary 5.4.3. *Assume (O) and (5.38). Then, for any $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ satisfying: $\mathscr{I}_h(u^h) \leq Ch^4$, there exists a sequence of rotation-valued maps $R_h \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^{3\times 3})$, such that with P_0 defined in (5.41) we have:*

$$\begin{aligned}&\frac{1}{h}\int_{\Omega_h} \Big|\nabla u^h(z) - R_h(z')\Big(Q_0(z') + hP_0\big(z', \frac{z_3}{h}\big)\Big)\Big|^2\,\mathrm{d}z \leq Ch^4,\\ &\int_\omega |\nabla R_h(z')|^2\,\mathrm{d}z' \leq Ch^2.\end{aligned} \tag{5.48}$$

5.4.1 Compactness and Γ-limit under Ch^4 energy bound

In this section, we derive the Γ-convergence result for the energy functionals $\mathscr{I}_h$ in the von Karman scaling regime. The general form of the limiting energy $\mathscr{I}_4^O$ will be further discussed and split into the stretching, bending, curvature and excess components in Section 5.4.2. We begin by stating the compactness result, that is the higher order version of Theorem 5.2.1.

Theorem 5.4.4. *Assume (O) and (5.38). Fix y_0 solving (5.39) and normalize it to have: $\int_\omega y_0(x')\,\mathrm{d}x' = 0$. Then, for any sequence of deformations $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ satisfying:*

$$\mathscr{I}_h(u^h) \leq Ch^4, \tag{5.49}$$

there exists a sequence $\{\bar{R}_h\}_h \subseteq \mathrm{SO}(3)$ such that the following convergences (up to a subsequence) below, hold for $y^h \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$:

$$y^h(x', x_3) = \bar{R}_h^{\mathsf{T}}\Big(u^h(x', hx_3) - \fint_{\Omega_h} u^h\,\mathrm{d}x\Big).$$

(i) $y^h \to y_0$ strongly in $\mathrm{W}^{1,2}(\Omega, \mathbb{R}^3)$ and $\frac{1}{h}\partial_3 y^h \to \vec{b}_0$ strongly in $\mathrm{L}^2(\Omega, \mathbb{R}^3)$, as $h \to 0$.

(*ii*) *There exists* $V \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3)$ *and* $\mathbb{S} \in \mathrm{L}^2(\omega, \mathrm{Sym}(2))$ *such that, as* $h \to 0$*:*

$$V^h(x') = \frac{1}{h} \int_{-1/2}^{1/2} y^h(x', x_3) - \big(y_0(x') + h x_3 \vec{b}_0(x')\big) \, \mathrm{d}x_3 \to V \quad \textit{strongly in } \mathrm{W}^{1,2}(\omega, \mathbb{R}^3)$$
$$\frac{1}{h}\big((\nabla y_0)^{\mathsf{T}} \nabla V^h\big)_{\mathrm{sym}} \rightharpoonup \mathbb{S} \quad \textit{weakly in } \mathrm{L}^2(\omega, \mathbb{R}^{2\times 2}).$$

(*iii*) *The limiting displacement* V *satisfies:* $\big((\nabla y_0)^{\mathsf{T}} \nabla V\big)_{\mathrm{sym}} = 0$ *in* ω.

We omit the proof because it follows as in [LRR17, Theorem 3.1] in view of condition (5.43). We only recall the definitions used in the sequel. The rotations $\bar{R}_h$ are given by:

$$\bar{R}_h = \mathbb{P}_{\mathrm{SO}(3)} \fint_{\Omega_h} \nabla u^h(z) Q_0^{-1}(z') \, \mathrm{d}z$$

and (5.48) implies that they satisfy, for some limiting rotation $\bar{R}$:

$$\int_\omega |R_h(x') - \bar{R}_h|^2 \, \mathrm{d}x' \leq Ch^2 \qquad \text{and} \qquad \bar{R}_h \to \bar{R} \in \mathrm{SO}(3). \tag{5.50}$$

Consequently:

$$S_h = \frac{1}{h}\big(\bar{R}_h^{\mathsf{T}} R_h - \mathrm{I}_3\big) \rightharpoonup S \qquad \text{weakly in } \mathrm{W}^{1,2}(\omega, \mathbb{R}^{3\times 3}) \tag{5.51}$$

The field $S \in \mathrm{W}^{1,2}\big(\omega, \mathrm{Skew}(3)\big)$ is such that $(\nabla y_0)^{\mathsf{T}} \nabla V = \big(Q_0^{\mathsf{T}} S Q_0\big)_{2\times 2} \in \mathrm{Skew}(2)$, which allows for defining a new vector field $\vec{p} \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^3)$ through:

$$[\nabla V, \ \vec{p}] = S Q_0 \quad \text{or equivalently:} \ \ \vec{p}(x') = -Q_0^{-\mathsf{T}} \begin{bmatrix} \nabla V^{\mathsf{T}} \vec{b}_0 \\ 0 \end{bmatrix} (x') \quad \text{for all } x' \in \omega. \tag{5.52}$$

Finally, by (5.48) we note the uniform boundedness of the fields $\{Z_h\}_h \subseteq \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3})$ below, together with their convergence (up to as subsequence) as $h \to 0$:

$$Z_h(x) = \frac{1}{h^2}\Big(\nabla u^h(x', hx_3) - R_h(x')\big(Q_0(x') + hP_0(x', x_3)\big)\Big) \rightharpoonup Z(x) \quad \text{weakly in } \mathrm{L}^2(\Omega, \mathbb{R}^{3\times 3}). \tag{5.53}$$

Rearranging terms and using the previously established convergences, it can be shown that:

$$\mathbb{S}(x') = \left(Q_0^{\mathsf{T}}(x') \bar{R}^{\mathsf{T}} \int_{-1/2}^{1/2} Z(x', x_3) \, \mathrm{d}x_3\right)_{2\times 2, \mathrm{sym}} - \frac{1}{2} \nabla V(x')^{\mathsf{T}} \nabla V(x') \qquad \text{for all } x' \in \omega. \tag{5.54}$$

Theorem 5.4.5. *In the setting of Theorem 5.4.4,* $\liminf_{h\to 0} \frac{1}{h^4} \mathscr{I}_h(u^h)$ *is bounded below by:*

$$\mathscr{I}_4^O(V, \mathbb{S}) = \frac{1}{2} \int_\Omega \mathcal{Q}_2\Big(x', I(x') + x_3 III(x') + II(x)\Big) \, \mathrm{d}x = \frac{1}{2}\|I + x_3 III + II\|_{\mathcal{Q}_2}^2,$$

where:

$$\begin{aligned} I(x') &= \mathbb{S}(x') + \frac{1}{2} \nabla V(x')^{\mathsf{T}} \nabla V(x') - \nabla y_0(x')^{\mathsf{T}} \nabla \int_{-1/2}^{1/2} \vec{d}_0(x) \, \mathrm{d}x_3, \\ III(x') &= \nabla y_0(x')^{\mathsf{T}} \nabla \vec{p}(x') + \nabla V(x')^{\mathsf{T}} \nabla \vec{b}_0, \\ II(x) &= \frac{x_3^2}{2} \nabla \vec{b}_0(x')^{\mathsf{T}} \nabla \vec{b}_0(x') + \nabla y_0(x')^{\mathsf{T}} \nabla' \vec{d}_0(x) - \frac{1}{4} \mathcal{G}_{2\times 2}^2(x). \end{aligned} \tag{5.55}$$

Proof. STEP 1. Towards estimating $\mathscr{I}_h(u^h)$, we replace the argument $\nabla u^h(z)A_h^{-1}(z)$ of the frame invariant density W by:

$$(5.56)\quad \big(Q_0\bar{A}^{-1}\big)^{\mathsf{T}}(z')R_h^{\mathsf{T}}(z')\nabla u^h(z)A_h^{-1}(z) = \big(Q_0\bar{A}^{-1}\big)^{\mathsf{T}}(z')Q_0(z')A_h^{-1}(z) + h\bar{A}^{-1}(z')Q_0^{\mathsf{T}}(z')P_0\left(z',\frac{z_3}{h}\right)A_h^{-1}(z) + h^2 I_3^h\left(z',\frac{z_3}{h}\right),\qquad \text{for all } z\in\Omega^h,$$

where I_3^h is given in (5.59). Calculating the higher order expansion of (5.16) and writing it in terms of the rescaled variable $(x',x_3)\in\Omega$:

$$(5.57)\quad A_h^{-1}(x',hx_3) = \bar{A}^{-1}(x') + \bar{A}^{-1}(x')\Big(-hA_1(x',x_3) + h^2A_1(x',x_3)\bar{A}^{-1}(x')A_1(x',x_3) - \frac{h^2}{2}A_2(x',x_3)\Big)\bar{A}^{-1}(x') + o(h^2),$$

the expressions in (5.56) can be written as:

$$(5.58)\quad \begin{aligned}&\big(Q_0\bar{A}^{-1}\big)^{\mathsf{T}}(x')R_h^{\mathsf{T}}(x')\nabla u^h(x',hx_3)A_h^{-1}(x',hx_3)\\ &\qquad = \mathrm{I}_3 + hI_1(x',x_3) + h^2\Big(I_2(x',x_3) + I_3^h(x',x_3)\Big) + o(h^2),\end{aligned}$$

for all $x\in\Omega$, where $I_1:\Omega\to \mathrm{Skew}(3)$ and $I_2:\Omega\to\mathbb{R}^{3\times3}$ are smooth matrix fields, given by:

$$\begin{aligned}I_1(x) =& \bar{A}^{-1}(x')\Big(Q_0^{\mathsf{T}}(x')P_0(x) - \bar{A}(x')A_1(x)\Big)\bar{A}^{-1}(x')\\ I_2(x) =& \bar{A}^{-1}(x')\Big(\bar{A}(x')A_1(x)\bar{A}^{-1}(x')A_1(x) - \frac12\bar{A}(x')A_2(x)\\ &\qquad - Q_0^{\mathsf{T}}(x')P_0(x)\bar{A}^{-1}(x')A_1(x)\Big)\bar{A}^{-1}(x').\end{aligned}$$

The fact that $I_1(x)\in\mathrm{Skew}(3)$ follows from (5.40). Also, we have:

$$(5.59)\quad \begin{aligned}I_3^h(x) =& \bar{A}^{-1}(x')Q_0^{\mathsf{T}}(x')R_h^{\mathsf{T}}(x')Z_h(x)A_h^{-1}(x',hx_3)\\ &\rightharpoonup I_3(x) = \bar{A}^{-1}(x')Q_0^{\mathsf{T}}(x')\bar{R}^{\mathsf{T}}Z(x)\bar{A}^{-1}(x)\qquad \text{weakly in } \mathrm{L}^2(\Omega,\mathbb{R}^{3\times3}),\end{aligned}$$

where we used (5.56) and (5.50) to pass to the limit with $(R_h)^{\mathsf{T}}$. As in the proof of Theorem 5.2.3, we now identify the "good" sets $\{|I_3^h|^2\leq 1/h\}\subseteq\Omega$ and employ (5.58) to write there the following Taylor's expansion of $W(\nabla u^h A_h^{-1})$:

$$(5.60)\quad \begin{aligned}W\Big(\nabla u^h(x',hx_3)A_h^{-1}(x',hx_3)\Big) &= W\Big(\mathrm{I}_3 + hI_1(x) + h^2(I_2(x)+I_3^h(x)) + o(h^2)\Big)\\ &= W\Big(e^{-hI_1(x)}\big(\mathrm{I}_3 + hI_1(x) + h^2(I_2(x)+I_3^h(x))\big) + o(h^2)\Big)\\ &= W\Big(\mathrm{I}_3 + h^2\big(I_2 - \frac12 I_1^2 + I_3^h\big) + o(h^2)\Big)\\ &= \frac{h^4}{2}\mathcal{Q}_3\Big(\big(I_2 - \frac12 I_1^2 + I_3^h\big)_{\mathrm{sym}}\Big) + o(h^4).\end{aligned}$$

Above, we repeatedly used the frame invariance of W and the exponential formula:

$$e^{-hI_1} = \mathrm{I}_3 - hI_1 + \frac{h^2}{2}I_1^2 + O(h^3).$$

Since the weak convergence in (5.59) implies convergence of measures $\mathcal{L}^3\big(\Omega\backslash\{|I_3^h|^2\leq 1/h\}\big)\to 0$ as $h\to0$, with the help of (5.60) we finally arrive at:

$$(5.61)\quad \begin{aligned}\liminf_{h\to0}\frac{1}{h^4}\frac1h\int_{\Omega_h} W\big(\nabla u^h(z)A_h^{-1}(z)\big)\,\mathrm{d}z &\geq \liminf_{h\to0}\frac12\int_{\{|I_3^h|^2\leq1/h\}}\mathcal{Q}_3\left(I_2-\frac12I_1^2+I_3^h\right)\,\mathrm{d}x\\ &\geq \frac12\int_\Omega\mathcal{Q}_3\left(I_2-\frac12I_1^2+I_3\right)\,\mathrm{d}x = \frac12\int_\Omega\mathcal{Q}_2\Big(x',\Big(\bar{A}\big(I_2-\frac12I_1^2+I_3\big)\bar{A}\Big)_{2\times2}\Big)\,\mathrm{d}x.\end{aligned}$$

STEP 2. We now compute the effective integrand in (5.61). Firstly, by (5.5) a direct calculation yields:

$$(5.62)\qquad \big(I_2(x) - \frac{1}{2}I_1^2(x)\big)_{\mathrm{sym}} = \big(I_2\big)_{\mathrm{sym}} + \frac{1}{2}I_1^{\mathsf{T}}I_1 = \frac{1}{2}\bar{A}^{-1}(x')\Big(P_0^{\mathsf{T}}P_0(x) - \frac{1}{2}\mathcal{G}^2(x)\Big)\bar{A}^{-1}(x')$$

Secondly, to address the symmetric part of the limit I_3 in (5.59), consider functions $f^{s,h}:\Omega\to\mathbb{R}^3$:

$$f^{s,h}(x) = \fint_0^s \Big(h\bar{R}_h^{\mathsf{T}}Z_h(x', x_3+t) + S_h(x')\big(Q_0(x') + hP_0(x', x_3+t)\big)\Big)\mathsf{f}_3\,\mathrm{d}t.$$

By (5.51) it easily follows that:

$$(5.63)\qquad f^{s,h}\to S\vec{b}_0 = \vec{p} \quad\text{and}\quad \partial_3 f^{s,h}\to 0 \qquad \text{strongly in } \mathrm{L}^2(\Omega,\mathbb{R}^3), \quad \text{as } h\to 0.$$

On the other hand, we write an equivalent form of $f^{s,h}$ and compute the tangential derivatives:

$$\begin{aligned} f^{s,h}(x) &= \frac{1}{h^2 s}\big(y^h(x', x_3+s) - y^h(x', x_3)\big) - \frac{1}{h}\vec{b}_0(x') - \frac{1}{s}\Big(\vec{d}_0(x', x_3+s) - \vec{d}_0(x', x_3)\Big),\\ \partial_i f^{s,h}(x) &= \frac{1}{s}\bar{R}_h^{\mathsf{T}}\big(Z_h(x', x_3+s) - Z_h(x', x_3)\big)\mathsf{f}_i + S_h(x')\partial_i\vec{b}_0(x')\\ &\qquad - \frac{1}{s}\Big(\partial_i\vec{d}_0(x', x_3+s) - \partial_i\vec{d}_0(x', x_3)\Big)\end{aligned}$$

for $i = 1,2$. In view of (5.50) and (5.51), convergence in (5.63) can thus be improved to: $f^{s,h}\rightharpoonup\vec{p}$ weakly in $\mathrm{W}^{1,2}(\Omega,\mathbb{R}^3)$ as $h\to 0$. Equating the derivatives ∂_1,∂_2, results in:

$$\bar{R}^{\mathsf{T}}\big(Z(x', x_3) - Z(x', 0)\big)\mathsf{f}_i = x_3\big(\partial_i\vec{p}(x') - S(x')\partial_i\vec{b}_0(x')\big) + \partial_i\vec{d}_0(x', x_3) - \partial_i\vec{d}_0(x', 0).$$

