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Ph.D. Course in Mathematical Analysis, Modelling and Applications

Structural and geometric properties of RCD spaces

Ph.D. Thesis

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To my grandmother Fernanda,
who accompanied me along this path
until the day I began to write this thesis.

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"What is done is done
and there is more to come"

Ulver

Introduction

One of the essential principles of modern mathematics can be informally stated as follows: *it is sometimes convenient to provide nonsmooth generalisations of smooth objects, since to investigate the former can shed new light on the latter.* A key instance of such phenomenon is given by the theory of Sobolev spaces, which allows to study several properties of the smooth functions by looking at the behaviour of the weakly differentiable ones.

From a geometric standpoint, in these last decades the tendency has been to generalise different concepts of curvature from the classical Riemannian world to the realm of nonsmooth metric structures. The very first step toward this direction is represented by the *Alexandrov geometry*. This theory has been originally introduced by A. D. Alexandrov in the late '40s and considerably developed by Y. Burago, M. Gromov and G. Perel'man in [BGP92]. An Alexandrov space is a metric space whose sectional curvature is bounded on one side by some constant $k \in \mathbb{R}$ (when the curvature is bounded from above, these spaces are now commonly referred to as $\mathsf{CAT}(k)$ spaces; the acronym CAT – coined by Gromov in [Gro87] – stands for 'Cartan-Alexandrov-Toponogov'). The sectional curvature bound is imposed via comparison of the geodesic triangles in the space with the geodesic triangles in the model Riemannian surface having curvature constantly equal to k. We do not enter into further details about such theory, albeit related to the topics of this thesis, but we rather refer the reader to the monograph [BBI01] for a thorough account of it. We just mention the fact that the class of Alexandrov spaces with curvature bounded by some $k \in \mathbb{R}$ is closed under *Gromov-Hausdorff convergence* – a crucial notion of distance between metric spaces, which measures how far two spaces are from being isometric. In particular, this grants that any Gromov-Hausdorff limit of a sequence of Riemannian manifolds (with uniformly bounded sectional curvature) – which might not be a Riemannian manifold – certainly has an Alexandrov space structure.

More recently, even the notion of 'having Ricci curvature bounded from below' has been generalised to the nonsmooth framework. In this case, the correct setting to work in is that of metric measure spaces, the reason being that the interplay between distance and measure is necessary in order to encode the Ricci curvature bounds. The first proposal in this direction is given by the theory of *Ricci limits*, introduced by J. Cheeger and T. Colding in [CC96] and extensively studied – among many others – in [CC97, CC00a, CC00b, CN12]. Shortly said, a Ricci limit is achieved as limit of a sequence of Riemannian manifolds with a uniform lower bound on the Ricci curvature; such convergence is meant to be with respect to the *measured Gromov-Hausdorff distance*, which constitutes a variant of the Gromov-Hausdorff distance that takes also the behaviour of the reference measure into account. An alternative to this 'extrinsic' approach has been independently proposed by Lott-Villani and Sturm in the seminal papers [LV07, Stu06a, Stu06b], thus giving birth to the theory of *curvature-dimension conditions* for metric measure spaces. These structures are typically called $\mathsf{CD}(K, N)$ spaces, where $K \in \mathbb{R}$ indicates the synthetic bound from below on the Ricci curvature, while the constant $N \in [1, \infty]$ is the bound from above on the dimension. A key feature of these spaces is the intrinsic nature of their definition: the above-mentioned bounds can be imposed by requiring some form of convexity of suitable entropy functionals along Wasserstein geodesics, by using an abstract optimal transport language that does not appeal to any smooth structure. This represents the main novelty of the CD theory as opposed to the Ricci limits one, which cannot overlook the smooth geometry as it is based upon the approximation with Riemannian manifolds. Nevertheless, the class of $\mathsf{CD}(K, N)$ spaces does not perfectly capture the essence of 'being Riemannian', as it also contains Finsler manifolds. An advantage of such fact is that it permits the study of geometric and functional inequalities in metric structures that could be very distant from being Euclidean-like. For instance, some of the several inequalities that are currently available in the $\mathsf{CD}(K, N)$ setting are the Bishop-Gromov inequality [Stu06b], the Bonnet-Myers diameter estimate [Stu06b], a weak local $(1, 1)$-Poincaré inequality [Raj12], Laplacian comparison estimates for squared distance functions [Gig15] and the Lévy-Gromov inequality [CM17]; we refer to [BGL12] for many other useful inequalities.

On the other hand, a more restrictive curvature-dimension condition has been proposed in order to select those $\mathsf{CD}(K, N)$ spaces that resemble more a Riemannian manifold: the so-called $\mathsf{RCD}(K, N)$ condition, introduced in [AGS14b, AGMR15] for the case $N = \infty$ and in [Gig15] for N finite (the added letter R is the initial of 'Riemannian'). Roughly speaking, an RCD space is a CD space that looks like a Hilbert space at infinitesimal scales (indeed, the precise concept expressing such property is called *infinitesimal Hilbertianity*). In the RCD context, besides the formulation via optimal transport (à la Lott-Villani-Sturm, the 'Lagrangian' approach), it is possible to utilise the Γ-calculus language (à la Bakry-Émery, the 'Eulerian' approach). The first geometric result for RCD spaces that has been obtained is the Abresch-Gromoll inequality [GM14].

We point out a fundamental feature of the CD/RCD theories, which actually justifies their importance: given any constants $K \in \mathbb{R}$ and $N \in [1, \infty]$, it holds that both the family of all $\mathsf{CD}(K, N)$ spaces and that of all $\mathsf{RCD}(K, N)$ spaces are closed under measured Gromov-Hausdorff convergence; in particular, they contain all Ricci limits. Furthermore, many analytic and geometric properties of CD/RCD spaces are stable under measured Gromov-Hausdorff convergence, thus allowing to prove several rigidity results. Among them, we just mention the Cheeger-Gromoll splitting theorem [Gig13, Gig14b], the maximal diameter theorem [Ket15a], the Obata rigidity theorem [Ket15b], the 'volume cone to metric cone' theorem [DPG16] and the Bochner rigidity theorem for the first cohomology group [GR17]. For a general overview and many historical remarks about the theory of curvature-dimension conditions on metric measure spaces, we refer the reader to the nice surveys [Amb18, Vil16, Vil17].

The purpose of the present thesis is to investigate some structural properties of finite-dimensional RCD spaces. In a few words, the main objective has been to extract 'concrete' geometric properties from the 'abstract' theory of these spaces and therefore to deepen our knowledge of their shape. Some of the articles that are undoubtedly important contributions in this direction are the following ones:

- N. Gigli, A. Mondino and T. Rajala showed in [GMR15] that any $\mathsf{RCD}(K, N)$ space (with $N < \infty$) has at least one Euclidean tangent cone around almost all of its points.

- A. Mondino and A. Naber proved in [MN14] that any finite-dimensional RCD space is rectifiable as a metric space.

- A definition of *non-collapsed* RCD space has been introduced and analysed by G. De Philippis and N. Gigli in [DPG18].
- E. Brué and D. Semola proved in [BS18a] that any $\mathsf{RCD}(K,N)$ space (with $N<\infty$) has constant dimension, in a suitable sense.
- It has been shown by Y. Kitabeppu in [Kit18] that, calling $n\in\mathbb{N}$ the dimension of an RCD space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we have that X admits no k-regular points for any $k>n$.

The language we shall adopt in this thesis is that of *L^p-normed L^∞-modules*, which have been introduced by N. Gigli in [Gig17b]. Another ingredient that plays a fundamental role in our discussion is the notion of *Sobolev space* over general metric measure spaces, which allows for the development of a differential calculus in such abstract setting. Part of the work we shall carry out will be to provide some 'foundational' results (at the level of abstract metric measure spaces, with no curvature bounds) about normed modules, Sobolev spaces and the possible relations between them. More precisely, we pursue these plans:

- To investigate an axiomatic concept of Sobolev space – called *D-structure* and introduced in [GT01] by V. Gol'dshtein and M. Troyanov – and combine it with the language of normed modules (cf. Section 2.1 and Subsection 4.1.1).
- To prove that a certain class of normed modules can be represented as spaces of sections of some notion of measurable Banach bundle (see Section 3.2).
- To show that any Sobolev map from a metric measure space to a metric space is associated with a *differential* operator, which is a linear and continuous map between suitable tangent modules (cf. Chapter 8).

On the other hand, the results we obtained concerning the structure theory of RCD spaces can be summarised as follows:

- We prove that finite-dimensional RCD spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$ are rectifiable 'as metric measure spaces', meaning that the maps provided by Mondino-Naber in [MN14] to display the metric rectifiability of X keep under control the reference measure; see Section 5.2.
- We show that on any $\mathsf{RCD}(K,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ the abstract differential calculus developed by N. Gigli, which is of purely functional-analytic nature and thus a priori possibly unrelated to the structure of the underlying space, can be actually linked (in a canonical way) to the geometry of X, namely to the pointed measured Gromov-Hausdorff rescalings of X around its points; cf. Sections 5.1 and 5.3.
- We propose a notion of parallel transport for $\mathsf{RCD}(K,\infty)$ spaces, prove its basic properties (such as uniqueness, norm-preservation and geometric consequences) and show its existence under suitable assumptions on the space; see Chapter 6.
- We explain in which sense any Sobolev vector field (in the language of [Gig17b]) over an $\mathsf{RCD}(K,\infty)$ space admits a unique quasi-continuous representative, much like Sobolev functions do; cf. Chapter 7.

We now briefly describe the structure of the thesis. However, any chapter will begin with an introductory part, which will explain more in details the material contained therein.

Structure of the thesis

The contents of the thesis are subdivided into the various chapters in the following way:

Chapter 1: Prolegomena. In this chapter we collect – for the usefulness of the reader – the main well-known definitions and results about geometric analysis on metric measure spaces, which will be needed later on in the thesis.

Chapter 2: Sobolev calculus on metric measure spaces. First of all, we present (in Section 2.1) the axiomatic approach of [GT01] to the Sobolev calculus on metric measure spaces, which aims at unifying several different variants of Sobolev space that appeared in the literature throughout the last twenty years. We also suggest some new notions of locality for these axiomatic Sobolev spaces and we show the consequent calculus rules. Finally, in Section 2.2 we focus our attention on a precise notion of Sobolev space: the one obtained via *weak upper gradients*, cf. [Che99, Sha00, AGS14a]. This is the approach we will follow in order to develop a differential calculus in our metric measure context.

Chapter 3. The language of normed modules. From this point forward the whole thesis will be based upon the terminology of L^p-normed L^∞-modules, which constitute a convenient abstraction of the notion of 'space of p-integrable vector fields' over a given Riemannian manifold. Section 3.1 is devoted to an exhaustive description of such objects, along the lines of the presentation in [Gig17b] and [Gig17a]; more specifically, besides the definition of normed module and its basic properties, we discuss a natural concept of local dimension that arises in this context and explain how to construct new normed modules out of the old ones (by taking duals, pullbacks and tensor products). On the other hand, Section 3.2 contains a new result – called *Serre-Swan theorem* – about the representation of normed modules. It says that any 'locally finitely-generated' normed module is isomorphic to the space of sections of a suitable *measurable Banach bundle*, whose fibers have finite dimension. Moreover, such module-bundle correspondence is also observed from a categorical viewpoint.

Chapter 4. Differential calculus on RCD spaces. A combination of Sobolev calculus and normed modules theory yields a first-order differential structure over any abstract metric measure space, to which the whole Section 4.1 is dedicated. In this regard, the most important objects are *tangent module* and *cotangent module*, which generalise the concepts of vector field and 1-form, respectively. Another fundamental tool is the *differential* operator that can be associated to any morphism of metric measure spaces. The new contributions that can be found in this section are Subsection 4.1.3 (where we examine the tangent module over $\mathbb{R}^d$ endowed with a generic Radon measure) and Subsection 4.1.5 (where a special notion of differential for maps with Euclidean target is built). Furthermore, in presence of curvature bounds even a second-order differentiable calculus is possible, as described in Section 4.2. To begin with, we recall the definition of $\mathsf{RCD}(K,N)$ space and the path that led to it. Then we explain how to define Hessian and covariant derivative on RCD spaces via suitable integration-by-parts formulae; their well-posedness is granted by the presence of a sufficiently vast class of test functions, which is produced by exploiting the regularising effects of the heat flow on RCD spaces.

Chapter 5. Structure of strongly $\mathfrak{m}$-rectifiable spaces. We focus on a special class of metric measure spaces, which are said to be *strongly* $\mathfrak{m}$*-rectifiable*. Shortly said, these are spaces that are 'almost isometrically' rectifiable via maps under which the reference measure behaves well. In Section 5.2 we prove that any $\mathsf{RCD}(K,N)$ space (with $N<\infty$) is strongly $\mathfrak{m}$-rectifiable, thus motivating our interest in such a class of spaces. In Section 5.3 we show that the abstract tangent module associated to a strongly $\mathfrak{m}$-rectifiable space can be actually realised as the space of sections of the *Gromov-Hausdorff tangent bundle*, which is obtained by glueing together (in a canonical way) the blow-ups of the space around its points. This result raises a bridge between the analytic machinery of tangent/cotangent modules and the geometric aspects of the spaces under consideration.

Chapter 6. A notion of parallel transport for RCD spaces. We introduce a notion of *parallel transport* for the class of $\mathsf{RCD}(K,\infty)$ spaces. In view of some technical issues due to the nature of our spaces, we do not speak about parallel transport along a simple Lipschitz curve, but rather we consider a 'weighted selection' of curves at the same time. In Section 6.1 we set up the theory of those functional spaces wherein the parallel transport lives. In Subsection 6.2.1 we give the definition of parallel transport and we show that it is well-posed, it preserves the norm and it forces the constant dimension of the underlying space. However, in the current state of the art we are not able to prove existence of the parallel transport on any $\mathsf{RCD}(K,\infty)$ space; we just show (in Subsection 6.2.2) its existence for a special class of finite-dimensional spaces admitting a 'good Sobolev basis' of the tangent module.

Chapter 7. Quasi-continuous vector fields on RCD spaces. Since a second-order differential calculus is available in the RCD setting, we might wonder whether (and in which sense) it is possible to take *quasi-continuous representatives* of Sobolev vector fields. This is the problem we address in this chapter. It amounts to solving the following tasks: to build up a new notion of 'capacitary' tangent module (with the variational capacity in place of the usual reference measure), to declare what a quasi-continuous capacitary vector field is and to show that any Sobolev vector field admits a unique quasi-continuous representative. Nonetheless, the geometric consequences of such theory have not been investigated yet.

Chapter 8. Differential of metric-valued Sobolev maps. One of the several possible ways to define Sobolev maps from a metric measure space to a metric space is via post-composition with Lipschitz functions. In this chapter we explain how to associate a *differential* to any such map, which ought to be a linear and continuous operator between appropriate normed modules. Moreover, we prove its consistency with some previously known notions of differential, such as Kirchheim's differential for metric-valued Lipschitz maps defined on the Euclidean space. Finally, we build the differential operator even for maps that are just locally Sobolev, by means of a suitable inverse limit construction. The motivation behind such results is the following: we would like to provide a *Bochner-Eells-Sampson inequality* for maps between an $\mathsf{RCD}(K,N)$ space X and a $\mathsf{CAT}(0)$ space Y, with the aim of proving that any harmonic maps from X to Y is locally Lipschitz. To pursue such long-term plan, the very first step to make is precisely to define a notion of differential that fits into this framework.

List of included papers

(I) N. Gigli and E. Pasqualetto, *Behaviour of the reference measure on* RCD *spaces under charts.* Accepted at Communications in Analysis and Geometry, (2016).

(II) N. Gigli and E. Pasqualetto, *Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces.* Submitted, arXiv:1611.09645, (2016).

(III) D. Lučić and E. Pasqualetto, *The Serre-Swan theorem for normed modules.* Rend. Circ. Mat. Palermo, (2018).

(IV) N. Gigli and E. Pasqualetto, *On the notion of parallel transport on* RCD *spaces.* Submitted, arXiv:1803.05374, (2018).

(V) N. Gigli and E. Pasqualetto, *Differential structure associated to axiomatic Sobolev spaces.* Submitted, arXiv:1807.05417, (2018).

(VI) N. Gigli, E. Pasqualetto and E. Soultanis, *Differential of metric valued Sobolev maps.* Submitted, arXiv:1807.10063, (2018).

(VII) C. Debin, N. Gigli and E. Pasqualetto, *Quasi-continuous vector fields on* RCD *spaces.* In progress, (2018).

The new results contained in the above articles are distributed along the thesis according to the following scheme:

(I) Theorem 3.35, Remark 3.36, Definition 4.15, Proposition 4.16, Proposition 4.17, Subsection 4.1.3, Proposition 4.37, Proposition 4.38, Section 5.2.

(II) Lemma 1.37, Proposition 1.38, Proposition 1.41, Remark 4.11, Remark 4.12, Lemma 4.33, Proposition 4.36, Subsection 4.1.5, Section 5.1, Section 5.3.

(III) Definition 3.18, Lemma 3.20, Remark 3.30, Section 3.2.

(IV) Theorem 2.15, Remark 2.16, Subsection 4.1.2, Chapter 6.

(V) Subsection 2.1.2, Remark 2.30, Theorem 4.1, Theorem 4.2, Proposition 4.3.

(VI) Remark 3.11, Chapter 8.

(VII) Lemma 4.54, Chapter 7.

1

Prolegomena

Contents

In this chapter we collect some well-known definitions and results, which will be needed throughout the whole thesis. More specifically, the chapter is organised as follows:

- In Section 1.1 we will present the basics of metric geometry, with a particular accent on the concepts of absolute continuity, Lipschitz continuity and geodesic space; most of the material can be found, for instance, in [BBI01].

- In Section 1.2 we shall recall the main topics of measure theory. A special role will be played by the space $L^0(\mathfrak{m})$ of all equivalence classes (up to $\mathfrak{m}$-a.e. equality) of Borel functions, which will constitute a fundamental tool in Chapter 3. For a thorough treatise about this vast subject, we refer e.g. to the monography [Bog07].

- Finally, Section 1.3 will be devoted to the main aspects of geometric analysis on metric measure spaces. Among the several tools we will discuss, we just mention the notion of *pointed-measured-Gromov-Hausdorff convergence*, which will be used later on to study the blow-ups of finite-dimensional RCD spaces around its points (in Chapter 5).

1.1 Metric spaces

A *metric space* is any couple (X, d), where X is a set and the function $\mathsf{d} : \mathrm{X} \times \mathrm{X} \to [0, +\infty)$, which is called *distance* on X, is symmetric, vanishes precisely on the diagonal and satisfies the triangle inequality. Given a point $x \in \mathrm{X}$ and a radius $r > 0$, we shall denote by $B_r(x)$ the *open ball* centered at x of radius r, namely

$$B_r(x) \doteq \big\{ y \in \mathrm{X} \,:\, \mathsf{d}(x,y) < r \big\}. \tag{1.1}$$

We will write $B_r^{\mathsf{d}}(x)$ instead of $B_r(x)$ whenever it will seem necessary to emphasise the distance under consideration. More generally, given any subset A of X and any radius $r > 0$, we will write $B_r(A)$ or $B_r^{\mathsf{d}}(A)$ to indicate the *r-neighbourhood* of A, which is defined as

$$B_r(A) \doteq \bigcup_{x \in A} B_r(x). \tag{1.2}$$

The distance between a point $x \in \mathrm{X}$ and a non-empty set $A \subseteq \mathrm{X}$ is given by the quantity

$$\mathsf{d}(x, A) \doteq \inf \big\{ \mathsf{d}(x,y) \,:\, y \in A \big\}. \tag{1.3}$$

With this notation, we can equivalently express $B_r(A)$ as the set $\big\{ x \in \mathrm{X} \,:\, \mathsf{d}(x,A) < r \big\}$.

We define the *diameter* $\mathrm{diam}(A) \in [0, +\infty]$ of any non-empty set $A \subseteq \mathrm{X}$ as

$$\mathrm{diam}(A) \doteq \sup \big\{ \mathsf{d}(x,y) \,:\, x, y \in A \big\}, \tag{1.4}$$

while we let $\mathrm{diam}(\emptyset) \doteq 0$ by convention. Observe that the only sets having zero diameter are the singletons and the empty set. A subset A of X is said to be *bounded* provided it has finite diameter – or, equivalently, if there exist $x \in \mathrm{X}$ and $r > 0$ such that $A \subseteq B_r(x)$. We say that the metric space (X, d) is *proper* provided any bounded closed subset of X is compact (while the converse implication is always verified: every compact set is closed and bounded).

Fix a sequence $(x_n)_n$ in X. We say that $(x_n)_n$ is *Cauchy* provided $\lim_{n,m \to \infty} \mathsf{d}(x_n, x_m) = 0$, while it *converges* to some limit $x_\infty \in \mathrm{X}$, briefly $x_n \to x_\infty$, provided $\lim_{n \to \infty} \mathsf{d}(x_n, x_\infty) = 0$. Any converging sequence is Cauchy, but in general the converse implication does not hold. Whenever it does, we say that (X, d) is *complete.* Moreover, a subset D of X is *dense* in X if for every $r > 0$ it holds that $\mathrm{X} = B_r(D)$. We say that (X, d) is *separable* provided there exists a sequence $(x_n)_n \subseteq \mathrm{X}$ that is dense in X.

By *curve* in X we intend any continuous map $\gamma : [0,1] \to \mathrm{X}$. For brevity, we will often write γ_t instead of $\gamma(t)$. Given $x, y \in \mathrm{X}$, we say that γ *joins* x to y provided $(\gamma_0, \gamma_1) = (x, y)$. The family of all curves in X will be shortly indicated by

$$\Gamma(\mathrm{X}) \doteq C\big([0,1], \mathrm{X}\big). \tag{1.5}$$

A natural distance that can be defined on the set $\Gamma(\mathrm{X})$ is the *sup distance* $\mathsf{d}_{\Gamma(\mathrm{X})}$, given by

$$\mathsf{d}_{\Gamma(\mathrm{X})}(\gamma, \sigma) \doteq \max \big\{ \mathsf{d}(\gamma_t, \sigma_t) \,\big|\, t \in [0,1] \big\} \qquad \text{for every } \gamma, \sigma \in \Gamma(\mathrm{X}). \tag{1.6}$$

If (X, d) is complete and separable, then also $\big(\Gamma(\mathrm{X}), \mathsf{d}_{\Gamma(\mathrm{X})}\big)$ is complete and separable.

We call *evaluation map* the continuous map $\mathrm{e} : \Gamma(\mathrm{X}) \times [0,1] \to \mathrm{X}$, which is defined by

$$\mathrm{e}(\gamma, t) \doteq \gamma_t \qquad \text{for every } (\gamma, t) \in \Gamma(\mathrm{X}) \times [0,1]. \tag{1.7}$$

Given $t \in [0,1]$, we denote by $\mathrm{e}_t \doteq \mathrm{e}(\cdot,t) : \Gamma(\mathrm{X}) \to \mathrm{X}$ the *evaluation map at time* t, namely

$$\mathrm{e}_t(\gamma) \doteq \gamma_t \qquad \text{for every } \gamma \in \Gamma(\mathrm{X}). \tag{1.8}$$

Each mapping e_t is clearly continuous.

The metric space (X,d) is said to be *doubling* (or *metrically doubling*) provided there exists a constant $\mathsf{C}_\mathsf{d} \in \mathbb{N}$, called *metric doubling constant* of the space (X,d), such that any open ball of some radius $r > 0$ can be covered by C_d many open balls of radius $r/2$.

1.1.1 Absolute continuity

An important class of curves is that of absolutely continuous curves, which we now describe.

Definition 1.1 (Absolutely continuous curve) *Let (X,d) be any complete metric space. Fix an exponent $p \in [1,\infty]$. Then a curve $\gamma \in \Gamma(\mathrm{X})$ is said to be p-*absolutely continuous *provided there exists a function $f \in L^p(0,1)$ such that*

$$\mathsf{d}(\gamma_t,\gamma_s) \leq \int_s^t f(r)\,\mathrm{d}r \qquad \textit{for every } t,s \in [0,1] \textit{ with } s < t. \tag{1.9}$$

The family of all p-absolutely continuous curves in X is denoted by $AC^p\big([0,1],\mathrm{X}\big)$.

For the sake of brevity, the space $AC^1\big([0,1],\mathrm{X}\big)$ will be denoted by $AC\big([0,1],\mathrm{X}\big)$ and its elements will be called just *absolutely continuous* curves in X.

Remark 1.2 In the case in which (X,d) coincides with the real line $\mathbb{R}$ endowed with the Euclidean distance, the previous notion of absolute continuity coincides with the classical one, which is treated e.g. in [Roy88]. ■

Remark 1.3 Given any complete metric space (X,d), the following inclusions hold:

$$AC^q\big([0,1],\mathrm{X}\big) \subseteq AC^p\big([0,1],\mathrm{X}\big) \qquad \text{for every } p,q \in [1,\infty] \text{ with } p \leq q. \tag{1.10}$$

It is a direct consequence of this fact: given that the interval $[0,1]$ has finite $\mathcal{L}^1$-measure, we have $L^q(0,1) \subseteq L^p(0,1)$ whenever $p \leq q$ by Hölder inequality. ■

Definition 1.4 (Metric speed) *Let (X,d) be a metric space. Then we define the* metric speed *operator $\mathsf{ms} : \Gamma(\mathrm{X}) \times [0,1] \to [0,+\infty]$ as follows: given $\gamma \in \Gamma(\mathrm{X})$ and $t \in [0,1]$, let*

$$\mathsf{ms}(\gamma,t) \doteq \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h},\gamma_t)}{|h|} \qquad \textit{whenever such limit exists} \tag{1.11}$$

and $\mathsf{ms}(\gamma,t) \doteq +\infty$ otherwise. For the sake of brevity, we shall often write $|\dot{\gamma}_t|$ to indicate the quantity $\mathsf{ms}(\gamma,t)$. Given any curve $\gamma \in \Gamma(\mathrm{X})$, we denote by $|\dot{\gamma}| : [0,1] \to [0,+\infty]$ the function sending $t \in [0,1]$ to $|\dot{\gamma}_t|$ and we call it the metric speed *of γ.*

Remark 1.5 The map $\mathsf{ms} : \Gamma(\mathrm{X}) \times [0,1] \to [0,+\infty]$ can be proven to be Borel measurable; see for instance [Pas18, Remark 5.1]. ■

The next result states that the metric speed of a p-absolutely continuous curve is the a.e. minimal $L^p(0,1)$-function that can be chosen as f in the right hand side of (1.9). For its proof we refer to [AGS08, Theorem 1.1.2].

Theorem 1.6 *Let* (X,d) *be a complete metric space. Fix* $p\in[1,\infty]$ *and* $\gamma\in AC^p\big([0,1],\mathrm{X}\big)$*. Then the metric speed* $|\dot\gamma|$ *of* γ *belongs to* $L^p(0,1)$ *and satisfies*

$$\mathsf{d}(\gamma_t,\gamma_s)\leq\int_s^t|\dot\gamma_r|\,\mathrm{d}r\qquad\text{for every } t,s\in[0,1] \text{ with } s<t. \tag{1.12}$$

Moreover, given any function $f\in L^p(0,1)$ *satisfying property (1.9), it holds that* $|\dot\gamma_t|\leq f(t)$ *for* $\mathcal{L}^1$*-a.e.* $t\in[0,1]$*.*

In particular, if γ is a p-absolutely continuous curve then the limit $\lim_{h\to 0}\mathsf{d}(\gamma_{t+h},\gamma_t)/|h|$ exists and is finite for $\mathcal{L}^1$-a.e. $t\in[0,1]$.

Remark 1.7 It may happen that the metric speed of a curve γ belongs to the space $L^p(0,1)$ even if γ is not p-absolutely continuous. For instance, the Cantor function $c:[0,1]\to[0,1]$ is not absolutely continuous but $|\dot c_t|=0$ for $\mathcal{L}^1$-a.e. $t\in[0,1]$, see e.g. [DMRV06]. ■

Definition 1.8 (Kinetic energy) *Let* (X,d) *be a complete metric space. Fix* $p\in(1,\infty)$*. Then we define the* p-kinetic energy *functional* $\mathsf{KE}_p:\Gamma(\mathrm{X})\to[0,+\infty]$ *as follows:*

$$\mathsf{KE}_p(\gamma)\doteq\begin{cases}\int_0^1|\dot\gamma_t|^p\,\mathrm{d}t & \text{if } \gamma\in AC^p\big([0,1],\mathrm{X}\big),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.13}$$

It turns out that the map KE_p is lower semicontinuous with respect to the distance $\mathsf{d}_{\Gamma(\mathrm{X})}$, see e.g. [Pas18, Proposition 3.7]. A direct consequence of this lower semicontinuity is that the set of all p-absolutely continuous curves is Borel; cf. [Lis07] for a proof of such fact:

Corollary 1.9 *Let* (X,d) *be a complete metric space. Fix an exponent* $p\in(1,\infty)$*. Then the space* $AC^p\big([0,1],\mathrm{X}\big)$ *is a Borel subset of* $\Gamma(\mathrm{X})$*.*

1.1.2 Lipschitz continuity

Let us consider two metric spaces $(\mathrm{X},\mathsf{d}_\mathrm{X})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$. Then we say that a map $f:\mathrm{X}\to\mathrm{Y}$ is *Lipschitz* provided there exists a constant $\lambda\geq 0$ such that

$$\mathsf{d}_\mathrm{Y}\big(f(x),f(y)\big)\leq\lambda\,\mathsf{d}_\mathrm{X}(x,y)\qquad\text{for every } x,y\in\mathrm{X}. \tag{1.14}$$

To be more precise, we can say that f is λ-Lipschitz. Observe that the map f is continuous.

We denote by $\mathrm{LIP}(\mathrm{X},\mathrm{Y})$ the family of all Lipschitz maps from X to Y, while $\mathrm{LIP}_\lambda(\mathrm{X},\mathrm{Y})$ is the space of all λ-Lipschitz maps. In the case in which the target $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ coincides with the real line endowed with the Euclidean distance, we simply write $\mathrm{LIP}(\mathrm{X})$ instead of $\mathrm{LIP}(\mathrm{X},\mathbb{R})$. Moreover, if a map $f\in\mathrm{LIP}(\mathrm{X},\mathrm{Y})$ is invertible and f, f^{-1} are λ-Lipschitz, then we say that the map f is λ-*biLipschitz*.

Definition 1.10 (Lipschitz constants) *Let* $(\mathrm{X},\mathsf{d}_\mathrm{X})$ *and* $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ *be metric spaces. Fix any Lipschitz map* $f\in\mathrm{LIP}(\mathrm{X},\mathrm{Y})$*. Then we give the following definitions:*

i) *The* Lipschitz constant *of* f *is the quantity* $\mathrm{Lip}(f)\in[0,+\infty)$*, which is defined as*

$$\mathrm{Lip}(f)\doteq\sup_{\substack{x,y\in\mathrm{X}\\ x\neq y}}\frac{\mathsf{d}_\mathrm{Y}\big(f(x),f(y)\big)}{\mathsf{d}_\mathrm{X}(x,y)}. \tag{1.15}$$

Given any subset E *of* X*, we indicate by* $\mathrm{Lip}(f;E)$ *the Lipschitz constant of* $f|_E$*.*

ii) *The* local Lipschitz constant *of f is the function* $\mathrm{lip}(f):\mathrm{X}\to[0,+\infty)$*, defined as*

$$\mathrm{lip}(f)(x) \doteq \varlimsup_{\substack{y\to x\\ y\in\mathrm{X}\setminus\{x\}}} \frac{\mathsf{d}_{\mathrm{Y}}\big(f(x),f(y)\big)}{\mathsf{d}_{\mathrm{X}}(x,y)} \qquad \text{if } x\in\mathrm{X} \text{ is an accumulation point} \tag{1.16}$$

and $\mathrm{lip}(f)(x)\doteq 0$ *if* $x\in\mathrm{X}$ *is an isolated point.*

iii) *The* asymptotic Lipschitz constant *of f is the function* $\mathrm{lip}_a(f):\mathrm{X}\to[0,+\infty)$ *given by*

$$\mathrm{lip}_a(f)(x) \doteq \lim_{r\searrow 0}\mathrm{Lip}\big(f;B_r(x)\big) \qquad \text{for every } x\in\mathrm{X}. \tag{1.17}$$

Observe that the Lipschitz constant $\mathrm{Lip}(f)$ is the smallest $\lambda\geq 0$ such that f is λ-Lipschitz.

Moreover, it directly follows from the definitions (1.15), (1.16) and (1.17) above that

$$\mathrm{lip}(f)(x)\leq \mathrm{lip}_a(f)(x)\leq \mathrm{Lip}(f) \qquad \text{for every } x\in\mathrm{X}. \tag{1.18}$$

Standard verifications yield the inequalities

$$\begin{aligned}\mathrm{lip}(f\circ\varphi)&\leq \mathrm{Lip}(\varphi)\,\mathrm{lip}(f)\circ\varphi,\\ \mathrm{lip}_a(f\circ\varphi)&\leq \mathrm{Lip}(\varphi)\,\mathrm{lip}_a(f)\circ\varphi\end{aligned} \tag{1.19}$$

for any metric spaces $(\mathrm{X},\mathsf{d}_{\mathrm{X}})$, $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}})$ and Lipschitz maps $\varphi\in\mathrm{LIP}(\mathrm{X},\mathrm{Y})$, $f\in\mathrm{LIP}(\mathrm{Y})$.

Remark 1.11 Given any complete metric space (X,d), it clearly holds that

$$\mathrm{LIP}\big([0,1],\mathrm{X}\big)=AC^\infty\big([0,1],\mathrm{X}\big). \tag{1.20}$$

In particular, each Lipschitz curve in X is absolutely continuous. ■

We shall frequently use the following well-known fact:

$$\begin{aligned}&\text{Given a metric space } (\mathrm{X},\mathsf{d})\text{, a subset } E \text{ of X and } f\in\mathrm{LIP}(E),\\ &\text{there exists } \bar f\in\mathrm{LIP}(\mathrm{X}) \text{ such that } \bar f_{|E}=f \text{ and } \mathrm{Lip}(\bar f)=\mathrm{Lip}(f).\end{aligned} \tag{1.21}$$

An explicit expression for such a function $\bar f$ – called *McShane extension* – is given by the formula $\bar f(x)\doteq\inf\big\{f(y)+\mathrm{Lip}(f)\,\mathsf{d}(x,y)\;:\;y\in E\big\}$ for any $x\in\mathrm{X}$. Moreover, we recall that:

$$\begin{aligned}&\text{Given any metric space } (\mathrm{X},\mathsf{d})\text{, any subset } E \text{ of X and } f\in\mathrm{LIP}(E,\mathbb{R}^n),\\ &\text{there exists } \bar f\in\mathrm{LIP}(\mathrm{X},\mathbb{R}^n) \text{ such that } \bar f_{|E}=f \text{ and } \mathrm{Lip}(\bar f)\leq\sqrt{n}\,\mathrm{Lip}(f),\end{aligned}$$

as one can readily deduce from (1.21) by arguing componentwise.

We conclude the subsection by introducing a key class of Lipschitz curves: the geodesics. Let (X,d) be a complete separable metric space. A curve $\gamma\in\Gamma(\mathrm{X})$ is said to be a *geodesic* if

$$\mathsf{d}(\gamma_t,\gamma_s)=|t-s|\,\mathsf{d}(\gamma_0,\gamma_1) \qquad \text{for every } t,s\in[0,1], \tag{1.22}$$

in other words if $\gamma:\,[0,1]\to\mathrm{X}$ is an isometric embedding. In particular, $\gamma\in\mathrm{LIP}\big([0,1],\mathrm{X}\big)$ and the equality $|\dot\gamma_t|=\mathsf{d}(\gamma_0,\gamma_1)$ holds for every $t\in[0,1]$.

The set of all geodesic curves in X is denoted by $\mathrm{Geo}(\mathrm{X})$. The space (X,d) is called a *geodesic space* provided for any $x,y\in\mathrm{X}$ there exists $\gamma\in\mathrm{Geo}(\mathrm{X})$ such that $\gamma_0=x$ and $\gamma_1=y$.

Remark 1.12 Let (X, d) be a geodesic metric space (containing at least two points). Given any $x \in \mathrm{X}$, let us define the 1-Lipschitz function $\mathsf{d}_x : \mathrm{X} \to [0, +\infty)$ as $\mathsf{d}_x(y) \doteq \mathsf{d}(x, y)$ for every $y \in \mathrm{X}$. Then it holds that

$$\operatorname{lip}(\mathsf{d}_x)(y) = 1 \qquad \text{for every } y \in \mathrm{X}. \tag{1.23}$$

We already know from (1.18) that $\operatorname{lip}(\mathsf{d}_x) \leq 1$ everywhere, whence to prove (1.23) it suffices to show that $\operatorname{lip}(\mathsf{d}_x)(y) \geq 1$ for any $y \in \mathrm{X}$. In the case in which $y = x$, we trivially have that

$$\operatorname{lip}(\mathsf{d}_x)(x) = \varlimsup_{z \to x} \frac{\left|\mathsf{d}_x(z) - \mathsf{d}_x(x)\right|}{\mathsf{d}(z, x)} = \varlimsup_{z \to x} \frac{\mathsf{d}_x(z)}{\mathsf{d}(z, x)} = 1.$$

On the other hand, given any $y \in \mathrm{X} \setminus \{x\}$ we can choose a geodesic γ joining y to x, so that

$$\operatorname{lip}(\mathsf{d}_x)(y) \geq \lim_{t \to 0} \frac{\left|\mathsf{d}_x(\gamma_t) - \mathsf{d}_x(\gamma_0)\right|}{\mathsf{d}(\gamma_t, \gamma_0)} = \lim_{t \to 0} \frac{\left|\mathsf{d}(\gamma_1, \gamma_t) - \mathsf{d}(\gamma_1, \gamma_0)\right|}{t\,\mathsf{d}(x, y)} = \lim_{t \to 0} \frac{\left|(1-t) - 1\right|}{t} = 1.$$

This completes the proof of the claim. ■

1.2 Measure spaces

A *measurable space* is any couple $(\mathrm{X}, \mathcal{A})$, where X is a set and $\mathcal{A}$ is a σ-algebra on X. By *measure* on $(\mathrm{X}, \mathcal{A})$ we intend any σ-additive map $\mathfrak{m} : \mathcal{A} \to [0, +\infty]$ such that $\mathfrak{m}(\emptyset) = 0$. Then we say that $(\mathrm{X}, \mathcal{A}, \mathfrak{m})$ is a *measure space*. We say that $\mathfrak{m}$ is a *finite* measure if $\mathfrak{m}(\mathrm{X}) < +\infty$, while it is said to be σ*-finite* provided there exists a sequence $(A_n)_n \subseteq \mathcal{A}$ such that $\mathrm{X} = \bigcup_n A_n$ and $\mathfrak{m}(A_n) < +\infty$ for all $n \in \mathbb{N}$. In particular, any finite measure is σ-finite.

Remark 1.13 We say that a set $N \in \mathcal{A}$ is $\mathfrak{m}$*-negligible* provided $\mathfrak{m}(N) = 0$. This provides us with a natural equivalence relation on $\mathcal{A}$: given any $A, B \in \mathcal{A}$, we declare that A and B are $\mathfrak{m}$*-equivalent* if their symmetric difference $A \Delta B$ is $\mathfrak{m}$-negligible. The quotient set of $\mathcal{A}$ by such relation will be denoted by $\mathcal{A}/\mathfrak{m}$, while $[A]_{\mathfrak{m}}$ indicates the equivalence class of $A \in \mathcal{A}$. Another measure μ on $(\mathrm{X}, \mathcal{A})$ is *absolutely continuous* with respect to $\mathfrak{m}$ – briefly, $\mu \ll \mathfrak{m}$ – provided $\mu(N) = 0$ whenever $N \in \mathcal{A}$ satisfies $\mathfrak{m}(N) = 0$. Therefore we have a natural map from $\mathcal{A}/\mathfrak{m}$ to $\mathcal{A}/\mu$, which associates to any element $[A]_{\mathfrak{m}}$ the equivalence class $[A]_\mu$. ■

A function $f : \mathrm{X} \to \overline{\mathbb{R}}$ is *measurable* if $f^{-1}(U) \in \mathcal{A}$ for any open subset U of $\overline{\mathbb{R}}$. Given any two measurable functions $f, g : \mathrm{X} \to \overline{\mathbb{R}}$, we declare that $f \sim_{\mathfrak{m}} g$ provided $f = g$ *holds* $\mathfrak{m}$*-a.e.*, which means that the set $\{f \neq g\}$ is $\mathfrak{m}$-negligible. Then we define the space $L^0(\mathfrak{m})$ as

$$L^0(\mathfrak{m}) \doteq \big\{f : \mathrm{X} \to \mathbb{R} \text{ measurable}\big\}/ \sim_{\mathfrak{m}}, \tag{1.24}$$

while we set $L^0(\mathfrak{m})^+ \doteq \big\{f : \mathrm{X} \to [0, +\infty] \text{ measurable}\big\}/ \sim_{\mathfrak{m}}$. It holds that $L^0(\mathfrak{m})$ is both a vector space and a commutative ring with identity when endowed with the pointwise operations. Furthermore, $L^0(\mathfrak{m})$ can be equipped with a distance that metrizes the convergence in measure; we postpone the discussion about this topic to Subsection 1.2.1.

Fix any exponent $p \in [1, \infty]$. Hence we define the space $L^p(\mathfrak{m})$ as

$$L^p(\mathfrak{m}) \doteq \big\{f \in L^0(\mathfrak{m}) \bigm| \|f\|_{L^p(\mathfrak{m})} < +\infty\big\}, \tag{1.25}$$

where the quantity $\|f\|_{L^p(\mathfrak{m})}$ is given by

$$\|f\|_{L^p(\mathfrak{m})} \doteq \begin{cases} \left(\int |f|^p \,\mathrm{d}\mathfrak{m}\right)^{1/p} & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{\mathrm{X}} |f| & \text{if } p = \infty. \end{cases} \tag{1.26}$$

Therefore $\big(L^p(\mathfrak{m}), \|\cdot\|_{L^p(\mathfrak{m})}\big)$ is a Banach space. Moreover, $L^\infty(\mathfrak{m})$ is a subring of $L^0(\mathfrak{m})$.

Given any function $f \in L^0(\mathfrak{m})^+$, we define the measure $f\mathfrak{m}$ on $(\mathrm{X}, \mathcal{A})$ as

$$(f\mathfrak{m})(A) \doteq \int_A f \,\mathrm{d}\mathfrak{m} \qquad \text{for every } A \in \mathcal{A}. \tag{1.27}$$

Moreover, for any set $A \in \mathcal{A}$ we define the *restricted measure* $\mathfrak{m}|_A$ as

$$\mathfrak{m}|_A \doteq \chi_A \,\mathfrak{m}. \tag{1.28}$$

Observe that $\mathfrak{m}|_A(B) = \mathfrak{m}(A \cap B)$ for every $B \in \mathcal{A}$.

Theorem 1.14 (Radon-Nikodým) *Let* $(\mathrm{X}, \mathcal{A}, \mathfrak{m})$ *be a* σ*-finite measure space. Let* μ *be any* σ*-finite measure on* $\mathcal{A}$ *with* $\mu \ll \mathfrak{m}$*. Then there exists a unique function* $\mathrm{d}\mu/\mathrm{d}\mathfrak{m} \in L^0(\mathfrak{m})^+$*, called* density *of* μ *with respect to* $\mathfrak{m}$ *or* Radon-Nikodým derivative*, such that* $\mu = (\mathrm{d}\mu/\mathrm{d}\mathfrak{m})\,\mathfrak{m}$*. Moreover, it holds that* $(\mathrm{d}\mu/\mathrm{d}\mathfrak{m})(x) < +\infty$ *for* $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$*.*

Remark 1.15 (Properties of the density) We recall two useful properties of the density:

i) The density of a measure is linear: if $\mu, \nu \ll \mathfrak{m}$, then $\mu + \nu \ll \mathfrak{m}$ and

$$\frac{\mathrm{d}(\mu+\nu)}{\mathrm{d}\mathfrak{m}} = \frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}} + \frac{\mathrm{d}\nu}{\mathrm{d}\mathfrak{m}} \qquad \text{holds } \mathfrak{m}\text{-a.e. in X}. \tag{1.29}$$

ii) The density of a measure satisfies the chain rule: if $\mu \ll \nu$ and $\nu \ll \mathfrak{m}$, then $\mu \ll \mathfrak{m}$ and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathfrak{m}} = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\mathfrak{m}} \qquad \text{holds } \mathfrak{m}\text{-a.e. in X}. \tag{1.30}$$

Both facts are immediate consequences of the uniqueness of the density. ■

Now consider two measurable spaces $(\mathrm{X}, \mathcal{A}_{\mathrm{X}})$, $(\mathrm{Y}, \mathcal{A}_{\mathrm{Y}})$ and a map $\varphi : \mathrm{X} \to \mathrm{Y}$. We say that φ is *measurable* provided $\varphi^{-1}(B) \in \mathcal{A}_{\mathrm{X}}$ for all $B \in \mathcal{A}_{\mathrm{Y}}$. Notice that the function $f \circ \varphi$ is measurable for every choice of $f : \mathrm{Y} \to \overline{\mathbb{R}}$ measurable and $\varphi : \mathrm{X} \to \mathrm{Y}$ measurable. Given any measure μ on $(\mathrm{X}, \mathcal{A}_{\mathrm{X}})$, we define the *pushforward measure* $\varphi_*\mu$ on $(\mathrm{Y}, \mathcal{A}_{\mathrm{Y}})$ as

$$(\varphi_*\mu)(B) \doteq \mu\big(\varphi^{-1}(B)\big) \qquad \text{for every } B \in \mathcal{A}_{\mathrm{Y}}. \tag{1.31}$$

It holds that $(\varphi_*\mu)(\mathrm{Y}) = \mu(\mathrm{X})$, in particular μ is finite if and only if $\varphi_*\mu$ is finite. We point out that the pushforward measure $\varphi_*\mu$ satisfies the following change-of-variable formula:

$$\int f \,\mathrm{d}\varphi_*\mu = \int f \circ \varphi \,\mathrm{d}\mu \qquad \text{whenever } f \in L^1(\varphi_*\mu) \text{ or } f \in L^0(\varphi_*\mu)^+. \tag{1.32}$$

The previous formula makes sense, because $f \circ \varphi \in L^1(\mu)$ for every $f \in L^1(\varphi_*\mu)$.

Remark 1.16 We claim that for any $f \in L^0(\varphi_*\mu)^+$ it holds

$$\varphi_*(f \circ \varphi\, \mu) = f\, \varphi_*\mu. \tag{1.33}$$

Indeed, $\varphi_*(f \circ \varphi\, \mu)(B) = \int_{\varphi^{-1}(B)} f \circ \varphi \, \mathrm{d}\mu = \int_B f \, \mathrm{d}\varphi_*\mu = (f\, \varphi_*\mu)(B)$ for all $B \in \mathcal{A}$. ■

Remark 1.17 (Pushforward of σ-algebra) Given a measurable space $(\mathrm{X}, \mathcal{A})$, a set Y and a map $f : \mathrm{X} \to \mathrm{Y}$, we define the *pushforward* of $\mathcal{A}$ via f as the σ-algebra on Y given by

$$f_*\mathcal{A} \doteq \big\{E \subseteq \mathrm{Y} \,\big|\, f^{-1}(E) \in \mathcal{A}\big\}. \tag{1.34}$$

It turns out that $f_*\mathcal{A}$ can be characterised as the greatest σ-algebra $\mathcal{A}'$ on Y such that the map f is measurable from $(\mathrm{X}, \mathcal{A})$ to $(\mathrm{Y}, \mathcal{A}')$. ■

1.2.1 The space $L^0(\mathfrak{m})$

Let $(\mathrm{X}, \mathcal{A}, \mathfrak{m})$ be a given σ-finite measure space. Consider the space $L^0(\mathfrak{m})$, defined in (1.24).

Remark 1.18 We claim that:

$$\text{There exists a finite measure } \mathfrak{m}' \text{ on } (\mathrm{X}, \mathcal{A}) \text{ such that } \mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}. \tag{1.35}$$

The proof of such fact can be achieved via explicit construction. The case $\mathfrak{m}(\mathrm{X}) < +\infty$ is trivial, so suppose $\mathfrak{m}(\mathrm{X}) = +\infty$. Since $\mathfrak{m}$ is σ-finite, we can pick a partition $(E_n)_n \subseteq \mathcal{A}$ of X such that $0 < \mathfrak{m}(E_n) < +\infty$ for every $n \in \mathbb{N}$. Then

$$\mathfrak{m}' \doteq \sum_{n \in \mathbb{N}} \frac{\mathfrak{m}|_{E_n}}{2^n\, \mathfrak{m}(E_n)} \tag{1.36}$$

is a finite measure on $(\mathrm{X}, \mathcal{A})$ having the same null sets as $\mathfrak{m}$. ■

Given any measure $\mathfrak{m}'$ as in Remark 1.18, we define the distance $\mathsf{d}_{L^0(\mathfrak{m})}$ on $L^0(\mathfrak{m})$ as

$$\mathsf{d}_{L^0(\mathfrak{m})}(f, g) \doteq \int |f - g| \wedge 1 \, \mathrm{d}\mathfrak{m}' \qquad \text{for every } f, g \in L^0(\mathfrak{m}). \tag{1.37}$$

It turns out that $L^0(\mathfrak{m})$ is both a topological vector space and a topological ring if equipped with the topology induced by the distance $\mathsf{d}_{L^0(\mathfrak{m})}$. Furthermore, observe that $\mathsf{d}_{L^0(\mathfrak{m})}$ might depend on the choice of the measure $\mathfrak{m}'$, but that its induced topology does not:

Proposition 1.19 *Let $(f_n)_n \subseteq L^0(\mathfrak{m})$ be given. Then $(f_n)_n$ is $\mathsf{d}_{L^0(\mathfrak{m})}$-Cauchy if and only if*

$$\lim_{n,m} \mathfrak{m}\Big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\Big) = 0 \qquad \textit{for all } \varepsilon > 0 \textit{ and } E \in \mathcal{A} \textit{ with } \mathfrak{m}(E) < +\infty. \tag{1.38}$$

A proof of Proposition 1.19 can be found – for instance – in [Pas18, Proposition 15.1].

The distance $\mathsf{d}_{L^0(\mathfrak{m})}$ metrizes the convergence in measure, as shown by the following result:

Proposition 1.20 *Let $f \in L^0(\mathfrak{m})$ and $(f_n)_n \subseteq L^0(\mathfrak{m})$. Then the following are equivalent:*

i) *It holds that $\mathsf{d}_{L^0(\mathfrak{m})}(f_n, f) \to 0$ as $n \to \infty$.*

ii) *Any subsequence $(n_m)_m$ admits a further subsequence $(n_{m_k})_k$ such that $f_{n_{m_k}}(x) \to f(x)$ as $k \to \infty$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$.*

iii) *Given any* $\varepsilon > 0$ *and* $E \in \mathcal{A}$ *with* $\mathfrak{m}(E) < +\infty$*, we have* $\lim_n \mathfrak{m}\big(E \cap \{|f_n - f| > \varepsilon\}\big) = 0$.

We refer e.g. to [Pas18, Proposition 15.3] for a proof of Proposition 1.20.

Remark 1.21 Another distance on $L^0(\mathfrak{m})$ metrizing the convergence in measure is given by

$$\mathsf{d}'_{L^0(\mathfrak{m})}(f,g) \doteq \inf_{\delta>0} \Big(\delta + \mathfrak{m}'\big(\{|f-g| > \delta\}\big)\Big) \qquad \text{for every } f,g \in L^0(\mathfrak{m}), \tag{1.39}$$

where $\mathfrak{m}'$ is any measure as in (1.35). See [Bog07] for additional details about $\mathsf{d}'_{L^0(\mathfrak{m})}$. ■

As shown – for example – in [Pas18, Proposition 15.6], it holds that

$$\big(L^0(\mathfrak{m}), \mathsf{d}_{L^0(\mathfrak{m})}\big) \text{ is a complete and separable metric space.} \tag{1.40}$$

A consequence of Proposition 1.20 is that the completeness of $L^0(\mathfrak{m})$ is not affected by the particular choice of the measure $\mathfrak{m}'$. Another important property of $L^0(\mathfrak{m})$ is the following:

$$\text{The inclusion map } L^p(\mathfrak{m}) \hookrightarrow L^0(\mathfrak{m}) \text{ is continuous and has dense image,} \tag{1.41}$$

for any $p \in [1,\infty]$. Finally, we point out that the distance $\mathsf{d}_{L^0(\mathfrak{m})}$ is *translation-invariant*, i.e.

$$\mathsf{d}_{L^0(\mathfrak{m})}(f,g) = \mathsf{d}_{L^0(\mathfrak{m})}(f+h, g+h) \qquad \text{for every } f,g,h \in L^0(\mathfrak{m}), \tag{1.42}$$

but that it is not induced by any norm.

1.2.2 Essential image

Let $(\mathrm{X}, \mathcal{A}_\mathrm{X}, \mathfrak{m}_\mathrm{X}), (\mathrm{Y}, \mathcal{A}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be σ-finite measure spaces. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a measurable map such that $\mathfrak{m}_\mathrm{Y} = \varphi_* \mathfrak{m}_\mathrm{X}$. We then define the map $\mathsf{Pr}_\varphi : L^1(\mathfrak{m}_\mathrm{X}) + L^\infty(\mathfrak{m}_\mathrm{X}) \to L^0(\mathfrak{m}_\mathrm{Y})$ as

$$\mathsf{Pr}_\varphi(f) \doteq \frac{\mathrm{d}\varphi_*(f^+\mathfrak{m}_\mathrm{X})}{\mathrm{d}\mathfrak{m}_\mathrm{Y}} - \frac{\mathrm{d}\varphi_*(f^-\mathfrak{m}_\mathrm{X})}{\mathrm{d}\mathfrak{m}_\mathrm{Y}} \qquad \text{for every } f \in L^1(\mathfrak{m}_\mathrm{X}) + L^\infty(\mathfrak{m}_\mathrm{X}). \tag{1.43}$$

We say that the operator Pr_φ is the *projection for functions* through the map φ.

Remark 1.22 To be sure that $\mathsf{Pr}_\varphi(f)$ is well-defined, we show that $\varphi_*(f^\pm\,\mathfrak{m}_\mathrm{X})$ are σ-finite measures on $\mathcal{A}_\mathrm{Y}$, so that the Radon-Nikodým theorem can be applied: choose $f_1 \in L^1(\mathfrak{m}_\mathrm{X})$ and $f_\infty \in L^\infty(\mathfrak{m}_\mathrm{X})$ such that $f = f_1 + f_\infty$. The measure $|f_1|\mathfrak{m}_\mathrm{X}$ is finite, whence $\varphi_*\big(|f_1|\mathfrak{m}_\mathrm{X}\big)$ is finite as well. It also holds that $\varphi_*\big(|f_\infty|\mathfrak{m}_\mathrm{X}\big) \le \|f_\infty\|_{L^\infty(\mathfrak{m}_\mathrm{X})}\mathfrak{m}_\mathrm{Y}$, so $\varphi_*\big(|f_\infty|\mathfrak{m}_\mathrm{X}\big)$ is σ-finite. Since $(f_1 + f_\infty)^\pm \le |f_1| + |f_\infty|$, we conclude that the measures $\varphi_*(f^\pm\mathfrak{m}_\mathrm{X})$ are σ-finite. ■

Notice that $\mathsf{Pr}_\varphi(f) = \mathsf{Pr}_\varphi(f^+) - \mathsf{Pr}_\varphi(f^-)$ is satisfied for every $f \in L^1(\mathfrak{m}_\mathrm{X}) + L^\infty(\mathfrak{m}_\mathrm{X})$. Furthermore, if $f \ge 0$ holds $\mathfrak{m}_\mathrm{X}$-a.e. in X, then one has that

$$\mathsf{Pr}_\varphi(f) = 0 \quad \mathfrak{m}_\mathrm{Y}\text{-a.e.} \qquad \Longleftrightarrow \qquad f = 0 \quad \mathfrak{m}_\mathrm{X}\text{-a.e.}. \tag{1.44}$$

Remark 1.23 We claim that

$$\mathsf{Pr}_\varphi : L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X) \to L^0(\mathfrak{m}_Y) \qquad \text{is a linear operator.} \tag{1.45}$$

Indeed, given any $\lambda \in \mathbb{R}$ and $f \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$, we have $(\lambda f)^\pm = \lambda f^\pm$ when $\lambda \geq 0$ and $(\lambda f)^\pm = -\lambda f^\mp$ when $\lambda < 0$, which grants that $\mathsf{Pr}_\varphi(\lambda f) = \lambda\, \mathsf{Pr}_\varphi(f)$, proving that Pr_φ is 1-homogeneous. To prove that it is also additive, consider $f, g \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$ and denote $q \doteq f^+ + g^+ - (f+g)^+ = f^- + g^- - (f+g)^- \geq 0$. Then $\mathsf{Pr}_\varphi(f^+) + \mathsf{Pr}_\varphi(g^+)$ and $\mathsf{Pr}_\varphi(f^-) + \mathsf{Pr}_\varphi(g^-)$ coincide with $\mathsf{Pr}_\varphi\big((f+g)^+\big) + \mathsf{Pr}_\varphi(q)$ and $\mathsf{Pr}_\varphi\big((f+g)^-\big) + \mathsf{Pr}_\varphi(q)$, respectively, whence $\mathsf{Pr}_\varphi(f+g) = \mathsf{Pr}_\varphi(f^+) + \mathsf{Pr}_\varphi(g^+) - \mathsf{Pr}_\varphi(q) - \big(\mathsf{Pr}_\varphi(f^-) + \mathsf{Pr}_\varphi(g^-) - \mathsf{Pr}_\varphi(q)\big)$ is equal to $\mathsf{Pr}_\varphi(f) + \mathsf{Pr}_\varphi(g)$, as required. Therefore (1.45) is proved. ■

Remark 1.24 The operator Pr_φ satisfies the following two properties:

$$\begin{aligned} \mathsf{Pr}_\varphi(c) &= c \quad \mathfrak{m}_Y\text{-a.e.} && \text{for every constant } c \in \mathbb{R}, \\ \mathsf{Pr}_\varphi(f) &\leq \mathsf{Pr}_\varphi(g) \quad \mathfrak{m}_Y\text{-a.e.} && \text{for every } f, g \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X) \text{ with } f \leq g. \end{aligned} \tag{1.46}$$

Both facts can be easily deduced from the very definition of Pr_φ. ■

Proposition 1.25 (Jensen) *Let $u : \mathbb{R} \to \mathbb{R}$ be a convex function. Then it holds that*

$$u \circ \mathsf{Pr}_\varphi(f) \leq \mathsf{Pr}_\varphi(u \circ f) \quad \mathfrak{m}_Y\text{-a.e.} \qquad \text{whenever } f, u \circ f \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X). \tag{1.47}$$

Proof. Fix $f \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$ such that $u \circ f \in L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$. Linearity of Pr_φ and the first property in (1.46) grant that

$$v \circ \mathsf{Pr}_\varphi(f) = \mathsf{Pr}_\varphi(v \circ f) \quad \mathfrak{m}_Y\text{-a.e.} \qquad \text{for every affine function } v : \mathbb{R} \to \mathbb{R}. \tag{1.48}$$

Now choose a countable family $\mathcal{F}(u)$ of affine functions $v : \mathbb{R} \to \mathbb{R}$, which satisfy $v \leq u$, such that $u(t) = \sup\big\{v(t) \,:\, v \in \mathcal{F}(u)\big\}$ for every $t \in \mathbb{R}$. Hence (1.48) gives

$$u \circ \mathsf{Pr}_\varphi(f) = \operatorname{ess\,sup}\big\{\mathsf{Pr}_\varphi(v \circ f) \,:\, v \in \mathcal{F}(u)\big\}. \tag{1.49}$$

Given that $\mathsf{Pr}_\varphi(v \circ f) \leq \mathsf{Pr}_\varphi(u \circ f)$ holds $\mathfrak{m}_Y$-a.e. for every $v \in \mathcal{F}(u)$, we deduce from (1.49) that $u \circ \mathsf{Pr}_\varphi(f) \leq \mathsf{Pr}_\varphi(u \circ f)$ is satisfied $\mathfrak{m}_Y$-a.e., thus proving (1.47). □

The previous result ensures that the projection Pr_φ maps $L^p(\mathfrak{m}_X)$ to $L^p(\mathfrak{m}_Y)$ for any p:

Corollary 1.26 *Given any $p \in [1, \infty)$, it holds that*

$$\big|\mathsf{Pr}_\varphi(f)\big|^p \leq \mathsf{Pr}_\varphi\big(|f|^p\big) \quad \mathfrak{m}_Y\text{-a.e.} \qquad \text{for every } f \in L^p(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X). \tag{1.50}$$

In particular, the operator Pr_φ continuously maps $L^p(\mathfrak{m}_X)$ to $L^p(\mathfrak{m}_Y)$ for any $p \in [1, \infty]$, with

$$\big\|\mathsf{Pr}_\varphi(f)\big\|_{L^p(\mathfrak{m}_Y)} \leq \|f\|_{L^p(\mathfrak{m}_X)} \qquad \text{for every } f \in L^p(\mathfrak{m}_X). \tag{1.51}$$

Proof. First of all, fix $p \in [1, \infty)$ and $f \in L^p(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$. Clearly both f and $|f|^p$ belong to $L^1(\mathfrak{m}_X) + L^\infty(\mathfrak{m}_X)$, so that property (1.47) with $u = |\cdot|^p$ ensures that $\big|\mathsf{Pr}_\varphi(f)\big|^p \leq \mathsf{Pr}_\varphi\big(|f|^p\big)$ holds $\mathfrak{m}_Y$-a.e. in Y, obtaining (1.50). To prove the last statement, fix $p \in [1, \infty]$. In the case in which $p < \infty$, we have for any $f \in L^p(\mathfrak{m}_X)$ that

$$\int \big|\mathsf{Pr}_\varphi(f)\big|^p \,\mathrm{d}\mathfrak{m}_Y \overset{(1.50)}{\leq} \int \mathsf{Pr}_\varphi\big(|f|^p\big) \,\mathrm{d}\mathfrak{m}_Y = \int \frac{\mathrm{d}\varphi_*\big(|f|^p \mathfrak{m}_X\big)}{\mathrm{d}\mathfrak{m}_Y} \,\mathrm{d}\mathfrak{m}_Y = \int |f|^p \,\mathrm{d}\mathfrak{m}_X,$$

showing that $\|\mathsf{Pr}_\varphi(f)\|_{L^p(\mathfrak{m}_Y)} \leq \|f\|_{L^p(\mathfrak{m}_X)}$ for every $f \in L^p(\mathfrak{m}_X)$, so that Pr_φ continuously maps $L^p(\mathfrak{m}_X)$ to $L^p(\mathfrak{m}_Y)$. Finally, if $p = \infty$ then (1.46) and (1.50) give

$$\big|\mathsf{Pr}_\varphi(f)\big| \leq \mathsf{Pr}_\varphi|f| \leq \mathsf{Pr}_\varphi\big(\|f\|_{L^\infty(\mathfrak{m}_X)}\big) = \|f\|_{L^\infty(\mathfrak{m}_X)} \quad \mathfrak{m}_Y\text{-a.e.} \qquad \text{for every } f \in L^\infty(\mathfrak{m}_X),$$

proving that $\|\mathsf{Pr}_\varphi(f)\|_{L^\infty(\mathfrak{m}_Y)} \leq \|f\|_{L^\infty(\mathfrak{m}_X)}$ for every $f \in L^\infty(\mathfrak{m}_X)$, thus accordingly Pr_φ continuously maps $L^\infty(\mathfrak{m}_X)$ to $L^\infty(\mathfrak{m}_Y)$. This completes the proof. □

Lemma 1.27 *Let $p \in [1,\infty]$ be fixed. Then*

$$\mathsf{Pr}_\varphi(g \circ \varphi\, f) = g\,\mathsf{Pr}_\varphi(f) \quad \mathfrak{m}_Y\text{-a.e.} \qquad \text{for every } f \in L^\infty(\mathfrak{m}_X) \text{ and } g \in L^p(\mathfrak{m}_Y). \tag{1.52}$$

Proof. First of all, consider $f \in L^\infty(\mathfrak{m}_X)^+$ and $g \in L^p(\mathfrak{m}_Y)^+$. Given that $\varphi_*(g \circ \varphi\, f\mathfrak{m}_X)$ coincides with $g\,\varphi_*(f\mathfrak{m}_X)$ by (1.33), we deduce that

$$\mathsf{Pr}_\varphi(g \circ \varphi\, f) = \frac{\mathrm{d}\big(g\,\varphi_*(f\mathfrak{m}_X)\big)}{\mathrm{d}\mathfrak{m}_Y} \overset{(1.30)}{=} \frac{\mathrm{d}\big(g\,\varphi_*(f\mathfrak{m}_X)\big)}{\mathrm{d}\varphi_*(f\mathfrak{m}_X)}\,\frac{\mathrm{d}\varphi_*(f\mathfrak{m}_X)}{\mathrm{d}\mathfrak{m}_Y} = g\,\mathsf{Pr}_\varphi(f) \qquad \mathfrak{m}_Y\text{-a.e.},$$

proving (1.52) for f, g nonnegative. For general $f \in L^\infty(\mathfrak{m}_X)$ and $g \in L^p(\mathfrak{m}_Y)$, we thus have

$$\begin{aligned}\mathsf{Pr}_\varphi(g \circ \varphi\, f) &= \mathsf{Pr}_\varphi(g^+ \circ \varphi\, f^+) + \mathsf{Pr}_\varphi(g^- \circ \varphi\, f^-) - \mathsf{Pr}_\varphi(g^+ \circ \varphi\, f^-) - \mathsf{Pr}_\varphi(g^- \circ \varphi\, f^+)\\ &= (g^+ - g^-)\big(\mathsf{Pr}_\varphi(f^+) - \mathsf{Pr}_\varphi(f^-)\big) = g\,\mathsf{Pr}_\varphi(f)\end{aligned}$$

in the $\mathfrak{m}_Y$-a.e. sense, which proves (1.52). Hence the statement is achieved. □

Remark 1.28 It can be readily deduced from Corollary 1.26 that the map Pr_φ can be uniquely extended to a linear and continuous operator

$$\mathsf{Pr}_\varphi : L^0(\mathfrak{m}_X) \to L^0(\mathfrak{m}_Y). \tag{1.53}$$

By an approximation argument, one easily obtains that

$$\begin{aligned}\big|\mathsf{Pr}_\varphi(f)\big| &\leq \mathsf{Pr}_\varphi|f| &&\mathfrak{m}_Y\text{-a.e.} &&\text{for every } f \in L^0(\mathfrak{m}_X),\\ \mathsf{Pr}_\varphi(g \circ \varphi\, f) &= g\,\mathsf{Pr}_\varphi(f) &&\mathfrak{m}_Y\text{-a.e.} &&\text{for every } f \in L^0(\mathfrak{m}_X) \text{ and } g \in L^0(\mathfrak{m}_Y),\end{aligned} \tag{1.54}$$

as a consequence of (1.50) and Lemma 1.27, respectively. ■

With the projection operator Pr_φ at our disposal, we can readily introduce the notion of 'essential image' of a measurable set under the map φ, as we are going to describe (recall the terminology that has been introduced in Remark 1.13).

Definition 1.29 *We define the map $\mathsf{Im}_\varphi : \mathcal{A}_X/\mathfrak{m}_X \to \mathcal{A}_Y/\mathfrak{m}_Y$ as*

$$\mathsf{Im}_\varphi(A) \doteq \big\{\mathsf{Pr}_\varphi(\chi_A) > 0\big\} \in \mathcal{A}_Y/\mathfrak{m}_Y \qquad \text{for every } A \in \mathcal{A}_X/\mathfrak{m}_X. \tag{1.55}$$

We shall refer to $\mathsf{Im}_\varphi(A)$ as the essential image *of the set A under the map φ.*

Observe that φ naturally induces a mapping $\varphi^{-1} : \mathcal{A}_Y/\mathfrak{m}_Y \to \mathcal{A}_X/\mathfrak{m}_X$, as follows:

$$\varphi^{-1}\big([B]_{\mathfrak{m}_Y}\big) \doteq \big[\varphi^{-1}(B)\big]_{\mathfrak{m}_X} \qquad \text{for every } B \in \mathcal{A}_Y. \tag{1.56}$$

Such definition is well-posed, since we have $\mathfrak{m}_X\big(\varphi^{-1}(B)\Delta\varphi^{-1}(B')\big) = (\varphi_*\mathfrak{m}_X)(B\Delta B') = 0$ whenever $B, B' \in \mathcal{A}_Y$ are two measurable sets satisfying $\mathfrak{m}_Y(B\Delta B') = 0$.

Proposition 1.30 *The map* Im_φ *is a left inverse of* $\varphi^{-1} : \mathcal{A}_{\mathrm{Y}}/\mathfrak{m}_{\mathrm{Y}} \to \mathcal{A}_{\mathrm{X}}/\mathfrak{m}_{\mathrm{X}}$, *namely*

$$\mathsf{Im}_\varphi\big(\varphi^{-1}(B)\big) = B \qquad \text{holds for every } B \in \mathcal{A}_{\mathrm{Y}}/\mathfrak{m}_{\mathrm{Y}}. \tag{1.57}$$

Proof. Fix a measurable set $B \in \mathcal{A}_{\mathrm{Y}}$ and denote $A \doteq \varphi^{-1}(B) \in \mathcal{A}_{\mathrm{X}}$. Choose any representative $B' \in \mathcal{A}_{\mathrm{Y}}$ of $\mathsf{Im}_\varphi\big(\varphi^{-1}(B)\big)$. Then our aim is to show that $\mathfrak{m}_{\mathrm{Y}}(B \Delta B') = 0$. Since the equality $\mathsf{Pr}_\varphi(\chi_A) = 0$ holds $\mathfrak{m}_{\mathrm{Y}}$-a.e. in $\mathrm{Y} \setminus B'$, we deduce that

$$\mathfrak{m}_{\mathrm{Y}}(B \setminus B') = \int_{\mathrm{Y}\setminus B'} \chi_B \,\mathrm{d}\varphi_*\mathfrak{m}_{\mathrm{X}} \overset{(1.33)}{=} \varphi_*(\chi_A\,\mathfrak{m}_{\mathrm{X}})(\mathrm{Y} \setminus B') = \int_{\mathrm{Y}\setminus B'} \mathsf{Pr}_\varphi(\chi_A)\,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} = 0.$$

On the other hand, the measures $\mathfrak{m}_{\mathrm{Y}}|_{B'}$ and $\mathsf{Pr}_\varphi(\chi_A)\,\mathfrak{m}_{\mathrm{Y}}$ have the same null sets. Given that

$$\int_{\mathrm{Y}\setminus B} \mathsf{Pr}_\varphi(\chi_A)\,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} = \big(\varphi_*(\chi_A\,\mathfrak{m}_{\mathrm{X}})\big)(\mathrm{Y} \setminus B) = \mathfrak{m}_{\mathrm{X}}\big(A \cap \varphi^{-1}(\mathrm{Y} \setminus B)\big) = 0,$$

we thus conclude that also $\mathfrak{m}_{\mathrm{Y}}(B' \setminus B) = \mathfrak{m}_{\mathrm{Y}}|_{B'}(\mathrm{Y} \setminus B) = 0$. This completes the proof. □

1.3 Metric measure spaces

Given any metric space (X, d), we denote by $\mathscr{B}(\mathrm{X})$ the *Borel σ-algebra* on X, i.e. the smallest σ-algebra on X that contains every open d-ball of X. Measures over $\big(\mathrm{X}, \mathscr{B}(\mathrm{X})\big)$ are called *Borel measures* on X. We say that a Borel measure $\mathfrak{m}$ on X is a *Radon measure* provided it is finite on compact sets. For our purposes, a *metric measure space* is a triple $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, where

$$\begin{array}{ll} (\mathrm{X},\mathsf{d}) & \text{is a complete and separable metric space,} \\ \mathfrak{m} \neq 0 & \text{is a non-negative Borel measure on X, finite on balls.} \end{array} \tag{1.58}$$

The measure $\mathfrak{m}$ is called *reference measure*. Observe that $\mathfrak{m}$ is a σ-finite Radon measure. The *support* of $\mathfrak{m}$ – denoted by $\mathrm{spt}(\mathfrak{m})$ – is defined as the intersection of all closed subsets C of X such that $\mathfrak{m}(\mathrm{X} \setminus C) = 0$. In particular, the support is a closed set.

Given two metric measure spaces $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$, we will always implicitly endow the product space $\mathrm{X} \times \mathrm{Y}$ with the product distance $\mathsf{d}_{\mathrm{X}} \times \mathsf{d}_{\mathrm{Y}}$, given by

$$(\mathsf{d}_{\mathrm{X}} \times \mathsf{d}_{\mathrm{Y}})\big((x_1, y_1), (x_2, y_2)\big) \doteq \sqrt{\mathsf{d}_{\mathrm{X}}(x_1, x_2)^2 + \mathsf{d}_{\mathrm{Y}}(y_1, y_2)^2} \tag{1.59}$$

for every $(x_1, y_1), (x_2, y_2) \in \mathrm{X} \times \mathrm{Y}$, and with the product measure $\mathfrak{m}_{\mathrm{X}} \otimes \mathfrak{m}_{\mathrm{Y}}$.