Further, by (5.59), (5.52) and since $S\in\mathrm{Skew}(3)$, it follows that:

$$(5.64)\qquad \begin{aligned} \big(\bar{A}(x')I_3(x)\bar{A}(x')\big)_{2\times 2,\mathrm{sym}} &= \Big(Q_0^{\mathsf{T}}(x')\bar{R}^{\mathsf{T}}Z(x)\Big)_{2\times 2,\mathrm{sym}}\\ &= \Big(Q_0^{\mathsf{T}}(x')\bar{R}^{\mathsf{T}}Z(x', 0)\Big)_{2\times 2,\mathrm{sym}} + x_3\big(\nabla y_0(x')^{\mathsf{T}}\nabla\vec{p}(x')\big)_{\mathrm{sym}}\\ &\quad + x_3\big(\nabla V(x')^{\mathsf{T}}\nabla\vec{b}_0(x')\big)_{\mathrm{sym}} + \Big(Q_0^{\mathsf{T}}(x')\nabla\vec{d}_0(x)\Big)_{2\times 2,\mathrm{sym}}\end{aligned}$$

On the other hand, taking the x_3-average and recalling (5.54), we get:

(5.65)
$$\Big(Q_0^{\mathsf{T}}(x')\bar{R}^{\mathsf{T}}Z(x', 0)\Big)_{2\times 2,\mathrm{sym}} = \mathbb{S}(x') + \frac{1}{2}\nabla V(x')^{\mathsf{T}}\nabla V(x') - \Big(\nabla y_0(x')^{\mathsf{T}}\nabla\int_{-1/2}^{1/2}\vec{d}_0(x)\,\mathrm{d}x_3\Big)_{\mathrm{sym}}$$

STEP 3. We now finish the proof of Theorem 5.4.5. Combining (5.62), (5.64) and (5.65), we see that:

$$\Big(\bar{A}(x')(I_2 - \frac{1}{2}I_1^2 + I_3)\bar{A}(x')\Big)_{2\times 2,\mathrm{sym}} = \Big(I(x') + x_3 III(x') + II(x)\Big)_{\mathrm{sym}} \qquad \text{on } \Omega,$$

where I, II, III are as in (5.55). In virtue of (5.61), we obtain:

$$\liminf_{h\to 0}\frac{1}{h^4}\frac{1}{h}\int_{\Omega_h} W\big(\nabla u^h(z)A_h^{-1}(z)\big)\,\mathrm{d}z \geq \frac{1}{2}\int_\Omega \mathcal{Q}_2\Big(x', I(x') + x_3 III(x') + II(x)\Big)\,\mathrm{d}x.$$

This yields the claimed lower bound by $\mathscr{I}_4^O(V,\mathbb{S})$. □

For the upper bound statement, define the linear spaces:

$$\begin{aligned} \mathscr{V} &:= \Big\{V \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3) : \ \big(\nabla y_0(x')^{\mathsf{T}} \nabla V(x')\big)_{\mathrm{sym}} = 0 \ \text{ for all } x' \in \omega\Big\}, \\ \mathscr{S} &:= \mathrm{cl}_{\mathrm{L}^2(\omega,\mathbb{R}^{2\times 2})}\Big\{\big((\nabla y_0)^{\mathsf{T}} \nabla w\big)_{\mathrm{sym}} \ : \ w \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^3)\Big\}. \end{aligned} \tag{5.66}$$

We see that the limiting quantities V and $\mathbb{S}$ in Theorem 5.4.5 satisfy: $V \in \mathscr{V}$, $\mathbb{S} \in \mathscr{S}$. The space $\mathscr{V}$ consists of the first order infinitesimal isometries on the smooth minimizing immersion surface $y_0(\omega)$, i.e. those Sobolev-regular displacements V that preserve the metric on $y_0(\omega)$ up to first order. The tensor fields $\mathbb{S} \in \mathscr{S}$ are the finite strains on $y_0(\omega)$, eventually forcing the stretching term in the von Karman energy $\mathscr{I}_4^O$ to be of second order.

Theorem 5.4.6. *Assume that y_0 solves (5.39). Then, for every $(V, \mathbb{S}) \in \mathscr{V} \times \mathscr{S}$ there exists a sequence $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$ such that the rescaled sequence $\{y^h := u^h(\cdot, h\cdot)\}_h$ satisfies (i) and (ii) of Theorem 5.4.4, together with:*

$$\lim_{h\to 0} \frac{1}{h^4} \mathscr{I}_h(u^h) = \mathscr{I}_4^O(V, \mathbb{S}). \tag{5.67}$$

Proof. Step 1. Given admissible V and $\mathbb{S}$, we first define the ε-recovery sequence $\{u^h\}_h \subseteq \mathrm{W}^{1,\infty}(\Omega_h, \mathbb{R}^3)$. The ultimate argument for (5.67) will be obtained via a diagonal argument. We set:

$$\begin{aligned} u^h(z', z_3) = {} & y_0(z') + h v^h(z') + h^2 w^h(z') + z_3 \vec{b}_0(z') + h^2 \vec{d}_0\big(z', \frac{z_3}{h}\big) \\ & + h^3 \vec{k}_0\big(z', \frac{z_3}{h}\big) + h z_3 \vec{p}^h(z') + h^2 z_3 \vec{q}^h(z') + h^3 \vec{r}^h\big(z', \frac{z_3}{h}\big) \qquad \text{for all } (z', z_3) \in \Omega_h. \end{aligned}$$

The smooth vector fields $\vec{b}_0$ and $\vec{d}_0$ are as in (5.39), (5.40). We now introduce other terms in the above expansion. The sequence $\{w^h\}_h \subseteq C^\infty(\overline{\omega}, \mathbb{R}^3)$ is such that:

$$\begin{aligned} & \Big((\nabla y_0)^{\mathsf{T}} \nabla \big(w^h + \int_{-1/2}^{1/2} \vec{d}_0(\cdot, t)\, \mathrm{d}t\big)\Big)_{\mathrm{sym}} \to \mathbb{S} \quad \text{strongly in } \mathrm{L}^2(\omega, \mathbb{R}^{2\times 2}) \quad \text{as } h \to 0, \\ & \lim_{h\to 0} \sqrt{h} \|w^h\|_{\mathrm{W}^{2,\infty}(\omega,\mathbb{R}^3)} = 0. \end{aligned} \tag{5.68}$$

Existence of such a sequence is guaranteed by the fact that $\mathbb{S} \in \mathscr{S}$, where we "slow down" the approximations $\{w^h\}$ to guarantee the blow-up rate of order less that $h^{-1/2}$. Further, for a fixed small $\varepsilon > 0$, the truncated sequence $\{v^h\}_h \subseteq \mathrm{W}^{2,\infty}(\omega, \mathbb{R}^3)$ is chosen according to the standard construction in [FJM02] (see Theorem 2.2.1 with $\lambda = \lambda_h = \frac{c}{h}$, for some $c > 0$), in a way that:

$$\begin{aligned} & v^h \to V \quad \text{strongly in } \mathrm{W}^{2,2}(\omega, \mathbb{R}^3) \quad \text{as } h \to 0, \\ & h\|v^h\|_{W^{2,\infty}(\omega,\mathbb{R}^3)} \leq \varepsilon \quad \text{and} \quad \lim_{h\to 0} \frac{1}{h^2} \mathcal{L}^2\big(\{x' \in \omega : \ v^h(x') \neq V(x')\}\big) = 0. \end{aligned} \tag{5.69}$$

The vector field $\vec{k}_0 \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$ and sequences $\{\vec{p}^h\}_h$, $\{\vec{q}^h\}_h \subseteq \mathrm{W}^{1,\infty}(\omega, \mathbb{R}^3)$ and $\{r^h\}_h \subseteq$

$\mathrm{L}^\infty(\Omega, \mathbb{R}^3)$ are defined by:

$$
\begin{aligned}
Q_0^\top \vec{p}^h &= \begin{bmatrix} -(\nabla v^h)^\top \vec{b}_0 \\ 0 \end{bmatrix}, \\
Q_0^\top \vec{q}^h &= \ell\Big(x', \big((\nabla y_0)^\top \nabla w^h\big)_{\mathrm{sym}} + \frac{1}{2}(\nabla v^h)^\top \nabla v^h\Big) - \begin{bmatrix} (\nabla v^h)^\top \vec{p}^h \\ \frac{1}{2}|\vec{p}^h|^2 \end{bmatrix} - \begin{bmatrix} (\nabla w^h)^\top \vec{b}_0 \\ 0 \end{bmatrix}, \\
Q_0^\top \partial_3 \vec{k}_0 &= \ell\Big(x', \big((\nabla y_0)^\top \nabla' \vec{d}_0\big)_{\mathrm{sym}} + \frac{x_3^2}{2}(\nabla \vec{b}_0)^\top \nabla \vec{b}_0 - \frac{1}{4}(\mathcal{G}_2)_{2\times 2}\Big) \\
&\quad - \begin{bmatrix} x_3(\nabla \vec{b}_0)^\top \partial_3 \vec{d}_0 \\ \frac{1}{2}|\partial_3 \vec{d}_0|^2 \end{bmatrix} + \begin{bmatrix} (\nabla' \vec{d}_0)^\top \vec{b}_0 \\ 0 \end{bmatrix} + \frac{1}{2}\mathcal{G}_2 e_3 - \frac{1}{4}(\mathcal{G}_2)_{33} e_3, \\
Q_0^\top \tilde{r}^h &= x_3 \ell\Big(x', \big((\nabla y_0)^\top \nabla \vec{p}^h + (\nabla v^h)^\top \nabla \vec{b}_0\big)_{\mathrm{sym}}\Big) - \begin{bmatrix} (\nabla v^h)^\top \partial_3 \vec{d}_0 \\ \langle \vec{p}^h, \partial_3 \vec{d}_0 \rangle \end{bmatrix}.
\end{aligned}
\tag{5.70}
$$

Finally, we choose $\{\vec{r}^h\}_h \subseteq \mathrm{W}^{1,\infty}(\Omega, \mathbb{R}^3)$ to satisfy:

$$
\lim_{h\to 0} \|\partial_3 \vec{r}^h - \tilde{r}^h\|_{\mathrm{L}^2(\Omega,\mathbb{R}^3)} = 0 \quad \text{and} \quad \lim_{h\to 0} \sqrt{h}\|\vec{r}^h\|_{\mathrm{W}^{1,\infty}(\Omega,\mathbb{R}^3)} = 0. \tag{5.71}
$$

Step 2. Observe that for all $(x', x_3) \in \Omega$ there holds::

$$
\begin{aligned}
\nabla u^h(x', hx_3) = Q_0 &+ h\Big(\big[\nabla v^h,\ \vec{p}^h\big] + P_0\Big) \\
&+ h^2\Big(\big[\nabla w^h,\ \vec{q}^h\big] + \big[x_3 \nabla \vec{p}^h,\ \partial_3 r^h\big] + \big(\nabla' \vec{d}_0,\ \partial_3 \vec{k}_0\big]\Big) \\
&+ O(h^3)\Big(|\nabla \vec{k}_0| + |\nabla \vec{q}^h| + |\nabla \vec{r}^h|\Big).
\end{aligned}
$$

Consequently, by (5.57) it follows that:

$$
\big((\nabla u^h) A_h^{-1}\big)(x', hx_3) = Q_0 \bar{A}^{-1}\Big(\mathrm{I}_3 + h\bar{A}^{-1} J_1^h \bar{A}^{-1} + h^2 \bar{A}^{-1} J_2^h \bar{A}^{-1} + J_h^3\Big),
$$

where:

$$
\begin{aligned}
J_1^h &= Q_0^\top\Big(\big[\nabla v^h,\ \vec{p}^h\big] + P_0\Big) - \bar{A}A_1, \\
J_2^h &= Q_0^\top\Big(\big[\nabla w^h,\ \vec{q}^h\big] + \big[x_3 \nabla \vec{p}^h,\ \partial_3 \vec{r}^h\big] + \big[\nabla \vec{d}_0,\ \partial_3 \vec{k}_0\big]\Big) - J_1^h \bar{A}^{-1} A_1 - \frac{1}{2}\bar{A}A_2,
\end{aligned}
$$

and where J_1^h, J_2^h, J_3^h satisfy the uniform bounds (independent of ε):

$$
\begin{aligned}
&|J_1^h| \leq C\big(1 + |\nabla v^h|\big), \\
&|J_2^h| \leq C\big(1 + |\nabla w^h| + |\nabla v^h|^2 + |\nabla^2 v^h| + |\nabla \vec{r}^h|\big), \\
&|J_3^h| \leq Ch^3\big(1 + |\nabla w^h| + |\nabla^2 w^h| + |\nabla v^h|^2 + |\nabla^2 v^h| + |\nabla v^h| \cdot |\nabla^2 v^h| + |\nabla \vec{r}^h|\big) + o(h^2).
\end{aligned}
$$

In particular, the distance $\mathrm{dist}\big((\nabla u^h)A_h^{-1}, \mathrm{SO}(3)\big) \leq |(\nabla u^h)A_h^{-1} - Q_0\bar{A}^{-1}|$ is as small as one wishes, uniformly in $x \in \Omega$, for h sufficiently small. Thus, the argument $(\nabla u^h)A_h^{-1}$ of the frame invariant density W in $\mathscr{I}_h(u^h)$ may be replaced by its polar decomposition factor:

$$
\begin{aligned}
\sqrt{\big(\nabla u^h A_h^{-1}\big)^\top \big(\nabla u^h A_h^{-1}\big)} &= \sqrt{\mathrm{I}_3 + 2h^2 \bar{A}^{-1}\big((J_2^h)_{\mathrm{sym}} + \frac{1}{2}(J_1^h)^\top \bar{A}^{-2} J_1^h\big)\bar{A}^{-1} + \mathcal{R}^h} \\
&= \mathrm{I}_3 + h^2 \bar{A}^{-1}\big((J_2^h)_{\mathrm{sym}} + \frac{1}{2}(J_1^h)^\top \bar{A}^{-2} J_1^h\big)\bar{A}^{-1} + \mathcal{R}^h,
\end{aligned}
$$

where $\mathcal{R}^h$ stands for any quantity obeying the following bound:

$$\mathcal{R}^h = O(h)|(J_1^h)_{\rm sym}| + O(h^3)\big(1+|\nabla v^h|\big)\big(1+|\nabla w^h|+|\nabla v^h|^2+|\nabla^2 v^h|+|\nabla \vec{r}^h|\big) + O(h^3)|\nabla^2 w^h| + o(h^2).$$

In conclusion, Taylor's expansion of W at I_3 gives:

$$\begin{aligned} &\frac{1}{h^4}\int_\Omega W\Big((\nabla u^h)A_h^{-1}(x',hx_3)\Big)\,\mathrm{d}x \\ &\qquad = \frac{1}{h^4}\int_\Omega W\Big(\sqrt{(\nabla u^h A_h^{-1})^{\mathsf T}(\nabla u^h A_h^{-1})(x',hx_3)}\Big)\,\mathrm{d}x \\ &\qquad \leq \frac{1}{2}\int_\Omega \mathcal{Q}_3\Big(\bar A^{-1}\big((J_2^h)_{\rm sym}+\frac{1}{2}(J_1^h)^{\mathsf T}\bar A^{-2}J_1^h\big)\bar A^{-1}+\frac{1}{h^2}\mathcal{R}^h\Big)\,\mathrm{d}x \\ &\qquad\quad + O(h^2)\int_\Omega |J_2^h|^3+|J_1^h|^6\,\mathrm{d}x + \frac{O(1)}{h^4}\int_\Omega |\mathcal{R}^h|^3\,\mathrm{d}x. \end{aligned} \tag{5.72}$$

The residual terms above are estimated as in [LRR17], using (5.68), (5.69), (5.71). We have:

$$h^2\int_\Omega |J_2^h|^3+|J_1^h|^6\,\mathrm{d}x \leq h^2\int_\Omega 1+|\nabla w^h|^3+|\nabla v^h|^6+|\nabla^2 v^h|^3+|\nabla \vec{r}^h|^3\,\mathrm{d}x \leq o(1),$$

since

$$h^2\int_\Omega |\nabla v^h|^6\,\mathrm{d}x \leq Ch^2\|\nabla v^h\|^6_{\mathrm{W}^{1,2}(\omega)} = o(1), \quad h^2\int_\Omega |\nabla^2 v^h|^3\,\mathrm{d}x \leq \varepsilon h\int_\Omega |\nabla^2 v^h|^2\,\mathrm{d}x = o(1).$$