Definition 1.31 (Doubling measure) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then we say that the reference measure* $\mathfrak{m}$ *is* pointwise doubling *at a point* $x \in \mathrm{spt}(\mathfrak{m})$ *provided*

$$\varlimsup_{r \searrow 0} \frac{\mathfrak{m}\big(B_{2r}(x)\big)}{\mathfrak{m}\big(B_r(x)\big)} < +\infty. \tag{1.60}$$

Moreover, we say that the metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *– or just the measure* $\mathfrak{m}$ *– is* doubling *provided there exists a constant* $\mathsf{C}_{\mathfrak{m}} > 0$ *such that*

$$\mathfrak{m}\big(B_{2r}(x)\big) \leq \mathsf{C}_{\mathfrak{m}}\,\mathfrak{m}\big(B_r(x)\big) \qquad \text{for every } x \in \mathrm{X} \text{ and } r > 0. \tag{1.61}$$

The least such constant $\mathsf{C}_{\mathfrak{m}}$ *is called the* doubling constant *of the space.*

Remark 1.32 It is immediate to see that a doubling measure is pointwise doubling at all points of the space. Moreover, if a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is doubling, then the underlying metric space (X,d) is metrically doubling. ■

Definition 1.33 (Vitali space) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then X is said to be a* Vitali space *provided the following condition is satisfied: given a Borel set $A\subseteq\mathrm{X}$ and a family $\mathcal{F}$ of closed balls in X such that $\inf\big\{r>0\,:\,\overline{B_r(x)}\in\mathcal{F}\big\}=0$ holds for $\mathfrak{m}$-a.e. $x\in A$, there exists a countable family $\mathcal{G}\subseteq\mathcal{F}$ of pairwise disjoint balls such that $\mathfrak{m}\big(A\setminus\bigcup_{B\in\mathcal{G}}B\big)=0$.*

By adapting the arguments in the proof of [Hei01, Theorem 1.6], one can prove that

$$\mathfrak{m}\text{ is pointwise doubling at }\mathfrak{m}\text{-a.e. }x\in\mathrm{X}\quad\Longrightarrow\quad(\mathrm{X},\mathsf{d},\mathfrak{m})\text{ is a Vitali space.}\tag{1.62}$$

A fundamental property of Vitali spaces is given by the Lebesgue differentiation theorem, whose proof can be found e.g. in [Hei01]:

Theorem 1.34 (Lebesgue differentiation theorem) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a Vitali space. Fix any function $f\in L^1_{\mathrm{loc}}(\mathfrak{m})$. Then*

$$f(x)=\lim_{r\searrow0}\frac{1}{\mathfrak{m}\big(B_r(x)\big)}\int_{B_r(x)}f\,\mathrm{d}\mathfrak{m}\qquad\textit{for }\mathfrak{m}\textit{-a.e. }x\in\mathrm{X}.\tag{1.63}$$

Definition 1.35 (Density of a point) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $E\subseteq\mathrm{X}$ be a Borel set. Then we say that a point $x\in\mathrm{spt}(\mathfrak{m})$ has* density $\lambda\in[0,1]$ *for E provided*

$$\exists\,D_E(x)\doteq\lim_{r\searrow0}\frac{\mathfrak{m}\big(E\cap B_r(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}=\lambda.\tag{1.64}$$

Corollary 1.36 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a Vitali space. Let $E\subseteq\mathrm{X}$ be a Borel set. Then*

$$D_E(x)=1\qquad\textit{holds for }\mathfrak{m}\textit{-a.e. point }x\in E.\tag{1.65}$$

Proof. Just apply Theorem 1.34 to the function $f\doteq\chi_E$. □

In Subsection 5.3.3, the following class of spaces will play a fundamental role:

$$\begin{aligned}&(\mathrm{X},\mathsf{d},\mathfrak{m})\text{ is a metric measure space with the following property:}\\&\text{for every Borel set }E\subseteq\mathrm{X}\text{ and for }\mathfrak{m}\text{-a.e. }\bar{x}\in E,\text{ it holds that}\\&\forall\varepsilon>0\quad\exists\,r>0:\quad\forall x\in B_r(\bar{x})\quad\exists\,y\in E:\quad\mathsf{d}(x,y)<\varepsilon\,\mathsf{d}(x,\bar{x}).\end{aligned}\tag{1.66}$$

A sufficient condition for the previous property to hold is given by the next result:

Lemma 1.37 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $A\subseteq\mathrm{X}$ be Borel. Suppose there exist constants $\bar{r},\mathsf{C}>0$ such that $\mathfrak{m}\big(B_{2r}(x)\big)\leq\mathsf{C}\,\mathfrak{m}\big(B_r(x)\big)$ for every $0<r<\bar{r}$ and $x\in A$. Then the metric measure space $\big(A,\mathsf{d}_{|A\times A},\mathfrak{m}_{|A}\big)$ satisfies property (1.66). In particular, any doubling metric measure space satisfies property (1.66).*

Proof. We argue by contradiction: assume the existence of $\varepsilon>0$ and of points $\{x_r\}_{r>0}\subseteq A$ with $\mathsf{d}(x_r,\bar{x})<r$ for every $r>0$, such that

$$E\cap B_{\varepsilon\,\mathsf{d}(x_r,\bar{x})}(x_r)=\emptyset\qquad\text{for every }r>0.\tag{1.67}$$

Fix $n \in \mathbb{N}$ such that $2^n \varepsilon \geq 2+\varepsilon$. Thus $B_{\varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r) \subseteq B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x}) \subseteq B_{2^n \varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r)$ for every $r>0$, hence in particular it holds that

$$\mathfrak{m}\big(B_{\varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r)\big) \geq \frac{\mathfrak{m}\big(B_{2^n \varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r)\big)}{\mathsf{C}^n} \geq \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x})\big)}{\mathsf{C}^n} \tag{1.68}$$

for every $r>0$ such that $r<\bar{r}/(2^{n-1}\varepsilon)$. Therefore

$$\begin{aligned} D_E(\bar{x}) &= \lim_{r\searrow 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x}) \cap E\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\ \text{(by (1.67))} \qquad &\leq \varlimsup_{r\searrow 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x}) \setminus B_{\varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r)\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\ &= \varlimsup_{r\searrow 0} \frac{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x})\big) - \mathfrak{m}\big(B_{\varepsilon\, \mathsf{d}(x_r,\bar{x})}(x_r)\big)}{\mathfrak{m}\big(B_{(1+\varepsilon)\, \mathsf{d}(x_r,\bar{x})}(\bar{x})\big)} \\ \text{(by (1.68))} \qquad &\leq 1 - \frac{1}{\mathsf{C}^n} < 1, \end{aligned}$$

which contradicts our assumption $D_E(\bar{x}) = 1$. Hence the statement follows. □

Given a metric space (X,d), a Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$ and a Borel set $E \subseteq \mathrm{X}$, we have that $\mathrm{lip}\big(f_{|E}\big)(x) \leq \mathrm{lip}(f)(x)$ is satisfied for every $x \in \mathrm{X}$, where $\mathrm{lip}\big(f_{|E}\big)$ is taken in the metric space $\big(E, \mathsf{d}_{|E\times E}\big)$. Simple examples show that in general equality does not hold; however, if we restrict to the case of a doubling metric measure space, then Lemma 1.37 grants that the equality holds at least on density points of E:

Proposition 1.38 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a doubling metric measure space. Fix a Borel set $E \subseteq \mathrm{X}$ and a Lipschitz function $f \in \mathrm{LIP}(\mathrm{X})$. Then*

$$\mathrm{lip}\big(f_{|E}\big)(x) = \mathrm{lip}(f)(x) \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x \in E. \tag{1.69}$$

Proof. It suffices to prove that $\mathrm{lip}(f)(x) \leq \mathrm{lip}\big(f_{|E}\big)(x)$ for every point $x \in E$ of density 1. Fix $x \in E$ with $D_E(x) = 1$. If x is an isolated point in X, then $\mathrm{lip}(f)(x) = \mathrm{lip}\big(f_{|E}\big)(x) = 0$. If x is an accumulation point, then take a sequence $(x_n)_n \subseteq \mathrm{X} \setminus \{x\}$ converging to x. Up to passing to a suitable subsequence, we can assume that $\varlimsup_n \big|f(x_n)-f(x)\big|/\mathsf{d}(x_n,x)$ is actually a limit. Moreover – possibly passing to a further subsequence – Lemma 1.37 provides the existence of a sequence $(y_n)_n \subseteq E$ satisfying $\mathsf{d}(x_n,y_n) < \mathsf{d}(x_n,x)/n$ for every $n \geq 1$. In particular, $\lim_n y_n = x$ and $y_n \neq x$ for every $n \geq 1$. Therefore

$$\begin{aligned} \lim_{n\to\infty} \frac{\big|f(x_n)-f(x)\big|}{\mathsf{d}(x_n,x)} &\leq \varlimsup_{n\to\infty} \frac{\big|f(x_n)-f(y_n)\big|}{\mathsf{d}(x_n,y_n)} \frac{\mathsf{d}(x_n,y_n)}{\mathsf{d}(x_n,x)} + \varlimsup_{n\to\infty} \frac{\big|f(y_n)-f(x)\big|}{\mathsf{d}(y_n,x)} \frac{\mathsf{d}(y_n,x)}{\mathsf{d}(x_n,x)} \\ &\leq \mathrm{Lip}(f) \lim_{n\to\infty} \frac{1}{n} + \varlimsup_{n\to\infty} \frac{\big|f(y_n)-f(x)\big|}{\mathsf{d}(y_n,x)} \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \leq \mathrm{lip}\big(f_{|E}\big)(x). \end{aligned}$$

The arbitrariness of $(x_n)_n$ gives the conclusion. □

1.3.1 Gromov-Hausdorff convergence

We say that $(\mathrm{X},\mathsf{d},\mathfrak{m},\bar{x})$ is a *pointed metric measure space* provided $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a metric measure space and the reference point $\bar{x} \in \mathrm{X}$ belongs to $\mathrm{spt}(\mathfrak{m})$. Two pointed metric measure spaces $(\mathrm{X},\mathsf{d}_{\mathrm{X}},\mathfrak{m}_{\mathrm{X}},\bar{y})$ and $(\mathrm{Y},\mathsf{d}_{\mathrm{Y}},\mathfrak{m}_{\mathrm{Y}},\bar{y})$ are said to be *isomorphic* if there exists an isometric embedding $\iota : \mathrm{spt}(\mathfrak{m}_{\mathrm{X}}) \to \mathrm{Y}$ such that $\iota_* \mathfrak{m}_{\mathrm{X}} = \mathfrak{m}_{\mathrm{Y}}$ and $\iota(\bar{x}) = \bar{y}$. The equivalence class of a given space $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x})$ under this isomorphism relation will be denoted by $[\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x}]$.

Given a complete separable metric space (X, d) and a sequence $(\mu_n)_{n \in \mathbb{N} \cup \{\infty\}}$ of non-negative Borel measures on X that are finite on bounded sets, we say that μ_n *weakly converges to* μ_∞ as $n \to \infty$, briefly $\mu_n \rightharpoonup \mu_\infty$, provided

$$\lim_{n \to \infty} \int f \, \mathrm{d}\mu_n = \int f \, \mathrm{d}\mu_\infty \qquad \text{for every } f \in C_{\mathrm{bs}}(\mathrm{X}), \tag{1.70}$$

where $C_{\mathrm{bs}}(\mathrm{X})$ is the space of all bounded continuous functions on X with bounded support.

Since we shall deal with possibly non-compact and non-doubling spaces, it will be convenient to work with the notion of pointed measured Gromov convergence. More precisely, we shall follow the so-called 'extrinsic approach', introduced in [GMS15, Definition 3.9]:

Definition 1.39 (Pointed measured Gromov convergence) *Fix a sequence of pointed metric measure spaces* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$*,* $n \in \mathbb{N} \cup \{\infty\}$*. Then* $[\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$ *is said to converge to* $[\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$ *in the* pointed measured Gromov sense*, or briefly* pmG-sense*, provided there exist a complete separable metric space* $(\mathrm{W}, \mathsf{d}_{\mathrm{W}})$ *and a sequence* $(\iota_n)_{n \in \mathbb{N} \cup \{\infty\}}$ *of isometric embeddings* $\iota_n : \mathrm{X}_n \to \mathrm{W}$ *such that*

$$\begin{aligned} \iota_n(\bar{x}_n) &\to \iota_\infty(\bar{x}_\infty) \in \mathrm{spt}\big((\iota_\infty)_* \mathfrak{m}_\infty\big), \\ (\iota_n)_* \mathfrak{m}_n &\rightharpoonup (\iota_\infty)_* \mathfrak{m}_\infty, \end{aligned} \tag{1.71}$$

as $n \to \infty$*.*

Let us fix a shorthand notation: given a pointed metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x})$ and any radius $r > 0$, we define the *normalised measure* $\mathfrak{m}_r^{\bar{x}}$ on X as

$$\mathfrak{m}_r^{\bar{x}} \doteq \frac{\mathfrak{m}}{\mathfrak{m}\big(B_r(\bar{x})\big)}. \tag{1.72}$$

We can now introduce the notion of tangent cone to a pointed metric measure space:

Definition 1.40 (Tangent cone) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x})$ *be a pointed metric measure space. Then we denote by* $\mathrm{Tan}[\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x}]$ *the family of all the classes* $[\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}}, \bar{y}]$ *that are obtained as pmG-limits of* $\big[\mathrm{X}, \mathsf{d}/r_n, \mathfrak{m}_{r_n}^{\bar{x}}, \bar{x}\big]$*, for a suitable sequence* $r_n \searrow 0$*. We refer to* $\mathrm{Tan}[\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x}]$ *as the* tangent cone *of* $[\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x}]$*.*

Proposition 1.41 (Locality of the tangent cone) *Fix a metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *and a Borel set* $A \subseteq \mathrm{X}$*. Let* $\bar{x} \in A$ *be a point of density* 1 *for* A *such that the reference measure* $\mathfrak{m}$ *is pointwise doubling at* $\bar{x}$*. Then*

$$\mathrm{Tan}[\mathrm{X}, \mathsf{d}, \mathfrak{m}, \bar{x}] = \mathrm{Tan}[A, \mathsf{d}_{|A \times A}, \mathfrak{m}_{|A}, \bar{x}]. \tag{1.73}$$

Proof. For the sake of simplicity, let us denote $\mathsf{d}' \doteq \mathsf{d}_{|A \times A}$ and $\mathfrak{m}' \doteq \mathfrak{m}_{|A}$. Suppose that the class $[\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}}, \bar{y}]$ is the pmG-limit of $\big[\mathrm{X}, \mathsf{d}/r_n, \mathfrak{m}_{r_n}^{\bar{x}}, \bar{x}\big]$ for some $r_n \searrow 0$. Then there exist a complete and separable metric space $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$, an isometric embedding $\iota_{\mathrm{Y}} : \mathrm{Y} \to \mathrm{Z}$ and a sequence $(\iota_n)_n$ of isometries $\iota_n : \big(\mathrm{X}, \mathsf{d}/r_n\big) \to (\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$ such that $\iota_n(\bar{x}) \to \iota_{\mathrm{Y}}(\bar{y}) \in \mathrm{spt}\big((\iota_{\mathrm{Y}})_* \mathfrak{m}_{\mathrm{Y}}\big)$ and $(\iota_n)_* \mathfrak{m}_{r_n}^{\bar{x}} \rightharpoonup (\iota_{\mathrm{Y}})_* \mathfrak{m}_{\mathrm{Y}}$. Hence let us define $\iota_n' \doteq \iota_{n|A}$ for every $n \in \mathbb{N}$. Clearly each map ι_n' is an isometry from $(A, \mathsf{d}'/r_n)$ to $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}})$. To conclude that $[\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}}, \bar{y}] \in \mathrm{Tan}[A, \mathsf{d}', \mathfrak{m}', \bar{x}]$, it is enough to show that $(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} \rightharpoonup (\iota_{\rm Y})_*\mathfrak{m}_{\rm Y}$. Thus fix $f \in C_{\rm bs}({\rm Z})$. Choose $R > 0$ such that ${\rm spt}(f) \subseteq B_R\big(\iota_{\rm Y}(\bar{y})\big)$, whence ${\rm spt}(f \circ \iota_n) \subseteq B_{2Rr_n}(\bar{x})$ for n big enough. Then

$$\int f\,{\rm d}(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} = \frac{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}\int f\,{\rm d}(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n} - \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}\int_{B_{2Rr_n}(\bar{x})\setminus A} f\circ\iota_n\,{\rm d}\mathfrak{m}.$$

Since $D_A(\bar{x}) = 1$ and $\mathfrak{m}$ is pointwise doubling at $\bar{x}$, one has

$$\frac{\Big|\int_{B_{2Rr_n}(\bar{x})\setminus A} f\circ\iota_n\,{\rm d}\mathfrak{m}\Big|}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)} \le \frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\setminus A\big)}{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}\,\frac{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\cap A\big)}\,\frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\,\max_{\rm Z}|f| \overset{n}{\longrightarrow} 0,$$

which grants that $\int f\,{\rm d}(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} \to \int f\,{\rm d}(\iota_n)_*\mathfrak{m}_{\rm Y}$, as required.

Conversely, let $[{\rm Y},{\sf d}_{\rm Y},\mathfrak{m}_{\rm Y},\bar{y}]$ be the pmG-limit of $[A,{\sf d}'/r_n,(\mathfrak{m}')^{\bar{x}}_{r_n},\bar{x}]$ for some $r_n \searrow 0$. Then take a complete separable metric space $({\rm W},{\sf d}_{\rm W})$, an isometric embedding $\iota'_{\rm Y} : {\rm Y} \to {\rm W}$ and a sequence of maps $\iota'_n : A \to {\rm W}$, which are isometries from $(A,{\rm d}'/r_n)$ to $({\rm W},{\sf d}_{\rm W})$, such that $\iota'_n(\bar{x}_n) \to \iota'_{\rm Y}(\bar{y}) \in {\rm spt}\big((\iota'_{\rm Y})_*\mathfrak{m}_{\rm Y}\big)$ and $(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} \rightharpoonup (\iota'_{\rm Y})_*\mathfrak{m}_{\rm Y}$. Hence there exist a complete separable metric space $({\rm Z},\mathfrak{m}_{\rm Z})$, an isometric embedding $\iota_{\rm W} : {\rm W} \to {\rm Z}$ and a sequence of maps $\iota_n : {\rm X} \to {\rm Z}$, which are isometries from $({\rm X},{\sf d}/r_n)$ to $({\rm Z},{\sf d}_{\rm Z})$, such that $\iota_n|_A = \iota_{\rm W}\circ\iota'_n$ holds for every $n \in \mathbb{N}$ – see for instance [GMS15, Proposition 3.10]. Denote $\iota_{\rm Y} \doteq \iota_{\rm W}\circ\iota'_{\rm Y}$. We clearly have that $\iota_n(\bar{x}) = \iota_{\rm W}\big(\iota'_n(\bar{x})\big) \to \iota_{\rm Y}(\bar{y}) \in {\rm spt}\big((\iota_{\rm Y})_*\mathfrak{m}_{\rm Y}\big)$ as $n \to \infty$, thus it only remains to prove that $(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n} \rightharpoonup (\iota_{\rm Y})_*\mathfrak{m}_{\rm Y}$ as $n \to \infty$. To this aim, fix $f \in C_{\rm bs}({\rm Z})$. Observe that

$$\int f\,{\rm d}(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n} = \frac{\mathfrak{m}\big(A\cap B_{r_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int f\circ\iota_{\rm W}\,{\rm d}(\iota'_n)_*(\mathfrak{m}')^{\bar{x}}_{r_n} + \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int_{{\rm X}\setminus A} f\circ\iota_n\,{\rm d}\mathfrak{m}.$$

The first addendum in the right hand side of the previous equation tends to $\int f\circ\iota_{\rm W}\,{\rm d}(\iota'_{\rm Y})_*\mathfrak{m}_{\rm Y}$, because $D_A(\bar{x}) = 1$ and $f\circ\iota_{\rm W} \in C_{\rm bs}({\rm W})$. To estimate the second one, take any $R > 0$ such that ${\rm spt}(f) \subseteq B_R\big(\iota_{\rm Y}(\bar{y})\big)$, so that ${\rm spt}(f\circ\iota_n) \subseteq B_{2Rr_n}(\bar{x})$ for n sufficiently big. Then

$$\left|\frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\int_{{\rm X}\setminus A} f\circ\iota_n\,{\rm d}\mathfrak{m}\right| \le \frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\setminus A\big)}{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}\,\frac{\mathfrak{m}\big(B_{2Rr_n}(\bar{x})\big)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)}\,\max_{\rm Z}|f| \longrightarrow 0.$$

Therefore $\int f\,{\rm d}(\iota_n)_*\mathfrak{m}^{\bar{x}}_{r_n} \to \int f\,{\rm d}(\iota_{\rm Y})_*\mathfrak{m}_{\rm Y}$, proving that $[{\rm Y},{\sf d}_{\rm Y},\mathfrak{m}_{\rm Y},\bar{y}] \in {\rm Tan}[{\rm X},{\sf d},\mathfrak{m},\bar{x}]$ and accordingly the statement. □

The previous result will allow us to concentrate our attention only on those spaces that satisfy property (1.66). In such context, it is easier to study the blow-ups of the space by means of a different notion of convergence (see, for instance, [GMS15, Definition 3.24]):

Definition 1.42 (Pointed measured Gromov-Hausdorff convergence) *Consider any sequence of pointed metric measure spaces* $({\rm X}_n,{\sf d}_n,\mathfrak{m}_n,\bar{x}_n)$*, with* $n \in \mathbb{N}\cup\{\infty\}$*. Then we say that* $({\rm X}_n,{\sf d}_n,\mathfrak{m}_n,\bar{x}_n)$ *converges to* $({\rm X}_\infty,{\sf d}_\infty,\mathfrak{m}_\infty,\bar{x}_\infty)$ *in the* pointed measured Gromov-Hausdorff sense*, or briefly* pmGH-sense*, provided for any fixed* $\varepsilon, R > 0$ *with* $\varepsilon < R$ *there exist* $\bar{n} \in \mathbb{N}$ *and a sequence* $(f_n)_{n\ge\bar{n}}$ *of Borel maps* $f_n : B^{{\sf d}_n}_R(\bar{x}_n) \to {\rm X}_\infty$ *such that*

i) $f_n(\bar{x}_n) = \bar{x}_\infty$ *for every* $n \ge \bar{n}$,

ii) $\Big|{\sf d}_\infty\big(f_n(x),f_n(y)\big) - {\sf d}_n(x,y)\Big| \le \varepsilon$ *for every* $n \ge \bar{n}$ *and* $x,y \in B^{{\sf d}_n}_R(\bar{x}_n)$,

iii) *the* ε*-neighbourhood of* $f_n\big(B^{{\sf d}_n}_R(\bar{x}_n)\big)$ *contains* $B^{{\sf d}_\infty}_{R-\varepsilon}(\bar{x}_\infty)$ *for every* $n \ge \bar{n}$,

iv) $(f_n)_*\big(\mathfrak{m}_n|_{B_R^{\mathsf{d}_n}(\bar{x}_n)}\big) \rightharpoonup \mathfrak{m}_\infty|_{B_R^{\mathsf{d}_\infty}(\bar{x}_\infty)}$ *as* $n \to \infty$ *for a.e.* $R > 0$.

As shown in [GMS15, Proposition 3.30], the relation between the two notions of convergence (for pointed metric measure spaces) introduced so far is the following:

Proposition 1.43 (From pmGH to pmG) *Let* $(\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n)$ *be a sequence of pointed metric measure spaces that converges to some limit* $(\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty)$ *in the pmGH-sense. Then the sequence of classes* $[\mathrm{X}_n, \mathsf{d}_n, \mathfrak{m}_n, \bar{x}_n]$ *pmG-converges to* $[\mathrm{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, \bar{x}_\infty]$.

1.3.2 Optimal transport

We report here just few basic notions of optimal transport theory – the ones that are enough for our purposes. For a complete treatise of this argument, we refer for instance to the monographs [Vil09] and [AG13], while the following short discussion is taken from [GRS16].

Let (X, d) be a complete and separable metric space. Then we denote by $\mathscr{P}_2(\mathrm{X})$ the set of all Borel probability measures μ on X having *finite second moment*, namely that satisfy the inequality $\int \mathsf{d}(\overline{x}, \cdot)^2 \, \mathrm{d}\mu < +\infty$ for some (thus any) point $\overline{x} \in \mathrm{X}$.

Suppose the space (X, d) is proper (i.e. closed bounded sets are compact) and geodesic. Then it is possible to formulate the dynamical version of the optimal transport problem:

$$W_2(\mu, \nu) \doteq \sqrt{\inf \iint_0^1 |\dot{\gamma}_t|^2 \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma)} \qquad \text{for every } \mu, \nu \in \mathscr{P}_2(\mathrm{X}), \tag{1.74}$$

where the infimum is taken among all $\boldsymbol{\pi} \in \mathscr{P}\big(\Gamma(\mathrm{X})\big)$ such that $(\mathrm{e}_0)_*\boldsymbol{\pi} = \mu$ and $(\mathrm{e}_1)_*\boldsymbol{\pi} = \nu$. It turns out that W_2 is a distance on $\mathscr{P}_2(\mathrm{X})$, called *quadratic transportation distance*. We say that $\big(\mathscr{P}_2(\mathrm{X}), W_2\big)$ is the *Wasserstein space* over X. Moreover, the infimum in (1.74) is actually a minimum and we indicate with $\mathrm{OptGeo}(\mu, \nu)$ the set of all its minimizers. Given any $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu, \nu)$, one has that $\boldsymbol{\pi}$ is concentrated on the set $\mathrm{Geo}(\mathrm{X})$ (which can be easily proven to be a closed subset of the space $\Gamma(\mathrm{X})$ of all continuous curves). Moreover, it holds that the map $[0, 1] \ni t \mapsto (\mathrm{e}_t)_*\boldsymbol{\pi} \in \mathscr{P}_2(\mathrm{X})$ is a W_2-geodesic curve joining μ to ν.

2

Sobolev calculus on metric measure spaces

Contents

These last two decades have witnessed an increasing interest in the theory of weakly differentiable functions on metric measure spaces, which represents a fundamental passage towards a differential calculus on nonsmooth structures. The notion of Sobolev space for metric measure spaces that first made its appearance in the literature is the one proposed by P. Hajłasz in [Haj96]. Nevertheless, such space does not fit our needs because it is 'non-local', meaning that the gradient of a Sobolev function may not depend just on the local behaviour of the function itself. This does not happen for the alternative approaches we are going to describe, which surprisingly turned out to be equivalent. We divide them into three groups:

- Approximation via 'smooth' functions. Any Sobolev function in the Euclidean space can be approximated by smooth functions; such classical result can be adapted to the metric context, by replacing 'smooth functions' with 'Lipschitz functions'. More precisely, it is possible to select the Sobolev functions by looking at the relaxation of the local Lipschitz constant. Such idea – which originally comes from J. Cheeger's paper [Che99] – has been further developed in [AGS14a].

- Distributional derivatives. In analogy with the smooth case, one can also define the Sobolev space by means of a suitable integration-by-parts formula, where the role of vector field is played by some concept of *derivation* – inspired by N. Weaver's papers [Wea99, Wea00]. This goal has been achieved by S. Di Marino in [DM14], thus obtaining a notion that is equivalent to the one described in the previous item.

- GOOD BEHAVIOUR ALONG CURVES. The very first perspective on weakly differentiable functions, due to B. Levi [Lev06], relies upon the following idea: Sobolev functions in the Euclidean space can be characterised by checking their behaviour along lines. This approach has been carried on by B. Fuglede in [Fug57], who introduced a fundamental potential-theoretic notion called *p-modulus* and denoted by Mod_p. Roughly speaking, the p-Sobolev functions are those admitting a good behaviour along Mod_p-almost every curve. In the metric measure framework, N. Shanmugalingam adapted the notion of p-modulus and combined it with the concept of upper gradient (cf. [KM98]), thus obtaining the so-called *Newtonian space* (see [Sha00]). An equivalent formulation of this technique – based on the notion of *test plan* – can be found in [AGS14a]; since this is the approach we shall adopt in the second half of the thesis, we will describe it more in details at the end of this introductory part of the chapter.

The several definitions of Sobolev space illustrated so far present many common features, thus enabling an axiomatic approach to the subject. This plan has been pursued by V. Gol'dshtein and M. Troyanov in [GT01]. The key object in their construction is given by the *D-structure*: any locally p-integrable function u is associated with a family $D[u]$ of *pseudo-gradients*, which are non-negative Borel functions on X exerting some control from above on the variation of u. The pseudo-gradients are not explicitly specified, but they are rather supposed to fulfil a certain list of axioms. Therefore the p-Sobolev space associated to D can be defined as the space of all p-integrable functions admitting a p-integrable pseudo-gradient. Standard functional-analytic techniques allow us to select – for any Sobolev function u – a distinguished minimal object $\underline{D}u$, called *minimal pseudo-gradient*. In Section 2.1 we shall first report the main definitions and results of [GT01], then propose new notions of locality for D-structures (cf. Definition 2.6) and show that under such additional assumptions the minimal pseudo-gradient satisfies some useful calculus rules (cf. Proposition 2.13).

In Section 2.2 we will focus our attention on the version of Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$ over a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ proposed by L. Ambrosio, N. Gigli and G. Savaré in [AGS14a] (we restrict to the case $p=2$ for simplicity, since this is enough for our purposes). The key ingredient is the concept of *test plan*, which constitutes a 'probabilistic' tool that is needed to select curves in X; from a technical point of view, it represents an alternative to the 2-modulus in Shanmugalingam's approach. Test plans permit to relax the notion of upper gradient, thus leading to the so-called *weak upper gradients* (cf. Definition 2.17) and accordingly to the definition of the Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$ (cf. Definition 2.24). Such space satisfies the above-mentioned locality properties and calculus rules; it will play a fundamental role in Chapter 4, when we will describe how to build a differential structure over general metric measure spaces.

2.1 Axiomatic theory of Sobolev spaces

2.1.1 Definition of D-structure and its basic properties

In this subsection we summarise the content of [GT01]. We point out that in the mentioned paper a more general notion of locally p-integrable function is considered – built upon the concept of *$\mathcal{K}$-set*. We chose the present approach just for simplicity, but the whole discussion would remain unaltered if we replaced our definition of $L^p_{\mathrm{loc}}(\mathfrak{m})$ with the one of [GT01].

Definition 2.1 (D-structure) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix any $p\in[1,\infty)$. Then a D-*structure *on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is any map D associating to each function $u\in L^p_{\rm loc}(\mathrm{X})$ a family $D[u]\subseteq L^0(\mathfrak{m})^+$ of* pseudo-gradients *of u, which satisfies the following axioms:*

- **A1** Non triviality. *It holds that $\mathrm{Lip}(u)\,\chi_{\{u>0\}}\in D[u]$ for every $u\in L^p_{\rm loc}(\mathrm{X})^+\cap\mathrm{LIP}(\mathrm{X})$.*
- **A2** Upper linearity. *Let $u_1,u_2\in L^p_{\rm loc}(\mathrm{X})$ be fixed. Consider $g_1\in D[u_1]$ and $g_2\in D[u_2]$. Suppose that a function $g\in L^0(\mathfrak{m})^+$ satisfies the inequality $g\geq|\alpha_1|\,g_1+|\alpha_2|\,g_2$ in the $\mathfrak{m}$-a.e. sense for some $\alpha_1,\alpha_2\in\mathbb{R}$. Then $g\in D[\alpha_1\,u_1+\alpha_2\,u_2]$.*
- **A3** Leibniz rule. *Fix a function $u\in L^p_{\rm loc}(\mathrm{X})$ and a pseudo-gradient $g\in D[u]$ of u. Then for every $\varphi\in\mathrm{LIP}_b(\mathrm{X})$ it holds that $g\,\|\varphi\|_{L^\infty(\mathfrak{m})}+\mathrm{Lip}(\varphi)\,|u|\in D[\varphi\,u]$.*
- **A4** Lattice property. *Fix $u_1,u_2\in L^p_{\rm loc}(\mathrm{X})$. Given any $g_1\in D[u_1]$ and $g_2\in D[u_2]$, one has that $g_1\vee g_2\in D[u_1\vee u_2]\cap D[u_1\wedge u_2]$.*
- **A5** Completeness. *Consider two sequences $(u_n)_n\subseteq L^p_{\rm loc}(\mathrm{X})$ and $(g_n)_n\subseteq L^p(\mathfrak{m})$ that satisfy $g_n\in D[u_n]$ for every $n\in\mathbb{N}$. Suppose that there exist $u\in L^p_{\rm loc}(\mathrm{X})$ and $g\in L^p(\mathfrak{m})$ such that $u_n\to u$ in $L^p_{\rm loc}(\mathrm{X})$ and $g_n\to g$ in $L^p(\mathfrak{m})$. Then $g\in D[u]$.*

Remark 2.2 It directly follows from axioms **A1** and **A2** that $0\in D[c]$ for every constant function $c\in\mathbb{R}$. Moreover, axiom **A2** grants that the set $D[u]\cap L^p(\mathfrak{m})$ is convex in $L^p(\mathfrak{m})$ and that $D[\alpha\,u]=|\alpha|\,D[u]$ for every $u\in L^p_{\rm loc}(\mathrm{X})$ and $\alpha\in\mathbb{R}\setminus\{0\}$, while axiom **A5** implies that the set $D[u]\cap L^p(\mathfrak{m})$ is closed in $L^p(\mathfrak{m})$. ■

Given any function $u\in L^p_{\rm loc}(\mathrm{X})$ and any Borel set $B\subseteq\mathrm{X}$, let us define the *p-Dirichlet energy* $\mathcal{E}_p(u\,|B)$ of u on B in the following way:

$$\mathcal{E}_p(u\,|B)\doteq\inf\left\{\int_B g^p\,\mathrm{d}\mathfrak{m}\;\middle|\;g\in D[u]\right\}\in[0,+\infty]. \tag{2.1}$$

For the sake of brevity, we shall use the shorthand notation $\mathcal{E}_p(u)$ to indicate $\mathcal{E}_p(u\,|\mathrm{X})$.

Definition 2.3 (Sobolev space) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p\in[1,\infty)$ be fixed. Given any D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we define the* Sobolev class *associated to D as*

$$\mathrm{S}^p(\mathrm{X})=\mathrm{S}^p(\mathrm{X};D)\doteq\left\{u\in L^p_{\rm loc}(\mathrm{X})\;\middle|\;\mathcal{E}_p(u)<+\infty\right\}. \tag{2.2}$$

Moreover, we define the Sobolev space *associated to D as*

$$W^{1,p}(\mathrm{X})=W^{1,p}(\mathrm{X};D)\doteq L^p(\mathfrak{m})\cap\mathrm{S}^p(\mathrm{X};D). \tag{2.3}$$

It turns out that $W^{1,p}(\mathrm{X};D)$ has a natural structure of Banach space, as we are going to show in the next result; cf. also [GT01, Theorem 1.5].

Theorem 2.4 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be any metric measure space and let $p\in[1,\infty)$. Consider any D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then $W^{1,p}(\mathrm{X})$ is a Banach space if endowed with the norm*

$$\|u\|_{W^{1,p}(\mathrm{X})}\doteq\Big(\|u\|^p_{L^p(\mathfrak{m})}+\mathcal{E}_p(u)\Big)^{1/p}\qquad\text{for every } u\in W^{1,p}(\mathrm{X}). \tag{2.4}$$

Proof. First of all, we have that $0 \in W^{1,p}(\mathrm{X})$ by Remark 2.2. Given any $u_1, u_2 \in W^{1,p}(\mathrm{X})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, we have $\alpha_1 u_1 + \alpha_2 u_2 \in W^{1,p}(\mathrm{X})$ by axiom **A2**: indeed, if $g_1 \in D[u_1] \cap L^p(\mathfrak{m})$ and $g_2 \in D[u_2] \cap L^p(\mathfrak{m})$, then $|\alpha_1|\, g_1 + |\alpha_2|\, g_2 \in D[\alpha_1 u_1 + \alpha_2 u_2] \cap L^p(\mathfrak{m})$. Therefore $W^{1,p}(\mathrm{X})$ is a vector subspace of $L^p(\mathfrak{m})$. We now prove that $\|\cdot\|_{W^{1,p}(\mathrm{X})}$ is actually a norm on $W^{1,p}(\mathrm{X})$:

- If $u \in W^{1,p}(\mathrm{X})$ satisfies $\|u\|_{W^{1,p}(\mathrm{X})} = 0$, then in particular $\|u\|_{L^p(\mathfrak{m})} = 0$, which grants that $u = 0$ holds $\mathfrak{m}$-a.e. in X.
- Fix $u \in W^{1,p}(\mathrm{X})$ and $\alpha \in \mathbb{R}$. It follows from Remark 2.2 that $\mathcal{E}_p(\alpha\, u) = |\alpha|^p\, \mathcal{E}_p(u)$, whence $\|\alpha\, u\|_{W^{1,p}(\mathrm{X})} = |\alpha|\, \|u\|_{W^{1,p}(\mathrm{X})}$.
- Let $u_1, u_2 \in W^{1,p}(\mathrm{X})$. Given any $g_1 \in D[u_1] \cap L^p(\mathfrak{m})$ and $g_2 \in D[u_2] \cap L^p(\mathfrak{m})$, one has that $g_1 + g_2 \in D[u_1 + u_2] \cap L^p(\mathfrak{m})$ by axiom **A2**, so accordingly

$$\begin{aligned}\|u_1 + u_2\|_{W^{1,p}(\mathrm{X})} &= \Big(\|u_1 + u_2\|^p_{L^p(\mathfrak{m})} + \mathcal{E}_p(u_1 + u_2)\Big)^{1/p} \\ &\leq \Big(\|u_1 + u_2\|^p_{L^p(\mathfrak{m})} + \|g_1 + g_2\|^p_{L^p(\mathfrak{m})}\Big)^{1/p} \\ &\leq \Big(\|u_1\|^p_{L^p(\mathfrak{m})} + \|g_1\|^p_{L^p(\mathfrak{m})}\Big)^{1/p} + \Big(\|u_2\|^p_{L^p(\mathfrak{m})} + \|g_2\|^p_{L^p(\mathfrak{m})}\Big)^{1/p}.\end{aligned}$$

  By arbitrariness of $g_1 \in D[u_1] \cap L^p(\mathfrak{m})$ and $g_2 \in D[u_2] \cap L^p(\mathfrak{m})$, we deduce from the previous estimates that $\|u_1 + u_2\|_{W^{1,p}(\mathrm{X})} \leq \|u_1\|_{W^{1,p}(\mathrm{X})} + \|u_2\|_{W^{1,p}(\mathrm{X})}$.

In order to conclude the proof, it only remains to show that $W^{1,2}(\mathrm{X})$ is complete. Let $(u_n)_n$ be a fixed Cauchy sequence in $W^{1,p}(\mathrm{X})$, in particular $u_n \to u$ in $L^p(\mathfrak{m})$ for some $u \in L^p(\mathfrak{m})$. With no loss of generality, we can also suppose that $\|u_n - u_{n+1}\|_{W^{1,p}(\mathrm{X})} < 1/2^n$ for all $n \in \mathbb{N}$. Then there exists a sequence $(h_n)_n \subseteq L^p(\mathfrak{m})$ such that $h_n \in D[u_n - u_{n+1}]$ and $\|h_n\|_{L^p(\mathfrak{m})} \leq 1/2^n$ for every $n \in \mathbb{N}$. Now call $v_n^k \doteq u_n - u_k$ for every $k > n \geq 0$. Given any $n \in \mathbb{N}$, one clearly has that $v_n^k \to u_n - u$ in $L^p(\mathfrak{m})$ as $k \to \infty$. Since $v_n^k = \sum_{i=n}^{k-1} u_i - u_{i+1}$ for every $k > n \geq 0$, we deduce from axiom **A2** that $g_n^k \doteq \sum_{i=n}^{k-1} h_i \in D[v_n^k] \cap L^p(\mathfrak{m})$. Moreover, notice that

$$\|g_n^k\|_{L^p(\mathfrak{m})} \leq \sum_{i=n}^{k-1} \|h_i\|_{L^p(\mathfrak{m})} \leq \sum_{i=n}^{k-1} \frac{1}{2^i} < \frac{1}{2^{n-1}} \qquad \text{for every } k > n \geq 0. \tag{2.5}$$

Given that the sequence $(g_n^k)_k$ is Cauchy in $L^p(\mathfrak{m})$ by construction, there exists $g_n \in L^p(\mathfrak{m})$ such that $g_n^k \to g_n$ in $L^p(\mathfrak{m})$ as $k \to \infty$. Hence $g_n \in D[u_n - u]$ for all $n \in \mathbb{N}$ by axiom **A5**. This grants that $u_0 - u \in W^{1,p}(\mathrm{X})$, thus accordingly $u = u_0 - (u_0 - u) \in W^{1,p}(\mathrm{X})$. Finally, we deduce from (2.5) that $\|g_n\|_{L^p(\mathfrak{m})} = \lim_k \|g_n^k\|_{L^p(\mathfrak{m})} \leq 1/2^{n-1}$, so that

$$\|u_n - u\|^p_{W^{1,p}(\mathrm{X})} = \|u_n - u\|^p_{L^p(\mathfrak{m})} + \mathcal{E}_p(u_n - u) \leq \|u_n - u\|^p_{L^p(\mathfrak{m})} + \|g_n\|^p_{L^p(\mathfrak{m})} \overset{n}{\longrightarrow} 0,$$

proving that $u_n \to u$ in $W^{1,p}(\mathrm{X})$. This completes the proof of the statement. □

In the case $p > 1$, there is a natural way to select for any Sobolev function a specific pseudo-gradient, minimal in an integral sense; cf. [GT01, Proposition 1.22]. More precisely:

Proposition 2.5 (Minimal pseudo-gradient) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $p \in (1, \infty)$*. Consider any D-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Let* $u \in \mathrm{S}^p(\mathrm{X})$ *be given. Then there exists a unique element* $\underline{D}u \in D[u]$ *such that* $\mathcal{E}_p(u) = \|\underline{D}u\|^p_{L^p(\mathfrak{m})}$*. The element* $\underline{D}u$ *is called* minimal pseudo-gradient *of* u*.*

Proof. Observe that $D[u] \cap L^p(\mathfrak{m}) \neq \emptyset$ and that $\mathcal{E}_p(u) = \inf\big\{\|g\|^p_{L^p(\mathfrak{m})} : g \in D[u] \cap L^p(\mathfrak{m})\big\}$. Recall also that the set $D[u] \cap L^p(\mathfrak{m})$ is convex and closed by Remark 2.2. Since any nonempty convex closed subset of a uniformly convex Banach space admits a unique element of minimal norm, we get the statement. □

2.1.2 Local D-structures and calculus rules

In order to provide some calculus rules for minimal pseudo-gradients, we need to be sure that they depend only on the local behaviour of the function, in a suitable sense. For this reason, we propose various notions of locality for D-structures; some of them can be found in [GT01].

Definition 2.6 (Locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Fix* $p \in (1,\infty)$*. Then we define five notions of locality for* D*-structures on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*:*

L1 *If* $B \subseteq \mathrm{X}$ *is Borel and* $u \in \mathrm{S}^p(\mathrm{X})$ *is* $\mathfrak{m}$*-a.e. constant in* B*, then* $\mathcal{E}_p(u\,|B) = 0$*.*

L2 *If* $B \subseteq \mathrm{X}$ *is Borel and* $u \in \mathrm{S}^p(\mathrm{X})$ *is* $\mathfrak{m}$*-a.e. constant in* B*, then* $\underline{D}u = 0$ $\mathfrak{m}$*-a.e. in* B*.*

L3 *If* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $g \in D[u]$*, then* $\chi_{\{u>0\}}\, g \in D[u^+]$*.*

L4 *If* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $g_1, g_2 \in D[u]$*, then* $g_1 \wedge g_2 \in D[u]$*.*

L5 *If* $u \in \mathrm{S}^p(\mathrm{X})$*, then* $\underline{D}u \leq g$ $\mathfrak{m}$*-a.e. in* X *for every* $g \in D[u]$*.*

Remark 2.7 In the language of [GT01, Definition 1.11], the properties **L1** and **L3** correspond to *locality* and *strict locality*, respectively. ■

We now discuss the relations among the several notions of locality for D-structures:

Proposition 2.8 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $p \in (1,\infty)$*. Fix a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then the following implications hold:*

$$\begin{aligned} \mathbf{L3} \quad &\Longrightarrow \quad \mathbf{L2} \quad \Longrightarrow \quad \mathbf{L1}, \\ \mathbf{L4} \quad &\Longleftrightarrow \quad \mathbf{L5}, \\ \mathbf{L1}+\mathbf{L5} \quad &\Longrightarrow \quad \mathbf{L2}+\mathbf{L3}. \end{aligned} \tag{2.6}$$

Proof. We divide the proof into several steps:
L2 ⟹ L1. Suppose that a function $u \in \mathrm{S}^p(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant on some $B \subseteq \mathrm{X}$ Borel. Then it holds that $\mathcal{E}_p(u\,|B) \leq \int_B (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} = 0$ by **L2**, thus proving **L1**.
L3 ⟹ L2. Suppose that a function $u \in \mathrm{S}^p(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant on some Borel set $B \subseteq \mathrm{X}$. Take that constant $c \in \mathbb{R}$ for which $u = c$ holds $\mathfrak{m}$-a.e. in B. Since $\underline{D}u \in D[u-c] \cap D[c-u]$ by axiom **A2** and Remark 2.2, we deduce from **L3** that

$$\begin{aligned} \chi_{\{u>c\}}\, \underline{D}u &\in D\big[(u-c)^+\big], \\ \chi_{\{u<c\}}\, \underline{D}u &\in D\big[(c-u)^+\big]. \end{aligned}$$

Given that $u - c = (u-c)^+ - (c-u)^+$, by applying again axiom **A2** we see that

$$\chi_{\{u\neq c\}}\, \underline{D}u = \chi_{\{u>c\}}\, \underline{D}u + \chi_{\{u<c\}}\, \underline{D}u \in D[u-c] = D[u].$$

Hence the minimality of $\underline{D}u$ grants that

$$\int (\underline{D}u)^p \, d\mathfrak{m} \leq \int_{\{u \neq c\}} (\underline{D}u)^p \, d\mathfrak{m},$$

which implies that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $\{u = c\}$, so in particular $\mathfrak{m}$-a.e. in B. This means that the D-structure satisfies property **L2**, as required.
L4 $\Longrightarrow$ L5. We argue by contradiction: suppose the existence of $u \in \mathrm{S}^p(\mathrm{X})$ and $g \in D[u]$ such that $\mathfrak{m}\big(\{\underline{D}u > g\}\big) > 0$, whence $h \doteq \underline{D}u \wedge g \in L^p(\mathfrak{m})$ satisfies $\int h^p \, d\mathfrak{m} < \int (\underline{D}u)^p \, d\mathfrak{m}$. Since $h \in D[u]$ by **L4**, we deduce that $\mathcal{E}_p(u) < \int (\underline{D}u)^p \, d\mathfrak{m}$, thus getting a contradiction. Therefore property **L5** is verified.
L5 $\Longrightarrow$ L4. Let $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$ be fixed. Since $\underline{D}u \leq g_1$ and $\underline{D}u \leq g_2$ hold $\mathfrak{m}$-a.e. by **L5**, we see that $\underline{D}u \leq g_1 \wedge g_2$ holds $\mathfrak{m}$-a.e. as well. Therefore $g_1 \wedge g_2 \in D[u]$ by **A2**, whence accordingly **L4** is proven to be satisfied.
L1+L5 $\Longrightarrow$ L2+L3. First of all, we prove **L2**. Fix $u \in \mathrm{S}^p(\mathrm{X})$ and suppose that u is $\mathfrak{m}$-a.e. constant on some Borel subset B of X. Property **L1** grants the existence of $(g_n)_n \subseteq D[u]$ such that $\int_B (g_n)^p \, d\mathfrak{m} \to 0$. Hence **L5** tells us that $\int_B (\underline{D}u)^p \, d\mathfrak{m} \leq \lim_n \int_B (g_n)^p \, d\mathfrak{m} = 0$, which implies that the equality $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in B, thus yielding property **L2**.

We now show the validity of **L3**. Let $u \in \mathrm{S}^p(\mathrm{X})$ and $g \in D[u]$ be fixed. Given that one has $h = h \vee 0 \in D[u \vee 0] = D[u^+]$ for every $h \in D[u]$ by **A4** and Remark 2.2, we have that the inclusion $D[u] \subseteq D[u^+]$ is verified, thus in particular $u^+ \in \mathrm{S}^p(\mathrm{X})$. Given that $u^+ = 0$ $\mathfrak{m}$-a.e. in the set $\{u \leq 0\}$, one has that $\underline{D}u^+ = 0$ holds $\mathfrak{m}$-a.e. in $\{u \leq 0\}$ by **L2**. Hence for any $g \in D[u]$ we have $\underline{D}u^+ \leq \chi_{\{u>0\}} \, g$ by **L5**, which implies that $\chi_{\{u>0\}} \, g \in D[u^+]$ by **A2**. Therefore property **L3** is proved, as required. □

Definition 2.9 (Pointwise locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and fix any exponent* $p \in (1, \infty)$*. Then a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* pointwise local *provided it satisfies* **L1** *and* **L5** *(thus also* **L2**, **L3** *and* **L4** *by Proposition 2.8).*

We now recall other two notions of locality for D-structures that appeared in the literature:

Definition 2.10 (Strong locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Consider a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then we give the following definitions:*

i) *We say that* D *is* strongly local in the sense of Timoshin *provided*

$$\chi_{\{u_1<u_2\}} \, g_1 + \chi_{\{u_2<u_1\}} \, g_2 + \chi_{\{u_1=u_2\}} \, (g_1 \wedge g_2) \in D[u_1 \wedge u_2] \tag{2.7}$$

whenever $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$*,* $g_1 \in D[u_1]$ *and* $g_2 \in D[u_2]$*.*

ii) *We say that* D *is* strongly local in the sense of Shanmugalingam *provided*

$$\chi_B \, g_1 + \chi_{\mathrm{X} \setminus B} \, g_2 \in D[u_2] \quad \text{for every } g_1 \in D[u_1] \text{ and } g_2 \in D[u_2] \tag{2.8}$$

whenever $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$ *satisfy* $u_1 = u_2$ $\mathfrak{m}$*-a.e. on some Borel set* $B \subseteq \mathrm{X}$*.*

The above two notions of strong locality have been proposed in [Tim06] and [Sha09], respectively. We now prove that they are both equivalent to our pointwise locality property:

Lemma 2.11 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Fix any* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then the following are equivalent:*

i) *D is pointwise local.*

ii) *D is strongly local in the sense of Shanmugalingam.*

iii) *D is strongly local in the sense of Timoshin.*

Proof. We divide the proof into several steps:
i) $\Longrightarrow$ ii) Fix $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$ such that $u_1 = u_2$ $\mathfrak{m}$-a.e. on some $E \subseteq \mathrm{X}$ Borel. Pick $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Observe that $\underline{D}(u_2 - u_1) + g_1 \in D\big[(u_2 - u_1) + u_1\big] = D[u_2]$ by **A2**, so that we have $\big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2 \in D[u_2]$ by **L4**. Since $\underline{D}(u_2 - u_1) = 0$ $\mathfrak{m}$-a.e. on B by **L2**, we see that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \geq \big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2$ holds $\mathfrak{m}$-a.e. in X, whence accordingly we conclude that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u_2]$ by **A2**. This shows the validity of ii).
ii) $\Longrightarrow$ i) First of all, let us prove **L1**. Let $u \in \mathrm{S}^p(\mathrm{X})$ and $c \in \mathbb{R}$ satisfy $u = c$ $\mathfrak{m}$-a.e. on some Borel set $B \subseteq \mathrm{X}$. Given any $g \in D[u]$, we deduce from ii) that $\chi_{\mathrm{X}\setminus B}\, g \in D[u]$, thus accordingly $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\mathrm{X}\setminus B}\, g)^p \,\mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Let us denote $B \doteq \{g_1 \leq g_2\}$. Therefore ii) grants that $g_1 \wedge g_2 = \chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u]$, thus obtaining **L4**. By recalling Proposition 2.8, we conclude that D is pointwise local.
i) + ii) $\Longrightarrow$ iii) Fix $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$, $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Recall that $g_1 \vee g_2 \in D[u_1 \wedge u_2]$ by axiom **A4**. Hence by using property ii) twice we obtain that

$$\begin{aligned} &\chi_{\{u_1 \leq u_2\}}\, g_1 + \chi_{\{u_1 > u_2\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2],\\ &\chi_{\{u_2 \leq u_1\}}\, g_2 + \chi_{\{u_2 > u_1\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2]. \end{aligned} \tag{2.9}$$

The pointwise minimum between the two functions that are written in (2.9) – namely given by $\chi_{\{u_1 < u_2\}}\, g_1 + \chi_{\{u_2 < u_1\}}\, g_2 + \chi_{\{u_1 = u_2\}}\, (g_1 \wedge g_2)$ – belongs to the class $D[u_1 \wedge u_2]$ as well by property **L4**, thus showing iii).
iii) $\Longrightarrow$ i) First of all, let us prove **L1**. Fix a function $u \in \mathrm{S}^p(\mathrm{X})$ that is $\mathfrak{m}$-a.e. equal to some constant $c \in \mathbb{R}$ on a Borel set $B \subseteq \mathrm{X}$. By using iii) and the fact that $0 \in D[0]$, we have that

$$\begin{aligned} \chi_{\{u<c\}}\, g &\in D\big[(u - c) \wedge 0\big] = D\big[-(u - c)^+\big] = D\big[(u - c)^+\big],\\ \chi_{\{u>c\}}\, g &\in D\big[(c - u) \wedge 0\big] = D\big[-(c - u)^+\big] = D\big[(c - u)^+\big]. \end{aligned} \tag{2.10}$$

Since $u - c = (u - c)^+ - (c - u)^+$, we know from **A2** and (2.10) that

$$\chi_{\{u \neq c\}}\, g = \chi_{\{u<c\}}\, g + \chi_{\{u>c\}}\, g \in D[u - c] = D[u],$$

whence $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\{u \neq c\}}\, g)^p \,\mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Hence (2.7) with $u_1 = u_2 \doteq u$ simply reads as $g_1 \wedge g_2 \in D[u]$, which gives **L4**. This proves that D is pointwise local. □

Remark 2.12 (L1 does not imply L2) As we are going to show in the following example, it can happen that a D-structure satisfies property **L1** but not property **L2**.

Let $G = (V, E)$ be a locally finite connected graph. The distance $\mathsf{d}(x, y)$ between two vertices $x, y \in V$ is defined as the minimum length of a path joining x to y, while as a reference measure $\mathfrak{m}$ on V we choose the counting measure. Notice that any function $u : V \to \mathbb{R}$ is locally Lipschitz and that any bounded subset of V consists of finitely many points. Then we define a D-structure on the metric measure space $(V, \mathsf{d}, \mathfrak{m})$ in the following way:

$$D[u] \doteq \Big\{ g : V \to [0, +\infty] \;\Big|\; \big|u(x) - u(y)\big| \leq g(x) + g(y) \text{ for any } x, y \in V \text{ with } x \sim y \Big\} \tag{2.11}$$

for every $u : V \to \mathbb{R}$, where the notation $x \sim y$ indicates that x and y are adjacent vertices, i.e. that there exists an edge in E joining x to y.

We claim that D fulfils **L1**. To prove it, suppose that some function $u : \mathrm{X} \to \mathbb{R}$ is constant on some set $B \subseteq V$, say $u(x) = c$ for every $x \in B$. Define the function $g : V \to [0, +\infty)$ as

$$g(x) \doteq \begin{cases} 0 & \text{if } x \in B, \\ |c| + \big|u(x)\big| & \text{if } x \in V \setminus B. \end{cases}$$

Hence $g \in D[u]$ and $\int_B g^p \, d\mathfrak{m} = 0$, so that $\mathcal{E}_p(u\,|B) = 0$. This proves the validity of **L1**.

On the other hand, if V contains more than one vertex, then property **L2** is not satisfied. Indeed, consider any non-constant function $u : V \to \mathbb{R}$. Clearly any pseudo-gradient $g \in D[u]$ of u is not identically zero, thus there exists $x \in V$ such that $\underline{D}u(x) > 0$. Since u is trivially constant on the set $\{x\}$, we then conclude that property **L2** does not hold. ■

Hereafter, we shall focus our attention on the pointwise local D-structures. Under these locality assumptions, one can show the following calculus rules for minimal pseudo-gradients, whose proof is suitably adapted from analogous results that have been proved in [AGS14a]:

Proposition 2.13 (Calculus rules for $\underline{D}u$) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $p \in (1, \infty)$. *Consider a pointwise local* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then the following hold:*

i) *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and let* $N \subseteq \mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N) = 0$. *Then the equality* $\underline{D}u = 0$ *holds* $\mathfrak{m}$*-a.e. in* $u^{-1}(N)$.

ii) CHAIN RULE. *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}(\mathbb{R})$. *Then* $|\varphi'| \circ u \, \underline{D}u \in D[\varphi \circ u]$. *More precisely,* $\varphi \circ u \in \mathrm{S}^p(\mathrm{X})$ *and* $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$ *holds* $\mathfrak{m}$*-a.e. in* X.

iii) LEIBNIZ RULE. *Let* $u, v \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. *Then* $|u|\, \underline{D}v + |v|\, \underline{D}u \in D[uv]$. *In other words,* $uv \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ *and* $\underline{D}(uv) \leq |u|\, \underline{D}v + |v|\, \underline{D}u$ *holds* $\mathfrak{m}$*-a.e. in* X.

Proof. First of all, we just briefly describe the plan of the ensuing proof:

- STEP 1. We show the first statement of ii) for $u \in \mathrm{S}^p(\mathrm{X})$ and φ piecewise affine.
- STEP 2. We prove the first statement of ii) for $u \in \mathrm{S}^p(\mathrm{X})$ and $\varphi \in \mathrm{LIP}(\mathbb{R}) \cap C^1(\mathbb{R})$, by using STEP 1 together with an approximation argument.
- STEP 3. We get i) for N compact, as a consequence of STEP 2.
- STEP 4. We prove i) for N general, by using STEP 3 and an approximation argument.
- STEP 5. We obtain the first statement of ii) in full generality, again by using STEP 3.
- STEP 6. The second statement of ii) follows from i) and the first statement of ii).
- STEP 7. We deduce iii) from ii) when $u, v \geq c$ holds $\mathfrak{m}$-a.e. for some constant $c > 0$.
- STEP 8. We prove iii) in full generality, as a consequence of i) and STEP 7.

With this said, we can now pass to the proof of the statement:
STEP 1. First, consider φ affine, say $\varphi(t) = \alpha\, t + \beta$. Then $|\varphi'| \circ u \, \underline{D}u = |\alpha|\, \underline{D}u \in D[\varphi \circ u]$ by Remark 2.2 and **A2**. Now suppose that the function φ is piecewise affine, i.e. there exists

a sequence $(a_k)_{k\in\mathbb{Z}}\subseteq\mathbb{R}$, with $a_k<a_{k+1}$ for all $k\in\mathbb{Z}$ and $a_0=0$, such that each $\varphi|_{[a_k,a_{k+1}]}$ is an affine function. Let us denote $A_k\doteq u^{-1}\big([a_k,a_{k+1})\big)$ and $u_k\doteq(u\vee a_k)\wedge a_{k+1}$ for every index $k\in\mathbb{Z}$. By combining **L3** with the axioms **A2** and **A5**, we can see that $\chi_{A_k}\,\underline{D}u\in D[u_k]$ for every $k\in\mathbb{Z}$. Calling $\varphi_k:\mathbb{R}\to\mathbb{R}$ the affine function coinciding with φ on $[a_k,a_{k+1})$, we deduce from the previous case that $|\varphi_k'|\circ u_k\,\underline{D}u_k\in D[\varphi_k\circ u_k]=D[\varphi\circ u_k]$, whence we have that $|\varphi'|\circ u_k\,\chi_{A_k}\,\underline{D}u\in D[\varphi\circ u_k]$ by **L5**, **A2** and **L2**. Let us define $(v_n)_n\subseteq\mathrm{S}^p(\mathrm{X})$ as

$$v_n\doteq\varphi(0)+\sum_{k=0}^{n}\big(\varphi\circ u_k-\varphi(a_k)\big)+\sum_{k=-n}^{-1}\big(\varphi\circ u_k-\varphi(a_{k+1})\big)\qquad\text{for every } n\in\mathbb{N}.$$

Hence $g_n\doteq\sum_{k=-n}^{n}|\varphi'|\circ u_k\,\chi_{A_k}\,\underline{D}u\in D[v_n]$ for all $n\in\mathbb{N}$ by **A2** and Remark 2.2. Given that one has $v_n\to\varphi\circ u$ in $L^p_{\mathrm{loc}}(\mathfrak{m})$ and $g_n\to|\varphi'|\circ u\,\underline{D}u$ in $L^p(\mathfrak{m})$ as $n\to\infty$, we finally conclude that $|\varphi'|\circ u\,\underline{D}u\in D[\varphi\circ u]$, as required.

Step 2. We aim to prove the chain rule for $\varphi\in C^1(\mathbb{R})\cap\mathrm{LIP}(\mathbb{R})$. For any $n\in\mathbb{N}$, let us denote by φ_n the piecewise affine function interpolating the points $\big(k/2^n,\varphi(k/2^n)\big)$ with $k\in\mathbb{Z}$. We denote by $D\subseteq\mathbb{R}$ the countable set $\big\{k/2^n\,:\,k\in\mathbb{Z},\,n\in\mathbb{N}\big\}$. Therefore φ_n uniformly converges to φ and $\varphi_n'(t)\to\varphi'(t)$ for all $t\in\mathbb{R}\setminus D$. In particular, the functions $g_n\doteq|\varphi_n'|\circ u\,\underline{D}u$ converge $\mathfrak{m}$-a.e. to $|\varphi'|\circ u\,\underline{D}u$ by **L2**. Moreover, $\mathrm{Lip}(\varphi_n)\le\mathrm{Lip}(\varphi)$ for every $n\in\mathbb{N}$ by construction, so that $(g_n)_n$ is a bounded sequence in $L^p(\mathfrak{m})$. This implies that (up to a not relabeled subsequence) $g_n\rightharpoonup|\varphi'|\circ u\,\underline{D}u$ weakly in $L^p(\mathfrak{m})$. Now apply Mazur lemma: for any $n\in\mathbb{N}$, there exists $(\alpha_i^n)_{i=n}^{N_n}\subseteq[0,1]$ such that $\sum_{i=n}^{N_n}\alpha_i^n=1$ and $h_n\doteq\sum_{i=n}^{N_n}\alpha_i^n\,g_i\overset{n}{\to}|\varphi'|\circ u\,\underline{D}u$ strongly in $L^p(\mathfrak{m})$. Given that $g_n\in D[\varphi_n\circ u]$ for every $n\in\mathbb{N}$ by Step 1, we deduce from axiom **A2** that $h_n\in D[\psi_n\circ u]$ for every $n\in\mathbb{N}$, where $\psi_n\doteq\sum_{i=n}^{N_n}\alpha_i^n\,\varphi_i$. Finally, it clearly holds that $\psi_n\circ u\to\varphi\circ u$ in $L^p_{\mathrm{loc}}(\mathrm{X})$, whence $|\varphi'|\circ u\,\underline{D}u\in D[\varphi\circ u]$ by axiom **A5**.

Step 3. We claim that

$$\underline{D}u=0\quad\mathfrak{m}\text{-a.e. in } u^{-1}(K),\qquad\text{for every } K\subseteq\mathbb{R}\text{ compact with }\mathcal{L}^1(K)=0.\tag{2.12}$$

For any $n\in\mathbb{N}\setminus\{0\}$, define $\psi_n\doteq n\,\mathsf{d}(\cdot,K)\wedge 1$ and denote by φ_n the primitive of ψ_n such that $\varphi_n(0)=0$. Since each ψ_n is continuous and bounded, any function φ_n is of class C^1 and Lipschitz. By applying the dominated convergence theorem we see that the $\mathcal{L}^1$-measure of the ε-neighbourhood of K converges to 0 as $\varepsilon\searrow 0$, thus accordingly φ_n uniformly converges to $\mathrm{id}_{\mathbb{R}}$ as $n\to\infty$. This implies that $\varphi_n\circ u\to u$ in $L^p_{\mathrm{loc}}(\mathrm{X})$. Moreover, we know from Step 2 that $|\psi_n|\circ u\,\underline{D}u\in D[\varphi_n\circ u]$, thus also $\chi_{\mathrm{X}\setminus u^{-1}(K)}\,\underline{D}u\in D[\varphi_n\circ u]$. Hence $\chi_{\mathrm{X}\setminus u^{-1}(K)}\,\underline{D}u\in D[u]$ by **A5**, which forces the equality $\underline{D}u=0$ to hold $\mathfrak{m}$-a.e. in $u^{-1}(K)$, proving (2.12).

Step 4. We are in a position to prove i). Choose any $\mathfrak{m}'\in\mathscr{P}(\mathrm{X})$ such that $\mathfrak{m}\ll\mathfrak{m}'\ll\mathfrak{m}$ and call $\mu\doteq u_*\mathfrak{m}'$. Then μ is a Radon measure on $\mathbb{R}$, in particular it is inner regular. We can thus find an increasing sequence of compact sets $K_n\subseteq N$ such that $\mu\big(N\setminus\bigcup_n K_n\big)=0$. We already know from Step 3 that $\underline{D}u=0$ holds $\mathfrak{m}$-a.e. in $\bigcup_n u^{-1}(K_n)$. Since $u^{-1}(N)\setminus\bigcup_n u^{-1}(K_n)$ is $\mathfrak{m}$-negligible by definition of μ, we conclude that $\underline{D}u=0$ holds $\mathfrak{m}$-a.e. in $u^{-1}(N)$. This shows the validity of property i).

Step 5. We now prove ii). Let us fix $\varphi\in\mathrm{LIP}(\mathbb{R})$. Choose some standard Euclidean convolution kernels $(\rho_n)_n$, i.e. a sequence of nonnegative-valued functions $\rho_n\in C^\infty_c(\mathbb{R})$ such that $\mathrm{spt}(\rho_n)\subseteq B_{1/n}(0)$ and $\int\rho_n(t)\,\mathrm{d}t=1$. Define $\varphi_n\doteq\varphi*\rho_n$ for all $n\in\mathbb{N}$. Then $\varphi_n\to\varphi$ uniformly and $\varphi_n'\to\varphi'$ pointwise $\mathcal{L}^1$-a.e., whence accordingly

$$\begin{aligned}\varphi_n\circ u&\to\varphi\circ u&&\text{in } L^p_{\mathrm{loc}}(\mathfrak{m}),\\|\varphi_n'|\circ u\,\underline{D}u&\to|\varphi'|\circ u\,\underline{D}u&&\text{pointwise }\mathfrak{m}\text{-a.e. in X.}\end{aligned}$$

Since $|\varphi_n'|\circ u\,\underline{D}u \leq \mathrm{Lip}(\varphi)\,\underline{D}u$ holds for all $n\in\mathbb{N}$, there exists a (not relabeled) subsequence such that $|\varphi_n'|\circ u\,\underline{D}u \rightharpoonup |\varphi'|\circ u\,\underline{D}u$ weakly in $L^p(\mathfrak{m})$. We know that $|\varphi_n'|\circ u\,\underline{D}u \in D[\varphi_n\circ u]$ for all $n\in\mathbb{N}$ because the chain rule holds for all $\varphi_n\in C^1(\mathbb{R})\cap\mathrm{LIP}(\mathbb{R})$, hence by combining Mazur lemma and **A5** as in STEP 2 we obtain that $|\varphi'|\circ u\,\underline{D}u\in D[\varphi\circ u]$, so that $\varphi\circ u\in \mathrm{S}^p(\mathrm{X})$ and the inequality $\underline{D}(\varphi\circ u)\leq|\varphi'|\circ u\,\underline{D}u$ holds $\mathfrak{m}$-a.e. in X.

STEP 6. We conclude the proof of ii) by showing that one actually has $\underline{D}(\varphi\circ u)=|\varphi'|\circ u\,\underline{D}u$. We can suppose without loss of generality that $\mathrm{Lip}(\varphi)=1$. Let us define the functions $\psi_\pm$ as $\psi_\pm(t)\doteq\pm t-\varphi(t)$ for all $t\in\mathbb{R}$, respectively. Then it $\mathfrak{m}$-a.e. holds in $u^{-1}\big(\{\pm\varphi'\geq 0\}\big)$ that

$$\underline{D}u=\underline{D}(\pm u)\leq\underline{D}(\varphi\circ u)+\underline{D}(\psi_\pm\circ u)\leq\big(|\varphi'|\circ u+|\psi_\pm'|\circ u\big)\,\underline{D}u=\underline{D}u,$$

which forces the equality $\underline{D}(\varphi\circ u)=\pm\varphi'\circ u\,\underline{D}u$ to hold $\mathfrak{m}$-a.e. in the set $u^{-1}\big(\{\pm\varphi'\geq0\}\big)$. This grants the validity of $\underline{D}(\varphi\circ u)=|\varphi'|\circ u\,\underline{D}u$, thus completing the proof of item ii).

STEP 7. We show iii) for the case in which $u,v\geq c$ is satisfied $\mathfrak{m}$-a.e. in X, for some $c>0$. Call $\varepsilon\doteq\min\{c,c^2\}$ and note that the function log is Lipschitz on the interval $[\varepsilon,+\infty)$, then choose any Lipschitz function $\varphi:\mathbb{R}\to\mathbb{R}$ that coincides with log on $[\varepsilon,+\infty)$. Now call C the constant $\log\big(\|uv\|_{L^\infty(\mathfrak{m})}\big)$ and choose a Lipschitz function $\psi:\mathbb{R}\to\mathbb{R}$ such that $\psi=\exp$ on the interval $[\log\varepsilon,C]$. By applying twice the chain rule ii), we thus deduce that $uv\in\mathrm{S}^p(\mathrm{X})$ and the $\mathfrak{m}$-a.e. inequalities

$$\begin{aligned}\underline{D}(uv)&\leq|\psi'|\circ\varphi\circ(uv)\,\underline{D}\big(\varphi\circ(uv)\big)\leq|uv|\,\big(\underline{D}\log u+\underline{D}\log v\big)\\&=|uv|\left(\frac{\underline{D}u}{|u|}+\frac{\underline{D}v}{|v|}\right)=|u|\,\underline{D}v+|v|\,\underline{D}u.\end{aligned}$$

Therefore the Leibniz rule iii) is verified under the additional assumption that $u,v\geq c>0$.

STEP 8. We conclude by proving item iii) for general $u,v\in\mathrm{S}^p(\mathrm{X})\cap L^\infty(\mathfrak{m})$. Given any $n\in\mathbb{N}$ and $k\in\mathbb{Z}$, let us denote $I_{n,k}\doteq\big[k/n,(k+1)/n\big)$. Call $\varphi_{n,k}:\mathbb{R}\to\mathbb{R}$ the continuous function that is the identity on $I_{n,k}$ and constant elsewhere. For any $n\in\mathbb{N}$, let us define

$$\begin{aligned}u_{n,k}&\doteq u-\frac{k-1}{n},&\tilde u_{n,k}&\doteq\varphi_{n,k}\circ u-\frac{k-1}{n}&&\text{for all }k\in\mathbb{Z},\\v_{n,\ell}&\doteq v-\frac{\ell-1}{n},&\tilde v_{n,\ell}&\doteq\varphi_{n,\ell}\circ v-\frac{\ell-1}{n}&&\text{for all }\ell\in\mathbb{Z}.\end{aligned}$$

Notice that the equalities $u_{n,k}=\tilde u_{n,k}$ and $v_{n,\ell}=\tilde v_{n,\ell}$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively. Hence $\underline{D}u_{n,k}=\underline{D}\tilde u_{n,k}=\underline{D}u$ and $\underline{D}v_{n,\ell}=\underline{D}\tilde v_{n,\ell}=\underline{D}v$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively, but we also have that

$$\underline{D}(u_{n,k}\,v_{n,\ell})=\underline{D}(\tilde u_{n,k}\,\tilde v_{n,\ell})\quad\text{ is verified }\mathfrak{m}\text{-a.e. in }u^{-1}(I_{n,k})\cap v^{-1}(I_{n,\ell}).$$

Moreover, we have the $\mathfrak{m}$-a.e. inequalities $1/n\leq\tilde u_{n,k},\tilde v_{n,\ell}\leq 2/n$ by construction. Therefore for any $k,\ell\in\mathbb{Z}$ it holds $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})\cap v^{-1}(I_{n,\ell})$ that

$$\begin{aligned}\underline{D}(uv)&\leq\underline{D}(\tilde u_{n,k}\,\tilde v_{n,\ell})+\frac{|k-1|}{n}\,\underline{D}v_{n,\ell}+\frac{|\ell-1|}{n}\,\underline{D}u_{n,k}\\&\leq|\tilde v_{n,\ell}|\,\underline{D}\tilde u_{n,k}+|\tilde u_{n,k}|\,\underline{D}\tilde v_{n,\ell}+\frac{|k-1|}{n}\,\underline{D}v_{n,\ell}+\frac{|\ell-1|}{n}\,\underline{D}u_{n,k}\\&\leq\left(|v|+\frac{4}{n}\right)\underline{D}u+\left(|u|+\frac{4}{n}\right)\underline{D}v,\end{aligned}$$

where the second inequality is a consequence of the case $u,v\geq c>0$, treated in STEP 7. This implies that the inequality $\underline{D}(uv)\leq|u|\,\underline{D}v+|v|\,\underline{D}u+4\,(\underline{D}u+\underline{D}v)/n$ holds $\mathfrak{m}$-a.e. in X. Given that $n\in\mathbb{N}$ is arbitrary, the Leibniz rule iii) follows. □

2.2 Sobolev spaces via test plans

The aim of the present section is to briefly illustrate the notion of Sobolev space $W^{1,2}(\mathrm{X})$ for metric measure spaces that has been introduced in [AGS14a]; unless otherwise specified, the whole material of this section can be found therein.

A key ingredient in the definition of the space $W^{1,2}(\mathrm{X})$ is the concept of *test plan*, which is a 'weighted selection' of absolutely continuous curves – with bounded kinetic energy – that do not concentrate mass too much (see Definition 2.14). Nevertheless, test plans are not mere auxiliary tools that are used to define the Sobolev space; they rather play a central role in the structure theory of RCD spaces. For instance, in Chapter 6 they will constitute – in a sense – a generalisation of the notion of Lipschitz curve.