Further:

$$\begin{aligned} \frac{1}{h^4}\int_\Omega |\mathcal{R}^h|^2\,\mathrm{d}x &\leq \frac{1}{h^2}\int_\Omega \big|\big((\nabla y_0)^{\mathsf T}\nabla v^h\big)_{\rm sym}\big|^2\,\mathrm{d}x \\ &\quad + O(h^2)\int_\Omega \big(1+|\nabla v^h|^2\big)\big(1+|\nabla w^h|^2+|\nabla v^h|^4+|\nabla^2 v^h|^2+|\nabla \vec{r}^h|^2\big)\,\mathrm{d}x \\ &\quad + O(h^2)\int_\Omega |\nabla^2 w^h|^2\,\mathrm{d}x + o(1) \\ &= o(1) + O(h^2)\int_\Omega |\nabla v^h|\cdot|\nabla^2 v^h|^2 \leq C\varepsilon, \end{aligned}$$

because the last condition in (5.69) implies:

$$\begin{aligned} \frac{1}{h^2}\int_\Omega \big|\big((\nabla y_0)^{\mathsf T}\nabla v^h\big)_{\rm sym}\big|^2\,\mathrm{d}x &\leq \frac{C}{h^2}\|\nabla^2 v^h\|_{\mathrm{L}^\infty(\omega)}\int_{\{v^h\neq V\}} \mathrm{dist}^2(x',\{v^h=V\})\,\mathrm{d}x' \\ &\leq \frac{C\varepsilon^2}{h^4}\int_{\{v^h\neq V\}} \mathrm{dist}^2(x',\{v^h=V\})\,\mathrm{d}x' \leq C\varepsilon^2\frac{1}{h^2}\big|\{v^h\neq V\}\big| = o(1). \end{aligned} \tag{5.73}$$

From the two estimates above it also follows that $\frac{1}{h^4}\int_\Omega |\mathcal{R}^h|^3\,\mathrm{d}x = o(1)$. Consequently, (5.72) yields:

$$\limsup_{h\to 0}\frac{1}{h^4}\mathscr{I}_h(u^h) \leq C\varepsilon + \limsup_{h\to 0}\frac{1}{2}\int_\Omega \mathcal{Q}_3\Big(\bar A^{-1}\big((J_2^h)_{\rm sym}+\frac{1}{2}(J_1^h)^{\mathsf T})\bar A^{-2}J_1^h\big)\bar A^{-1}\Big)\,\mathrm{d}x. \tag{5.74}$$

Step 3. Observe now that:

$$\begin{aligned} (J_2^h)_{\rm sym}+\frac{1}{2}(J_1^h)^{\mathsf T}\bar A^{-2}J_1^h = &-\Big(\big((\nabla y_0)^{\mathsf T}\nabla v^h\big)^*_{\rm sym}\bar A^{-1}A_1\Big)_{\rm sym} \\ &+\Big(Q_0^{\mathsf T}\big[\nabla w^h,\ \vec{q}^h\big]+Q_0^{\mathsf T}\big[x_3\nabla\vec{p}^h,\ \partial_3\vec{r}^h\big]+Q_0^{\mathsf T}\big[\nabla\vec{d}_0,\ \partial_3\vec{k}_0\big]\Big)_{\rm sym} \\ &+\frac{1}{2}\big[\nabla v^h,\ \vec{p}^h\big]^{\mathsf T}\big[\nabla v^h,\ \vec{p}^h\big]+\Big(\big[\nabla v^h,\ \vec{p}^h\big]^{\mathsf T}P_0\Big)_{\rm sym} \\ &+\frac{1}{2}P_0^{\mathsf T}P_0-\frac{1}{4}\mathcal{G}^2. \end{aligned}$$

Replacing $\partial_3\vec{r}^h$ by $\tilde{r}^h$ and using (5.70), it follows that:

$$\begin{aligned}\Big(\int_\Omega \mathcal{Q}_3\Big(\bar{A}^{-1}\big((J_2^h)_{\mathrm{sym}} + \frac{1}{2}(J_1^h)^{\mathsf{T}}\bar{A}^{-2}J_1^h\big)\bar{A}^{-1}\Big)\,\mathrm{d}x\Big)^{1/2} \\ \leq \Big(\int_\Omega \mathcal{Q}_2\Big(x', \big((\nabla y_0)^{\mathsf{T}}\nabla w^h\big)_{\mathrm{sym}} + \frac{1}{2}(\nabla v^h)^{\mathsf{T}}\nabla v^h + x_3\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{p}^h + (\nabla v^h)^{\mathsf{T}}\nabla\vec{b}_0\big)_{\mathrm{sym}} \\ + \frac{x_3^2}{2}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{4}\mathcal{G}^2_{2\times2}\Big)\,\mathrm{d}x\Big)^{1/2} \\ + \|\big((\nabla y_0)^{\mathsf{T}}\nabla v^h\big)_{\mathrm{sym}}\|_{\mathrm{L}^2(\Omega)} + \|\partial_3\vec{r}^h - \tilde{r}^h\|_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

The second term above converges to 0 by (5.73) and the third term also converges to 0, by (5.71). On the other hand, the first term can be split into the integral on the set $\{v^h = V\}$, whose limit as $h \to 0$ is estimated by $\mathscr{I}_4^O(V, \mathbb{S})$, and the remaining integral that is bounded by:

$$\begin{aligned}C\int_{\{v^h\neq V\}\times(-\frac{1}{2},\frac{1}{2})} 1 + |\nabla w^h|^2 + |\nabla v^h|^4 + |\nabla^2 v^h|^2 + |\nabla\vec{r}^h|^3\,\mathrm{d}x \\ \leq C\epsilon^2\frac{1}{h^2}|\{v^h \neq V\}| + C\int_{\{v^h\neq V\}} |\nabla v^h|^4\,\mathrm{d}x' \leq o(1) + C|\{v^h \neq V\}|^{1/2}\|\nabla v^h\|^4_{L^8} = o(1).\end{aligned}$$

In conclusion, (5.74) becomes (with a uniform constant C that does not depend on ε):

$$\limsup_{h\to0}\frac{1}{h^4}\mathscr{I}_h(u^h) \leq C\varepsilon + \mathscr{I}_4^O(V, \mathbb{S}).$$

A diagonal argument applied to the indicated ε-recovery sequence $\{u^h\}_h$ completes the proof. □

Corollary 5.4.7. *The functional $\mathscr{I}_4^O$ attains its infimum and there holds:*

$$\lim_{h\to0}\frac{1}{h^4}\inf\mathscr{I}_h = \min\ \mathscr{I}_4^O.$$

The infima in the left hand side are taken over the set $\mathrm{W}^{1,2}(\Omega_h, \mathbb{R}^3)$, whereas the minimum in the right hand side is taken over admissible displacement-strain couples $(V, \mathbb{S}) \in \mathscr{V}\times\mathscr{S}$ in (5.66).

5.4.2 Discussion on $\mathscr{I}_4^O$ and reduction to the non-oscillatory case

In this section, we identify the appropriate components of the integrand in the energy $\mathscr{I}_4^O$ as: stretching, bending, curvature and the order-4 excess, the latter quantity being the projection of the entire integrand on the orthogonal complement of $\mathbb{E}_2$ in $\mathbb{E}$. This superposition is in the same spirit, as the integrand of $\mathscr{I}_2^O$ in Theorem 5.2.3 decoupling into bending and the order-2 excess, defined as the projection on the orthogonal complement of $\mathbb{E}_1$. There, the assumed condition $\int_{-1/2}^{1/2}\mathcal{G}^1(\cdot, t)\,\mathrm{d}t = 0$ served as the compatibility criterion, assuring that the 2-excess being null results in $\mathscr{I}_4^O$ coinciding with the non-oscillatory limiting energy $\mathscr{I}_4$, written for the effective metric $\bar{G}$ in (EF). Below, we likewise derive the parallel version $\mathscr{I}_4$ of $\mathscr{I}_4^O$, corresponding to the non-oscillatory case, and show that the vanishing of the 4-excess reduces $\mathscr{I}_4^O$ to $\mathscr{I}_4$ (for the effective metric (EF)), under two new further compatibility conditions (5.83) on $\mathcal{G}^2_{2\times2}$.

The following formulas will be useful in the sequel:

Lemma 5.4.8. *In the non-oscillatory setting (NO), let y_0, $\vec{b}_0$ be as in (5.39) and $\tilde{d}_0$ as in (5.42). Then for $i,j=1,2$ it holds:*

$$\big[\partial_{ij}y_0,\ \partial_i\vec{b}_0,\ \tilde{d}_0\big](x') = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\big](x')\cdot\begin{bmatrix}\Gamma^1_{ij} & \Gamma^1_{i3} & \Gamma^1_{33}\\ \Gamma^2_{ij} & \Gamma^2_{i3} & \Gamma^2_{33}\\ \Gamma^3_{ij} & \Gamma^3_{i3} & \Gamma^3_{33}\end{bmatrix}(x',0), \tag{5.75}$$

for all $x'\in\omega$. Consequently, for any smooth vector field $\vec{q}:\omega\to\mathbb{R}^3$ there holds:

$$\Big[\nabla y_0(x')^{\mathsf{T}}\nabla\big(Q_0(x')^{-\mathsf{T}}\vec{q}(x')\big)\Big]_{i,j=1,2} = \big(\nabla\vec{q}\big)_{2\times2}(x') - \Big[\Big\langle\vec{q}(x'),[\Gamma^1_{ij},\ \Gamma^2_{ij},\ \Gamma^3_{ij}](x',0)\Big\rangle\Big]_{i,j=1,2}.$$

Above, $\{\Gamma^k_{ij}\}$ are the Christoffel symbols of the metric G and the expression in the right and side represents the tangential part of the covariant derivative of the $(0,1)$ tensor field $\vec{q}$ with respect to G.

Proof. In view of $\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{b}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G_{2\times2}(x',0)$ in (5.39) and recalling (5.23), we get:

$$\langle\partial_{ij}y_0,\vec{b}_0\rangle = \frac{1}{2}\big(\partial_i G_{j3}+\partial_j G_{i3}-\partial_3 G_{ij}\big)(x',0)\qquad\text{for all } i,j=1,2,$$

which easily results in:

$$\langle\partial_i\vec{b}_0,\partial_j y_0\rangle = \frac{1}{2}\big(\partial_i G_{j3}-\partial_j G_{i3}+\partial_3 G_{ij}\big)(x',0)\quad\text{and}\quad\langle\partial_i\vec{b}_0,\vec{b}_0\rangle = \frac{1}{2}\partial_i G_{33}(x',0).$$

Thus (5.24) and the above allow for computing the coordinates in the basis $\partial_1 y_0,\partial_2 y_0,\vec{b}_0$ as claimed in (5.75); see also [LRR17, Theorem 6.2] for more details. The second formula results from:

$$\begin{aligned}\big\langle\partial_i y_0,\partial_j\big(Q_0^{-\mathsf{T}}\vec{q}\big)\big\rangle &= \big\langle\partial_i y_0,\partial_j\big(Q_0^{-\mathsf{T}}\big)\vec{q}\big\rangle + \big\langle\partial_i y_0,Q_0^{-\mathsf{T}}\partial_j\vec{q}\big\rangle\\ &= -\big\langle\partial_i y_0,Q_0^{-\mathsf{T}}\partial_j\big(Q_0^{\mathsf{T}}\big)Q_0^{-\mathsf{T}}\vec{q}\big\rangle + \big\langle Q_0^{-1}\partial_i y_0,\partial_j\vec{q}\big\rangle\\ &= -\big\langle Q_0^{-1}\partial_j\big(Q_0\big)\mathsf{f}_i,\vec{q}\big\rangle + \big\langle\mathsf{f}_i,\partial_j\vec{q}\big\rangle,\end{aligned}$$

which together with (5.75) yields the Lemma. □

Lemma 5.4.9. *In the non-oscilaltory setting (NO), let y_0, $\vec{b}_0$ be as in (5.39) and $\tilde{d}_0$ as in (5.42). Then the metric-related term II in (5.55) has the form $II = \frac{x_3^2}{2}\bar{II}(x')$ and for all $x'\in\omega$ we have:*

$$\bar{II} = (\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 + \big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} - \frac{1}{2}\partial_{33}G_{2\times2}(x',0) = \begin{bmatrix}R_{1313} & R_{1323}\\ R_{1323} & R_{2323}\end{bmatrix}(x',0). \tag{5.76}$$

Above, R_{ijkl} are the Riemann curvatures of the metric G, evaluated at the midplate points $(x',0)\in\omega\times\{0\}$.

Proof. We argue as in the proof of [LRR17, Theorem 6.2]. Using (5.40) we arrive at:

$$\begin{aligned}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} =& -\big[\langle\partial_{ij}y_0,\tilde{d}_0\rangle\big]_{i,j=1,2} + \frac{1}{2}\partial_{33}G_{2\times2}(x',0)\\ &+\Big[R_{i2j3} - G_{np}\big(\Gamma^n_{i3}\Gamma^p_{j3} - \Gamma^n_{ij}\Gamma^p_{33}\big)\Big]_{i,j=1,2}(x',0).\end{aligned} \tag{5.77}$$

Directly from (5.75) we hence obtain:

$$\langle\partial_{ij}\vec{y}_0,\tilde{d}_0\rangle = G_{np}\Gamma^n_{ij}\Gamma^p_{33},\qquad\langle\partial_i\vec{b}_0,\partial_j\vec{b}_0\rangle = G_{np}\Gamma^n_{i3}\Gamma^p_{j3}, \tag{5.78}$$

which together with (5.77) yields (5.76). □

With the use of Lemma 5.4.9, it is quite straightforward to derive the ultimate form of the energy $\mathscr{I}_4^O$ in the non-oscillatory setting. In particular, the proof of the following result is a special case of the proof of Theorem 5.4.12 below.

Theorem 5.4.10. *Assume (NO) and (5.38). The expression (5.80) becomes:*

$$\begin{aligned}\mathscr{I}_4(V,\mathbb{S}) = \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x', \mathbb{S}(x') + \frac{1}{2}\nabla V^{\mathsf{T}}\nabla V(x') + \frac{1}{24}\nabla\vec{b}_0^{\mathsf{T}}\nabla\vec{b}_0(x') - \frac{1}{48}\partial_{33}G_{2\times 2}(x',0)\Big)\,\mathrm{d}x' \\ + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', \nabla y_0(x')^{\mathsf{T}}\nabla\vec{p}(x') + \nabla V(x')^{\mathsf{T}}\nabla\vec{b}_0(x')\Big)\,\mathrm{d}x' \\ + \frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', \begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{bmatrix}(x',0)\Big)\,\mathrm{d}x',\end{aligned}$$

where R_{ijkl} stand for the Riemann curvatures of the metric G.

Remark 5.4.11. In the particular, "flat" case of $G = \mathrm{I}_3$ the functional $\mathscr{I}_4$ reduces to the classical von Kármán energy below. Indeed, the unique solution to (5.39) is: $y_0 = id$, $\vec{b}_0 = \mathsf{f}_3$ and further:

$$\begin{aligned}\mathscr{V} &= \big\{V(x) = (\alpha x^\perp + \vec{\beta}, v(x)) :\ \alpha\in\mathbb{R},\ \vec{\beta}\in\mathbb{R}^2,\ v\in \mathrm{W}^{2,2}(\omega)\big\}, \\ \mathscr{S} &= \big\{\nabla_{\mathrm{sym}} w :\ w\in \mathrm{W}^{1,2}(\omega,\mathbb{R}^2)\big\}.\end{aligned}$$

Given $V\in\mathscr{V}$, we have $\vec{p} = (-\nabla v, 0)$ and thus:

$$\mathscr{I}_4(V,\mathbb{S}) = \frac{1}{2}\int_\omega \mathcal{Q}_2\big(x', \nabla_{\mathrm{sym}} w + \frac{1}{2}(\alpha^2\mathrm{I}_2 + \nabla v\otimes\nabla v)\big)\,\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', \nabla^2 v\big)\,\mathrm{d}x'.$$

Absorbing the stretching $\alpha^2\mathrm{I}_2$ into $\nabla_{\mathrm{sym}} w$, the above energy can be expressed in a familiar form:

$$\mathscr{I}_4(v,w) = \frac{1}{2}\int_\omega \mathcal{Q}_2\big(x', \nabla_{\mathrm{sym}} w + \frac{1}{2}\nabla v\otimes\nabla v\big)\,\mathrm{d}x' + \frac{1}{24}\int_\omega \mathcal{Q}_2\big(x', \nabla^2 v\big)\,\mathrm{d}x', \tag{5.79}$$

as a function of the out-of-plane scalar displacement v and the in-plane vector displacement w. ■

As done for the Kirchhoff energy $\mathscr{I}_2^O$ in Theorem 5.2.3, we now identify conditions allowing $\mathscr{I}_4^O$ to coincide with $\mathscr{I}_4$ of the effective metric $\bar{G}$, modulo the introduced below order-4 excess term.

Theorem 5.4.12. *In the setting of Theorem 5.4.5, we have:*

(5.80)

$$\begin{aligned}\mathscr{I}_4^O(V,\mathbb{S}) = \frac{1}{2}\int_\omega \mathcal{Q}_2\Big(x', \mathbb{S} + \frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V + B_0\Big)\,\mathrm{d}x' \\ + \frac{1}{24}\int_\omega \mathcal{Q}_2\Big(x', (\nabla V)^{\mathsf{T}}\nabla\vec{b}_0 + (\nabla y_0)^{\mathsf{T}}\nabla\vec{p} + 12B_1\Big)\,\mathrm{d}x' \\ + \frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 + (\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0 - \frac{1}{2}\bar{\mathcal{G}}^2_{2\times 2}\Big)\,\mathrm{d}x' + \mathrm{dist}^2\Big(II_{\mathrm{sym}}, \mathbb{E}_2\Big),\end{aligned}$$

where $\bar{\mathcal{G}}^1$ and $\bar{\mathcal{G}}^2$ are given in (EF), inducing $\tilde{d}_0$ via (5.42) for $\partial_3 G = \bar{\mathcal{G}}^1$, and where we introduce the following purely metric-related quantities:

(5.81)

$$\begin{aligned}\mathrm{dist}^2\big(II_{\mathrm{sym}}, \mathbb{E}_2\big) = \mathrm{dist}^2\Big(\int_0^{x_3}\nabla(\mathcal{G}^1\mathsf{f}_3)_{2\times 2,\mathrm{sym}}\,\mathrm{d}t - \Big[\Big\langle\int_0^{x_3}\mathcal{G}^1\mathsf{f}_3\,\mathrm{d}t, [\Gamma^1_{ij},\ \Gamma^2_{ij},\ \Gamma^3_{ij}](x',0)\Big\rangle\Big]_{i,j=1,2} \\ + \frac{1}{2}\int_0^{x_3}(\mathcal{G}^1)_{33}\,\mathrm{d}t\,\big[\Gamma^3_{ij}(x',0)\big]_{i,j=1,2} - \frac{1}{4}\mathcal{G}^2_{2\times 2}, \mathbb{E}_2\Big).\end{aligned}$$

$$
\begin{aligned}
B_0 &= \frac{1}{24}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{4}\int_{-1/2}^{1/2}\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3\\
&= \frac{1}{24}\Big[\sum_{n,p=1\dots 3}\bar{\mathcal{G}}_{np}\Gamma^n_{i3}\Gamma^p_{j3}\Big]_{i,j=1,2} - \frac{1}{4}\int_{-1/2}^{1/2}\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3
\end{aligned}
$$

(5.82)
$$
\begin{aligned}
B_1 &= (\nabla y_0)^{\mathsf{T}}\nabla\Big(\int_{-1/2}^{1/2}x_3\vec{d}_0\,\mathrm{d}x_3\Big) - \frac{1}{4}\int_{-1/2}^{1/2}x_3(\mathcal{G}^2)_{2\times 2}\,\mathrm{d}x_3\\
&= -\nabla\Big(\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1\mathsf{f}_3\,\mathrm{d}x_3\Big)_{2\times 2} + \Big[\Big\langle\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1\mathsf{f}_3\,\mathrm{d}x_3, [\Gamma^1_{ij},\ \Gamma^2_{ij},\ \Gamma^3_{ij}](x',0)\Big\rangle\Big]_{i,j=1,2}\\
&\quad - \frac{1}{2}\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1_{33}\,\mathrm{d}x_3\Big[\Gamma^3_{ij}(x',0)\Big]_{i,j=1,2} - \frac{1}{4}\int_{-1/2}^{1/2}x_3\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3,
\end{aligned}
$$

By $\{\Gamma^k_{ij}\}$ we denote the Christoffel symbols of the metric $\bar{G}$ in (EF). The third term in (5.80) equals the scaled norm of the Riemann curvatures of the effective metric G:

$$
\frac{1}{1440}\int_\omega \mathcal{Q}_2\Big(x', \begin{bmatrix} R_{1313} & R_{1323}\\ R_{1323} & R_{2323}\end{bmatrix}(x',0)\Big)\,\mathrm{d}x'.
$$

The first three terms in $\mathcal{I}_4^O$ coincide with $\mathcal{I}_4$ in Theorem 5.4.10 for the effective metric $\bar{G}$ in (EF), provided that the following compatibility conditions hold:

(5.83)
$$
\begin{aligned}
&\int_{-1/2}^{1/2}\Big(15x_3^2 - \frac{9}{4}\Big)\mathcal{G}^2_{2\times 2}(x',x_3)\,\mathrm{d}x_3 = 0,\\
&\frac{1}{4}\int_{-1/2}^{1/2}x_3\mathcal{G}^2_{2\times 2}(x',x_3)\,\mathrm{d}x_3 + \nabla\Big(\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1\mathsf{f}_3\,\mathrm{d}x_3\Big)_{2\times 2,\mathrm{sym}}\\
&\qquad - \Big[\Big\langle\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1\mathsf{f}_3\,\mathrm{d}x_3, [\Gamma^1_{ij},\ \Gamma^2_{ij},\ \Gamma^3_{ij}](x',0)\Big\rangle\Big]_{i,j=1,2}\\
&\qquad + \frac{1}{2}\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1_{33}\,\mathrm{d}x_3\Big[\Gamma^3_{ij}(x',0)\Big]_{i,j=1,2} = 0.
\end{aligned}
$$

Proof. We write:

$$
\mathcal{I}_4^O(V,\mathbb{S}) = \frac{1}{2}\|I + x_3III + II\|^2_{\mathcal{Q}_2} = \frac{1}{2}\|I + x_3III + \mathbb{P}_2(II)\|^2_{\mathcal{Q}_2} + \frac{1}{2}\mathrm{dist}^2\big(II_{\mathrm{sym}}, \mathbb{E}_2\big),
$$

and further decompose the first term above along the Legendre projections:

$$
\begin{aligned}
\|I + x_3III + \mathbb{P}_2(II)\|^2_{\mathcal{Q}_2} &= \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_0(x_3)\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2}\\
&\quad + \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_1(x_3)\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2} + \Big\|\int_{-1/2}^{1/2}(I + x_3III + II)p_2(x_3)\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2}\\
&= \underbrace{\Big\|I + \int_{-1/2}^{1/2}II\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2}}_{\text{Stretching}} + \frac{1}{12}\underbrace{\Big\|III + 12\int_{-1/2}^{1/2}x_3II\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2}}_{\text{Bending}} + \underbrace{\Big\|\int_{-1/2}^{1/2}p_2(x_3)II\,\mathrm{d}x_3\Big\|^2_{\mathcal{Q}_2}}_{\text{Curvature}}.
\end{aligned}
$$

To identify the four indicated terms in $\mathcal{I}_4^O$, observe that

$$
\int_{-1/2}^{1/2}x_3\int_0^{x_3}\mathcal{G}^1\,\mathrm{d}x_3 = -\int_{-1/2}^{1/2}\frac{x_3^2}{2}\mathcal{G}^1\,\mathrm{d}x_3
$$

and that:

$$\operatorname{dist}^2\big(II_{\rm sym}, \mathbb{E}_2\big) = \operatorname{dist}^2\Big(\Big((\nabla y_0)^{\sf T}\nabla'\big(Q_0^{-\sf T}\int_0^{x_3}\mathcal{G}^1 \mathrm{f}_3\,\mathrm{d}t - \frac{1}{2}Q_0^{-\sf T}\int_0^{x_3}\mathcal{G}_{33}^1\,\mathrm{d}t\,\mathrm{f}_3\big)\Big)_{\rm sym} - \frac{1}{4}\mathcal{G}^2_{2\times 2}, \mathbb{E}_2\Big).$$

Thus the formulas in (5.81) and (5.82) follow directly from Lemma 5.4.8 and (5.78). There also holds:

$$\begin{aligned}
\text{Stretching} &= \int_\omega \mathcal{Q}_2\Big(x', \mathbb{S} + \frac{1}{2}(\nabla V)^{\sf T}\nabla V + \frac{1}{24}(\nabla\vec{b}_0)^{\sf T}\nabla\vec{b}_0 - \frac{1}{4}\int_{-1/2}^{1/2}\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3\Big)\,\mathrm{d}x',\\
\text{Bending} &= \frac{1}{12}\int_\omega \mathcal{Q}_2\Big(x', (\nabla V)^{\sf T}\nabla\vec{b}_0 + (\nabla y_0)^{\sf T}\nabla\vec{p}\\
&\qquad + 12(\nabla y_0)^{\sf T}\nabla\big(\int_{-1/2}^{1/2}x_3\vec{d}_0\,\mathrm{d}x_3\big) - 3\int_{-1/2}^{1/2}x_3\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3\Big),\,\mathrm{d}x'\\
\text{Curvature} &= \frac{1}{720}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec{b}_0)^{\sf T}\nabla\vec{b}_0 + 60(\nabla y_0)^{\sf T}\nabla\big(\int_{-1/2}^{1/2}(6x_3^2 - \frac{1}{2})\vec{d}_0\,\mathrm{d}x_3\big)\\
&\qquad - 15\int_{-1/2}^{1/2}(6x_3^2 - \frac{1}{2})\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3\Big)\,\mathrm{d}x'.
\end{aligned}$$

It is easy to check that with the choice of the effective metric components $\bar{\mathcal{G}}^1\mathrm{f}_3$ and $\bar{\mathcal{G}}^2_{2\times 2}$ and denoting $\tilde{d}_0$ the corresponding vector in (5.42), we have:

$$\text{Curvature} = \frac{1}{720}\int_\omega \mathcal{Q}_2\Big(x', (\nabla\vec{b}_0)^{\sf T}\nabla\vec{b}_0 + (\nabla y_0)^{\sf T}\nabla\tilde{d}_0 - \frac{1}{2}\bar{\mathcal{G}}^2_{2\times 2}\Big)\,\mathrm{d}x'.$$

This proves (5.80). Equivalence of the constraints (5.83) with:

$$\int_{-1/2}^{1/2}\mathcal{G}^2_{2\times 2}\,\mathrm{d}x_3 = \frac{1}{12}\bar{\mathcal{G}}^2_{2\times 2} \quad \text{and} \quad (B_1)_{\rm sym} = 0 \quad \text{in } \omega,$$

follows by a direct inspection. We now invoke Lemma 5.4.9 to complete the proof. □

Remark 5.4.13. Observe that the vanishing of the 4-excess and curvature terms in $\mathscr{I}_4^O$:

$$II_{\rm sym} \in \mathbb{E}_2 \quad \text{and} \quad \text{Curvature} = 0,$$

are the necessary conditions for $\min \mathscr{I}_4^O = 0$ and they are equivalent to $II_{\rm sym} \in \mathbb{E}_1$. Consider now a particular case scenario of $\bar{\mathcal{G}} = \mathrm{I}_3$ and $\mathcal{G}^1 = 0$, where the spaces $\mathscr{V}$ and $\mathscr{S}$ are given in Remark 5.4.11, together with $\vec{d}_0 = 0$. Then, the above necessary condition reduces to: $\mathcal{G}^2_{2\times 2} \in \mathbb{E}_1$, namely:

$$\mathcal{G}^2_{2\times 2}(x', x_3) = x_3\mathcal{F}_1(x') + \mathcal{F}_0(x') \qquad \text{for all } x = (x', x_3) \in \overline{\Omega}.$$

It is straightforward that, on a simply connected midplate ω, both terms:

$$\begin{aligned}
\text{Stretching} &= \int_\omega \mathcal{Q}_2\Big(x', \nabla_{\rm sym} w + \frac{1}{2}\nabla v\otimes\nabla v - \frac{1}{4}\mathcal{F}_0\Big)\,\mathrm{d}x',\\
\text{Bending} &= \int_\omega \mathcal{Q}_2\Big(x', \nabla^2 v + \frac{1}{4}\mathcal{F}_1\Big)\,\mathrm{d}x',
\end{aligned}$$

can be equated to 0 by choosing appropriate displacements v and w, if and only if there holds:

$$\operatorname{curl}\mathcal{F}_1 = 0, \qquad \operatorname{curl}^{\sf T}\operatorname{curl}\mathcal{F}_0 + \frac{1}{4}\det\mathcal{F}_1 = 0 \quad \text{in } \omega. \tag{5.84}$$

Note that these are precisely the linearised Gauss-Codazzi-Mainardi (see (1.4) and Lemma 1.7) equations corresponding to the metric $\mathrm{I}_2 + 2h^2\mathcal{F}_0$ and shape operator $\frac{1}{2}h\mathcal{F}_1$ on ω. We see that these conditions are automatically satisfied in presence of (5.83), when $\mathcal{G}^2_{2\times 2} \in \mathbb{E}_1$ actually results in $\mathcal{G}^2_{2\times 2} = 0$. An integrability criterion similar to (5.84) can be derived also in the general case, under $II_{\mathrm{sym}} \in \mathbb{E}_1$ and again it automatically holds with (5.83). This last statement will be pursued in the next section. ■

5.4.3 Identification of the Ch^4 scaling regime

Theorem 5.4.14. *The energy scaling beyond the von Kármán regime:*

$$\lim_{h\to 0} \frac{1}{h^4} \inf \mathscr{I}_h = 0$$

is equivalent to the following condition:

(i) in the oscillatory case (O), in presence of the compatibility conditions (5.83)

(5.85) *$II_{\mathrm{sym}} \in \mathbb{E}_1$ and (5.86) holds with G replaced by the effective metric $\bar{G}$ in (EF). This condition involves $\bar{\mathcal{G}}$, $\bar{\mathcal{G}}^1$ and $\bar{\mathcal{G}}^2_{2\times 2}$ terms of $\bar{G}$.*

(ii) in the non-oscillatory case (NO)

(5.86) *All the Riemann curvatures of the metric G vanish on $\omega \times \{0\}$:*

$$R_{ijkl}(x', 0) = 0, \qquad \text{for all } x' \in \omega \text{ and all } i, j, k, l = 1, \ldots, 3.$$

Proof. By Corollary 5.4.7, it suffices to determine the equivalent conditions for $\min \mathscr{I}_4 = 0$. Clearly, $\min \mathscr{I}_4 = 0$ implies (5.86). Vice versa, if (5.86) holds, then:

$$\frac{1}{24}(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\vec{b}_0 - \frac{1}{48}\partial_{33}G(x', 0) = -\frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}},$$

by Lemma 5.4.9. Taking $V = \vec{p} = 0$ and $\mathbb{S} = \frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{d}_0\big)_{\mathrm{sym}} \in \mathscr{S}$, we get $\mathscr{I}_4(V, \mathbb{S}) = 0$. □

5.5 Coercivity of the limiting energy $\mathscr{I}_4$

We further have the following counterpart of the essential uniqueness of the minimizing isometric immersion y_0 statement in Theorem 5.2.7:

Theorem 5.5.1. *In the non-oscillatory setting (NO), assume (5.86). Then $\mathscr{I}_4(V, \mathbb{S}) = 0$ if and only if:*

$$V = Sy_0 + \mathsf{c} \quad \text{and} \quad \mathbb{S} = \frac{1}{2}\Big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0 + \frac{1}{12}\tilde{d}_0\big)\Big)_{\mathrm{sym}} \quad \text{on } \omega, \tag{5.87}$$

for some skew-symmetrix matrix $S \in \mathrm{Skew}(3)$ and a vector $\mathsf{c} \in \mathbb{R}^3$.