We adopt the following notation: the 1-dimensional Lebesgue measure restricted to the unit interval $[0,1]$ will be denoted by

$$\mathcal{L}_1 \doteq \mathcal{L}^1|_{[0,1]}. \tag{2.13}$$

We can now give the definition of test plan:

Definition 2.14 (Test plan) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then we say that a Borel probability measure* $\boldsymbol{\pi}\in\mathscr{P}\big(\Gamma(\mathrm{X})\big)$ *is a* test plan *on* X *provided the following hold:*

i) *There exists a constant* $\mathsf{C}>0$ *such that*

$$(\mathrm{e}_t)_*\boldsymbol{\pi}\leq \mathsf{C}\,\mathfrak{m} \qquad \text{for every } t\in[0,1]. \tag{2.14}$$

We denote by $\mathsf{C}(\boldsymbol{\pi})$ *the smallest constant* $\mathsf{C}>0$ *for which* (2.14) *is satisfied.*

ii) *The measure* $\boldsymbol{\pi}$ *has* finite kinetic energy, *i.e. it is concentrated on* $AC\big([0,1],\mathrm{X}\big)$ *and*

$$\mathsf{ms}\in L^2(\boldsymbol{\pi}\times\mathcal{L}_1), \tag{2.15}$$

where ms *denotes the metric speed operator defined in Definition 1.4.*

Theorem 2.15 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Let* $\boldsymbol{\pi}$ *be a test plan on* X. *Then*

$$[0,1]\ni t\longmapsto f\circ\mathrm{e}_t\in L^1(\boldsymbol{\pi}) \text{ is continuous} \qquad \text{for every } f\in L^1(\mathfrak{m}). \tag{2.16}$$

In particular, the function $[0,1]\ni t\mapsto\int f\circ\mathrm{e}_t\,\mathrm{d}\boldsymbol{\pi}$ *is continuous for every* $f\in L^1(\mathfrak{m})$.

Proof. First of all, we claim that

$$\lim_{s\to t}\int|f\circ\mathrm{e}_s-f\circ\mathrm{e}_t|\,\mathrm{d}\boldsymbol{\pi}=0 \qquad \text{for all } f\in C_{\mathrm{b}}(\mathrm{X})\cap L^1(\mathfrak{m}) \text{ and } t\in[0,1]. \tag{2.17}$$

To prove it, note that $|f\circ\mathrm{e}_s-f\circ\mathrm{e}_t|(\gamma)\leq 2\,\|f\|_{L^\infty(\mathfrak{m})}$ for every $\gamma\in\Gamma(\mathrm{X})$ and $t,s\in[0,1]$. Moreover, $\big|f(\gamma_s)-f(\gamma_t)\big|\to 0$ as $s\to t$ by continuity, for every $\gamma\in\Gamma(\mathrm{X})$ and $t\in[0,1]$. Hence we obtain (2.17) as a consequence of the dominated convergence theorem. Observe also that

$$L^1(\mathfrak{m})\ni f\longmapsto f\circ\mathrm{e}_t\in L^1(\boldsymbol{\pi}) \text{ is linear and continuous} \qquad \text{for every } t\in[0,1]. \tag{2.18}$$

Indeed, $\int |f \circ \mathrm{e}_t| \,\mathrm{d}\boldsymbol{\pi} \leq \mathsf{C}(\boldsymbol{\pi}) \int |f| \,\mathrm{d}\mathfrak{m}$ is satisfied for every $f \in L^1(\mathfrak{m})$. Now fix $f \in L^1(\mathfrak{m})$. Choose a sequence $(f_n)_n \subseteq C_{\mathrm{b}}(\mathrm{X}) \cap L^1(\mathfrak{m})$ that converges to f with respect to the L^1-norm. Given any $t \in [0,1]$ and $n \in \mathbb{N}$, we have that (2.17) and (2.18) yield

$$\begin{aligned} \varlimsup_{s \to t} \int |f \circ \mathrm{e}_s - f \circ \mathrm{e}_t| \,\mathrm{d}\boldsymbol{\pi} &\leq 2\,\mathsf{C}(\boldsymbol{\pi})\, \|f - f_n\|_{L^1(\mathfrak{m})} + \varlimsup_{s \to t} \int |f_n \circ \mathrm{e}_s - f_n \circ \mathrm{e}_t| \,\mathrm{d}\boldsymbol{\pi} \\ &= 2\,\mathsf{C}(\boldsymbol{\pi})\, \|f - f_n\|_{L^1(\mathfrak{m})}. \end{aligned} \tag{2.19}$$

By letting $n \to \infty$ in (2.19) we finally conclude that $\int |f \circ \mathrm{e}_s - f \circ \mathrm{e}_t| \,\mathrm{d}\boldsymbol{\pi} \to 0$ as $s \to t$, which proves (2.16). To prove the last statement, just observe that the operator

$$L^1(\boldsymbol{\pi}) \ni g \longmapsto \int g \,\mathrm{d}\boldsymbol{\pi} \in \mathbb{R}$$

is (linear and) continuous. This completes the proof of the theorem. □

Remark 2.16 Consider the continuous map $\mathrm{e} : (\gamma, t) \mapsto \gamma_t$, introduced in (1.7). Given any Borel function $f : \mathrm{X} \to \mathbb{R}$, we have that $f \circ \mathrm{e}$ is Borel as well. Moreover, observe that

$$(\boldsymbol{\pi} \times \mathcal{L}_1)\big(\mathrm{e}^{-1}(A)\big) = \int_0^1 \boldsymbol{\pi}\big(\mathrm{e}_t^{-1}(A)\big) \,\mathrm{d}t \leq \mathsf{C}(\boldsymbol{\pi})\, \mathfrak{m}(A) \qquad \text{for every } A \in \mathscr{B}(\mathrm{X})$$

by Fubini theorem, in other words it holds that $\mathrm{e}_*(\boldsymbol{\pi} \times \mathcal{L}_1) \leq \mathsf{C}(\boldsymbol{\pi})\, \mathfrak{m}$. Therefore one has that the composition $f \circ \mathrm{e} \in L^0(\boldsymbol{\pi} \times \mathcal{L}_1)$ is well-defined for any $f \in L^0(\mathfrak{m})$. ■

With the notion of test plan at disposal, we can give the definition of weak upper gradient:

Definition 2.17 (Weak upper gradient) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Fix a Borel function* $f : \mathrm{X} \to \mathbb{R}$. *Then we say that* $G \in L^2(\mathfrak{m})^+$ *is a* weak upper gradient *of* f *if*

$$\int \big|f(\gamma_1) - f(\gamma_0)\big| \,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \iint_0^1 G(\gamma_t)\, |\dot{\gamma}_t| \,\mathrm{d}t \,\mathrm{d}\boldsymbol{\pi}(\gamma) \qquad \textit{for every test plan } \boldsymbol{\pi} \textit{ on } \mathrm{X}. \tag{2.20}$$

Remark 2.18 The term 'gradient' for G is slightly inappropriate, since it is an object in duality with curves and accordingly it behaves like the 'modulus of the differential'. ■

The following theorem – for whose proof we refer e.g. to [Pas18, Theorem 7.7] – provides us with several equivalent definitions of weak upper gradient:

Theorem 2.19 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Fix a Borel function* $f : \mathrm{X} \to \mathbb{R}$. *Let* $G \in L^2(\mathfrak{m})^+$ *be given. Then the following are equivalent:*

i) *The function* G *is a weak upper gradient of* f.

ii) *For every test plan* $\boldsymbol{\pi}$ *on* X *and for every* $t, s \in [0,1]$ *with* $s < t$, *it holds that*

$$\big|f(\gamma_t) - f(\gamma_s)\big| \leq \int_s^t G(\gamma_r)\, |\dot{\gamma}_r| \,\mathrm{d}r \qquad \textit{for } \boldsymbol{\pi}\textit{-a.e. } \gamma. \tag{2.21}$$

iii) *Given any test plan* $\boldsymbol{\pi}$ *on* X, *it holds that:*

a) *The map* $[0,1] \ni t \mapsto f \circ \mathrm{e}_t - f \circ \mathrm{e}_0 \in L^1(\boldsymbol{\pi})$ *is absolutely continuous.*

b) *For a.e.* $t \in [0,1]$, *the* $L^1(\boldsymbol{\pi})$*-limit* $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t \doteq \lim_{h \to 0} \frac{f \circ \mathrm{e}_{t+h} - f \circ \mathrm{e}_t}{h}$ *exists.*

c) *The inequality* $\left|\mathsf{Der}_{\boldsymbol{\pi}}(f)_t\right| \leq G(\gamma_t)\,|\dot\gamma_t|$ *is satisfied for* $(\boldsymbol{\pi}\times\mathcal{L}_1)$*-a.e.* (γ,t).

iv) *Given any test plan* $\boldsymbol{\pi}$ *on* X, *we have for* $\boldsymbol{\pi}$*-a.e.* γ *that the function* $f\circ\gamma$ *belongs to the space* $W^{1,1}(0,1)$ *and satisfies the inequality* $\left|(f\circ\gamma)'_t\right| \leq G(\gamma_t)\,|\dot\gamma_t|$ *for a.e.* $t\in[0,1]$.

If the above hold, then the equality $\mathsf{Der}_{\boldsymbol{\pi}}(f)_t(\gamma) = (f\circ\gamma)'_t$ *is verified for* $(\boldsymbol{\pi}\times\mathcal{L}_1)$*-a.e.* (γ,t).

We introduce the notion of Sobolev class, following the original presentation of [AGS14a]:

Definition 2.20 (Sobolev class) *We define the* Sobolev class $\mathrm{S}^2(\mathrm{X})$ *as the space of all Borel functions* $f:\mathrm{X}\to\mathbb{R}$ *admitting a weak upper gradient.*

It immediately follows from item iii) of Theorem 2.19 that $G_1\wedge G_2$ is a weak upper gradient of f whenever G_1, G_2 are weak upper gradients of f. This grants that there exists a (unique) weak upper gradient of f that is minimal in the $\mathfrak{m}$-a.e. sense:

Definition 2.21 (Minimal weak upper gradient) *Let* $f\in\mathrm{S}^2(\mathrm{X})$ *be a Sobolev function. Then we denote by* $|Df|\in L^2(\mathfrak{m})$ *the* minimal weak upper gradient *of* f.

In the following result, we collect the main properties of minimal weak upper gradients:

Theorem 2.22 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then the following hold:*

i) Lower semicontinuity. *Let* $(f_n)_n\subseteq\mathrm{S}^2(\mathrm{X})$ *satisfy* $f_n\to f$ *in the* $\mathfrak{m}$*-a.e. sense, for some Borel function* $f:\mathrm{X}\to\mathbb{R}$. *Suppose* $|Df_n|\rightharpoonup G$ *in the weak topology of* $L^2(\mathfrak{m})$, *for some* $G\in L^2(\mathfrak{m})$. *Then* $f\in\mathrm{S}^2(\mathrm{X})$ *and the inequality* $|Df|\leq G$ *holds* $\mathfrak{m}$*-a.e. in* X.

ii) Subadditivity. *Let* $f,g\in\mathrm{S}^2(\mathrm{X})$ *and* $\alpha,\beta\in\mathbb{R}$ *be given. Then* $\alpha\,f+\beta\,g\in\mathrm{S}^2(\mathrm{X})$ *and*

$$\left|D(\alpha\,f+\beta\,g)\right| \leq |\alpha|\,|Df| + |\beta|\,|Dg| \qquad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{2.22}$$

iii) Behaviour of Lipschitz functions. *Let* $f\in\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$. *Then* $f\in\mathrm{S}^2(\mathrm{X})$ *and*

$$|Df|\leq \mathrm{lip}(f) \qquad \textit{holds } \mathfrak{m}\textit{-a.e. in } \mathrm{X}. \tag{2.23}$$

Remark 2.23 Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a *PI space*, i.e. a doubling metric measure space supporting a weak $(1,2)$-Poincaré inequality. Then

$$|Df| = \mathrm{lip}(f) \quad \mathfrak{m}\text{-a.e. in X} \qquad \text{for every } f\in\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})\subseteq\mathrm{S}^2(\mathrm{X}). \tag{2.24}$$

This deep result has been proved in [Che99]. We will use it in Section 5.2. ■

By considering the class of all 2-integrable functions admitting a weak upper gradient, we obtain the notion of Sobolev space $W^{1,2}(\mathrm{X})$:

Definition 2.24 (Sobolev space) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then we define*

$$W^{1,2}(\mathrm{X}) \doteq L^2(\mathfrak{m})\cap\mathrm{S}^2(\mathrm{X}). \tag{2.25}$$

We endow the Sobolev space $W^{1,2}(\mathrm{X})$ *with the following norm:*

$$\|f\|_{W^{1,2}(\mathrm{X})} \doteq \sqrt{\|f\|^2_{L^2(\mathfrak{m})} + \big\||Df|\big\|^2_{L^2(\mathfrak{m})}} \qquad \textit{for every } f\in W^{1,2}(\mathrm{X}). \tag{2.26}$$

It turns out that $W^{1,2}(\mathrm{X})$ is a Banach space, but in general it is not a Hilbert space.

Definition 2.25 (Infinitesimal Hilbertianity) *A metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* infinitesimally Hilbertian *provided its associated Sobolev space* $W^{1,2}(\mathrm{X})$ *is Hilbert.*

Remark 2.26 (Consistency on Finsler manifolds) If X is a smooth Finsler manifold, then $W^{1,2}(\mathrm{X})$ concides with the Sobolev space defined via charts and $|Df|$ is a.e. equal to the norm of the distributional differential of f. ■

As it has been proved in [AGS13], Lipschitz functions are dense in the Sobolev space:

Theorem 2.27 (Density in energy) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then the space* $\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ *is* dense in energy *in* $W^{1,2}(\mathrm{X})$*, i.e. for every* $f \in W^{1,2}(\mathrm{X})$ *there exists a sequence* $(f_n)_n \subseteq \mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ *such that* $f_n \to f$ *and* $\mathrm{lip}_a(f_n) \to |Df|$ *in* $L^2(\mathfrak{m})$.

In particular, if the Sobolev space $W^{1,2}(\mathrm{X})$ *is reflexive, then* $\mathrm{LIP}_{\mathrm{bs}}(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})$ *with respect to the* $W^{1,2}(\mathrm{X})$*-norm.*

By recalling item i) of Theorem 2.22, one can readily prove that the sequence $(f_n)_n$ in Theorem 2.27 can be chosen so that also $|Df_n| \to |Df|$ and $\mathrm{lip}(f_n) \to |Df|$ in $L^2(\mathfrak{m})$.

A sufficient condition for the reflexivity of the Sobolev space is provided by the following theorem, which has been proved in [ACDM12]:

Theorem 2.28 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space such that* (X, d) *is doubling. Then the Sobolev space* $W^{1,2}(\mathrm{X})$ *is reflexive.*

Furthermore, it is proven – again in the paper [ACDM12] – that the implication

$$W^{1,2}(\mathrm{X}) \ \text{reflexive} \quad \Longrightarrow \quad W^{1,2}(\mathrm{X}) \ \text{separable} \tag{2.27}$$

occurs on any metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$.

Remark 2.29 By suitably adapting the definition of weak upper gradient, it is possible to define the Sobolev space $W^{1,p}(\mathrm{X})$ for any exponent $p \in (1, \infty)$. We point out that – as shown in [DMS15] – the minimal weak upper gradient might depend on p.

Nevertheless, this cannot happen whenever $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is an RCD space (whose definition will be given later on, namely in Subsection 4.2.1), as proven in [GH16]. ■

Remark 2.30 (Relation with the theory of D-structures) It can be readily checked that the notion of Sobolev space via test plan fits in the framework of D-structures, described in Section 2.1. By looking at weak upper gradients, we can indeed define a pointwise local D-structure – denoted by D_{wug} – on any metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. In particular, for any $f \in \mathrm{S}^2(\mathrm{X}) \cap L^2_{\mathrm{loc}}(\mathrm{X})$ we have that $|Df|$ is the minimal pseudo-gradient of f and that

$$D_{\mathrm{wug}}[f] = \Big\{ g \in L^0(\mathfrak{m})^+ \ \Big| \ g \geq |Df| \ \ \mathfrak{m}\text{-a.e. on X} \Big\}.$$

Hence the following calculus rules for minimal weak upper gradients can be deduced from Proposition 2.13 (at least for functions in $\mathrm{S}^2(\mathrm{X}) \cap L^2_{\mathrm{loc}}(\mathrm{X})$; for the general case of functions in $\mathrm{S}^2(\mathrm{X})$ the proof has been carried out in [AGS14a]). ■

Proposition 2.31 (Calculus rules for $|Df|$) *The following properties hold:*

i) LOCALITY. *Let $f \in \mathrm{S}^2(\mathrm{X})$ and let $N \subseteq \mathbb{R}$ be a Borel set with $\mathcal{L}^1(N) = 0$. Then the equality $|Df| = 0$ holds $\mathfrak{m}$-a.e. in $f^{-1}(N)$.*

ii) CHAIN RULE. *Let $f \in \mathrm{S}^2(\mathrm{X})$ and $\varphi \in \mathrm{LIP}(\mathbb{R})$ be given. Then $\varphi \circ f \in \mathrm{S}^2(\mathrm{X})$ and the equality $\big|D(\varphi \circ f)\big| = |\varphi'| \circ f\,|Df|$ holds $\mathfrak{m}$-a.e. in X.*

iii) LEIBNIZ RULE. *Let $f, g \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ be given. Then $fg \in \mathrm{S}^2(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and the inequality $\big|D(fg)\big| \leq |f|\,|Dg| + |g|\,|Df|$ holds $\mathfrak{m}$-a.e. in X.*

3

The language of normed modules

Contents

In order to develop a differential structure over an abstract metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, a crucial role is played by the concept of *normed module*. Such notion – which can be regarded as a generalisation of the 'space of measurable sections of some vector bundle' – has been proposed by N. Gigli in [Gig17b] and further refined in [Gig17a]. Technically speaking, it is a variant of a similar notion that has been introduced by N. Weaver [Wea00], who was in turn inspired by the papers [Sau89, Sau90] of J.-L. Sauvageot.

In an informal way, an $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module* $\mathscr{M}$ over the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is an object that is composed of the following structures:

- Algebraic. It is a module over the commutative ring $L^\infty(\mathfrak{m})$.
- Geometric. It is endowed with a *pointwise norm* operator $|\cdot|:\mathscr{M}\to L^2(\mathfrak{m})$, which 'fiberwise' behaves like a norm, in some suitable $\mathfrak{m}$-a.e. sense.
- Analytic. It is a Banach space if endowed with the norm $\mathscr{M}\ni v\mapsto \big\||v|\big\|_{L^2(\mathfrak{m})}$.

A fundamental example of L^2-normed L^∞-module – which actually served as a motivation for its axiomatisation – is the space $L^2(TM)$ of 2-integrable vector fields on a Riemannian manifold (M, g, Vol), i.e. the space of all L^2-sections of the tangent bundle TM of M. In this case, the normed module structure is the following: given any $f \in L^\infty(\mathrm{Vol})$ and $v \in L^2(TM)$, we define $f \cdot v \in L^2(TM)$ as $(f \cdot v)(x) \doteq f(x)\, v(x) \in T_xM$ for Vol-a.e. $x \in M$; moreover, the pointwise norm of $v \in L^2(TM)$ is given by $|v|(x) \doteq \big\|v(x)\big\|_{T_xM}$ for Vol-a.e. $x \in M$.

Sometimes it will be convenient to work with objects not satisfying any integrability requirement; this led to the notion of $L^0(\mathfrak{m})$-*normed* $L^0(\mathfrak{m})$-*module*. Anyway, the two concepts of normed modules are strictly related (cf. Propositions 3.7, 3.8 and Theorem 3.69).

The chapter is organised as follows: in Section 3.1 we introduce the normed modules and their basic properties, we show that on normed modules there is a well-defined notion of *local dimension* (the $L^0(\mathfrak{m})$-module structure is enough for this; cf. [LP18, Subsection 1.1]), finally we explain how one can build duals, pullbacks and tensor products of normed modules and investigate their main features. Such discussion is mostly taken from [Gig17a].

In Section 3.2 we shall prove that any 'locally finitely-generated' normed module (called *proper*, see Definition 3.18 for the precise formulation of such concept) can be viewed as the space of sections of some 'measurable Banach bundle' with finite-dimensional fibers. We will give the definition of such an object (cf. Definition 3.42) and discuss its properties, then we will equip the space of its measurable sections with an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module structure (thus obtaining the so-called 'section functor', see Definition 3.50), finally we will prove the equivalence between proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules and measurable Banach bundles via the section functor (the *Serre-Swan theorem*, cf. Theorem 3.54). The whole material contained in this section can be found in the paper [LP18].

3.1 Abstract theory of normed modules

3.1.1 Definition of normed module

Consider a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, which will remain fixed for the whole subsection.

Definition 3.1 (L^2-normed L^∞-module) *We define an* $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module *as any structure* $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot, |\cdot|\big)$ *with these properties:*

i) $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}\big)$ *is a Banach space.*

ii) $(\mathscr{M}, \cdot)$ *is a module over the commutative ring* $L^\infty(\mathfrak{m})$*, i.e.* $\cdot : L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ *is a bilinear operator – called* multiplication by $L^\infty(\mathfrak{m})$-functions *– which satisfies*

$$\begin{aligned} f \cdot (g \cdot v) &= (fg) \cdot v && \text{for every } f, g \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \\ \hat{1} \cdot v &= v && \text{for every } v \in \mathscr{M}, \end{aligned} \tag{3.1}$$

where $\hat{1} \in L^\infty(\mathfrak{m})$ *denotes the function identically equal to* 1*.*

iii) *The operator* $|\cdot| : \mathscr{M} \to L^2(\mathfrak{m})^+$*, called* pointwise norm*, satisfies*

$$\begin{aligned} |f \cdot v| &= |f||v| && \mathfrak{m}\text{-a.e. for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \\ \|v\|_{\mathscr{M}} &= \big\||v|\big\|_{L^2(\mathfrak{m})} && \text{for every } v \in \mathscr{M}. \end{aligned} \tag{3.2}$$

For the sake of simplicity, we shall typically write fv instead of $f \cdot v$. Given any $v, w \in \mathscr{M}$ and a Borel set $E \subseteq \mathrm{X}$, we say that $v = w$ holds $\mathfrak{m}$-a.e. on E provided $\chi_E (v - w) = 0$, or equivalently $|v - w| = 0$ is satisfied $\mathfrak{m}$-a.e. on E. Moreover, we define the space $\mathscr{M}_{|E}$ as

$$\mathscr{M}_{|E} \doteq \{\chi_E \, v \mid v \in \mathscr{M}\}. \tag{3.3}$$

Then $\mathscr{M}_{|E}$ is an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module as well and is called *restriction* of $\mathscr{M}$ to E.

It readily follows from the very definition of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module that one has

$$\begin{aligned} |\lambda v| &= |\lambda||v|, \\ |v + w| &\leq |v| + |w| \end{aligned} \tag{3.4}$$

in the $\mathfrak{m}$-a.e. sense for every $\lambda \in \mathbb{R}$ and $v, w \in \mathscr{M}$.

Proposition 3.2 *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then $\cdot : L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ and $|\cdot| : \mathscr{M} \to L^2(\mathfrak{m})$ are continuous maps. In particular, $\mathscr{M}$ is a topological $L^\infty(\mathfrak{m})$-module.*

Proof. It follows from (3.2) that $\|fv\|_{\mathscr{M}} \leq \|f\|_{L^\infty(\mathfrak{m})} \|v\|_{\mathscr{M}}$ and $\big\||v - w|\big\|_{L^2(\mathfrak{m})} = \|v - w\|_{\mathscr{M}}$ hold for every $f \in L^\infty(\mathfrak{m})$ and $v, w \in \mathscr{M}$, thus proving continuity of $\cdot : L^\infty(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ and $|\cdot| : \mathscr{M} \to L^2(\mathfrak{m})$, respectively. □

It is often convenient to deal with objects having no integrability assumption:

Definition 3.3 (L^0-normed L^0-module) *We define an* $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module *as any structure $\big(\mathscr{M}^0, \tau, \cdot, |\cdot|\big)$ with these properties:*

i) *$(\mathscr{M}^0, \tau)$ is a topological vector space.*

ii) *$(\mathscr{M}^0, \cdot)$ is a module over the commutative ring $L^0(\mathfrak{m})$, i.e. $\cdot : L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}^0$ is a bilinear operator – called* multiplication by $L^0(\mathfrak{m})$-functions *– which satisfies*

$$\begin{aligned} f \cdot (g \cdot v) &= (fg) \cdot v && \text{for every } f, g \in L^0(\mathfrak{m}) \text{ and } v \in \mathscr{M}^0, \\ \hat{1} \cdot v &= v && \text{for every } v \in \mathscr{M}^0. \end{aligned} \tag{3.5}$$

iii) *The operator $|\cdot| : \mathscr{M}^0 \to L^0(\mathfrak{m})$, called* pointwise norm, *satisfies*

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathscr{M}^0, \text{ with equality if and only if } v = 0, \\ |f \cdot v| &= |f||v| && \text{for every } f \in L^0(\mathfrak{m}) \text{ and } v \in \mathscr{M}^0, \\ |v + w| &\leq |v| + |w| && \text{for every } v, w \in \mathscr{M}^0, \end{aligned} \tag{3.6}$$

where all equalities and inequalities have to be intended in the $\mathfrak{m}$-a.e. sense.

iv) *For some (thus any) Borel probability measure $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$, it holds that the distance $\mathsf{d}_{\mathscr{M}^0}$ on $\mathscr{M}^0$, defined as*

$$\mathsf{d}_{\mathscr{M}^0}(v, w) \doteq \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}' \qquad \text{for every } v, w \in \mathscr{M}^0, \tag{3.7}$$

is complete and induces the topology τ.

Remark 3.4 As pointed out in item iv) of Definition 3.3, the particular choice of $\mathfrak{m}'$ does not affect neither the completeness of $\mathsf{d}_{\mathscr{M}^0}$ nor its induced topology τ. ■

Remark 3.5 The space $L^0(\mathfrak{m})$ depends only on the negligible sets of $\mathfrak{m}$: given any σ-finite Borel measure $\mathfrak{m}'$ on X that is mutually absolutely continuous with respect to $\mathfrak{m}$, it holds that $L^0(\mathfrak{m}) = L^0(\mathfrak{m}')$ as topological rings. Then we can unambiguously make use of the notation $L^0(\mathcal{N})$ to indicate the space $L^0(\mathfrak{m})$, where $\mathcal{N}$ is the σ-ideal of negligible sets of $\mathfrak{m}$. Accordingly, one could speak about $L^0(\mathcal{N})$-normed $L^0(\mathcal{N})$-modules, in order to underline their dependence just on the family of null sets. ■

A proof of the following result – which represents the analogue of Proposition 3.2 – can be found, for instance, in [Pas18, Remark 15.12].

Proposition 3.6 *Let $\mathscr{M}^0$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then $\cdot : L^0(\mathfrak{m}) \times \mathscr{M}^0 \to \mathscr{M}^0$ and $|\cdot| : \mathscr{M}^0 \to L^0(\mathfrak{m})$ are continuous maps. In particular, $\mathscr{M}^0$ is a topological $L^0(\mathfrak{m})$-module.*

The relation between L^2-normed L^∞-modules and L^0-normed L^0-modules is the object of the following two results. The former is proven in [Gig17b, Subsection 1.3], while for the latter we refer to [Gig17a, Theorem/Definition 1.7].

Proposition 3.7 (L^2-restriction) *Let $\mathscr{M}^0$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then*

$$\mathscr{M} \doteq \big\{ v \in \mathscr{M}^0 \;\big|\; |v| \in L^2(\mathfrak{m}) \big\} \tag{3.8}$$

has a natural structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Proposition 3.8 (L^0-completion) *Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then there exists a unique couple $(\mathscr{M}^0, \iota)$, where $\mathscr{M}^0$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and the map $\iota : \mathscr{M} \to \mathscr{M}^0$ is a linear operator with dense image that preserves the pointwise norm.*

Uniqueness is intended up to unique isomorphism: given any other such couple $(\mathscr{N}^0, \iota')$, there exists a unique isomorphism $\Phi : \mathscr{M}^0 \to \mathscr{N}^0$ of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules such that

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\ \iota\ } & \mathscr{M}^0 \\ & \searrow^{\iota'} & \downarrow \Phi \\ & & \mathscr{N}^0 \end{array} \tag{3.9}$$

is a commutative diagram.

Remark 3.9 As described in Subsection 3.2.4, the operations of taking the L^0-completion and the L^2-restriction can be actually made into an equivalence of categories, between the category of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules and that of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules. ■

Let $\mathscr{M}^0$ be any $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and let E be any Borel subset of X. Let us define $\mathscr{M}^0|_E \doteq \big\{ \chi_E\, v \;:\; v \in \mathscr{M}^0 \big\}$. It then turns out that $\mathscr{M}^0|_E$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Moreover, if $\mathscr{M}$ denotes the L^2-restriction of $\mathscr{M}^0$, then the module $\mathscr{M}^0|_E$ can be canonically identified with the L^0-completion of $\mathscr{M}|_E$.

Remark 3.10 (L^p-normed L^∞-module) By replacing $p = 2$ with any other $p \in [1, \infty]$ in Definition 3.1, we obtain the notion of $L^p(\mathfrak{m})$-*normed* $L^\infty(\mathfrak{m})$-*module*. All the properties illustrated so far – and many others that we shall see in the sequel – can be suitably adapted to deal with the class of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules. ■

Remark 3.11 We now present a simple construction that we shall frequently make use of. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a given metric measure space. Consider any Borel set $E \subseteq \mathrm{X}$ and call $\nu \doteq \mathfrak{m}_{|E}$. Then to any $L^0(\nu)$-normed $L^0(\nu)$-module $\mathscr{M}$ we can canonically associate an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module – called *extension* of $\mathscr{M}$ and denoted by $\mathrm{Ext}(\mathscr{M})$ – in the following way. First of all, we notice that we have a natural projection/restriction operator

$$\mathrm{proj} : \; L^0(\mathfrak{m}) \longrightarrow L^0(\nu)$$

given by the passage to the quotient up to ν-a.e. equality and a natural 'extension' operator

$$\mathrm{ext} : \; L^0(\nu) \longrightarrow L^0(\mathfrak{m})$$

that sends $f \in L^0(\nu)$ to the function $\mathfrak{m}$-a.e. equal to f on E and to 0 on $\mathrm{X} \setminus E$. Then for a generic $L^0(\nu)$-normed $L^0(\nu)$-module $\mathscr{M}$ we put $\mathrm{Ext}(\mathscr{M}) \doteq \mathscr{M}$ as set, while the multiplication of $v \in \mathrm{Ext}(\mathscr{M})$ by $f \in L^0(\mathfrak{m})$ is defined as $\mathrm{proj}(f)v \in \mathscr{M} = \mathrm{Ext}(\mathscr{M})$ and the pointwise norm of v as $\mathrm{ext}\big(|v|\big) \in L^0(\mathfrak{m})$. Further, we shall denote by $\mathrm{ext} : \mathscr{M} \to \mathrm{Ext}(\mathscr{M})$ the identity map. Notice that one trivially has that

$$\mathrm{Ext}(\mathscr{M}^*) \sim \mathrm{Ext}(\mathscr{M})^* \qquad \text{via the coupling } \; \mathrm{ext}(L)\big(\mathrm{ext}(v)\big) \doteq \mathrm{ext}\big(L(v)\big). \tag{3.10}$$

In what follows, we shall always implicitly make this identification. ■

In the theory of normed modules, a fundamental role is played by the following class:

Definition 3.12 (Hilbert module) *Let $\mathscr{H}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then we say that $\mathscr{H}$ is a* Hilbert module *provided $\big(\mathscr{H}, \|\cdot\|_{\mathscr{H}}\big)$ is a Hilbert space, or equivalently if*

$$|v+w|^2 + |v-w|^2 = 2\,|v|^2 + 2\,|w|^2 \qquad \text{holds } \mathfrak{m}\text{-a.e. in } \mathrm{X} \tag{3.11}$$

for every $v, w \in \mathscr{H}$. Such formula is referred to as the pointwise parallelogram identity*.*

The pointwise norm of a Hilbert module $\mathscr{H}$ induces – by polarisation – a *pointwise scalar product* $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H} \to L^1(\mathfrak{m})$ on $\mathscr{H}$, which is the $L^\infty(\mathfrak{m})$-bilinear map given by

$$\langle v, w\rangle \doteq \frac{|v+w|^2 - |v|^2 - |w|^2}{2} \qquad \text{for every } v, w \in \mathscr{H}. \tag{3.12}$$

It is easy to check that $\langle v, v\rangle = |v|^2$ is satisfied $\mathfrak{m}$-a.e. for any $v \in \mathscr{H}$. Moreover, one has that

$$\big|\langle v, w\rangle\big| \leq |v||w| \qquad \text{holds } \mathfrak{m}\text{-a.e. for every } v, w \in \mathscr{H}. \tag{3.13}$$

The previous formula is called *pointwise Cauchy-Schwarz inequality.*

Remark 3.13 An $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{H}^0$ is said to be a *Hilbert module* provided the identity (3.11) is satisfied for every $v, w \in \mathscr{H}^0$. Then it turns out that an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{H}$ is Hilbert if and only if its L^0-completion $\mathscr{H}^0$ is Hilbert. It can be readily checked that the pointwise scalar product of $\mathscr{H}$ can be uniquely extended to an $L^0(\mathfrak{m})$-bilinear map $\langle\cdot,\cdot\rangle : \mathscr{H}^0 \times \mathscr{H}^0 \to L^0(\mathfrak{m})$ that satisfies the pointwise Cauchy-Schwarz inequality. ■

3.1.2 Local dimension of a normed module

Consider a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, which will remain fixed for the whole subsection.

Definition 3.14 (Local dimension) *Let $\mathscr{M}^0$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Let E be any Borel subset of X such that $\mathfrak{m}(E) > 0$. Then:*

i) *A set $S \subseteq \mathscr{M}^0$ is said to* generate *the module $\mathscr{M}^0$ on E provided the set of all finite sums of the form $\sum_{i=1}^{n} \chi_{E_i} v_i$, where $(E_i)_{i=1}^{n}$ is a Borel partition of E and $(v_i)_{i=1}^{n} \subseteq S$, is dense in the space $\mathscr{M}^0|_E$.*

ii) *Finitely many elements $v_1, \ldots, v_n \in \mathscr{M}^0$ are said to be* independent *on E provided any n-tuple $(f_1, \ldots, f_n) \in \big[L^0(\mathfrak{m})\big]^n$ for which $\sum_{i=1}^{n} f_i \cdot v_i = 0$ must vanish $\mathfrak{m}$-a.e. on E.*

iii) *We say that some elements $v_1, \ldots, v_n \in \mathscr{M}^0$ constitute a* local basis *for $\mathscr{M}^0$ on E provided they are independent on E and they generate $\mathscr{M}^0$ on E.*

iv) *We declare that the module $\mathscr{M}^0$ has* local dimension *equal to $n \in \mathbb{N}^+$ on E provided it admits a local basis $v_1, \ldots, v_n \in \mathscr{M}^0$ on E, while we say that it has local dimension equal to 0 on E if $\mathscr{M}^0|_E = \{0\}$. Such definition is well-posed, since any two local bases for $\mathscr{M}^0$ on E must have the same cardinality.*

Notice that some elements $v_1, \ldots, v_n \in \mathscr{M}^0$ generate the module $\mathscr{M}^0$ on E if and only if for any $v \in \mathscr{M}^0$ there exist $f_1, \ldots, f_n \in L^0(\mathfrak{m})$ such that $\chi_E v = \sum_{i=1}^{n} f_i v_i$.

Remark 3.15 The notions introduced in Definition 3.14 have been originally formulated for $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules, in Subsection 1.4 of [Gig17b]. For our purposes, it is sufficient to say that an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$ has local dimension equal to $n \in \mathbb{N}$ on a Borel set $E \subseteq \mathrm{X}$ provided its L^0-completion $\mathscr{M}^0$ has local dimension n on E. ■

Remark 3.16 (Orthonormal basis) Suppose that $\mathfrak{m}$ is a finite measure and consider a Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{H}$. Assume that $\mathscr{H}$ has local dimension $n \in \mathbb{N}^+$ on some Borel set $E \subseteq \mathrm{X}$. Then it turns out that there exist $v_1, \ldots, v_n \in \mathscr{H}$ such that

$$\langle v_i, v_j \rangle = \delta_{ij} \quad \mathfrak{m}\text{-a.e. in } E \qquad \text{for every } i, j = 1, \ldots, n. \tag{3.14}$$

Clearly $v_1, \ldots, v_n$ form a local basis for $\mathscr{H}$ on E – called *orthonormal basis*. ■

An important feature of the normed modules is that they admit a (unique) dimensional decomposition. For a proof of the following result, we refer e.g. to [LP18, Theorem 1.8].

Theorem 3.17 (Dimensional decomposition) *Fix an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}^0$. Then there exists a unique Borel partition $(E_n)_{n \in \mathbb{N} \cup \{\infty\}}$ of the space X, called* dimensional decomposition *of $\mathscr{M}^0$, such that the following properties hold:*

i) *$\mathscr{M}^0$ has local dimension equal to n on E_n, for any $n \in \mathbb{N}$ such that $\mathfrak{m}(E_n) > 0$.*

ii) *$\mathscr{M}^0$ does not admit any finite local basis on any Borel subset $E \subseteq E_\infty$ with $\mathfrak{m}(E) > 0$.*

Uniqueness is intended up to $\mathfrak{m}$-a.e. equality: given any other sequence $(F_n)_{n \in \mathbb{N} \cup \{\infty\}}$ with the same properties, it holds that $\mathfrak{m}(E_n \Delta F_n) = 0$ for every $n \in \mathbb{N} \cup \{\infty\}$.

Many important normed modules that arise in the study of the differential structure of finite-dimensional RCD spaces – which will be described in the sequel – are 'locally finitely-generated', in the following sense; the definition is taken from [LP18].

Definition 3.18 (Proper module) *Let $\mathscr{M}^0$ be any $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module, whose dimensional decomposition is denoted by $(E_n)_{n\in\mathbb{N}\cup\{\infty\}}$. Then we say that $\mathscr{M}^0$ is a* proper module *provided* $\mathfrak{m}(E_\infty) = 0$.

Remark 3.19 In Subsection 1.4 of [Gig17b], the dimensional decomposition is defined – and proven to exist – for any $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$. The dimensional decomposition of $\mathscr{M}$ coincides with the dimensional decomposition of its L^0-completion $\mathscr{M}^0$. ■

Lemma 3.20 *Let $\mathscr{M}^0$ be a proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then $\mathscr{M}^0$ is separable.*

Proof. Call $(E_n)_{n\in\mathbb{N}}$ the dimensional decomposition of $\mathscr{M}^0$. For any $n \in \mathbb{N}$, choose a local basis $v_1^n, \ldots, v_n^n \in \mathscr{M}^0|_{E_n}$ for $\mathscr{M}^0|_{E_n}$. Fix a countable dense subset D of $L^0(\mathfrak{m})$. Then

$$\left\{ \sum_{n=1}^{\infty}\sum_{i=1}^{n} f_i^n \cdot v_i^n \;\middle|\; (f_i^n)_{1\leq i\leq n} \subseteq D \right\} \subseteq \mathscr{M}^0,$$

which is countable by construction, is dense in $\mathscr{M}^0$ by Proposition 3.6. □

A *module morphism* $\Phi : \mathscr{M} \to \mathscr{N}$ between two $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules (resp. $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules) is any $L^0(\mathfrak{m})$-linear (resp. $L^\infty(\mathfrak{m})$-linear) operator such that

$$\big|\Phi(v)\big| \leq |v| \quad \mathfrak{m}\text{-a.e. in X} \qquad \text{for every } v \in \mathscr{M}. \tag{3.15}$$

Furthermore, in Section 3.2 we shall adopt the following categorical terminology:

Definition 3.21 (Categories of normed modules) *Let $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a given metric measure space and let $p \in [1, \infty]$. The category of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules is denoted by* $\mathbf{NMod}^p(\mathbb{X})$ *and that of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules by* $\mathbf{NMod}^0(\mathbb{X})$.

Moreover, those subcategories of $\mathbf{NMod}^p(\mathbb{X})$ *and* $\mathbf{NMod}^0(\mathbb{X})$ *that consist of all proper modules will be called* $\mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$ *and* $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$, *respectively.*

3.1.3 Construction of normed modules: dual

Fix a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. We introduce the notion of dual for normed modules:

Definition 3.22 (Dual of a L^2-normed L^∞-module) *Let us consider an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$. Then we define its* dual module *$\mathscr{M}^*$ as*

$$\mathscr{M}^* \doteq \Big\{ L : \mathscr{M} \to L^1(\mathfrak{m}) \;\Big|\; L \textit{ is } L^\infty(\mathfrak{m})\textit{-linear and continuous} \Big\}, \tag{3.16}$$

endowed with the following operations:

$$\begin{aligned} (L + L')(v) &\doteq L(v) + L'(v),\\ (f \cdot L)(v) &\doteq L(f \cdot v),\\ |L| &\doteq \operatorname{ess\,sup}\big\{L(v) \;\big|\; v \in \mathscr{M},\, |v| \leq 1 \;\mathfrak{m}\text{-a.e.}\big\},\\ \|L\|_{\mathscr{M}^*} &\doteq \big\||L|\big\|_{L^2(\mathfrak{m})} \end{aligned} \tag{3.17}$$

for every $f \in L^\infty(\mathfrak{m})$ and $L, L' \in \mathscr{M}^*$. The space $\mathscr{M}^*$ is an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Definition 3.23 (Dual of an L^0-normed L^0-module) *Let us consider an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}^0$. Then we define its* dual module $(\mathscr{M}^0)^*$ *as*

$$(\mathscr{M}^0)^* \doteq \Big\{ L : \mathscr{M}^0 \to L^0(\mathfrak{m}) \;\Big|\; L \text{ is } L^0(\mathfrak{m})\text{-linear and continuous} \Big\}, \tag{3.18}$$

endowed with the following operations:

$$\begin{aligned} (L + L')(v) &\doteq L(v) + L'(v), \\ (f \cdot L)(v) &\doteq L(f \cdot v), \\ |L| &\doteq \operatorname{ess\,sup} \big\{ L(v) \;\big|\; v \in \mathscr{M}^0,\ |v| \le 1\ \mathfrak{m}\text{-a.e.} \big\} \end{aligned} \tag{3.19}$$

for any $f \in L^0(\mathfrak{m})$ and $L, L' \in (\mathscr{M}^0)^*$. The space $(\mathscr{M}^0)^*$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module.

Remark 3.24 Let $\mathscr{M}$ be an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and let $\mathscr{M}^0$ be its L^0-completion. Then it can be readily checked that $(\mathscr{M}^0)^*$ is the L^0-completion of $\mathscr{M}^*$. In other words, the operations of taking the dual and taking the L^0-completion commute. ■

Given an $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $\mathscr{M}$, we denote by $\mathscr{M}'$ its dual as a Banach space. Then the integration provides a natural map $\textsc{Int}_{\mathscr{M}} : \mathscr{M}^* \to \mathscr{M}'$, which is defined as

$$\textsc{Int}_{\mathscr{M}}(L)(v) \doteq \int L(v)\,\mathrm{d}\mathfrak{m} \qquad \text{for every } L \in \mathscr{M}^* \text{ and } v \in \mathscr{M}. \tag{3.20}$$

It holds that $\textsc{Int}_{\mathscr{M}}$ is a bijective isometry, i.e. $\|L\|_{\mathscr{M}^*} = \big\|\textsc{Int}_{\mathscr{M}}(L)\big\|_{\mathscr{M}'}$ for every $L \in \mathscr{M}^*$.

Let us denote by $\mathcal{I}_{\mathscr{M}} : \mathscr{M} \hookrightarrow \mathscr{M}^{**}$ the canonical embedding in the bidual, given by

$$\mathcal{I}_{\mathscr{M}}(v)(L) \doteq L(v) \qquad \text{for every } v \in \mathscr{M} \text{ and } L \in \mathscr{M}^*. \tag{3.21}$$

Then $\mathcal{I}_{\mathscr{M}}$ is an $L^\infty(\mathfrak{m})$-linear operator that preserves the pointwise norm. We say that $\mathscr{M}$ is *reflexive* as a module provided the map $\mathcal{I}_{\mathscr{M}}$ is surjective. Actually, we have that $\mathscr{M}$ is reflexive as a module if and only if it is reflexive as a Banach space.

Theorem 3.25 (Riesz) *Let $\mathscr{H}$ be a Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then the map sending any $v \in \mathscr{H}$ to the element $L_v \in \mathscr{H}^*$, given by*

$$L_v(w) \doteq \langle v, w \rangle \qquad \text{for every } w \in \mathscr{H}, \tag{3.22}$$

is an isomorphism of modules and is called Riesz isomorphism. *In particular, $\mathscr{H}$ is reflexive.*

3.1.4 Construction of normed modules: pullback

Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be two fixed metric measure spaces.

Definition 3.26 (Map of bounded compression) *Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a Borel map. Then we say that φ has* bounded compression *provided there exists a constant $C > 0$ such that*

$$\varphi_* \mathfrak{m}_\mathrm{X} \le C\, \mathfrak{m}_\mathrm{Y}. \tag{3.23}$$

The smallest such constant C is called compression constant *of φ and is denoted by* $\mathrm{Comp}(\varphi)$.

Any map of bounded compression canonically induces a notion of pullback for L^2-normed L^∞-modules, as expressed by the following result; cf. [Gig17a, Theorem/Definition 1.23].

Theorem 3.27 (Pullback) *Let $\mathscr{M}$ be an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded compression. Then there exists a unique couple $(\varphi^*\mathscr{M}, \varphi^*)$, where $\varphi^*\mathscr{M}$ is an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{X}})$-module and $\varphi^* : \mathscr{M} \to \varphi^*\mathscr{M}$ is a linear continuous operator, such that the following properties are satisfied:*

i) *The identity $|\varphi^* v| = |v| \circ \varphi$ holds $\mathfrak{m}_{\mathrm{X}}$-a.e. for every $v \in \mathscr{M}$.*

ii) *The set $\{\varphi^* v : v \in \mathscr{M}\}$ generates $\varphi^*\mathscr{M}$ on the whole X in the sense of modules.*

Uniqueness is intended up to unique isomorphism: given any other such couple $(\mathscr{N}, T)$, there exists a unique isomorphism $\Phi : \varphi^*\mathscr{M} \to \mathscr{N}$ of $L^2(\mathfrak{m}_{\mathrm{X}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{X}})$-modules such that

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\varphi^*} & \varphi^*\mathscr{M} \\ & {\scriptstyle T}\searrow & \downarrow{\scriptstyle \Phi} \\ & & \mathscr{N} \end{array} \tag{3.24}$$

is a commutative diagram.

Remark 3.28 Observe that the space $L^2(\mathfrak{m}_{\mathrm{Y}})$ is an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module, whose local dimension is 1 on the whole Y. Moreover, we have that $\varphi^* L^2(\mathfrak{m}_{\mathrm{Y}}) \cong L^2(\mathfrak{m}_{\mathrm{X}})$, with pullback map $\varphi^* : L^2(\mathfrak{m}_{\mathrm{Y}}) \to L^2(\mathfrak{m}_{\mathrm{X}})$ given by $\varphi^* f = f \circ \varphi$ for every $f \in L^2(\mathfrak{m}_{\mathrm{Y}})$. ■

Remark 3.29 It can be readily checked – by exploiting item i) of Theorem 3.27 – that the pullback of a Hilbert module is a Hilbert module as well. ■

As described in Subsection 2.2 of [GP17], the concept of pullback module can be carried over to the class of L^0-normed L^0-modules in a natural way: given an $L^0(\mathfrak{m}_{\mathrm{Y}})$-normed $L^0(\mathfrak{m}_{\mathrm{Y}})$-module $\mathscr{M}^0$ and a map φ of bounded compression, we define the pullback $\varphi^*\mathscr{M}^0$ as the $L^0(\mathfrak{m}_{\mathrm{X}})$-completion of $\varphi^*\mathscr{M}$, where $\mathscr{M}$ is the $L^2(\mathfrak{m}_{\mathrm{Y}})$-restriction of $\mathscr{M}^0$, while the pullback map $\varphi^* : \mathscr{M}^0 \to \varphi^*\mathscr{M}^0$ is the unique linear continuous extension of $\varphi^* : \mathscr{M} \to \varphi^*\mathscr{M}$.

Furthermore, we have that $(\varphi^*\mathscr{M}^0, \varphi^*)$ can be characterised as the unique couple (up to unique isomorphism) such that the equality $|\varphi^* v| = |v| \circ \varphi$ holds $\mathfrak{m}_{\mathrm{X}}$-a.e. for every $v \in \mathscr{M}^0$ and the set $\{\varphi^* v : v \in \mathscr{M}^0\}$ generates $\varphi^*\mathscr{M}^0$ on X as an $L^0(\mathfrak{m}_{\mathrm{X}})$-normed $L^0(\mathfrak{m}_{\mathrm{X}})$-module.

Remark 3.30 Let $\mathscr{M}$ be an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module and let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded compression. The role of this assumption on φ is to ensure that $|\varphi^* v| \in L^2(\mathfrak{m}_{\mathrm{X}})$ for every $v \in \mathscr{M}$, as shown by the following estimates:

$$\int |\varphi^* v|^2 \,\mathrm{d}\mathfrak{m}_{\mathrm{X}} = \int |v|^2 \circ \varphi \,\mathrm{d}\mathfrak{m}_{\mathrm{X}} = \int |v|^2 \,\mathrm{d}\varphi_*\mathfrak{m}_{\mathrm{X}} \leq \mathrm{Comp}(\varphi) \int |v|^2 \,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} < +\infty.$$

However, the hypothesis of bounded compression seems unnaturally strong in the case of L^0-normed L^0-modules, as the integrability issue does not occur. Indeed, it turns out that the pullback $\varphi^*\mathscr{M}^0$ of an $L^0(\mathfrak{m}_{\mathrm{Y}})$-normed $L^0(\mathfrak{m}_{\mathrm{Y}})$-module $\mathscr{M}^0$ can be built whenever $\varphi : \mathrm{X} \to \mathrm{Y}$ is a Borel map for which $\varphi_*\mathfrak{m}_{\mathrm{X}} \ll \mathfrak{m}_{\mathrm{Y}}$. In order to prove such fact, let us define

$$\mathfrak{m}'_{\mathrm{Y}} \doteq (\varphi_*\mathfrak{m}_{\mathrm{X}})|_A + \mathfrak{m}_{\mathrm{Y}}|_{\mathrm{X}\setminus A}, \qquad \text{where we set } A \doteq \left\{ \frac{\mathrm{d}\varphi_*\mathfrak{m}_{\mathrm{X}}}{\mathrm{d}\mathfrak{m}_{\mathrm{Y}}} > 0 \right\}.$$

Hence we have $\mathfrak{m}'_{\mathrm{Y}} \ll \mathfrak{m}_{\mathrm{Y}} \ll \mathfrak{m}'_{\mathrm{Y}}$, which grants that $L^0(\mathfrak{m}'_{\mathrm{Y}}) = L^0(\mathfrak{m}_{\mathrm{Y}})$ and accordingly that $\mathscr{M}^0$ is an $L^0(\mathfrak{m}'_{\mathrm{Y}})$-normed $L^0(\mathfrak{m}'_{\mathrm{Y}})$-module. Given that $\varphi_*\mathfrak{m}_{\mathrm{X}} \leq \mathfrak{m}'_{\mathrm{Y}}$, i.e. the map φ has bounded compression when the target space Y is endowed with the measure $\mathfrak{m}'_{\mathrm{Y}}$, it makes sense to consider the pullback module $\varphi^*\mathscr{M}^0$. This proves the above claim. ■

The pullback module can be proven to satisfy the following universal property:

Proposition 3.31 (Universal property of the pullback) *Let $\mathscr{M}$ be an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module and $\varphi : \mathrm{X} \to \mathrm{Y}$ a map of bounded compression. Take an $L^2(\mathfrak{m}_{\mathrm{X}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{X}})$-module $\mathscr{N}$ and a linear operator $T : \mathscr{M} \to \mathscr{N}$ such that the inequality*

$$\big|T(v)\big| \leq C\,|v| \circ \varphi \quad \mathfrak{m}_{\mathrm{X}}\text{-a.e.} \qquad \text{for every } v \in \mathscr{M} \tag{3.25}$$

is satisfied for some constant $C > 0$. Then there exists a unique $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linear continuous operator $\hat{T} : \varphi^*\mathscr{M} \to \mathscr{N}$ such that

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\;T\;} & \mathscr{N} \\ {\scriptstyle \varphi^*}\big\downarrow & \nearrow_{\hat{T}} & \\ \varphi^*\mathscr{M} & & \end{array} \tag{3.26}$$

is a commutative diagram.

An analogous universal property is verified by the pullback of an L^0-normed L^0-module; cf. [GP17, Proposition 2.9] for the details.

Remark 3.32 (Functoriality) An important consequence of Proposition 3.31 is that the pullback of modules is *functorial*, in the sense we are now going to explain.

Let $(\mathrm{Z}, \mathsf{d}_{\mathrm{Z}}, \mathfrak{m}_{\mathrm{Z}})$ be a metric measure space. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ and $\psi : \mathrm{Y} \to \mathrm{Z}$ be maps of bounded compression. Consider any $L^2(\mathfrak{m}_{\mathrm{Z}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Z}})$-module $\mathscr{M}$. Then $(\psi \circ \varphi)^*\mathscr{M}$ can be canonically identified with $\varphi^*(\psi^*\mathscr{M})$. ■

Remark 3.33 Suppose that $\varphi : \mathrm{X} \to \mathrm{Y}$ and $\psi : \mathrm{Y} \to \mathrm{X}$ are maps of bounded compression such that the identities $\psi \circ \varphi = \mathrm{id}_{\mathrm{X}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathrm{Y}}$ hold $\mathfrak{m}_{\mathrm{X}}$-a.e. and $\mathfrak{m}_{\mathrm{Y}}$-a.e., respectively. Consider any $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module $\mathscr{M}$. Then Remark 3.32 grants that the pullback map $\varphi^* : \mathscr{M} \to \varphi^*\mathscr{M}$ is bijective. Since the right composition $f \mapsto f \circ \varphi$ is an isomorphism between $L^\infty(\mathfrak{m}_{\mathrm{Y}})$ and $L^\infty(\mathfrak{m}_{\mathrm{X}})$, we conclude that the modules $\mathscr{M}$ and $\varphi^*\mathscr{M}$ can be identified via the isomorphism φ^*. ■

A natural question is the following: do the operations of taking the dual and taking the pullback commute? As we shall see in the next result, the answer is in general no – unless we add some additional assumptions on the module under consideration.

Theorem 3.34 (Relation between $\varphi^*\mathscr{M}^*$ and $(\varphi^*\mathscr{M})^*$) *Let $\mathscr{M}$ be any $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded compression. Then there exists a unique $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linear and continuous map $\mathcal{I} : \varphi^*\mathscr{M}^* \to (\varphi^*\mathscr{M})^*$ such that*

$$\mathcal{I}(\varphi^* L)(\varphi^* v) = L(v) \circ \varphi \qquad \text{for every } L \in \mathscr{M}^* \text{ and } v \in \mathscr{M}. \tag{3.27}$$

It holds that the operator $\mathcal{I}$ preserves the pointwise norm.

Moreover, suppose that either $\mathscr{M}^*$ is separable (when viewed as a Banach space) or $\mathscr{M}$ is a Hilbert module. Then the operator $\mathcal{I}$ is surjective. In other words, the modules $(\varphi^*\mathscr{M})^*$ and $\varphi^*\mathscr{M}^*$ can be canonically identified.

The pullback map is in general not bijective. Nevertheless, by relying upon the machinery that has been developed in Subsection 1.2.2 it is possible – in the case in which φ^* preserves the reference measure – to select a special left inverse Pr_φ of the pullback map φ^*.

Theorem 3.35 (Projection for modules) *Let $\mathscr{M}$ be a $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a Borel map such that $\varphi_*\mathfrak{m}_{\mathrm{X}} = \mathfrak{m}_{\mathrm{Y}}$. Then there exists a unique linear and continuous operator $\mathsf{Pr}_\varphi : \varphi^*\mathscr{M} \to \mathscr{M}$ such that*

$$\mathsf{Pr}_\varphi(f\,\varphi^* v) = \mathsf{Pr}_\varphi(f)\,v \qquad \text{for every } f \in L^\infty(\mathfrak{m}_{\mathrm{X}}) \text{ and } v \in \mathscr{M}. \tag{3.28}$$

In particular, it holds that $\mathsf{Pr}_\varphi(\varphi^* v) = v$ for every $v \in \mathscr{M}$ and

$$\big|\mathsf{Pr}_\varphi(w)\big| \le \mathsf{Pr}_\varphi|w| \quad \mathfrak{m}_{\mathrm{Y}}\text{-a.e.} \qquad \text{for every } w \in \varphi^*\mathscr{M}. \tag{3.29}$$

Proof. Let us denote by V the set of all the elements $w \in \varphi^*\mathscr{M}$ that can be written in the form $w = \sum_{i=1}^n \chi_{E_i}\,\varphi^* v_i$, where $(E_i)_{i=1}^n$ is a Borel partition of X and $(v_i)_{i=1}^n \subseteq \mathscr{M}$. Recall that V is a dense vector subspace of $\varphi^*\mathscr{M}$. Then we are forced to define $\mathsf{Pr}_\varphi : V \to \mathscr{M}$ as

$$\mathsf{Pr}_\varphi(w) \doteq \sum_{i=1}^n \mathsf{Pr}_\varphi(\chi_{E_i})\,v_i \qquad \text{for every } w = \sum_{i=1}^n \chi_{E_i}\,\varphi^* v_i \in V. \tag{3.30}$$

Well-posedness of such definition is guaranteed by the following $\mathfrak{m}_{\mathrm{Y}}$-a.e. inequality:

$$\bigg|\sum_{i=1}^n \mathsf{Pr}_\varphi(\chi_{E_i})\,v_i\bigg| \le \sum_{i=1}^n \mathsf{Pr}_\varphi(\chi_{E_i})\,|v_i| \overset{(1.52)}{=} \sum_{i=1}^n \mathsf{Pr}_\varphi\big(\chi_{E_i}\,|v_i|\circ\varphi\big) = \mathsf{Pr}_\varphi\bigg|\sum_{i=1}^n \chi_{E_i}\,\varphi^* v_i\bigg|.$$

This shows that $\big|\mathsf{Pr}_\varphi(w)\big| \le \mathsf{Pr}_\varphi|w|$ holds $\mathfrak{m}_{\mathrm{Y}}$-a.e. for any $w \in V$. By integrating it, we get

$$\big\|\mathsf{Pr}_\varphi(w)\big\|_{\mathscr{M}} \le \big\|\mathsf{Pr}_\varphi|w|\big\|_{L^2(\mathfrak{m}_{\mathrm{Y}})} \overset{(1.51)}{\le} \big\||w|\big\|_{L^2(\mathfrak{m}_{\mathrm{X}})} = \|w\|_{\varphi^*\mathscr{M}} \qquad \text{for every } w \in V,$$

which grants that the map $\mathsf{Pr}_\varphi : V \to \mathscr{M}$ – that is linear by construction – is continuous. Therefore Pr_φ can be uniquely extended to a linear continuous operator $\mathsf{Pr}_\varphi : \varphi^*\mathscr{M} \to \mathscr{M}$. An approximation argument shows that property (3.29) is satisfied. Finally, for any $v \in \mathscr{M}$ it holds that $\mathsf{Pr}_\varphi(\varphi^* v) = \mathsf{Pr}_\varphi(1)\,v = v$ by (1.46), thus proving that $\mathsf{Pr}_\varphi \circ \varphi^* = \mathrm{id}_{\mathscr{M}}$. □

Remark 3.36 We claim that

$$g\,\mathsf{Pr}_\varphi(w) = \mathsf{Pr}_\varphi(g\circ\varphi\,w) \qquad \text{for every } g \in L^\infty(\mathfrak{m}_{\mathrm{Y}}) \text{ and } w \in \varphi^*\mathscr{M}. \tag{3.31}$$

Indeed, just observe that both sides of the identity are linear continuous with respect to the entry w and agree on those w's of the form $f\,\varphi^* v$, with $f \in L^\infty(\mathfrak{m}_{\mathrm{X}})$ and $v \in \mathscr{M}$. ■

As we are going to show in the next results, the projection map Pr_φ is a key tool in order to prove that the passage to the pullback 'preserves' the local dimension of the module.

Proposition 3.37 *Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a Borel map such that $\varphi_*\mathfrak{m}_{\mathrm{X}} = \mathfrak{m}_{\mathrm{Y}}$. Let $\mathscr{M}$ be any $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module. Let E be a Borel subset of Y such that $\mathfrak{m}_{\mathrm{Y}}(E) > 0$. Then:*

i) *Given any set $S \subseteq \mathscr{M}$ that generates $\mathscr{M}$ on E, it holds that the family $\big\{\varphi^* v : v \in S\big\}$ generates $\varphi^*\mathscr{M}$ on $\varphi^{-1}(E)$.*

ii) *Given any $v_1, \ldots, v_n \in \mathscr{M}$ independent on E, it holds that $\varphi^* v_1, \ldots, \varphi^* v_n \in \varphi^*\mathscr{M}$ are independent on $\varphi^{-1}(E)$.*

Proof. To prove item i), fix a set $S \subseteq \mathscr{M}$ generating $\mathscr{M}$ on E. This means that the set V of all finite sums $\sum_{i=1}^n \chi_{E_i} v_i$, where $(E_i)_{i=1}^n$ is a Borel partition of E and $(v_i)_{i=1}^n \subseteq S$, is dense in $\mathscr{M}|_E$. Therefore any element of $\varphi^*(\mathscr{M}|_E) \simeq (\varphi^*\mathscr{M})|_{\varphi^{-1}(E)}$ can be approximated by a sequence of elements of the form $\sum_{j=1}^m \chi_{F_j} \varphi^* w_j$, with $(F_j)_{j=1}^m$ Borel partition of $\varphi^{-1}(E)$ and $(w_j)_{j=1}^m \subseteq V$. This ensures that $\{\varphi^* v : v \in S\}$ generates $\varphi^*\mathscr{M}$ on $\varphi^{-1}(E)$, as required.

To prove item ii), fix some elements $v_1, \ldots, v_n \in \mathscr{M}$ that are independent on E. Let us consider any $f_1, \ldots, f_n \in L^\infty(\mathfrak{m}_{\mathrm{X}})$ such that $\chi_{\varphi^{-1}(E)} \sum_{i=1}^n f_i \varphi^* v_i = 0$. Given any subset S of $\{1, \ldots, n\}$, we define the Borel set $A_S \subseteq \mathrm{X}$ as

$$A_S \doteq \varphi^{-1}(E) \cap \bigcap_{i \in S} \{f_i \geq 0\} \cap \bigcap_{i \notin S} \{f_i < 0\}.$$

Then $\{A_S : S \subseteq \{1, \ldots, n\}\}$ constitutes a Borel partition of $\varphi^{-1}(E)$. Fix any $S \subseteq \{1, \ldots, n\}$. We have that $\sum_{i=1}^n \mathsf{Pr}_\varphi(\chi_{A_S} f_i) v_i = \mathsf{Pr}_\varphi\big(\sum_{i=1}^n \chi_{A_S} f_i \varphi^* v_i\big) = 0$, whence $\chi_E \mathsf{Pr}_\varphi(\chi_{A_S} f_i) = 0$ holds $\mathfrak{m}_{\mathrm{Y}}$-a.e. for all $i = 1, \ldots, n$ by independence of $v_1, \ldots, v_n$ on E. Moreover, it holds

$$\big|\chi_{E^c} \mathsf{Pr}_\varphi(\chi_{A_S} f_i)\big| \overset{(1.50)}{\leq} \chi_{E^c} \mathsf{Pr}_\varphi\big(\chi_{A_S} |f_i|\big) \overset{(1.46)}{\leq} \chi_{E^c} \mathsf{Pr}_\varphi(\chi_{A_S}) \, \|f_i\|_{L^\infty(\mathfrak{m}_{\mathrm{X}})} \overset{(1.57)}{=} 0$$

in the $\mathfrak{m}_{\mathrm{Y}}$-a.e. sense. Then $\mathsf{Pr}_\varphi(\chi_{A_S} f_i) = 0$ for all i, so that $\chi_{A_S} f_i = 0$ for every i by (1.44). Therefore $\chi_{\varphi^{-1}(E)} f_i = \sum_{S \subseteq \{1,\ldots,n\}} \chi_{A_S} f_i = 0$, proving the statement. □

Theorem 3.38 *Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a Borel map such that $\varphi_* \mathfrak{m}_{\mathrm{X}} = \mathfrak{m}_{\mathrm{Y}}$. Let $\mathscr{M}$ be an $L^2(\mathfrak{m}_{\mathrm{Y}})$-normed $L^\infty(\mathfrak{m}_{\mathrm{Y}})$-module. Let E be a Borel subset of Y with $\mathfrak{m}_{\mathrm{Y}}(E) > 0$. Then $\mathscr{M}$ has local dimension $n \in \mathbb{N}$ on E if and only if $\varphi^*\mathscr{M}$ has local dimension n on $\varphi^{-1}(E)$.*

Proof. First of all, assume that $\mathscr{M}$ has local dimension n on E and let us choose any local basis $v_1, \ldots, v_n \in \mathscr{M}$ for $\mathscr{M}$ on E. Hence the elements $\varphi^* v_1, \ldots, \varphi^* v_n$ constitute a local basis for $\varphi^*\mathscr{M}$ on $\varphi^{-1}(E)$ by Proposition 3.37, thus $\varphi^*\mathscr{M}$ has local dimension n on $\varphi^{-1}(E)$. The converse implication follows from the well-posedness of the notion of local dimension. □

3.1.5 Construction of normed modules: tensor product

Let us fix any metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$.

Given two Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules $\mathscr{H}_1$, $\mathscr{H}_2$, we denote by $\mathscr{H}_1 \otimes_{\mathsf{Alg}} \mathscr{H}_2$ their tensor product as $L^\infty(\mathfrak{m})$-modules – i.e. the space of formal finite sums of objects of the kind $v_1 \otimes v_2$, with $(v_1, v_2) \mapsto v_1 \otimes v_2$ being $L^\infty(\mathfrak{m})$-bilinear. Then let us define

(3.32) $$(v_1 \otimes v_2) : (w_1 \otimes w_2) \doteq \langle v_1, w_1 \rangle_1 \, \langle v_2, w_2 \rangle_2 \qquad \text{for any } v_1, w_1 \in \mathscr{H}_1 \text{ and } v_2, w_2 \in \mathscr{H}_2,$$

where $\langle \cdot, \cdot \rangle_i$ denotes the pointwise scalar product of $\mathscr{H}_i$ for $i = 1, 2$. We can extend it to a unique $L^\infty(\mathfrak{m})$-bilinear symmetric operator : from $(\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2) \times (\mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2)$ to $L^0(\mathfrak{m})$.

It turns out that the inequality $A : A \geq 0$ is verified $\mathfrak{m}$-a.e. for any $A \in \mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$. Moreover, given any Borel set $E \subseteq \mathrm{X}$ we have that

(3.33) $$A : A = 0 \quad \mathfrak{m}\text{-a.e. on } E \qquad \Longleftrightarrow \qquad A = 0 \quad \mathfrak{m}\text{-a.e. on } E.$$

In other words, the map : is 'pointwise positive definite'. The *Hilbert-Schmidt pointwise norm* of any tensor $A \in \mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2$ is given by

(3.34) $$|A|_{\mathsf{HS}} \doteq \sqrt{A : A} \in L^0(\mathfrak{m}),$$

thus we define the tensor product norm as

$$\|A\|_{\mathscr{H}_1\otimes\mathscr{H}_2} \doteq \left(\int |A|_{\mathsf{HS}}^2 \,\mathrm{d}\mathfrak{m}\right)^{1/2} \in [0,+\infty] \qquad \text{for every } A \in \mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2. \tag{3.35}$$

We are now in a position to define the 'analytical' tensor product between $\mathscr{H}_1$ and $\mathscr{H}_2$.

Definition 3.39 (Tensor product) *We define the* tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ *as the completion of the space* $\{A \in \mathscr{H}_1 \otimes_{\mathrm{Alg}} \mathscr{H}_2 \,;\, \|A\|_{\mathscr{H}_1\otimes\mathscr{H}_2} < +\infty\}$ *with respect to the norm* $\|\cdot\|_{\mathscr{H}_1\otimes\mathscr{H}_2}$.

It can be readily proved that $\mathscr{H}_1 \otimes \mathscr{H}_2$ has a natural structure of Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module, whose pointwise norm will be denoted by $|\cdot|_{\mathsf{HS}}$.

Remark 3.40 Let us suppose that the modules $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable. Then their tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ is separable as well. ■

Remark 3.41 Consider any two Hilbert $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules $\mathscr{H}_1^0$ and $\mathscr{H}_2^0$. Let us denote by $\mathscr{H}_1$ and $\mathscr{H}_2$ the $L^2(\mathfrak{m})$-restrictions of $\mathscr{H}_1^0$ and $\mathscr{H}_2^0$, respectively. Then we define the tensor product $\mathscr{H}_1^0 \otimes \mathscr{H}_2^0$ as the $L^0(\mathfrak{m})$-completion of $\mathscr{H}_1 \otimes \mathscr{H}_2$, thus accordingly the space $\mathscr{H}_1^0 \otimes \mathscr{H}_2^0$ is a Hilbert $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. ■

Let $\mathscr{H}$ be a Hilbert $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then for any $n \in \mathbb{N}^+$ we set

$$\mathscr{H}^{\otimes n} \doteq \underbrace{\mathscr{H} \otimes \ldots \otimes \mathscr{H}}_{n \text{ times}}. \tag{3.36}$$

We conclude by pointing out that the operator

$$\mathscr{H} \otimes_{\mathrm{Alg}} \mathscr{H} \ni v_1 \otimes v_2 \longmapsto v_2 \otimes v_1 \in \mathscr{H} \otimes_{\mathrm{Alg}} \mathscr{H}$$

induces an automorphism of $\mathscr{H}^{\otimes 2}$, which will be called *transposition* and denoted by $A \mapsto A^{\mathsf{t}}$. Then a tensor $A \in \mathscr{H}^{\otimes 2}$ is said to be *symmetric* provided $A^{\mathsf{t}} = A$.

3.2 Serre-Swan theorem for proper normed modules

3.2.1 Measurable Banach bundles and section functor

The first part of this subsection is devoted to propose a notion of measurable Banach bundle – or briefly MBB – over a given metric measure space $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$. An alternative definition of MBB, which does not perfectly fit into our framework, can be found in [GP16b].

Definition 3.42 (MBB) *We define a* measurable Banach bundle *over the space* $\mathbb{X}$ *as any quadruplet* $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$*, where:*

i) *The sequence* $\underline{E} = (E_n)_{n\in\mathbb{N}}$ *is a Borel partition of* X.

ii) *The set* $T \doteq \bigsqcup_{n\in\mathbb{N}} E_n \times \mathbb{R}^n$ *is called* total space *and is always implicitly endowed with the* σ*-algebra* $\bigcap_{n\in\mathbb{N}} (\iota_n)_*\mathscr{B}(E_n \times \mathbb{R}^n)$*, where* $\iota_n : E_n \times \mathbb{R}^n \hookrightarrow T$ *denotes the inclusion map for every* $n \in \mathbb{N}$*. Recall Remark 1.17 for the definition of* $(\iota_n)_*\mathscr{B}(E_n \times \mathbb{R}^n)$.

iii) *The map sending any element* $(x, v) \in T$ *to its base point* $x \in \mathrm{X}$ *is denoted by* $\pi : T \to \mathrm{X}$ *and is called* projection map.

iv) *The measurable function* $\mathsf{n} : T \to [0, +\infty)$ *has the property that for any* $n \in \mathbb{N}$ *it holds that* $\mathsf{n}(x, \cdot)$ *is a norm on* $\mathbb{R}^n$ *for* $\mathfrak{m}$*-a.e. point* $x \in E_n$*.*

Given $n \in \mathbb{N}$ and $x \in E_n$, we say that $(\overline{\mathbb{T}})_x \doteq \pi^{-1}\{x\} = \{x\} \times \mathbb{R}^n$ is the *fiber* of $\overline{\mathbb{T}}$ over x. We will often implicitly identify the fiber $(\overline{\mathbb{T}})_x$ with the vector space $\mathbb{R}^n$ itself.

Remark 3.43 It is immediate to check that a subset S of the total space T of an MBB $\overline{\mathbb{T}}$ is measurable if and only if $S \cap (E_n \times \mathbb{R}^n)$ is a Borel subset of $E_n \times \mathbb{R}^n$ for any $n \in \mathbb{N}$. ■

We now describe which are the (pre-)morphisms between any two given MBB's.

Definition 3.44 (MBB pre-morphisms) *Let* $\overline{\mathbb{T}}_1 = (T_1, \underline{E}^1, \pi_1, \mathsf{n}_1)$*,* $\overline{\mathbb{T}}_2 = (T_2, \underline{E}^2, \pi_2, \mathsf{n}_2)$ *be MBB's over* $\mathbb{X}$*. Then a measurable map* $\overline{\varphi} : T_1 \to T_2$ *is said to be an* MBB pre-morphism *from* $\overline{\mathbb{T}}_1$ *to* $\overline{\mathbb{T}}_2$ *provided it holds that the diagram*

$$\begin{array}{ccc} T_1 & \xrightarrow{\overline{\varphi}} & T_2 \\ & \underset{\pi_1}{\searrow} & \downarrow{\scriptstyle \pi_2} \\ & & \mathrm{X} \end{array} \tag{3.37}$$

commutes and that for $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *the operator*

$$\overline{\varphi}|_{(\overline{\mathbb{T}}_1)_x} : \big((\overline{\mathbb{T}}_1)_x, \mathsf{n}_1(x, \cdot)\big) \to \big((\overline{\mathbb{T}}_2)_x, \mathsf{n}_2(x, \cdot)\big)$$

is linear and 1*-Lipschitz.*

We declare two MBB pre-morphisms $\overline{\varphi}, \overline{\varphi}' : T_1 \to T_2$ to be equivalent, briefly $\overline{\varphi} \sim \overline{\varphi}'$, if

$$\overline{\varphi}|_{(\overline{\mathbb{T}}_1)_x} = \overline{\varphi}'|_{(\overline{\mathbb{T}}_1)_x} \qquad \text{holds for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \tag{3.38}$$

We are now finally in a position to define the category of measurable Banach bundles over $\mathbb{X}$:

Definition 3.45 (The category of MBB's) *The collection of measurable Banach bundles over* $\mathbb{X}$ *and of equivalence classes of MBB pre-morphisms form a category, which we shall denote by* $\mathbf{MBB}(\mathbb{X})$*.*

Once a notion of measurable Banach bundle is given, it is natural to consider its 'measurable sections', namely those maps which assign (in a measurable way) to almost every point of the underlying metric measure space an element of the fiber over such point.

It will turn out that the space $\Gamma_0(\mathbb{T})$ of all measurable sections of a measurable Banach bundle $\mathbb{T}$ is a proper L^0-normed L^0-module. The correspondence $\mathbb{T} \mapsto \Gamma_0(\mathbb{T})$ can be made into a functor, called 'section functor', from the category of measurable Banach bundles to the category of proper L^0-normed L^0-modules.