Proof. We first observe that the bending term III in (5.55) is already symmetric, because:

$$\begin{aligned}
\Big[\langle \partial_i y_0, \partial_j \vec{p}\rangle + \langle \partial_i V, \partial_j \vec{b}_0\rangle\Big]_{i,j=1,2} & \\
&= \Big[\partial_j\big(\langle \partial_i y_0, \vec{p}\rangle + \langle \partial_i V, \vec{b}_0\rangle\big)\Big]_{i,j=1,2} - \Big[\langle \partial_{ij} y_0, \vec{p}\rangle + \langle \partial_{ij} V, \vec{b}_0\rangle\Big]_{i,j=1,2} \\
&= -\Big[\langle \partial_{ij} y_0, \vec{p}\rangle + \langle \partial_{ij} V, \vec{b}_0\rangle\Big]_{i,j=1,2} \in \mathrm{Sym}(2),
\end{aligned}$$

where we used the definition of $\vec p$ in (5.52). Recalling (5.76), we see that $\mathscr{I}_4(V,\mathbb{S})=0$ if and only if:

$$\begin{aligned}&\mathbb{S}+\frac{1}{2}(\nabla V)^{\mathsf{T}}\nabla V-\frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde d_0\big)_{\mathrm{sym}}=0,\\&(\nabla y_0)^{\mathsf{T}}\nabla\vec p+(\nabla V)^{\mathsf{T}}\nabla\vec b_0=0.\end{aligned}\tag{5.88}$$

Consider the matrix field $S=\big[\nabla V,\ \vec p\big]Q_0^{-1}\in \mathrm{W}^{1,2}(\omega,\mathrm{Skew}(3))$ as in (5.52). Note that:

$$\begin{aligned}&\partial_i S=\big[\nabla\partial_i V,\ \partial_i\vec p\big]Q_0^{-1}-\big[\nabla V,\ \vec p\big]Q_0^{-1}(\partial_i Q_0)Q_0^{-1}=Q_0^{-1,\mathsf{T}}\bar S^iQ_0^{-1}\qquad\text{for }\ i=1,2\\&\qquad\text{where}\quad \bar S^i=Q_0^{\mathsf{T}}\big[\nabla\partial_iV,\ \partial_i\vec p\big]+\big[\nabla V,\ \vec p\big]^{\mathsf{T}}(\partial_iQ_0)\in \mathrm{L}^2(\omega,\mathrm{Skew}(3)).\end{aligned}\tag{5.89}$$

Then we have:

$$\langle\bar S^ie_1,e_2\rangle=\partial_i\big(\langle\partial_2y_0,\partial_iV\rangle+\langle\partial_2V,\partial_iy_0\rangle\big)-\big(\langle\partial_{12}y_0,\partial_iV\rangle+\langle\partial_{12}V,\partial_iy_0\rangle\big)=0,$$

because the first term in the right hand side above equals 0 in view of $V\in\mathscr{V}$, whereas the second term equals $\partial_2\langle\partial_1y_0,\partial_1V\rangle$ for $i=1$ and $\partial_1\langle\partial_2y_0,\partial_2V\rangle$ for $i=2$, both expression being null again in view of $V\in\mathscr{V}$. We now claim that $\{\bar S^i\}_{i=1,2}=0$ is actually equivalent to the second condition in (5.88). It suffices to examine the only possibly nonzero components:

$$\langle\bar S^i\mathsf{f}_3,\mathsf{f}_j\rangle=\langle\partial_jy_0,\partial_i\vec p\rangle+\langle\partial_jV,\partial_i\vec b_0\rangle=\big((\nabla y_0)^{\mathsf{T}}\nabla\vec p+(\nabla V)^{\mathsf{T}}\nabla\vec b_0\big)_{ij},\qquad\text{for }\ i,j=1,2,\tag{5.90}$$

proving the claim.

Consequently, the second condition in (5.88) is equivalent to S being constant, to the effect that $\nabla V=\nabla(Sy_0)$, or equivalently that $V-Sy_0$ is a constant vector. In this case:

$$\mathbb{S}=\frac{1}{2}(\nabla y_0)^{\mathsf{T}}\nabla\big(S^2y_0\big)+\frac{1}{24}\big((\nabla y_0)^{\mathsf{T}}\nabla\tilde d_0\big)_{\mathrm{sym}}=\frac{1}{2}\Big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2y_0+\frac{1}{12}\tilde d_0\big)\Big)_{\mathrm{sym}}$$

is equivalent to the first condition in (5.88), as $(\nabla V)^{\mathsf{T}}\nabla V=-(\nabla y_0)^{\mathsf{T}}S^2\nabla y_0$. The proof is done. □

From Theorem 5.5.1 we deduce its quantitative version, that is a counterpart of Theorem 5.3.1 in the present von Kármán regime:

Theorem 5.5.2. *In the non-oscillatory setting (NO), assume (5.86). Then for all $V\in\mathscr{V}$ there holds:*

$$\begin{aligned}&\mathrm{dist}^2_{\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)}\Big(V,\ \big\{Sy_0+\mathsf{c}:\ S\in\mathrm{Skew}(3),\ \mathsf{c}\in\mathbb{R}^3\big\}\Big)\\&\qquad\qquad\le C\int_\omega\mathcal{Q}_2\big(x',(\nabla y_0)^{\mathsf{T}}\nabla\vec p+(\nabla V)^{\mathsf{T}}\nabla\vec b_0\big)\,\mathrm{d}x'\end{aligned}\tag{5.91}$$

with a constant $C>0$ that depends on G,ω and W but it is independent of V.

Proof. We argue by contradiction. Since $\mathscr{V}_{lin}:=\{Sy_0+\mathsf{c}:\ S\in\mathrm{Skew}(3),\ \mathsf{c}\in\mathbb{R}^3\}$ is a linear subspace of $\mathscr{V}$ and likewise the expression III in (5.55) is linear in V, with its kernel equal to $\mathscr{V}_{lin}$ in virtue of Theorem 5.5.1, it suffices to take a sequence $\{V_n\in\mathscr{V}\}_{n\to\infty}$ such that:

$$\begin{aligned}&\|V_n\|_{\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)}=1,\qquad V_n\perp_{\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)}\mathscr{V}_{lin}\qquad\text{for all }n,\\&\text{and:}\quad(\nabla y_0)^{\mathsf{T}}\nabla\vec p_n+(\nabla V_n)^{\mathsf{T}}\nabla\vec b_0\to0\quad\text{strongly in }\mathrm{L}^2(\omega,\mathbb{R}^{2\times2}),\quad\text{as }\ n\to\infty.\end{aligned}\tag{5.92}$$

Passing to a subsequence if necessary and using the definition of $\vec{p}$ in (5.52), it follows that:

$$V_n \rightharpoonup V \quad \text{weakly in } \mathrm{W}^{2,2}(\omega,\mathbb{R}^3), \qquad \vec{p}_n \rightharpoonup \vec{p} \quad \text{weakly in } \mathrm{W}^{1,2}(\omega,\mathbb{R}^3). \tag{5.93}$$

Clearly, $Q_0^{\mathsf{T}}\big[\nabla V,\ \vec{p}\big] \in \mathrm{L}^2(\omega, \mathrm{Skew}(3))$ so that $V \in \mathscr{V}$, but also $(\nabla y_0)^{\mathsf{T}}\nabla\vec{p} + (\nabla V)^{\mathsf{T}}\nabla\vec{b}_0 = 0$. Thus, Theorem 5.5.1 and the perpendicularity assumption in (5.92) imply: $V = \vec{p} = 0$. We will now prove:

$$V_n \to 0 \quad \text{strongly in } \ \mathrm{W}^{2,2}(\omega,\mathbb{R}^3), \tag{5.94}$$

which will contradict the first (normalisation) condition in (5.91).

As in (5.89), the assumption $V_n \in \mathscr{V}$ implies that for each $x' \in \omega$ and $i = 1, 2$, the following matrix (denoted previously by $\bar{S}^i$) is skew-symmetric:

$$Q_0^{\mathsf{T}}\big[\nabla\partial_i V_n,\ \partial\vec{p}_n\big] + \big[\nabla V_n,\ \vec{p}_n\big]^{\mathsf{T}}(\partial_i Q_0) \in \mathrm{Skew}(3).$$

Equating tangential entries and observing (5.92), yields for every $i, j, k = 1, 2$:

$$\langle\partial_j y_0, \partial_{ik}V_n\rangle + \langle\partial_k y_0, \partial_{ij}V_n\rangle = -\Big(\langle\partial_j V_n, \partial_{ik}y_0\rangle + \langle\partial_k V_n, \partial_{ij}y_0\rangle\Big) \to 0 \quad \text{strongly in } \mathrm{L}^2(\omega).$$

Permuting i, j, k we eventually get:

$$\langle\partial_j y_0, \partial_{ik}V_n\rangle \to 0 \quad \text{strongly in } \mathrm{L}^2(\omega) \qquad \text{for all } \ i, j, k = 1, 2.$$

On the other hand, equating off-tangential entries, we get by (5.92) and (5.93) that for each $i = 1, 2$:

$$\langle\vec{b}_0, \partial_{ij}V_n\rangle = -\big((\nabla y_0)^{\mathsf{T}}\nabla\vec{p}_n + (\nabla V_n)^{\mathsf{T}}\nabla\vec{b}_0\big)_{ij} - \langle\vec{p}_n, \partial_{ij}y_0\rangle \to 0 \quad \text{strongly in } \mathrm{L}^2(\omega).$$

Consequently, $\{Q_0^{\mathsf{T}}\partial_{ij}V_n \to 0\}_{i,j=1,2}$ in $\mathrm{L}^2(\omega,\mathbb{R}^3)$, which implies convergence (5.94) as claimed. This ends the proof of (5.91). □

Remark 5.5.3. Although the kernel of the (nonlinear) energy $\mathscr{I}_4$, displayed in Theorem 5.5.1, is finite dimensional, the full coercivity estimate of the form below is *false*:

(5.95)
$$\min_{S\in\mathrm{Skew}(3), \mathsf{c}\in\mathbb{R}^3}\Big(\|V - (Sy_0 + \mathsf{c})\|^2_{\mathrm{W}^{2,2}(\omega,\mathbb{R}^3)} + \|\mathbb{S} - \frac{1}{2}\big((\nabla y_0)^{\mathsf{T}}\nabla\big(S^2 y_0 - \frac{1}{12}\tilde{d}_0\big)\big)_{\mathrm{sym}}\|^2_{\mathrm{L}^2(\omega,\mathbb{R}^{2\times 2})}\Big)$$
$$\leq C\mathscr{I}_4(V,\mathbb{S}), \qquad \text{for all } \ (V,\mathbb{S}) \in \mathscr{V}\times\mathscr{S}.$$

For a counterexample, consider the particular case of classical von Kármán functional (5.79), specified in Remark 5.4.11. Clearly, $\mathscr{I}_4(v, w) = 0$ if an only if $v(x) = \langle\mathsf{v}, x\rangle + \alpha$ and $w(x) = \beta x^{\perp} - \frac{1}{2}\langle\mathsf{v}, x\rangle\mathsf{v} + \gamma$, for some $\mathsf{v} \in \mathbb{R}^2$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Note that (5.91) reflects then the Poincaré inequality: $\int_\omega |\nabla v - \fint_\omega \nabla v|^2\,\mathrm{d}x' \leq C\int_\omega |\nabla^2 v|^2\,\mathrm{d}x'$, whereas (5.95) takes the form:

$$\min_{\mathsf{v}\in\mathbb{R}^2}\Big(\int_\omega |\nabla v - \mathsf{v}|^2\,\mathrm{d}x' + \int_\omega |\nabla_{\mathrm{sym}}w - \frac{1}{2}\mathsf{v}\otimes\mathsf{v}|^2\,\mathrm{d}x'\Big) \leq C\mathscr{I}_4(v, w). \tag{5.96}$$

Let $\omega = B_1(0)$. Given $v \in \mathrm{W}^{2,2}(\omega)$ such that $\det\nabla^2 v = 0$, let w satisfy: $\nabla_{\mathrm{sym}}w = -\frac{1}{2}\nabla v\otimes\nabla v$, which results in vanishing of the first term in (5.79). Neglecting the first term in the left hand side of (5.96), leads in this context to the following weaker form, which we below disprove:

$$\min_{a\in\mathbb{R}^2}\int_\omega |\nabla v\otimes\nabla v - \mathsf{v}\otimes\mathsf{v}|^2\,\mathrm{d}x' \leq C\int_\omega |\nabla^2 v|^2\,\mathrm{d}x'. \tag{5.97}$$

Define $v_n(x) = n(x_1+x_2) + \frac{1}{2}(x_1+x_2)^2$ for all $x=(x_1,x_2)\in\omega$. Then we have that $\nabla v_n = (n+x_1+x_2)(1,1)$ and $\det\nabla^2 v_n = 0$. Minimization in (5.97) becomes:

$$\min_{\mathsf{v}\in\mathbb{R}^2}\int_\omega \big|(n+x_1+x_2)^2(1,1)^\mathsf{T}\otimes(1,1)^\mathsf{T} - \mathsf{v}\otimes\mathsf{v}\big|^2\,\mathrm{d}x'$$

and an easy explict calculation yields the necessary form of the minimizer: $\mathsf{v}=\delta(1,1)^\mathsf{T}$. Thus, the same minimization can be equivalently written and estimated in:

$$4\cdot\min_{\delta\in\mathbb{R}}\int_\omega \big|(n+x_1+x_2)^2-\delta^2\big|^2\,\mathrm{d}x' \sim 4n^2\to\infty \qquad \text{as } n\to\infty.$$

On the other hand, $|\nabla^2 v_n|^2 = 4$ at each $x'\in\omega$. Therefore, the estimate (5.97) cannot hold. ■

5.6 Beyond the von Kármán regime: an example

Given a function $\phi\in C^\infty\big((-\frac{1}{2},\frac{1}{2})\big)$, consider the conformal metric:

$$G(z',z_3) = e^{2\phi(z_3)}\mathrm{I}_3 \qquad \text{for all } z=(z',z_3)\in\Omega_h.$$

The midplate metric $\bar{\mathcal{G}}_{2\times2} = e^{2\phi(0)}\mathrm{I}_2$ has a smooth isometric immersion $y_0 = e^{\phi(0)}id_2:\omega\to\mathbb{R}^2$ and thus by Theorem 5.2.4 there must be:

$$\inf\mathscr{I}_h \leq Ch^2.$$

By a computation, we get that the only possibly non-zero Christoffel symbols of G are: $\Gamma_{11}^3=\Gamma_{22}^3=-\phi'(z_3)$ and $\Gamma_{13}^1=\Gamma_{23}^2=\Gamma_{33}^3=\phi'(z_3)$, while the only possibly nonzero Riemann curvatures are:

$$R_{1212} = -\phi'(z_3)^2e^{2\phi(z_3)}, \qquad R_{1313}=R_{2323} = -\phi''(z_3)e^{2\phi(z_3)}. \tag{5.98}$$

Consequently, the results of this paper provide the following hierarchy of possible energy scalings:

(a) $\{ch^2\leq\inf\mathscr{I}_h\leq Ch^2\}_{h\to0}$ with $c,C>0$. This scenario is equivalent to $\phi'(0)\neq0$. The functionals $\frac{1}{h^2}\mathscr{I}_h$ as in Theorems 5.2.1, 5.2.3 and 5.2.4 exhibit the indicated compactness properties and Γ-converge to the following energy $\mathscr{I}_2$ defined on the set of deformations $y\in W^{2,2}_{\mathrm{iso}}(\omega)$:

$$\mathscr{I}_2(y) = \frac{1}{24}\int_\omega \mathcal{Q}_2\big(\Pi_y - \phi'(0)\mathrm{I}_2\big)\,\mathrm{d}x'.$$

Here $\mathcal{Q}_2(F_{2\times2}) = \min\big\{D^2W(\mathrm{I}_3)[\tilde{F}]^2;\ \tilde{F}\in\mathbb{R}^{3\times3} \text{ with } \tilde{F}_{2\times2}=F_{2\times2}\big\}$.