Definition 3.46 (Sections of an MBB) *Let* $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ *be (a representative of) an MBB over* $\mathbb{X}$*. Then we call* (measurable) section *of* $\overline{\mathbb{T}}$ *any measurable right inverse of the projection* π*, i.e. any measurable map* $\overline{s} : \mathrm{X} \to T$ *such that* $\pi \circ \overline{s} = \mathrm{id}_{\mathrm{X}}$*.*

Two given sections $\overline{s}_1, \overline{s}_2 : \mathrm{X} \to T$ are equivalent provided $\overline{s}_1(x) = \overline{s}_2(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. The space of all equivalence classes of sections of $\overline{\mathbb{T}}$ will be denoted by $\Gamma_0(\overline{\mathbb{T}})$. We add some structure to the set $\Gamma_0(\overline{\mathbb{T}})$, in order to get an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module:

i) Vector space. Let $s_1, s_2 \in \Gamma_0(\overline{\mathbb{T}})$ and $\lambda \in \mathbb{R}$. Pick a representative $\overline{s}_i : \mathrm{X} \to T$ of s_i for each $i = 1, 2$. Then we can pointwise define the sections $\overline{s}_1 + \overline{s}_2$ and $\lambda\,\overline{s}_1$ of $\overline{\mathbb{T}}$ as

$$(3.39) \qquad \begin{aligned} (\overline{s}_1 + \overline{s}_2)(x) &\doteq \overline{s}_1(x) + \overline{s}_2(x) \\ (\lambda\,\overline{s}_1)(x) &\doteq \lambda\,\overline{s}_1(x) \end{aligned} \qquad \text{for every } x \in \mathrm{X}.$$

Therefore we define $s_1 + s_2 \in \Gamma_0(\overline{\mathbb{T}})$ and $\lambda\, s_1 \in \Gamma_0(\overline{\mathbb{T}})$ as the equivalence classes of $\overline{s}_1 + \overline{s}_2$ and $\lambda\,\overline{s}_1$, respectively. It can be readily seen that these operations are well-defined and give to $\Gamma_0(\overline{\mathbb{T}})$ a vector space structure.

ii) Multiplication by L^0-functions. Let $s \in \Gamma_0(\overline{\mathbb{T}})$ and $f \in L^0(\mathfrak{m})$ be fixed. Choose a representative $\overline{s} : \mathrm{X} \to T$ of s and a Borel version $\overline{f} : \mathrm{X} \to \mathbb{R}$ of f. Then it holds that the map $\overline{f} \cdot \overline{s} : \mathrm{X} \to T$, given by

$$(3.40) \qquad (\overline{f} \cdot \overline{s})(x) \doteq \overline{f}(x)\,\overline{s}(x) \in (\overline{\mathbb{T}})_x \qquad \text{for every } x \in \mathrm{X},$$

is a section of $\overline{\mathbb{T}}$. Hence we define $f \cdot s \in \Gamma_0(\overline{\mathbb{T}})$ as the equivalence class of $\overline{f} \cdot \overline{s}$. This yields a well-posed bilinear operator $\cdot : L^0(\mathfrak{m}) \times \Gamma_0(\overline{\mathbb{T}}) \to \Gamma_0(\overline{\mathbb{T}})$.

iii) Pointwise norm. Consider any section $s \in \Gamma_0(\overline{\mathbb{T}})$. Pick a representative $\overline{s} : \mathrm{X} \to T$ of s. Define the Borel function $|\overline{s}| : \mathrm{X} \to [0, +\infty)$ as

$$(3.41) \qquad |\overline{s}|(x) \doteq \mathsf{n}\big(\overline{s}(x)\big) \qquad \text{for every } x \in \mathrm{X}.$$

Then we denote by $|s| \in L^0(\mathfrak{m})$ the equivalence class of the function $|\overline{s}|$. This provides us a well-defined operator $|\cdot| : \Gamma_0(\overline{\mathbb{T}}) \to L^0(\mathfrak{m})$.

iv) Topology on $\Gamma_0(\overline{\mathbb{T}})$. Pick a Borel probability measure $\mathfrak{m}'$ on X with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. Then we define the distance $\mathsf{d}_{\Gamma_0(\overline{\mathbb{T}})}$ on $\Gamma_0(\overline{\mathbb{T}})$ as follows:

$$(3.42) \qquad \mathsf{d}_{\Gamma_0(\overline{\mathbb{T}})}(s_1, s_2) \doteq \int |s_1 - s_2| \wedge 1 \,\mathrm{d}\mathfrak{m}' \qquad \text{for every } s_1, s_2 \in \Gamma_0(\overline{\mathbb{T}}).$$

We denote by τ the topology induced by $\mathsf{d}_{\Gamma_0(\overline{\mathbb{T}})}$.

It turns out that $\Gamma_0(\overline{\mathbb{T}})$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Furthermore, given a measurable Banach bundle $\mathbb{T}$ over the space $\mathbb{X}$, we define

$$(3.43) \qquad \Gamma_0(\mathbb{T}) \doteq \Gamma_0(\overline{\mathbb{T}}) \qquad \text{for one (thus any) representative } \overline{\mathbb{T}} \text{ of } \mathbb{T}.$$

Well-posedness of such definition is granted by the fact that $\Gamma_0(\overline{\mathbb{T}}_1)$ and $\Gamma_0(\overline{\mathbb{T}}_2)$ are isomorphic as $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules whenever $\overline{\mathbb{T}}_1$ and $\overline{\mathbb{T}}_2$ are equivalent bundles.

Remark 3.47 (Constant sections) In the forthcoming discussion, a key role will be played by those sections of $\mathbb{T}$ that are obtained in this way: for any $n \in \mathbb{N}$ and any vector $v \in \mathbb{R}^n$, we consider the section $\boldsymbol{v} \in \Gamma_0(\mathbb{T})$ that is identically equal to v on E_n and null elsewhere.

More precisely, for any $n \in \mathbb{N}$ and any vector $v \in \mathbb{R}^n$, we define $\boldsymbol{v} \in \Gamma_0(\mathbb{T})$ as the equivalence class of the section $\overline{\boldsymbol{v}} : \mathrm{X} \to T$, given by

$$(3.44) \qquad \overline{\boldsymbol{v}}(x) \doteq \begin{cases} (x, v) & \text{if } x \in E_n, \\ (x, 0) & \text{if } x \in \mathrm{X} \setminus E_n, \end{cases}$$

where $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ is any chosen representative of $\mathbb{T}$. ■

Proposition 3.48 *The space $\Gamma_0(\mathbb{T})$ is a proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. More precisely, for any representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ of the bundle $\mathbb{T}$ it holds that $\underline{E} = (E_n)_{n\in\mathbb{N}}$ constitutes a dimensional decomposition of $\Gamma_0(\mathbb{T})$.*

Proof. Fix $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n}) \in \mathbb{T}$ and $n \in \mathbb{N}$. Denote by $\mathsf{e}_1, \dots, \mathsf{e}_n$ the canonical basis of $\mathbb{R}^n$. Then consider the sections $\mathbf{e}_1, \dots, \mathbf{e}_n \in \Gamma_0(\mathbb{T})$ defined in Remark 3.47. We claim that

$$\mathbf{e}_1, \dots, \mathbf{e}_n \quad \text{is a local basis for } \Gamma_0(\mathbb{T}) \text{ on } E_n. \tag{3.45}$$

Take any $s \in \Gamma_0(\mathbb{T})$, with representative $\overline{s} : \mathrm{X} \to T$. Since the map $\overline{s}_{|E_n} : E_n \to E_n \times \mathbb{R}^n$ is Borel measurable by Remark 3.43, there exists a Borel function $\overline{c} = (\overline{c}_1, \dots, \overline{c}_n) : E_n \to \mathbb{R}^n$ such that $\overline{s}(x) = \big(x, \overline{c}(x)\big)$ holds for every $x \in E_n$. Now extend each $\overline{c}_i$ to the whole X by declaring it equal to 0 on the complement of E_n. Hence $\chi_{E_n} \cdot \overline{s} = \sum_{i=1}^n \overline{c}_i \cdot \overline{\mathbf{e}}_i$, where $\overline{\mathbf{e}}_1, \dots, \overline{\mathbf{e}}_n$ are defined as in (3.44). Calling $c_i \in L^0(\mathfrak{m})$ the equivalence class of $\overline{c}_i$ for every $i = 1, \dots, n$, we deduce that $\chi_{E_n} \cdot s = \sum_{i=1}^n c_i \cdot \mathbf{e}_i$, which grants that $\mathbf{e}_1, \dots, \mathbf{e}_n$ generate $\Gamma_0(\mathbb{T})$ on E_n.

Now suppose that $\sum_{i=1}^n c_i \cdot \mathbf{e}_i = 0$ for some $c_1, \dots, c_n \in L^0(\mathfrak{m})$. Choose a Borel representative $\overline{c}_i : \mathrm{X} \to \mathbb{R}$ of each c_i, whence $\big(\overline{c}_1(x), \dots, \overline{c}_n(x)\big) = \big(\sum_{i=1}^n \overline{c}_i \cdot \overline{\mathbf{e}}_i\big)(x) = 0$ holds for $\mathfrak{m}$-a.e. $x \in E_n$, in other words $\chi_{E_n} c_1, \dots, \chi_{E_n} c_n = 0$. Therefore the sections $\mathbf{e}_1, \dots, \mathbf{e}_n$ are independent on E_n. This yields (3.45) and accordingly the statement. □

In order to define the functor Γ_0 from $\mathbf{MBB}(\mathbb{X})$ to $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$, it only remains to declare how it behaves on morphisms, namely to associate to any MBB morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ a suitable morphism $\Gamma_0(\varphi) : \Gamma_0(\mathbb{T}_1) \to \Gamma_0(\mathbb{T}_2)$ of proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules.

Let $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathbf{n}_i)$ be representatives of MBB's over $\mathbb{X}$ for $i = 1, 2$. Take a section $\overline{s}$ of $\overline{\mathbb{T}}_1$ and a pre-morphism $\overline{\varphi} : T_1 \to T_2$. Since $\overline{\varphi} \circ \overline{s} : \mathrm{X} \to T_2$ is measurable as composition of measurable maps and $\pi_2 \circ \overline{\varphi} \circ \overline{s} = \pi_1 \circ \overline{s} = \mathrm{id}_{\mathrm{X}}$, we conclude that $\overline{\varphi} \circ \overline{s}$ is a section of $\overline{\mathbb{T}}_2$.

Now let us call $\mathbb{T}_1$, $\mathbb{T}_2$ and φ the equivalence classes of $\overline{\mathbb{T}}_1$, $\overline{\mathbb{T}}_2$ and $\overline{\varphi}$, respectively. Then we define $\Gamma_0(\varphi) : \Gamma_0(\mathbb{T}_1) \to \Gamma_0(\mathbb{T}_2)$ as follows: given any $s \in \Gamma_0(\mathbb{T}_1)$, we set

$$\Gamma_0(\varphi)(s) \doteq \text{the equivalence class of } \overline{\varphi} \circ \overline{s}, \text{ where } \overline{s} \text{ is any representative of } s. \tag{3.46}$$

In the next result, we shall prove that $\Gamma_0(\varphi)$ is actually a module morphism:

Lemma 3.49 *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two measurable Banach bundles over $\mathbb{X}$ and let $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$. Then $\Gamma_0(\varphi) \in \mathrm{Mor}\big(\Gamma_0(\mathbb{T}_1), \Gamma_0(\mathbb{T}_2)\big)$.*

Proof. It suffices to show that for any $s_1, s_2 \in \Gamma_0(\mathbb{T}_1)$ and $f_1, f_2 \in L^0(\mathfrak{m})$ one has

$$\begin{gathered} \Gamma_0(\varphi)(f_1 \cdot s_1 + f_2 \cdot s_2) = f_1 \cdot \Gamma_0(\varphi)(s_1) + f_2 \cdot \Gamma_0(\varphi)(s_2), \\ \big|\Gamma_0(\varphi)(s_1)\big| \leq |s_1| \qquad \mathfrak{m}\text{-a.e. in X}. \end{gathered} \tag{3.47}$$

Choose representatives $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathbf{n}_i)$ of $\mathbb{T}_i$, $\overline{\varphi} : T_1 \to T_2$ of φ and $\overline{s}_i : \mathrm{X} \to T_1$ of s_i for each $i = 1, 2$. Further, choose Borel functions $\overline{f}_1, \overline{f}_2 : \mathrm{X} \to \mathbb{R}$ that are representatives of f_1 and f_2, respectively. Hence for $\mathfrak{m}$-a.e. point $x \in \mathrm{X}$ it holds that

$$\begin{aligned} \big(\overline{\varphi} \circ (\overline{f}_1 \cdot \overline{s}_1 + \overline{f}_2 \cdot \overline{s}_2)\big)(x) &= \overline{\varphi}\big(\overline{f}_1(x)\, \overline{s}_1(x) + \overline{f}_2(x)\, \overline{s}_2(x)\big) \\ &= \overline{f}_1(x)\, (\overline{\varphi} \circ \overline{s}_1)(x) + \overline{f}_2(x)\, (\overline{\varphi} \circ \overline{s}_2)(x), \end{aligned}$$

whence $\Gamma_0(\varphi)(f_1 \cdot s_1 + f_2 \cdot s_2) = f_1 \cdot \Gamma_0(\varphi)(s_1) + f_2 \cdot \Gamma_0(\varphi)(s_2)$, i.e. the first in (3.47).

To prove the second one, observe that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ one has that

$$|\overline{\varphi} \circ \overline{s}_1|(x) = \mathbf{n}_2\big((\overline{\varphi} \circ \overline{s}_1)(x)\big) = (\mathbf{n}_2 \circ \overline{\varphi})\big(\overline{s}_1(x)\big) \leq \mathbf{n}_1\big(\overline{s}_1(x)\big) = |\overline{s}_1|(x),$$

so that $\big|\Gamma_0(\varphi)(s_1)\big| \leq |s_1|$ holds $\mathfrak{m}$-a.e. in X. Therefore the thesis is achieved. □

Definition 3.50 (Section functor) *The covariant functor* $\Gamma_0 : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^0_{\rm pr}(\mathbb{X})$, *which associates to any object* $\mathbb{T}$ *of* $\mathbf{MBB}(\mathbb{X})$ *the object* $\Gamma_0(\mathbb{T})$ *of* $\mathbf{NMod}^0_{\rm pr}(\mathbb{X})$ *and to any morphism* $\varphi : \mathbb{T}_1 \to \mathbb{T}_2$ *the morphism* $\Gamma_0(\varphi) : \Gamma_0(\mathbb{T}_1) \to \Gamma_0(\mathbb{T}_2)$, *is called* section functor.

3.2.2 Serre-Swan theorem

We now prove that the section functor is actually an equivalence of categories. We shall refer to such result as the *Serre-Swan theorem for normed modules*. First, we prove a technical lemma that provides us with a suitable dense subset of the space of all measurable sections of a measurable Banach bundle. Then such density result (Lemma 3.51) will be needed to show that the section functor is 'essentially surjective' (Proposition 3.52) and fully faithful (Proposition 3.53). Finally, the Serre-Swan theorem (Theorem 3.54) will immediately follow.

Given a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$ and any $n \in \mathbb{N}$, we set

$$\mathsf{S}(\mathbb{T}, n) \doteq \bigg\{ \sum_{i \in \mathbb{N}} \chi_{A_i} \cdot \boldsymbol{q}^i \;\bigg|\; (A_i)_{i\in\mathbb{N}} \text{ is a Borel partition of } E_n,\ (q^i)_{i\in\mathbb{N}} \subseteq \mathbb{Q}^n \bigg\}, \tag{3.48}$$

where the 'constant sections' $\boldsymbol{q}^i \in \Gamma_0(\mathbb{T})$ are defined as in Remark 3.47. Note that any element of the form $\sum_{i\in\mathbb{N}} \chi_{A_i} \cdot \boldsymbol{q}^i \in \Gamma_0(\mathbb{T})$ is well-defined since the sets A_i's are pairwise disjoint.

Then we define the family $\mathsf{S}(\mathbb{T}) \subseteq \Gamma(\mathbb{T})$ of *simple sections* of $\mathbb{T}$ as follows:

$$\mathsf{S}(\mathbb{T}) \doteq \big\{ t \in \Gamma_0(\mathbb{T}) \;\big|\; \chi_{E_n} \cdot t \in \mathsf{S}(\mathbb{T}, n) \text{ for every } n \in \mathbb{N} \big\}. \tag{3.49}$$

We now show that such class of sections, which is a $\mathbb{Q}$-vector space, is actually dense in $\Gamma_0(\mathbb{T})$:

Lemma 3.51 *Let* $\mathbb{T}$ *be a measurable Banach bundle over* $\mathbb{X}$. *Then* $\mathsf{S}(\mathbb{T})$ *is dense in* $\Gamma_0(\mathbb{T})$.

Proof. Let $s \in \Gamma_0(\mathbb{T})$ and $\varepsilon > 0$ be fixed. Choose any Borel probability measure $\mathfrak{m}'$ on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and define the distance $\mathsf{d}_{\Gamma_0(\mathbb{T})}$ on $\Gamma_0(\mathbb{T})$ as in (3.42). We aim to construct a simple section $t \in \mathsf{S}(\mathbb{T})$ that satisfies the inequality $\mathsf{d}_{\Gamma_0(\mathbb{T})}(s,t) \leq \varepsilon$. In order to do so, choose representatives $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ and $\overline{s} : \mathrm{X} \to T$ of $\mathbb{T}$ and s, respectively. We can clearly suppose without loss of generality that $\mathbf{n}(x,\cdot)$ is a norm for every $x \in \mathrm{X}$. Given any $n \in \mathbb{N}$, let us define

$$E_{n,k} \doteq \bigg\{ x \in E_n \;\bigg|\; k-1 < \sup_{q \in \mathbb{Q}^n \setminus \{0\}} \frac{\mathbf{n}(x,q)}{|q|} \leq k \bigg\} \qquad \text{for every } k \in \mathbb{N}.$$

Since $E_n \ni x \mapsto \mathbf{n}(x,q)/|q|$ is Borel for every $q \in \mathbb{Q}^n \setminus \{0\}$, we know that each $E_{n,k}$ is Borel. Moreover, the fact that any two norms on $\mathbb{R}^n$ are equivalent grants that the supremum in the definition of $E_{n,k}$ is finite for every $x \in E_n$, whence for all $n \in \mathbb{N}$ we have that $(E_{n,k})_{k\in\mathbb{N}}$ constitutes a Borel partition of E_n. For any $n,k \in \mathbb{N}$, call $\overline{s}_{n,k} : E_{n,k} \to \mathbb{R}^n$ that Borel map for which $\overline{s}(x) = \big(x, \overline{s}_{n,k}(x)\big)$ for every $x \in E_{n,k}$. It is well-known that there exists a Borel map $\overline{t}_{n,k} : E_{n,k} \to \mathbb{R}^n$ whose image is a finite subset of $\mathbb{Q}^n$ and satisfying

$$\int_{E_{n,k}} \big| k\,\overline{s}_{n,k}(x) - k\,\overline{t}_{n,k}(x) \big| \wedge 1 \,\mathrm{d}\mathfrak{m}'(x) \leq \frac{\varepsilon}{2^{n+k}}. \tag{3.50}$$

Given that $\mathsf{n}(x,c) \leq k\,|c|$ holds for every $x \in E_{n,k}$ and $c \in \mathbb{R}^n$, we deduce from (3.50) that

$$\int_{E_{n,k}} \mathsf{n}\big(x, \bar{s}_{n,k}(x) - \bar{t}_{n,k}(x)\big) \wedge 1 \, \mathrm{d}\mathfrak{m}'(x) \leq \frac{\varepsilon}{2^{n+k}}. \tag{3.51}$$

Now let us denote by $\bar{t} : \mathrm{X} \to T$ the measurable map such that $\bar{t}|_{E_{n,k}} = (\mathrm{id}_{E_{n,k}}, \bar{t}_{n,k})$ holds for every $n, k \in \mathbb{N}$, which is meaningful since $(E_{n,k})_{n,k\in\mathbb{N}}$ is a partition of X. Call $t \in \Gamma_0(\mathbb{T})$ the equivalence class of $\bar{t}$. Notice that $t \in \mathsf{S}(\mathbb{T})$ by construction. Property (3.51) yields

$$\begin{aligned} \mathsf{d}_{\Gamma(\mathbb{T})}(s,t) &= \int |s-t| \wedge 1 \, \mathrm{d}\mathfrak{m}' = \sum_{n,k\in\mathbb{N}} \int_{E_{n,k}} \mathsf{n}\big(x, \bar{s}_{n,k}(x) - \bar{t}_{n,k}(x)\big) \wedge 1 \, \mathrm{d}\mathfrak{m}'(x) \\ &\leq \sum_{n,k\in\mathbb{N}} \frac{\varepsilon}{2^{n+k}} = \varepsilon, \end{aligned}$$

which gives the statement. □

We briefly recall the notion of Carathéodory map, which will be needed in the proofs of the next two results. Given three metric spaces W, Y and Z, we say that a map $F : \mathrm{W} \times \mathrm{Y} \to \mathrm{Z}$ is *Carathéodory* provided the following hold:

i) $F(\cdot, y) : \mathrm{W} \to \mathrm{Z}$ is Borel measurable for every $y \in \mathrm{Y}$,

ii) $F(w, \cdot) : \mathrm{Y} \to \mathrm{Z}$ is continuous for every $w \in \mathrm{W}$.

It is well-known that the Carathéodory map F is (jointly) Borel measurable as soon as the metric space Y is separable.

Proposition 3.52 *Let $\mathscr{M}$ be a proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then there exists a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$ such that $\Gamma_0(\mathbb{T})$ is isomorphic to $\mathscr{M}$.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a dimensional decomposition of the module $\mathscr{M}$. Set $\underline{E} \doteq (E_n)_{n\in\mathbb{N}}$ and take T, π as in the definition of MBB. In order to define n, fix a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}$ such that the elements $v_1, \ldots, v_n$ form a local basis for $\mathscr{M}$ on E_n for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We define the linear and continuous operator $P_n : \mathbb{R}^n \to \mathscr{M}$ in the following way:

$$P_n(c) \doteq \chi_{E_n} \cdot (c_1\, v_1 + \ldots + c_n\, v_n) \in \mathscr{M} \qquad \text{for every } c = (c_1, \ldots, c_n) \in \mathbb{R}^n.$$

For any $q \in \mathbb{Q}^n$, choose any Borel representative $\overline{|P_n(q)|} : \mathrm{X} \to [0, +\infty)$ of $|P_n(q)| \in L^0(\mathfrak{m})$. Hence there is a Borel set $N_n \subseteq E_n$, with $\mathfrak{m}(N_n) = 0$, such that for any $x \in E_n \setminus N_n$ it holds

$$\begin{aligned} \overline{|P_n(q^1) + P_n(q^2)|}(x) &\leq \overline{|P_n(q^1)|}(x) + \overline{|P_n(q^2)|}(x) && \text{for every } q^1, q^2 \in \mathbb{Q}^n, \\ \overline{|P_n(\lambda\, q)|}(x) &= |\lambda|\, \overline{|P_n(q)|}(x) && \text{for every } \lambda \in \mathbb{Q} \text{ and } q \in \mathbb{Q}^n, \\ \overline{|P_n(q)|}(x) &> 0 && \text{for every } q \in \mathbb{Q}^n \setminus \{0\}. \end{aligned} \tag{3.52}$$

Then let us define

$$\mathsf{n}(x, q) \doteq \overline{|P_n(q)|}(x) \qquad \text{for every } x \in E_n \setminus N_n \text{ and } q \in \mathbb{Q}^n. \tag{3.53}$$

We deduce from (3.52) that $\mathsf{n}(x, \cdot)$ is a norm on $\mathbb{Q}^n$ for every $x \in E_n \setminus N_n$. In particular it is uniformly continuous, whence it can be uniquely extended to a uniformly continuous map on the whole $\mathbb{R}^n$, still denoted by $\mathsf{n}(x, \cdot)$. By approximation, we see that such extension is actually a norm on $\mathbb{R}^n$. Finally, we set $\mathsf{n}(x,c) \doteq 0$ for every $x \in N_n$ and $c \in \mathbb{R}^n$. We thus built a function $\mathsf{n} : T \to [0,+\infty)$. We claim that

$$\mathsf{n}|_{E_n \times \mathbb{R}^n} \text{ is a Carathéodory function for every } n \in \mathbb{N}, \tag{3.54}$$

which grants that each $\mathsf{n}|_{E_n \times \mathbb{R}^n}$ is Borel, so accordingly that n is measurable by Remark 3.43. First of all, fix $n \in \mathbb{N}$ and notice that the function $\mathsf{n}(x,\cdot) : \mathbb{R}^n \to [0,+\infty)$ is continuous for every $x \in E_n$. Moreover, given any $c \in \mathbb{R}^n$ and a sequence $(q^k)_{k\in\mathbb{N}} \subseteq \mathbb{Q}^n$ converging to c, we have that $\mathsf{n}(x,c) = \lim_k \mathsf{n}(x,q^k) = \lim_k \overline{|P_n(q^k)|}(x)$ for every $x \in E_n \setminus N_n$, whence the function $\mathsf{n}(\cdot,c) : E_n \to [0,+\infty)$ is Borel as pointwise limit of a sequence of Borel functions. Therefore the claim (3.54) is proved. We thus deduce that $\overline{\mathbb{T}} \doteq (T, \underline{E}, \pi, \mathsf{n})$ is an MBB over the space $\mathbb{X}$. Then let us denote by $\mathbb{T}$ the equivalence class of $\overline{\mathbb{T}}$.

In order to get the statement, we want to exhibit a module isomorphism $I : \Gamma_0(\mathbb{T}) \to \mathscr{M}$, namely an $L^0(\mathfrak{m})$-linear map preserving the pointwise norm. We proceed as follows: given any $s \in \Gamma_0(\mathbb{T})$, choose a representative $\overline{s} : \mathrm{X} \to T$. For any $n \in \mathbb{N}$, pick $\overline{c}^n : \mathrm{X} \to \mathbb{R}^n$ Borel such that $\overline{s}(x) = \big(x, \overline{c}^n(x)\big)$ for every $x \in E_n$ and call $c_1^n, \ldots, c_n^n \in L^0(\mathfrak{m})$ those elements for which $(c_1^n, \ldots, c_n^n)$ is the equivalence class of $\overline{c}^n$. Now let us define

$$I(s) \doteq \sum_{n\in\mathbb{N}} \chi_{E_n} \cdot (c_1^n \cdot v_1 + \ldots + c_n^n \cdot v_n) \in \mathscr{M}. \tag{3.55}$$

One can easily see that the resulting map $I : \Gamma_0(\mathbb{T}) \to \mathscr{M}$ is a (well-defined) $L^0(\mathfrak{m})$-linear and continuous operator. We show that it is surjective: fix any $v \in \mathscr{M}$, whence for each $n \in \mathbb{N}$ there exist $c_1^n, \ldots, c_n^n \in L^0(\mathfrak{m})$ such that $\chi_{E_n} \cdot v = \chi_{E_n} \cdot (c_1^n \cdot v_1 + \ldots + c_n^n \cdot v_n)$. Pick any Borel representative $\overline{c}^n : \mathrm{X} \to \mathbb{R}^n$ of $(c_1^n, \ldots, c_n^n)$ and define $\overline{s} : \mathrm{X} \to T$ as $\overline{s}(x) \doteq \big(x, \overline{c}^n(x)\big)$ for every $n \in \mathbb{N}$ and $x \in E_n$. Hence the equivalence class $s \in \Gamma_0(\mathbb{T})$ of $\overline{s}$ satisfies $I(s) = v$, thus proving that the map I is surjective. It only remains to prove that $\big|I(s)\big| = |s|$ holds $\mathfrak{m}$-a.e. in X for every $s \in \Gamma_0(\mathbb{T})$. First of all, for any $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$ one has that $I(\boldsymbol{q}) = P_n(q)$, where the definition of $\boldsymbol{q}$ is taken from Remark 3.47. Therefore

$$\big|I(\boldsymbol{q})\big| = \big|P_n(q)\big| \overset{(3.53)}{=} \mathsf{n} \circ \boldsymbol{q} \overset{(3.41)}{=} |\boldsymbol{q}| \qquad \text{holds } \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{3.56}$$

We then directly deduce from (3.56) and the $L^0(\mathfrak{m})$-linearity of I that the equality $\big|I(t)\big| = |t|$ is verified $\mathfrak{m}$-a.e. for every simple section $t \in \mathsf{S}(\mathbb{T})$. Recall that $\mathsf{S}(\mathbb{T})$ is dense in $\Gamma_0(\mathbb{T})$, as seen in Lemma 3.51. Since both I and the pointwise norm are continuous operators, we finally conclude that $\big|I(s)\big| = |s|$ holds $\mathfrak{m}$-a.e. for every $s \in \Gamma_0(\mathbb{T})$. Therefore I preserves the pointwise norm, thus completing the proof. □

Proposition 3.53 *The section functor $\Gamma : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ satisfies the following:*

i) Γ *is* full, *i.e. given two objects $\mathbb{T}_1, \mathbb{T}_2$ in $\mathbf{MBB}(\mathbb{X})$ and a morphism $\Phi : \Gamma(\mathbb{T}_1) \to \Gamma(\mathbb{T}_2)$, there exists a morphism $\varphi : \mathbb{T}_1 \to \mathbb{T}_2$ such that $\Phi = \Gamma(\varphi)$.*

ii) Γ *is* faithful, *i.e. given two objects $\mathbb{T}_1, \mathbb{T}_2$ in $\mathbf{MBB}(\mathbb{X})$ and morphisms $\varphi, \psi : \mathbb{T}_1 \to \mathbb{T}_2$ with $\Gamma(\varphi) = \Gamma(\psi)$, it holds that $\varphi = \psi$.*

Proof. We divide the proof into two steps:

FAITHFUL. Fix two measurable Banach bundles $\mathbb{T}_1$ and $\mathbb{T}_2$. Let $\varphi, \psi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ be two different bundle morphisms. Choose a representative $\overline{T}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ of $\mathbb{T}_i$ for $i = 1, 2$, then representatives $\overline{\varphi}, \overline{\psi} : T_1 \to T_2$ of φ and ψ, respectively. Hence there exist $n \in \mathbb{N}$ and a Borel set $E \subseteq E^1_n$, with $\mathfrak{m}(E) > 0$, such that $\overline{\varphi}|_{(\mathbb{T}_1)_x} \neq \overline{\psi}|_{(\mathbb{T}_1)_x}$ for every $x \in E$. Let us denote by $\mathsf{e}_1, \ldots, \mathsf{e}_n$ the canonical basis of $\mathbb{R}^n$. Therefore there exists $k \in \{1, \ldots, n\}$ such that

$$\mathfrak{m}\big(\big\{x \in E \,\big|\, \overline{\varphi}(x, \mathsf{e}_k) \neq \overline{\psi}(x, \mathsf{e}_k)\big\}\big) > 0.$$

This means that $\overline{\varphi} \circ \bar{\mathbf{e}}_k$ is not $\mathfrak{m}$-a.e. coincident with $\overline{\psi} \circ \bar{\mathbf{e}}_k$, where the section $\bar{\mathbf{e}}_k$ is defined as in Remark 3.47, whence $\Gamma_0(\varphi)(\mathbf{e}_k) \neq \Gamma_0(\psi)(\mathbf{e}_k)$. This implies that $\Gamma_0(\varphi) \neq \Gamma_0(\psi)$, thus proving that the functor Γ_0 is faithful.
FULL. Fix measurable Banach bundles $\mathbb{T}_1$, $\mathbb{T}_2$ and a module morphism $\Phi : \Gamma_0(\mathbb{T}_1) \to \Gamma_0(\mathbb{T}_2)$. We aim to show that there exists a bundle morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ such that $\Phi = \Gamma_0(\varphi)$. Since the ideas of the proof are similar in spirit to those that have been used for proving Proposition 3.52, we shall omit some of the details. For $i = 1, 2$, let us choose any representative $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathbf{n}_i)$ of $\mathbb{T}_i$. We define the Borel sets $F_{n,m} \subseteq \mathrm{X}$ as

$$F_{n,m} \doteq E^1_n \cap E^2_m \qquad \text{for every } n, m \in \mathbb{N}.$$

For any $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$, consider the section $\boldsymbol{q} \in \Gamma_0(\mathbb{T}_1)$ as in Remark 3.47 and choose a representative $\overline{\Phi(\boldsymbol{q})} : \mathrm{X} \to T_2$ of $\Phi(\boldsymbol{q}) \in \Gamma_0(\mathbb{T}_2)$. Given any $n, m \in \mathbb{N}$, there exists a Borel subset $N_{n,m}$ of $F_{n,m}$, with $\mathfrak{m}(N_{n,m}) = 0$, such that for every $x \in F_{n,m} \setminus N_{n,m}$ it holds

$$\begin{aligned} \overline{\Phi(\boldsymbol{q}^1 + \boldsymbol{q}^2)}(x) &= \overline{\Phi(\boldsymbol{q}^1)}(x) + \overline{\Phi(\boldsymbol{q}^2)}(x) && \text{for every } q^1, q^2 \in \mathbb{Q}^n, \\ \overline{\Phi(\lambda\, \boldsymbol{q})}(x) &= \lambda\, \overline{\Phi(\boldsymbol{q})}(x) && \text{for every } \lambda \in \mathbb{Q} \text{ and } q \in \mathbb{Q}^n, \\ \mathbf{n}_2\big(\overline{\Phi(\boldsymbol{q})}(x)\big) &\leq \mathbf{n}_1\big(\overline{\boldsymbol{q}}(x)\big) && \text{for every } q \in \mathbb{Q}^n. \end{aligned} \tag{3.57}$$

Then let us define

$$\overline{\varphi}(x, q) \doteq \begin{cases} \overline{\Phi(\boldsymbol{q})}(x) & \text{for every } x \in F_{n,m} \setminus N_{n,m} \text{ and } q \in \mathbb{Q}^n, \\ 0_{\mathbb{R}^m} & \text{for every } x \in N_{n,m} \text{ and } q \in \mathbb{Q}^n. \end{cases} \tag{3.58}$$

Property (3.57) grants that $\overline{\varphi}(x, \cdot) : \big(\mathbb{Q}^n, \mathbf{n}_1(x, \cdot)\big) \to \big(\mathbb{R}^m, \mathbf{n}_2(x, \cdot)\big)$ is a $\mathbb{Q}$-linear 1-Lipschitz operator for all $x \in F_{n,m}$, whence it can be uniquely extended to an $\mathbb{R}$-linear 1-Lipschitz operator $\overline{\varphi}(x, \cdot) : \big(\mathbb{R}^n, \mathbf{n}_1(x, \cdot)\big) \to \big(\mathbb{R}^m, \mathbf{n}_2(x, \cdot)\big)$. This defines a map $\overline{\varphi} : T_1 \to T_2$. To show that such map is an MBB pre-morphism, it only remains to check its measurability, which amounts to proving that $\overline{\varphi}|_{F_{n,m} \times \mathbb{R}^n} : F_{n,m} \times \mathbb{R}^n \to F_{n,m} \times \mathbb{R}^m$ is Borel for every $n, m \in \mathbb{N}$. We actually show that each $\overline{\varphi}|_{F_{n,m} \times \mathbb{R}^n}$ is a Carathéodory map: for any $x \in F_{n,m}$ we have that $\overline{\varphi}(x, \cdot)$ is continuous by its very construction, while for any vector $c \in \mathbb{R}^n$ we have that the map $F_{n,m} \ni x \mapsto \overline{\varphi}(x, c) \in \mathbb{R}^m$ is Borel as pointwise limit of the Borel maps $\chi_{F_{n,m}} \overline{\Phi(\boldsymbol{q}^k)}$, where $(q^k)_{k \in \mathbb{N}} \subseteq \mathbb{Q}^n$ is any sequence converging to c. Hence let us define $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ as the equivalence class of the MBB pre-morphism $\overline{\varphi}$.

We conclude by proving that $\Gamma_0(\varphi) = \Phi$. For any $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$, we have that a representative of $\Gamma_0(\varphi)(\boldsymbol{q})$ is given by the map $\overline{\varphi} \circ \overline{\boldsymbol{q}}$, which $\mathfrak{m}$-a.e. coincides in E^1_n with $\overline{\Phi(\boldsymbol{q})}$, whence $\Gamma_0(\varphi)(\boldsymbol{q}) = \Phi(\boldsymbol{q})$. Since both $\Gamma_0(\varphi)$ and Φ are $L^0(\mathfrak{m})$-linear, we thus immediately deduce that $\Gamma_0(\varphi)(t) = \Phi(t)$ for every $t \in \mathsf{S}(\mathbb{T}_1)$. Finally, the density of $\mathsf{S}(\mathbb{T}_1)$ in $\Gamma_0(\mathbb{T}_1)$ – proven in Lemma 3.51 – together with the continuity of $\Gamma_0(\varphi)$ and Φ, grant that $\Gamma_0(\varphi) = \Phi$, as required. Therefore the section functor Γ_0 is full. □

We now collect the last two results, thus obtaining the main theorem of this section:

Theorem 3.54 (Serre-Swan) *Let* $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then the section functor* $\Gamma_0 : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$ *on* $\mathbb{X}$ *is an equivalence of categories.*

Proof. By [Awo06, Proposition 7.25] it suffices to prove that the functor Γ_0 is fully faithful and 'essentially surjective', the latter meaning that for each object $\mathscr{M}$ of $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$ there exists an object $\mathbb{T}$ of $\mathbf{MBB}(\mathbb{X})$ such that $\Gamma_0(\mathbb{T})$ and $\mathscr{M}$ are isomorphic. Therefore we have that Proposition 3.52 and Proposition 3.53 yield the statement. □

Remark 3.55 (Comparison with the classical Serre-Swan theorem) We now point out which are the main analogies and differences between our result and the Serre-Swan theorem for smooth manifolds, for whose presentation we refer to [Nes03, Chapter 11]. The result in the smooth case can be informally stated in the following way: *the category of smooth vector bundles over a connected manifold M is equivalent to the category of finitely-generated projective $C^\infty(M)$-modules.* In our non-smooth setting we had to replace 'smooth' with 'measurable' – in a sense – and this led to these discrepancies with the case of manifolds:

i) The fibers of a measurable Banach bundle need not have the same dimension (still, they are finite dimensional), while on a connected manifold any smooth vector bundle must have constant dimension for topological reasons.

ii) In the definition of measurable Banach bundle we do not speak about the analogue of the 'trivialising diffeomorphisms', the reason being that one can always patch together countably many measurable maps still obtaining a measurable map. Hence there is no loss of generality in requiring the total space to be of the form $\bigsqcup_{n\in\mathbb{N}} E_n \times \mathbb{R}^n$ and its measurable subsets to be those sets whose intersection with each $E_n \times \mathbb{R}^n$ is Borel.

iii) Given that we want to correlate the measurable Banach bundles with the $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules, which are naturally equipped with a pointwise norm $|\cdot|$, we also require the existence of a function n that assigns a norm to (almost) every fiber of our bundle. A similar structure is not treated in the smooth case.

iv) The Serre-Swan theorem for smooth manifolds deals with modules that are finitely-generated and projective. In our context, any finitely-generated module is automatically projective, as proven in [LP18, Proposition 1.5]. Moreover, the flexibility of $L^0(\mathfrak{m})$ actually allowed us to extend the result to all proper modules, that are not necessarily 'globally' finitely-generated but only 'locally' finitely-generated, in a sense.

We refer to [LP18] for a more detailed discussion about this topic. ■

3.2.3 Measurable Hilbert bundles, pullbacks and duals

Let $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We denote by $\mathbf{HNMod}^0_{\mathrm{pr}}(\mathbb{X})$ the subcategory of $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$ made of those modules that are Hilbert modules. Our goal is to characterise those measurable Banach bundles that correspond to the Hilbert modules via the section functor Γ_0. As one might expect, such bundles are precisely the following ones:

Definition 3.56 (Measurable Hilbert bundle) *Let $\mathbb{T}$ be a measurable Banach bundle over the space $\mathbb{X}$. Then we say that $\mathbb{T}$ is a* measurable Hilbert bundle*, or briefly* MHB*, provided for one (thus any) representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ of $\mathbb{T}$ it holds that $\mathsf{n}(x, \cdot)$ is a norm induced by a scalar product for $\mathfrak{m}$-a.e. point $x \in \mathrm{X}$.*

Given any such point $x \in \mathrm{X}$, we denote the associated scalar product on $(\overline{\mathbb{T}})_x$ by

$$\big\langle (x,v),(x,w)\big\rangle_x \doteq \frac{\mathsf{n}(x,v+w)^2 - \mathsf{n}(x,v)^2 - \mathsf{n}(x,w)^2}{2} \tag{3.59}$$

for every $(x,v),(x,w) \in (\overline{\mathbb{T}})_x$.

We shall denote by $\mathbf{MHB}(\mathbb{X})$ the subcategory of $\mathbf{MBB}(\mathbb{X})$ made of those bundles that are measurable Hilbert bundles. Therefore we can easily prove that:

Proposition 3.57 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Then $\mathbb{T}$ is a measurable Hilbert bundle if and only if $\Gamma_0(\mathbb{T})$ is a Hilbert module.*

Proof. Choose any representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ of the measurable Banach bundle $\mathbb{T}$.
NECESSITY. Suppose that $\mathbb{T}$ is a measurable Hilbert bundle. This means that $\mathsf{n}(x,\cdot)$ satisfies the parallelogram rule for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Now let $s_1, s_2 \in \Gamma_0(\mathbb{T})$ be fixed and choose some representatives $\overline{s}_1, \overline{s}_2 : \mathrm{X} \to T$. Hence for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ it holds that

$$\begin{aligned}
|\overline{s}_1 + \overline{s}_2|^2(x) + |\overline{s}_1 - \overline{s}_2|^2(x) &= \big(\mathsf{n} \circ (\overline{s}_1 + \overline{s}_2)\big)^2(x) + \big(\mathsf{n} \circ (\overline{s}_1 - \overline{s}_2)\big)^2(x) \\
&= 2\,(\mathsf{n} \circ \overline{s}_1)^2(x) + 2\,(\mathsf{n} \circ \overline{s}_2)^2(x) \\
&= 2\,|\overline{s}_1|^2(x) + 2\,|\overline{s}_2|^2(x),
\end{aligned}$$

which grants that $|s_1 + s_2|^2 + |s_1 - s_2|^2 = 2\,|s_1|^2 + 2\,|s_2|^2$ holds $\mathfrak{m}$-a.e. in X. Therefore $\Gamma_0(\mathbb{T})$ is a Hilbert module by arbitrariness of $s_1, s_2 \in \Gamma_0(\mathbb{T})$.
SUFFICIENCY. Suppose that $\Gamma_0(\mathbb{T})$ is Hilbert module. Let $n \in \mathbb{N}$ be fixed. For any $q \in \mathbb{Q}^n$, consider $\overline{\boldsymbol{q}} : \mathrm{X} \to T$ and $\boldsymbol{q} \in \Gamma_0(\mathbb{T})$ as in Remark 3.47. Then there exists an $\mathfrak{m}$-negligible Borel subset N_n of E_n such that $\mathsf{n}(x,\cdot)$ is a norm and the equality

$$|\overline{\boldsymbol{q}}_1 + \overline{\boldsymbol{q}}_2|^2(x) + |\overline{\boldsymbol{q}}_1 - \overline{\boldsymbol{q}}_2|^2(x) = 2\,|\overline{\boldsymbol{q}}_1|^2(x) + 2\,|\overline{\boldsymbol{q}}_2|^2(x) \qquad \text{for every } q_1, q_2 \in \mathbb{Q}^n$$

is satisfied for every point $x \in E_n \setminus N_n$. This implies that

$$\mathsf{n}(x, q_1 + q_2)^2 + \mathsf{n}(x, q_1 - q_2)^2 = 2\,\mathsf{n}(x, q_1)^2 + 2\,\mathsf{n}(x, q_2)^2 \qquad \text{for all } x \in E_n \setminus N_n \text{ and } q_1, q_2 \in \mathbb{Q}^n.$$

Therefore $\mathsf{n}(x,\cdot)$ satisfies the parallelogram rule for every $x \in E_n \setminus N_n$ by continuity, so that accordingly $\mathbb{T}$ is a measurable Hilbert bundle. □

As a consequence of Proposition 3.57, we can conclude that:

Theorem 3.58 (Serre-Swan for Hilbert modules) *The section functor Γ_0 restricts to an equivalence of categories between $\mathbf{MHB}(\mathbb{X})$ and $\mathbf{HNMod}^0_{\rm pr}(\mathbb{X})$.*

Remark 3.59 It has been proved in [Gig17b, Theorem 1.4.11] that any separable Hilbert module (thus in particular any proper Hilbert module by Lemma 3.20) is the space of sections of a suitable measurable Hilbert bundle. Moreover – as pointed out in [Gig17b, Remark 1.4.12] – this theory of Hilbert modules coincides with that of direct integral of Hilbert spaces (cf. [Tak79]). More precisely, under this identification a Hilbert module corresponds to a measurable field of Hilbert spaces. ■

Let $\mathscr{H}_1$, $\mathscr{H}_2$ be two given Hilbert modules over $\mathbb{X}$. Then we can consider their tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$, which is a Hilbert module over $\mathbb{X}$ as well (cf. Subsection 3.1.5).

Remark 3.60 Suppose $\mathscr{H}_1$ and $\mathscr{H}_2$ are proper, with dimensional decomposition $(E^1_n)_{n\in\mathbb{N}}$ and $(E^2_m)_{m\in\mathbb{N}}$, respectively. Then it can be readily checked that the module $\mathscr{H}_1\otimes\mathscr{H}_2$ has dimension equal to nm on $E^1_n\cap E^2_m$ for any $n,m\in\mathbb{N}$. In particular, the dimensional decomposition $(E_k)_{k\in\mathbb{N}}$ of $\mathscr{H}_1\otimes\mathscr{H}_2$ is given by

$$E_k \doteq \bigcup_{\substack{n,m\in\mathbb{N}:\\ nm=k}} E^1_n\cap E^2_m \qquad \text{for every } k\in\mathbb{N}, \tag{3.60}$$

so that $\mathscr{H}_1\otimes\mathscr{H}_2$ is a proper module as well. ■

On the other hand, we now define the tensor product of two MHB's in the following way:

Definition 3.61 (Tensor product of MHB's) *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be measurable Hilbert bundles over $\mathbb{X}$. Choose two representatives $\overline{\mathbb{T}}_1$ and $\overline{\mathbb{T}}_2$, say $\overline{\mathbb{T}}_i=(T_i,\underline{E}^i,\pi_i,\mathsf{n}_i)$ for $i=1,2$. Let us define $\underline{E}\doteq(E_k)_{k\in\mathbb{N}}$ as in (3.60) and T, π accordingly. Given $n,m\in\mathbb{N}$ and $x\in E^1_n\cap E^2_m$ such that $\mathsf{n}_1(x,\cdot)$, $\mathsf{n}_2(x,\cdot)$ are norms induced by a scalar product, we define*

$$\mathsf{n}(x,c)\doteq\left(\sum_{j,j'=1}^{n}\sum_{\ell,\ell'=1}^{m} c_{(j-1)m+\ell}\,c_{(j'-1)m+\ell'}\,\big\langle(x,\mathsf{e}_j),(x,\mathsf{e}_{j'})\big\rangle_{1,x}\big\langle(x,\mathsf{f}_\ell),(x,\mathsf{f}_{\ell'})\big\rangle_{2,x}\right)^{1/2} \tag{3.61}$$

for every $c=(c_1,\dots,c_{nm})\in\mathbb{R}^{nm}$, where $\mathsf{e}_1,\dots,\mathsf{e}_n$ and $\mathsf{f}_1,\dots,\mathsf{f}_m$ denote the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, while $\langle\cdot,\cdot\rangle_{i,x}$ stands for the scalar product on $(\overline{\mathbb{T}}_i)_x$ as in (3.59). Then we define the tensor product *$\mathbb{T}_1\otimes\mathbb{T}_2$ as the equivalence class of $(T,\underline{E},\pi,\mathsf{n})$, which turns out to be a measurable Hilbert bundle over $\mathbb{X}$.*

Given any real number $\lambda\in\mathbb{R}$, we shall write $\lceil\lambda\rceil\in\mathbb{Z}$ to indicate the smallest integer number that is greater than or equal to λ.

Theorem 3.62 *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be measurable Hilbert bundles over $\mathbb{X}$. Then*

$$\Gamma_0(\mathbb{T}_1)\otimes\Gamma_0(\mathbb{T}_2)=\Gamma_0(\mathbb{T}_1\otimes\mathbb{T}_2). \tag{3.62}$$

Proof. We build an operator $\iota:\Gamma_0(\mathbb{T}_1)\otimes\Gamma_0(\mathbb{T}_2)\to\Gamma_0(\mathbb{T}_1\otimes\mathbb{T}_2)$ in the following way: first of all, let us fix $s^1\in\Gamma_0(\mathbb{T}_1)$ and $s^2\in\Gamma_0(\mathbb{T}_2)$. Choose any representatives $\overline{\mathbb{T}}_i=(T_i,\underline{E}^i,\pi_i,\mathsf{n}_i)$ and $\overline{s}^i:\mathrm{X}\to T_i$ for $i=1,2$. Given any natural numbers $n,m\in\mathbb{N}$, any point $x\in E^1_n\cap E^2_m$ and called $\overline{s}^1(x)=(x,v)$, $\overline{s}^2(x)=(x,w)$, we define

$$\overline{s}(x)\doteq(x,c),\qquad \text{where } c_k\doteq v_{\lceil k/m\rceil}\,w_{k-m\lceil k/m\rceil+m}\ \text{ for all } k=1,\dots,nm.$$

Hence the equivalence class $\iota(s^1\otimes s^2)$ of $\overline{s}$ is a section of $\mathbb{T}_1\otimes\mathbb{T}_2$. Simple computations yield

$$\big|\iota(s^1\otimes s^2)\big|=\sqrt{|s^1|\,|s^2|}=|s^1\otimes s^2|\qquad \mathfrak{m}\text{-a.e. on } \mathrm{X}.$$

Therefore ι can be uniquely extended to the whole $\Gamma_0(\mathbb{T}_1)\otimes\Gamma_0(\mathbb{T}_2)$ by linearity and continuity, thus obtaining an $L^0(\mathfrak{m})$-linear operator $\iota:\Gamma_0(\mathbb{T}_1)\otimes\Gamma_0(\mathbb{T}_2)\to\Gamma_0(\mathbb{T}_1\otimes\mathbb{T}_2)$ that preserves the pointwise norm. In order to conclude, it only remains to check that such ι is surjective. Fix $n,m\in\mathbb{N}$ and call $(\mathsf{e}_i)_{i=1}^n$, $(\mathsf{f}_j)_{j=1}^m$ and $(\mathsf{g}_k)_{k=1}^{nm}$ the canonical bases of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{nm}$, respectively. Denote by $\mathbf{e}_i\in\Gamma_0(\mathbb{T}_1)$, $\mathbf{f}_j\in\Gamma_0(\mathbb{T}_2)$ and $\mathbf{g}_k\in\Gamma_0(\mathbb{T}_1\otimes\mathbb{T}_2)$ the associated constant sections. It is then easy to realise that

$$\chi_{E^1_n\cap E^2_m}\cdot\mathbf{g}_k=\iota\big((\chi_{E^1_n\cap E^2_m}\cdot\mathbf{e}_{\lceil k/m\rceil})\otimes(\chi_{E^1_n\cap E^2_m}\cdot\mathbf{f}_{k-m\lceil k/m\rceil+m})\big)\qquad\text{for all } k=1,\dots,nm.$$

Hence the set $(\chi_{E^1_n\cap E^2_m}\cdot\mathbf{g}_k)_{k=1}^{nm}$, which forms a local basis for $\Gamma_0(\mathbb{T}_1\otimes\mathbb{T}_2)$ on $E^1_n\cap E^2_m$, is contained in the range of the map ι. This grants that ι is surjective, as required. □

Now let $\mathbb{X} = (\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $\mathbb{Y} = (\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be two metric measure spaces. Fix a Borel map $f : \mathrm{X} \to \mathrm{Y}$ such that $f_*\mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$. As described in Remark 3.30, we can consider the pullback $f^*\mathscr{M}$ of any $L^0(\mathfrak{m}_\mathrm{Y})$-normed $L^0(\mathfrak{m}_\mathrm{Y})$-module $\mathscr{M}$. On the other hand, we can define what is the pullback of a measurable Banach bundle over $\mathbb{Y}$:

Definition 3.63 (Pullback of an MBB) *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{Y}$. Choose a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ of $\mathbb{T}$. Let us set*

$$\begin{aligned} \underline{E}' &\doteq \big(f^{-1}(E_n)\big)_{n\in\mathbb{N}} && \text{and } T', \pi' \text{ accordingly,} \\ \mathsf{n}'(x,v) &\doteq \mathsf{n}\big(f(x),v\big) && \text{for every } (x,v) \in T'. \end{aligned} \tag{3.63}$$

Then we define the pullback bundle *$f^*\mathbb{T}$ as the equivalence class of $(T', \underline{E}', \pi', \mathsf{n}')$, which turns out to be a measurable Banach bundle over $\mathbb{X}$.*

Theorem 3.64 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{Y}$. Then*

$$f^*\Gamma_0(\mathbb{T}) = \Gamma_0(f^*\mathbb{T}). \tag{3.64}$$

Proof. We aim to build a linear map $f^* : \Gamma_0(\mathbb{T}) \to \Gamma_0(f^*\mathbb{T})$ such that

$$\begin{aligned} |f^*s| &= |s| \circ f && \text{for every } s \in \Gamma_0(\mathbb{T}), \\ \big\{f^*s \,:\, s \in \Gamma_0(\mathbb{T})\big\} && & \text{generates } \Gamma_0(f^*\mathbb{T}). \end{aligned} \tag{3.65}$$

Pick a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ of $\mathbb{T}$ and define $(T', \underline{E}', \pi', \mathsf{n}')$ as in (3.63). Take any section $s \in \Gamma_0(\mathbb{T})$, with representative $\overline{s} : \mathrm{Y} \to T$. Given any $x \in \mathrm{X}$, we define $\overline{s}'(x) \doteq (x, v)$, where v is the unique vector for which $\overline{s}\big(f(x)\big) = \big(f(x), v\big)$. It clearly holds that $\overline{s}' : \mathrm{X} \to T'$ is a section of the MBB $(T', \underline{E}', \pi', \mathsf{n}')$. Then we define f^*s as the equivalence class of $\overline{s}'$. We thus built a map $f^* : \Gamma_0(\mathbb{T}) \to \Gamma_0(f^*\mathbb{T})$, which is linear and satisfies the first claim in (3.65).

Now fix $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$. Denote by $\boldsymbol{q} \in \Gamma_0(\mathbb{T})$ and $\boldsymbol{q}' \in \Gamma_0(f^*\mathbb{T})$ the constant sections associated to q. It is then easy to check that $\boldsymbol{q}' = f^*\boldsymbol{q}$. This grants that

$$\mathsf{S}(f^*\mathbb{T}) \subseteq \left\{ \sum_{i\in\mathbb{N}} \chi_{A_i} \cdot f^*s_i \;\middle|\; (A_i)_i \text{ is a Borel partition of } \mathrm{X},\ (s_i)_i \subseteq \Gamma_0(\mathbb{T}) \right\}.$$

Since $\mathsf{S}(f^*\mathbb{T})$ is dense in $\Gamma_0(f^*\mathbb{T})$ by Lemma 3.51, we finally conclude that the second condition in (3.65) is verified as well. Therefore the statement is achieved. □

We finally introduce the notion of dual bundle:

Definition 3.65 (Dual bundle) *Let $\mathbb{T}$ be a measurable Banach bundle over some metric measure space $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Choose a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ of $\mathbb{T}$. Let us set*

$$\mathsf{n}^*(x,v) \doteq \begin{cases} \sup_{w\in(\overline{\mathbb{T}})_x\setminus\{0\}} \frac{|v\cdot w|}{\mathsf{n}(x,w)} & \text{if } \mathsf{n}(x,\cdot) \text{ is a norm,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.66}$$

Then we define the dual bundle *$\mathbb{T}^*$ as the equivalence class of $(T, \underline{E}, \pi, \mathsf{n}^*)$, which turns out to be a measurable Banach bundle over $\mathbb{X}$.*

Theorem 3.66 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Then*

$$\Gamma_0(\mathbb{T})^* = \Gamma_0(\mathbb{T}^*). \tag{3.67}$$

Proof. Consider the operator $\iota : \Gamma_0(\mathbb{T}^*) \to \Gamma_0(\mathbb{T})^*$ defined as follows: given any $s^* \in \Gamma_0(\mathbb{T}^*)$, we call $\iota(s^*) : \Gamma_0(\mathbb{T}) \to L^0(\mathfrak{m})$ the map sending (the equivalence class of) any section $\bar{s}$ to the function $\mathrm{X} \ni x \mapsto \bar{s}^*(x) \cdot \bar{s}(x) \in \mathbb{R}$, where $\bar{s}^*$ is any representative of s^*. One can easily deduce from its very construction that ι is a module morphism that preserves the pointwise norm. To conclude, it only remains to show that the map ι is surjective. Let $T \in \Gamma_0(\mathbb{T})^*$ be fixed. For any $n \in \mathbb{N}$, denote by $\mathsf{e}_1^n, \ldots, \mathsf{e}_n^n$ the canonical basis of $\mathbb{R}^n$ and by $\mathbf{e}_1^n, \ldots, \mathbf{e}_n^n \in \Gamma_0(\mathbb{T})$ the associated constant sections. Hence let us define $s^* \in \Gamma_0(\mathbb{T}^*)$ as

$$s^*(x) \doteq \sum_{n \in \mathbb{N}} \Big(x, \big(T\mathbf{e}_1^n(x), \ldots, T\mathbf{e}_n^n(x) \big) \Big) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Simple computations show that $\iota(s^*) = T$. Hence ι is surjective, concluding the proof. □

3.2.4 A variant for L^p-normed L^∞-modules

Our choice of using the language of L^0-normed L^0-modules – instead of that of L^p-normed L^∞-modules – is only a matter of practicality, not due to any theoretical reason. Indeed, we now show that the results obtained so far can be reformulated for L^p-normed L^∞-modules.

Consider any two $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules $\mathscr{M}^p$, $\mathscr{N}^p$ and pick any module morphism $\Phi : \mathscr{M}^p \to \mathscr{N}^p$. Then there exists a unique module morphism $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$ extending Φ, where $\mathscr{M}^0$ and $\mathscr{N}^0$ denote the L^0-completions of $\mathscr{M}^p$ and $\mathscr{N}^p$, respectively.

Definition 3.67 (L^0-completion functor) *We define the* L^0-completion functor *as the functor* $\mathsf{C}^p : \mathbf{NMod}^p(\mathbb{X}) \to \mathbf{NMod}^0(\mathbb{X})$ *that assigns to any* $\mathscr{M}^p$ *its* L^0*-completion* $\mathscr{M}^0$ *and to any module morphism* $\Phi : \mathscr{M}^p \to \mathscr{N}^p$ *its unique extension* $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$.

Conversely, given any $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}^0$, one has that

$$\mathscr{M}^p \doteq \big\{ v \in \mathscr{M}^0 \;\big|\; |v| \in L^p(\mathfrak{m}) \big\} \qquad \text{is an } L^p(\mathfrak{m})\text{-normed } L^\infty(\mathfrak{m})\text{-module.} \tag{3.68}$$

Moreover, it holds that the L^0-completion of $\mathscr{M}^p$ is the original module $\mathscr{M}^0$.

Definition 3.68 (L^p-restriction functor) *The* L^p-restriction functor *is defined as that functor* $\mathsf{R}^p : \mathbf{NMod}^0(\mathbb{X}) \to \mathbf{NMod}^p(\mathbb{X})$ *that assigns to any* $\mathscr{M}^0$ *its 'restriction'* $\mathscr{M}^p$*, as in* (3.68)*, and to any module morphism* $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$ *its restriction* $\Phi \doteq \widetilde{\Phi}|_{\mathscr{M}^p} : \mathscr{M}^p \to \mathscr{N}^p$*, which turns out to be a morphism of* $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-modules.*

We can finally collect all of the properties described so far in the following statement:

Theorem 3.69 ($\mathbf{NMod}^p(\mathbb{X})$ is equivalent to $\mathbf{NMod}^0(\mathbb{X})$) *Both the functors* C^p *and* R^p *are equivalence of categories, one the inverse of the other.*

It is trivial to check that $\mathscr{M}^p$ and $\mathsf{C}^p(\mathscr{M}^p)$ have the same dimensional decomposition, thus in particular the above functors naturally restrict to $\mathsf{C}^p_{\mathrm{pr}} : \mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X}) \to \mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$ and $\mathsf{R}^p_{\mathrm{pr}} : \mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X}) \to \mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$. Therefore:

Corollary 3.70 ($\mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$ is equivalent to $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$) *It holds that the two functors* $\mathsf{C}^p_{\mathrm{pr}}$ *and* $\mathsf{R}^p_{\mathrm{pr}}$ *are equivalence of categories, one the inverse of the other.*

Now fix a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$. Then let us define

$$\Gamma_p(\mathbb{T}) \doteq \big\{ s \in \Gamma_0(\mathbb{T}) \;\big|\; |s| \in L^p(\mathfrak{m}) \big\}. \tag{3.69}$$

The space $\Gamma_p(\mathbb{T})$ can be viewed as an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Moreover, given any two measurable Banach bundles $\mathbb{T}_1$, $\mathbb{T}_2$ over $\mathbb{X}$ and a bundle morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$, let us define $\Gamma_p(\varphi) \in \mathrm{Mor}\big(\Gamma_p(\mathbb{T}_1), \Gamma_p(\mathbb{T}_2)\big)$ as

$$\Gamma_p(\varphi) \doteq \Gamma_0(\varphi)|_{\Gamma_p(\mathbb{T}_1)} : \Gamma_p(\mathbb{T}_1) \to \Gamma_p(\mathbb{T}_2). \tag{3.70}$$

Hence such construction induces an L^p-section functor $\Gamma_p : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$. Then

$$\begin{array}{ccc} \mathbf{MBB}(\mathbb{X}) & \xrightarrow{\ \Gamma_0\ } & \mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X}) \\ & \searrow^{\Gamma_p} & \downarrow \mathsf{R}^p_{\mathrm{pr}} \\ & & \mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X}) \end{array} \tag{3.71}$$

is a commutative diagram.

We can thus finally conclude that

Theorem 3.71 (Serre-Swan for L^p-normed L^∞-modules) *It holds that the L^p-section functor $\Gamma_p : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$ on $\mathbb{X}$ is an equivalence of categories.*

Proof. It follows from Theorem 3.54, from Corollary 3.70 and from property (3.71). □

4

Differential calculus on RCD spaces

Contents

The aim of this chapter is to illustrate the *differential structure* for RCD spaces that has been introduced by N. Gigli in [Gig17b]. Roughly speaking, such theory consists of a combination of the Sobolev calculus (discussed in Chapter 2) with the language of normed modules (discussed in Chapter 3). More precisely, we shall proceed in the following way:

- First-order calculus. As described in Section 4.1, a first-order differential calculus can be developed over any metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$. The key tool is given by the *cotangent module* $L^0(T^*\mathrm{X})$, which constitutes a convenient abstraction of the notion of 'measurable 1-form' over the space X. Shortly said, $L^0(T^*\mathrm{X})$ is the smallest $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module containing any $L^0(\mathfrak{m})$-linear combination of 'differentials of Sobolev functions' (this is the link with the Sobolev calculus). Therefore the *tangent module* $L^0(T\mathrm{X})$ can be defined as the module dual of the cotangent one and its elements are called *vector fields* over X. In a similar fashion, one can introduce other differential operators, such as (measure-valued) divergence and Laplacian. We point out that this formal calculus is fully consistent with the classical one on the Euclidean space $\mathbb{R}^d$, even if the Lebesgue measure is replaced by any other Radon measure μ: as we shall see in Subsection 4.1.3, the 'abstract' tangent module $L^2_\mu(T\mathbb{R}^d)$ associated to the weighted Euclidean space $\big(\mathbb{R}^d, |\cdot|, \mu\big)$ can be isometrically embedded into the space of all $L^2(\mu)$-integrable 'concrete' vector fields on $\mathbb{R}^d$ (cf. Theorem 4.27). Finally, we will discuss how to build a notion of differential for sufficiently regular maps between nonsmooth structures; namely, any map $\varphi : \mathrm{X} \to \mathrm{Y}$ of *bounded deformation* between two metric measure spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ – i.e. the map φ is Lipschitz and satisfies $\varphi_*\mathfrak{m}_\mathrm{X} \leq C\,\mathfrak{m}_\mathrm{Y}$ for some constant $C > 0$ – induces in a natural way a differential operator $\mathrm{d}\varphi : L^2(T\mathrm{X}) \to \varphi^*L^2(T\mathrm{Y})$. This topic will be treated in Subsection 4.1.4. In Subsection 4.1.5 we will see a technical variant of the differential for maps of bounded deformation, tailored for the situation in which the target space is the Euclidean one.

- SECOND-ORDER CALCULUS. Section 4.2 will be entirely devoted to a fundamental class of metric measure spaces: the RCD spaces. The acronym RCD stands for *Riemannian Curvature-Dimension condition*, indeed these spaces are Riemannian-like structures with prescribed lower bounds on the Ricci curvature and upper bounds on the dimension (in a synthetic way). The definition of this condition and its main properties are recalled in Subsection 4.2.1. An advantage of working within this framework is given by the possibility to develop even a second-order calculus on top of the first-order one. The necessity of calling into play the curvature bounds is due to the fact that the latter grant the presence of a sufficiently vast class of *test functions*, which are 'twice differentiable'. More specifically, by testing against such special functions it is possible to introduce Hessian and covariant derivative via suitable integration-by-parts formulae, thus leading to the definition of the Sobolev spaces $W^{2,2}(\mathrm{X})$ and $W^{1,2}_C(T\mathrm{X})$, respectively. The related discussion can be found in Subsection 4.2.2.

4.1 First-order differential structure of metric measure spaces

4.1.1 Cotangent and tangent modules

We begin with the definition of *cotangent module*, which is the object that will play a central role throughout the whole thesis. The ensuing three results are taken from [GP18].

Theorem 4.1 (Cotangent module associated to a D-structure) *Let $p \in (1, \infty)$ and let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Consider a pointwise local D-structure on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Then there exists a unique couple $\big(L^p(T^*\mathrm{X}; D), \mathrm{d}\big)$, where $L^p(T^*\mathrm{X}; D)$ is an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module and $\mathrm{d} : \mathrm{S}^p(\mathrm{X}; D) \to L^p(T^*\mathrm{X}; D)$ is a linear map, such that the following hold:*

i) *The equality $|\mathrm{d}u| = \underline{D}u$ is satisfied $\mathfrak{m}$-a.e. in X for every $u \in \mathrm{S}^p(\mathrm{X}; D)$.*

ii) *The vector space $\mathcal{V}$ of all elements of the form $\sum_{i=1}^n \chi_{B_i}\,\mathrm{d}u_i$, where $(B_i)_i$ is a Borel partition of X and $(u_i)_i \subseteq \mathrm{S}^p(\mathrm{X}; D)$, is dense in the space $L^p(T^*\mathrm{X}; D)$.*

Uniqueness has to be intended up to unique isomorphism: given another such couple $(\mathscr{M}, \mathrm{d}')$, there is a unique isomorphism $\Phi : L^p(T^*\mathrm{X}; D) \to \mathscr{M}$ such that the diagram

$$\begin{array}{ccc} \mathrm{S}^p(\mathrm{X}; D) & \xrightarrow{\ \mathrm{d}\ } & L^p(T^*\mathrm{X}; D) \\ & \searrow^{\mathrm{d}'} & \downarrow \Phi \\ & & \mathscr{M} \end{array} \tag{4.1}$$

commutes. The space $L^p(T^*\mathrm{X};D)$ *is called* cotangent module*, while* d *is called* differential.

Proof. We first prove the uniqueness part of the statement and then the existence one.
UNIQUENESS. Consider any element $\omega\in\mathcal{V}$ written as $\omega=\sum_{i=1}^n\chi_{B_i}\,\mathrm{d}u_i$, with $(B_i)_i$ Borel partition of X and $u_1,\ldots,u_n\in S^p(\mathrm{X};D)$. Notice that the requirements that $\Phi\circ\mathrm{d}=\mathrm{d}'$ and that Φ is $L^\infty(\mathfrak{m})$-linear force the definition $\Phi(\omega)\doteq\sum_{i=1}^n\chi_{B_i}\,\mathrm{d}'u_i$. The $\mathfrak{m}$-a.e. equality

$$\big|\Phi(\omega)\big|=\sum_{i=1}\chi_{B_i}\,|\mathrm{d}'u_i|=\sum_{i=1}^n\chi_{B_i}\,\underline{D}u_i=\sum_{i=1}^n\chi_{B_i}\,|\mathrm{d}u_i|=|\omega|$$

grants that $\Phi(\omega)$ is well-defined, in the sense that it does not depend on the particular way of representing ω, and that $\Phi:\mathcal{V}\to\mathscr{M}$ preserves the pointwise norm. In particular, one has that the map $\Phi:\mathcal{V}\to\mathscr{M}$ is (linear and) continuous. Since $\mathcal{V}$ is dense in $L^p(T^*\mathrm{X};D)$, we can uniquely extend Φ to a linear and continuous map $\Phi:L^p(T^*\mathrm{X};D)\to\mathscr{M}$, which also preserves the pointwise norm. Moreover, we deduce from the very definition of Φ that the identity $\Phi(h\,\omega)=h\,\Phi(\omega)$ holds for every $\omega\in\mathcal{V}$ and $h\in\mathsf{Sf}(\mathrm{X})$, whence the $L^\infty(\mathfrak{m})$-linearity of Φ follows by an approximation argument. Finally, the image $\Phi(\mathcal{V})$ is dense in $\mathscr{M}$, which implies that Φ is surjective. Therefore Φ is the unique isomorphism satisfying $\Phi\circ\mathrm{d}=\mathrm{d}'$.
EXISTENCE. First of all, let us define the *pre-cotangent module* as

$$\mathsf{Pcm}\doteq\left\{\big\{(B_i,u_i)\big\}_{i=1}^n\;\middle|\;\begin{array}{l}n\in\mathbb{N},\ u_1,\ldots,u_n\in\mathrm{S}^p(\mathrm{X};D),\\(B_i)_{i=1}^n\text{ Borel partition of X}\end{array}\right\}.$$

We define an equivalence relation on Pcm as follows: we declare that $\big\{(B_i,u_i)\big\}_i\sim\big\{(C_j,v_j)\big\}_j$ provided $\underline{D}(u_i-v_j)=0$ holds $\mathfrak{m}$-a.e. on $B_i\cap C_j$ for every i,j. The equivalence class of an element $\big\{(B_i,u_i)\big\}_i$ of Pcm will be denoted by $[B_i,u_i]_i$. We can endow the quotient $\mathsf{Pcm}/\sim$ with a vector space structure:

$$\begin{aligned}[B_i,u_i]_i+[C_j,v_j]_j&\doteq[B_i\cap C_j,u_i+v_j]_{i,j},\\ \lambda\,[B_i,u_i]_i&:\doteq[B_i,\lambda\,u_i]_i,\end{aligned}\tag{4.2}$$

for every $[B_i,u_i]_i,[C_j,v_j]_j\in\mathsf{Pcm}/\sim$ and $\lambda\in\mathbb{R}$. We only check that the sum operator is well-defined; the proof of the well-posedness of the multiplication by scalars follows along the same lines. Suppose that $\big\{(B_i,u_i)\big\}_i\sim\big\{(B'_k,u'_k)\big\}_k$ and $\big\{(C_j,v_j)\big\}_j\sim\big\{(C'_\ell,v'_\ell)\big\}_\ell$, in other words $\underline{D}(u_i-u'_k)=0$ $\mathfrak{m}$-a.e. on $B_i\cap B'_k$ and $\underline{D}(v_j-v'_\ell)=0$ $\mathfrak{m}$-a.e. on $C_j\cap C'_\ell$ for every choice of i,j,k,ℓ, whence accordingly

$$\underline{D}\big((u_i+v_j)-(u'_k+v'_\ell)\big)\overset{\mathbf{L5}}{\leq}\underline{D}(u_i-u'_k)+\underline{D}(v_j-v'_\ell)=0\qquad\mathfrak{m}\text{-a.e. on }(B_i\cap C_j)\cap(B'_k\cap C'_\ell).$$

This shows that $\big\{(B_i\cap C_j,u_i+v_j)\big\}_{i,j}\sim\big\{(B'_k\cap C'_\ell,u'_k+v'_\ell)\big\}_{k,\ell}$, thus proving that the sum operator defined in (4.2) is well-posed. Now let us define

$$\big\|[B_i,u_i]_i\big\|_{L^p(T^*\mathrm{X};D)}\doteq\sum_{i=1}^n\bigg(\int_{B_i}(\underline{D}u_i)^p\,\mathrm{d}\mathfrak{m}\bigg)^{1/p}\qquad\text{for every }[B_i,u_i]_i\in\mathsf{Pcm}/\sim.\tag{4.3}$$

Such definition is well-posed: if $\big\{(B_i,u_i)\big\}_i\sim\big\{(C_j,v_j)\big\}_j$ then for all i,j it holds that

$$|\underline{D}u_i-\underline{D}v_j|\overset{\mathbf{L5}}{\leq}\underline{D}(u_i-v_j)=0\qquad\mathfrak{m}\text{-a.e. on }B_i\cap C_j,$$

i.e. that the equality $\underline{D}u_i = \underline{D}v_j$ is satisfied $\mathfrak{m}$-a.e. on $B_i \cap C_j$. Therefore one has that

$$\sum_i \left(\int_{B_i} (\underline{D}u_i)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} = \sum_{i,j} \left(\int_{B_i \cap C_j} (\underline{D}u_i)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} = \sum_{i,j} \left(\int_{B_i \cap C_j} (\underline{D}v_j)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p}$$
$$= \sum_j \left(\int_{C_j} (\underline{D}v_j)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p},$$

which grants that $\| \cdot \|_{L^p(T^*\mathrm{X};D)}$ in (4.3) is well-defined. The fact that it is a norm on $\mathsf{Pcm}/\sim$ easily follows from standard verifications. Hence let us define

$$L^p(T^*\mathrm{X};D) \doteq \text{completion of } \big(\mathsf{Pcm}/\sim, \| \cdot \|_{L^p(T^*\mathrm{X};D)}\big),$$
$$\mathrm{d} : \mathrm{S}^p(\mathrm{X};D) \to L^p(T^*\mathrm{X};D), \quad \mathrm{d}u := [\mathrm{X},u] \text{ for every } u \in \mathrm{S}^p(\mathrm{X};D).$$

Observe that $L^p(T^*\mathrm{X};D)$ is a Banach space and that d is a linear operator. Furthermore, given any $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$ and $h = \sum_j \lambda_j \, \chi_{C_j} \in \mathsf{Sf}(\mathrm{X})$, where $(\lambda_j)_j \subseteq \mathbb{R}$ and $(C_j)_j$ is a Borel partition of X, we set

$$\big|[B_i,u_i]_i\big| \doteq \sum_i \chi_{B_i} \, \underline{D}u_i,$$
$$h\,[B_i,u_i]_i \doteq [B_i \cap C_j, \lambda_j \, u_i]_{i,j}.$$

One can readily prove that such operations, which are well-posed again by the pointwise locality of D, can be uniquely extended to a pointwise norm $| \cdot | : L^p(T^*\mathrm{X};D) \to L^p(\mathfrak{m})^+$ and to a multiplication by L^∞-functions $L^\infty(\mathfrak{m}) \times L^p(T^*\mathrm{X};D) \to L^p(T^*\mathrm{X};D)$, respectively. Therefore the space $L^p(T^*\mathrm{X};D)$ turns out to be an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module when equipped with the operations described so far. In order to conclude, it suffices to notice that

$$|\mathrm{d}u| = \big|[\mathrm{X},u]\big| = \underline{D}u \quad \text{holds } \mathfrak{m}\text{-a.e.} \qquad \text{for every } u \in \mathrm{S}^p(\mathrm{X};D)$$

and that $[B_i,u_i]_i = \sum_i \chi_{B_i} \, \mathrm{d}u_i$ for all $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$, giving i) and ii), respectively. □

An important property of the cotangent module is the closure of the differential operator:

Theorem 4.2 (Closure of the differential) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $p \in (1,\infty)$. Consider a pointwise local D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then the differential operator* d *is* closed, *i.e. if a sequence $(u_n)_n \subseteq \mathrm{S}^p(\mathrm{X};D)$ converges in $L^p_{\mathrm{loc}}(\mathfrak{m})$ to $u \in L^p_{\mathrm{loc}}(\mathfrak{m})$ and $\mathrm{d}u_n \rightharpoonup \omega$ weakly in $L^p(T^*\mathrm{X};D)$ for a suitable limit $\omega \in L^p(T^*\mathrm{X};D)$, then $u \in \mathrm{S}^p(\mathrm{X};D)$ and $\mathrm{d}u = \omega$.*

Proof. Since d is linear, we can assume with no loss of generality that $\mathrm{d}u_n \to \omega$ in $L^p(T^*\mathrm{X};D)$ by Mazur lemma, so that $\mathrm{d}(u_n - u_m) \to \omega - \mathrm{d}u_m$ in $L^p(T^*\mathrm{X};D)$ for any $m \in \mathbb{N}$. In particular, one has $u_n - u_m \to u - u_m$ in $L^p_{\mathrm{loc}}(\mathfrak{m})$ and $\underline{D}(u_n - u_m) = \big|\mathrm{d}(u_n - u_m)\big| \to |\omega - \mathrm{d}u_m|$ in $L^p(\mathfrak{m})$ as $n \to \infty$ for all $m \in \mathbb{N}$, whence $u - u_m \in \mathrm{S}^p(\mathrm{X};D)$ and $\underline{D}(u - u_m) \leq |\omega - \mathrm{d}u_m|$ holds $\mathfrak{m}$-a.e. for all $m \in \mathbb{N}$ by **A5** and **L5**. Therefore $u = (u - u_0) + u_0 \in \mathrm{S}^p(\mathrm{X};D)$ and

$$\varlimsup_{m\to\infty} \|\mathrm{d}u - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)} = \varlimsup_{m\to\infty} \big\|\underline{D}(u - u_m)\big\|_{L^p(\mathfrak{m})} \leq \varlimsup_{m\to\infty} \|\omega - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)}$$
$$= \varlimsup_{m\to\infty} \lim_{n\to\infty} \|\mathrm{d}u_n - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)} = 0,$$

which grants that $\mathrm{d}u_m \to \mathrm{d}u$ in $L^p(T^*\mathrm{X};D)$ as $m \to \infty$ and accordingly that $\mathrm{d}u = \omega$. □

Furthermore, the differential operator satisfies the following calculus rules:

Proposition 4.3 (Calculus rules for $\mathrm{d}u$**)** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any metric measure space and let* $p\in(1,\infty)$*. Consider a pointwise local* D*-structure on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$*. Then the following hold:*

i) *Let* $u\in\mathrm{S}^p(\mathrm{X};D)$ *and let* $N\subseteq\mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N)=0$*. Then* $\chi_{u^{-1}(N)}\,\mathrm{d}u=0$*.*

ii) Chain rule. *Let* $u\in\mathrm{S}^p(\mathrm{X};D)$ *and* $\varphi\in\mathrm{LIP}(\mathbb{R})$ *be given. Recall that* $\varphi\circ u\in\mathrm{S}^p(\mathrm{X})$ *by Proposition 2.13. Then*

$$\mathrm{d}(\varphi\circ u)=\varphi'\circ u\,\mathrm{d}u. \tag{4.4}$$

iii) Leibniz rule. *Let* $u,v\in\mathrm{S}^p(\mathrm{X};D)\cap L^\infty(\mathfrak{m})$ *be given. Recall that* $uv\in\mathrm{S}^p(\mathrm{X};D)\cap L^\infty(\mathfrak{m})$ *by Proposition 2.13. Then*

$$\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u. \tag{4.5}$$

Proof.
i) We have that $|\mathrm{d}u|=\underline{D}u=0$ holds $\mathfrak{m}$-a.e. on $u^{-1}(N)$ by item i) of Proposition 2.13, thus accordingly $\chi_{u^{-1}(N)}\,\mathrm{d}u=0$, as required.
ii) If φ is an affine function, say $\varphi(t)=\alpha\,t+\beta$, then $\mathrm{d}(\varphi\circ u)=\mathrm{d}(\alpha\,u+\beta)=\alpha\,\mathrm{d}u=\varphi'\circ u\,\mathrm{d}u$. Now suppose that φ is a piecewise affine function. Say that $(I_n)_n$ is a sequence of intervals whose union covers the whole real line $\mathbb{R}$ and that $(\psi_n)_n$ is a sequence of affine functions such that $\varphi|_{I_n}=\psi_n$ holds for every $n\in\mathbb{N}$. Since φ' and ψ_n' coincide $\mathcal{L}^1$-a.e. in the interior of I_n, we have that $\mathrm{d}(\varphi\circ f)=\mathrm{d}(\psi_n\circ f)=\psi_n'\circ f\,\mathrm{d}f=\varphi'\circ f\,\mathrm{d}f$ holds $\mathfrak{m}$-a.e. on $f^{-1}(I_n)$ for all n, so that $\mathrm{d}(\varphi\circ u)=\varphi'\circ u\,\mathrm{d}u$ is verified $\mathfrak{m}$-a.e. on $\bigcup_n u^{-1}(I_n)=\mathrm{X}$.

To prove the case of a general Lipschitz function $\varphi:\mathbb{R}\to\mathbb{R}$, we want to approximate φ with a sequence of piecewise affine functions: for any $n\in\mathbb{N}$, let us denote by φ_n the function that coincides with φ at $\{k/2^n\,:\,k\in\mathbb{Z}\}$ and that is affine on the interval $\big[k/2^n,(k+1)/2^n\big]$ for every $k\in\mathbb{Z}$. It is clear that $\mathrm{Lip}(\varphi_n)\leq\mathrm{Lip}(\varphi)$ for all $n\in\mathbb{N}$. Moreover, one can readily check that, up to a not relabeled subsequence, $\varphi_n\to\varphi$ uniformly on $\mathbb{R}$ and $\varphi_n'\to\varphi'$ pointwise $\mathcal{L}^1$-almost everywhere. The former grants that $\varphi_n\circ u\to\varphi\circ u$ in $L^p_{\mathrm{loc}}(\mathfrak{m})$. Given that $|\varphi_n'-\varphi'|^p\circ u\,(\underline{D}u)^p\leq 2^p\,\mathrm{Lip}(\varphi)^p\,(\underline{D}u)^p\in L^1(\mathfrak{m})$ for all $n\in\mathbb{N}$ and $|\varphi_n'-\varphi'|^p\circ u\,(\underline{D}u)^p\to 0$ pointwise $\mathfrak{m}$-a.e. by the latter above together with i), we obtain $\int|\varphi_n'-\varphi'|^p\circ u\,(\underline{D}u)^p\,\mathrm{d}\mathfrak{m}\to 0$ as $n\to\infty$ by the dominated convergence theorem. In other words, $\varphi_n'\circ u\,\mathrm{d}u\to\varphi'\circ u\,\mathrm{d}u$ in the strong topology of $L^p(T^*\mathrm{X};D)$. Hence Theorem 2.22 ensures that $\mathrm{d}(\varphi\circ u)=\varphi'\circ u\,\mathrm{d}u$, thus proving the chain rule ii) for any $\varphi\in\mathrm{LIP}(\mathbb{R})$.
iii) In the case $u,v\geq 1$, we argue as in the proof of Proposition 2.13 to deduce from ii) that

$$\frac{\mathrm{d}(uv)}{uv}=\mathrm{d}\log(uv)=\mathrm{d}\big(\log(u)+\log(v)\big)=\mathrm{d}\log(u)+\mathrm{d}\log(v)=\frac{\mathrm{d}u}{u}+\frac{\mathrm{d}v}{v},$$

whence we get $\mathrm{d}(uv)=u\,\mathrm{d}v+v\,\mathrm{d}u$ by multiplying both sides by uv.

In the general case $u,v\in L^\infty(\mathfrak{m})$, choose a constant $C>0$ so big that $u+C,v+C\geq 1$. By the case treated above, we know that

$$\begin{aligned}\mathrm{d}\big((u+C)(v+C)\big)&=(u+C)\,\mathrm{d}(v+C)+(v+C)\,\mathrm{d}(u+C)\\&=(u+C)\,\mathrm{d}v+(v+C)\,\mathrm{d}u\\&=u\,\mathrm{d}v+v\,\mathrm{d}u+C\,\mathrm{d}(u+v),\end{aligned} \tag{4.6}$$

while a direct computation yields

$$\mathrm{d}\big((u+C)(v+C)\big) = \mathrm{d}\big(uv + C(u+v) + C^2\big) = \mathrm{d}(uv) + C\,\mathrm{d}(u+v). \tag{4.7}$$

By subtracting (4.7) from (4.6), we finally obtain that $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$, as required. This completes the proof of the Lebniz rule iii). □

Hereafter, we shall only consider the D-structure obtained by considering weak upper gradients and $p = 2$. In this case, the cotangent module of $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ will be just denoted by

$$L^2(T^*\mathrm{X}) \doteq L^2(T^*\mathrm{X}; D_{\mathrm{wug}}). \tag{4.8}$$

In order to stress the dependence on the measure $\mathfrak{m}$, we shall sometimes write $L^2_{\mathfrak{m}}(T^*\mathrm{X})$. Analogously, the differential $\mathrm{d}f$ of a Sobolev function $f \in \mathrm{S}^2(\mathrm{X})$ will be sometimes denoted by $\mathrm{d}_{\mathfrak{m}} f$. From now on, we shall follow [Gig17b] unless otherwise specified. Let us define

$$\begin{aligned} &L^0(T^*\mathrm{X}) \doteq L^0\text{-completion of } L^2(T^*\mathrm{X}), \\ &L^p(T^*\mathrm{X}) \doteq \big\{\omega \in L^0(T^*\mathrm{X}) \;\big|\; |\omega| \in L^p(\mathfrak{m})\big\} \qquad \text{for every } p \in [1,\infty]. \end{aligned} \tag{4.9}$$

It turns out that $L^p(T^*\mathrm{X})$ has a natural structure of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module.

Remark 4.4 Observe that the modules $L^p(T^*\mathrm{X})$ and $L^p(T^*\mathrm{X}; D_{\mathrm{wug}})$ might be different, the reason being that – as alredy pointed out in Remark 2.29 – the Sobolev class $\mathrm{S}^p(\mathrm{X})$ could depend on the exponent p. ■

Remark 4.5 If M is a smooth Finsler manifold, then $L^2(T^*M)$ can be identified – in light of Remark 2.26 – with the space of L^2-sections of the cotangent bundle T^*M of M. ■

A standard cut-off and truncation argument gives the following result:

Proposition 4.6 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Then*

$$\big\{\mathrm{d}f \;\big|\; f \in W^{1,2}(\mathrm{X})\big\} \;\; \textit{generates } L^2(T^*\mathrm{X}) \textit{ on } \mathrm{X}. \tag{4.10}$$

In particular, if $W^{1,2}(\mathrm{X})$ is separable, then $L^2(T^*\mathrm{X})$ is separable as well.

By duality we can thus introduce the space of '2-integrable vector fields' over X:

Definition 4.7 (Tangent module) *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be any metric measure space. Then we define the* tangent module $L^2(T\mathrm{X})$ *of* X *as the dual of $L^2(T^*\mathrm{X})$ in the sense of modules. Its elements are called* vector fields. *We shall sometimes write $L^2_{\mathfrak{m}}(T\mathrm{X})$ instead of $L^2(T\mathrm{X})$.*

An L^2*-derivation* on X is any linear map $L : \mathrm{S}^2(\mathrm{X}) \to L^1(\mathfrak{m})$ for which there exists some function $g \in L^2(\mathfrak{m})$ satisfying $\big|L(f)\big| \le g\,|Df|$ in the $\mathfrak{m}$-a.e. sense for every $f \in \mathrm{S}^2(\mathrm{X})$. The relation between L^2-derivations and vector fields is explained by the following result, for whose proof we refer to [Gig17a, Theorem 1.20]:

Proposition 4.8 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. Given any $v \in L^2(T\mathrm{X})$, it holds that $v \circ \mathrm{d} : \mathrm{S}^2(\mathrm{X}) \to L^1(\mathfrak{m})$ is an L^2-derivation on X. Conversely, given any L^2-derivation L on X, there exists a unique vector field $v \in L^2(T\mathrm{X})$ such that $L = v \circ \mathrm{d}$.*

A metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian if and only if its associated cotangent module $L^2(T^*\mathrm{X})$ is a Hilbert module. Hence the following definition is meaningful:

Definition 4.9 (Gradient) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Let $f\in \mathrm{S}^2(\mathrm{X})$ be given. Then we denote by $\nabla f\in L^2(T\mathrm{X})$ the element corresponding to $\mathrm{d}f\in L^2(T^*\mathrm{X})$ via the Riesz isomorphism (cf. Theorem 3.25) and we call it* gradient *of f.*

Remark 4.10 (Calculus rules for ∇f) We immediately deduce from the calculus rules for the differential (see Proposition 4.3) that

$$\begin{aligned} \nabla(\varphi\circ f) &= \varphi'\circ f\,\nabla f && \text{for every } f\in \mathrm{S}^2(\mathrm{X}) \text{ and } \varphi\in \mathrm{LIP}(\mathbb{R}),\\ \nabla(fg) &= f\,\nabla g+g\,\nabla f && \text{for every } f,g\in \mathrm{S}^2(\mathrm{X})\cap L^\infty(\mathfrak{m}), \end{aligned} \tag{4.11}$$

which are called *chain rule* and *Leibniz rule* for the gradient, respectively. ■

The following two remarks are taken from the paper [GP16b].

Remark 4.11 (Localisation of the cotangent module) Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix an open set $\Omega\subseteq \mathrm{X}$ and define $\widetilde{\mathfrak{m}}\doteq \mathfrak{m}_{|\Omega}$. Then the cotangent module $L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})$ can be canonically identified with $L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$, in the following sense: there exists a (unique) linear isomorphism $\iota:\,L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})\to L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$ such that

$$\begin{aligned} \big|\iota(v)\big| &= |v| \quad \widetilde{\mathfrak{m}}\text{-a.e.} && \text{for every } v\in L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X}),\\ \iota(\mathrm{d}_{\widetilde{\mathfrak{m}}}f) &= \mathrm{d}_{\mathfrak{m}}f && \text{for every } f\in W^{1,2}_{\mathfrak{m}}(\mathrm{X}) \text{ with } \mathsf{d}\big(\mathrm{spt}(f),\mathrm{X}\setminus\Omega\big)>0. \end{aligned} \tag{4.12}$$

First of all, observe that the second line in (4.12) makes sense, because any map $f\in W^{1,2}_{\mathfrak{m}}(\mathrm{X})$ with $\mathsf{d}\big(\mathrm{spt}(f),\mathrm{X}\setminus\Omega\big)>0$ belongs to $W^{1,2}_{\widetilde{\mathfrak{m}}}(\mathrm{X})$ and satisfies $|Df|_{\widetilde{\mathfrak{m}}}=|Df|_{\mathfrak{m}}$ in the $\widetilde{\mathfrak{m}}$-a.e. sense. This can be proved by arguing as in [AGS14b, Theorem 4.19] and [Gig15, Proposition 2.6].

Let us denote by $\mathcal{F}$ the family of all maps f as above. Then the set $\mathcal{F}$ is dense in $W^{1,2}_{\widetilde{\mathfrak{m}}}(\mathrm{X})$, as follows by a standard cut-off argument, so that accordingly $L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})$ is generated by the 1-forms $\mathrm{d}_{\widetilde{\mathfrak{m}}}f$ with $f\in\mathcal{F}$. Analogously, the set $\{\mathrm{d}_{\mathfrak{m}}f\,:\,f\in\mathcal{F}\}$ generates $L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$. Now let us define $\iota\big(\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}_{\widetilde{\mathfrak{m}}}f_i\big)\doteq\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}_{\mathfrak{m}}f_i$ for every 1-form $\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}_{\widetilde{\mathfrak{m}}}f_i$, where $(A_i)_{i=1}^n$ is a Borel partition of Ω and $f_1,\ldots,f_n\in\mathcal{F}$. Hence ι can be uniquely extended to a linear isomorphism $\iota:\,L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X})\to L^2_{\mathfrak{m}}(T^*\mathrm{X})_{|\Omega}$ satisfying (4.12), which proves the above claim.

Therefore it immediately follows that there exists a (uniquely determined) linear and continuous isomorphism $\iota:\,L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X})\to L^2_{\mathfrak{m}}(T\mathrm{X})_{|\Omega}$ such that

$$\iota(\omega)\big(\iota(v)\big)=\omega(v)\quad \widetilde{\mathfrak{m}}\text{-a.e. in }\mathrm{X},\qquad \text{for every } \omega\in L^2_{\widetilde{\mathfrak{m}}}(T^*\mathrm{X}) \text{ and } v\in L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X}). \tag{4.13}$$

In particular, the equality $\big|\iota(v)\big|=|v|$ is satisfied $\widetilde{\mathfrak{m}}$-a.e. in X for every $v\in L^2_{\widetilde{\mathfrak{m}}}(T\mathrm{X})$. ■

Remark 4.12 (Cotangent module on the Euclidean space) Fix any $k\in\mathbb{N}$ and let us consider the Euclidean space $(\mathbb{R}^k,\mathsf{d}_{\mathrm{Eucl}},\mathcal{L}^k)$. We denote by $L^2(\mathbb{R}^k,\mathbb{R}^k)$ the standard space of 2-integrable vector fields on $\mathbb{R}^k$ and by $L^2(\mathbb{R}^k,(\mathbb{R}^k)^*)$ its dual, i.e. the space of 2-integrable 1-forms on $\mathbb{R}^k$. Notice that the module dual of $L^2(\mathbb{R}^k,(\mathbb{R}^k)^*)$ is $L^2(\mathbb{R}^k,\mathbb{R}^k)$.

We know that the Sobolev space $W^{1,2}(\mathbb{R}^k)$ coincides with the classically defined one via distributional derivatives. For any $f\in W^{1,2}(\mathbb{R}^k)$, we denote by $\underline{\mathrm{d}}f$ its distributional differential, which naturally belongs to $L^2(\mathbb{R}^k,(\mathbb{R}^k)^*)$. Its pointwise norm $|\underline{\mathrm{d}}f|$ coincides a.e. with the minimal weak upper gradient $|Df|$ of f (see [AGS14a]). It is readily verified that the

1-forms of the kind $\sum_{i=1}^n \chi_{A_i} \underline{\mathrm{d}} f_i$ – with $(A_i)_{i=1}^n$ Borel partition of $\mathbb{R}^k$ and $(f_i)_{i=1}^n \subset W^{1,2}(\mathbb{R}^k)$ – are dense in $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$. Thanks to Theorem 4.1, these facts are sufficient to conclude that the 'concrete' space of 2-integrable 1-forms $L^2(\mathbb{R}^k, (\mathbb{R}^k)^*)$ and the abstract cotangent module $L^2(T^*\mathbb{R}^k)$ can be canonically identified (via the isomorphism sending $\underline{\mathrm{d}} f$ to $\mathrm{d} f$).

Once this identification is done, it also follows that the space of L^2 vector fields $L^2(\mathbb{R}^k, \mathbb{R}^k)$ can be canonically identified with the tangent module $L^2(T\mathbb{R}^k)$. Such identification allows us to identify, for a given Borel set $E \subseteq \mathbb{R}^k$, the restricted module $L^2(T\mathbb{R}^k)_{|E}$ with $L^2(E, \mathbb{R}^k)$.

Finally, we point out that for every function $f \in \mathrm{LIP}(\mathbb{R}^k) \cap W^{1,2}(\mathbb{R}^k)$ it holds that

$$|\mathrm{d} f| = \mathrm{lip}(f) \qquad \text{is satisfied } \mathcal{L}^k\text{-a.e. in } \mathbb{R}^k, \tag{4.14}$$

which represents a reinforcement of property (2.23). ■

The notion of divergence can be obtained by taking the adjoint of the differential.

Definition 4.13 (Divergence) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then we say that a vector field* $v \in L^2(T\mathrm{X})$ *has divergence in* $L^2(\mathfrak{m})$ *– briefly,* $v \in D(\mathrm{div})$ *– provided there exists a function* $g \in L^2(\mathfrak{m})$ *such that*

$$\int f g \, \mathrm{d}\mathfrak{m} = -\int \mathrm{d} f(v) \, \mathrm{d}\mathfrak{m} \qquad \textit{for every } f \in W^{1,2}(\mathrm{X}). \tag{4.15}$$

The function g*, which is uniquely determined, is denoted by* $\mathrm{div}(v)$.

Remark 4.14 Given any $v \in D(\mathrm{div})$ and $f \in \mathrm{LIP_b}(\mathrm{X})$, it holds that $gv \in D(\mathrm{div})$ and

$$\mathrm{div}(fv) = \mathrm{d} f(v) + f \, \mathrm{div}(v). \tag{4.16}$$

Indeed: take $g \in W^{1,2}(\mathrm{X})$ and call $g_n \doteq (g \wedge n) \vee (-n) \in W^{1,2}(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ for $n \in \mathbb{N}$. Then

$$\int g_n \big[\mathrm{d} f(v) + f \, \mathrm{div}(v)\big] \mathrm{d}\mathfrak{m} \overset{(4.15)}{=} \int \big[g_n \, \mathrm{d} f(v) - \mathrm{d}(f g_n)(v)\big] \mathrm{d}\mathfrak{m} \overset{(4.5)}{=} -\int g_n \, \mathrm{d} f(v) \, \mathrm{d}\mathfrak{m}. \tag{4.17}$$

By letting $n \to \infty$ in (4.17), we conclude that formula (4.16) is satisfied. ■

Nevertheless, we will mainly work with a more general notion: that of measure-valued divergence (cf. [GP16a]). For an earlier approach to this sort of definition, we refer to [GM13].

Definition 4.15 (Measure-valued divergence) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with* (X, d) *proper. Let* $\Omega \subseteq \mathrm{X}$ *be an open set. Then we say that a vector field* $v \in L^2(T\mathrm{X})$ *has* measure-valued divergence *in* Ω *– briefly,* $v \in D(\mathbf{div}_\mathfrak{m}, \Omega)$ *– provided there exists a Radon measure* μ *on* Ω *such that*

$$\int \mathrm{d} f(v) \, \mathrm{d}\mathfrak{m} = -\int f \, \mathrm{d}\mu \qquad \textit{for every } f \in \mathrm{LIP_c}(\Omega). \tag{4.18}$$

The measure μ*, which is uniquely determined, is denoted by* $\mathbf{div}_\mathfrak{m}|_\Omega(v)$*. In the case* $\Omega = \mathrm{X}$*, we shall simply write* $D(\mathbf{div}_\mathfrak{m})$ *and* $\mathbf{div}_\mathfrak{m}(v)$ *instead of* $D(\mathbf{div}_\mathfrak{m}, \Omega)$ *and* $\mathbf{div}_\mathfrak{m}|_\Omega(v)$*, respectively.*

We have the following two basic calculus rules for the divergence, which are both consequences of the Leibniz rule for the differential.

Proposition 4.16 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* (X,d) *proper. Let* $\Omega\subseteq\mathrm{X}$ *be an open set and let* $v\in D(\mathbf{div}_{\mathfrak{m}},\Omega)$*. Fix any* $g\in\mathrm{LIP_b}(\mathrm{X})$*. Then* $gv\in D(\mathbf{div}_{\mathfrak{m}},\Omega)$ *and*

$$\mathbf{div}_{\mathfrak{m}|_\Omega}(gv)=g\,\mathbf{div}_{\mathfrak{m}|_\Omega}(v)+\mathrm{d}g(v)\,\mathfrak{m}|_\Omega. \tag{4.19}$$

Proof. Observe that for any $f\in\mathrm{LIP_c}(\Omega)$ it holds that

$$-\int f\,\mathrm{d}\big(g\,\mathbf{div}_{\mathfrak{m}|_\Omega}(v)+\mathrm{d}g(v)\,\mathfrak{m}|_\Omega\big)\overset{(4.18)}{=}\int\mathrm{d}(fg)(v)-f\mathrm{d}g(v)\,\mathrm{d}\mathfrak{m}\overset{(4.5)}{=}\int\mathrm{d}f(gv)\,\mathrm{d}\mathfrak{m},$$

whence the statement follows. □

Proposition 4.17 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* (X,d) *proper. Let* Ω_1,Ω_2 *be open subsets of* X *and let* $v\in D(\mathbf{div}_{\mathfrak{m}},\Omega_1)\cap D(\mathbf{div}_{\mathfrak{m}},\Omega_1)$ *be given. Then*

$$\big(\mathbf{div}_{\mathfrak{m}|_{\Omega_1}}(v)\big)|_{\Omega_1\cap\Omega_2}=\big(\mathbf{div}_{\mathfrak{m}|_{\Omega_2}}(v)\big)|_{\Omega_1\cap\Omega_2}. \tag{4.20}$$

Moreover, it holds that $v\in D(\mathbf{div}_{\mathfrak{m}},\Omega_1\cup\Omega_2)$ *and*

$$\big(\mathbf{div}_{\mathfrak{m}|_{\Omega_1\cup\Omega_2}}(v)\big)|_{\Omega_i}=\mathbf{div}_{\mathfrak{m}|_{\Omega_i}}(v)\qquad\textit{for } i=1,2. \tag{4.21}$$

Proof. To prove (4.20), it is sufficient to consider Lipschitz functions with support in $\Omega_1\cap\Omega_2$ – that are dense in $C_c(\Omega_1\cap\Omega_2)$ – in the definition of $\mathbf{div}_{\mathfrak{m}|_{\Omega_1}}(v)$ and $\mathbf{div}_{\mathfrak{m}|_{\Omega_2}}(v)$. In order to show (4.21), take any Lipschitz function $f:\mathrm{X}\to\mathbb{R}$ having compact support contained in $\Omega\doteq\Omega_1\cup\Omega_2$ and a Lipschitz partition of unity $\chi_1,\chi_2:\mathrm{X}\to[0,1]$ of the space $\mathrm{spt}(f)$ subordinate to the cover $\{\Omega_1,\Omega_2\}$. By letting μ be the measure defined by (4.21), one has

$$\begin{aligned}-\int f\,\mathrm{d}\mu&=-\int f\chi_1\,\mathrm{d}\big(\mathbf{div}_{\mathfrak{m}|_{\Omega_1}}(v)\big)-\int f\chi_2\,\mathrm{d}\big(\mathbf{div}_{\mathfrak{m}|_{\Omega_2}}(v)\big)\\&=\int\big(\mathrm{d}(f\chi_1)+\mathrm{d}(f\chi_2)\big)(v)\,\mathrm{d}\mathfrak{m}\\&=\int\mathrm{d}f(v)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

where we used the fact that $\mathrm{d}(\chi_1+\chi_2)=\mathrm{d}1=0$. This proves the validity of (4.21). □

We finally conclude the present subsection by introducing the notion of Laplacian.

Definition 4.18 (Laplacian) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be an infinitesimally Hilbertian metric measure space and* $f\in W^{1,2}(\mathrm{X})$*. Then we say that* f has Laplacian in $L^2(\mathfrak{m})$ *– briefly,* $f\in D(\Delta)$ *– provided there exists a function* $h\in L^2(\mathfrak{m})$ *such that*

$$\int hg\,\mathrm{d}\mathfrak{m}=-\int\langle\nabla f,\nabla g\rangle\,\mathrm{d}\mathfrak{m}\qquad\textit{for every } g\in W^{1,2}(\mathrm{X}). \tag{4.22}$$

The function h*, which is uniquely determined, is denoted by* Δf*.*

It can be readily checked that for any $f\in W^{1,2}(\mathrm{X})$ one has

$$f\in D(\Delta)\quad\Longleftrightarrow\quad\nabla f\in D(\mathrm{div}). \tag{4.23}$$

In this case, it holds that $\Delta f=\mathrm{div}(\nabla f)$. In particular, $D(\mathrm{div})$ is dense in $L^2(T\mathrm{X})$ by (4.16).

Remark 4.19 (Calculus rules for Δf) One can easily prove the following calculus rules:

$$\begin{aligned}\Delta(\varphi\circ f)&=\varphi'\circ f\,\Delta f+\varphi''\circ f\,|\nabla f|^2,\\\Delta(fg)&=f\,\Delta g+g\,\Delta f+2\,\langle\nabla f,\nabla g\rangle,\end{aligned} \tag{4.24}$$

for every $f,g\in D(\Delta)\cap\mathrm{LIP_b}(\mathrm{X})$ and $\varphi\in C^2(\mathbb{R})$. ■

4.1.2 Speed of a test plan

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a given metric measure space. Let us fix a test plan $\boldsymbol{\pi}$ on X. Then it holds that $\big(\Gamma(\mathrm{X}),\mathsf{d}_{\Gamma(\mathrm{X})},\boldsymbol{\pi}\big)$ is a metric measure space as well. Moreover, each evaluation map e_t has bounded compression from $\big(\Gamma(\mathrm{X}),\mathsf{d}_{\Gamma(\mathrm{X})},\boldsymbol{\pi}\big)$ to $(\mathrm{X},\mathsf{d},\mathfrak{m})$, whence it makes sense to consider the pullback $\mathrm{e}_t^*L^2(T\mathrm{X})$ of the tangent module – thus obtaining an $L^2(\boldsymbol{\pi})$-normed $L^\infty(\boldsymbol{\pi})$-module. Observe that $\mathrm{e}_t^*L^2(T\mathrm{X})$ is a Hilbert module as soon as the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian (by Remark 3.29).

Under suitable assumptions on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we have at our disposal a notion of 'speed' $\boldsymbol{\pi}'_t$ of a test plan $\boldsymbol{\pi}$ at time t, as described by the following result. For the proof of such fact, we refer to [Gig17b, Theorem 2.3.18] or to [Gig17a, Theorem/Definition 1.32].

Theorem 4.20 (Speed of a test plan) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space such that the tangent module $L^2(T\mathrm{X})$ is separable. Let $\boldsymbol{\pi}$ be a test plan on X. Then there exists a unique (up to $\mathcal{L}_1$-a.e. equality) family $\boldsymbol{\pi}'_t \in \mathrm{e}_t^*L^2(T\mathrm{X})$, $t\in[0,1]$, such that*

$$\lim_{h\to 0}\frac{f\circ \mathrm{e}_{t+h}-f\circ\mathrm{e}_t}{h} = (\mathrm{e}_t^*\mathrm{d}f)(\boldsymbol{\pi}'_t) \qquad \text{for } \mathcal{L}_1\text{-a.e. } t\in[0,1] \tag{4.25}$$

for every $f\in W^{1,2}(\mathrm{X})$, where the limit is intended in the strong topology of $L^1(\boldsymbol{\pi})$. Moreover, the Borel function $(\gamma,t)\mapsto|\boldsymbol{\pi}'_t|(\gamma)$ satisfies for $\mathcal{L}_1$-a.e. $t\in[0,1]$ the following property:

$$|\boldsymbol{\pi}'_t|(\gamma) = |\dot\gamma_t| \qquad \text{for } \boldsymbol{\pi}\text{-a.e. } \gamma. \tag{4.26}$$

The following technical fact will be needed in Chapter 6:

Proposition 4.21 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space such that $L^2(T\mathrm{X})$ is separable, let $\boldsymbol{\pi}$ be a test plan on X and let $f\in W^{1,2}(\mathrm{X})$. Then the almost everywhere defined map*

$$[0,1]\ni t\longmapsto (\mathrm{e}_t^*\mathrm{d}f)(\boldsymbol{\pi}'_t)\in L^1(\boldsymbol{\pi}) \tag{4.27}$$

is a.e. equivalent to a Borel map.

Proof. For every $h\in(0,1)$, the map $[0,1-h]\ni t\mapsto (f\circ\mathrm{e}_{t+h}-f\circ\mathrm{e}_t)/h\in L^1(\boldsymbol{\pi})$ is continuous. Thus by classical arguments the set of t's for which the limit as $h\to 0$ exists is Borel. In addition, the limit function – set to 0 when the limit does not exist – is Borel. □

In the sequel, we will mostly focus our attention on those test plans $\boldsymbol{\pi}$ that are concentrated on an equiLipschitz family of curves. As illustrated by the next definition, we will refer to them as 'Lipschitz test plans'. The ensuing discussion is taken from [GP17].

Definition 4.22 (Lipschitz test plan) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $\boldsymbol{\pi}$ be a test plan on X. Then we say that $\boldsymbol{\pi}$ is a* Lipschitz test plan *on X provided $\mathsf{ms}\in L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)$. Let us define $\mathsf{L}(\boldsymbol{\pi})\doteq\|\mathsf{ms}\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)}$. In other words, $\mathsf{L}(\boldsymbol{\pi})$ is the smallest constant $\mathsf{L}\geq 0$ such that $\boldsymbol{\pi}$ is concentrated on the family of all L-Lipschitz curves in X.*

Remark 4.23 Given any metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ with $L^2(T\mathrm{X})$ separable and any Lipschitz test plan $\boldsymbol{\pi}$ on X, it holds that

$$|\boldsymbol{\pi}'_t|\leq\mathsf{L}(\boldsymbol{\pi}) \quad \boldsymbol{\pi}\text{-a.e. on } \Gamma(\mathrm{X}) \qquad \text{for a.e. } t\in[0,1], \tag{4.28}$$

as one can immediately infer from the very definition of Lipschitz test plan. ■

Whenever the test plan $\boldsymbol{\pi}$ is Lipschitz, one has $(\mathrm{e}_t^*\mathrm{d}f)(\boldsymbol{\pi}'_t) \in L^2(\boldsymbol{\pi})$ for all $f \in W^{1,2}(\mathrm{X})$. One is led to wonder whether in this case the $L^1(\boldsymbol{\pi})$-limit in (4.25) takes place also in $L^2(\boldsymbol{\pi})$. The answer is affirmative, as shown by the following simple result:

Proposition 4.24 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with* $L^2(T\mathrm{X})$ *separable. Let* $\boldsymbol{\pi}$ *be a Lipschitz test plan on* X. *Let* $f \in W^{1,2}(\mathrm{X})$. *Then* $t \mapsto f \circ \mathrm{e}_t \in L^2(\boldsymbol{\pi})$ *is Lipschitz and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \mathrm{e}_t) = (\mathrm{e}_t^*\mathrm{d}f)(\boldsymbol{\pi}'_t) \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1], \tag{4.29}$$

where the derivative is taken in the Banach space $L^2(\boldsymbol{\pi})$.

Proof. Given any $t, s \in [0,1]$ with $s < t$, one has that

$$\begin{aligned}
\big\|f \circ \mathrm{e}_t - f \circ \mathrm{e}_s\big\|^2_{L^2(\boldsymbol{\pi})} &= \int \big|f(\gamma_t) - f(\gamma_s)\big|^2 \,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\leq \int \bigg(\int_s^t |Df|(\gamma_r)\,|\dot{\gamma}_r|\,\mathrm{d}r\bigg)^2 \mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\leq (t-s)\,\mathsf{L}(\boldsymbol{\pi})^2 \iint_s^t |Df|^2(\gamma_r)\,\mathrm{d}r\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\leq \mathsf{C}(\boldsymbol{\pi})\,\mathsf{L}(\boldsymbol{\pi})^2\,\|f\|^2_{W^{1,2}(\mathrm{X})}\,(t-s)^2,
\end{aligned}$$

which shows that $t \mapsto f \circ \mathrm{e}_t \in L^2(\boldsymbol{\pi})$ is Lipschitz. In particular, it is differentiable at almost every $t \in [0,1]$ by Theorem C.4, so that (4.29) follows from (4.25). □

4.1.3 Tangent module on the weighted Euclidean space

The material contained in this subsection is entirely taken from [GP16a].

Let $d \in \mathbb{N} \setminus \{0\}$ be fixed. Consider the Euclidean space $\mathbb{R}^d$, equipped with the Euclidean distance $\mathsf{d}_{\mathrm{Eucl}}$ and with any non-negative Radon measure μ. Then we have (at least) two different ways of speaking about '$L^2(\mu)$-vector fields' on $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$:

- the tangent module associated to $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$, which will be denoted by $L^2_\mu(T\mathbb{R}^d)$,
- the space $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ of all $L^2(\mu)$-maps from $\mathbb{R}^d$ to itself.

Our main concern here is to give an answer to the following question:

Which is the relation between $L^2_\mu(T\mathbb{R}^d)$ and $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$?

In general, these two spaces are different: for instance, if μ is a Dirac delta then $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ has dimension d, while $L^2_\mu(T\mathbb{R}^d)$ reduces to the zero space. Nevertheless, there is always a canonical way to isometrically embed $L^2_\mu(T\mathbb{R}^d)$ into $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$, as we are going to prove.

It can be readily seen that the spaces $L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$ and $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ have a natural structure of $L^2(\mu)$-normed $L^\infty(\mu)$-module. More precisely, they are both Hilbert modules, one the dual of the other. In order to keep a distinguished notation, their elements will typically be underlined, while those of $L^2_\mu(T^*\mathbb{R}^d)$ and $L^2_\mu(T\mathbb{R}^d)$ will be not.

For instance, given any function $f \in C^1_c(\mathbb{R}^d)$, we shall denote by $\underline{\mathrm{d}}f \in L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$ its 'classical' differential and by $\mathrm{d}f \in L^2_\mu(T^*\mathbb{R}^d)$ its differential in the sense of modules. One has

$$|\mathrm{d}f| \leq \mathrm{lip}(f) = |\underline{\mathrm{d}}f| \qquad \mu\text{-a.e. in } \mathbb{R}^d, \tag{4.30}$$

as a consequence of property (2.23).

Proposition 4.25 *There exists a unique $L^\infty(\mu)$-linear and continuous operator*

$$P:\, L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu)\longrightarrow L^2_\mu(T^*\mathbb{R}^d) \tag{4.31}$$

such that $P(\underline{\mathrm{d}}f)=\mathrm{d}f$ for every $f\in C^1_{\mathrm{c}}(\mathbb{R}^d)$. Moreover, it holds that

$$\big|P(\underline{\omega})\big|\leq|\underline{\omega}|\quad \mu\text{-a.e. in } \mathbb{R}^d \qquad \text{for every } \underline{\omega}\in L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu). \tag{4.32}$$

Proof. First of all, let us define the vector space $V\subseteq L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu)$ as

$$V\doteq\bigg\{\sum_{i=1}^n\chi_{A_i}\,\underline{\mathrm{d}}f_i\;\bigg|\;n\in\mathbb{N}^+,\ (A_i)_{i=1}^n \text{ Borel partition of } \mathbb{R}^d,\ (f_i)_{i=1}^n\subseteq C^1_{\mathrm{c}}(\mathbb{R}^d)\bigg\}.$$

We define the operator $P:\,V\to L^2_\mu(T^*\mathbb{R}^d)$ in the following way:

$$P\bigg(\sum_{i=1}^n\chi_{A_i}\,\underline{\mathrm{d}}f_i\bigg)\doteq\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}f_i \qquad \text{for every } \sum_{i=1}^n\chi_{A_i}\,\underline{\mathrm{d}}f_i\in V.$$

In particular, $P(\underline{\mathrm{d}}f)=\mathrm{d}f$ for all $f\in C^1_{\mathrm{c}}(\mathbb{R}^d)$. Note that (4.30) gives the μ-a.e. inequality

$$\bigg|\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}f_i\bigg|=\sum_{i=1}^n\chi_{A_i}\,|\mathrm{d}f_i|\leq\sum_{i=1}^n\chi_{A_i}\,|\underline{\mathrm{d}}f_i|=\bigg|\sum_{i=1}^n\chi_{A_i}\,\underline{\mathrm{d}}f_i\bigg|. \tag{4.33}$$

This grants that the map P is well-defined, linear and continuous. Given that the space V is dense in $L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu)$, we deduce that P can be uniquely extended to a linear continuous operator $P:\,L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu)\to L^2_\mu(T^*\mathbb{R}^d)$. The fact that such extension is $L^\infty(\mu)$-linear can be checked by first noticing that P properly behaves with respect to multiplication by simple functions, then arguing by approximation. Finally, property (4.33) says that (4.32) holds for every $\underline{\omega}\in V$, whence also for any $\underline{\omega}\in L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu)$ by density of V. □

By duality with the map P, it holds that there exists a unique $L^\infty(\mu)$-linear continuous operator $\iota:\,L^2_\mu(T\mathbb{R}^d)\to L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$ satisfying

$$\underline{\omega}\big(\iota(v)\big)=P(\underline{\omega})(v) \qquad \text{for every } v\in L^2_\mu(T\mathbb{R}^d) \text{ and } \underline{\omega}\in L^2(\mathbb{R}^d,(\mathbb{R}^d)^*;\mu). \tag{4.34}$$

Furthermore, the bound in (4.32) grants that

$$\big|\iota(v)\big|\leq|v|\ \text{ holds } \mu\text{-a.e. in } \mathbb{R}^d \qquad \text{for every } v\in L^2_\mu(T\mathbb{R}^d). \tag{4.35}$$

We want to prove that equality holds in (4.35), i.e. that ι is actually an isometric embedding.

This will be achieved by showing that P is a *quotient map*, more specifically that it is surjective and that it satisfies the following property:

$$\forall\,\omega\in L^2_\mu(T^*\mathbb{R}^d)\quad \exists\,\underline{\omega}\in P^{-1}(\omega):\quad |\omega|=|\underline{\omega}|\quad \mu\text{-a.e. in } \mathbb{R}^d. \tag{4.36}$$

We shall need the following lemma about the structure of Sobolev spaces over weighted $\mathbb{R}^d$.

Lemma 4.26 (Density in energy of C^1 functions) *Let $f\in W^{1,2}(\mathbb{R}^d,\mathsf{d}_{\mathrm{Eucl}},\mu)$ be given. Then there exists a sequence $(f_n)_n\subseteq C^1_{\mathrm{c}}(\mathbb{R}^d)$ such that $f_n\to f$ and $|\underline{\mathrm{d}}f_n|\to|\mathrm{d}f|$ in $L^2(\mu)$.*

Proof. We know from Theorem 2.27 that there exists a sequence $(g_n)_n$ of compactly supported Lipschitz functions $g_n : \mathbb{R}^d \to \mathbb{R}$ such that $g_n \to f$ and $\mathrm{lip}_a(g_n) \to |\mathrm{d}f|$ in $L^2(\mu)$. Choose any standard family $(\rho_k)_{k\in\mathbb{N}^+}$ of mollifiers such that $\mathrm{spt}(\rho_k) \subseteq B_{1/k}(0)$. Now let $n \in \mathbb{N}$ be fixed. We define $g_n^k \in C_c^1(\mathbb{R}^d)$ for any $k \in \mathbb{N}^+$ as follows:

$$g_n^k(x) \doteq (g_n * \rho_k)(x) = \int g_n(z)\,\rho_k(x-z)\,\mathrm{d}z \qquad \text{for every } x \in \mathbb{R}^d.$$

It is well-known that $\mathrm{spt}(g_n^k) \subseteq B_{1/k}\big(\mathrm{spt}(g_n)\big)$ for every $k \in \mathbb{N}^+$ and that g_n^k uniformly converges to g_n as $k \to \infty$, whence accordingly $g_n^k \to g_n$ in $L^2(\mu)$ since μ is a Radon measure. Moreover, if we choose $r > \frac{1}{k}$ and $x, y \in \mathbb{R}^d$ such that $0 < |x-y| < r$, then it holds that

$$\begin{aligned}\frac{\big|g_n^k(x) - g_n^k(y)\big|}{|x-y|} &\leq \frac{1}{|x-y|}\int_{B_r(0)} \big|g_n(x-z) - g_n(y-z)\big|\,\rho_k(z)\,\mathrm{d}z \leq \mathrm{Lip}\big(g_n|_{B_{2r}(x)}\big)\int \rho_k(z)\,\mathrm{d}z \\ &= \mathrm{Lip}\big(g_n|_{B_{2r}(x)}\big).\end{aligned}$$

By first letting $y \to x$ and then $k \to \infty$, we deduce that $\overline{\lim}_k\,|\underline{\mathrm{d}}\,g_n^k|(x) \leq \mathrm{Lip}\big(g_n|_{B_{2r}(x)}\big)$ for every $x \in \mathbb{R}^d$ and $r > 0$. By taking the infimum among all $r > 0$ we thus obtain that

$$\overline{\lim_{k\to\infty}}\,|\underline{\mathrm{d}}\,g_n^k|(x) \leq \mathrm{lip}_a(g_n)(x) \qquad \text{for every } x \in \mathbb{R}^d. \tag{4.37}$$

Observe that also $|\underline{\mathrm{d}}\,g_n^k| \leq \mathrm{Lip}(g_n)\,\chi_{B_1(\mathrm{spt}(g_n))}$ for every $k \in \mathbb{N}^+$, so that by applying the reverse Fatou lemma to (4.37) we get the inequality $\overline{\lim}_k\,\big\||\underline{\mathrm{d}}\,g_n^k|\big\|_{L^2(\mu)} \leq \big\|\mathrm{lip}_a(g_n)\big\|_{L^2(\mu)}$. Hence a diagonalisation argument gives the existence of a sequence $(k_n)_n$ such that $f_n \doteq g_n^{k_n}$ satisfies

$$\lim_{n\to\infty}\|f_n - f\|_{L^2(\mu)} = 0 \qquad \text{and} \qquad \overline{\lim_{n\to\infty}}\,\big\||\underline{\mathrm{d}}f_n|\big\|_{L^2(\mu)} \leq \big\||\mathrm{d}f|\big\|_{L^2(\mu)}. \tag{4.38}$$

In particular, the sequence $\big(|\underline{\mathrm{d}}f_n|\big)_n$ is bounded in $L^2(\mu)$. By recalling (4.30), we see that the sequence $\big(|\mathrm{d}f_n|\big)_n$ is bounded in $L^2(\mu)$ as well. Therefore (up to passing to a not relabeled subsequence) it holds that

$$\begin{aligned}|\mathrm{d}f_n| &\rightharpoonup G && \text{weakly in } L^2(\mu),\\ |\underline{\mathrm{d}}f_n| &\rightharpoonup G' && \text{weakly in } L^2(\mu),\\ f_n(x) &\to f(x) && \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,\end{aligned}$$

for suitable functions $G, G' \in L^2(\mu)$ such that $G \leq G'$ holds μ-a.e. in $\mathbb{R}^d$. Note that item i) of Theorem 2.22 grants that $|\mathrm{d}f| \leq G$, whence the second property in (4.38) gives

$$\big\||\mathrm{d}f|\big\|_{L^2(\mu)} \leq \|G'\|_{L^2(\mu)} \leq \underline{\lim_{n\to\infty}}\,\big\||\underline{\mathrm{d}}f_n|\big\|_{L^2(\mu)} \leq \overline{\lim_{n\to\infty}}\,\big\||\underline{\mathrm{d}}f_n|\big\|_{L^2(\mu)} \leq \big\||\mathrm{d}f|\big\|_{L^2(\mu)},$$

which forces the equality $\lim_n \big\||\underline{\mathrm{d}}f_n|\big\|_{L^2(\mu)} = \big\||\mathrm{d}f|\big\|_{L^2(\mu)} = \|G'\|_{L^2(\mu)}$. Hence $G' = |\mathrm{d}f|$ and accordingly $|\underline{\mathrm{d}}f_n| \to |\mathrm{d}f|$ in $L^2(\mu)$, as required. □

Theorem 4.27 **($L^2_\mu(T\mathbb{R}^d)$ isometrically embeds into $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$)** *The operator P as in Proposition 4.25 is a quotient map and its adjoint operator $\iota : L^2_\mu(T\mathbb{R}^d) \to L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ is a module morphism preserving the pointwise norm.*

Proof. First of all, we show that $\mathrm{d}f$ belongs to the image of P for every $f \in W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$. To this aim, pick $(f_n)_n \subseteq C^1_{\mathrm{c}}(\mathbb{R}^d)$ as in Lemma 4.26 and notice that $(\underline{\mathrm{d}}f_n)_n$ is a bounded sequence in $L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$. Being such space reflexive, we have (up to a not relabeled subsequence) that $\underline{\mathrm{d}}f_n \rightharpoonup \underline{\omega}$ for some $\underline{\omega} \in L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$. Since the map P is linear and continuous, we know that $\mathrm{d}f_n = P(\underline{\mathrm{d}}f_n) \rightharpoonup P(\underline{\omega})$ weakly in $L^2_\mu(T^*\mathbb{R}^d)$, whence the closure of the differential (cf. Theorem 2.22) grants that $\mathrm{d}f = P(\underline{\omega})$. Hence $\mathrm{d}f$ is in the image of P.

We claim that $|\underline{\omega}| \leq |\mathrm{d}f|$ holds μ-a.e. in $\mathbb{R}^d$: if not, there exists a Borel set $A \subseteq \mathbb{R}^d$ such that $\mu(A) > 0$ and $|\mathrm{d}f|(x) < |\underline{\omega}|(x)$ for μ-a.e. $x \in A$, whence the fact that $\chi_A \, \underline{\mathrm{d}}f_n \rightharpoonup \chi_A \, \underline{\omega}$ weakly in $L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$ and the strong $L^2(\mu)$-convergence of $|\underline{\mathrm{d}}f_n|$ to $|\mathrm{d}f|$ (that is granted by Lemma 4.26) yield the inequalities

$$\int_A |\mathrm{d}f|^2 \, \mathrm{d}\mu < \int_A |\underline{\omega}|^2 \, \mathrm{d}\mu \leq \varliminf_{n\to\infty} \int_A |\underline{\mathrm{d}}f_n|^2 \, \mathrm{d}\mu = \int_A |\mathrm{d}f|^2 \, \mathrm{d}\mu,$$

which lead to a contradiction. Therefore $|\underline{\omega}| \leq |\mathrm{d}f|$ is satisfied μ-a.e. in $\mathbb{R}^d$, as claimed above. Since we have that $|\mathrm{d}f| = \big|P(\underline{\omega})\big| \leq |\underline{\omega}|$ holds μ-a.e. in $\mathbb{R}^d$ by (4.32), we see that $|\underline{\omega}| = |\mathrm{d}f|$, which shows that property (4.36) is satisfied for $\omega \doteq \mathrm{d}f$.

Denote by V the space of all $\omega \in L^2_\mu(T^*\mathbb{R}^d)$ of the form $\sum_{i=1}^n \chi_{A_i} \, \mathrm{d}f_i$, where $(A_i)_i$ is a Borel partition of $\mathbb{R}^d$ and $f_1, \ldots, f_n \in W^{1,2}(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mu)$. Now let $\omega = \sum_{i=1}^n \chi_{A_i} \, \mathrm{d}f_i \in V$ be fixed. As proven in the previous paragraph, there exist $\underline{\omega}_1, \ldots, \underline{\omega}_n \in L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$ such that $P(\underline{\omega}_i) = \mathrm{d}f_i$ and $|\mathrm{d}f_i| = |\underline{\omega}_i|$ for all $i = 1, \ldots, n$. Given that P is a module morphism, we deduce that $P\big(\sum_{i=1}^n \chi_{A_i} \, \underline{\omega}_i\big) = \omega$ and $\big|\sum_{i=1}^n \chi_{A_i} \, \underline{\omega}_i\big| = |\omega|$, thus proving that the vector space V is contained in the image of P and that property (4.36) holds for any $\omega \in V$.

Now fix $\omega \in L^2_\mu(T^*\mathbb{R}^d)$. Since V is dense in $L^2_\mu(T^*\mathbb{R}^d)$, there exists a sequence $(\omega_n)_n \subseteq V$ that $L^2_\mu(T^*\mathbb{R}^d)$-converges to ω. By the previous case, we know that for any $n \in \mathbb{N}$ we can pick $\underline{\omega}_n \in P^{-1}(\omega_n)$ such that $|\underline{\omega}_n| = |\omega_n|$. In particular, the sequence $(\underline{\omega}_n)_n$ is bounded in the space $L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$, whence (up to a not relabeled subsequence) it holds that $\underline{\omega}_n$ weakly converges to some $\underline{\omega} \in L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$. By Mazur lemma, we can also suppose that $\underline{\omega}_n \to \underline{\omega}$ strongly in $L^2(\mathbb{R}^d, (\mathbb{R}^d)^*; \mu)$, thus accordingly $\omega_n = P(\underline{\omega}_n) \to P(\underline{\omega})$ strongly in $L^2_\mu(T^*\mathbb{R}^d)$, which yields $P(\underline{\omega}) = \omega$. Up to a further subsequence, we can even assume that $|\underline{\omega}_n| \to |\underline{\omega}|$ and $|\omega_n| \to |\omega|$ pointwise μ-a.e. in $\mathbb{R}^d$, so that

$$|\omega|(x) = \lim_{n\to\infty} |\omega_n|(x) = \lim_{n\to\infty} |\underline{\omega}_n|(x) = |\underline{\omega}|(x) \qquad \text{holds for } \mu\text{-a.e. } x \in \mathbb{R}^d.$$

Therefore the operator P is surjective and property (4.36) is verified.

We turn to the last part of the statement. It suffices to prove that $\big|\iota(v)\big| = |v|$ holds μ-a.e. for every $v \in L^2_\mu(T\mathbb{R}^d)$. Let $v \in L^2_\mu(T\mathbb{R}^d)$ and $\varepsilon > 0$ be fixed. Then there exists $\omega \in L^2_\mu(T^*\mathbb{R}^d)$ such that $\|\omega\|_{L^2_\mu(T^*\mathbb{R}^d)} = 1$ and $\int \omega(v) \, \mathrm{d}\mu \geq \|v\|_{L^2_\mu(T\mathbb{R}^d)} - \varepsilon$. Hence, what previously proved grants the existence of some $\underline{\omega} \in P^{-1}(\omega)$ for which $|\underline{\omega}| = |\omega|$ holds μ-a.e. in $\mathbb{R}^d$. Therefore

$$\big\|\iota(v)\big\|_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \geq \int \underline{\omega}\big(\iota(v)\big) \, \mathrm{d}\mu \overset{(4.34)}{=} \int P(\underline{\omega})(v) \, \mathrm{d}\mu = \int \omega(v) \, \mathrm{d}\mu \geq \|v\|_{L^2_\mu(T\mathbb{R}^d)} - \varepsilon.$$

By letting $\varepsilon \searrow 0$ in the previous formula, we see that $\big\|\iota(v)\big\|_{L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)} \geq \|v\|_{L^2_\mu(T\mathbb{R}^d)}$. Since we also have the μ-a.e. inequality $\big|\iota(v)\big| \leq |v|$ by (4.35), we finally conclude that the map ι preserves the pointwise norm, as required. □

Corollary 4.28 *Let $\iota : L^2_\mu(T\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ be as in Theorem 4.27. Consider some vector fields $v_1, \ldots, v_n \in L^2_\mu(T\mathbb{R}^d)$ that are independent on a Borel set $E \subseteq \mathbb{R}^d$. Then the vectors $\iota(v_1)(x), \ldots, \iota(v_n)(x) \in \mathbb{R}^d$ are linearly independent for μ-a.e. point $x \in E$.*

Proof. Let us choose some Borel representatives $\overline{\iota(v_1)}, \ldots, \overline{\iota(v_n)} : \mathbb{R}^d \to \mathbb{R}^d$ of $\iota(v_1), \ldots, \iota(v_n)$, respectively. Since $v_1, \ldots, v_n$ are independent on E by hypothesis – whence $\iota(v_1), \ldots, \iota(v_n)$ are independent on E by Theorem 4.27 – there exists $N \subseteq E$ Borel such that $\mu(N) = 0$ and

$$\sum_{i=1}^{n} q_i \, \overline{\iota(v_i)}(x) \neq 0 \qquad \text{for every } (q_1, \ldots, q_n) \in \mathbb{Q}^n \setminus \{0\} \text{ and } x \in E \setminus N.$$

This clearly grants that $\overline{\iota(v_1)}(x), \ldots, \overline{\iota(v_n)}(x)$ are linearly independent for every $x \in E \setminus N$, thus yielding the statement. □

Corollary 4.29 *Let μ be any Radon measure on the Euclidean space $\mathbb{R}^d$. Then $L^2_\mu(T\mathbb{R}^d)$ is a separable Hilbert module.*

Proof. It can be readily checked that $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ is a separable Hilbert module. Then the statement immediately follows from Theorem 4.27. □

We briefly recall few basic notions about 1-dimensional currents in the Euclidean space. By 1-*dimensional current* T in $\mathbb{R}^d$ we mean any linear continuous functional on the space of all smooth compactly-supported differential 1-forms on $\mathbb{R}^d$. The *boundary* ∂T of T is the 0-current (i.e. distribution) defined as $\langle \partial T, f \rangle \doteq \langle T, \underline{\mathrm{d}} f \rangle$ for any smooth function $f : \mathbb{R}^d \to \mathbb{R}$ having compact support. The *mass* of T is given by the supremum of $\langle T, \underline{\omega} \rangle$ among all the 1-forms $\underline{\omega}$ on $\mathbb{R}^d$ such that $|\underline{\omega}| \leq 1$ everywhere. By applying Theorem 1.14, we can represent any 1-dimensional current T with finite mass as $T = \vec{T}\,\|T\|$, where $\|T\| \geq 0$ is a finite Borel measure on $\mathbb{R}^d$ and $\vec{T} \in L^1\big(\mathbb{R}^d, \mathbb{R}^d; \|T\|\big)$ satisfies $\big|\vec{T}(x)\big| = 1$ for $\|T\|$-a.e. $x \in \mathbb{R}^d$. Then

$$\langle T, \underline{\omega} \rangle = \int \big\langle \underline{\omega}(x), \vec{T}(x) \big\rangle \,\mathrm{d}\|T\|(x) \tag{4.39}$$

for any smooth compactly-supported 1-form $\underline{\omega}$ on $\mathbb{R}^d$. Finally, we say that a 1-dimensional current T is *normal* provided both T and ∂T have finite mass. We refer to [Fed69] for more information about currents in the Euclidean space.

Now consider a Radon measure μ on $\mathbb{R}^d$ and the embedding $\iota : L^2_\mu(T\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$, whose existence has been proved in Theorem 4.27. We can further proceed by associating to each vector field $v \in L^2_\mu(T\mathbb{R}^d)$ a 1-dimensional current $\mathcal{I}(v)$ in $\mathbb{R}^d$, defined as follows:

$$\big\langle \mathcal{I}(v), \underline{\omega} \big\rangle \doteq \int \underline{\omega}\big(\iota(v)\big)\,\mathrm{d}\mu = \int P(\underline{\omega})(v)\,\mathrm{d}\mu \tag{4.40}$$

for every smooth compactly-supported 1-form $\underline{\omega}$ on $\mathbb{R}^d$ (with P given by Proposition 4.25). It is clear that $\mathcal{I}(v)$ has locally finite mass and that – since ι preserves the pointwise norm – the mass measure $\big\|\mathcal{I}(v)\big\|$ is given by $|v|\mu$. The boundary of $\mathcal{I}(v)$ acts on $f \in C^\infty_c(\mathbb{R}^d)$ as

$$\big\langle \partial\mathcal{I}(v), f \big\rangle = \big\langle \mathcal{I}(v), \underline{\mathrm{d}} f \big\rangle \overset{(4.40)}{=} \int \underline{\mathrm{d}} f\big(\iota(v)\big)\,\mathrm{d}\mu \overset{(4.34)}{=} \int \mathrm{d}f(v)\,\mathrm{d}\mu. \tag{4.41}$$

By looking at the third expression in this chain of equalities, we see that $\partial\mathcal{I}(v)$ has locally finite mass if and only if the distributional divergence of $\iota(v)\mu$ is a Radon measure. In this case, such measure coincides with $-\partial\mathcal{I}(v)$. Finally, by looking at the last term in (4.41) and comparing it with Definition 4.15, we obtain the following result:

Corollary 4.30 *Let $v \in L^2_\mu(T\mathbb{R}^d)$ be such that $\iota(v)$ has compact support. Then $\mathcal{I}(v)$ is a normal current if and only if $v \in D(\mathbf{div}_\mu)$. In this case, it holds that $\partial\mathcal{I}(v) = -\mathbf{div}_\mu(v)$.*

4.1.4 Maps of bounded deformation

Let us fix two metric measure spaces $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$ and $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$.

Definition 4.31 (Map of bounded deformation) *Let $\varphi:\mathrm{X}\to\mathrm{Y}$ be a given Borel map. Then we say that φ is a* map of bounded deformation *provided it is of bounded compression (cf. Definition 3.26) and Lipschitz.*

Maps of bounded deformation satisfy the following fundamental property:

Proposition 4.32 *Let $\varphi:\mathrm{X}\to\mathrm{Y}$ be a map of bounded deformation. Let $f\in\mathrm{S}^2(\mathrm{Y})$ be fixed. Then it holds that $f\circ\varphi\in\mathrm{S}^2(\mathrm{X})$ and*

$$|\mathrm{d}(f\circ\varphi)|\leq\mathrm{Lip}(\varphi)\,|\mathrm{d}f|\circ\varphi\qquad\mathfrak{m}_\mathrm{X}\text{-a.e. in }\mathrm{X}. \tag{4.42}$$

In particular, if $f\in W^{1,2}(\mathrm{Y})$ then $f\circ\varphi\in W^{1,2}(\mathrm{X})$.

Proof. Consider the map $\hat\varphi:\Gamma(\mathrm{X})\to\Gamma(\mathrm{Y})$ defined as $\gamma\mapsto\varphi\circ\gamma$. Given any $\gamma\in AC\big([0,1],\mathrm{X}\big)$, it turns out that $\hat\varphi(\gamma)\in AC\big([0,1],\mathrm{Y}\big)$ and that $\mathsf{ms}\big(\hat\varphi(\gamma),t\big)\leq\mathrm{Lip}(\varphi)\,\mathsf{ms}(\gamma,t)$ for a.e. $t\in[0,1]$, whence $\hat\varphi_*\boldsymbol{\pi}$ is a test plan on Y whenever $\boldsymbol{\pi}$ is a test plan on X. Then for $f\in\mathrm{S}^2(\mathrm{Y})$ one has

$$\begin{aligned}\int\big|(f\circ\varphi)(\gamma_1)-(f\circ\varphi)(\gamma_0)\big|\,\mathrm{d}\boldsymbol{\pi}(\gamma)&=\int\big|f(\sigma_1)-f(\sigma_0)\big|\,\mathrm{d}(\hat\varphi_*\boldsymbol{\pi})(\sigma)\\&\leq\iint_0^1|\mathrm{d}f|(\sigma_t)\,\mathsf{ms}(\sigma,t)\,\mathrm{d}t\,\mathrm{d}(\hat\varphi_*\boldsymbol{\pi})(\sigma)\\&=\iint_0^1\big(|\mathrm{d}f|\circ\varphi\big)(\gamma_t)\,\mathsf{ms}\big(\hat\varphi(\gamma),t\big)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\&\leq\mathrm{Lip}(\varphi)\iint_0^1\big(|\mathrm{d}f|\circ\varphi\big)(\gamma_t)\,|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma).\end{aligned}$$

By arbitrariness of $\boldsymbol{\pi}$, we conclude that $f\circ\varphi\in\mathrm{S}^2(\mathrm{X})$ and that property (4.42) holds. □

In the right hand side of (4.42) the quantity $\mathrm{Lip}(\varphi)$ appears. The following result shows that the global Lipschitz constant can be replaced by a 'more local' object; see [GP16b].

Lemma 4.33 *Let $\varphi:\mathrm{X}\to\mathrm{Y}$ be a map of bounded deformation. Let $E\subseteq\mathrm{X}$ be a Borel set. Then for any $f\in W^{1,2}(\mathrm{Y})$ we have that*

$$\chi_E\,\big|\mathrm{d}(f\circ\varphi)\big|\leq\mathrm{Lip}(\varphi;E)\,\chi_E\,|\mathrm{d}f|\circ\varphi\qquad\text{holds }\mathfrak{m}_\mathrm{X}\text{-a.e. in }\mathrm{X}. \tag{4.43}$$

Proof. Choose a sequence $(f_n)_n\subseteq\mathrm{LIP_{bs}}(\mathrm{Y})$ such that $f_n\to f$ and $\mathrm{lip}_a(f_n)\to|\mathrm{d}f|$ in $L^2(\mathfrak{m}_\mathrm{Y})$. Let $n\in\mathbb{N}$ be fixed. Given any $x\in E$ and $r>0$, there exists a Lipschitz map $g\in\mathrm{LIP}(\mathrm{X})$ such that $\mathrm{Lip}(g)=\mathrm{Lip}\big(f_n\circ\varphi;E\cap B_r(x)\big)$ and $g=f_n\circ\varphi$ on $E\cap B_r(x)$. Since $\varphi\big(E\cap B_r(x)\big)$ is contained in the ball $B_{\mathrm{Lip}(\varphi)r}\big(\varphi(x)\big)$, we have that

$$\chi_{E\cap B_r(x)}\,\big|\mathrm{d}(f_n\circ\varphi)\big|=\chi_{E\cap B_r(x)}\,|\mathrm{d}g|\leq\mathrm{Lip}(g)\leq\mathrm{Lip}(\varphi;E)\,\mathrm{Lip}\big(f_n;B_{\mathrm{Lip}(\varphi)r}(\varphi(x))\big) \tag{4.44}$$

holds $\mathfrak{m}_\mathrm{X}$-a.e.. Given that $B_{\mathrm{Lip}(\varphi)r}\big(\varphi(x)\big)\subseteq B_{2\mathrm{Lip}(\varphi)r}\big(\varphi(y)\big)$ is satisfied for every $y\in B_r(x)$, we deduce from (4.44) and Lindelöf lemma that

$$\big|\mathrm{d}(f_n\circ\varphi)\big|(x)\leq\mathrm{Lip}(\varphi;E)\,\mathrm{Lip}\Big(f_n;B_{2\mathrm{Lip}(\varphi)r}\big(\varphi(x)\big)\Big)\qquad\text{holds for }\mathfrak{m}_\mathrm{X}\text{-a.e. }x\in E. \tag{4.45}$$

By letting $r \searrow 0$ in (4.45), we thus obtain the $\mathfrak{m}_{\mathrm{X}}$-a.e. inequality

$$\chi_E \left|\mathrm{d}(f_n \circ \varphi)\right| \leq \mathrm{Lip}(\varphi; E)\, \chi_E \,\mathrm{lip}_a(f_n) \circ \varphi \qquad \text{for every } n \in \mathbb{N}. \tag{4.46}$$

Notice that $\left|\mathrm{d}(f_n \circ \varphi)\right| \leq \mathrm{Lip}(\varphi)\,|\mathrm{d}f_n| \circ \varphi \leq \mathrm{Lip}(\varphi)\,\mathrm{lip}_a(f_n) \circ \varphi$ is satisfied $\mathfrak{m}_{\mathrm{X}}$-a.e., thus accordingly the set of all functions $\left|\mathrm{d}(f_n \circ \varphi)\right|$, with $n \in \mathbb{N}$, is norm bounded in $L^2(\mathfrak{m}_{\mathrm{X}})$. In particular, possibly passing to a (not relabeled) subsequence, one has that $\left|\mathrm{d}(f_n \circ \varphi)\right| \rightharpoonup h$ weakly in $L^2(\mathfrak{m}_{\mathrm{X}})$ for a suitable map $h \in L^2(\mathfrak{m}_{\mathrm{X}})$. By lower semicontinuity of minimal weak upper gradients, we deduce that $\left|\mathrm{d}(f \circ \varphi)\right| \leq h$ holds $\mathfrak{m}_{\mathrm{X}}$-a.e.. Since $\mathrm{lip}_a(f_n) \circ \varphi \rightharpoonup |\mathrm{d}f| \circ \varphi$ weakly in $L^2(\mathfrak{m}_{\mathrm{X}})$, we finally conclude by recalling (4.46) that

$$\chi_E \left|\mathrm{d}(f \circ \varphi)\right| \leq \chi_E\, h \leq \mathrm{Lip}(\varphi; E)\, \chi_E \,|\mathrm{d}f| \circ \varphi \qquad \text{holds } \mathfrak{m}_{\mathrm{X}}\text{-a.e. in X},$$

yielding (4.43) and accordingly the statement. □

Theorem 4.34 (Differential of maps of bounded deformation) *Suppose that $L^2(T\mathrm{Y})$ is separable. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation. Then there exists a unique $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linear continuous map $\mathrm{d}\varphi : L^2(T\mathrm{X}) \to \varphi^* L^2(T\mathrm{Y})$, called* differential *of φ, such that*

$$(\varphi^* \mathrm{d}f)\big(\mathrm{d}\varphi(v)\big) = \mathrm{d}(f \circ \varphi)(v) \qquad \textit{for every } f \in W^{1,2}(\mathrm{Y}) \textit{ and } v \in L^2(T\mathrm{X}). \tag{4.47}$$

Moreover, it holds that

$$\left|\mathrm{d}\varphi(v)\right| \leq \mathrm{Lip}(\varphi)\,|v| \quad \mathfrak{m}_{\mathrm{X}}\textit{-a.e.} \qquad \textit{for every } v \in L^2(T\mathrm{X}). \tag{4.48}$$

Proof. Fix $v \in L^2(T\mathrm{X})$. Let us define $L_v(\varphi^* \mathrm{d}f) \doteq \mathrm{d}(f \circ \varphi)(v)$ for any $f \in W^{1,2}(\mathrm{Y})$. Since

$$\left|\mathrm{d}(f \circ \varphi)(v)\right| \overset{(4.42)}{\leq} \mathrm{Lip}(\varphi)\,|\mathrm{d}f| \circ \varphi\,|v| = \mathrm{Lip}(\varphi)\,|\varphi^* \mathrm{d}f|\,|v| \qquad \mathfrak{m}_{\mathrm{X}}\text{-a.e.},$$

we see that $L_v(\varphi^* \mathrm{d}f)$ is well-posed and that L_v can be uniquely extended to an $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linear and continuous map $L_v : \varphi^* L^2(T^*\mathrm{Y}) \to L^1(\mathfrak{m}_{\mathrm{X}})$. Theorem 3.34 grants that L_v corresponds to an element $\mathrm{d}\varphi(v) \in \varphi^* L^2(T\mathrm{Y})$, which clearly satisfies properties (4.47) and (4.48). □

Remark 4.35 Without the separability assumption on $L^2(T\mathrm{Y})$, it is still possible to build a differential $\mathrm{d}\varphi : L^2(T\mathrm{X}) \to \big(\varphi^* L^2(T^*\mathrm{Y})\big)^*$, as shown in the proof of Theorem 4.34. ■

It is possible to refine of inequality (4.48) by using Lemma 4.33 (cf. [GP16b]):

Proposition 4.36 *Suppose that $L^2(T\mathrm{Y})$ is separable. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation. Fix any Borel set $E \subseteq \mathrm{X}$. Then for any $v \in L^2(T\mathrm{X})_{|E}$ we have that*

$$\left|\mathrm{d}\varphi(v)\right| \leq \mathrm{Lip}(\varphi; E)\,|v| \qquad \textit{holds } \mathfrak{m}_{\mathrm{X}}\textit{-a.e. in } E. \tag{4.49}$$

Proof. Given any $f \in W^{1,2}(\mathrm{Y})$, it $\mathfrak{m}_{\mathrm{X}}$-a.e. holds that

$$\begin{aligned}\chi_E \left|(\varphi^* \mathrm{d}f)\big(\mathrm{d}\varphi(v)\big)\right| &= \chi_E \left|\mathrm{d}(f \circ \varphi)(v)\right| \leq \chi_E \left|\mathrm{d}(f \circ \varphi)\right|\,|v| \overset{(4.43)}{\leq} \mathrm{Lip}(\varphi; E)\, \chi_E\, |\mathrm{d}f| \circ \varphi\, |v| \\ &= \mathrm{Lip}(\varphi; E)\, \chi_E\, |\varphi^* \mathrm{d}f|\,|v|,\end{aligned}$$

which grants that (4.49) holds by $L^\infty(\mathfrak{m}_{\mathrm{X}})$-linearity and continuity of $\mathrm{d}\varphi(v)$. □

Let $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$, $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ be metric measure spaces with $L^2(T\mathrm{Y})$ separable, so that the module dual of $\varphi^*L^2(T^*\mathrm{Y})$ can be identified with $\varphi^*L^2(T\mathrm{Y})$ for any map $\varphi:\mathrm{X}\to\mathrm{Y}$ of bounded deformation (recall Theorem 3.34). We now follow [GP16a]: we claim that

$$\omega\big(\mathsf{Pr}_\varphi(w)\big)=\mathsf{Pr}_\varphi\big((\varphi^*\omega)(w)\big)\qquad\text{for every }\omega\in L^2(T^*\mathrm{Y})\text{ and }w\in\varphi^*L^2(T\mathrm{Y}).\tag{4.50}$$

To prove it, notice that both sides of the identity are linear continuous with respect to w and agree on those w's of the form $f\,\varphi^*v$, with $f\in L^\infty(\mathfrak{m}_\mathrm{X})$ and $v\in L^2(T\mathrm{Y})$.