(b) $\{ch^4\leq\inf\mathscr{I}_h\leq Ch^4\}_{h\to0}$ with $c,C>0$. This scenario is equivalent to $\phi'(0)=0$ and $\phi''(0)\neq0$. The unique (up to rigid motions) minimizing isometric immersion is then $id_2:\omega\to\mathbb{R}^2$ and the functionals $\frac{1}{h^4}\mathscr{I}_h$ have the compactness and Γ-convergence properties as in Theorem 5.4.4, Theorem 5.4.5 and Theorem 5.4.6. The following limiting functional $\mathscr{I}_4$ is defined on the set of displacements $\{(v,w)\in W^{2,2}(\omega,\mathbb{R})\times W^{1,2}(\omega,\mathbb{R}^2)\}$ as in Remark 5.4.11:

$$\begin{aligned}\mathscr{I}_4(v,w) = \frac{1}{2}\int_\omega &\mathcal{Q}_2\big(\nabla_{\mathrm{sym}}w + \frac{1}{2}\nabla v\otimes\nabla v - \frac{1}{24}\phi''(0)\mathrm{I}_2\big)\,\mathrm{d}x'\\ &+\frac{1}{24}\int_\omega \mathcal{Q}_2(\nabla^2 v)\,\mathrm{d}x' + \frac{1}{1440}\phi''(0)^2|\omega|\mathcal{Q}_2(\mathrm{I}_2).\end{aligned}$$

(c) $\{\inf \mathscr{I}_h \leq Ch^6\}_{h\to 0}$ with $C > 0$. This scenario is equivalent to $\phi'(0) = 0$ and $\phi''(0) = 0$ (in agreement with Lemma 5.7.1) and in fact we have the following more precise result below.

Theorem 5.6.1. *Let $G(z', z_3) = e^{2\phi(z_3)}\mathrm{I}_3$, where $\phi^{(k)}(0) = 0$ for $k = 1 \dots n-1$ up to some $n > 2$. Then: $\inf \mathscr{I}_h \leq Ch^{2n}$ and:*

$$\lim_{h\to 0} \frac{1}{h^{2n}} \inf \mathscr{I}_h \geq c_n\, \phi^{(n)}(0)^2 |\omega| \mathcal{Q}_2(\mathrm{I}_2), \tag{5.99}$$

where $c_n > 0$. In particular, if $\phi^{(n)}(0) \neq 0$ then we have: $ch^{2n} \leq \inf \mathscr{I}_h \leq Ch^{2n}$ with $c, C > 0$.

Proof. Step 1. For the upper bound, we compute:

$$\begin{aligned}\mathscr{I}_h\big(e^{\phi(0)} id_3\big) &= \frac{1}{h}\int_{\Omega^h} W\big(e^{\phi(0)-\phi(z_3)}\mathrm{I}_3\big)\,\mathrm{d}z = \frac{1}{2h}\int_{\Omega^h} \mathcal{Q}_3\big(\phi^{(n)}(0)\frac{z_3^n}{n!}\mathrm{I}_3\big) + O(h^{2n+2})\,\mathrm{d}z \\ &= h^{2n}\bigg(\frac{\phi^{(n)}(0)^2}{(n!)^2}\frac{1}{(2n+1)2^{2n+1}}|\omega|\mathcal{Q}_3(\mathrm{I}_3) + o(1)\bigg) \leq Ch^{2n},\end{aligned}$$

where we used the fact that $e^{\phi(0)-\phi(z_3)} = 1 - \phi^{(n)}(0)\frac{z_3^n}{n!} + O(|z_3|^{n+1})$.

Step 2. To prove the lower bound (5.99), let $\{u^h\}_h \subseteq \mathrm{W}^{1,2}(\Omega^h, \mathbb{R}^3)$ be such that $\mathscr{I}_h(u^h) \leq Ch^{2n}$. Then:

$$\begin{aligned}\mathscr{I}_h(u^h) &\geq \frac{c}{h}\int_{\Omega^h} \mathrm{dist}^2\big(\nabla u^h, e^{\phi(z_3)}\mathrm{SO}(3)\big)\,\mathrm{d}z \\ &\geq \frac{c}{h}\int_{\Omega^h} \mathrm{dist}^2\big(\nabla u^h, e^{\phi(0)}\mathrm{SO}(3)\big)\,\mathrm{d}z - \frac{\bar{c}}{h}\int_{\Omega^h} \big|\phi^{(n)}(0)\frac{z_3^n}{n!} + O(h^{n+1})\big|^2\,\mathrm{d}z,\end{aligned}$$

which results in: $\frac{1}{h}\int_{\Omega^h} \mathrm{dist}^2\big(e^{-\phi(0)}\nabla u^h, \mathrm{SO}(3)\big)\,\mathrm{d}x \leq Ch^{2n}$. Similarly as in Lemma 5.4.2 and Corollary 5.4.3, it follows that there exist a sequence of approximating rotation fields $\{R_h\}_h \subseteq \mathrm{W}^{1,2}(\omega, \mathbb{R}^{3\times 3})$ such that:

$$\frac{1}{h}\int_{\Omega^h} |\nabla u^h - e^{\phi(0)}R_h|^2\,\mathrm{d}z \leq Ch^{2n}, \qquad \int_\omega |\nabla R_h|^2\,\mathrm{d}x' \leq Ch^{2n-2}. \tag{5.100}$$

As in sections 5.2 and 5.4, we define the following displacement and deformation fields:

$$y^h(x', x_3) = (\bar{R}_h)^{\mathsf{T}}\Big(u^h(x', hx_3) - \fint_{\Omega^h} u^h\,\mathrm{d}z\Big) \in \mathrm{W}^{1,2}(\Omega, \mathbb{R}^3), \quad \bar{R}_h = \mathbb{P}\fint_{\Omega^h} e^{-\phi(0)}\nabla u^h(z)\,\mathrm{d}z,$$

$$V^h(x') = \frac{1}{h^{n-1}}\int_{-1/2}^{1/2} y^h(x', x_3) - e^{\phi(0)}\big(id_2 + hx_3\mathrm{f}_3\big)\,\mathrm{d}x_3 \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^3).$$

In view of (5.100), we obtain then the following convergences (up to a not relabelled subsequence):

$$\begin{aligned}&y^h \to e^{\phi(0)}id_2 \text{ in } \mathrm{W}^{1,2}(\omega, \mathbb{R}^3), \qquad \frac{1}{h}\partial_3 y^h \to e^{\phi(0)}\mathrm{f}_3 \text{ in } \mathrm{L}^2(\omega, \mathbb{R}^3), \\ &V^h \to V \in \mathrm{W}^{2,2}(\omega, \mathbb{R}^3) \text{ in } \mathrm{W}^{1,2}(\omega, \mathbb{R}^3), \\ &\frac{1}{h}(\nabla V^h)_{2\times 2,\mathrm{sym}} \rightharpoonup \nabla_{\mathrm{sym}} w \text{ weakly in } \mathrm{L}^2(\omega, \mathbb{R}^{2\times 2}).\end{aligned}$$

This allows to conclude the claimed lower bound:

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{h^{2n}}\mathscr{I}_h(u^h) &\geq \frac{1}{2}\big\|e^{-\phi(0)}\nabla_{\mathrm{sym}} w - x_3 e^{-\phi(0)}\nabla^2 V^3 - \phi^{(n)}(0)\frac{x_3^n}{n!}\mathrm{I}_2\big\|_{\mathcal{Q}_2}^2 \\ &\geq \frac{1}{2}\Big\|\phi^{(n)}(0)\frac{x_3^n}{n!}\mathrm{I}_2 - \mathbb{P}_1\Big(\phi^{(n)}(0)\frac{x_3^n}{n!}\mathrm{I}_2\Big)\Big\|_{\mathcal{Q}_2}^2 \\ &= \frac{1}{2}\frac{\phi^{(n)}(0)^2}{(n!)^2}\cdot\int_{-1/2}^{1/2}\big(x_3^n - \mathbb{P}_1(x_3^n)\big)^2\,\mathrm{d}x_3 \cdot |\omega|\mathcal{Q}_2(\mathrm{I}_2),\end{aligned}$$

as in(5.99), with the following constant c_n:

$$c_n = \frac{1}{2^{2n+1}(n!)^2}\begin{cases} \frac{(n-1)^2}{(2n+1)(n+2)^2} & \text{for } n \text{ odd} \\ \frac{n^2}{(2n+1)(n+1)^2} & \text{for } n \text{ even.} \end{cases}$$

Observe that $c_2 = \frac{1}{1440}$, consistently with the previous direct application of Theorem 5.4.5. □

5.7 Beyond the von Kármán regime: preliminary results for the Ch^6 scaling regime

In this section, we focus on the non-oscillatory case in the general setting (NO) and derive the equivalent of Lemma 5.4.9 at the next order scaling, which turns out to be Ch^6. Our findings are consistent with those of the example in Section 5.6. Similarly to Lemma 5.4.1 we first construct the recovery sequence with energy smaller than that in the von Kármán regime:

Lemma 5.7.1. *Assume (NO) and write:*

$$G(z) = G(z',0) + z_3\partial_3 G(z',0) + \frac{z_3^2}{2}\partial_{33}G(z',0) + O(|z_3|^3) \qquad \text{for all } z=(z',z_3)\in\Omega_h.$$

If there holds: $\lim_{h\to 0}\frac{1}{h^4}\mathscr{I}_h = 0$, *then we automatically have:*

$$\inf \mathscr{I}_h \le Ch^6.$$

Proof. Under the assumption (5.86) we set the sequence of deformations $u^h : \overline{\Omega}_h \to \mathbb{R}^3$ to be:

$$u^h(z',z_3) = y_0(z') + z_3\vec{b}_0(z') + \frac{z_3^2}{2}\tilde{d}_0(z') + \frac{z_3^3}{6}\tilde{e}_0(z'), \qquad z=(z',z_3)\in\overline{\Omega}_h,$$

where $y_0, \vec{b}_0$ are as in (5.39) and $\tilde{d}_0$ as in (5.42). The new vector field $\tilde{e}_0 : \bar{\omega} \to \mathbb{R}^3$ is defined through the last formula below, in view of $\bar{II} = 0$ in (5.76):

$$\begin{aligned} &Q_0^{\mathsf T}Q_0 = G(z',0), \qquad \big(Q_0^{\mathsf T}\tilde{P}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_3 G(z',0), \\ &\tilde{P}_0^{\mathsf T}\tilde{P}_0 + \big(Q_0^{\mathsf T}\tilde{D}_0\big)_{\mathrm{sym}} = \frac{1}{2}\partial_{33}G(z',0). \end{aligned} \tag{5.101}$$

We will use the following matrix fields definitions:

$$Q_0 = \big[\partial_1 y_0,\ \partial_2 y_0,\ \vec{b}_0\big], \qquad \tilde{P}_0 = \big[\partial_1\vec{b}_0,\ \partial_2\vec{b}_0,\ \tilde{d}_0\big], \qquad \tilde{D}_0 = \big[\partial_1\tilde{d}_0,\ \partial_2\tilde{d}_0,\ \tilde{e}_0\big]. \tag{5.102}$$

Consequently:

$$\nabla u^h = Q_0 + z_3\tilde{P}_0 + \frac{z_3^2}{2}\tilde{D}_0 + \frac{z_3^3}{6}\big[\partial_1\tilde{e}_0,\ \partial_2\tilde{e}_0,\ 0\big],$$

whereas writing $A = G^{1/2}$, the expansion (5.57) becomes:

$$\begin{aligned} A^{-1}(z) = {} & \bar{A}(z')^{-1} \\ & + \bar{A}^{-1}(z')\Big(- z_3\bar{\partial}_3 A(z',0) + z_3^2\partial_3 A(z',0)\bar{A}^{-1}(z')\partial_3 A(z',0) - \frac{z_3^2}{2}\partial_{33}A(z',0)\Big)\bar{A}^{-1}(z') \\ & + O(|z_3|^3) \qquad \text{for all } z=(z',z_3)\in\Omega_h. \end{aligned}$$

We thus obtain the following expression:

$$\nabla u^h(z)A^{-1}(z) = (Q_0\bar{A}^{-1})(z')\big(\mathrm{I}_3 + z_3 S_1(z') + \frac{z_3^2}{2}S_2(z')\big) + O(|z_3|^3), \tag{5.103}$$

with:

$$S_1 = \bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)\bar{A}^{-1},$$
$$S_2 = \bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{D}_0 - 2Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}\partial_3 A + 2\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A - \bar{A}\partial_{33}A\big)\bar{A}^{-1}.$$

As in the proof of Lemma 5.4.1, we now get:

(5.104)
$$\begin{aligned} W\big(\nabla u^h(z)A^{-1}(z)\big) &= W\Big(\mathrm{I}_3 + z_3 S_1(z')_{\mathrm{sym}} + \frac{z_3^2}{2}\big(S_2(z')_{\mathrm{sym}} + S_1(z')^{\mathsf{T}}S_1(z')\big) + O(|z_3|^3)\Big) \\ &= W\big(\mathrm{I}_3 + O(|z_3|^3)\big) = O(h^6). \end{aligned}$$

The final equality follows from:

$$\begin{aligned} (S_1)_{\mathrm{sym}} &= \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{P}_0)_{\mathrm{sym}} - (\bar{A}\partial_3 A)_{\mathrm{sym}}\Big)\bar{A}^{-1} = \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{D}_0)_{\mathrm{sym}} - \frac{1}{2}\partial_3 G\Big)\bar{A}^{-1} = 0, \\ (S_2)_{\mathrm{sym}} + S_1^{\mathsf{T}}S_1 &= \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{P}_0)_{\mathrm{sym}} + 2\big(\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A\big)_{\mathrm{sym}} - \big(\bar{A}\partial_{33}A\big)_{\mathrm{sym}} \\ &\qquad - 2\big(Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}\partial_3 A\big)_{\mathrm{sym}} + \big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)^{\mathsf{T}}\bar{A}^{-2}\big(Q_0^{\mathsf{T}}\tilde{P}_0 - \bar{A}\partial_3 A\big)\Big)\bar{A}^{-1}, \end{aligned}$$

so that:

$$\begin{aligned} (S_2)_{\mathrm{sym}} + S_1^{\mathsf{T}}S_1 &= \bar{A}^{-1}\Big((Q_0^{\mathsf{T}}\tilde{D}_0)_{\mathrm{sym}} + \tilde{P}_0^{\mathsf{T}}\tilde{P}_0 - \frac{1}{2}\partial_{33}G + (\partial_3 A)^2 - 2\big(Q_0^{\mathsf{T}}\tilde{P}_0\big)_{\mathrm{sym}}\bar{A}^{-1}\partial_3 A \\ &\qquad - 2(\partial_3 A)\bar{A}^{-1}\big(Q_0^{\mathsf{T}}\tilde{P}_0\big)_{\mathrm{sym}} + 2\big(\bar{A}(\partial_3 A)\bar{A}^{-1}\partial_3 A\big)_{\mathrm{sym}}\Big)\bar{A}^{-1} \\ &= \bar{A}^{-1}\Big((\partial_3 A)^2 - 2\big((\partial_3 G)\bar{A}^{-1}\partial_3 A + \bar{A}(\partial_3 A)\bar{A}^{-1}(\partial_3 A)\big)_{\mathrm{sym}}\Big)\bar{A}^{-1} = 0, \end{aligned}$$

where we have repeatedly used the assumption (5.101). □

As in Section 5.4, using the change of variable by the smooth deformation $Y = u^h$ in the proof of Theorem 5.7.1, one obtains the following approximations, by adjusting the proofs of Lemma 5.4.2 and Corollary 5.4.3:

Lemma 5.7.2. *Assume (NO) and (5.86). If y_0 is injective on an open, Lipschitz subset $\mathcal{V} \subseteq \omega$, then for every $u^h \in \mathrm{W}^{1,2}(\mathcal{V}_h, \mathbb{R}^3)$ there exists $\bar{R}_h \in \mathrm{SO}(3)$ such that:*

$$\frac{1}{h}\int_{\mathcal{V}_h}\Big|\nabla u^h(z) - \bar{R}_h\Big(Q_0(z') + z_3\tilde{P}_0(z') + \frac{x_3^2}{2}\tilde{D}_0(z')\Big)\Big|^2\,\mathrm{d}z \le C\big(\mathscr{I}_h(u^h, \mathcal{V}_h) + h^5|\mathcal{V}_h|\big),$$

with the smooth correction matrix fields $\tilde{P}_0, \tilde{D}_0$ in (5.102). The constant C above is uniform for all subdomains $\mathcal{V}_h \subset \Omega_h$ which are bi-Lipschitz equivalent with controlled Lipschitz constants.