Proposition 4.37 *Let $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}_\mathrm{X})$, $(\mathrm{Y},\mathsf{d}_\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ be metric measure spaces such that $L^2(T\mathrm{Y})$ is separable. Let $\varphi:\mathrm{X}\to\mathrm{Y}$ be a map of bounded deformation with $\varphi_*\mathfrak{m}_\mathrm{X}=\mathfrak{m}_\mathrm{Y}$. Suppose that for some Borel set $E\subseteq\mathrm{X}$ we have that $\varphi|_E$ is injective with $(\varphi|_E)^{-1}$ Lipschitz. Assume also that Lipschitz functions on X are dense in $W^{1,2}(\mathrm{X})$. Then the map*

$$L^2(T\mathrm{X})|_E\ni v\longmapsto\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\in L^2(T\mathrm{Y})\tag{4.51}$$

is injective. In particular, if $v_1,\ldots,v_n\in L^2(T\mathrm{X})$ are independent on the set E, then the vector fields $\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(\chi_E\,v_1)\big),\ldots,\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(\chi_E\,v_n)\big)\in L^2(T\mathrm{Y})$ are independent on $\mathsf{Im}_\varphi(E)\subseteq\mathrm{Y}$.

Proof. By inner regularity of $\mathfrak{m}_\mathrm{X}$, we can assume that E is compact. The assumption that Lipschitz functions on X are dense in $W^{1,2}(\mathrm{X})$ grants that $\big\{\mathrm{d}f\,:\,f\in\mathrm{LIP}(\mathrm{X})\cap W^{1,2}(\mathrm{X})\big\}$ is dense in $\big\{\mathrm{d}f\,:\,f\in W^{1,2}(\mathrm{X})\big\}$ with respect to the $L^2(T^*\mathrm{X})$ topology. Recalling that $L^2(T^*\mathrm{X})$ is generated by the differentials of functions in $W^{1,2}(\mathrm{X})$, we therefore deduce that

$$V\doteq\bigg\{\chi_E\sum_{i=1}^n h_i\,\mathrm{d}f_i\;\bigg|\;n\in\mathbb{N},\,(f_i)_{i=1}^n\subseteq\mathrm{LIP}(\mathrm{X})\cap W^{1,2}(\mathrm{X}),\,(h_i)_{i=1}^n\subseteq L^\infty(\mathfrak{m}_\mathrm{X})\bigg\}$$

is dense in $L^2(T^*\mathrm{X})|_E$. Now let $f\in\mathrm{LIP}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$ be fixed. Consider the Lipschitz function $f\circ(\varphi|_E)^{-1}$ defined on $\varphi(E)$ and extend it to a Lipschitz function g on Y with bounded support. Then we have $g\in W^{1,2}(\mathrm{Y})$ and $g\circ\varphi=f$ on E. This identity and the locality of the differential imply that $\chi_E\,\mathrm{d}f=\chi_E\,\mathrm{d}(g\circ\varphi)$, so that

$$V\subseteq W\doteq\big\{L^\infty(\mathfrak{m}_\mathrm{X})\text{-linear combinations of }\chi_E\,\mathrm{d}(g\circ\varphi),\,g\in W^{1,2}(\mathrm{Y})\big\},$$

thus accordingly

$$W\text{ is dense in }L^2(T^*\mathrm{X})|_E.\tag{4.52}$$

Moreover, we claim that for any $f\in L^1(\mathfrak{m}_\mathrm{X})$ concentrated on E we have

$$\mathsf{Pr}_\varphi(f)=0\quad\mathfrak{m}_\mathrm{Y}\text{-a.e.}\qquad\Longleftrightarrow\qquad f=0\quad\mathfrak{m}_\mathrm{X}\text{-a.e.}.\tag{4.53}$$

To prove it, it suffices to define $g\in L^1(\mathfrak{m}_\mathrm{Y})$ as $g\doteq\chi_{\varphi(E)}\,\mathrm{sgn}\big(f\circ(\varphi|_E)^{-1}\big)$ and to notice that

$$0=\int g\,\mathsf{Pr}_\varphi(f)\,\mathrm{d}\mathfrak{m}_\mathrm{Y}=\int g\,\mathrm{d}\varphi_*(f\mathfrak{m}_\mathrm{X})=\int g\circ\varphi\,f\,\mathrm{d}\mathfrak{m}_\mathrm{X}=\int|f|\,\mathrm{d}\mathfrak{m}_\mathrm{X}.$$

The injectivity claim follows from this fact: given any $v\in L^2(T\mathrm{X})|_E$, it holds that

$$\begin{aligned}
&\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)=0 &&\Longleftrightarrow\quad \omega\big(\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\big)=0\ \text{ for all }\omega\in L^2(T^*\mathrm{Y})\\
&\text{(by (4.50))} &&\Longleftrightarrow\quad \mathsf{Pr}_\varphi\big((\varphi^*\omega)(\mathrm{d}\varphi(v))\big)=0\ \text{ for all }\omega\in L^2(T^*\mathrm{Y})\\
&\text{(by (4.53))} &&\Longleftrightarrow\quad (\varphi^*\omega)\big(\mathrm{d}\varphi(v)\big)=0\ \text{ for all }\omega\in L^2(T^*\mathrm{Y})\\
& &&\Longleftrightarrow\quad (\varphi^*\mathrm{d}g)\big(\mathrm{d}\varphi(v)\big)\ \text{ for all }g\in W^{1,2}(\mathrm{Y})\\
&\text{(by (4.47))} &&\Longleftrightarrow\quad \mathrm{d}(g\circ\varphi)(v)=0\ \text{ for all }g\in W^{1,2}(\mathrm{Y})\\
&\text{(by (4.52))} &&\Longleftrightarrow\quad v=0.
\end{aligned}$$

In order to prove the last claim, just observe that for any $f_1, \ldots, f_n \in L^\infty(\mathfrak{m}_{\mathrm{Y}})$ we have

$$\sum_{i=1}^n f_i \,\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(\chi_E\, v_i)\big) \overset{(3.31)}{=} \sum_{i=1}^n \mathsf{Pr}_\varphi\big(f_i \circ \varphi\,\mathrm{d}\varphi(\chi_E\, v_i)\big) = \mathsf{Pr}_\varphi\bigg(\mathrm{d}\varphi\bigg(\chi_E \sum_{i=1}^n f_i \circ \varphi\, v_i\bigg)\bigg),$$

therefore

$$\begin{aligned}
\sum_{i=1}^n f_i \,\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(\chi_E\, v_i)\big) = 0 \qquad &\Longleftrightarrow \qquad \chi_E \sum_{i=1}^n f_i \circ \varphi\, v_i = 0 \\
&\Longleftrightarrow \qquad f_i \circ \varphi = 0 \quad \mathfrak{m}_{\mathrm{X}}|_E\text{-a.e. for all } i = 1, \ldots, n \\
&\Longleftrightarrow \qquad f_i = 0 \quad \varphi_*\big(\mathfrak{m}_{\mathrm{X}}|_E\big)\text{-a.e. for all } i = 1, \ldots, n \\
&\Longleftrightarrow \qquad f_i = 0 \quad \mathfrak{m}_{\mathrm{Y}}\text{-a.e. on } \mathsf{Im}_\varphi(E) \text{ for all } i = 1, \ldots, n.
\end{aligned}$$

Hence also the last statement is achieved, thus completing the proof. □

We conclude the present subsection by pointing out how the measure-valued divergence (introduced in Definition 4.15) is transformed under maps of bounded deformation:

Proposition 4.38 *Let $(\mathrm{X}, \mathsf{d}_{\mathrm{X}}, \mathfrak{m}_{\mathrm{X}})$, $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}}, \mathfrak{m}_{\mathrm{Y}})$ be metric measure spaces such that $(\mathrm{X}, \mathsf{d}_{\mathrm{X}})$ and $(\mathrm{Y}, \mathsf{d}_{\mathrm{Y}})$ are proper. Let $\varphi : \mathrm{X} \to \mathrm{Y}$ be a map of bounded deformation with $\mathfrak{m}_{\mathrm{Y}} = \varphi_*\mathfrak{m}_{\mathrm{X}}$. Suppose that the map φ is proper, i.e. the set $\varphi^{-1}(K)$ is compact whenever $K \subseteq \mathrm{Y}$ is compact. Then for any $v \in L^2(T\mathrm{X})$ and $f \in W^{1,2}(\mathrm{Y})$ it holds that*

$$\int \mathrm{d}f\big(\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\big)\,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} = \int \mathrm{d}(f \circ \varphi)(v)\,\mathrm{d}\mathfrak{m}_{\mathrm{X}}. \tag{4.54}$$

In particular, if $v \in D(\mathbf{div}_{\mathfrak{m}_{\mathrm{X}}})$, then $\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big) \in D(\mathbf{div}_{\mathfrak{m}_{\mathrm{Y}}})$ and

$$\mathbf{div}_{\mathfrak{m}_{\mathrm{Y}}}\big(\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\big) = \varphi_*\big(\mathbf{div}_{\mathfrak{m}_{\mathrm{X}}}(v)\big). \tag{4.55}$$

Proof. Fix $f \in \mathrm{LIP_c}(\mathrm{Y})$. Recalling (4.50) and the definition of $\mathrm{d}\varphi(v)$, we have that

$$\mathrm{d}f\big(\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\big) = \mathsf{Pr}_\varphi\big((\varphi^*\mathrm{d}f)\big(\mathrm{d}\varphi(v)\big)\big) = \mathsf{Pr}_\varphi\big(\mathrm{d}(f \circ \varphi)(v)\big).$$

By integrating with respect to $\mathfrak{m}_{\mathrm{Y}}$ and using the trivial identity $\int \mathsf{Pr}_\varphi(g)\,\mathrm{d}\mathfrak{m}_{\mathrm{Y}} = \int g\,\mathrm{d}\mathfrak{m}_{\mathrm{X}}$, which is valid for any $g \in L^1(\mathfrak{m}_{\mathrm{X}})$, we deduce that

$$\begin{aligned}
\int \mathrm{d}f\big(\mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v)\big)\big)\,\mathrm{d}\varphi_*\mathfrak{m}_{\mathrm{X}} &= \int \mathrm{d}(f \circ \varphi)(v)\,\mathrm{d}\mathfrak{m}_{\mathrm{X}} = -\int f \circ \varphi\,\mathrm{d}\big(\mathbf{div}_{\mathfrak{m}_{\mathrm{X}}}(v)\big) \\
&= -\int f\,\mathrm{d}\varphi_*\big(\mathbf{div}_{\mathfrak{m}_{\mathrm{X}}}(v)\big).
\end{aligned}$$

Since the function f has been arbitrarily chosen, we get the statement. □

4.1.5 An alternative notion of differential for $\mathbb{R}^d$-valued maps

In Chapter 5 we shall deal with maps φ defined on some Borel set $E \subseteq \mathrm{X}$ and taking values into the Euclidean space $\mathbb{R}^k$. In addition, the map $\varphi : E \to \varphi(E)$ under consideration will be of bounded deformation, invertible and with inverse of bounded deformation. Thanks to the high regularity of the target space $\mathbb{R}^k$ and to the invertibility of φ, it will be possible to associate to any element $v \in L^2(T\mathrm{X})|_E$ a 'concrete' vector field $\widehat{\mathrm{d}}\varphi(v)$ in $L^2\big(\varphi(E), \mathbb{R}^k\big)$.

Such a new notion of differential $\widehat{\mathrm{d}}\varphi$ – tailored for this kind of maps φ – is described by the following result, which has been originally proved in [GP16b].

Theorem 4.39 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space with* $W^{1,2}(\mathrm{X})$ *reflexive. Let* $E\subseteq\mathrm{X}$ *be a Borel set and* $\varphi: E\to\mathbb{R}^k$ *be a Lipschitz map. Suppose there exist* $\mathsf{L},\mathsf{C}>1$ *such that*

$$\begin{aligned}&\varphi: E\to\varphi(E)\quad\textit{is }\mathsf{L}\textit{-biLipschitz},\\&\mathsf{C}^{-1}\,\mathcal{L}^k|_{\varphi(E)}\leq\varphi_*\big(\mathfrak{m}|_E\big)\leq\mathsf{C}\,\mathcal{L}^k|_{\varphi(E)}.\end{aligned}\tag{4.56}$$

Then there exists a unique linear and continuous operator $\widehat{\mathrm{d}}\varphi: L^2(T\mathrm{X})|_E\to L^2\big(\varphi(E),\mathbb{R}^k\big)$, *called* differential *of* φ, *which satisfies the following conditions for any* $v\in L^2(T\mathrm{X})|_E$*:*

$$\begin{aligned}\mathrm{d}g\big(\widehat{\mathrm{d}}\varphi(v)\big)&=\big(\mathrm{d}(g\circ\overline{\varphi})(v)\big)\circ\varphi^{-1}&&\textit{for every } g\in\mathrm{LIP_c}(\mathbb{R}^k),\\\widehat{\mathrm{d}}\varphi(fv)&=f\circ\varphi^{-1}\,\widehat{\mathrm{d}}\varphi(v)&&\textit{for every } f\in L^\infty(\mathfrak{m}),\end{aligned}\tag{4.57}$$

where $\overline{\varphi}:\mathrm{X}\to\mathbb{R}^k$ *is any Lipschitz extension of* φ*. Moreover, we have that*

$$\mathsf{L}^{-1}\,|v|\circ\varphi^{-1}\leq\big|\widehat{\mathrm{d}}\varphi(v)\big|\leq\mathsf{L}\,|v|\circ\varphi^{-1}\qquad\textit{holds } \mathcal{L}^k\textit{-a.e. in } \varphi(E)\tag{4.58}$$

for every vector field $v\in L^2(T\mathrm{X})|_E$*.*

Proof. Fix any Lipschitz extension $\overline{\varphi}:\mathrm{X}\to\mathbb{R}^k$ of φ. We divide the proof into several steps:
STEP 1. We claim that it is enough to prove the statement for $\mathfrak{m}$ finite. Indeed, suppose the thesis holds for finite measures and consider any (not necessarily finite) reference measure $\mathfrak{m}$ on X. There is a sequence $(K_n)_n$ of disjoint compact subsets of E with $\mathfrak{m}\big(E\setminus\bigcup_n K_n\big)=0$, by inner regularity of $\mathfrak{m}$. Given that $\mathfrak{m}$ is also outer regular, we can find a sequence $(\Omega_n)_n$ of open subsets of X such that $K_n\subseteq\Omega_n$ and $\mathfrak{m}(\Omega_n)<+\infty$ for every $n\in\mathbb{N}$. Fix any $n\in\mathbb{N}$ and call $\mathfrak{m}_n\doteq\mathfrak{m}|_{\Omega_n}$. Hence we can apply the theorem to the map $\varphi|_{K_n}$, thus obtaining a linear and continuous operator $T_n: L^2_{\mathfrak{m}_n}(T\mathrm{X})|_{K_n}\to L^2\big(\varphi(K_n),\mathbb{R}^k\big)$ such that the following conditions are satisfied $\mathcal{L}^k$-a.e. in $\varphi(K_n)$ for any $v\in L^2_{\mathfrak{m}_n}(T\mathrm{X})|_{K_n}$:

$$\begin{aligned}\mathrm{d}g\big(T_n(v)\big)&=\big(\mathrm{d}(g\circ\overline{\varphi})(v)\big)\circ\big(\varphi|_{K_n}\big)^{-1}&&\text{for every } g\in\mathrm{LIP_c}(\mathbb{R}^k),\\T_n(fv)&=f\circ\big(\varphi|_{K_n}\big)^{-1}\,T_n(v)&&\text{for every } f\in L^\infty(\mathfrak{m}_n),\\\mathsf{L}^{-1}\,|v|\circ\big(\varphi|_{K_n}\big)^{-1}\leq\big|T_n(v)\big|&\leq\mathsf{L}\,|v|\circ\big(\varphi|_{K_n}\big)^{-1}.\end{aligned}\tag{4.59}$$

Denote by $\iota_n: L^2_{\mathfrak{m}_n}(T\mathrm{X})\to L^2_{\mathfrak{m}}(T\mathrm{X})|_{\Omega_n}$ the isomorphism built in Remark 4.11. Therefore we can 'glue' together the functions T_n obtained above – thanks to the third line in (4.59) – in the sense that there exists a unique map $\widehat{\mathrm{d}}\varphi: L^2_{\mathfrak{m}}(T\mathrm{X})|_E\to L^2\big(\varphi(E),\mathbb{R}^k\big)$ such that

$$\chi_{\varphi(K_n)}\,\widehat{\mathrm{d}}\varphi(v)=T_n\big(\iota_n^{-1}(\chi_{\Omega_n}v)\big)\quad\mathcal{L}^k\text{-a.e. in }\varphi(K_n),\qquad\text{for all } v\in L^2_{\mathfrak{m}}(T\mathrm{X})|_E\text{ and } n\in\mathbb{N}.$$

We then deduce from (4.59) that $\widehat{\mathrm{d}}\varphi$ is a linear and continuous operator satisfying both (4.57) and (4.58), as required.
STEP 2. From now on, suppose that $\mathfrak{m}$ is a finite measure. Define $\mu\doteq\overline{\varphi}_*\mathfrak{m}$, which is a finite Borel measure on $\mathbb{R}^k$. In particular, we have that $\mathrm{LIP_c}(\mathbb{R}^k)\subseteq W^{1,2}_\mu(\mathbb{R}^k)$. The tangent module $L^2_\mu(T\mathbb{R}^k)$ turns out to be isometrically embedded into the space $L^2(\mathbb{R}^k,\mathbb{R}^k;\mu)$ of all the $L^2(\mu)$-vector fields from $\mathbb{R}^k$ to itself, as proved in Theorem 4.27, thus $L^2_\mu(T\mathbb{R}^k)$ is separable (Corollary 4.29). Since $\overline{\varphi}$ is of bounded deformation when viewed as a function from $(\mathrm{X},\mathsf{d},\mathfrak{m})$ to $\big(\mathbb{R}^k,|\cdot|,\mu\big)$, we can then consider its differential $\mathrm{d}\overline{\varphi}: L^2(T\mathrm{X})\to\overline{\varphi}^*L^2_\mu(T\mathbb{R}^k)$. Now let us fix any vector field $v\in L^2(T\mathrm{X})|_E$. The family of all finite sums of the form $\sum_{i=1}^n\chi_{A_i}\,\mathrm{d}g_i$ –

where $(A_i)_{i=1}^n$ is a Borel partition of $\varphi(E)$ and $(g_i)_{i=1}^n \subseteq \mathrm{LIP_c}(\mathbb{R}^k)$ – is a dense vector subspace of $L^2\big(\varphi(E),(\mathbb{R}^k)^*\big)$. Given any such simple 1-form $\omega=\sum_{i=1}^n \chi_{A_i}\,\mathrm{d}g_i \in L^2\big(\varphi(E),(\mathbb{R}^k)^*\big)$, let

$$T_v(\omega) \doteq \sum_{i=1}^n \chi_{A_i}\,(\overline{\varphi}^*\mathrm{d}_\mu g_i)\big(\mathrm{d}\overline{\varphi}(v)\big)\circ\varphi^{-1} \in L^1\big(\varphi(E)\big). \tag{4.60}$$

The operator T_v is well-defined, as granted by the following $\mathcal{L}^k|_{\varphi(E)}$-a.e. inequalities:

$$\begin{aligned}\big|T_v(\omega)\big| &= \sum_{i=1}^n \chi_{A_i}\Big|(\varphi^*\mathrm{d}_\mu g_i)\big(\mathrm{d}\overline{\varphi}(v)\big)\Big|\circ\varphi^{-1} \leq \big|\mathrm{d}\overline{\varphi}(v)\big|\circ\varphi^{-1}\sum_{i=1}^n \chi_{A_i}\,|\mathrm{d}_\mu g_i|\circ\varphi\circ\varphi^{-1}\\ &\leq \big|\mathrm{d}\overline{\varphi}(v)\big|\circ\varphi^{-1}\sum_{i=1}^n \chi_{A_i}\,\mathrm{lip}(g_i) = \big|\mathrm{d}\overline{\varphi}(v)\big|\circ\varphi^{-1}\,|\omega|.\end{aligned} \tag{4.61}$$

Another consequence of property (4.61) is that the operator T_v can be uniquely extended to a vector field $\widehat{\mathrm{d}}\varphi(v) \in L^2\big(\varphi(E),\mathbb{R}^k\big)$, for which $\big|\widehat{\mathrm{d}}\varphi(v)\big| \leq \big|\mathrm{d}\overline{\varphi}(v)\big|\circ\varphi^{-1}$ holds $\mathcal{L}^k$-a.e. in $\varphi(E)$. Furthermore, it can be readily verified that $\widehat{\mathrm{d}}\varphi$ is the unique operator satisfying (4.57).

Step 3. In order to conclude the proof, it only remains to show (4.58). Let $v \in L^2(T\mathrm{X})|_E$. It immediately follows from Proposition 4.36 that $\big|\widehat{\mathrm{d}}\varphi(v)\big| \leq \mathsf{L}\,|v|\circ\varphi^{-1}$ holds $\mathcal{L}^k$-a.e. in $\varphi(E)$. To prove the other inequality in (4.58), we need a more refined argument: fix $\varepsilon>0$. Given that $|v| = \operatorname{ess\,sup}\omega(v)$, where the essential supremum is taken among all the $\omega \in L^2(T^*\mathrm{X})$ with $|\omega| \leq 1$ $\mathfrak{m}_\mathrm{X}$-a.e., there exists $\omega \in L^2(T^*\mathrm{X})|_E$ such that $|\omega|=1$ and $\omega(v) \geq (1-\varepsilon)\,|v|$ are verified $\mathfrak{m}_\mathrm{X}$-a.e. in E. Since the simple forms $\sum_i \chi_{A_i}\,\mathrm{d}f_i \in L^2(T^*\mathrm{X})$ are dense in $L^2(T^*\mathrm{X})$, we can apply Egorov theorem to obtain a partition $(K^n)_{n\in\mathbb{N}}$ of E (up to $\mathfrak{m}_\mathrm{X}$-negligible sets) into compact sets and a sequence $(f^n)_n \subseteq W^{1,2}(\mathrm{X})$ such that $|\mathrm{d}f^n|<1$ and $\mathrm{d}f^n(v) \geq (1-\varepsilon)^2\,|v|$ hold $\mathfrak{m}_\mathrm{X}$-a.e. in K^n for every $n\in\mathbb{N}$. By using the reflexivity of $W^{1,2}(\mathrm{X})$, Theorem 2.27 and Egorov theorem, we can find a partition $(K^n_m)_{m\in\mathbb{N}}$ of K^n (up to $\mathfrak{m}_\mathrm{X}$-negligible sets) into compact sets and a sequence of functions $(f^n_m)_m \subseteq \mathrm{LIP}(\mathrm{X})\cap W^{1,2}(\mathrm{X})$ such that $\mathrm{lip}_a(f^n_m) \leq 1$ and $\mathrm{d}f^n_m(v) \geq (1-\varepsilon)^3\,|v|$ are satisfied $\mathfrak{m}_\mathrm{X}$-a.e. in K^n_m for every $m\in\mathbb{N}$. Let us denote by ψ^n_m the inverse of the map $\varphi|_{K^n_m} : K^n_m \to \varphi(K^n_m)$ and pick any compactly supported Lipschitz map $h^n_m \in \mathrm{LIP_c}(\mathbb{R}^k)$ such that $h^n_m|_{\varphi(K^n_m)} = f^n_m\circ\psi^n_m$. Observe that the following statement is satisfied $\mathcal{L}^k$-a.e. in the set $\varphi(K^n_m)$:

$$|\mathrm{d}h^n_m| \overset{(4.14)}{=} \mathrm{lip}(h^n_m) \overset{(1.69)}{=} \mathrm{lip}\big(h^n_m|_{\varphi(K^n_m)}\big) \overset{(1.19)}{\leq} \mathrm{Lip}(\psi^n_m)\,\mathrm{lip}(f^n_m)\circ\psi^n_m \leq \mathsf{L}. \tag{4.62}$$

Moreover, the fact that $f^n_m|_{K^n_m} = h^n_m\circ\overline{\varphi}|_{K^n_m}$ yields $\chi_{K^n_m}\,\mathrm{d}f^n_m = \chi_{K^n_m}\,\mathrm{d}(h^n_m\circ\overline{\varphi})$, so that

$$\begin{aligned}\Big|\big(\chi_{\varphi(K^n_m)}\,\mathrm{d}h^n_m\big)\big(\widehat{\mathrm{d}}\varphi(v)\big)\Big| &= \chi_{\varphi(K^n_m)}\Big|(\overline{\varphi}^*\mathrm{d}_\mu h^n_m)\big(\widehat{\mathrm{d}}\varphi(v)\big)\Big|\circ\varphi^{-1}\\ &\geq \chi_{\varphi(K^n_m)}\,\big(\mathrm{d}(h^n_m\circ\overline{\varphi})(v)\big)\circ\varphi^{-1}\\ &= \chi_{\varphi(K^n_m)}\,\big(\mathrm{d}f^n_m(v)\big)\circ\varphi^{-1}\\ &\geq (1-\varepsilon)^3\,\chi_{\varphi(K^n_m)}\,|v|\circ\varphi^{-1} \qquad \text{holds } \mathcal{L}^k\text{-a.e. in } \varphi(K^n_m).\end{aligned}$$

In particular, (4.62) grants that $\big|\widehat{\mathrm{d}}\varphi(v)\big| \geq (1-\varepsilon)^3\,\mathsf{L}^{-1}\,|v|\circ\varphi^{-1}$ is satisfied $\mathcal{L}^k$-a.e. in $\varphi(K^n_m)$ for any $n,m\in\mathbb{N}$, hence also $\mathscr{L}^k$-a.e. in all of $\varphi(E)$. By letting $\varepsilon\searrow 0$, we finally obtain that the inequality $\big|\widehat{\mathrm{d}}\varphi(v)\big| \geq \mathsf{L}^{-1}\,|v|\circ\varphi^{-1}$ holds $\mathcal{L}^k$-a.e. in $\varphi(E)$, concluding the proof of (4.58). Therefore the statement is finally achieved. □

4.2 Second-order differential structure of RCD spaces

4.2.1 Definition of RCD space

The theory of synthetic Ricci curvature bounds for nonsmooth spaces has seen impressive developments in the last few years. In this subsection we just briefly recall the most important ideas behind these notions, referring to the surveys [Amb18, Vil16, Vil17] for a thorough account about their history and their main properties.

The first concept of lower Ricci curvature bounds for possibly nonsmooth structures is the so-called *Ricci limit*, which has been introduced by J. Cheeger and T. Colding in [CC96]. Such spaces are obtained as Gromov-Hausdorff limits of a sequence of smooth Riemannian manifolds with a uniform lower bound on the Ricci curvature. The structural properties of Ricci limits are deeply investigated in the papers [CC97, CC00a, CC00b, CN12].

Besides this 'extrinsic' approach (i.e. obtained via approximation with smooth objects), several 'intrinsic' approaches (i.e. obtained without referring to Riemannian manifolds) made their appearance in the literature during the last decade. In general, such notions are usually referred to as *curvature-dimension conditions*. As we are going to briefly describe, the corresponding theories can be divided into two main groups:

- LAGRANGIAN APPROACH. One can impose lower Ricci curvature bounds on metric measure spaces by requiring convexity of suitable integral functionals along geodesics in the Wasserstein space (cf. Subsection 1.3.2 for this optimal transport terminology). The idea of looking at convexity along W_2-geodesics is due to McCann [McC97]. Sturm [Stu06a, Stu06b] and Lott-Villani [LV07] independently proposed the first definition of curvature-dimension condition – called $\mathsf{CD}(K,N)$ *condition* – by taking inspiration from the results about Riemannian manifolds achieved in [CEMS01, OV00, vRS09]. The term $\mathsf{CD}(K,N)$ indicates that, in some generalised sense, the Ricci curvature is bounded from below by $K \in \mathbb{R}$ and the dimension is bounded from above by $N \in [1,\infty]$. In order to have better tensorisation and globalisation properties, a weaker curvature-dimension condition – called *reduced curvature-dimension condition* or $\mathsf{CD}^*(K,N)$ *condition* – has been introduced by K. Bacher and K. T. Sturm in [BS10]. Moreover, another variant of CD space is the *entropic* $\mathsf{CD}^e(K,N)$ *space*, introduced in [EKS14].

  Nevertheless, we point out that all the several notions of CD space mentioned so far allow for Finsler structures. The intention to select a class of spaces that rules out all Finsler manifolds led to the definition of $\mathsf{RCD}(K,N)$ *space*, see [AGS14b, AGMR15, Gig15], where the added letter R stands for 'Riemannian'. Shortly said, an RCD space is an infinitesimally Hilbertian CD space (recall Definition 2.25). In a similar fashion, one can define the class of $\mathsf{RCD}^*(K,N)$ spaces. It has been proven by F. Cavalletti and E. Milman in [CM16] that the $\mathsf{RCD}(K,N)$ condition and the $\mathsf{RCD}^*(K,N)$ condition are actually equivalent (whenever the reference measure is finite).

- EULERIAN APPROACH. It is also possible to define a curvature-dimension condition at the level of Dirichlet forms and Γ-calculus, thus leading to the *Bakry-Émery theory*. This method – originally proposed by D. Bakry and M. Émery in [Bak85, BÉ85] – has been motivated by the study of hypercontractivity for diffusion processes. In our metric measure space setting, the key concept is the so-called $\mathsf{BE}(K,N)$ *condition*, introduced by L. Ambrosio, N. Gigli and G. Savaré in [AGS15]. Informally speaking, this approach consists of a suitable weak formulation of the Bochner inequality, which is capable of encoding both the Ricci curvature bound and the dimension one at the same time. The BE condition and the RCD one are equivalent, as shown in [EKS14] and [AMS15]; a precursor of the proof of such equivalence is given by K. Kuwada's paper [Kuw10]. This is the language we shall adopt in the present thesis (cf. Definitions 4.40 and 4.41).

A fundamental feature of the class of $\mathsf{RCD}(K,N)$ spaces is that it is closed under measured Gromov-Hausdorff convergence, thus in particular it contains all the Ricci limit spaces. We refer to [Amb18] for a comprehensive discussion about the several geometric properties and functional inequalities that are available in the CD/RCD framework.

We now give the definition of $\mathsf{RCD}(K,\infty)$ space (following the axiomatisation of [AGS15], but formulated in terms of the language of normed modules).

Definition 4.40 **($\mathsf{RCD}(K,\infty)$ space)** *Let $K\in\mathbb{R}$. Then a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be an* $\mathsf{RCD}(K,\infty)$ space *provided the following properties are satisfied:*

i) *The space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian.*

ii) *There exist $C>0$ and $\bar{x}\in\mathrm{X}$ such that $\mathfrak{m}\big(B_r(\bar{x})\big)\leq e^{Cr^2}$ for every $r>0$.*

iii) Sobolev-to-Lipschitz property. *Every function $f\in W^{1,2}(\mathrm{X})$ with $|Df|\in L^\infty(\mathfrak{m})$ admits a Lipschitz representative $\bar{f}$ such that $\mathrm{Lip}(\bar{f})=\big\||Df|\big\|_{L^\infty(\mathfrak{m})}$.*

iv) Weak Bochner inequality. *Given any $f\in D(\Delta)$ and $g\in D(\Delta)\cap L^\infty(\mathfrak{m})^+$ such that $\Delta f\in W^{1,2}(\mathrm{X})$ and $\Delta g\in L^\infty(\mathfrak{m})$, it holds that*

$$\frac{1}{2}\int|Df|^2\,\Delta g\,\mathrm{d}\mathfrak{m}\geq\int\Big[\langle\nabla f,\nabla\Delta f\rangle+K\,|Df|^2\Big]g\,\mathrm{d}\mathfrak{m}. \tag{4.63}$$

By building on top on the previous definition, we can also introduce the finite-dimensional refinement of the RCD condition (again taken from [AGS15]).

Definition 4.41 **($\mathsf{RCD}(K,N)$ space)** *Let $K\in\mathbb{R}$ and $N\in[1,\infty)$. Then a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be an* $\mathsf{RCD}(K,N)$ space *provided it is an $\mathsf{RCD}(K,\infty)$ space and it satisfies the following variant of the weak Bochner inequality:*

$$\frac{1}{2}\int|Df|^2\,\Delta g\,\mathrm{d}\mathfrak{m}\geq\int\Big[(\Delta f)^2/N+\langle\nabla f,\nabla\Delta f\rangle+K\,|Df|^2\Big]g\,\mathrm{d}\mathfrak{m} \tag{4.64}$$

for every $f\in D(\Delta)$ and $g\in D(\Delta)\cap L^\infty(\mathfrak{m})^+$ such that $\Delta f\in W^{1,2}(\mathrm{X})$ and $\Delta g\in L^\infty(\mathfrak{m})$.

Remark 4.42 Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,N)$ space, for some $K\in\mathbb{R}$ and $N\in[1,\infty)$. Let us consider any two measures $\mu,\nu\in\mathscr{P}_2(\mathrm{X})$ with bounded support such that $\mu,\nu\leq\mathsf{C}\mathfrak{m}$. Then the family $\mathrm{OptGeo}(\mu,\nu)$ consists of a unique element $\boldsymbol{\pi}$, which satisfies

$$(\mathrm{e}_t)_*\boldsymbol{\pi}\leq\lambda\mathsf{C}\mathfrak{m}\qquad\text{for every }t\in[0,1], \tag{4.65}$$

where the constant $\lambda\geq 1$ depends just on K, N and $\mathrm{diam}\big(\mathrm{spt}(\mu)\cup\mathrm{spt}(\nu)\big)$. More generally, it holds that $\boldsymbol{\pi}$ is a test plan. For a proof of these facts, see [GRS16, Lemma 3.2]. ■

4.2.2 Hessian and covariant derivative

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a fixed $\mathsf{RCD}(K,\infty)$ space, for some $K\in\mathbb{R}$. A fundamental notion that we need in order to develop a second-order differential calculus on X is that of *test function*, which has been introduced in [Sav14]:

$$\mathrm{TestF}(\mathrm{X}) \doteq \Big\{ f\in D(\Delta)\cap \mathrm{LIP}(\mathrm{X})\cap L^\infty(\mathfrak{m}) \;\Big|\; \Delta f\in W^{1,2}(\mathrm{X})\Big\}. \tag{4.66}$$

The presence of many test functions is granted by the regularising properties of the heat flow, which we are now going to briefly describe. The *heat flow* on $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is the gradient flow (h_t) of the Dirichlet energy $f\mapsto \mathcal{E}_2(f)\doteq\frac{1}{2}\int|Df|^2\,\mathrm{d}\mathfrak{m}$ (cf. the monography [AGS08]), meaning that for any $f\in L^2(\mathfrak{m})$ it holds that $[0,+\infty)\ni t\mapsto \mathsf{h}_t f\in L^2(\mathfrak{m})$ is the unique continuous curve with $\mathsf{h}_0 f=f$ that is absolutely continuous in $(0,+\infty)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathsf{h}_t f=\Delta\mathsf{h}_t f \qquad \text{for a.e. } t>0. \tag{4.67}$$

The heat flow fulfils the *weak maximum principle*: if a function $f\in L^2(\mathfrak{m})$ satisfies $f\leq C$ in the $\mathfrak{m}$-a.e. sense for some constant $C\in\mathbb{R}$, then for any $t\geq 0$ the inequality $\mathsf{h}_t f\leq C$ holds $\mathfrak{m}$-a.e.. This grants that the heat flow operator can be extended to the space $L^1(\mathfrak{m})+L^\infty(\mathfrak{m})$. Two fundamental properties of the heat flow on RCD spaces are the following:

- BAKRY-ÉMERY ESTIMATE. Given any $f\in W^{1,2}(\mathrm{X})$ and $t\geq 0$, it holds that

$$|D\mathsf{h}_t f|^2\leq e^{-2Kt}\,\mathsf{h}_t\big(|Df|^2\big) \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{4.68}$$

- L^∞-TO-LIPSCHITZ REGULARISATION. If $f\in L^\infty(\mathfrak{m})$ and $t>0$, then $\mathsf{h}_t f\in \mathrm{LIP}(\mathrm{X})$ and

$$\left(2\int_0^t e^{2Ks}\,\mathrm{d}s\right)^{1/2}\mathrm{Lip}(\mathsf{h}_t f)\leq \|f\|_{L^\infty(\mathfrak{m})}. \tag{4.69}$$

We now collect those properties of the family $\mathrm{TestF}(\mathrm{X})$ that will be needed in the sequel:

Proposition 4.43 *The following properties hold:*

i) *The family* $\mathrm{TestF}(\mathrm{X})$ *is dense in* $W^{1,2}(\mathrm{X})$.

ii) *It holds that* $\langle\nabla f,\nabla g\rangle\in W^{1,2}(\mathrm{X})$ *whenever* $f,g\in\mathrm{TestF}(\mathrm{X})$.

iii) *The space* $\mathrm{TestF}(\mathrm{X})$ *is an algebra.*

In order to introduce the Hessian of a Sobolev function, we need the notion of tensor product of Hilbert modules – already described in Subsection 3.1.5. For the sake of brevity, we shall make use of the following shorthand notation:

$$\begin{aligned} L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)&\doteq L^2(T^*\mathrm{X})^{\otimes 2}=L^2(T^*\mathrm{X})\otimes L^2(T^*\mathrm{X}),\\ L^2(T^{\otimes 2}\mathrm{X})&\doteq L^2(T\mathrm{X})^{\otimes 2}=L^2(T\mathrm{X})\otimes L^2(T\mathrm{X}). \end{aligned} \tag{4.70}$$

Then the space $W^{2,2}(\mathrm{X})$ is defined as follows:

Definition 4.44 (Hessian) *Let $f \in W^{1,2}(\mathrm{X})$ be given. Then we say that the function f belongs to the space $W^{2,2}(\mathrm{X})$ provided there exists a tensor $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$ such that*

$$\begin{aligned} &2\int h\,A(\nabla g_1,\nabla g_2)\,\mathrm{d}\mathfrak{m} \\ &= -\int \langle\nabla f,\nabla g_1\rangle\,\mathrm{div}(h\nabla g_2)+\langle\nabla f,\nabla g_2\rangle\,\mathrm{div}(h\nabla g_1)+h\,\big\langle\nabla f,\nabla\langle\nabla g_1,\nabla g_2\rangle\big\rangle\,\mathrm{d}\mathfrak{m} \end{aligned} \tag{4.71}$$

holds for every $g_1,g_2 \in \mathrm{TestF}(\mathrm{X})$ and $h \in \mathrm{LIP_b}(\mathrm{X})$. The element $A \in L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)$, which is uniquely determined, is called Hessian *of f and denoted by* $\mathrm{Hess}(f)$. *Moreover, we endow the vector space $W^{2,2}(\mathrm{X})$ with the norm $\|\cdot\|_{W^{2,2}(\mathrm{X})}$, given by*

$$\|f\|_{W^{2,2}(\mathrm{X})} \doteq \sqrt{\|f\|^2_{L^2(\mathfrak{m})}+\|\mathrm{d}f\|^2_{L^2(T^*\mathrm{X})}+\big\|\mathrm{Hess}(f)\big\|_{L^2((T^*)^{\otimes 2}\mathrm{X})}} \tag{4.72}$$

for every $f \in W^{2,2}(\mathrm{X})$.

The main properties of $W^{2,2}(\mathrm{X})$ and of the Hessian are given by the following result:

Proposition 4.45 *The following properties hold:*

i) *It holds that $W^{2,2}(\mathrm{X})$ is a separable Hilbert space.*

ii) *The Hessian is a closed operator, i.e.*

$$\Big\{\big(f,\mathrm{Hess}(f)\big)\;\Big|\;f \in W^{2,2}(\mathrm{X})\Big\} \qquad \text{is closed in } W^{1,2}(\mathrm{X})\times L^2\big((T^*)^{\otimes 2}\mathrm{X}\big). \tag{4.73}$$

iii) *The Hessian is symmetric, i.e.* $\mathrm{Hess}(f)^{\mathsf{t}} = \mathrm{Hess}(f)$ *for every $f \in W^{2,2}(\mathrm{X})$.*

The fact that the space $W^{2,2}(\mathrm{X})$ is sufficiently vast is ensured by the following theorem:

Theorem 4.46 *It holds that $D(\Delta) \subseteq W^{2,2}(\mathrm{X})$ and*

$$\int \big|\mathrm{Hess}(f)\big|^2_{\mathsf{HS}}\,\mathrm{d}\mathfrak{m} \leq \int (\Delta f)^2 - K\,|\nabla f|^2\,\mathrm{d}\mathfrak{m} \qquad \text{for every } f \in D(\Delta). \tag{4.74}$$

In particular, we have that $\mathrm{TestF}(\mathrm{X})$ is contained in $W^{2,2}(\mathrm{X})$. Nevertheless, we do not know whether it is dense in $W^{2,2}(\mathrm{X})$, whence the following definition is meaningful:

Definition 4.47 *We define the space $H^{2,2}(\mathrm{X})$ as the $W^{2,2}(\mathrm{X})$-closure of* $\mathrm{TestF}(\mathrm{X})$.

We conclude the discussion about the Hessian by recalling some calculus rules:

Proposition 4.48 (Calculus rules for the Hessian) *The following hold:*

i) Product rule for functions. *Let $f,g \in W^{2,2}(\mathrm{X})\cap\mathrm{LIP_b}(\mathrm{X})$ be given. Then it holds that $fg \in W^{2,2}(\mathrm{X})$ and*

$$\mathrm{Hess}(fg) = f\,\mathrm{Hess}(g)+g\,\mathrm{Hess}(f)+\mathrm{d}f\otimes\mathrm{d}g+\mathrm{d}g\otimes\mathrm{d}f \qquad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{4.75}$$

ii) Chain rule. *Let $f \in W^{2,2}(\mathrm{X})\cap\mathrm{LIP}(\mathrm{X})$ be given. Take $\varphi \in C^2(\mathbb{R})$ with φ', φ'' bounded and $\varphi(0)=0$. Then it holds that $\varphi\circ f \in W^{2,2}(\mathrm{X})$ and*

$$\mathrm{Hess}(\varphi\circ f) = \varphi''\circ f\,\mathrm{d}f\otimes\mathrm{d}f+\varphi'\circ f\,\mathrm{Hess}(f) \qquad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{4.76}$$

iii) PRODUCT RULE FOR GRADIENTS. *Let $f,g\in H^{2,2}(\mathrm{X})\cap \mathrm{LIP}(\mathrm{X})$ be given. Then it holds that $\langle\nabla f,\nabla g\rangle\in W^{1,2}(\mathrm{X})$ and*

$$\mathrm{d}\langle\nabla f,\nabla g\rangle=\mathrm{Hess}(f)(\nabla g,\cdot)+\mathrm{Hess}(g)(\nabla f,\cdot)\qquad \mathfrak{m}\text{-a.e. in }\mathrm{X}. \tag{4.77}$$

With the notion of Hessian at disposal, we can give the definition of covariant derivative:

Definition 4.49 (Covariant derivative) *Let $v\in L^2(T\mathrm{X})$ be given. Then we say that v belongs to the space $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ provided there exists a tensor $T\in L^2(T^{\otimes 2}\mathrm{X})$ such that*

$$\int h\,T:(\nabla f\otimes\nabla g)\,\mathrm{d}\mathfrak{m}=-\int\langle v,\nabla g\rangle\,\mathrm{div}(h\nabla f)-h\,\mathrm{Hess}(g)(v,\nabla f)\,\mathrm{d}\mathfrak{m} \tag{4.78}$$

for every $f,g\in\mathrm{TestF}(\mathrm{X})$ and $h\in\mathrm{LIP}_{\mathrm{b}}(\mathrm{X})$. The element $T\in L^2(T^{\otimes 2}\mathrm{X})$, which is uniquely determined, is called covariant derivative *of v and denoted by ∇v. Moreover, we endow the vector space $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ with the norm $\|\cdot\|_{W^{1,2}_{\mathrm{C}}(T\mathrm{X})}$, given by*

$$\|v\|_{W^{1,2}_{\mathrm{C}}(T\mathrm{X})}\doteq\sqrt{\|v\|^2_{L^2(T\mathrm{X})}+\|\nabla v\|^2_{L^2(T^{\otimes 2}\mathrm{X})}}\qquad\text{for every }v\in W^{1,2}_{\mathrm{C}}(T\mathrm{X}). \tag{4.79}$$

We also introduce the class $\mathrm{TestV}(\mathrm{X})\subseteq L^2(T\mathrm{X})$ of *test vector fields* on X:

$$\mathrm{TestV}(\mathrm{X})\doteq\bigg\{\sum_{i=1}^n g_i\,\nabla f_i\;\bigg|\;(f_i)_{i=1}^n,(g_i)_{i=1}^n\subseteq\mathrm{TestF}(\mathrm{X})\bigg\}. \tag{4.80}$$

It can be readily proved that $\mathrm{TestV}(\mathrm{X})$ is dense in $L^2(T\mathrm{X})$.

Proposition 4.50 *The following properties hold:*

i) *It holds that $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ is a separable Hilbert space.*

ii) *The covariant derivative is a closed operator, i.e.*

$$\big\{(v,\nabla v)\;\big|\;v\in W^{1,2}_{\mathrm{C}}(T\mathrm{X})\big\}\qquad\text{is closed in }L^2(T\mathrm{X})\times L^2(T^{\otimes 2}\mathrm{X}). \tag{4.81}$$

iii) *Given any $f\in W^{2,2}(\mathrm{X})$, we have that $\nabla f\in W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ and*

$$\nabla(\nabla f)=\mathrm{Hess}(f)^\sharp, \tag{4.82}$$

where $L^2\big((T^*)^{\otimes 2}\mathrm{X}\big)\ni A\mapsto A^\sharp\in L^2(T^{\otimes 2}\mathrm{X})$ denotes the Riesz isomorphism.

iv) *It holds that $\mathrm{TestV}(\mathrm{X})\subseteq W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ and*

$$\nabla v=\sum_{i=1}^n\nabla g_i\otimes\nabla f_i+g_i\,\mathrm{Hess}(f_i)^\sharp\qquad\text{for every }v=\sum_{i=1}^n g_i\,\nabla f_i\in\mathrm{TestV}(\mathrm{X}). \tag{4.83}$$

In particular, the space $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ is dense in $L^2(T\mathrm{X})$.

Since it is not known whether $\mathrm{TestV}(\mathrm{X})$ is dense in $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$, we give this definition:

Definition 4.51 *We define the space $H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ as the $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$-closure of* $\mathrm{TestV}(\mathrm{X})$.

We now introduce a useful notation: given any $v \in W^{1,2}_{\rm C}(T{\rm X})$ and $z \in L^0(T{\rm X})$, we define the vector field $\nabla_z v \in L^0(T{\rm X})$ as the unique element of $L^0(T{\rm X})$ satisfying

$$\langle \nabla_z v, w \rangle = \nabla v : (z \otimes w) \quad \mathfrak{m}\text{-a.e. in X} \qquad \text{for every } w \in L^0(T{\rm X}). \tag{4.84}$$

Therefore the $L^0(\mathfrak{m})$-linear continuous operator $L^0(T{\rm X}) \ni z \mapsto \nabla_z v \in L^0(T{\rm X})$ satisfies

$$|\nabla_z v| \leq |\nabla v|_{\sf HS}\, |z| \quad \mathfrak{m}\text{-a.e. in X} \qquad \text{for every } z \in L^0(T{\rm X}). \tag{4.85}$$

In particular, one has that $\nabla_z v \in L^2(T{\rm X})$ whenever $z \in L^\infty(T{\rm X})$.

We now collect the main calculus rules for the covariant derivative:

Proposition 4.52 (Calculus rules for the covariant derivative) *The following hold:*

i) Leibniz rule. *Let $v \in W^{1,2}_{\rm C}(T{\rm X}) \cap L^\infty(T{\rm X})$ and $f \in W^{1,2}({\rm X}) \cap L^\infty(\mathfrak{m})$ be given. Then it holds that $fv \in W^{1,2}_{\rm C}(T{\rm X})$ and*

$$\nabla(fv) = \nabla f \otimes v + f\, \nabla v. \tag{4.86}$$

ii) Compatibility with the metric. *Let us consider $v \in W^{1,2}_{\rm C}(T{\rm X})$ and $w \in H^{1,2}_{\rm C}(T{\rm X})$ such that $v, w \in L^\infty(T{\rm X})$. Then $\langle v, w \rangle \in W^{1,2}({\rm X})$ and*

$$\mathrm{d}\langle v, w \rangle(z) = \langle \nabla_z v, w \rangle + \langle v, \nabla_z w \rangle \quad \mathfrak{m}\text{-a.e. in X} \qquad \text{for every } z \in L^2(T{\rm X}). \tag{4.87}$$

iii) Torsion-free identity. *Let $f \in H^{2,2}({\rm X}) \cap {\rm LIP}({\rm X})$ and $v, w \in W^{1,2}_{\rm C}(T{\rm X}) \cap L^\infty(T{\rm X})$. Then it holds that $\mathrm{d}f(v), \mathrm{d}f(w) \in W^{1,2}({\rm X})$ and*

$$\mathrm{d}\big(\mathrm{d}f(w)\big)(v) - \mathrm{d}\big(\mathrm{d}f(v)\big)(w) = \mathrm{d}f(\nabla_v w - \nabla_w v) \qquad \mathfrak{m}\text{-a.e. in X}. \tag{4.88}$$

Remark 4.53 We define the *Lie bracket* of two vector fields $v, w \in W^{1,2}_{\rm C}(T{\rm X})$ as

$$[v, w] \doteq \nabla_v w - \nabla_w v \in L^1(T{\rm X}). \tag{4.89}$$

Moreover, we shall denote $v(f) \doteq \mathrm{d}f(v)$ for any $f \in W^{1,2}({\rm X})$ and $v \in L^2(T{\rm X})$. Hence the torsion-free identity (4.88) can be rewritten in the following compact form:

$$v\big(w(f)\big) - w\big(v(f)\big) = [v, w](f) \qquad \mathfrak{m}\text{-a.e. in X} \tag{4.90}$$

for every $f \in H^{2,2}({\rm X}) \cap {\rm LIP}({\rm X})$ and $v, w \in W^{1,2}_{\rm C}(T{\rm X}) \cap L^\infty(T{\rm X})$. ■

In particular, item ii) of Proposition 4.52 grants that for any $v \in H^{1,2}_{\rm C}(T{\rm X}) \cap L^\infty(T{\rm X})$ one has $|v|^2 \in W^{1,2}({\rm X})$ and

$$\mathrm{d}|v|^2(w) = 2\, \langle \nabla_w v, v \rangle \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } w \in L^0(T{\rm X}), \tag{4.91}$$

whence $\big|D|v|^2\big| \leq 2\, |\nabla v|_{\sf HS}\, |v|$ holds $\mathfrak{m}$-a.e. in X. This in turn implies the following fact (that is proven, for instance, in the paper [DGP18]):

Lemma 4.54 *Let $v \in H^{1,2}_{\rm C}(T{\rm X})$ be fixed. Then $|v| \in W^{1,2}({\rm X})$ and*

$$\big|D|v|\big| \leq |\nabla v|_{\sf HS} \qquad \textit{holds } \mathfrak{m}\textit{-a.e. in X}. \tag{4.92}$$

Proof. First of all, we prove the statement for $v \in \mathrm{TestV}(\mathrm{X})$. Given any $\varepsilon > 0$, let us define the Lipschitz function $\varphi_\varepsilon : [0, +\infty) \to \mathbb{R}$ as $\varphi_\varepsilon(t) \doteq \sqrt{t+\varepsilon}$ for any $t \geq 0$. Hence by applying the chain rule for minimal weak upper gradients we see that $\varphi_\varepsilon \circ |v|^2 \in \mathrm{S}^2(\mathrm{X})$ and

$$\big|D(\varphi_\varepsilon \circ |v|^2)\big| = \varphi'_\varepsilon \circ |v|^2 \, \big|D|v|^2\big| = \frac{\big|D|v|^2\big|}{2\sqrt{|v|^2+\varepsilon}} \leq \frac{|v|}{\sqrt{|v|^2+\varepsilon}} \, |\nabla v|_{\mathsf{HS}} \leq |\nabla v|_{\mathsf{HS}}.$$

This grants the existence of $G \in L^2(\mathfrak{m})$ and a sequence $\varepsilon_j \searrow 0$ such that $\big|D(\varphi_{\varepsilon_j} \circ |v|^2)\big| \rightharpoonup G$ weakly in $L^2(\mathfrak{m})$ as $j \to \infty$ and $G \leq |\nabla v|_{\mathsf{HS}}$ in the $\mathfrak{m}$-a.e. sense. Since $\varphi_{\varepsilon_j} \circ |v|^2 \to |v|$ pointwise $\mathfrak{m}$-a.e. as $j \to \infty$, we deduce from the lower semicontinuity of minimal weak upper gradients that $|v| \in W^{1,2}(\mathrm{X})$ and that $\big|D|v|\big| \leq |\nabla v|_{\mathsf{HS}}$ holds $\mathfrak{m}$-a.e. in X.

Now fix any $v \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$. Pick a sequence $(v_n)_n \subseteq \mathrm{TestV}(\mathrm{X})$ that $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$-converges to v. In particular, $|v_n| \to |v|$ and $|\nabla v_n|_{\mathsf{HS}} \to |\nabla v|_{\mathsf{HS}}$ in $L^2(\mathfrak{m})$. By the first part of the proof we know that $|v_n| \in W^{1,2}(\mathrm{X})$ and $\big|D|v_n|\big| \leq |\nabla v_n|_{\mathsf{HS}}$ for all $n \in \mathbb{N}$, thus accordingly (up to a not relabeled subsequence) we have that $\big|D|v_n|\big| \rightharpoonup H$ weakly in $L^2(\mathfrak{m})$, for some $H \in L^2(\mathfrak{m})$ such that $H \leq |\nabla v|_{\mathsf{HS}}$ holds $\mathfrak{m}$-a.e. in X. Again by lower semicontinuity of minimal weak upper gradients, we conclude that $|v| \in W^{1,2}(\mathrm{X})$ with $\big|D|v|\big| \leq |\nabla v|_{\mathsf{HS}}$ in the $\mathfrak{m}$-a.e. sense, proving the statement. □

We conclude by pointing out that many other second-order notions can be built over an RCD space: for instance, the *exterior derivative*, the *de Rham cohomology* and the *Ricci curvature tensor*. Since we will not need such theories in this thesis, we do not add any further detail and we refer to [Gig17a] for the related discussion.

5

Structure of strongly $\mathfrak{m}$-rectifiable spaces

Contents

In the context of metric geometry, there is a well-established notion of tangent space at a point: the pointed-Gromov-Hausdorff limit of the rescalings of the space around the chosen point. On the other hand, an abstract concept of tangent bundle can be given on any metric measure space, as seen in Chapter 4 (recall the notion of tangent module). However, without any regularity assumption the 'geometric' approach and the 'analytical' one might be totally unrelated. For instance, the pGH limits of the blow-ups can fail to exist or the tangent module can be trivial (e.g. if the space admits no non-constant Lipschitz curves, then the Sobolev space is trivial and gives no information about the underlying space). The purpose of this chapter is to introduce and study a class of metric measure spaces – called 'strongly $\mathfrak{m}$-rectifiable spaces' – for which the two approaches are in fact equivalent.

A metric measure space $({\rm X},{\sf d},\mathfrak{m})$ is said to be *strongly* $\mathfrak{m}$*-rectifiable* provided its associated Sobolev space $W^{1,2}({\rm X})$ is reflexive (that is a technical assumption) and for every $\varepsilon>0$ it is possible to cover $\mathfrak{m}$-almost all of X with a sequence $(U_i)_i$ of Borel sets having the following property: there exists a map $\varphi_i:U_i\to\mathbb{R}^{k_i}$, for some $k_i\in\mathbb{N}$, which is $(1+\varepsilon)$-biLipschitz with its image and satisfies $(\varphi_i)_*\big(\mathfrak{m}_{|U_i}\big)\ll\mathcal{L}^{k_i}$. Any such couple (U_i,φ_i) is called ε*-chart*, while the collection $\big\{(U_i,\varphi_i)\big\}_i$ is said to be an ε*-atlas*, with a clear reminiscence of the differential geometric language. Section 5.1 will be devoted to the definition of strong $\mathfrak{m}$-rectifiability and to its basic properties.

In order to describe the crucial structural property of strongly $\mathfrak{m}$-rectifiable spaces – which is the main topic of Section 5.3 – we first need to introduce some notation. Given $(\mathrm{X},\mathsf{d},\mathfrak{m})$ as above and calling $(E_k)_{k\in\mathbb{N}}$ the dimensional decomposition of its tangent module $L^2(T\mathrm{X})$, it is possible to show (by using the ε-charts) that for $\mathfrak{m}$-a.e. point $x\in E_k$ the pmG-tangent cone at x contains exactly the (normalised) k-dimensional Euclidean space. By 'patching together' these Euclidean blow-ups of X around its points, we obtain the *Gromov-Hausdorff tangent bundle* $T_{\mathrm{GH}}\mathrm{X}$; more precisely, we have that $T_{\mathrm{GH}}\mathrm{X}$ is a measurable Hilbert bundle in the sense of Definition 3.56. Then we can consider the space $L^2(T_{\mathrm{GH}}\mathrm{X})$ of all 2-integrable sections of $T_{\mathrm{GH}}\mathrm{X}$ (which is denoted by $\Gamma_2(T_{\mathrm{GH}}\mathrm{X})$ in the language of Section 3.2). Hence the key property of strongly $\mathfrak{m}$-rectifiable spaces (see Theorem 5.21 below) is the following: the normed modules $L^2(T_{\mathrm{GH}}\mathrm{X})$ and $L^2(T\mathrm{X})$ are isometrically isomorphic.

We point out that the chosen structure on the Gromov-Hausdorff tangent bundle $T_{\mathrm{GH}}\mathrm{X}$ is canonical, in a sense that is illustrated by the ensuing discussion.

Suppose to have a metric space (X,d) such that for every $x\in\mathrm{X}$ the tangent space at the point x (in the sense of pointed-Gromov-Hausdorff limit) is the Euclidean space of a certain fixed dimension k. Then obviously all such tangent spaces would be isometric and we might want to identify all of them with a given fixed $\mathbb{R}^k$. Once these identifications are chosen, given $x\in\mathrm{X}$ and $\mathsf{v}\in\mathbb{R}^k$ we might think of v as an element of the tangent space at x. Therefore a vector field should be thought of as a map from X to $\mathbb{R}^k$. However, the choice of the identifications/isometries of the abstract tangent spaces with the fixed $\mathbb{R}^k$ is highly arbitrary and affects the structure that one is building. In fact, in general there is no solution to this problem, in the sense that there is no canonical choice of these identifications. The problem is that – by its very definition – a pointed-Gromov-Hausdorff limit is the isometric class of a metric space rather than a 'concrete' one.

As we shall see, the situation changes if we work on a strongly $\mathfrak{m}$-rectifiable metric measure space: much like in the smooth setting the charts of a manifold are used to give structure to the tangent bundle, in this case the presence of charts

- allows for a canonical identification of the tangent spaces,
- ensures existence and uniqueness of a measurable structure on the resulting bundle.

Let us remark that – while the initial definition of the Gromov-Hausdorff tangent bundle (and in particular of its measurable structure) is simply given by a product – in fact (as we will show in Subsection 5.3.3) such measurable structure is natural, because it is compatible with 'taking all pGH-limits at the same time'; see Theorem 5.23 for the details.

The motivating examples of strongly $\mathfrak{m}$-rectifiable space are the finite-dimensional RCD spaces, as we are now going to describe; such results will be presented in Section 5.2.

It has been proved by A. Mondino and A. Naber in the paper [MN14] that any $\mathsf{RCD}(K,N)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, with $K\in\mathbb{R}$ and $N\in(1,\infty)$, is 'rectifiable as a metric space', in the following sense: given any $\varepsilon>0$, there exists a sequence $(U_i,\varphi_i)_{i\in\mathbb{N}}$ such that $(U_i)_{i\in\mathbb{N}}$ is a Borel partition of X (up to $\mathfrak{m}$-negligible sets) and each map $\varphi_i:U_i\to\mathbb{R}^{k_i}$ (for some $k_i\leq N$) is $(1+\varepsilon)$-biLipschitz with its image. Nevertheless, in [MN14] the behaviour of the reference measure $\mathfrak{m}$ under the maps φ_i's is not investigated. The main result of Section 5.2 (namely Theorem 5.16) says that the maps φ_i's of Mondino-Naber satisfy $(\varphi_i)_*(\mathfrak{m}_{|U_i})\ll\mathcal{L}^{k_i}$, thus proving that any $\mathsf{RCD}(K,N)$ space X is strongly $\mathfrak{m}$-rectifiable. Notice that our result is equivalent to the fact that the restriction of $\mathfrak{m}$ to U_i is absolutely continuous with respect to the k_i-dimensional Hausdorff measure. We point out that similar structural results have been independently achieved in [DPMR17] and [KM18].

In the case of Ricci limit spaces, the analogue of Theorem 5.16 was already known from the work of Cheeger-Colding [CC00a]. However, the technique used therein is not applicable to our setting, the problem being that in [CC00a] the spaces under consideration are limits of manifolds equipped with the volume measure – a fact leading to some cancellations that do not occur in the weighted case. Specifically, the key lemma [CC00a, Lemma 1.14] does not hold on weighted Riemannian manifolds, thus a fortiori cannot hold on RCD spaces.

Our proof combines a deep result on Radon measures in the Euclidean space – obtained by G. De Philippis and F. Rindler in [DPR16] – with the construction by Mondino-Naber and the Laplacian comparison estimates for distance functions obtained by N. Gigli in [Gig15].

The results of Sections 5.1 and 5.3 are taken from [GP16b], while the content of Section 5.2 can be found in [GP16a].

5.1 Definition and basic properties of strong $\mathfrak{m}$-rectifiability

We introduce a new class of metric measure spaces, called 'strongly $\mathfrak{m}$-rectifiable' spaces. Roughly speaking, these spaces can be partitioned (up to negligible sets) into countably many Borel sets, which are biLipschitz equivalent to suitable subsets of the Euclidean space, by means of maps that also keep under control the measure. Our interest in this class of spaces is mainly motivated by the fact that – as we shall prove in Subsection 5.2 – any finite-dimensional RCD space turns out to be strongly $\mathfrak{m}$-rectifiable.

For the sake of simplicity, it is convenient to use the following notation: given any measure space $(\mathrm{X},\mathcal{A},\mathfrak{m})$, we say that $(E_i)_{i\in\mathbb{N}}\subseteq\mathcal{A}$ is an $\mathfrak{m}$-*partition* of $E\in\mathcal{A}$ provided it is a partition of some $F\in\mathcal{A}$ such that $F\subseteq E$ and $\mathfrak{m}(E\setminus F)=0$. Moreover, given two $\mathfrak{m}$-partitions $(E_i)_i$ and $(F_j)_j$ of E, we say that $(F_j)_j$ is a *refinement* of $(E_i)_i$ if for every $j\in\mathbb{N}$ with $F_j\neq\emptyset$ there exists (a unique) $i\in\mathbb{N}$ such that $F_j\subseteq E_i$.

Definition 5.1 (Strongly $\mathfrak{m}$-rectifiable space) *A metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be* $\mathfrak{m}$-rectifiable *provided it can be written as a countable disjoint union $\bigcup_{k\in\mathbb{N}}A_k$ of suitable sets $(A_k)_k\subseteq\mathscr{B}(\mathrm{X})$, such that the following condition is satisfied: given any $k\in\mathbb{N}$, there exists an $\mathfrak{m}$-partition $(U_i)_{i\in\mathbb{N}}\subseteq\mathscr{B}(\mathrm{X})$ of A_k and a sequence $(\varphi_i)_{i\in\mathbb{N}}$ of maps $\varphi_i:U_i\to\mathbb{R}^k$ such that*

$$
\begin{aligned}
&\varphi_i:U_i\to\varphi_i(U_i) \qquad \textit{is biLipschitz,}\\
&(\varphi_i)_*\big(\mathfrak{m}_{|U_i}\big)\ll\mathcal{L}^k
\end{aligned}
\tag{5.1}
$$

for every $i\in\mathbb{N}$. The partition $\mathrm{X}=\bigcup_{k\in\mathbb{N}}A_k$ – which is clearly unique up to modification of negligible sets – is called dimensional decomposition *of* X.

Moreover, the space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is said to be strongly $\mathfrak{m}$-rectifiable *provided for every $\varepsilon>0$ the (U_i,φ_i)'s can be chosen so that the φ_i's are $(1+\varepsilon)$-biLipschitz.*

Remark 5.2 Given an $\mathfrak{m}$-rectifiable space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ with dimensional decomposition $(A_k)_k$, we have that each set A_k is countably $\mathcal{H}^k$-rectifiable. Moreover, it follows from (B.6) and

(5.1) that there exists a sequence $(N_k)_k$ of Borel sets $N_k \subseteq A_k$ with $\mathfrak{m}(N_k) = 0$ such that

$$\mathfrak{m}|_{A_k \setminus N_k} = \theta_k\, \mathcal{H}^k|_{A_k \setminus N_k} \qquad \text{for every } k \in \mathbb{N}, \tag{5.2}$$

where the density θ_k is a suitable Borel map $\theta_k : A_k \setminus N_k \to (0, +\infty)$. ■

When working on $\mathfrak{m}$-rectifiable spaces, it is natural to adopt the following terminology, which is inspired by the language of differential geometry:

Definition 5.3 (Charts and atlases) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathfrak{m}$*-rectifiable space. A* chart *on* X *is a couple* (U, φ)*, where* $U \in \mathscr{B}(A_k)$ *for some* $k \in \mathbb{N}$ *and* $\varphi : U \to \mathbb{R}^k$ *satisfies*

$$\begin{aligned} &\varphi : U \to \varphi(U) \qquad \textit{is biLipschitz,} \\ &C^{-1}\, \mathcal{L}^k|_{\varphi(U)} \leq \varphi_*\big(\mathfrak{m}|_U\big) \leq C\, \mathcal{L}^k|_{\varphi(U)}, \end{aligned} \tag{5.3}$$

for a suitable constant $C \geq 1$*. An* atlas *on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is a family* $\mathscr{A} = \bigcup_{k \in \mathbb{N}} \big\{(U_i^k, \varphi_i^k)\big\}_{i \in \mathbb{N}}$ *of charts on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *such that* $(U_i^k)_{i \in \mathbb{N}}$ *is an* $\mathfrak{m}$*-partition of* A_k *for every* $k \in \mathbb{N}$*.*

Moreover, the chart (U, φ) *is said to be an* ε-chart *provided the map* $\varphi : U \to \varphi(U)$ *is* $(1+\varepsilon)$*-biLipschitz and an atlas is said to be an* ε-atlas *provided all of its charts are* ε*-charts.*

We collect few simple facts about atlases which we shall frequently use in what follows:

i) Any $\mathfrak{m}$-rectifiable space admits an atlas and any strongly $\mathfrak{m}$-rectifiable space admits an ε-atlas for every $\varepsilon > 0$. Indeed, given any (U_i, φ_i) as in (5.1), we can consider the density ρ_i of $\varphi_*\big(\mathfrak{m}|_{U_i}\big)$ with respect to the Lebesgue measure and the sets

$$U_{i,j} \doteq \varphi_i^{-1}\big(\{2^j \leq \rho_i < 2^{j+1}\}\big) \qquad \text{for every } j \in \mathbb{Z}.$$

It is clear that $\big(U_{ij}, \varphi_i|_{U_{i,j}}\big)$ is a chart for every $j \in \mathbb{Z}$ and that the U_{ij}'s provide an $\mathfrak{m}$-partition of U_i, so that repeating the construction for every i yields the desired atlas.

ii) Let $(U_i, \varphi_i)_{i \in \mathbb{N}}$ be an atlas on X. Given any $i \in \mathbb{N}$, take an $\mathfrak{m}$-partition $(U_{ij})_{j \in \mathbb{N}}$ of U_i. Then $\big(U_{ij}, \varphi_i|_{U_{ij}}\big)_{i,j \in \mathbb{N}}$ is an atlas as well. In particular – by inner regularity of $\mathfrak{m}$ – every $\mathfrak{m}$-rectifiable space admits an atlas whose charts are defined on compact sets.

A first property of $\mathfrak{m}$-rectifiable spaces, whose proof is based upon the notion of differential introduced in Theorem 4.39, is given by the following result.

Theorem 5.4 (Dimensional decomposition of the tangent module) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathfrak{m}$*-rectifiable space, with* $W^{1,2}(\mathrm{X})$ *reflexive. Let* $(A_k)_k$ *be its dimensional decomposition. Then for every* $k \in \mathbb{N}$ *such that* $\mathfrak{m}(A_k) > 0$ *we have that* $L^2(T\mathrm{X})$ *has dimension* k *on* A_k*.*

Proof. Let $\mathscr{A} = \big\{(U_i^k, \varphi_i^k)\big\}_{k,i}$ be an atlas on X. The claim is equivalent to the fact that for every U_i^k with $\mathfrak{m}(U_i^k) > 0$ the dimension of $L^2(T\mathrm{X})$ on U_i^k is k. Consider the differential

$$\widehat{\mathrm{d}}\varphi_i^k : L^2(T\mathrm{X})|_{U_i^k} \longrightarrow L^2\big(\varphi_i^k(U_i^k), \mathbb{R}^k\big)$$

in the sense of Theorem 4.39. Such operator is continuous, invertible, with continuous inverse and sends $h\mathsf{v}$ to $h \circ (\varphi_i^k)^{-1}\, \widehat{\mathrm{d}}\varphi_i^k(\mathsf{v})$. It is then clear that $L^2(T\mathrm{X})|_{U_i^k}$ and $L^2\big(\varphi_i^k(U_i^k), \mathbb{R}^k\big)$ have the same dimension. Since the latter has dimension k, the conclusion follows. □

Remark 5.5 Using the finite dimensionality results obtained by Cheeger in [Che99], it is not hard to see that the dimensional decomposition $(A_k)_k$ of a PI space (i.e. a doubling metric measure space supporting a weak $(1,2)$-Poincaré inequality) that is also $\mathfrak{m}$-rectifiable must be so that $\mathfrak{m}(A_k)=0$ for all k sufficiently large. ■

Proposition 5.6 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathfrak{m}$-rectifiable space. Then X is a Vitali space (recall Definition 1.33). In particular, given any Borel set $E\subseteq\mathrm{X}$, it holds that $\mathfrak{m}$-a.e. point $x\in E$ is of density 1 for E (by Corollary 1.36).*

Proof. By recalling (1.62), it is sufficient to prove that $\mathfrak{m}$ is pointwise doubling at $\mathfrak{m}$-almost every point of X. To this aim, call $(A_k)_k$ the dimensional decomposition of X and fix $k\in\mathbb{N}$. Let $(N_k)_k$ be as in Remark 5.2 and call $A'_k\doteq A_k\setminus N_k$ for all $k\in\mathbb{N}$. We claim that

$$\lim_{r\to 0}\frac{\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)}{\omega_k\,2^k\,r^k}=0 \qquad \text{holds for } \mathcal{H}^k\text{-a.e. } x\in A'_k. \tag{5.4}$$

We argue by contradiction: if not, there exist a Borel set $P\subseteq A'_k$ with $\mathcal{H}^k(P)>0$ and a constant $\lambda>0$ such that $\overline{\lim}_{r\searrow 0}\,\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)/(\omega_k\,2^k\,r^k)\geq\lambda$ holds for any point $x\in P$. Hence (B.7) with $\mu\doteq\mathfrak{m}|_{\mathrm{X}\setminus A'_k}$ yields $\lambda\,\mathcal{H}^k(P)\leq\mathfrak{m}(P\setminus A'_k)=0$, which leads to a contradiction.

Therefore (B.8) and (5.4) grant that for $\mathcal{H}^k$-a.e. $x\in A'_k$ (thus also $\mathfrak{m}$-a.e. $x\in A'_k$) it holds

$$\overline{\lim_{r\searrow 0}}\,\frac{\mathfrak{m}\big(B_{2r}(x)\big)}{\mathfrak{m}\big(B_r(x)\big)}\leq\lim_{r\searrow 0}\frac{\mathfrak{m}\big(B_{2r}(x)\cap A'_k\big)}{\mathfrak{m}\big(B_r(x)\cap A'_k\big)}+\lim_{r\searrow 0}\frac{\mathfrak{m}\big(B_{2r}(x)\setminus A'_k\big)}{\mathfrak{m}\big(B_r(x)\cap A'_k\big)}=2^k,$$

thus proving the statement. □

When we restrict our attention to the smaller class of strongly $\mathfrak{m}$-rectifiable spaces, we have a stronger geometric characterisation of the tangent module. Subsection 5.3.2 will be entirely devoted to describe such result. In order to further develop our theory in that direction, we need to provide any strongly $\mathfrak{m}$-rectifiable space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ with a special sequence of atlases, which are aligned in a suitable sense.

Definition 5.7 (Aligned family of atlases) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a strongly $\mathfrak{m}$-rectifiable space. Let $\varepsilon_n\searrow 0$ and $\delta_n\searrow 0$. Let $(\mathscr{A}_n)_{n\in\mathbb{N}}$ be a sequence of atlases on X. Then we say that $(\mathscr{A}_n)_n$ is an* aligned family of atlases *of parameters ε_n and δ_n provided the following conditions hold:*

i) *Each $\mathscr{A}_n=\big\{(U^{k,n}_i,\varphi^{k,n}_i)\big\}_{k,i}$ is an ε_n-atlas and the domains $U^{k,n}_i$ are compact.*

ii) *The family $(U^{k,n}_i)_{k,i}$ is a refinement of $(U^{k,n-1}_j)_{k,j}$ for any $n\in\mathbb{N}^+$.*

iii) *If $n\in\mathbb{N}^+$, $k\in\mathbb{N}$ and $i,j\in\mathbb{N}$ satisfy $U^{k,n}_i\subseteq U^{k,n-1}_j$, then*

$$\Big\|\mathrm{d}\Big(\mathrm{id}_{\mathbb{R}^k}-\varphi^{k,n-1}_j\circ\big(\varphi^{k,n}_i\big)^{-1}\Big)(y)\Big\|\leq\delta_n \qquad \text{for } \mathcal{L}^k\text{-a.e. } y\in\varphi^{k,n}_i(U^{k,n}_i). \tag{5.5}$$

The discussions made before grant that any strongly $\mathfrak{m}$-rectifiable space admits atlases satisfying i) and ii). In fact, as we shall see in a moment, also iii) can be fulfilled by an appropriate choice of atlases, but in order to show this we need a small digression.