Corollary 5.7.3. *Assume (NO). Then, for any sequence $\{u^h\}_h \subseteq W^{1,2}(\Omega_h, \mathbb{R}^3)$ satisfying $\mathscr{I}_h(u^h) \le Ch^6$, there exists a sequence of rotation-valued maps $R_h \in \mathrm{W}^{1,2}(\omega, \mathbb{R}^{3\times 3})$, such that:*

$$\frac{1}{h}\int_{\Omega_h}\Big|\nabla u^h(z) - R_h(z')\Big(Q_0(z') + z_3\tilde{P}_0(z') + \frac{z_3^2}{2}\tilde{D}_0(z')\Big)\Big|^2\,\mathrm{d}z \le Ch^6,$$
$$\int_\omega |\nabla R_h(z')|^2\,\mathrm{d}z' \le Ch^4.$$

We now make the following observation. Writing the non-oscillatory metric G in its proper third order Taylor's expansion:

$$G(z) = G(z',0) + z_3\partial_3 G(z',0) + \frac{z_3^2}{2}\partial_{33}G(z',0) + \frac{z_3^3}{6}\partial_{333}G(z',0) + o(z_3^4),$$

one can readily check that the term $O(|z_3|^3)$ in the right hand side of the formula (5.103) can be explicitly written as: $(Q_0\bar{A}^{-1})(z)\frac{z_3^3}{6}S_3(z') + o(|z_3|^3)$, where:

$$\begin{aligned} S_3 = \bar{A}^{-1}\Big(&Q_0^{\mathsf{T}}[\partial_1\tilde{e}_0,\ \partial_2\tilde{e}_2,\ 0] - 3Q_0^{\mathsf{T}}\tilde{D}_0\bar{A}^{-1}\partial_3 A - 3Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}\partial_{33}A \\ &+ 6Q_0^{\mathsf{T}}\tilde{P}_0\bar{A}^{-1}(\partial_3 A)\bar{A}^{-1}\partial_3 A\Big)\bar{A}^{-1} + \bar{A}\partial_{333}A^{-1}(x',0). \end{aligned}$$

Consequently, (5.104) becomes:

$$W\big(\nabla u^h(z)A(z)^{-1}\big) = W\Big(\mathrm{I}_3 + \frac{z_3^3}{6}\big(S_3(z') + 3S_1(z')^{\mathsf{T}}S_2(z')\big)_{\mathrm{sym}} + o(|z_3|^3)\Big). \tag{5.105}$$

A tedious but direct inspection shows now that:

$$\big(S_3 + 3(S_1)^{\mathsf{T}}S_2\big)_{\mathrm{sym}} = \bar{A}^{-1}\Big(Q_0^{\mathsf{T}}[\partial_1\tilde{e}_0,\ \partial_2\tilde{e}_0,\ 0] + 3\tilde{P}_0^{\mathsf{T}}\tilde{D}_0 - \frac{1}{2}\partial_{333}G(x',0)\Big)_{\mathrm{sym}}\bar{A}^{-1},$$

and we see that the tensor playing the role similar to the curvature term $\bar{II}_{\mathrm{sym}}$, at the present h^6 scaling regime, which equals the 2×2 minor of the right hand side above after discarding the external multiplying factors $\bar{A}^{-1}$, has the form:

$$\Big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{e}_0 + 3(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\tilde{d}_0\Big)_{\mathrm{sym}} - \frac{1}{2}\partial_{333}G(x',0)_{2\times 2}$$

With the eye on future applications, we now identify this tensor in terms of the components R_{ijkl}. Recall that in Section 5.6, the relevant curvature quantity corresponding to $n=3$ was: $-\phi'''(0)e^{2\phi(0)}\mathrm{I}_2$, equal to $\partial_3\big[R_{i3j3}\big]_{i,j=1,2}(x',0)$ in view of (5.98). We have:

Theorem 5.7.4. *Assume (NO) and (5.86). Let $y_0, \vec{b}_0, \tilde{d}_0, \tilde{e}_0$ be as in (5.101), (5.102). Then for all $x'\in\omega$ we have: $\tilde{e}_0 = Q_0\big[\partial_3\Gamma^i_{33} + \Gamma^i_{p3}\Gamma^p_{33}\big]_{i=1\dots3}(x',0)$ and:*

$$\Big((\nabla y_0)^{\mathsf{T}}\nabla\tilde{e}_0 + 3(\nabla\vec{b}_0)^{\mathsf{T}}\nabla\tilde{d}_0\Big)_{\mathrm{sym}}(x') - \frac{1}{2}\partial_{333}G(x',0)_{2\times 2} = \partial_3\begin{bmatrix} R_{1313} & R_{1323} \\ R_{1323} & R_{2323}\end{bmatrix}(x',0).$$

Proof. STEP 1. Recall that existence of smooth vector fields $y_0, \vec{b}_0, \tilde{d}_0, \tilde{e}_0$ satisfying condition (5.101) is equivalent to the vanishing of the entire Riemann curvature tensor of the metric G on $\omega\times\{0\}$. Below, all equalities are valid at points $(x',0)$. Using (5.75) and the third identity in (5.101), we obtain:

$$\begin{aligned} \big(Q_0^{\mathsf{T}}\tilde{e}_0\big)_i &= \partial_{33}G_{i3} - \frac{1}{2}\partial_{i3}G_{33} - G_{pq}\Gamma^p_{33}\Gamma^q_{i3} = \partial_3\big(G_{pi}\Gamma^p_{33}\big) - G_{pq}\Gamma^p_{33}\Gamma^q_{i3} \\ &= G_{pi}\partial_3\Gamma^p_{33} + \big(\partial_3 G_{pi} - G_{pq}\Gamma^q_{i3}\big)\Gamma^p_{33} = G_{pi}\partial_3\Gamma^p_{33} + G_{qi}\Gamma^q_{p3}\Gamma^p_{33} \qquad \text{for } i = 1\dots3, \end{aligned}$$

by the Levi-Civita connection's compatibility in: $\nabla_3 G_{pi} = 0$. Consequently, it follows that:

$$Q_0^{\mathsf{T}}\tilde{e}_0 = G\big[\partial_3\Gamma^i_{33} + \Gamma^i_{p3}\Gamma^p_{33}\big]_{i=1\dots3}.$$

By the first equation in (5.101), we deduce the claimed formula for $\tilde{e}_0$.

STEP 2. Similarly, by (5.75), we obtain for all $i,j=1,2$:

$$\begin{aligned}\langle \partial_i y_0, \partial_j \tilde e_0\rangle &= \partial_j \langle \partial_i y_0, \tilde e_0\rangle - \langle \partial_{ij} y_0, \tilde e_0\rangle \\ &= \partial_j\big(G_{pi}\partial_3\Gamma^p_{33} + G_{qi}\Gamma^q_{p3}\Gamma^p_{33}\big) - G_{pq}\Gamma^p_{ij}\big(\partial_3\Gamma^q_{33} + \Gamma^q_{t3}\Gamma^t_{33}\big) \\ &= G_{is}\Big(\partial_{j3}\Gamma^s_{33} + \Gamma^s_{jq}\partial_3\Gamma^q_{33} + \partial_j\big(\Gamma^s_{p3}\Gamma^p_{33}\big) + \Gamma^s_{tj}\Gamma^t_{q3}\Gamma^q_{33}\Big) \\ &= G_{is}\Big(\partial_3\big(\partial_j\Gamma^s_{33} + \Gamma^s_{jp}\Gamma^p_{33}\big) + \big(\partial_j\Gamma^p_{33} + \Gamma^p_{qj}\Gamma^q_{33}\big)\Gamma^s_{3p} - \Gamma^q_{33}R^s_{q3j}\Big) \\ &= G_{is}\Big(\partial_3\big(\partial_j\Gamma^s_{33} + \Gamma^s_{jp}\Gamma^p_{33}\big) + \big(\partial_j\Gamma^p_{33} + \Gamma^p_{qj}\Gamma^q_{33}\big)\Gamma^s_{3p}\Big)\end{aligned}$$

where we have used $\nabla_j G_{pi}=0$ and the assumed condition $R^s_{q3j}=0$. Further:

$$\langle \partial_i \vec b_0, \partial_j \tilde d_0\rangle = G_{ps}\Gamma^p_{i3}\big(\partial_j\Gamma^s_{33} + \Gamma^s_{jq}\Gamma^q_{33}\big).$$

Consequently, it follows that:

$$\begin{aligned}&\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0 + 3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{ij} \\ =&G_{is}\partial_3\big(\partial_j\Gamma^s_{33} + \Gamma^s_{jq}\Gamma^q_{33}\big) + \big(\partial_j\Gamma^p_{33} + \Gamma^p_{jq}\Gamma^q_{33}\big)\big(G_{is}\Gamma^s_{3p} + 3G_{ps}\Gamma^s_{i3}\big) \\ =&G_{is}\partial_3\big(\partial_j\Gamma^s_{33} + \Gamma^s_{jq}\Gamma^q_{33}\big) + \big(\partial_3\Gamma^p_{j3} + \Gamma^q_{j3}\Gamma^p_{q3}\big)\big(2\partial_3 G_{ip} + \partial_i G_{3p} - \partial_p G_{3i}\big),\end{aligned}$$

by $R^p_{3j3}=0$. Observe also that: $\frac12\partial_{333}G_{ij} = \frac12\partial_{33}\big(G_{si}\Gamma^s_{3j} + G_{sj}\Gamma^s_{3i}\big)$. Expanding:

$$\begin{aligned}\partial_{33}\big(G_{si}\Gamma^s_{3j}\big) &= \partial_3\Big(G_{si}\partial_3\Gamma^s_{3j} + G_{si}\Gamma^s_{p3}\Gamma^p_{sj} + G_{sp}\Gamma^p_{i3}\Gamma^s_{3j}\Big) \\ &= G_{si}\partial_3\big(\partial_3\Gamma^s_{3j} + \Gamma^s_{3p}\Gamma^p_{3j}\big) + (\partial_3 G_{si})\big(\partial_3\Gamma^s_{j3} + \Gamma^s_{3p}\Gamma^p_{3j}\big) + \partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big)\end{aligned}$$

we finally obtain:

$$\begin{aligned}&\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0 + 3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{ij} - \partial_{33}\big(G_{si}\Gamma^s_{3j}\big) \\ &\qquad= G_{si}\partial_3\Big(\partial_j\Gamma^s_{33} + \Gamma^s_{jq}\Gamma^q_{33} - \partial_3\Gamma^s_{3j} - \Gamma^s_{3p}\Gamma^p_{3j}\Big) \\ &\qquad\quad + \big(\partial_3\Gamma^p_{j3} + \Gamma^q_{j3}\Gamma^p_{3q}\big)\big(\partial_3 G_{pi} + \partial_i G_{p3} - \partial_p G_{si}\big) - \partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big) \\ &\qquad= G_{is}\partial_3 R^s_{3j3} + S_{ij},\end{aligned} \tag{5.106}$$

It now follows that:

$$\begin{aligned}S_{ij} + S_{ji} &= 2\big(\partial_3\Gamma^s_{j3} + \Gamma^q_{j3}\Gamma^s_{3q}\big)G_{ps}\Gamma^p_{3i} - \partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big) \\ &\quad + 2\big(\partial_3\Gamma^s_{i3} + \Gamma^q_{i3}\Gamma^s_{3q}\big)G_{ps}\Gamma^p_{3j} - \partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big) \\ &= 2G_{ps}\Big(\partial_3\big(\Gamma^s_{j3}\Gamma^p_{i3}\big) + \big(\Gamma^q_{j3}\Gamma^p_{i3} + \Gamma^q_{i3}\Gamma^p_{j3}\big)\Gamma^s_{3q}\Big) - 2\partial_3\big(G_{sp}\Gamma^p_{i3}\Gamma^s_{j3}\big) \\ &= 2G_{ps}\Gamma^s_{3q}\big(\Gamma^q_{j3}\Gamma^p_{i3} + \Gamma^q_{i3}\Gamma^p_{j3}\big) - 2\Gamma^p_{i3}\Gamma^s_{j3}\partial_3 G_{sp} = 0.\end{aligned}$$

Hence, (5.106) results in:

$$\begin{aligned}&\Big((\nabla y_0)^{\mathsf T}\nabla\tilde e_0 + 3(\nabla\vec b_0)^{\mathsf T}\nabla\tilde d_0\Big)_{\mathrm{sym}\,,ij} - \frac12\partial_{333}G_{ij} \\ &\qquad= \frac12\Big(G_{is}\partial_3 R^s_{3j3} + G_{js}\partial_3 R^s_{3i3}\Big) = \frac12\partial_3\big(G_{is}R^s_{3j3} + G_{js}R^s_{3i3}\big) \\ &\qquad= \frac12\partial_3\big(R_{i3j3} + R_{j3i3}\big) = \partial_3 R_{i3j3},\end{aligned}$$

by $R^s_{3js} = R^s_{3i3} = 0$. This completes the proof. □

Bibliography

[AB99] Charalambos D. Aliprantis and Kim C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Studies in Economic Theory. Springer, 1999.

[AD14] Marino Arroyo and Antonio DeSimone. Shape control of active surfaces inspired by the movement of euglenids. *J. Mech. Phys. Solids*, 62:99 – 112, 2014. Sixtieth anniversary issue in honor of Professor Rodney Hill.

[AD15] Virginia Agostiniani and Antonio DeSimone. Rigorous derivation of active plate models for thin sheets of nematic elastomers. *Mathematics and Mechanics of Solids*, 2015.

[AD17] Virginia Agostiniani and Antonio DeSimone. Dimension reduction via Γ - convergence for soft active materials. *Meccanica*, 52(14):3457–3470, Nov 2017.

[ADK16] Virginia Agostiniani, Antonio DeSimone, and Konstantinos Koumatos. Shape programming for narrow ribbons of nematic elastomers. *Journal of Elasticity*, 127, 2016.

[ADLL] Virginia Agostiniani, Antonio DeSimone, Alessandro Lucantonio, and Danka Lučić. Foldable structures made of hydrogel bilayers. Preprint.

[AEKS11] Shahaf Armon, Efi Efrati, Raz Kupferman, and Eran Sharon. Geometry and mechanics in the opening of chiral seed pods. *Science*, 333(6050):1726–1730, 2011.

[AHMD12] Marino Arroyo, Luca Heltai, Daniel Millán, and Antonio DeSimone. Reverse engineering the euglenoid movement. *PNAS*, 109(44):17874 – 17879, 2012.

[ALL17] Virginia Agostiniani, Alessandro Lucantonio, and Danka Lučić. Heterogeneous elastic plates with in-plane modulation of the target curvature and applications to thin gel sheets. *ArXiv e-prints*, 2017.

[ASK14] Hillel Aharoni, Eran Sharon, and Raz Kupferman. Geometry of thin nematic elastomer sheets. *Phys. Rev. Lett.*, 113:257801, Dec 2014.

[BBN15] Sören Bartels, Andrea Bonito, and Ricardo H Nochetto. Bilayer Plates: Model Reduction, Γ-Convergent Finite Element Approximation, and Discrete Gradient Flow. *Communications on Pure and Applied Mathematics*, 2015.

[BK14] Peter Bella and Robert V. Kohn. Metric-induced wrinkling of a thin elastic sheet. *Journal of Nonlinear Science*, 24(6):1147–1176, 2014.

[BLS16] Kaushik Bhattacharya, Marta Lewicka, and Mathias Schäffner. Plates with incompatible prestrain. *Archive for Rational Mechanics and Analysis*, 221(1):143–181, 2016.

[Bra02] Andrea Braides. Γ-*convergence for Beginners.* Oxford Lecture Series in Mathe. Oxford University Press, 2002.

[Bre10] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media, 2010.

[CC05] Philipe G. Ciarlet and Patrick Ciarlet. Another approach to linearized elasticity and a new proof of Korn's inequality. *Mathematical Models and Methods in Applied Sciences*, 15(02):259–271, 2005.

[CD11] Pierluigi Cesana and Antonio DeSimone. Quasiconvex envelopes of energies for nematic elastomers in the small strain regime and applications. *J. Mech. Phys. Solids*, 59(4):787–803, 2011.

[CDD02a] Sergio Conti, Antonio DeSimone, and Georg Dolzmann. Semisoft elasticity and director reorientation in stretched sheets of nematic elastomers. *Phys. Rev. E*, 66:061710, Dec 2002.

[CDD02b] Sergio Conti, Antonio DeSimone, and Georg Dolzmann. Soft elastic response of stretched sheets of nematic elastomers: a numerical study. *J. Mech. Phys. Solids*, 50(7):1431 – 1451, 2002.

[Cia06] Philippe G. Ciarlet. *An Introduction to Differential Geometry with Applications to Elasticity.* Springer, 2006.

[CRS17] Marco Cicalese, Matthias Ruf, and Francesco Solombrino. On global and local minimizers of prestrained thin elastic rods. *Calculus of Variations and Partial Differential Equations*, 56:115, 08 2017.

[DD02] Antonio DeSimone and Georg Dolzmann. Macroscopic response of nematic elastomers via relaxation of a class of SO(3)-invariant energies. *Arch. Ration. Mech. Anal.*, 161(3):181–204, 2002.

[DeS99] Antonio DeSimone. Energetics of fine domain structures. *Ferroelectrics*, 222:275–284, 1999.

[DeS18] Antonio DeSimone. Spontaneous bending of pre-stretched bilayers. *Meccanica*, 53(3):511–518, Feb 2018.

[DF75] Ennio De Giorgi and Tullio Franzoni. Su un tipo di convergenza variazionale. *Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat.*, 58:842–850, 1975.

[DFMS17] Bernard Dacorogna, Nicola Fusco, Stefan Müller, and Vladimir Sverak. *Vector-Valued Partial Differential Equations and Applications*, volume 2179. 01 2017.

[dHSSB+12] Laurens T. de Haan, Carlos Sańchez-Somolinos, Cees M. W. Bastiaansen, Albertus P. H. J. Schenning, and Dirk J. Broer. Engineering of Complex Order and the Macroscopic Deformation of Liquid Crystal Polymer Networks. *Angewandte Chemie International Edition*, 51(50):12469–12472, December 2012.

[DM12] Gianni Dal Maso. *An introduction to* Γ*-convergence*, volume 8. Springer Science & Business Media, 2012.

[Doi09] Masao Doi. Gel dynamics. *Journal of the Physical Society of Japan*, 78(5):052001–052001, 2009.

[DS11] Hui-Hui Dai and Zilong Song. Some analytical formulas for the equilibrium states of a swollen hydrogel shell. *Soft Matter*, 7:8473–8483, 2011.

[DT09] Antonio DeSimone and Luciano Teresi. Elastic energies for nematic elastomers. *The European Physical Journal E*, 29(2):191–204, Jun 2009.

[DVR97] Colin Dawson, Julian F. V. Vincent, and Anne-Marie Rocca. How pine cones open. *Nature*, 290:668, 1997.

[ESK09] Efi Efrati, Eran Sharon, and Raz Kupferman. Elastic theory of unconstrained non-Euclidean plates. *Journal of the Mechanics and Physics of Solids*, 57(4):762–775, 2009.

[FHMP16] Lorenzo Freddi, Peter Hornung, Maria Giovanna Mora, and Roberto Paroni. A corrected Sadowsky functional for inextensible elastic ribbons. *Journal of Elasticity*, 123, 2016.

[FJM02] Gero Friesecke, Richard D. James, and Stefan Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11):1461–1506, 2002.

[FJM06] Gero Friesecke, Richard D. James, and Stefan Müller. A hierarchy of plate models derived from nonlinear elasticity by Γ-convergence. *Archive for rational mechanics and analysis*, 180(2):183–236, 2006.

[FRS93] D. D. Fox, Annie Raoult, and Juan C. Simo. A justification of nonlinear properly invariant plate theories. *Archive for Rational Mechanics and Analysis*, 124(2):157–199, Jun 1993.

[GLZ13] Wei Guo, Meie Li, and Jinxiong Zhou. Modeling programmable deformation of self-folding all-polymer structures with temperature-sensitive hydrogels. *Smart Materials and Structures*, 22(11):115028, 2013.

[Gur82] Morton E. Gurtin. *An introduction to continuum mechanics*, volume 158. Academic press, 1982.

[HAB$^+$10] Elliot Hawkes, B. An, Nadia M. Benbernou, H. Tanaka, Sangbae Kim, E. D. Demaine, D. Rus, and Robert J. Wood. Programmable matter by folding. *Proceedings of the National Academy of Sciences*, 107(28):12441–12445, 2010.

[HH06] Qing Han and Jia-Xing Hong. *Isometric embedding of Riemannian manifolds in Euclidean spaces*, volume 13. American Mathematical Soc., 2006.

[HMVV16] Walid Hamouche, Corrado Maurini, Angela Vincenti, and Stefano Vidoli. Basic criteria to design and produce multistable shells. *Meccanica*, 51(10):2305–2320, Oct 2016.

[HMVV17] Walid Hamouche, Corrado Maurini, Angela Vincenti, and Stefano Vidoli. Multiparameter actuation of a neutrally stable shell: a flexible gear-less motor. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 473(2204), 2017.

[Hor11a] Peter Hornung. Approximation of flat $W^{2,2}$ - isometric immersions by smooth ones. *Archive for rational mechanics and analysis*, 199(3):1015–1067, 2011.

[Hor11b] Peter Hornung. Fine level set structure of flat isometric immersions. *Archive for rational mechanics and analysis*, 199(3):943–1014, 2011.

[HZL95] Zhibing Hu, Xiaomin Zhang, and Yong Li. Synthesis and application of modulated polymer gels. *Science*, 269(5223):525–527, 1995.

[Ion11] Leonid Ionov. Soft microorigami: self-folding polymer films. *Soft Matter*, 7(15):6786–6791, 2011.

[JM15] Gareth Wyn Jones and Lakshminarayanan Mahadevan. Optimal control of plates using incompatible strains. *Nonlinearity*, 28(9):3153, 2015.

[KES07] Yael Klein, Efi Efrati, and Eran Sharon. Shaping of elastic sheets by prescription of non-Euclidean metrics. *Science*, 315(5815):1116–1120, 2007.

[KHB+12] Jungwook Kim, James A. Hanna, Myunghwan Byun, Christian D. Santangelo, and Ryan C. Hayward. Designing responsive buckled surfaces by halftone gel lithography. *Science*, 335(6073):1201–1205, 2012.

[KM14] Raz Kupferman and Cy Maor. A Riemannian approach to the membrane limit in non-euclidean elasticity. *Communications in Contemporary Mathematics*, 16(05):1350052, 2014.

[KMS18] Raz Kupferman, Cy Maor, and Asaf Shachar. Reshetnyak rigidity for Riemannian manifolds. *Archive for Rational Mechanics and Analysis*, Jul 2018.

[KO17] Robert V. Kohn and Ethan O'Brien. On the bending and twisting of rods with misfit. *Journal of Elasticity*, pages 1–29, 3 2017.

[KS14] Raz Kupferman and Jake P. Solomon. A Riemannian approach to reduced plate, shell, and rod theories. *Journal of Functional Analysis*, 266(5):2989–3039, 2014.

[Kui55] Nicolaas H. Kuiper. On C^1-isometric imbeddings. I. In *Indagationes Mathematicae (Proceedings)*, volume 58, pages 545–556. Elsevier, 1955.

[LDR95] Hervé Le Dret and Annie Raoult. The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 74, 01 1995.

[Lew11] Marta Lewicka. Reduced theories in nonlinear elasticity. In *Nonlinear Conservation Laws and Applications*, pages 393–403. Springer, 2011.

[LL18] Marta Lewicka and Danka Lučić. Dimension reduction for thin films with transversally varying prestrain: the oscillatory and the non-oscillatory case. *ArXiv e-prints*, July 2018.

[LMP10] Marta Lewicka, Lakshminarayanan Mahadevan, and Mohammad Reza Pakzad. The Föppl-von Kármán equations for plates with incompatible strains. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 467(2126):402–426, 2010.

[LNS14] Alessandro Lucantonio, Paola Nardinocchi, and Howard A. Stone. Swelling dynamics of a thin elastomeric sheet under uniaxial pre-stretch. *Journal of Applied Physics*, 115(8):083505, 2014.

[LOP+15] Marta Lewicka, Pablo Ochoa, Mohammad Reza Pakzad, et al. Variational models for prestrained plates with Monge-Ampere constraint. *Differential and Integral Equations*, 28(9/10):861–898, 2015.

[LP11] Marta Lewicka and Mohammad Reza Pakzad. Scaling laws for non-Euclidean plates and the $W^{2,2}$ - isometric immersions of Riemannian metrics. *ESAIM: Control, Optimisation and Calculus of Variations*, 17(4):1158–1173, 2011.

[LP16] Marta Lewicka and Mohammad Reza Pakzad. Prestrained elasticity: from shape formation to Monge-Ampere anomalies. *Notices of the AMS*, 2016.

[LRR17] Marta Lewicka, Annie Raoult, and Diego Ricciotti. Plates with incompatible prestrain of high order. In *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*. Elsevier, 2017.

[LSDG17] Ying Liu, Brandi Shaw, Michael D. Dickey, and Jan Genzer. Sequential self-folding of polymer sheets. *Science Advances*, 3(3), 2017.

[LT97] Fon-Che Liu and Wei-Shyan Tai. Lusin properties and interpolation of Sobolev spaces. *Topol. Methods Nonlinear Anal.*, 9(1):163–177, 1997.

[LTD16] Alessandro Lucantonio, Giuseppe Tomassetti, and Antonio DeSimone. Large-strain poroelastic plate theory for polymer gels with applications to swelling-induced morphing of composite plates. *Composites Part B: Engineering*, 2016.

[Mar03] Sorin Mardare. The fundamental theorem of surface theory for surfaces with little regularity. *Journal of elasticity*, 73(1-3):251–290, 2003.

[Mar04] Sorin Mardare. On isometric immersions of a Riemannian space with little regularity. *Analysis and applications*, 2(03):193–226, 2004.

[Mos15] Cyrus Mostajeran. Curvature generation in nematic surfaces. *Phys. Rev. E*, 91:062405, Jun 2015.

[MP05] Stefan Müller and Mohammad Reza Pakzad. Regularity properties of isometric immersions. *Mathematische Zeitschrift*, 251(2):313–331, Oct 2005.

[MS18] Cy Maor and Asaf Shachar. On the role of curvature in the elastic energy of non-Euclidean thin bodies. *ArXiv e-prints*, January 2018.

[MSG15] Giovanni Battista Maggiani, Riccardo Scala, and Nicolas Van Goethem. A compatible-incompatible decomposition of symmetric tensors in L^p with application to elasticity. *Mathematical Methods in the Applied Sciences*, 38(18):5217–5230, 2015.

[MYI$^+$15] Yiqi Mao, Kai Yu, Michael S Isakov, Jiangtao Wu, Martin L Dunn, and H Jerry Qi. Sequential self-folding structures by 3d printed digital shape memory polymers. *Scientific reports*, 5:13616, 2015.

[Nas54] John Nash. C^1 - isometric imbeddings. *Annals of mathematics*, pages 383–396, 1954.

[NEB$^+$] Jun-Hee Na, Arthur A. Evans, Jinhye Bae, Maria C. Chiappelli, Christian D. Santangelo, Robert J. Lang, Thomas C. Hull, and Ryan C. Hayward. Programming reversibly self-folding origami with micropatterned photo-crosslinkable polymer trilayers. *Advanced Materials*, 27(1):79–85.

[Nol68] Walter Noll. *A Mathematical Theory of the Mechanical Behavior of Continuous Media*, pages 25–54. Springer Berlin Heidelberg, Berlin, Heidelberg, 1968.

[Pak04] Mohammad Reza Pakzad. On the Sobolev space of isometric immersions. *Journal of Differential Geometry*, 66(1):47–69, 2004.

[Ped97] Pablo Pedregal. *Parametrized measures and variational principles.* Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser Verlag, Basel, 1997.

[PSNH15] Matteo Pezzulla, Steven A. Shillig, Paola Nardinocchi, and Douglas P. Holmes. Morphing of geometric composites via residual swelling. *Soft Matter*, 11:5812–5820, 2015.

[PSNH16] Matteo Pezzulla, Gabriel P. Smith, Paola Nardinocchi, and Douglas P. Holmes. Geometry and mechanics of thin growing bilayers. *Soft Matter*, 12:4435–4442, 2016.

[Sch07a] Bernd Schmidt. Minimal energy configurations of strained multi-layers. *Calculus of Variations and Partial Differential Equations*, 30(4):477–497, 2007.

[Sch07b] Bernd Schmidt. Plate theory for stressed heterogeneous multilayers of finite bending energy. *Journal de mathématiques pures et appliquées*, 88(1):107–122, 2007.

[SE10] Eran Sharon and Efi Efrati. The mechanics of non-Euclidean plates. *Soft Matter*, 6(22):5693, 2010.

[Spi75] Michael Spivak. *A Comprehensive Introduction to Differential Geometry.* Number v. 3 in A Comprehensive Introduction to Differential Geometry. Publish or Perish, Incorporated, 1975.

[SRS07] Eran Sharon, Benoît Roman, and Harry L. Swinney. Geometrically driven wrinkling observed in free plastic sheets and leaves. *Physical review. E, Statistical, nonlinear, and soft matter physics*, 75 4 Pt 2:046211, 2007.

[SUT+10] Yoshiki Sawa, Kenji Urayama, Toshikazu Takigawa, Antonio DeSimone, and Luciano Teresi. Thermally driven giant bending of liquid crystal elastomer films with hybrid alignment. *Macromolecules*, 43:4362–4369, May 2010.

[SYU+11] Yoshiki Sawa, Fangfu Ye, Kenji Urayama, Toshikazu Takigawa, Vianney Gimenez-Pinto, Robin L. B. Selinger, and Jonathan V. Selinger. Shape selection of twist-nematic-elastomer ribbons. *PNAS*, 108(16):6364–6368, 2011.

[SZT+12] Georgi Stoychev, Svetlana Zakharchenko, Sébastien Turcaud, John WC Dunlop, and Leonid Ionov. Shape-programmed folding of stimuli-responsive polymer bilayers. *ACS nano*, 6(5):3925–3934, 2012.

[Top06] Flemming Topsok. Some bounds for the logarithmic function. *Inequality theory and applications*, 4:137, 2006.

[Wan66] Chao-Cheng Wang. On the stored-energy functions of hyperelastic materials. *Archive for Rational Mechanics and Analysis*, 23(1):1–14, Jan 1966.

[Wan68] Chao-Cheng Wang. *On the Geometric Structures of Simple Bodies, a Mathematical Foundation for the Theory of Continuous Distributions of Dislocations*, pages 87–148. Springer Berlin Heidelberg, Berlin, Heidelberg, 1968.

[WMG+13] Zi Liang Wu, Michael Moshe, Jesse Greener, Heloise Therien-Aubin, Zhihong Nie, Eran Sharon, and Eugenia Kumacheva. Three-dimensional shape transformations of hydrogel sheets induced by small-scale modulation of internal stresses. *Nature Communications*, 4:1586, March 2013.

[YXP+14] ChangKyu Yoon, Rui Xiao, JaeHyun Park, Jaepyeong Cha, Thao D Nguyen, and David H Gracias. Functional stimuli responsive hydrogel devices by self-folding. *Smart Materials and Structures*, 23(9):094008, 2014.