Recall that $O(\mathbb{R}^k)$ denotes the group of linear isometries of $\mathbb{R}^k$. For $\varepsilon>0$, let us define

$$O^\varepsilon(\mathbb{R}^k)\doteq\Big\{T:\mathbb{R}^k\to\mathbb{R}^k \text{ linear, invertible and such that } \|T\|,\|T^{-1}\|\leq 1+\varepsilon\Big\}. \tag{5.6}$$

Notice that the space $O^\varepsilon(\mathbb{R}^k)$ – being closed and bounded – is compact for every $\varepsilon>0$ and that $O(\mathbb{R}^k)=\bigcap_{\varepsilon>0}O^\varepsilon(\mathbb{R}^k)$. Then we have the following simple result:

Proposition 5.8 *Let $k \in \mathbb{N}$ and $\delta > 0$ be given. Then there exist a constant $\varepsilon > 0$ and a Borel mapping $R : O^\varepsilon(\mathbb{R}^k) \to O(\mathbb{R}^k)$ with finite image such that*

$$\big\|T - R(T)\big\| \leq \delta \qquad \text{for every } T \in O^\varepsilon(\mathbb{R}^k). \tag{5.7}$$

Proof. Since the space $O(\mathbb{R}^k)$ is compact, there exist $T_1, \ldots, T_n \in O(\mathbb{R}^k)$ such that

$$O(\mathbb{R}^k) \subseteq U_\delta \doteq \bigcup_{i=1}^n B_\delta(T_i).$$

We claim that there exists $\varepsilon > 0$ such that $O^\varepsilon(\mathbb{R}^k) \subset U_\delta$ and argue by contradiction. If not, the compact set $K^\varepsilon \doteq O^\varepsilon(\mathbb{R}^k) \setminus U_\delta$ would be not empty for every $\varepsilon > 0$. Since $K^\varepsilon \subset K^{\varepsilon'}$ for any $\varepsilon \leq \varepsilon'$, the family K^ε has the finite intersection property, but on the other hand the identity $O(\mathbb{R}^k) = \bigcap_{\varepsilon>0} O^\varepsilon(\mathbb{R}^k)$ yields $\bigcap_{\varepsilon>0} K^\varepsilon = \emptyset$, which leads to a contradiction. Thus there exists $\varepsilon > 0$ such that $O^\varepsilon(\mathbb{R}^k) \subseteq U_\delta$. For such ε, we define $R : O^\varepsilon(\mathbb{R}^k) \to O(\mathbb{R}^k)$ to be equal to T_1 on $B_\delta(T_1)$, then recursively to be equal to T_n on $B_\delta(T_n) \setminus \bigcup_{i<n} B_\delta(T_i)$. □

By using Proposition 5.8, it is possible to show that any strongly $\mathfrak{m}$-rectifiable space admits an aligned family of atlases:

Theorem 5.9 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a strongly $\mathfrak{m}$-rectifiable metric measure space. Let $\varepsilon_n \searrow 0$ and $\delta_n \searrow 0$ be two given sequences. Then X admits an aligned family $(\mathscr{A}_n)_n$ of atlases of parameters ε_n and δ_n.*

Proof. Let $(A_k)_k$ be the dimensional decomposition of X and notice that to conclude it is sufficient to build, for every $k \in \mathbb{N}$, aligned charts as in iii) of Definition 5.7 covering $\mathfrak{m}$-almost all of A_k. For any given $k, n \in \mathbb{N}$, let $\varepsilon'_{n,k} > 0$ be associated to δ_n and k as in Proposition 5.8 and choose $\bar\varepsilon_{n,k} > 0$ such that

$$\bar\varepsilon_{n,k} \leq \varepsilon_n \quad \text{and} \quad (1+\bar\varepsilon_{n-1,k})(1+\bar\varepsilon_{n,k}) \leq 1+\varepsilon'_{n,k} \qquad \text{for every } k, n \in \mathbb{N}. \tag{5.8}$$

We now construct the required aligned family $(\mathscr{A}_n)_n$ of atlases by recursion: start by observing that, since $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is strongly $\mathfrak{m}$-rectifiable, there exists an atlas $\mathscr{A}_0$ such that the charts with domain included in A_k are $\bar\varepsilon_{0,k}$-biLipschitz. Now assume that for some $n \in \mathbb{N}$ we have already defined $\mathscr{A}_0, \ldots, \mathscr{A}_{n-1}$ satisfying the alignment conditions and say that $\mathscr{A}_{n-1} = \big\{(U_i^k, \varphi_i^k)\big\}_{k,i}$. Again using the strong $\mathfrak{m}$-rectifiability of X, find an atlas $\big\{(V_j^k, \psi_j^k)\big\}_{k,j}$ whose domains $(V_j^k)_{k,j}$ constitute a refinement of the domains $(U_i^k)_{k,i}$ of $\mathscr{A}_{n-1}$ and such that those charts with domain included in A_k are $\bar\varepsilon_{n,k}$-biLipschitz.

Fix $k, j \in \mathbb{N}$ and let $i \in \mathbb{N}$ be the unique index such that $V_j^k \subseteq U_i^k$. For the sake of brevity, let us denote by τ the transition map $\varphi_i^k \circ (\psi_j^k)^{-1} : \psi_j^k(V_j^k) \to \varphi_i^k(V_j^k)$ and observe that it is $(1+\varepsilon'_{n,k})$-biLipschitz by (5.8). Hence its differential $\mathrm{d}\tau$ satisfies $\big\|\mathrm{d}\tau(y)\big\|, \big\|\mathrm{d}\tau(y)^{-1}\big\| \leq 1+\varepsilon'_{n,k}$, or equivalently $\mathrm{d}\tau(y) \in O^{\varepsilon'_{n,k}}(\mathbb{R}^k)$, for $\mathcal{L}^k$-a.e. point $y \in \psi_j^k(V_j^k)$.

Let $R : O^{\varepsilon'_{n,k}}(\mathbb{R}^k) \to O(\mathbb{R}^k)$ be given by Proposition 5.8 (with $\delta \doteq \delta_n$) and let us denote by $F_j^k \subseteq O(\mathbb{R}^k)$ its finite image. For any $T \in F_j^k$, we define $P_T \doteq (R \circ \mathrm{d}\tau)^{-1}(T) \subseteq \mathbb{R}^k$, so that $(P_T)_{T \in F_j^k}$ is an $\mathcal{L}^k$-partition of $\psi_j^k(V_j^k)$. For $\mathcal{L}^k$-a.e. point $y \in T(P_T) \subseteq \mathbb{R}^k$ we have that

$$
\begin{aligned}
\left\| \mathrm{d}\big(\varphi_i^k \circ (T \circ \psi_j^k)^{-1} - \mathrm{id}_{\mathbb{R}^k}\big)(y) \right\| &= \left\| \mathrm{d}\big(\tau \circ T^{-1} - \mathrm{id}_{\mathbb{R}^k}\big)(y) \right\| \\
&= \left\| \mathrm{d}\big((\tau - T) \circ T^{-1}\big)(y) \right\| \\
&\leq \left\| \mathrm{d}\tau\big(T^{-1}(y)\big) - T \right\| \left\| T^{-1} \right\| \\
&= \left\| \mathrm{d}\tau\big(T^{-1}(y)\big) - T \right\| \\
\text{(because } T^{-1}(y) \in P_T) \qquad &= \left\| \mathrm{d}\tau\big(T^{-1}(y)\big) - R\big(\mathrm{d}\tau\big(T^{-1}(y)\big)\big) \right\| \\
\text{(by definition of } R) \qquad &\leq \delta_n.
\end{aligned} \tag{5.9}
$$

We therefore define

$$
\overline{U}_{j,T}^k \doteq (\psi_j^k)^{-1}(P_T) \quad \text{and} \quad \overline{\varphi}_{j,T}^k \doteq T \circ \psi_j^k|_{\overline{U}_{j,T}^k} \qquad \text{for every } T \in F_j^k, \tag{5.10}
$$

so that accordingly

$$
\mathscr{A}_n \doteq \left\{ \big(\overline{U}_{j,T}^k, \overline{\varphi}_{j,T}^k\big) \;\middle|\; k, j \in \mathbb{N},\, T \in F_j^k \right\} \tag{5.11}
$$

is an atlas on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, which fulfills ii), iii) of Definition 5.7 and such that the charts with domain included in A_k are $\bar{\varepsilon}_{n,k}$-biLipschitz. Up to a further refining, we can assume that the charts in $\mathscr{A}_n$ have compact domain. Given that $\bar{\varepsilon}_{n,k} \leq \varepsilon_n$ holds for every $k, n \in \mathbb{N}$, the statement is proved. □

5.2 Finite-dimensional RCD spaces are strongly $\mathfrak{m}$-rectifiable

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a fixed $\mathsf{RCD}(K, N)$ space, for some $K \in \mathbb{R}$ and $N \in (1, \infty)$. The aim of this section is to prove that the space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is strongly $\mathfrak{m}$-rectifiable.

We shall use the shorthand notation d_x to indicate the distance function $\mathrm{X} \ni y \mapsto \mathsf{d}(x, y)$ from a given point $x \in \mathrm{X}$. We start with a simple statement concerning the minimal weak upper gradient of the distance function from a point:

Proposition 5.10 *Let $x \in \mathrm{X}$ be given. Then it holds that*

$$
|D\mathsf{d}_x| = 1 \qquad \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{5.12}
$$

Proof. Any $\mathsf{RCD}(K, N)$ space is doubling and supports a weak $(1, 2)$-Poincaré inequality (as proven in [Stu06b] and [Raj12], respectively). Moreover, recall that the local Lipschitz constant of d_x is identically equal to 1, as it follows from Remark 1.12 and the fact any $\mathsf{RCD}(K, N)$ space is geodesic. Hence the statement is a consequence of Remark 2.23. □

Remark 5.11 We refer to the proof of [GP16a, Proposition 3.1] for an alternative argument, which does not use the results in [Che99], but relies instead on the additional regularity of both the space and the function under consideration. ■

The 'metric rectifiability' of finite-dimensional RCD spaces has been proved in [MN14]. We now recall the main result of such paper; some of the claims we make are implicit in the various proofs of [MN14], thus we highlight where such passages appear.

Theorem 5.12 (Rectifiability of RCD spaces) *There exists a Borel partition $(A_i)_{i=1}^n$ of the space X – for some $n \in \mathbb{N}$ with $n \leq N$ – such that the following property holds: given any $i = 1, \ldots, n$ and $\varepsilon > 0$, there exist an $\mathfrak{m}$-partition $(U^\varepsilon_{i,j})_{j\in\mathbb{N}} \subseteq \mathscr{B}(\mathrm{X})$ of the set A_i and a family $\{x^\varepsilon_{i,j,k} : j \in \mathbb{N},\, k = 1, \ldots, i\}$ of points of X such that*

$$\big|\big\langle \nabla \mathsf{d}_{x^\varepsilon_{i,j,k}}, \nabla \mathsf{d}_{x^\varepsilon_{i,j,k'}} \big\rangle\big| \leq \varepsilon \quad \mathfrak{m}\text{-a.e. on } U^\varepsilon_{i,j} \qquad \text{for every } k \neq k' \tag{5.13}$$

and so that each map $\varphi^\varepsilon_{i,j} : \mathrm{X} \to \mathbb{R}^i$, given by $\varphi^\varepsilon_{i,j}(x) \doteq \big(\mathsf{d}_{x^\varepsilon_{i,j,1}}(x), \ldots, \mathsf{d}_{x^\varepsilon_{i,j,i}}(x)\big)$, satisfies

$$\varphi^\varepsilon_{i,j}|_{U^\varepsilon_{i,j}} : U^\varepsilon_{i,j} \longrightarrow \varphi^\varepsilon_{i,j}(U^\varepsilon_{i,j}) \qquad \text{is } (1+\varepsilon)\text{-biLipschitz.} \tag{5.14}$$

Proof. The fact that X can be covered by Borel charts $(1+\varepsilon)$-biLipschitz to subsets of the Euclidean space is the main result in [MN14]. The fact that the coordinates of the charts are distance functions is part of the construction, see [MN14, Theorem 6.5]; more precisely, in [MN14] the coordinates are distance functions plus well-chosen constants, so that 0 is always an element of the image, but this has no effect for our discussion.

Thus we are left to prove inequality (5.13). Looking at the construction of the sets $U^\varepsilon_{i,j}$ in [MN14], we see that they are contained in the set of x's such that

$$\sup_{r' \in (0,r)} \frac{1}{\mathfrak{m}\big(B_{r'}(x)\big)} \int_{B_{r'}(x)} \sum_{1 \leq k \leq k' \leq i} \left| D\left(\frac{\mathsf{d}_{x^\varepsilon_{i,j,k}} + \mathsf{d}_{x^\varepsilon_{i,j,k'}}}{\sqrt{2}} - \mathsf{d}_{x^\varepsilon_{i,j,k,k'}} \right) \right|^2 \mathrm{d}\mathfrak{m} \leq \varepsilon_1, \tag{5.15}$$

where $r, \varepsilon_1 > 0$ are bounded from above in terms of K, N, ε and the points $x^\varepsilon_{i,j,k,k'}$ are built together with the $x^\varepsilon_{i,j,k}$'s (in [MN14] the points $x_{i,j,k}, x_{i,j,k'}, x_{i,j,k,k'}$ are called $p_i, p_j, p_i + p_j$, respectively). We remark that the choice of r, ε_1 affects the construction of the sets $U^\varepsilon_{i,j}$ and of the points $x^\varepsilon_{i,j,k}$. In any case, ε_1 can be chosen to be smaller than $\varepsilon^2/(\sqrt{2}+1)^2$.

Notice that in [MN14] the distance in (5.15) is scaled by a factor r, whose only effect is that r' varies in $(0,1)$ rather than in $(0,r)$. The validity of (5.15) comes from:

- the definition of maximal function – called M^k – in [MN14, Equation/Definition (67)],
- the fact that the sets $U^k_{\varepsilon_1,\delta_1}$, introduced in [MN14, Equation/Definition (70)], are contained in $\{M^k \leq \varepsilon_1\}$ by definition,
- the fact that the charts in [MN14, Theorem 6.5] are defined on $B^{\tilde{\mathsf{d}}}_{\delta_1} \cap U^k_{\varepsilon_1,\delta_1} \subseteq U^k_{\varepsilon_1,\delta_1}$.

We come back to the proof of (5.13). Recall that – being the measure $\mathfrak{m}$ doubling – the Lebesgue differentiation theorem holds. Hence from (5.15) and the discussion thereafter we see that, up to properly choosing ε_1 and thus $U^\varepsilon_{i,j}, x^\varepsilon_{i,j,k}, x^\varepsilon_{i,j,k,k'}$, we can assume that

$$\left| D\left(\frac{\mathsf{d}_{x^\varepsilon_{i,j,k}} + \mathsf{d}_{x^\varepsilon_{i,j,k'}}}{\sqrt{2}} - \mathsf{d}_{x^\varepsilon_{i,j,k,k'}} \right) \right|^2 \leq \left| \frac{\varepsilon}{\sqrt{2}+1} \right|^2 \qquad \text{holds } \mathfrak{m}\text{-a.e. on } U^\varepsilon_{i,j}.$$

Thus to conclude it is sufficient to prove that, given any $x_1, x_2, y \in \mathrm{X}$, we have that

$$\left| D\left(\frac{\mathsf{d}_{x_1} + \mathsf{d}_{x_2}}{\sqrt{2}} - \mathsf{d}_y \right) \right| \leq \frac{\varepsilon}{\sqrt{2}+1} \qquad \Longrightarrow \qquad \big|\langle \nabla \mathsf{d}_{x_1}, \nabla \mathsf{d}_{x_2} \rangle\big| \leq \varepsilon.$$

This follows with minor algebraic manipulations from the identity (5.12):

$$
\begin{aligned}
\big|\langle\nabla \mathsf{d}_{x_1},\nabla \mathsf{d}_{x_2}\rangle\big| &= \left|\left|D\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}\right)\right|^2-\frac{|D\mathsf{d}_{x_1}|^2+|D\mathsf{d}_{x_2}|^2}{2}\right| \\
&= \left|\left|D\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}\right)\right|^2-1\right| \\
&= \left|\left|D\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}\right)\right|^2-|D\mathsf{d}_y|^2\right| \\
&= \left|\left\langle\nabla\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}+\mathsf{d}_y\right),\nabla\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}+\mathsf{d}_y\right)\right\rangle\right| \\
&\leq (\sqrt{2}+1)\left|D\left(\frac{\mathsf{d}_{x_1}+\mathsf{d}_{x_2}}{\sqrt{2}}-\mathsf{d}_y\right)\right|.
\end{aligned}
$$

Hence the statement is achieved. □

A key tool we shall make use of is the following structural theorem about Radon measures on the Euclidean space, proven in [DPR16]. We point out that such statement is only one of the several consequences of the main deep result in [DPR16]. Recall that the language of 1-dimensional currents in $\mathbb{R}^d$ has been briefly illustrated in the last part of Subsection 4.1.3.

Theorem 5.13 *Let $(T_i)_{i=1}^d$ be 1-dimensional normal currents in $\mathbb{R}^d$, written as $T_i=\vec{T_i}\,\|T_i\|$. Let μ be a Radon measure on $\mathbb{R}^d$. Suppose that:*

i) *We have $\mu \ll \|T_i\|$ for every $i=1,\dots,d$.*

ii) *The vectors $\vec{T_1}(x),\dots,\vec{T_d}(x)\in\mathbb{R}^d$ are linearly independent for μ-a.e. $x\in\mathbb{R}^d$.*

Then μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

The last result we shall need is the Laplacian comparison estimate for distance functions, obtained in [Gig15]. Such result holds in the sharp form, but we recall it in its qualitative form, which is sufficient for our purposes:

Theorem 5.14 *Let $x\in\mathrm{X}$ be fixed. Then the distributional Laplacian of d_x in $\mathrm{X}\setminus\{x\}$ is a measure, i.e. there exists a Radon measure μ on X such that*

$$\int \langle\nabla f,\nabla \mathsf{d}_x\rangle\,\mathrm{d}\mathfrak{m} = -\int f\,\mathrm{d}\mathfrak{m} \qquad \textit{for every } f\in \mathrm{LIP_{bs}}(\mathrm{X}) \textit{ with } \mathrm{spt}(f)\subseteq \mathrm{X}\setminus\{x\}. \tag{5.16}$$

When read in terms of measure-valued divergence, the above theorem yields:

Corollary 5.15 *Let $x\in\mathrm{X}$ be fixed. Choose any $\psi\in\mathrm{LIP_c}(\mathrm{X})$ satisfying $\mathrm{spt}(\psi)\subseteq\mathrm{X}\setminus\{x\}$. Then the vector field $\psi\nabla\mathsf{d}_x\in L^2(T\mathrm{X})$ belongs to $D(\mathbf{div}_\mathfrak{m})$.*

Proof. Since $|\psi\nabla\mathsf{d}_x|\leq|\psi|$, it is clear that $\psi\nabla\mathsf{d}_x\in L^2(T\mathrm{X})$. Theorem 5.14 above, the very definition of measure-valued divergence and the Leibniz rule given in Proposition 4.16 ensure that $\psi\nabla\mathsf{d}_x\in D\big(\mathbf{div}_\mathfrak{m},\mathrm{X}\setminus\{x\}\big)$. On the other hand, the vector field $\psi\nabla\mathsf{d}_x$ is equal to 0 on some neighbourhood of x by construction, therefore it has null measure-valued divergence in such neighbourhood. Then the conclusion comes from Proposition 4.17. □

We now have at our disposal all the ingredients to prove the main result of this section:

Theorem 5.16 *Given any $\varepsilon \in (0,1/N)$, let us consider $(U^\varepsilon_{i,j},\varphi^\varepsilon_{i,j})_{i,j}$ as in Theorem 5.12. Then it holds that*

$$(\varphi^\varepsilon_{i,j})_*(\mathfrak{m}|_{U^\varepsilon_{i,j}}) \ll \mathcal{L}^i \qquad \textit{for every } i,j. \tag{5.17}$$

In particular, any $\mathsf{RCD}(K,N)$ *space – with $K \in \mathbb{R}$ and $N \in (1,\infty)$ – is strongly* $\mathfrak{m}$*-rectifiable.*

Proof. By inner regularity of $\mathfrak{m}$ applied to the sets $U^\varepsilon_{i,j} \setminus \{x^\varepsilon_{i,j,1},\ldots,x^\varepsilon_{i,j,i}\}$, we can assume that the $U^\varepsilon_{i,j}$'s are compact and that $x^\varepsilon_{i,j,k} \notin U^\varepsilon_{i,j}$ for every $k=1,\ldots,i$. Now fix i,j. For the sake of brevity, we write $\varphi,U,x_1,\ldots,x_i$ in place of $\varphi^\varepsilon_{i,j},U^\varepsilon_{i,j},x^\varepsilon_{i,j,1},\ldots,x^\varepsilon_{i,j,i}$, respectively.
STEP 1. Let $(\psi^\delta)_{\delta>0}$ be a family of Lipschitz $[0,1]$-valued maps on X with compact support, which pointwise converge to χ_U as $\delta \searrow 0$. Consider the vector fields

$$v^\delta_k \doteq \psi^\delta \nabla \mathsf{d}_{x_k} \in L^2(T\mathrm{X}) \qquad \text{for every } k=1,\ldots,i.$$

By Corollary 5.15, we know that $v^\delta_k \in D(\mathbf{div}_\mathfrak{m})$. Now observe that $\varphi : \mathrm{X} \to \mathbb{R}^i$ is a Lipschitz and proper map (i.e. the preimage of a compact set is compact), thus $\mu \doteq \varphi_*\mathfrak{m}$ is a Radon measure on $\mathbb{R}^i$. If we equip $\mathbb{R}^i$ with such measure we have that $\varphi : \mathrm{X} \to \mathbb{R}^i$ has bounded deformation. By Proposition 4.38 we know that the vector fields

$$u^\delta_k \doteq \mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(v^\delta_k)\big) \in L^2_\mu(T\mathbb{R}^i)$$

belong to $D(\mathbf{div}_\mu)$. Let us consider the embedding $\iota : L^2_\mu(T\mathbb{R}^d) \to L^2(\mathbb{R}^d,\mathbb{R}^d;\mu)$ introduced in Theorem 4.27 and the map $\mathcal{I}$ defined in (4.40). Since the $\iota(u^\delta_k)$'s have compact support, we see from Corollary 4.30 that the 1-dimensional currents

$$\mathcal{I}(u^\delta_k) = \overrightarrow{\mathcal{I}(u^\delta_k)}\,\big\|\mathcal{I}(u^\delta_k)\big\| = \frac{\iota(u^\delta_k)}{|u^\delta_k|}\,(|u^\delta_k|\,\mu) \qquad \text{for any } k=1,\ldots,i$$

are normal. We also notice that it trivially holds

$$\mu|_{\{|u^\delta_k|>0\}} \ll |u^\delta_k|\mu = \big\|\mathcal{I}(u^\delta_k)\big\| \qquad \text{for every } k=1,\ldots,i. \tag{5.18}$$

STEP 2. We claim that:

$$\text{The vector fields } \nabla\mathsf{d}_{x_1},\ldots,\nabla\mathsf{d}_{x_i} \in L^2_{\mathrm{loc}}(T\mathrm{X}) \text{ are independent on } U. \tag{5.19}$$

To prove it we shall use the hypothesis $\varepsilon < 1/N$. Let $f_1,\ldots,f_i \in L^\infty(\mathfrak{m})$ be chosen so that the identity $\sum_{k=1}^i f_k\,\nabla\mathsf{d}_{x_k} = 0$ is verified $\mathfrak{m}$-a.e. on U and observe that

$$0 = \Big\langle \nabla\mathsf{d}_{x_k}, \sum_{k'=1}^i f_{k'}\,\nabla\mathsf{d}_{x_{k'}} \Big\rangle = f_k\,|D\mathsf{d}_{x_k}|^2 + \sum_{k'\neq k} f_{k'}\,\langle \nabla\mathsf{d}_{x_k},\nabla\mathsf{d}_{x_{k'}}\rangle \qquad \mathfrak{m}\text{-a.e. on } U.$$

From (5.12), (5.13) and the fact that $\varepsilon < 1/N$, we deduce that

$$|f_k| = |f_k|\,|D\mathsf{d}_{x_k}|^2 \le \sum_{k'\neq k} |f_{k'}|\,\big|\langle \nabla\mathsf{d}_{x_k},\nabla\mathsf{d}_{x_{k'}}\rangle\big| \le \frac{1}{N}\sum_{k'\neq k}|f_{k'}| \qquad \mathfrak{m}\text{-a.e. on } U.$$

By adding up in $k=1,\ldots,i$, we infer that $\sum_k |f_k| \le \frac{i-1}{N}\sum_k |f_k|$ holds $\mathfrak{m}$-a.e. on U. Since by Theorem 5.12 we have $i \le N$, this forces $\sum_k |f_k| = 0$ to hold $\mathfrak{m}$-a.e. on U, which is (5.19).

Now notice that Theorem 5.12 grants that $\varphi : \mathrm{X} \to \mathbb{R}^i$ is of bounded deformation (having equipped $\mathbb{R}^i$ with the measure $\mu = \varphi_* \mathfrak{m}$), partially invertible on U and such that $\left(\varphi|_U\right)^{-1}$ is Lipschitz. Recall that Lipschitz functions are dense in $W^{1,2}(\mathrm{X})$, cf. Theorem 2.27. Therefore Proposition 4.37 grants that the vector fields

$$u_k^0 \doteq \mathsf{Pr}_\varphi\big(\mathrm{d}\varphi(\chi_U \nabla \mathsf{d}_{x_k})\big) \in L^2_\mu(T\mathbb{R}^i) \qquad \text{for any } k = 1, \ldots, i$$

are independent on $\mathsf{Im}_\varphi(U)$, which by Corollary 4.28 implies that

$$\iota(u_1^0)(x), \ldots, \iota(u_i^0)(x) \in \mathbb{R}^i \text{ are linearly independent for } \mu\text{-a.e. } x \in \mathsf{Im}_\varphi(U). \tag{5.20}$$

Step 3. The fact that the family $(\psi^\delta)_{\delta>0}$ is equibounded in $L^\infty(\mathfrak{m})$ and pointwise converging to χ_U implies that $v_k^\delta \to \chi_U \nabla \mathsf{d}_{x_k}$ in $L^2(T\mathrm{X})$ for every $k = 1, \ldots, i$. By continuity of $\mathrm{d}\varphi$ and of Pr_φ, we deduce that for any $k = 1, \ldots, i$ we have $u_k^\delta \to u_k^0$ in $L^2_\mu(T\mathbb{R}^i)$ as $\delta \searrow 0$, thus Theorem 4.27 grants that $\iota(u_k^\delta) \to \iota(u_k^0)$ in $L^2(\mathbb{R}^i, \mathbb{R}^i; \mu)$ as $\delta \searrow 0$. For any $\delta \geq 0$, let us define the Borel set $A^\delta \subseteq \mathbb{R}^i$ – up to μ-negligible sets – as

$$A^\delta \doteq \Big\{ x \in \mathbb{R}^i \;\Big|\; ; \iota(u_1^\delta)(x), \ldots, \iota(u_i^\delta)(x) \text{ are linearly independent} \Big\}.$$

Since being an independent family is an open condition, the convergence just proved ensures that for any $\delta_n \searrow 0$ we have

$$\mu\Big(A^0 \setminus \bigcup_{n \in \mathbb{N}} A^{\delta_n}\Big) = 0. \tag{5.21}$$

For any given $\delta > 0$, we apply Theorem 5.13 to the currents $\mathcal{I}(u_k^\delta)$: since the vectors $\iota(u_k^\delta)$ are all nonzero μ-a.e. on A^δ, we have that $\mu|_{A^\delta} \ll \mu|_{\{|u_k^\delta|>0\}}$ for every $k = 1, \ldots, i$, whence

$$\mu|_{A^{\delta_n}} \ll \mathcal{L}^i \qquad \text{for every } n \in \mathbb{N}$$

by (5.18) and Theorem 5.13. Then from (5.21) we deduce that

$$\mu|_{A^0} \ll \mathcal{L}^i.$$

On the other hand, property (5.20) grants that (up to μ-negligible sets) it holds $A_0 \supseteq \mathsf{Im}_\varphi(U)$, which together with the above implies that

$$\mathsf{Pr}_\varphi(\chi_U)\mu \ll \mathcal{L}^i.$$

Given that $\varphi_*\big(\mathfrak{m}|_U\big) = \mathsf{Pr}_\varphi(\chi_U)\mu$, the proof is finally achieved. □

5.3 Equivalence of two different notions of tangent module

5.3.1 Gromov-Hausdorff tangent bundle $T_{\mathrm{GH}}\mathrm{X}$

Let us fix a strongly $\mathfrak{m}$-rectifiable metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Denote by $\underline{A} = (A_k)_{k \in \mathbb{N}}$ its dimensional decomposition. Then we define the Gromov-Hausdorff tangent bundle on X as:

Definition 5.17 (GH tangent bundle) *We define the* Gromov-Hausdorff tangent bundle *of* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *as the measurable Hilbert bundle* $\mathbb{T}_{\rm GH}\mathrm{X}=\big(T_{\rm GH}\mathrm{X},\underline{A},\pi,\mathsf{n}\big)$ *on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$*, given by*

$$T_{\rm GH}\mathrm{X}\doteq\bigsqcup_{k\in\mathbb{N}}A_k\times\mathbb{R}^k\ \textit{ and }\ \mathsf{n}(x,v)\doteq|v|_{\mathbb{R}^k}\ \textit{ for every } k\in\mathbb{N} \textit{ and } (x,v)\in A_k\times\mathbb{R}^k. \tag{5.22}$$

The module $\Gamma_2(\mathbb{T}_{\rm GH}\mathrm{X})$ *of all* L^2*-sections of such bundle is called* Gromov-Hausdorff tangent module *and is denoted by* $L^2(T_{\rm GH}\mathrm{X})$*.*

The choice of the measurable structure $\mathcal{M}_{\rm GH}(\mathrm{X})\doteq\bigcap_k(\iota_k)_*\mathscr{B}(A_k\times\mathbb{R}^k)$ on $T_{\rm GH}\mathrm{X}$ – where by $\iota_k:\,A_k\times\mathbb{R}^k\hookrightarrow T_{\rm GH}\mathrm{X}$ we mean the inclusion – could seem to be naïve, but we now prove that it is the only one coherent with some (thus any) atlas on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, in the sense that we are now going to describe in details.

Fix an ε-atlas $\mathscr{A}=\big\{(U_i^k,\varphi_i^k)\big\}_{k,i}$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. For any $k,i\in\mathbb{N}$, choose $C_i^k\geq 1$ such that

$$(C_i^k)^{-1}\,\mathcal{L}^k|_{\varphi_i^k(U_i^k)}\leq(\varphi_i^k)_*\big(\mathfrak{m}|_{U_i^k}\big)\leq C_i^k\,\mathcal{L}^k|_{\varphi_i^k(U_i^k)}. \tag{5.23}$$

Fix any sequence of radii $r_j\searrow 0$ and define $\widehat{\varphi}_{ij}^k:\,U_i^k\times U_i^k\to A_k\times\mathbb{R}^k$ as

$$\widehat{\varphi}_{ij}^k(\bar{x},x)\doteq\left(\bar{x}\,,\,\frac{\varphi_i^k(x)-\varphi_i^k(\bar{x})}{r_j}\right)\qquad\text{for every } (\bar{x},x)\in U_i^k\times U_i^k. \tag{5.24}$$

For the sake of brevity, for $k,i,j\in\mathbb{N}$ let us call

$$\begin{aligned}W_{ij}^k&\doteq\widehat{\varphi}_{ij}^k(U_i^k\times U_i^k),\\ W^k&\doteq\bigcup\nolimits_{i,j\in\mathbb{N}}W_{ij}^k\end{aligned} \tag{5.25}$$

and notice that simple computations yield

$$\begin{gathered}\widehat{\varphi}_{ij}^k:\,U_i^k\times U_i^k\to W_{ij}^k\qquad\text{is }\sqrt{1+(1+\varepsilon)^2/(r_j)^2}\text{-biLipschitz},\\ \frac{(r_j)^k}{C_i^k}\,(\mathfrak{m}\otimes\mathcal{L}^k)|_{W_{ij}^k}\leq\big(\widehat{\varphi}_{ij}^k\big)_*\big((\mathfrak{m}\otimes\mathfrak{m})|_{U_i^k\times U_i^k}\big)\leq(r_j)^k\,C_i^k\,(\mathfrak{m}\otimes\mathcal{L}^k)|_{W_{ij}^k}.\end{gathered} \tag{5.26}$$

In particular, $W_{ij}^k\in\mathscr{B}(A_k\times\mathbb{R}^k)$ for every k,i,j, thus accordingly also $W^k\in\mathscr{B}(A_k\times\mathbb{R}^k)$. Finally, let us define $N_k\doteq(A_k\times\mathbb{R}^k)\setminus W^k$.

Lemma 5.18 *With the notation just introduced, for every* $k\in\mathbb{N}$ *we have that*

$$(\mathfrak{m}\otimes\mathcal{L}^k)(N_k)=0. \tag{5.27}$$

Proof. For $k\in\mathbb{N}$, we put

$$D_k\doteq\bigcup_{i\in\mathbb{N}}\Big\{x\in U_i^k\;\Big|\;\varphi_i^k(x)\text{ is a point of density 1 for }\varphi_i^k(U_i^k)\Big\}.$$

From (5.23) and (1.65), we see that $\mathfrak{m}(A_k\setminus D_k)=0$, hence for every $i,m,h\in\mathbb{N}$ and $\bar{x}\in D_k$ there is $j\in\mathbb{N}$ such that

$$1\geq\frac{\mathcal{L}^k\Big(\frac{\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})}{r_j}\cap B_m(0)\Big)}{\mathcal{L}^k\big(B_m(0)\big)}=\frac{\mathcal{L}^k\Big(\varphi_i^k(U_i^k)\cap B_{mr_j}\big(\varphi_i^k(\bar{x})\big)\Big)}{\mathcal{L}^k\Big(B_{mr_j}\big(\varphi_i^k(\bar{x})\big)\Big)}>1-\frac{1}{h},$$

whence $\mathcal{L}^k\Big(B_m(0)\setminus\bigcup_j\big(\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})\big)/r_j\Big)=0$ for all $i,m\in\mathbb{N}$ and $\bar{x}\in D_k$. Therefore by Fubini theorem we deduce that

$$\begin{aligned}(\mathfrak{m}\otimes\mathcal{L}^k)\Big(\big(A_k\times B_m(0)\big)\setminus W^k\Big)&=\sum_{i\in\mathbb{N}}(\mathfrak{m}\otimes\mathcal{L}^k)\Big(\big(U_i^k\times B_m(0)\big)\setminus W^k\Big)\\&\leq\sum_{i\in\mathbb{N}}\int_{D_k}\mathcal{L}^k\Big(B_m(0)\setminus\bigcup_j\big(\varphi_i^k(U_i^k)-\varphi_i^k(\bar{x})\big)/r_j\Big)\,\mathrm{d}\mathfrak{m}(\bar{x})=0,\end{aligned}$$

so that $(\mathfrak{m}\otimes\mathcal{L}^k)(N_k)=\lim_m(\mathfrak{m}\otimes\mathcal{L}^k)\big(\big(A_k\times B_m(0)\big)\setminus W^k\big)=0$. □

We now endow $T_{\mathrm{GH}}\mathrm{X}$ with a new σ-algebra $\mathcal{M}\big(\mathscr{A},(r_j)\big)$, depending on the atlas $\mathscr{A}$ and the sequence $(r_j)_j$. Denote by $\bar{\iota}_k:N^k\hookrightarrow T_{\mathrm{GH}}\mathrm{X}$ the inclusion maps, then let us define

$$\mathcal{M}\big(\mathscr{A},(r_j)\big)\doteq\bigcap_{k\in\mathbb{N}}\bigg((\bar{\iota}_k)_*\mathscr{B}(N^k)\cap\bigcap_{i,j\in\mathbb{N}}(\iota_k\circ\widehat{\varphi}_{ij}^k)_*\mathscr{B}(U_i^k\times U_i^k)\bigg).\tag{5.28}$$

Equivalently, a subset E of $T_{\mathrm{GH}}\mathrm{X}$ belongs to $\mathcal{M}\big(\mathscr{A},(r_j)\big)$ if and only if $E\cap N^k\in\mathscr{B}(N^k)$ for every $k\in\mathbb{N}$ and $(\widehat{\varphi}_{ij}^k)^{-1}\big(E\cap(A_k\times\mathbb{R}^k)\big)\in\mathscr{B}(U_i^k\times U_i^k)$ for every k,i,j.

Finally, the fact that our choice of the σ-algebra $\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ on $T_{\mathrm{GH}}\mathrm{X}$ is canonical is encoded in the following proposition:

Proposition 5.19 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable metric measure space, let* $\mathscr{A}$ *an* ε*-atlas and let* $r_j\searrow 0$ *be a given sequence. Then*

$$\mathcal{M}_{\mathrm{GH}}(\mathrm{X})=\mathcal{M}\big(\mathscr{A},(r_j)\big).\tag{5.29}$$

Proof. If $E\in\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$ then $\iota_k^{-1}(E)\in\mathscr{B}(A_k\times\mathbb{R}^k)$ for every $k\in\mathbb{N}$, so accordingly $E\cap N^k$ belongs to $\mathscr{B}(N^k)$ and $(\widehat{\varphi}_{ij}^k)^{-1}\big(\iota_k^{-1}(E)\big)$ belongs to $\mathscr{B}(U_i^k\times U_i^k)$ for every k,i,j, which proves that $E\in\mathcal{M}\big(\mathscr{A},(r_j)\big)$.

Conversely, let $E\in\mathcal{M}\big(\mathscr{A},(r_j)\big)$. Hence $E\cap N^k\in\mathscr{B}(N^k)\subseteq\mathscr{B}(A_k\times\mathbb{R}^k)$, while

$$F_{ij}^k\doteq(\widehat{\varphi}_{ij}^k)^{-1}\big(\iota_k^{-1}(E)\big)\in\mathscr{B}(U_i^k\times U_i^k)$$

implies that $E\cap W_{ij}^k=\widehat{\varphi}_{ij}^k(F_{ij}^k)\in\mathscr{B}(A_k\times\mathbb{R}^k)$. Thus

$$\iota_k^{-1}(E)=(E\cap N^k)\cup\bigcup_{i,j}(E\cap W_{ij}^k)\in\mathscr{B}(A_k\times\mathbb{R}^k)$$

for every $k\in\mathbb{N}$, which is equivalent to saying that $E\in\mathcal{M}_{\mathrm{GH}}(\mathrm{X})$. □

Remark 5.20 This last proposition does not use the strong $\mathfrak{m}$-rectifiability of the space but only its $\mathfrak{m}$-rectifiability, as seen by the fact that we did not consider any sequence of ε_n-atlases. We chose this presentation because the reason for the introduction of the Gromov-Hausdorff tangent module is in the statement contained in Subsection 5.3.2, which grants that the space of its sections is isometric to the abstract tangent module $L^2(T\mathrm{X})$ – a result that we can have only for strongly $\mathfrak{m}$-rectifiable spaces. ■

5.3.2 Equivalence between $L^2(T\mathrm{X})$ and $L^2(T_{\mathrm{GH}}\mathrm{X})$

The main result of this section is the following: the two different notions of tangent modules described so far – namely the 'analytic' tangent module $L^2(T\mathrm{X})$ and the 'geometric' Gromov-Hausdorff tangent module $L^2(T_{\mathrm{GH}}\mathrm{X})$ – can be actually identified. More precisely, given a strongly $\mathfrak{m}$-rectifiable space X whose associated Sobolev space is reflexive, there exists an isomorphism between $L^2(T\mathrm{X})$ and $L^2(T_{\mathrm{GH}}\mathrm{X})$ which preserves the pointwise norm. Moreover, such isomorphism can be canonically chosen once an aligned sequence of atlases is given.

Notice that Theorem 5.4 (which is valid on more general $\mathfrak{m}$-rectifiable spaces) is equivalent to the fact that there exists a morphism of $L^2(T\mathrm{X})$ into $L^2(T_{\mathrm{GH}}\mathrm{X})$ with continuous inverse, thus in particular changing the pointwise norm by a bounded factor. Thus Theorem 5.21 below can be seen as the improvement of Theorem 5.4, which shows that for strongly $\mathfrak{m}$-rectifiable spaces such factor can be taken to be 1.

Theorem 5.21 (Equivalence of $L^2(T\mathrm{X})$ and $L^2(T_{\mathrm{GH}}\mathrm{X})$) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any strongly* $\mathfrak{m}$-*rectifiable space such that* $W^{1,2}(\mathrm{X})$ *is reflexive. Then there exists an isometric isomorphism of modules* $\mathscr{I}: L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$, *so that in particular it holds that*

$$\big|\mathscr{I}(\mathsf{v})\big| = |\mathsf{v}| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X} \qquad \text{for every } \mathsf{v}\in L^2(T\mathrm{X}). \tag{5.30}$$

Proof. Consider an aligned family $(\mathscr{A}_n)_n$ of atlases $\mathscr{A}_n = \big\{(U_i^{k,n},\varphi_i^{k,n})\big\}_{k,i}$ on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, of parameters $\varepsilon_n \doteq 1/2^n$ and $\delta_n \doteq 1/2^n$, whose existence is guaranteed by Theorem 5.9. Now let $\mathsf{v}\in L^2(T\mathrm{X})$ and $n\in\mathbb{N}$ be fixed. For $k,i\in\mathbb{N}$, we put $V_i^{k,n} \doteq \varphi_i^{k,n}(U_i^{k,n}) \in \mathscr{B}(\mathbb{R}^k)$. Recall that $\varphi_i^{k,n}: U_i^{k,n}\to V_i^{k,n}$ and its inverse are maps of bounded deformation. Thus it makes sense to consider $\widehat{\mathrm{d}}\varphi_i^{k,n}(\chi_{U_i^{k,n}}\mathsf{v}) \in L^2(V_i^{k,n},\mathbb{R}^k)$ (recall Theorem 4.39) and we can define

$$\mathsf{w}_i^{k,n}(x) \doteq \begin{cases} \big(\widehat{\mathrm{d}}\varphi_i^{k,n}(\chi_{U_i^{k,n}}\mathsf{v})\big)\big(\varphi_i^{k,n}(x)\big) & \text{for } \mathfrak{m}\text{-a.e. } x\in U_i^{k,n},\\ 0 & \text{for } \mathfrak{m}\text{-a.e. } x\in \mathrm{X}\setminus U_i^{k,n}. \end{cases}$$

The bound (4.58) gives

$$|\mathsf{w}_i^{k,n}|(x) \le \mathrm{Lip}(\varphi_i^{k,n})\,|\mathsf{v}|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in U_i^{k,n}, \tag{5.31}$$

so that $\|\mathsf{w}_i^{k,n}\|_{L^2(T_{\mathrm{GH}}\mathrm{X})} \le (1+2^{-n})\big\||\mathsf{v}|\big\|_{L^2(U_i^{k,n})}$. In particular, the series $\sum_{i,k}\mathsf{w}_i^{k,n}$ converges in $L^2(T_{\mathrm{GH}}\mathrm{X})$ to some vector field $\mathscr{I}_n(\mathsf{v})$ whose norm is bounded by $(1+2^{-n})\big\||\mathsf{v}|\big\|_{L^2(\mathrm{X})}$ and that satisfies

$$\chi_{U_i^{k,n}}\,\mathscr{I}_n(\mathsf{v}) = \mathsf{w}_i^{k,n} \qquad \text{for every } k,i\in\mathbb{N}. \tag{5.32}$$

It is then clear that $\mathscr{I}_n: L^2(T\mathrm{X})\to L^2(T_{\mathrm{GH}}\mathrm{X})$ is $L^\infty(\mathfrak{m})$-linear, continuous and satisfying the inequality $\big|\mathscr{I}_n(\mathsf{v})\big| \le \big(1+2^{-n}\big)\,|\mathsf{v}|$ in the $\mathfrak{m}$-a.e. sense for all $\mathsf{v}\in L^2(T\mathrm{X})$. We claim that:

(5.33) The sequence $(\mathscr{I}_n)_n$ is Cauchy with respect to the operator norm.

In order to prove this, let us take $\mathsf{v}\in L^2(T\mathrm{X})$ and $k,i,j\in\mathbb{N}$ with $U_i^{k,n+1}\subseteq U_j^{k,n}$. For $\mathfrak{m}$-a.e.

point $x \in U_i^{k,n+1}$, putting for brevity $y \doteq \varphi_i^{k,n+1}(x)$, it holds that

$$\begin{aligned}
\big|\mathscr{I}_{n+1}(\mathsf{v}) - \mathscr{I}_n(\mathsf{v})\big|(x) &= \Big|\Big(\widehat{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big)\big(\varphi_i^{k,n+1}(x)\big) - \Big(\widehat{\mathrm{d}}\varphi_j^{k,n}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big)\big(\varphi_j^{k,n}(x)\big)\Big| \\
((4.57),(4.58)) \qquad &\leq \Big\|\mathrm{d}\Big(\mathrm{id}_{V_i^{k,n+1}} - \varphi_j^{k,n}\circ\big(\varphi_i^{k,n+1}\big)^{-1}\Big)(y)\Big\|\,\Big|\widehat{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big|(y) \\
(\delta_{n+1} = 2^{-n-1}) \qquad &\leq \frac{1}{2^{n+1}}\,\Big|\widehat{\mathrm{d}}\varphi_i^{k,n+1}\big(\chi_{U_i^{k,n+1}}\mathsf{v}\big)\Big|\big(\varphi_i^{k,n+1}(x)\big) \\
(\varepsilon_{n+1} = 2^{-n-1}) \qquad &\leq \frac{1}{2^{n+1}}\left(1 + \frac{1}{2^{n+1}}\right)|\mathsf{v}|(x) \leq \frac{1}{2^n}\,|\mathsf{v}|(x).
\end{aligned}$$

It follows that $\big\|\mathscr{I}_{n+1}(\mathsf{v}) - \mathscr{I}_n(\mathsf{v})\big\|_{L^2(T_{\mathrm{GH}}\mathrm{X})} \leq 2^{-n}\,\|\mathsf{v}\|_{L^2(T\mathrm{X})}$, which by arbitrariness of v means that it holds

$$\big\|\mathscr{I}_{n+1} - \mathscr{I}_n\big\| \leq \frac{1}{2^n},$$

where the norm in the left hand side is the operator one. Hence $\sum_{n=0}^{\infty}\big\|\mathscr{I}_{n+1} - \mathscr{I}_n\big\| < +\infty$ and the claim (5.33) is proved.

Let $\mathscr{I} : L^2(T\mathrm{X}) \to L^2(T_{\mathrm{GH}}\mathrm{X})$ be the limit of $(\mathscr{I}_n)_n$ and notice that – being the limit of $L^\infty(\mathfrak{m})$-linear maps – it is also $L^\infty(\mathfrak{m})$-linear. Moreover, the fact that $\mathscr{I}_n(\mathsf{v}) \to \mathscr{I}(\mathsf{v})$ strongly in $L^2(T_{\mathrm{GH}}\mathrm{X})$ implies that $\big|\mathscr{I}_n(\mathsf{v})\big| \to \big|\mathscr{I}(\mathsf{v})\big|$ in $L^2(\mathrm{X})$, whence – up to subsequences – we have that

$$\big|\mathscr{I}(\mathsf{v})\big|(x) = \lim_{n\to\infty}\big|\mathscr{I}_n(\mathsf{v})\big|(x) \leq \lim_{n\to\infty}\left(1 + \frac{1}{2^n}\right)|\mathsf{v}|(x) = |\mathsf{v}|(x) \qquad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \tag{5.34}$$

In order to prove that $\mathscr{I}$ is actually an isometric isomorphism that preserves the pointwise norm, we explicitly exhibit its inverse functional $\mathscr{J}$. In analogy with the construction just done, for any $\mathsf{w} \in L^2(T_{\mathrm{GH}}\mathrm{X})$ and $n \in \mathbb{N}$ one can build a unique $\mathscr{J}_n(\mathsf{w}) \in L^2(T\mathrm{X})$ such that

$$\chi_{U_i^{k,n}}\,\mathscr{J}_n(\mathsf{w}) = \big(\widehat{\mathrm{d}}\varphi_i^{k,n}\big)^{-1}\Big(\mathsf{w}\circ\big(\varphi_i^{k,n}\big)^{-1}\Big) \qquad \text{for every } k,i \in \mathbb{N}. \tag{5.35}$$

By means of the same arguments used above, we can prove that $\mathscr{J}_n : L^2(T_{\mathrm{GH}}\mathrm{X}) \to L^2(T\mathrm{X})$ is $L^\infty(\mathfrak{m})$-linear continuous and converges to a limit functional $\mathscr{J} : L^2(T_{\mathrm{GH}}\mathrm{X}) \to L^2(T\mathrm{X})$ in the operator norm as $n \to \infty$. The operator $\mathscr{J}$ is $L^\infty(\mathfrak{m})$-linear and satisfies

$$\big|\mathscr{J}(\mathsf{w})\big| \leq |\mathsf{w}| \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } \mathsf{w} \in L^2(T_{\mathrm{GH}}\mathrm{X}). \tag{5.36}$$

Our aim is now to prove that $\mathscr{I}$, $\mathscr{J}$ are one the inverse of the other.

Let $\mathsf{v} \in L^2(T\mathrm{X})$ and $n \in \mathbb{N}$ be fixed. For any $k,i \in \mathbb{N}$, we have that (5.32) and (5.35) give

$$\begin{aligned}
\chi_{U_i^{k,n}}\,\mathscr{J}_n\big(\mathscr{I}_n(\mathsf{v})\big) &= \chi_{U_i^{k,n}}\,\mathscr{J}_n\big(\chi_{U_i^{k,n}}\,\mathscr{I}_n(\mathsf{v})\big) = \chi_{U_i^{k,n}}\,\mathscr{J}_n\Big(\widehat{\mathrm{d}}\varphi_i^{k,n}\big(\chi_{U_i^{k,n}}\mathsf{v}\big)\circ\varphi_i^{k,n}\Big) \\
&= \big(\widehat{\mathrm{d}}\varphi_i^{k,n}\big)^{-1}\Big(\widehat{\mathrm{d}}\varphi_i^{k,n}\big(\chi_{U_i^{k,n}}\mathsf{v}\big)\Big) = \chi_{U_i^{k,n}}\,\mathsf{v},
\end{aligned}$$

therefore $\mathscr{J}_n \circ \mathscr{I}_n = \mathrm{id}_{L^2(T\mathrm{X})}$. In an analogous way, also $\mathscr{I}_n \circ \mathscr{J}_n = \mathrm{id}_{L^2(T_{\mathrm{GH}}\mathrm{X})}$. Thus for every $n \in \mathbb{N}$ we have that

$$\begin{aligned}
\big\|\mathscr{J}\circ\mathscr{I} - \mathrm{id}_{L^2(T\mathrm{X})}\big\| &= \big\|\mathscr{J}\circ\mathscr{I} - \mathscr{J}_n\circ\mathscr{I}_n\big\| \\
&\leq \big\|\mathscr{J}\circ(\mathscr{I} - \mathscr{I}_n)\big\| + \big\|(\mathscr{J} - \mathscr{J}_n)\circ\mathscr{I}_n\big\| \\
&\leq \|\mathscr{J}\|\,\big\|\mathscr{I} - \mathscr{I}_n\big\| + \big\|\mathscr{J} - \mathscr{J}_n\big\|\,\sup_n\|\mathscr{I}_n\|,
\end{aligned}$$

so that by letting $n \to \infty$ we conclude that $\mathscr{J} \circ \mathscr{I} = \mathrm{id}_{L^2(T\mathrm{X})}$. A symmetric argument yields also the identity $\mathscr{I} \circ \mathscr{J} = \mathrm{id}_{L^2(T_{\mathrm{GH}}\mathrm{X})}$. To conclude, note that for any $\mathsf{v} \in L^2(T\mathrm{X})$ we have

$$|\mathsf{v}| = \big|\mathscr{J}\big(\mathscr{I}(\mathsf{v})\big)\big| \overset{(5.36)}{\leq} \big|\mathscr{I}(\mathsf{v})\big| \overset{(5.34)}{\leq} |\mathsf{v}| \qquad \mathfrak{m}\text{-a.e. in } \mathrm{X}.$$

Hence all inequalities are actually equalities, yielding (5.30) and the conclusion. □

Corollary 5.22 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable space such that* $W^{1,2}(\mathrm{X})$ *is reflexive. Then* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is infinitesimally Hilbertian.*

Proof. One can readily show that any couple of vector fields $v, w \in L^2(T_{\mathrm{GH}}\mathrm{X})$ satisfies (3.11). Hence the space $L^2(T\mathrm{X})$ is a Hilbert module, as a consequence of Theorem 5.21. Therefore the metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is infinitesimally Hilbertian, as required. □

5.3.3 Geometric interpretation of $T_{\mathrm{GH}}\mathrm{X}$

Let us now focus on metric measure spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$ satisfying the following properties:

$$\begin{aligned}&(\mathrm{X},\mathsf{d},\mathfrak{m}) \text{ is a strongly } \mathfrak{m}\text{-rectifiable space which satisfies (1.66)},\\ &\text{having constant dimension } k \in \mathbb{N} \text{ and whose reference measure is}\\ &\text{given by } \mathfrak{m} = \theta\,\mathcal{H}^k, \text{ for some continuous density } \theta : \mathrm{X} \to (0,+\infty).\end{aligned} \tag{5.37}$$

Consider a family $\mathscr{A}_n = \big\{(U_i^n, \varphi_i^n)\big\}_{i\in\mathbb{N}}$ of ε_n-atlases on $(\mathrm{X},\mathsf{d},\mathfrak{m})$, with compact domains U_i^n. We can use the atlases to build Borel maps $\Psi_n : \mathrm{X} \times \big(\frac{1}{r_n}\mathrm{X}\big) \to T_{\mathrm{GH}}\mathrm{X}$ which are 'bundle maps', i.e. which fix the first coordinate and that are approximate isometries as maps on the second variable, in the following way. We first recall that for any closed subset U of X there exists a Borel map $P_U : \mathrm{X} \to U$ such that

$$\mathsf{d}\big(x, P_U(x)\big) \leq 2\,\mathsf{d}(x,U) \qquad \text{for every } x \in \mathrm{X}.$$

This can be built – for instance – by first considering a countable dense subset $(x_n)_n$ of U, then by declaring $P_U(x) \doteq x$ for $x \in U$ and

$$P_U(x) \doteq x_n, \qquad \text{where } n \text{ is the least number such that } \mathsf{d}(x,x_n) \leq 2\,\mathsf{d}(x,U),$$

for $x \notin U$. Then given a sequence $r_n \searrow 0$ we set

$$\Phi_n(x,y) \doteq \frac{\varphi_i^n\big(P_{U_i^n}(y)\big) - \varphi_i^n(x)}{r_n} \in \mathbb{R}^k \qquad \text{for every } x \in U_i^n \text{ and } y \in \mathrm{X}, \tag{5.38}$$

while $\Phi_n(x,y) \doteq 0_{\mathbb{R}^k}$ if $x \notin \bigcup_i U_i^n$. Finally, we define

$$\Psi_n(x,y) \doteq \big(x, \Phi_n(x,y)\big) \qquad \text{for every } x,y \in \mathrm{X}.$$

Notice that the map Ψ_n is Borel for every $n \in \mathbb{N}$. In the next theorem we show that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ the maps $y \mapsto \Phi_n(x,y)$ provide approximate measured isometries from X rescaled by a factor $\frac{1}{r_n}$ to $\mathbb{R}^k$, thus showing not only that the tangent space of X at x is $\mathbb{R}^k$, but also that there is a 'compatible' choice of approximate isometries making the resulting global maps (i.e. Ψ_n) Borel.

Theorem 5.23 *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a space satisfying* (5.37)*. Fix* $\varepsilon_n \searrow 0$*. Let* $\mathscr{A}_n = \big\{(U_i^n,\varphi_i^n)\big\}_i$ *be a family of* ε_n*-atlases with compact domains* U_i^n*. Then there exists a sequence* $r_n \searrow 0$ *such that, defining* Φ_n *as in* (5.38)*, for* $\mathfrak{m}$*-a.e.* $x\in\mathrm{X}$ *the following holds: for every* $R>\varepsilon>0$ *there is* $\bar n\in\mathbb{N}$ *so that for every* $n\geq\bar n$ *we have*

$$
\begin{aligned}
&\left|\big|\Phi_n(x,y_0)-\Phi_n(x,y_1)\big|_{\mathbb{R}^k}-\frac{\mathsf{d}(y_0,y_1)}{r_n}\right|\leq\varepsilon &&\text{for every } y_0,y_1\in B_{r_nR}(x),\\
&B_{R-\varepsilon}(0_{\mathbb{R}^k})\subseteq B_\varepsilon\big(\big\{\Phi_n(x,y)\,:\,y\in B_{r_nR}(x)\big\}\big),\\
&\Phi_n(x,\cdot)_*\big(\mathfrak{m}^x_{r_n}|_{B_{r_nR}(x)}\big)\rightharpoonup\omega_k^{-1}\,\mathcal{L}^k|_{B_R(0)} &&\text{as } n\to\infty.
\end{aligned}
\tag{5.39}
$$

In particular, the space $\big(\mathrm{X},\mathsf{d}/r_n,\mathfrak{m}^x_{r_n},x\big)$ *pmGH-converges to* $\big(\mathbb{R}^k,\mathsf{d}_{\mathbb{R}^k},\mathcal{L}^k/\omega_k,0\big)$ *as* $n\to\infty$*.*

Proof. For any $i,n\in\mathbb{N}$, let us set $V_i^n\doteq\varphi_i^n(U_i^n)$. From (5.3) we see that for $\mathfrak{m}$-a.e. $x\in U_i^n$ the point $\varphi_i^n(x)$ is of density 1 for V_i^n. Let us call D' the set of all the points $x\in\mathrm{X}$ that satisfy $\mathcal{H}^k\big(B_r(x)\cap U^n_{i(n)}\big)/(\omega_k\,r^k)\to 1$ as $r\searrow 0$ for every $n\in\mathbb{N}$, where $i(n)\in\mathbb{N}$ is chosen so that $x\in U^n_{i(n)}$. Given that each domain U_i^n is countably $\mathcal{H}^k$-rectifiable, we deduce from Theorem B.2 that $\mathcal{H}^k(\mathrm{X}\setminus D')=0$. Hence the set

$$
D\doteq D'\cap\bigcap_n\bigcup_i\Big\{x\in U_i^n\;\Big|\;x,\ \varphi_i^n(x)\text{ are points of density 1 for } U_i^n,\ V_i^n,\text{ respectively}\Big\}
$$

is Borel and $\mathfrak{m}(\mathrm{X}\setminus D)=0$. Fix $\bar x\in D$ and $R>\varepsilon>0$. Let $i(n)\in\mathbb{N}$ be such that $\bar x\in U^n_{i(n)}$. For brevity, we call $B_n\doteq B_{r_nR}(\bar x)$, $U_n\doteq U^n_{i(n)}$, $V_n\doteq V^n_{i(n)}$ and $\varphi_n\doteq\varphi^n_{i(n)}$. Let us denote

$$
\mathrm{avg}_n\doteq\frac{1}{\mathcal{H}^k(B_n\cap U_n)}\int_{B_n\cap U_n}\theta\,\mathrm{d}\mathcal{H}^k\qquad\text{for every } n\in\mathbb{N}.
$$

STEP 1. Fix $\bar\varepsilon<\varepsilon/\max\{4R,R-\varepsilon\}$ positive and repeatedly apply property (1.66) to $\bar x$, U_n and to $\varphi_n(\bar x)$, V_n, with $\bar\varepsilon$ in place of ε, to find a sequence $r_n\searrow 0$ such that for $n\in\mathbb{N}$ it holds

$$
\begin{aligned}
\mathsf{d}\big(y,P_{U_n}(y)\big)&\leq 2\,\bar\varepsilon\,r_n\,R &&\text{for every } y\in B_n,\\
\mathsf{d}_{\mathbb{R}^k}(z,V_n)&\leq\bar\varepsilon\,\big|z-\varphi_n(\bar x)\big| &&\text{for every } z\in B_{r_nR}\big(\varphi_n(\bar x)\big).
\end{aligned}
\tag{5.40}
$$

Furthermore, since $\bar x\in D$ and the map θ is continuous, we can also require that

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{\mathcal{H}^k(B_n\cap U_n)}{\omega_k\,r_n^k\,R^k}=\lim_{n\to\infty}\frac{(1+\varepsilon_n)^k\,\mathcal{L}^k\Big(V_n\cap B_{r_nR/(1+\varepsilon_n)}\big(\varphi_n(\bar x)\big)\Big)}{\omega_k\,r_n^k\,R^k}=1,\\
&\big|\theta(x)-\mathrm{avg}_n\big|\leq\frac{1}{n}\qquad\text{for every } n\in\mathbb{N}^+\text{ and } x\in B_n\cap U_n,\\
&\lim_{n\to\infty}\frac{\mathfrak{m}(B_n\cap U_n)}{\mathfrak{m}\big(B_{r_n}(\bar x)\big)}=R^k.
\end{aligned}
\tag{5.41}
$$

From the fact that φ_n is $(1+\varepsilon_n)$-biLipschitz, we see that for any $y_0,y_1\in B_n$ it holds that

$$
\begin{aligned}
\big|\Phi_n(\bar x,y_0)-\Phi_n(\bar x,y_1)\big|_{\mathbb{R}^k}&\leq\frac{1+\varepsilon_n}{r_n}\,\mathsf{d}\big(P_{U_n}(y_0),P_{U_n}(y_1)\big)\\
\text{(by (5.40))}\qquad&\leq\frac{1+\varepsilon_n}{r_n}\,\big(\mathsf{d}(y_0,y_1)+4\,\bar\varepsilon\,r_n\,R\big).
\end{aligned}
$$

Similarly, we get $\big|\Phi_n(\bar x,y_0)-\Phi_n(\bar x,y_1)\big|_{\mathbb{R}^k}\geq\frac{1}{(1+\varepsilon_n)r_n}\big(\mathsf{d}(y_0,y_1)-4\,\bar\varepsilon\,r_n\,R\big)$, thus

$$
\left|\big|\Phi_n(\bar x,y_0)-\Phi_n(\bar x,y_1)\big|-\frac{\mathsf{d}(y_0,y_1)}{r_n}\right|\leq 2\,R\,\max\left\{2(1+\varepsilon_n)\bar\varepsilon+\varepsilon_n,\frac{2\bar\varepsilon+\varepsilon_n}{1+\varepsilon_n}\right\}
$$

for every $y_0, y_1 \in B_{r_n R}(\bar{x})$. Since $\bar{\varepsilon} < \varepsilon/(4R)$, this is sufficient to show that the first claim in (5.39) is fulfilled for n large enough.

STEP 2. For the second condition in (5.39), pick any $w \in \mathbb{R}^k$ such that $|w| < R - \varepsilon$ and let us put $z_n \doteq \varphi_n(\bar{x}) + r_n w$. Thus the point z_n belongs to $B_{r_n R}\big(\varphi_n(\bar{x})\big)$. From the second line in (5.40) and the compactness of U_n, we deduce that there exists $y_n \in U_n$ such that

$$\big|z_n - \varphi_n(y_n)\big| \leq \bar{\varepsilon}\, r_n |w|. \tag{5.42}$$

Since the right hand side is bounded from above by $\bar{\varepsilon}\, r_n\, R$, for n sufficiently large it is bounded above by ε, so that to conclude it suffices to show that – independently on the choice of w – for all n sufficiently large it holds that $y_n \in B_n$. To see this, recall that the inverse of the map φ_n is $(1+\varepsilon_n)$-Lipschitz to get that

$$\begin{aligned} \mathsf{d}(\bar{x}, y_n) &\leq (1+\varepsilon_n)\, \big|\varphi_n(\bar{x}) - \varphi_n(y_n)\big| \leq (1+\varepsilon_n)\Big(\big|\varphi_n(\bar{x}) - z_n\big| + \big|z_n - \varphi_n(y_n)\big|\Big) \\ \text{(by (5.42))} \qquad &\leq r_n(1+\varepsilon_n)(1+\bar{\varepsilon})\,|w| \leq r_n(1+\varepsilon_n)(1+\bar{\varepsilon})(R-\varepsilon). \end{aligned}$$

Since $\bar{\varepsilon} < \varepsilon/(R-\varepsilon)$ we have that $(1+\bar{\varepsilon})(R-\varepsilon) < R$, therefore for n sufficiently large we have that $r_n(1+\varepsilon_n)(1+\bar{\varepsilon})(R-\varepsilon) < r_n R$, which concludes the proof of the second condition in (5.39).

STEP 3. Let us now denote $\psi_n \doteq \varphi_n \circ P_{U_n} - \varphi_n(\bar{x})$, so that $\Phi_n(\bar{x}, \cdot) = \psi_n / r_n$. We have that

$$\mathcal{L}^k\left(\frac{\psi_n(B_n \cap U_n)}{r_n}\, \Delta\, B_R(0)\right) \longrightarrow 0 \qquad \text{when } n \to \infty, \tag{5.43}$$

as one can easily prove by using (5.41), which grants that $\mathcal{H}^k(B_n \cap U_n)/\mathcal{H}^k(B_n) \to 1$.

To prove the third claim in (5.39), fix $f \in C_c(\mathbb{R}^k)$. Observe that $\int f \,\mathrm{d}\Phi_n(\bar{x}, \cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big)$ can be written as $Q_1(n)\,\mathfrak{m}(B_n \cap U_n)/\mathfrak{m}\big(B_{r_n}(\bar{x})\big) + Q_2(n) + Q_3(n)$, where

$$\begin{aligned} Q_1(n) &\doteq \frac{1}{\mathcal{H}^k(B_n \cap U_n)} \int f(\,\cdot\,/r_n)\, \mathrm{d}(\psi_n)_*\big(\mathcal{H}^k|_{B_n \cap U_n}\big), \\ Q_2(n) &\doteq \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \int_{B_n \cap U_n} f \circ \Phi_n(\bar{x}, \cdot)\, (\theta - \mathrm{avg}_n)\, \mathrm{d}\mathcal{H}^k, \\ Q_3(n) &\doteq \frac{1}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \int_{B_n \setminus U_n} f \circ \Phi_n(\bar{x}, \cdot)\, \mathrm{d}\mathfrak{m}. \end{aligned}$$

First of all, it directly follows from the last two statements in (5.41) that

$$\begin{aligned} \big|Q_2(n)\big| &\leq \frac{1}{n}\, \frac{\mathfrak{m}(B_n \cap U_n)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \max_{\mathbb{R}^k} |f| \longrightarrow 0, \\ \big|Q_3(n)\big| &\leq \frac{\mathfrak{m}(B_n \setminus U_n)}{\mathfrak{m}\big(B_{r_n}(\bar{x})\big)} \max_{\mathbb{R}^k} |f| \longrightarrow 0. \end{aligned} \tag{5.44}$$

Moreover, (B.6) yields $(1+\varepsilon_n)^{-k}\, \mathcal{L}^k|_{\psi_n(B_n \cap U_n)} \leq (\psi_n)_*\big(\mathcal{H}^k|_{B_n \cap U_n}\big) \leq (1+\varepsilon_n)^k\, \mathcal{L}^k|_{\psi_n(B_n \cap U_n)}$, thus accordingly it holds that

$$\frac{(1+\varepsilon_n)^{-k}\, r_n^k}{\mathcal{H}^k(B_n \cap U_n)} \int_{\frac{\psi_n(B_n \cap U_n)}{r_n}} f \,\mathrm{d}\mathcal{L}^k \leq Q_1(n) \leq \frac{(1+\varepsilon_n)^k\, r_n^k}{\mathcal{H}^k(B_n \cap U_n)} \int_{\frac{\psi_n(B_n \cap U_n)}{r_n}} f \,\mathrm{d}\mathcal{L}^k. \tag{5.45}$$

Finally, by recalling (5.43) we can immediately deduce that

$$\left| \int_{\frac{\psi_n(B_n \cap U_n)}{r_n}} f \,\mathrm{d}\mathcal{L}^k - \int_{B_R(0)} f \,\mathrm{d}\mathcal{L}^k \right| \leq \mathcal{L}^k\left(\frac{\psi_n(B_n \cap U_n)}{r_n}\, \Delta\, B_R(0)\right) \max_{\mathbb{R}^k} |f| \longrightarrow 0.$$

Therefore the first line in (5.41) gives $Q_1(n) \to (\omega_k\, R^k)^{-1}\int_{B_R(0)} f\,\mathrm{d}\mathcal{L}^k$, which together with (5.44) and the third line in (5.41) grant that $\omega_k^{-1}\int_{B_R(0)} f\,\mathrm{d}\mathcal{L}^k = \lim_n \int f\,\mathrm{d}\Phi_n(\bar{x},\cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big)$. This means that $\Phi_n(x,\cdot)_*\big(\mathfrak{m}^{\bar{x}}_{r_n}|_{B_n}\big) \rightharpoonup \omega_k^{-1}\,\mathcal{L}^k|_{B_R(0)}$, thus proving the statement. □

By putting together several results obtained so far, it is then easy to prove the following:

Theorem 5.24 (Euclidean tangent cone) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a strongly* $\mathfrak{m}$*-rectifiable space, whose dimensional decomposition is denoted by* $(A_k)_{k\in\mathbb{N}}$*. Then for every* $k\in\mathbb{N}$ *it holds that*

$$\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},x] = \Big\{\big[\mathbb{R}^k,\mathsf{d}_{\mathbb{R}^k},\mathcal{L}^k/\omega_k,0\big]\Big\} \qquad \textit{for } \mathfrak{m}\textit{-a.e. } x\in A_k. \tag{5.46}$$

Proof. Let the sequence $(N_k)_k$ be as in Remark 5.2 and define $A'_k \doteq A_k\setminus N_k$ for every $k\in\mathbb{N}$. Fix $k\in\mathbb{N}$ and write $\mathfrak{m}|_{A'_k} = \theta_k\,\mathcal{H}^k|_{A'_k}$ for a suitable Borel density $\theta_k : A'_k \to (0,+\infty)$. Let

$$A^i_k \doteq \big\{x\in A'_k \;\big|\; 2^i \le \theta_k(x) < 2^{i+1}\big\} \qquad \text{for every } i\in\mathbb{Z},$$

then $(A^i_k)_i$ constitutes a Borel partition of A'_k. Thus fix $i\in\mathbb{Z}$. By arguing as in the proof of Proposition 5.6, one can see that $\lim_{r\to 0}\mathfrak{m}\big(B_r(x)\big)/(\omega_k\,r^k) = \theta_k(x)$ for $\mathfrak{m}$-a.e. $x\in A^i_k$. By applying Lusin theorem and Egorov theorem, we can cover $\mathfrak{m}$-a.a. of A^i_k with countably many compact sets $A^{ij}_k \subseteq A^i_k$, where $j\in\mathbb{N}$, in such a way that the maps $\theta_k|_{A^{ij}_k}$ are continuous and

$$\left|\frac{\mathfrak{m}\big(B_r(x)\big)}{\omega_k\,r^k} - \theta_k(x)\right| < 2^{i-1} \qquad \text{for every } x\in A^{ij}_k \text{ and } r>0 \text{ smaller than some } r^{ij}_k > 0.$$

In particular, it holds that

$$\omega_k\,r^k\,2^{i-1} < \mathfrak{m}\big(B_r(x)\big) < 5\,\omega_k\,r^k\,2^{i-1} \qquad \text{for every } x\in A^{ij}_k \text{ and } r<r^{ij}_k.$$

Therefore A^{ij}_k fulfills the hypotheses of Lemma 1.37, so accordingly each space A^{ij}_k (with the restricted distance and measure) satisfies (5.37). So Theorem 5.23 and Proposition 1.43 give

$$\mathrm{Tan}\Big[A^{ij}_k,\,\mathsf{d}|_{A^{ij}_k\times A^{ij}_k},\,\mathfrak{m}|_{A^{ij}_k},\,x\Big] = \Big\{\big[\mathbb{R}^k,\,\mathsf{d}_{\mathbb{R}^k},\,\mathcal{L}^k/\omega_k,\,0\big]\Big\} \qquad \text{for } \mathfrak{m}\text{-a.e. } x\in A^{ij}_k,$$

since $r_n \searrow 0$ in Theorem 5.23 can be actually chosen among the subsequences of any fixed sequence converging to 0 and the pmG topology is metrizable, cf. [GMS15, Theorem 3.15]. Given that $\mathfrak{m}$-a.e. point of A^{ij}_k is of density 1 for A^{ij}_k itself and $\mathfrak{m}$ is pointwise doubling at $\mathfrak{m}$-a.e point by Proposition 5.6, we deduce from Proposition 1.41 that $\big[\mathbb{R}^k,\mathsf{d}_{\mathbb{R}^k},\mathcal{L}^k/\omega_k,0\big]$ is the unique element of $\mathrm{Tan}[\mathrm{X},\mathsf{d},\mathfrak{m},x]$ for $\mathfrak{m}$-a.e. $x\in A^{ij}_k$. By arbitrariness of i and j, we finally conclude that (5.46) is satisfied, thus proving the statement. □

6

A notion of parallel transport for RCD spaces

Contents

As already seen in Section 4.2, a second-order differential calculus can be built over any $\mathsf{RCD}(K,\infty)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$. In particular, there is a well-defined notion of covariant derivative for vector fields on X. In the classical smooth Riemannian framework, covariant derivative and parallel transport are two closely related concepts, thus – given the existence of covariant derivative on RCD spaces – it is natural to ask: is there a notion of parallel transport in the same setting? In this chapter we address such question, our main results being the following:

- We provide a precise framework and give a rigorous meaning to the 'PDE' defining the parallel transport (see Definitions 6.19, 6.22 and 6.26).
- By the nature of our definition, norm-preservation and linearity of the parallel transport can be immediately derived; these in turn will give uniqueness (see Corollary 6.28).
- On RCD spaces satisfying a certain regularity property, we are able to prove existence of the parallel transport (see Subsection 6.2.2). The regularity condition that we need

concerns the existence of Sobolev vector fields with bounded covariant derivative (see Definition 6.34 for the precise assumption).

We believe that in fact the parallel transport exists on any RCD space, but we are currently unable to get the full proof – so that the theory of parallel transport on RCD spaces is still incomplete. An insight on why this should not be too easy to prove is the following: on a space where the parallel transport exists, the dimension of the tangent module must be constant (see Theorem 6.32) and thanks to the results of Chapter 5 this would in turn imply that the dimension of the pmGH-limits of the rescaled space is constant. This very same result has been extremely elusive even in the context of Ricci limit spaces and has been obtained by Colding-Naber in [CN12]; in the RCD setting, the problem of the constant dimension has been brilliantly solved by E. Brué and D. Semola in [BS18a].

Moreover, we remark that the assumptions we shall make in order to obtain existence of the parallel transport are rather *ad hoc* and not really interesting from a geometric perspective: the intent with our existence result is just to show that the approach we propose is non-void. Let us also mention that in Subsection 6.3 (see Theorem 6.39) we prove that any $\mathsf{RCD}(K,\infty)$ space admits a basis of the tangent module made of vector fields in $H^{1,2}_{\mathrm{C}}(T\mathrm{X})$. This means that if we relax the condition of 'bounded covariant derivative' into '2-integrable covariant derivative', then every RCD space meets the requirement.

Let us now briefly describe our approach. The crucial idea is that we do not study the problem of parallel transport along a single Lipschitz curve, but rather we investigate the problem along $\boldsymbol{\pi}$-a.e. curve at the same time, where $\boldsymbol{\pi}$ is a given Lipschitz test plan over X (recall Definition 4.22). The advantage of working with these plans rather than with single curves is that they are naturally linked to the Sobolev calculus and thus to the functional-analytic machinery described in Chapter 4.

Let us now pretend – for the sake of this introduction – that our space X is in fact a smooth Riemannian manifold. In this case, a time-dependent vector field (v_t) along $\boldsymbol{\pi}$ is (roughly said) given by a choice of time-dependent vector fields (v_t^γ) on X for $\boldsymbol{\pi}$-a.e. curve γ. Then we say that (v_t) is a parallel transport along $\boldsymbol{\pi}$ provided for $\boldsymbol{\pi}$-a.e. curve γ the vector field $t \mapsto v_t^\gamma(\gamma_t)$ is a parallel transport along γ. This happens if and only if

$$\text{for } \boldsymbol{\pi}\text{-a.e. } \gamma \ \text{ we have that } \partial_t v_t^\gamma + \nabla_{\gamma_t'} v_t^\gamma = 0 \ \text{ for a.e. } t.$$

A relevant part of this chapter is devoted to showing that the above PDE can be stated even in our non-smooth setting, the key point being that it is possible to define a closed operator acting on 2-integrable vector fields along $\boldsymbol{\pi}$ that plays the role of $(\partial_t + \nabla_{\gamma_t'})$; for more details, see Definitions 6.17, 6.19 and Proposition 6.20.

We conclude by recalling that A. Petrunin proved in [Pet98] that a certain notion of parallel transport exists along geodesics on Alexandrov spaces (i.e. metric spaces having a synthetic notion of sectional curvature bounded from below, see for instance [BBI01]), while uniqueness for his construction is still an open problem. It would be also interesting to compare our notion of parallel transport with the one proposed by Petrunin (we point out that any Alexandrov space is an RCD space, cf. [Pet10]).

The whole discussion contained in this chapter is taken from the paper [GP17].

6.1 Introduction of appropriate functional spaces

Throughout all this section, $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a given $\mathsf{RCD}(K,\infty)$ space for some $K\in\mathbb{R}$, while $\boldsymbol{\pi}$ is any fixed test plan over $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Recall the classes $\mathrm{TestF}(\mathrm{X})$ of test functions and $\mathrm{TestV}(\mathrm{X})$ of test vector fields, defined in (4.66) and (4.80), respectively.

6.1.1 Test vector fields along $\boldsymbol{\pi}$

To begin with, we define the space of *vector fields along* $\boldsymbol{\pi}$ as

$$\mathrm{VF}(\boldsymbol{\pi})\doteq\prod_{t\in[0,1]}\mathrm{e}_t^*L^2(T\mathrm{X}). \tag{6.1}$$

In other words, $\mathrm{VF}(\boldsymbol{\pi})$ is the collection of all maps assigning to each $t\in[0,1]$ an element of the module $\mathrm{e}_t^*L^2(T\mathrm{X})$; it is a vector space with respect to the natural pointwise operations.

To each $V\in\mathrm{VF}(\boldsymbol{\pi})$ we associate the function $[\![V]\!]:\,[0,1]\to[0,+\infty)$, defined by

$$[\![V]\!]_t\doteq\|V_t\|_{\mathrm{e}_t^*L^2(T\mathrm{X})}\qquad\text{for every } t\in[0,1]. \tag{6.2}$$

The subspace $\mathrm{TestVF}(\boldsymbol{\pi})\subset\mathrm{VF}(\boldsymbol{\pi})$ of *test vector fields along* $\boldsymbol{\pi}$ is defined as

$$\mathrm{TestVF}(\boldsymbol{\pi})\doteq\left\{t\mapsto\sum_{i=1}^n\varphi_i(t)\,\chi_{A_i}\,\mathrm{e}_t^*v_i\;\middle|\;\begin{array}{l} n\in\mathbb{N}^+,\ A_i\in\mathscr{B}\big(\Gamma(\mathrm{X})\big),\ \varphi_i\in\mathrm{LIP}\big([0,1]\big)\\ \text{and } v_i\in\mathrm{TestV}(\mathrm{X}) \text{ for every } i=1,\dots,n\end{array}\right\}. \tag{6.3}$$

Since $\mathrm{TestV}(\mathrm{X})\subseteq L^\infty(T\mathrm{X})$ by definition, we deduce that for any $V\in\mathrm{TestVF}(\boldsymbol{\pi})$ the function $(\gamma,t)\mapsto|V_t|(\gamma)$ belongs to $L^\infty(\mathcal{L}_1\times\boldsymbol{\pi})$.

Proposition 6.1 (Continuity of test vector fields along $\boldsymbol{\pi}$) *For $V,W\in\mathrm{TestVF}(\boldsymbol{\pi})$ it holds that the mapping*

$$[0,1]\ni t\longmapsto\langle V_t,W_t\rangle\in L^1(\boldsymbol{\pi})\quad\textit{is continuous.} \tag{6.4}$$

In particular, the function $[\![V]\!]:\,[0,1]\to[0,+\infty)$ is continuous for every $V\in\mathrm{TestVF}(\boldsymbol{\pi})$.

Proof. By linearity, it is clear that it is sufficient to prove the claim for vector fields V, W of the form $V=\chi_A\,\mathrm{e}_t^*v$, $W=\chi_B\,\mathrm{e}_t^*w$ – where $v,w\in\mathrm{TestV}(\mathrm{X})$. In this case, the claim (6.4) is a direct consequence of the equality

$$\langle V_t,W_t\rangle=\chi_{A\cap B}\,\langle v,w\rangle\circ\mathrm{e}_t$$

and Theorem 2.15. The last statement follows by choosing $W=V$. □

We now define two norms on the space $\mathrm{TestVF}(\boldsymbol{\pi})$:

$$\begin{aligned}\|V\|_{\mathscr{L}^2(\boldsymbol{\pi})}&\doteq\sqrt{\int_0^1[\![V]\!]_t^2\,\mathrm{d}t},\\ \|V\|_{\mathscr{C}(\boldsymbol{\pi})}&\doteq\max_{t\in[0,1]}[\![V]\!]_t.\end{aligned} \tag{6.5}$$

Proposition 6.1 ensures that $t\mapsto[\![V]\!]_t$ is Borel, thus $\|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi})}$ is well-defined; routine computations show that $\|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi})},\|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}$ are norms on $\mathrm{TestVF}(\boldsymbol{\pi})$ with $\|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi})}\le\|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}$.

We now want to show that $\big(\mathrm{TestVF}(\boldsymbol{\pi}), \|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}\big)$ is separable by exhibiting a countable dense subset. To this aim, we first choose three countable families $\mathcal{F}_1 \subseteq \big\{\text{open sets of } \Gamma(\mathrm{X})\big\}$, $\mathcal{F}_2 \subseteq \mathrm{LIP}\big([0,1]\big)$ and $\mathcal{F}_3 \subseteq \mathrm{TestV}(\mathrm{X})$ such that:

$$
\begin{aligned}
&\text{Given } A \subseteq \Gamma(\mathrm{X}) \text{ Borel and } \varepsilon > 0, \text{ there exists } U \in \mathcal{F}_1 \text{ with } \boldsymbol{\pi}(A\Delta U) < \varepsilon,\\
&\mathcal{F}_2 \text{ is dense in } C([0,1]) \text{ and stable by product and } \mathbb{Q}\text{-linear combinations},\\
&\mathcal{F}_3 \text{ is a } \mathbb{Q}\text{-vector space of functions in } W^{1,2}(\mathrm{X}) \text{ whose gradients generate } L^2(T\mathrm{X}).
\end{aligned}
$$

We proceed in the following way:

$\mathcal{F}_1$: Since $\Gamma(\mathrm{X})$ is separable, there exists a countable family $\widetilde{\mathcal{F}}_1$ of open subsets of $\Gamma(\mathrm{X})$ that is a neighbourhood basis for each point $\gamma \in \Gamma(\mathrm{X})$. Let us denote by $\mathcal{F}_1$ the set of finite unions of elements of $\widetilde{\mathcal{F}}_1$, so that $\mathcal{F}_1$ is countable. Fix $A \in \mathscr{B}\big(\Gamma(\mathrm{X})\big)$ and $\varepsilon > 0$. The measure $\boldsymbol{\pi}$ is regular, since $\big(\Gamma(\mathrm{X}), \mathsf{d}_{\Gamma(\mathrm{X})}\big)$ is complete and separable. By inner regularity of $\boldsymbol{\pi}$, there exists a compact subset $K \subseteq A$ such that $\boldsymbol{\pi}(A \setminus K) < \varepsilon/2$. By outer regularity of $\boldsymbol{\pi}$, there exists $V \subseteq \Gamma(\mathrm{X})$ open such that $K \subseteq V$ and $\boldsymbol{\pi}(V \setminus K) < \varepsilon/2$. We can then associate to any $\gamma \in K$ a set $U_\gamma \in \widetilde{\mathcal{F}}_1$ such that $\gamma \in U_\gamma \subseteq V$. By compactness of K, one has $K \subseteq U_{\gamma_1} \cup \ldots \cup U_{\gamma_n} \subseteq V$ for some finite choice $\gamma_1, \ldots, \gamma_n \in K$. Therefore let us call $U \doteq U_{\gamma_1} \cup \ldots \cup U_{\gamma_n} \in \mathcal{F}_1$. We have that

$$
\boldsymbol{\pi}(A\Delta U) = \boldsymbol{\pi}(A \setminus U) + \boldsymbol{\pi}(U \setminus A) \leq \boldsymbol{\pi}(A \setminus K) + \boldsymbol{\pi}(V \setminus K) < \varepsilon.
$$

$\mathcal{F}_2$: The existence of such $\mathcal{F}_2$ stems from the separability of $C\big([0,1]\big)$.

$\mathcal{F}_3$: Since $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian, we have that $W^{1,2}(\mathrm{X})$ is reflexive and therefore separable by (2.27). Let $\mathcal{F}_3$ be any countable dense $\mathbb{Q}$-vector subspace of $W^{1,2}(\mathrm{X})$. Since gradients of functions in $W^{1,2}(\mathrm{X})$ generate the tangent module $L^2(T\mathrm{X})$, the same holds for functions in $\mathcal{F}_3$.

We now define the class $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ of *test vector fields along* $\boldsymbol{\pi}$ as

$$
\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi}) \doteq \left\{ t \mapsto \sum_{i=1}^{n} \psi_i(t)\, \chi_{U_i}\, \mathrm{e}_t^* \nabla f_i \;\middle|\; \begin{array}{c} n \in \mathbb{N}^+ \text{ and } U_i \in \mathcal{F}_1,\, \psi_i \in \mathcal{F}_2, \\ f_i \in \mathcal{F}_3 \text{ for every } i = 1, \ldots, n \end{array} \right\}. \tag{6.6}
$$

Clearly $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ is a countable subset of $\mathrm{TestVF}(\boldsymbol{\pi})$. Notice that the inequalities

$$
\begin{aligned}
[\![\chi_A\, \mathrm{e}_t^* v - \chi_A\, \mathrm{e}_t^* w]\!]_t &\leq \sqrt{\mathsf{C}(\boldsymbol{\pi})}\, \|v - w\|_{L^2(T\mathrm{X})},\\
[\![\chi_A\, \mathrm{e}_t^* v - \chi_U\, \mathrm{e}_t^* v]\!]_t &\leq \sqrt{\boldsymbol{\pi}(A\Delta U)}\, \|v\|_{L^\infty(T\mathrm{X})},
\end{aligned}
$$

are valid for any $t \in [0,1]$, $A, U \subseteq \Gamma(\mathrm{X})$ Borel and $v, w \in L^2(T\mathrm{X})$, whence by recalling the very definition of pullback module we see that:

$$
\text{For any } t \in [0,1], \text{ the set } \big\{ W_t \;\big|\; W \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi}) \big\} \text{ is dense in } \mathrm{e}_t^* L^2(T\mathrm{X}). \tag{6.7}
$$

Lemma 6.2 (Separability of $\mathrm{TestVF}(\boldsymbol{\pi})$) *The family* $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ *is dense in* $\mathrm{TestVF}(\boldsymbol{\pi})$ *with respect to the norm* $\|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}$*, thus also with respect to the norm* $\|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi})}$.

Proof. Let $V \in \mathrm{TestVF}(\boldsymbol{\pi})$ be arbitrary. Let $\varepsilon > 0$ and $t_0 \in [0,1]$. There is $W \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ such that $[\![V - W]\!]_{t_0} < \varepsilon$ by (6.7). Since $t \mapsto [\![V - W]\!]_t^2 = [\![V]\!]_t^2 + [\![W]\!]_t^2 - 2\int \langle V_t, W_t\rangle \, \mathrm{d}\boldsymbol{\pi}$ is a continuous function, we see that $[\![V - W]\!]_t < \varepsilon$ for every t in a neighbourhood of t_0. By compactness of $[0,1]$, we can find a finite number of open intervals $I_1, \ldots, I_n$ covering $[0,1]$ and elements $W_1, \ldots, W_n \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ such that

$$[\![V - W_i]\!]_t < \varepsilon \qquad \text{for every } i = 1, \ldots, n \text{ and } t \in I_i \cap [0,1]. \tag{6.8}$$

By multiplying W_i by an appropriate function in $\mathcal{F}_2$, we can also assume that

$$[\![W_i]\!]_t < \|V\|_{\mathscr{C}(\boldsymbol{\pi})} + 2\varepsilon \qquad \text{for every } t \in [0,1]. \tag{6.9}$$

Let $(\phi_i)_i$ be a Lipschitz partition of the unity subordinate to the cover made with the I_i's. Moreover, for any i let $\psi_i \in \mathcal{F}_2$ be such that $\big|\phi_i(t) - \psi_i(t)\big| < \varepsilon$ for every $t \in [0,1]$. Then we have that $\big(t \mapsto W_t \doteq \sum_i \psi_i(t)\, W_{i,t}\big) \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ and

$$[\![V - W]\!]_t \leq [\![V - \textstyle\sum_i \phi_i(t)\, W_i]\!]_t + [\![\textstyle\sum_i \big(\psi_i(t) - \phi_i(t)\big) W_i]\!]_t \overset{(6.8),(6.9)}{\leq} \varepsilon + \varepsilon(\|V\|_{\mathscr{C}(\boldsymbol{\pi})} + 2\varepsilon)$$

for any $t \in [0,1]$. The conclusion follows from the arbitrariness of $\varepsilon > 0$. □

6.1.2 The space $\mathscr{L}^2(\boldsymbol{\pi})$

Let us define the class of Borel vector fields along the test plan $\boldsymbol{\pi}$:

Definition 6.3 (Borel vector fields along $\boldsymbol{\pi}$) *We say that $V \in \mathrm{VF}(\boldsymbol{\pi})$ is* Borel *provided*

$$[0,1] \ni t \longmapsto \int \langle V_t, W_t\rangle \, \mathrm{d}\boldsymbol{\pi} \quad \textit{is a Borel function} \tag{6.10}$$

for every $W \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$.

Notice that – thanks to Lemma 6.2 – this notion would remain unaltered if we required the identity (6.10) to hold for any $W \in \mathrm{TestVF}(\boldsymbol{\pi})$. Furthermore, Proposition 6.1 ensures that test vector fields are Borel. We have the following basic result:

Proposition 6.4 *Let $V \in \mathrm{VF}(\boldsymbol{\pi})$ be Borel. Then the map $[\![V]\!] : [0,1] \to [0,\infty)$ is Borel.*

Proof. From (6.7) we deduce that

$$[\![V]\!]_t^2 = \sup_{W \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})} \left(2\int \langle V_t, W_t\rangle \, \mathrm{d}\boldsymbol{\pi} - [\![W]\!]_t^2 \right) \qquad \text{for every } t \in [0,1],$$

whence the statement follows. □

We can now define the space $\mathscr{L}^2(\boldsymbol{\pi})$ in the following way:

Definition 6.5 (The space $\mathscr{L}^2(\boldsymbol{\pi})$) *The space $\mathscr{L}^2(\boldsymbol{\pi})$ is the space of all $V \in \mathrm{VF}(\boldsymbol{\pi})$ with*

$$\|V\|_{\mathscr{L}^2(\boldsymbol{\pi})}^2 \doteq \int_0^1 [\![V]\!]_t^2 \, \mathrm{d}t = \int_0^1 \!\! \int |V_t|^2 \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t < +\infty, \tag{6.11}$$

where we identify $V, \tilde{V} \in \mathrm{VF}(\boldsymbol{\pi})$ if $V_t = \tilde{V}_t$ for a.e. $t \in [0,1]$.

Clearly $\big(\mathscr{L}^2(\boldsymbol{\pi}), \|\cdot\|_{\mathscr{L}^2(\boldsymbol{\pi})}\big)$ is a normed space, wherein $\mathrm{TestVF}(\boldsymbol{\pi})$ is embedded. By adapting the classical arguments concerning the standard L^2-spaces, we obtain the following:

Proposition 6.6 *The normed space $\mathscr{L}^2(\boldsymbol{\pi})$ is a Hilbert space. Moreover, if $V_n \to V$ strongly in $\mathscr{L}^2(\boldsymbol{\pi})$, then there is a subsequence such that $V_{n,t} \to V_t$ in $\mathrm{e}_t^* L^2(TX)$ for a.e. $t \in [0,1]$.*

Proof. It is clear that the $\mathscr{L}^2(\boldsymbol{\pi})$-norm comes from the scalar product

$$\langle V, W\rangle_{\mathscr{L}^2(\boldsymbol{\pi})} \doteq \int_0^1 \int \langle V_t, W_t\rangle \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t.$$

To conclude the proof, we show that if $(V_n)_n$ is a sequence of Borel vector fields in $\mathscr{L}^2(\boldsymbol{\pi})$ such that $\sum_n \|V_{n+1} - V_n\|_{\mathscr{L}^2(\boldsymbol{\pi})} < \infty$, then such sequence has a limit $V \in \mathscr{L}^2(\boldsymbol{\pi})$ and for almost every $t \in [0,1]$ it holds that $V_{n,t} \to V_t$ in $\mathrm{e}_t^* L^2(TX)$.

Define the Borel function $g : [0,1] \to [0,+\infty]$ as $g \doteq \sum_n [\![V_{n+1} - V_n]\!]$. Notice that, since

$$\Big\| \sum_{n=1}^N [\![V_{n+1} - V_n]\!] \Big\|_{L^2(0,1)} \le \sum_{n=1}^N \|V_{n+1} - V_n\|_{\mathscr{L}^2(\boldsymbol{\pi})} \le \sum_{n=1}^\infty \|V_{n+1} - V_n\|_{\mathscr{L}^2(\boldsymbol{\pi})} < \infty$$

for every $N \in \mathbb{N}$, we have that $g \in L^2(0,1)$. Let $N \doteq \big\{t \in [0,1] \,:\, g(t) = +\infty\big\}$ and observe that for every $t \in [0,1] \setminus N$ it holds that

$$\sum_{n=1}^\infty \|V_{n+1,t} - V_{n,t}\|_{\mathrm{e}_t^* L^2(TX)} = \sum_{n=1}^\infty [\![V_{n+1} - V_n]\!]_t = g(t) < \infty, \tag{6.12}$$

proving that $(V_{n,t})_n$ is a Cauchy sequence in $\mathrm{e}_t^* L^2(TX)$. Then define

$$V_t \doteq \begin{cases} \lim_n V_{n,t} \in \mathrm{e}_t^* L^2(TX) & \text{if } t \in [0,1] \setminus N, \\ 0 \in \mathrm{e}_t^* L^2(TX) & \text{if } t \in N. \end{cases}$$

For every $W \in \mathrm{TestVF}(\boldsymbol{\pi})$ we have $\int \langle V_t, W_t\rangle \,\mathrm{d}\boldsymbol{\pi} = \lim_n \int \langle V_{n,t}, W_t\rangle \,\mathrm{d}\boldsymbol{\pi}$ for all $t \in [0,1] \setminus N$, hence the function $[0,1] \ni t \mapsto \int \langle V_t, W_t\rangle \,\mathrm{d}\boldsymbol{\pi}$ is Borel and – by arbitrariness of W – this shows that V is Borel. Since trivially we have $[\![V]\!]_t \le [\![V_1]\!]_t + \sum_{n=1}^\infty [\![V_{n+1} - V_n]\!]_t$, by (6.12) we see that $V \in \mathscr{L}^2(\boldsymbol{\pi})$. To check that $V_n \to V$ in $\mathscr{L}^2(\boldsymbol{\pi})$ we use the fact that, again by (6.12), the sequence $[\![V - V_n]\!]_t$ is dominated in $L^2(0,1)$ and that for every $t \in [0,1] \setminus N$ it holds

$$\lim_{n\to\infty} [\![V - V_n]\!]_t \le \lim_{n\to\infty} \lim_{m\to\infty} \sum_{i=n}^m [\![V_{i+1} - V_i]\!]_t \overset{(6.12)}{=} 0,$$

so that the conclusion follows by the dominated convergence theorem. □

Proposition 6.7 (Density of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ in $\mathscr{L}^2(\boldsymbol{\pi})$) *It holds that the space $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ is dense in $\mathscr{L}^2(\boldsymbol{\pi})$. In particular, the space $\mathscr{L}^2(\boldsymbol{\pi})$ is separable.*

Proof. Let $(Z_k)_k$ be an enumeration of the elements of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$. Now pick a Borel vector field $V \in \mathscr{L}^2(\boldsymbol{\pi})$ and choose $\varepsilon > 0$. Then let $\widetilde{G}_k \doteq \big\{t \in [0,1] \,:\, [\![V - Z_k]\!]_t < \varepsilon\big\}$ for all $k \in \mathbb{N}$. We put $G_1 \doteq \widetilde{G}_1$ and $G_k \doteq \widetilde{G}_k \setminus (\widetilde{G}_1 \cup \ldots \cup \widetilde{G}_{k-1})$ for $k > 1$. Then (6.7) grants that $(G_k)_{k\ge 1}$ constitutes a Borel partition of $[0,1]$.

For $m \in \mathbb{N} \cup \{\infty\}$ define $W_m \in \mathscr{L}^2(\boldsymbol{\pi})$ as $W_{m,t} \doteq \sum_{k=1}^m \chi_{G_k}(t)\, Z_{k,t}$. Observe that we have $\|W_\infty - V\|_{\mathscr{L}^2(\boldsymbol{\pi})} < \varepsilon$ by definition of G_k. Moreover, for each $m \geq 1$ one has that

$$\|W_m - W_\infty\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} = \sum_{k=m+1}^{\infty} \int_{G_k} [\![Z_k]\!]_t^2 \,\mathrm{d}t \leq \int_{\bigcup_{k>m} G_k} 2\left([\![V]\!]_t^2 + \varepsilon^2\right) \mathrm{d}t,$$

so that accordingly $\lim_{m\to\infty} \|W_m - W_\infty\|_{\mathscr{L}^2(\boldsymbol{\pi})} = 0$ by dominated convergence theorem.

Hence to conclude it is sufficient to show that each W_m belongs to the $\mathscr{L}^2(\boldsymbol{\pi})$-closure of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$. In turn, this would follow if we proved that for any $Z \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ and any Borel set $G \subseteq [0,1]$ the vector field $\chi_G\, Z$ belongs to the $\mathscr{L}^2(\boldsymbol{\pi})$-closure of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$. To see this, simply let $(\varphi_n)_n \subseteq \mathrm{LIP}\big([0,1]\big)$ be uniformly bounded and a.e. converging to χ_G. Note that $\varphi_n\, Z \in \mathrm{TestVF}(\boldsymbol{\pi})$ and that an application of the dominated convergence theorem shows that $\varphi_n\, Z \to \chi_G\, Z$ in $\mathscr{L}^2(\boldsymbol{\pi})$. This completes the proof. □

Now consider the speed $\boldsymbol{\pi}'_t$, associated to any test plan $\boldsymbol{\pi}$ by Theorem 4.20.

Proposition 6.8 *The (equivalence class up to a.e. equality of the) map $t \mapsto \boldsymbol{\pi}'_t$ is an element of the space $\mathscr{L}^2(\boldsymbol{\pi})$.*

Proof. We have that $\boldsymbol{\pi}'_t \in \mathrm{e}_t^* L^2(T\mathrm{X})$ for a.e. $t \in [0,1]$. Moreover,

$$\int_0^1 \int |\boldsymbol{\pi}'_t|^2 \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \overset{(4.26)}{=} \int_0^1 \int |\dot{\gamma}_t|^2 \,\mathrm{d}\boldsymbol{\pi}(\gamma)\,\mathrm{d}t < +\infty$$

by the very definition of test plan. Hence we only need to show that $t \mapsto \boldsymbol{\pi}'_t$ has a Borel representative, in the sense of Definition 6.3.

Notice that for any $f \in W^{1,2}(\mathrm{X})$ by Proposition 4.21 we have that the map $t \mapsto (\mathrm{e}_t^*\mathrm{d}f)(\boldsymbol{\pi}'_t)$ admits a Borel representative. Therefore the same holds for the map $t \mapsto \psi(t)\chi_U \langle \mathrm{e}_t^*\nabla f, \boldsymbol{\pi}'_t\rangle$ for every $\psi \in \mathrm{LIP}\big([0,1]\big)$ and $U \subseteq \Gamma(\mathrm{X})$ Borel. Hence there is a Borel negligible set $N \subseteq [0,1]$ such that for every $V \in \mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ the function $t \mapsto \int \langle V_t, \boldsymbol{\pi}'_t\rangle \,\mathrm{d}\boldsymbol{\pi}$, set to 0 on N, is Borel. This is sufficient to conclude. □

We conclude the subsection by pointing out that $\mathscr{L}^2(\boldsymbol{\pi})$ can be also seen as the pullback of $L^2(T\mathrm{X})$ via the evaluation map $\mathrm{e} : \Gamma(\mathrm{X}) \times [0,1] \to \mathrm{X}$, defined as $\mathrm{e}(\gamma, t) \doteq \gamma_t$. To this aim, let us start by defining the following operations:

i) Given any $f \in L^\infty(\boldsymbol{\pi} \times \mathcal{L}_1)$ and $V \in \mathscr{L}^2(\boldsymbol{\pi})$, we define $fV \in \mathscr{L}^2(\boldsymbol{\pi})$ as

$$(fV)_t \doteq f(\cdot, t)\, V_t \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.13}$$

ii) To each $V \in \mathscr{L}^2(\boldsymbol{\pi})$ we associate the function $|V| \in L^2(\boldsymbol{\pi} \times \mathcal{L}_1)$, defined by

$$|V|(\gamma, t) \doteq |V_t|(\gamma) \qquad \text{for } (\boldsymbol{\pi} \times \mathcal{L}_1)\text{-a.e. } (\gamma, t) \in \Gamma(\mathrm{X}) \times [0,1]. \tag{6.14}$$

It is clear that these operations give $\mathscr{L}^2(\boldsymbol{\pi})$ the structure of an $L^2(\boldsymbol{\pi} \times \mathcal{L}_1)$-normed module.

We define the linear continuous operator $\Phi :\, L^2(T\mathrm{X}) \to \mathscr{L}^2(\boldsymbol{\pi})$ as

$$\Phi(v)_t \doteq \mathrm{e}_t^* v \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.15}$$

We then have the following identification:

Proposition 6.9 ($\mathscr{L}^2(\boldsymbol{\pi})$ as pullback) *It holds that* $\big(\mathscr{L}^2(\boldsymbol{\pi}),\Phi\big)\cong\big(\mathrm{e}^*L^2(T\mathrm{X}),\mathrm{e}^*\big)$, *i.e.*

$$\begin{aligned} \big|\Phi(v)\big| &= |v|\circ \mathrm{e} && \textit{holds } (\boldsymbol{\pi}\times\mathcal{L}_1)\textit{-a.e. for any } v\in L^2(T\mathrm{X}),\\ \big\{\Phi(v)\;\big|\;v\in L^2(T\mathrm{X})\big\} && & \textit{generates } \mathscr{L}^2(\boldsymbol{\pi}) \textit{ as a module.}\end{aligned} \tag{6.16}$$

Proof. The first in (6.16) follows by noticing that $\big|\Phi(v)\big|(\gamma,t)=|\mathrm{e}_t^*v|(\gamma)=\big(|v|\circ\mathrm{e}\big)(\gamma,t)$ holds for $(\boldsymbol{\pi}\times\mathcal{L}_1)$-a.e. (γ,t), the second one stems from the density of $\mathrm{TestVF}(\boldsymbol{\pi})$ in $\mathscr{L}^2(\boldsymbol{\pi})$. □

Notice that the notion of pullback module $\big(\mathrm{e}^*L^2(T\mathrm{X}),\mathrm{e}^*\big)$ makes no (explicit) reference to the concept of 'test vector field', as defined in Subsection 6.1.1. Thus this last proposition is also telling us that the choice of using these test objects to check Borel regularity – which a priori might seem arbitrary – leads in fact to a canonical interpretation of $\mathscr{L}^2(\boldsymbol{\pi})$.

Remark 6.10 ($\mathscr{L}^2(\boldsymbol{\pi})$ as direct integral) The construction of $\mathscr{L}^2(\boldsymbol{\pi})$ can be summarized by saying that such space is the direct integral of the $\mathrm{e}_t^*L^2(T\mathrm{X})$'s, the space of Borel vector fields being the so-called 'measurable sections' and the set $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ being the one used to check measurability. ■

6.1.3 The space $\mathscr{C}(\boldsymbol{\pi})$

Here we introduce and briefly study those vector fields in $\mathrm{VF}(\boldsymbol{\pi})$ which are 'continuous in time'. We start with the following definition:

Definition 6.11 (The space $\mathscr{C}(\boldsymbol{\pi})$) *Let* $V\in\mathrm{VF}(\boldsymbol{\pi})$ *be given. Then we say that* V *is a* continuous vector field along $\boldsymbol{\pi}$ *provided*

$$[0,1]\ni t\longmapsto\int\langle V_t,W_t\rangle\,\mathrm{d}\boldsymbol{\pi}\quad\textit{is continuous} \tag{6.17}$$

for every $W\in\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ *and*

$$[0,1]\ni t\longmapsto [\![V]\!]_t\quad\textit{is continuous.} \tag{6.18}$$

We denote the family of all continuous vector fields by $\mathscr{C}(\boldsymbol{\pi})$*. For every* $V\in\mathscr{C}(\boldsymbol{\pi})$*, we set*

$$\|V\|_{\mathscr{C}(\boldsymbol{\pi})}\doteq\max_{t\in[0,1]}[\![V]\!]_t. \tag{6.19}$$

Lemma 6.2 ensures that this definition would be unaltered if we required (6.17) to hold for any $W\in\mathrm{TestVF}(\boldsymbol{\pi})$. Moreover, Proposition 6.1 grants that $\mathrm{TestVF}(\boldsymbol{\pi})\subseteq\mathscr{C}(\boldsymbol{\pi})$.

It is not obvious that $\mathscr{C}(\boldsymbol{\pi})$ is a vector space, the problem being in checking that (6.18) holds for any linear combination. This will be a consequence of the density of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ in $\mathscr{C}(\boldsymbol{\pi})$, which is part of the content of the next result:

Proposition 6.12 *It holds that the space* $\big(\mathscr{C}(\boldsymbol{\pi}),\|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}\big)$ *is a separable Banach space, wherein the set* $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ *is dense.*

Proof. Let $V_1,V_2\in\mathscr{C}(\boldsymbol{\pi})$ be given. Notice that – by using (6.17), (6.18) and by arguing exactly as in the proof of Lemma 6.2 – we can find $(W_{1,n})_n,(W_{2,n})_n\subseteq\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ such that the functions $t\mapsto[\![V_i-W_{i,n}]\!]_t$ uniformly converge to 0 as $n\to\infty$ for $i=1,2$.

Now observe that – since $W_{1,n}+W_{2,n}\in\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ – the function $t\mapsto[\![W_{1,n}+W_{2,n}]\!]_t$ is continuous and for every $t\in[0,1]$ we have

$$\big|[\![V_1+V_2]\!]_t-[\![W_{1,n}+W_{2,n}]\!]_t\big|\le[\![V_1-W_{1,n}+V_2-W_{2,n}]\!]_t\le[\![V_1-W_{1,n}]\!]_t+[\![V_2-W_{2,n}]\!]_t.$$

Hence $t \mapsto [\![V_1 + V_2]\!]_t$ is the uniform limit of continuous functions and thus continuous itself. Since trivially $\mathscr{C}(\boldsymbol{\pi})$ is closed by multiplication by scalars, we proved that it is a vector space. That $\|\cdot\|_{\mathscr{C}(\boldsymbol{\pi})}$ is a complete norm on it is trivial and the density of $\mathrm{TestVF}_{\mathbb{N}}(\boldsymbol{\pi})$ has already been shown. Hence the proof is finished. □

A useful consequence of the density of test vector fields is the following strengthening of the continuity property:

Corollary 6.13 *Let $V \in \mathscr{C}(\boldsymbol{\pi})$ be given. Then the map $t \mapsto |V_t|^2 \in L^1(\boldsymbol{\pi})$ is continuous.*

Proof. For $V \in \mathrm{TestVF}(\boldsymbol{\pi})$ the claim has been proved in Proposition 6.1. Now notice that for $V, W \in \mathscr{C}(\boldsymbol{\pi})$ we have

$$\int \big||V_t|^2 - |W_t|^2\big| \,\mathrm{d}\boldsymbol{\pi} \le \int |V_t + W_t|\,|V_t - W_t| \,\mathrm{d}\boldsymbol{\pi} \le \big(\|V\|_{\mathscr{C}(\boldsymbol{\pi})} + \|W\|_{\mathscr{C}(\boldsymbol{\pi})}\big)\,[\![V - W]\!]_t,$$

so if $V_n \to V$ in $\mathscr{C}(\boldsymbol{\pi})$ then $\big(t \mapsto |V_{n,t}|^2\big) \in L^1(\boldsymbol{\pi})$ uniformly converge to $\big(t \mapsto |V_t|^2\big) \in L^1(\boldsymbol{\pi})$. The conclusion then follows from the density of $\mathrm{TestVF}(\boldsymbol{\pi})$ in $\mathscr{C}(\boldsymbol{\pi})$. □

6.1.4 The spaces $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ and $\mathscr{H}^{1,2}(\boldsymbol{\pi})$

Throughout all this section, we shall further make the assumption that the test plan $\boldsymbol{\pi}$ is Lipschitz (in the sense of Definition 4.22).

Let $v \in W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ be given. Notice that the map from $L^0(T\mathrm{X})$ to $\mathrm{e}_t^* L^0(T\mathrm{X})$ defined by

$$w \longmapsto \mathrm{e}_t^*(\nabla_w v) \tag{6.20}$$

satisfies the inequality

$$\big|\mathrm{e}_t^*(\nabla_w v)\big| \le |\nabla v|_{\mathsf{HS}} \circ \mathrm{e}_t \, |w| \circ \mathrm{e}_t \qquad \text{in the } \boldsymbol{\pi}\text{-a.e. sense.} \tag{6.21}$$

Hence by the universal property of the pullback – given in Proposition 3.31 – we know that there exists a unique $L^0(\boldsymbol{\pi})$-linear and continuous operator from $\mathrm{e}_t^* L^0(T\mathrm{X})$ to $\mathrm{e}_t^* L^0(T\mathrm{X})$, which we shall call $\mathrm{Cov}(v, \cdot)$, such that

$$\mathrm{Cov}(v, \mathrm{e}_t^* w) = \mathrm{e}_t^*(\nabla_w v) \qquad \text{for every } w \in L^0(T\mathrm{X}) \tag{6.22}$$

and such operator satisfies the bound

$$\big|\mathrm{Cov}(v, W)\big| \le |\nabla v|_{\mathsf{HS}} \circ \mathrm{e}_t |W| \qquad \text{in the } \boldsymbol{\pi}\text{-a.e. sense.} \tag{6.23}$$

We shall be interested in such covariant differentiation along the speed of our test plan: for every $t \in [0,1]$ such that $\boldsymbol{\pi}'_t$ exists, we define the map $\mathrm{Cov}_t : W^{1,2}_{\mathrm{C}}(T\mathrm{X}) \to \mathrm{e}_t^* L^0(T\mathrm{X})$ as

$$\mathrm{Cov}_t(v) \doteq \mathrm{Cov}(v, \boldsymbol{\pi}'_t). \tag{6.24}$$

We point out the following simple fact:

Proposition 6.14 *For every $t \in [0,1]$ such that $\boldsymbol{\pi}'_t$ exists, the operator Cov_t is linear and continuous from $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ to $\mathrm{e}_t^* L^2(T\mathrm{X})$. Moreover, for every $v \in W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ the (equivalence class up to a.e. equality of the) map $t \mapsto \mathrm{Cov}_t(v) \in \mathrm{e}_t^* L^2(T\mathrm{X})$ is an element of $\mathscr{L}^2(\boldsymbol{\pi})$.*

Proof. The continuity of Cov_t as map from $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ to $\mathrm{e}_t^* L^2(T\mathrm{X})$ is a direct consequence of the bounds (6.23) and our assumption that $\boldsymbol{\pi}$ is a Lipschitz test plan:

$$\|\mathrm{Cov}_t(v)\|^2_{\mathrm{e}_t^* L^2(T\mathrm{X})} = \int \big|\mathrm{Cov}_t(v)\big|^2 \,\mathrm{d}\boldsymbol{\pi} \overset{(6.23)}{\leq} \int |\nabla v|^2_{\mathsf{HS}} \circ \mathrm{e}_t \,|\boldsymbol{\pi}'_t|^2 \,\mathrm{d}\boldsymbol{\pi} \leq \mathsf{C}(\boldsymbol{\pi})\,\mathsf{L}(\boldsymbol{\pi})^2\, \|v\|^2_{W^{1,2}_{\mathrm{C}}(T\mathrm{X})}.$$

Thanks to this bound, to conclude it suffices to show that for any vector field $v \in W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ the map $t \mapsto \mathrm{Cov}_t(v) = \mathrm{Cov}(v, \boldsymbol{\pi}'_t)$ is a.e. equal to a Borel element of $\mathrm{VF}(\boldsymbol{\pi})$. Taking into account that $\big(t \mapsto \boldsymbol{\pi}'_t\big) \in \mathscr{L}^2(\boldsymbol{\pi})$ by Proposition 6.8, that $\mathrm{TestVF}(\boldsymbol{\pi})$ is dense in $\mathscr{L}^2(\boldsymbol{\pi})$, the second claim in Proposition 6.6 and the bound (6.23), we see that to conclude it is sufficient to show that $t \mapsto \mathrm{Cov}(v, V_t)$ is a Borel vector field in $\mathrm{VF}(\boldsymbol{\pi})$ for any $V \in \mathrm{TestVF}(\boldsymbol{\pi})$.

Fix such V, say $V_t = \sum_i \phi_i(t)\,\chi_{A_i}\,\mathrm{e}_t^* v_i$. Let $\big(t \mapsto W_t = \sum_j \psi_j(t)\,\chi_{B_j}\,\mathrm{e}_t^* w_j\big) \in \mathrm{TestVF}(\boldsymbol{\pi})$ be arbitrary. Notice that, since $|v_i|, |w_j| \in L^2(\mathfrak{m}) \cap L^\infty(\mathfrak{m})$, we have that $\langle v_i, \nabla_{w_j} v\rangle \in L^1(\mathfrak{m})$ and thus by Theorem 2.15 we deduce that the map $t \mapsto \langle v_i, \nabla_{w_j} v\rangle \circ \mathrm{e}_t \in L^1(\boldsymbol{\pi})$ is continuous for every choice of i, j. Therefore

$$t \longmapsto \int \big\langle V_t, \mathrm{Cov}_t(v, W_t)\big\rangle \,\mathrm{d}\boldsymbol{\pi} = \sum_{i,j} \varphi_i(t)\,\psi_j(t) \int \chi_{A_i \cap B_j}\,\langle v_i, \nabla_{w_j} v\rangle \circ \mathrm{e}_t \,\mathrm{d}\boldsymbol{\pi}$$

is continuous, establishing – as W is arbitrary – the Borel regularity of $t \mapsto \mathrm{Cov}(v, V_t)$. □

The 'compatibility with the metric' of the covariant derivative yields the following simple but crucial lemma:

Lemma 6.15 *Let $v, w \in \mathrm{TestV}(\mathrm{X})$ be given. Then the map $\big(t \mapsto \langle v, w\rangle \circ \mathrm{e}_t\big) \in L^2(\boldsymbol{\pi})$, which is Lipschitz by Proposition 4.24, satisfies the identity*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle v, w\rangle \circ \mathrm{e}_t = \big\langle \mathrm{Cov}_{\boldsymbol{\pi}}(v)_t, \mathrm{e}_t^* w\big\rangle + \big\langle \mathrm{e}_t^* v, \mathrm{Cov}_{\boldsymbol{\pi}}(w)_t\big\rangle \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1], \tag{6.25}$$

where the derivative is intended in the strong topology of $L^2(\boldsymbol{\pi})$.

Proof. Recall from item ii) of Proposition 4.52 that it holds

$$\mathrm{d}\langle v, w\rangle(z) = \langle \nabla_z v, w\rangle + \langle v, \nabla_z w\rangle \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } z \in L^0(T\mathrm{X}).$$

From the defining property of pointwise norm in the pullback and by polarization, we obtain that $\langle \mathrm{e}_t^* v_1, \mathrm{e}_t^* v_2\rangle = \langle v_1, v_2\rangle \circ \mathrm{e}_t$ for every $v_1, v_2 \in L^0(T\mathrm{X})$. Thus we have that the identity

$$\big(\mathrm{e}_t^* \mathrm{d}\langle v, w\rangle\big)(Z) = \big\langle \mathrm{Cov}_t(v, Z), \mathrm{e}_t^* w\big\rangle + \big\langle \mathrm{e}_t^* v, \mathrm{Cov}_t(w, Z)\big\rangle \tag{6.26}$$

holds for every $Z \in \mathrm{e}_t^* L^2(T\mathrm{X})$ of the form $Z_t = \mathrm{e}_t^* z$ for some $z \in L^2(T\mathrm{X})$. Since both sides of this identity are $L^\infty(\boldsymbol{\pi})$-linear and continuous in Z, we see that (6.26) holds for $Z \in \mathrm{e}_t^* L^2(T\mathrm{X})$ generic. The conclusion comes by picking $Z = \boldsymbol{\pi}'_t$ and recalling Proposition 4.24. □

We want to introduce a new differential operator, initially defined only on $\mathrm{TestVF}(\boldsymbol{\pi})$ and then extended to more general vector fields. To this aim, the following lemma will be useful:

Lemma 6.16 *Let $(\varphi_i)_i, (\psi_j)_j \subseteq \mathrm{LIP}\big([0,1]\big)$, let $(A_i)_i, (B_j)_j$ be Borel partitions of $\Gamma(\mathrm{X})$ and let $(v_i)_i, (w_j)_j \subseteq \mathrm{TestV}(\mathrm{X})$, where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Assume that*

$$\sum_{i=1}^{n} \chi_{A_i} \varphi_i(t)\, \mathrm{e}_t^* v_i = \sum_{j=1}^{m} \chi_{B_j} \psi_j(t)\, \mathrm{e}_t^* w_j \qquad \text{for every } t \in [0,1]. \tag{6.27}$$

Then for a.e. $t \in [0,1]$ it holds that

$$\begin{aligned} \sum_{i=1}^{n} \chi_{A_i} \varphi_i'(t)\, \mathrm{e}_t^* v_i &= \sum_{j=1}^{m} \chi_{B_j} \psi_j'(t)\, \mathrm{e}_t^* w_j, \\ \sum_{i=1}^{n} \chi_{A_i} \varphi_i(t)\, \mathrm{Cov}_t(v_i) &= \sum_{j=1}^{m} \chi_{B_j} \psi_j(t)\, \mathrm{Cov}_t(w_j). \end{aligned} \tag{6.28}$$

Proof. For the first claim in (6.28), note that our assumption (6.27) and Proposition 6.9 yield

$$\sum_i \chi_{A_i \times [0,1]}(\cdot,t)\, \varphi_i(t)\, \mathrm{e}^* v_i = \sum_j \chi_{B_j \times [0,1]}(\cdot,t)\, \psi_j(t)\, \mathrm{e}^* w_j$$

as elements of $\mathrm{e}^* L^2(T\mathrm{X}) \cong \mathscr{L}^2(\boldsymbol{\pi})$, thus we can differentiate in time and conclude by using again Proposition 6.9.

For the second claim in (6.28), start by noticing that our assumption (6.27) and the very definition of pullback imply that – for any i,j and $t \in [0,1]$– it holds $\chi_C\, \varphi_i(t)\, v_i = \chi_C\, \psi_j(t)\, w_j$, where we set $C \doteq \{\mathrm{d}(\mathrm{e}_t)_*(\chi_{A_i \cap B_j}\boldsymbol{\pi})/\mathrm{d}(\mathrm{e}_t)_*\boldsymbol{\pi} > 0\}$. This identity and the locality of the covariant derivative give that $\chi_C\, \varphi_i(t)\, \nabla_z v_i = \chi_C\, \psi_j(t)\, \nabla_z w_j$ for every $z \in L^2(T\mathrm{X})$. By applying the pullback map on both sides and noticing that $\chi_C \circ \mathrm{e}_t \geq \chi_{A_i \cap B_j}$, we deduce that

$$\chi_{A_i \cap B_j}\, \varphi_i(t)\, \mathrm{Cov}(v_i, Z) = \chi_{A_i \cap B_j}\, \psi_j(t)\, \mathrm{Cov}(w_j, Z)$$

for every Z of the form $Z_t = \mathrm{e}_t^* z$. From the $L^\infty(\boldsymbol{\pi})$-linearity in Z of both sides and the arbitrariness of i,j, the conclusion follows. □

We can now define the convective derivative of a test vector field:

Definition 6.17 (Convective derivative along a test plan) *We define the* convective derivative *operator* $\widetilde{\mathrm{D}}_{\boldsymbol{\pi}} : \mathrm{TestVF}(\boldsymbol{\pi}) \to \mathscr{L}^2(\boldsymbol{\pi})$ *as follows: to the element* $V \in \mathrm{TestVF}(\boldsymbol{\pi})$*, of the form* $V_t = \sum_{i=1}^n \varphi_i(t)\, \chi_{A_i}\, \mathrm{e}_t^* v_i$*, we associate the vector field* $\widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V \in \mathscr{L}^2(\boldsymbol{\pi})$ *given by*

$$(\widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V)_t \doteq \sum_{i=1}^{n} \chi_{A_i} \Big(\varphi_i'(t)\, \mathrm{e}_t^* v_i + \varphi_i(t)\, \mathrm{Cov}_t(v_i) \Big) \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.29}$$

For the sake of simplicity, we will briefly write $\widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V_t$ *instead of* $(\widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V)_t$.

Notice that Lemma 6.16 ensures that the right hand side of (6.29) depends only on V and not on the way we write it as $V_t = \sum_{i=1}^n \varphi_i(t)\, \chi_{A_i}\, \mathrm{e}_t^* v_i$. The fact that the right hand side of (6.29) defines a Borel vector field in $\mathrm{VF}(\boldsymbol{\pi})$ follows directly from Proposition 6.14; to see that it belongs to $\mathscr{L}^2(\boldsymbol{\pi})$, notice that $(t \mapsto \mathrm{e}_t^* v_i), \mathrm{Cov}_{\boldsymbol{\pi}}(v_i) \in \mathscr{L}^2(\boldsymbol{\pi})$ for every i and that the functions φ_i's are Lipschitz. Hence the definition is well-posed and $\widetilde{\mathrm{D}}_{\boldsymbol{\pi}}$ is a linear operator.

The convective derivative has the following simple and crucial property, which is a direct consequence of Lemma 6.15.

Proposition 6.18 *Let* $V, W \in \mathrm{TestVF}(\boldsymbol{\pi})$*. Then the map* $t \mapsto \langle V_t, W_t \rangle \in L^2(\boldsymbol{\pi})$ *is Lipschitz and satisfies the identity*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle V_t, W_t \rangle = \langle \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V_t, W_t \rangle + \langle V_t, \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} W_t \rangle \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1], \tag{6.30}$$

where the derivative is intended in the strong topology of $L^2(\boldsymbol{\pi})$.

Proof. By bilinearity, in order to prove (6.30) it suffices to consider the case $V_t = \varphi(t)\chi_A\,\mathrm{e}_t^* v$ and $W_t = \psi(t)\,\chi_B\,\mathrm{e}_t^* w$ for $v, w \in \mathrm{TestV}(\mathrm{X})$. Lemma 6.15 ensures that

$$[0,1] \ni t \longmapsto \langle \mathrm{e}_t^* v, \mathrm{e}_t^* w\rangle = \langle v, w\rangle \circ \mathrm{e}_t \in L^2(\boldsymbol{\pi}) \quad \text{is a Lipschitz map}$$

and it is thus clear that $t \mapsto \langle V_t, W_t\rangle = \chi_{A\cap B}\,\varphi(t)\,\psi(t)\,\langle \mathrm{e}_t^* v, \mathrm{e}_t^* w\rangle$ is Lipschitz as well. The identity (6.30) now follows from (6.25) and the Leibniz rule. □

This last proposition will allow us to 'integrate by parts' and to extend the definition of convective derivative to 'Sobolev vector fields along $\boldsymbol{\pi}$'.

Let us define the *support* $\mathrm{spt}(V)$ of a test vector field $V \in \mathrm{TestVF}(\boldsymbol{\pi})$ as the closure of the set of t's with $V_t \neq 0$. We introduce the space of sections with compact support in $(0,1)$:

$$\mathrm{TestVF_c}(\boldsymbol{\pi}) \doteq \big\{ V \in \mathrm{TestVF}(\boldsymbol{\pi}) \,\big|\, \mathrm{spt}(V) \subseteq (0,1) \big\}. \tag{6.31}$$

A simple cut-off argument shows that $\mathrm{TestVF_c}(\boldsymbol{\pi})$ is $\mathscr{L}^2(\boldsymbol{\pi})$-dense in $\mathrm{TestVF}(\boldsymbol{\pi})$, whence accordingly also in $\mathscr{L}^2(\boldsymbol{\pi})$. With this said, we can give the following definition:

Definition 6.19 (The space $\mathscr{W}^{1,2}(\boldsymbol{\pi})$) *The* Sobolev space $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ *is the vector subspace of $\mathscr{L}^2(\boldsymbol{\pi})$ consisting of all those $V \in \mathscr{L}^2(\boldsymbol{\pi})$ for which there exists $Z \in \mathscr{L}^2(\boldsymbol{\pi})$ satisfying*

$$\int_0^1\!\!\int \langle V_t, \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} W_t\rangle \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t = -\int_0^1\!\!\int \langle Z_t, W_t\rangle \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \qquad \textit{for every } W \in \mathrm{TestVF_c}(\boldsymbol{\pi}). \tag{6.32}$$

In this case the section Z, whose uniqueness is granted by density of $\mathrm{TestVF_c}(\boldsymbol{\pi})$ in $\mathscr{L}^2(\boldsymbol{\pi})$, can be unambiguously denoted by $\mathrm{D}_{\boldsymbol{\pi}} V$ and called convective derivative *of V. We endow the space $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ with the norm $\|\cdot\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})}$, defined by*

$$\|V\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})} \doteq \sqrt{\|V\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} + \|\mathrm{D}_{\boldsymbol{\pi}} V\|^2_{\mathscr{L}^2(\boldsymbol{\pi})}} \qquad \textit{for every } V \in \mathscr{W}^{1,2}(\boldsymbol{\pi}). \tag{6.33}$$

As we are going to see, this choice of terminology is consistent with that Definition 6.17:

Proposition 6.20 *Let $V \in \mathrm{TestVF}(\boldsymbol{\pi})$ be given. Then $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ and $\mathrm{D}_{\boldsymbol{\pi}} V = \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V$.*

Proof. Fix $W \in \mathrm{TestVF_c}(\boldsymbol{\pi})$. We know from Proposition 6.18 that $[0,1] \ni t \mapsto \int \langle V_t, W_t\rangle\,\mathrm{d}\boldsymbol{\pi}$ is an absolutely continuous function, so that (6.30) gives, after integration, that

$$0 = \int \langle V_1, W_1\rangle \,\mathrm{d}\boldsymbol{\pi} - \int \langle V_0, W_0\rangle \,\mathrm{d}\boldsymbol{\pi} = \int_0^1\!\!\int \langle \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V_t, W_t\rangle \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t + \int_0^1\!\!\int \langle V_t, \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} W_t\rangle \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t.$$

This proves that V satisfies (6.32) with $Z = \widetilde{\mathrm{D}}_{\boldsymbol{\pi}} V$. □

Proposition 6.21 (Basic properties of $\mathscr{W}^{1,2}(\boldsymbol{\pi})$) *The following hold:*

i) *The operator $\mathrm{D}_{\boldsymbol{\pi}}$ is closed from $\mathscr{L}^2(\boldsymbol{\pi})$ into itself, i.e. its graph is closed in the product space $\mathscr{L}^2(\boldsymbol{\pi}) \times \mathscr{L}^2(\boldsymbol{\pi})$.*

ii) *The space $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ is a separable Hilbert space.*

iii) *Let* $V, Z \in \mathscr{L}^2(\boldsymbol{\pi})$ *be given. Then we have* $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ *and* $Z = \mathrm{D}_{\boldsymbol{\pi}} V$ *if and only if the map* $t \mapsto \langle V_t, W_t \rangle$ *belongs to* $W^{1,1}\big([0,1], L^1(\boldsymbol{\pi})\big)$*, with derivative given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle V_t, W_t \rangle = \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle + \langle Z_t, W_t \rangle \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1], \tag{6.34}$$

for every $W \in \mathrm{TestVF}(\boldsymbol{\pi})$.

Proof. We divide the proof into some steps:
i) Let $(V_n)_n \subseteq \mathscr{W}^{1,2}(\boldsymbol{\pi})$ be a sequence such that $V_n \to V$ and $\mathrm{D}_{\boldsymbol{\pi}} V_n \to Z$ in $\mathscr{L}^2(\boldsymbol{\pi})$, for suitable vector fields $V, Z \in \mathscr{L}^2(\boldsymbol{\pi})$. Then for arbitrary $W \in \mathrm{TestVF_c}(\boldsymbol{\pi})$ we have that

$$\begin{aligned}\int_0^1\!\!\int \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t &= \lim_{n\to\infty} \int_0^1\!\!\int \langle V_t^n, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t = -\lim_{n\to\infty} \int_0^1\!\!\int \langle \mathrm{D}_{\boldsymbol{\pi}} V_t^n, W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t \\ &= -\int_0^1\!\!\int \langle Z_t, W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t,\end{aligned}$$

proving that $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ with $\mathrm{D}_{\boldsymbol{\pi}} V = Z$, which was the claim.
ii) Consequence of what just proved and the fact that the map

$$\mathscr{W}^{1,2}(\boldsymbol{\pi}) \ni V \longmapsto (V, \mathrm{D}_{\boldsymbol{\pi}} V) \in \mathscr{L}^2(\boldsymbol{\pi}) \times \mathscr{L}^2(\boldsymbol{\pi})$$

is an isometry, provided we endow $\mathscr{L}^2(\boldsymbol{\pi}) \times \mathscr{L}^2(\boldsymbol{\pi})$ with the norm

$$\big\|(V, Z)\big\| \doteq \sqrt{\|V\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} + \|Z\|^2_{\mathscr{L}^2(\boldsymbol{\pi})}},$$

which is separable by Proposition 6.7.
iii) The 'if' trivially follows from (6.34) by integration. For the 'only if', fix $W \in \mathrm{TestVF}(\boldsymbol{\pi})$, let $\varphi \in C^1_{\mathrm{c}}(0,1)$ and let $\Gamma \subseteq \Gamma(\mathrm{X})$ be Borel. Then $t \mapsto \varphi(t)\, \chi_\Gamma\, W_t$ is in $\mathrm{TestVF_c}(\boldsymbol{\pi})$ and a direct computation shows that $\mathrm{D}_{\boldsymbol{\pi}}(\varphi\, \chi_\Gamma\, W)_t = \varphi'(t)\, \chi_\Gamma\, W_t + \varphi(t)\, \chi_\Gamma\, \mathrm{D}_{\boldsymbol{\pi}} W_t$. Hence by writing the defining property (6.32) (with $\varphi\, \chi_\Gamma\, W$ in place of W) we get – after rearrangement – that

$$\int_0^1 \varphi'(t) \int_\Gamma \langle V_t, W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t = -\int_0^1 \varphi(t) \int_\Gamma \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle + \langle Z_t, W_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t.$$

The arbitrariness of φ, Γ and Proposition C.3 yield the claim. □

We just proved that $\mathrm{TestVF}(\boldsymbol{\pi})$ is contained in $\mathscr{W}^{1,2}(\boldsymbol{\pi})$, but we do not know if it is dense. Therefore the following definition is meaningful:

Definition 6.22 (The space $\mathscr{H}^{1,2}(\boldsymbol{\pi})$) *We define* $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ *as the* $\mathscr{W}^{1,2}(\boldsymbol{\pi})$*-closure of the space* $\mathrm{TestVF}(\boldsymbol{\pi})$.

Clearly, $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ is a separable Hilbert space. A key feature of the elements of $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ is that they admit a continuous representative (much like Sobolev functions on intervals):

Theorem 6.23 *The inclusion* $\mathrm{TestVF}(\boldsymbol{\pi}) \hookrightarrow \mathscr{C}(\boldsymbol{\pi})$ *uniquely extends to a linear, continuous and injective operator* $\iota : \mathscr{H}^{1,2}(\boldsymbol{\pi}) \to \mathscr{C}(\boldsymbol{\pi})$.

Proof. We claim that

$$\|V\|_{\mathscr{C}(\boldsymbol{\pi})} \le \sqrt{2}\, \|V\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})} \qquad \text{for every } V \in \mathrm{TestVF}(\boldsymbol{\pi}). \tag{6.35}$$

By density of $\mathrm{TestVF}(\boldsymbol{\pi})$ in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$, this is enough to get existence of the operator ι. Thus let $V \in \mathrm{TestVF}(\boldsymbol{\pi})$, pick $W = V$ in (6.30) and integrate in $[t_1, t_2] \subseteq [0,1]$ with respect to $\boldsymbol{\pi}$ to obtain that

$$\begin{aligned}\Big| [\![V]\!]_{t_2}^2 - [\![V]\!]_{t_1}^2 \Big| &= 2 \left| \int_{t_1}^{t_2} \int \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} V_t \rangle \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t \right| \\ &\leq 2 \int_{t_1}^{t_2} \int |V_t| \, |\mathrm{D}_{\boldsymbol{\pi}} V_t| \, \mathrm{d}\boldsymbol{\pi} \, \mathrm{d}t \leq \|V\|_{\mathscr{L}^2(\boldsymbol{\pi})}^2 + \|\mathrm{D}_{\boldsymbol{\pi}} V\|_{\mathscr{L}^2(\boldsymbol{\pi})}^2.\end{aligned}$$

Hence for any $t \in [0,1]$ one has that

$$[\![V]\!]_t^2 = \int_0^1 [\![V]\!]_t^2 \, \mathrm{d}s \leq \int_0^1 \Big| [\![V]\!]_t^2 - [\![V]\!]_s^2 \Big| \, \mathrm{d}s + \|V\|_{\mathscr{L}^2(\boldsymbol{\pi})}^2 \leq 2 \, \|V\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})}^2,$$

which is our claim (6.35).

In order to prove injectivity, choose $V \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ such that $\iota(V) = 0$. Let us pick any sequence $(V_n)_n \subseteq \mathrm{TestVF}(\boldsymbol{\pi})$ which is $\mathscr{W}^{1,2}(\boldsymbol{\pi})$-converging to V and notice that – up to passing to a subsequence and by using Proposition 6.6 – we can assume that $V_t^n \to V_t$ for almost every $t \in [0,1]$. By continuity of the operator ι, one also has that

$$\|V^n\|_{\mathscr{C}(\boldsymbol{\pi})} = \big\| \iota(V^n) - \iota(V) \big\|_{\mathscr{C}(\boldsymbol{\pi})} \longrightarrow 0$$

and thus in particular $V_t^n \to 0$ for all $t \in [0,1]$. Therefore $V_t = 0$ for $\mathcal{L}_1$-a.e. $t \in [0,1]$, yielding the required injectivity of ι. □

Whenever we will consider an element V of $\mathscr{H}^{1,2}(\boldsymbol{\pi})$, we will always implicitly refer to its unique continuous representative $\iota(V) \in \mathscr{C}(\boldsymbol{\pi})$.

Among the several properties of the test sections that can be carried over to the elements of $\mathscr{H}^{1,2}(\boldsymbol{\pi})$, the most important one is the Leibniz formula for convective derivatives:

Proposition 6.24 (Leibniz formula for $\mathrm{D}_{\boldsymbol{\pi}}$) *Let $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ and $W \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$. Then the map $t \mapsto \langle V_t, W_t \rangle$ is in $W^{1,1}\big([0,1], L^1(\boldsymbol{\pi})\big)$ and its derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle V_t, W_t \rangle = \langle \mathrm{D}_{\boldsymbol{\pi}} V_t, W_t \rangle + \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.36}$$

Proof. For $W \in \mathrm{TestVF}(\boldsymbol{\pi})$ the claim is a direct consequence of point iii) of Proposition 6.21. The general case can be achieved by approximation, just noticing that the simple inequalities

$$\begin{aligned}\big\| \langle V_t, W_t \rangle \big\|_{L^1(\boldsymbol{\pi} \times \mathcal{L}_1)} &\leq \|V\|_{\mathscr{L}^2(\boldsymbol{\pi})} \, \|W\|_{\mathscr{L}^2(\boldsymbol{\pi})}, \\ \big\| \langle \mathrm{D}_{\boldsymbol{\pi}} V_t, W_t \rangle + \langle V_t, \mathrm{D}_{\boldsymbol{\pi}} W_t \rangle \big\|_{L^1(\boldsymbol{\pi} \times \mathcal{L}_1)} &\leq 2 \, \|V\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})} \, \|W\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})}\end{aligned}$$

allow us to pass to the limit in the distributional formulation of $\frac{\mathrm{d}}{\mathrm{d}t} \langle V_t, W_t \rangle$ as the element W varies in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$. □

In the next proposition we collect some examples of elements of $\mathscr{H}^{1,2}(\boldsymbol{\pi})$:

Proposition 6.25 *The following hold:*

i) *Given any $w \in H_{\mathrm{C}}^{1,2}(TX)$, we have that the vector field $t \mapsto W_t \doteq \mathrm{e}_t^* w$ belongs to the space $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and satifies*

$$\mathrm{D}_{\boldsymbol{\pi}} W_t = \mathrm{Cov}_t(w) \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.37}$$

ii) *Let* $W \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ *be such that* $|W|, |\mathrm{D}_{\boldsymbol{\pi}} W| \in L^\infty(\boldsymbol{\pi} \times \mathcal{L}_1)$ *and* $a \in W^{1,2}\big([0,1], L^2(\boldsymbol{\pi})\big)$. *Then* $aW \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ *and*

$$\mathrm{D}_{\boldsymbol{\pi}}(aW)_t = a'_t\, W_t + a_t\, \mathrm{D}_{\boldsymbol{\pi}} W_t \qquad \textit{for } \mathcal{L}_1\textit{-a.e. } t \in [0,1]. \tag{6.38}$$

Moreover, if $W \in \mathscr{C}(\boldsymbol{\pi})$ *and* $a \in AC^2\big([0,1], L^2(\boldsymbol{\pi})\big)$, *then* $aW \in \mathscr{C}(\boldsymbol{\pi})$.

Proof. We divide the proof into some steps:
i) If $w \in \mathrm{TestV(X)}$ then $W \in \mathrm{TestVF}(\boldsymbol{\pi})$ by definition and in this case formula (6.37) holds by the definition (6.29) and by Proposition 6.20. The general case can be obtained by approximating w with vector fields in $\mathrm{TestV(X)}$ with respect to the $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$ topology, by using the bounds

$$\begin{aligned} \int_0^1\!\!\int \big|\mathrm{e}_t^*(v)\big|^2 \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \;&=\; \int_0^1\!\!\int |v|^2 \circ \mathrm{e}_t \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \leq \mathsf{C}(\boldsymbol{\pi})\, \|v\|^2_{W^{1,2}_{\mathrm{C}}(T\mathrm{X})}, \\ \int_0^1\!\!\int \big|\mathrm{Cov}_{\boldsymbol{\pi}}(v)_t\big|^2 \,\mathrm{d}\boldsymbol{\pi}\,\mathrm{d}t \;&\overset{(6.23)}{\leq}\; \mathsf{C}(\boldsymbol{\pi})\, \mathsf{L}(\boldsymbol{\pi})^2\, \|v\|^2_{W^{1,2}_{\mathrm{C}}(T\mathrm{X})} \end{aligned}$$

and by recalling the closure of the operator $\mathrm{D}_{\boldsymbol{\pi}}$.
ii) The claim about continuity is obvious, so we concentrate on the other one. Assume at first that a belongs to the space $\mathcal{A}$, defined as

$$\mathcal{A} \doteq \left\{ \sum_{i=1}^n \varphi_i\, \chi_{E_i} \;\middle|\; n \in \mathbb{N},\ \varphi_i \in \mathrm{LIP}([0,1]),\ (E_i)_i \text{ Borel partition of } \Gamma(\mathrm{X}) \right\}$$

and that $W \in \mathrm{TestVF}(\boldsymbol{\pi})$. In this case aW belongs to $\mathrm{TestVF}(\boldsymbol{\pi})$ as well and formula (6.38) is a direct consequence of the definitions. Then by using the trivial bounds

$$\begin{aligned} \|aW\|_{\mathscr{L}^2(\boldsymbol{\pi})} &\leq \|a\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)}\, \|W\|_{\mathscr{L}^2(\boldsymbol{\pi})}, \\ \big\|a'\, W + a\, \mathrm{D}_{\boldsymbol{\pi}} W\big\|_{\mathscr{L}^2(\boldsymbol{\pi})} &\leq \Big(\|a\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)} + \|a'\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)} \Big)\, \|W\|_{\mathscr{W}^{1,2}(\boldsymbol{\pi})}, \end{aligned}$$

the $\mathscr{W}^{1,2}(\boldsymbol{\pi})$-density of $\mathrm{TestVF}(\boldsymbol{\pi})$ in $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ and the closure of the operator $\mathrm{D}_{\boldsymbol{\pi}}$, we conclude that $aW \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ for every $a \in \mathcal{A}$ and $W \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$, and that (6.38) holds in this case.

Now let W be as in the assumptions and observe that we also have the bounds

$$\begin{aligned} \|aW\|_{\mathscr{L}^2(\boldsymbol{\pi})} &\leq \|a\|_{L^2([0,1],L^2(\boldsymbol{\pi}))}\, \big\||W|\big\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)}, \\ \big\|a'\, W + a\, \mathrm{D}_{\boldsymbol{\pi}} W\big\|_{\mathscr{L}^2(\boldsymbol{\pi})} &\leq \|a\|_{W^{1,2}([0,1],L^2(\boldsymbol{\pi}))} \Big(\big\||W|\big\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)} + \big\||\mathrm{D}_{\boldsymbol{\pi}} W|\big\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)} \Big). \end{aligned}$$

Therefore – by using again the closure of $\mathrm{D}_{\boldsymbol{\pi}}$ – we see that to conclude it is sufficient to prove that $\mathcal{A}$ is dense in $W^{1,2}\big([0,1], L^2(\boldsymbol{\pi})\big)$. To this aim, we argue as follows: for every $n \in \mathbb{N}$, let us take a Borel partition $(E_i^n)_{i\in\mathbb{N}}$ of $\mathrm{spt}(\boldsymbol{\pi}) \subseteq \Gamma(\mathrm{X})$, made of sets with positive $\boldsymbol{\pi}$-measure and diameter $\leq \frac{1}{n}$. Then for every $n, N \in \mathbb{N}$ let $P_n^N : L^2(\boldsymbol{\pi}) \to L^2(\boldsymbol{\pi})$ be defined by

$$P_n^N(f) \doteq \sum_{i=1}^N \chi_{E_i^n}\, \frac{1}{\boldsymbol{\pi}(E_i^n)} \int_{E_i^n} f \,\mathrm{d}\boldsymbol{\pi}.$$

It is clear that P_n^N has operator norm ≤ 1 for every $n, N \in \mathbb{N}$ and an application of the dominated convergence theorem shows that

$$\lim_{n\to\infty} \lim_{N} P_n^N(f) = f \tag{6.39}$$

for every $f \in C_b\big(\Gamma(\mathrm{X})\big)$, the limits being intended in $L^2(\boldsymbol{\pi})$. Therefore (6.39) also holds for every $f \in L^2(\boldsymbol{\pi})$. The linearity and continuity of P_n^N grant that if $t \mapsto a_t$ belongs to the space $W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$, then $t \mapsto P_n^N(a)_t \doteq P_n^N(a_t)$ is in $W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$ as well and

$$\big(P_n^N(a)\big)'_t = P_n^N(a'_t) \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1]. \tag{6.40}$$

All these considerations imply that

$$\lim_{n\to\infty}\lim_{N\to\infty} P_n^N(a) = a \qquad \text{in } W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$$

for every $a \in W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$, thus to conclude it is sufficient to prove that $P_n^N(a)$ belongs to the $W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$-closure of $\mathcal{A}$ for every $n,N \in \mathbb{N}$ and $a \in W^{1,2}\big([0,1],L^2(\boldsymbol{\pi})\big)$.

Finally, it is clear by construction and by (6.40) that we can write $P_n^N(a) = \sum_{i=1}^N g_i\,\chi_{E_i^n}$ for some $g_i \in W^{1,2}(0,1)$. Given any $i = 1,\dots,N$, we can find a sequence $(g_{i,j})_j \subseteq \mathrm{LIP}\big([0,1]\big)$ that $W^{1,2}([0,1])$-converges to g_i. Note that

$$\bigg\|\sum_{i=1}^N (g_{i,j}-g_i)\,\chi_{E_i^n}\bigg\|^2_{W^{1,2}([0,1],L^2(\boldsymbol{\pi}))} = \sum_{i=1}^N \boldsymbol{\pi}(E_i^n)\,\|g_{i,j}-g_i\|^2_{W^{1,2}(0,1)} \longrightarrow 0 \qquad \text{as } j \to \infty.$$

Since $\sum_{i=1}^N g_{i,j}\,\chi_{E_i^n} \in \mathcal{A}$ for every j, the proof is completed. □

6.2 Parallel transport on RCD spaces

6.2.1 Definition and basic properties of parallel transport

By relying upon the machinery developed in the Section 6.1, we propose a notion of parallel transport for RCD spaces. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a fixed $\mathsf{RCD}(K,\infty)$ space, for some $K \in \mathbb{R}$.

We shall frequently use the fact that, since $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ is continuously embedded into $\mathscr{C}(\boldsymbol{\pi})$ by Theorem 6.23, any vector field $V \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ has pointwise values $V_t \in \mathrm{e}_t^*L^2(T\mathrm{X})$ defined at every time $t \in [0,1]$.

Definition 6.26 (Parallel transport) *Let $\boldsymbol{\pi}$ be a Lipschitz test plan on* X. *Then a* parallel transport along $\boldsymbol{\pi}$ *is an element $V \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ such that $\mathrm{D}_{\boldsymbol{\pi}}V = 0$.*

The linearity of the requirement $\mathrm{D}_{\boldsymbol{\pi}}V = 0$ ensures that the set of parallel transports forms a vector space. From Proposition 6.24 we deduce the following simple but crucial result:

Proposition 6.27 (Norm preservation) *Let V be a parallel transport along a Lipschitz test plan $\boldsymbol{\pi}$ on* X. *Then the map $t \mapsto |V_t|^2 \in L^1(\boldsymbol{\pi})$ is constant.*

Proof. We know from Corollary 6.13 that the map $t \mapsto |V_t|^2 \in L^1(\boldsymbol{\pi})$ is continuous. Hence the choice $W = V$ in Proposition 6.24 tells that such map is absolutely continuous, with derivative given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|V_t|^2 = 2\,\langle \mathrm{D}_{\boldsymbol{\pi}}V_t, V_t\rangle = 0 \qquad \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1].$$

This is sufficient to conclude. □

Linearity and norm preservation imply uniqueness:

Corollary 6.28 (Uniqueness of parallel transport) *Let $\boldsymbol{\pi}$ be a Lipschitz test plan on* X. *Let V_1, V_2 be parallel transports along $\boldsymbol{\pi}$ with $V_{1,t_0} = V_{2,t_0}$ for some $t_0 \in [0,1]$. Then $V_1 = V_2$.*

Proof. Since $\mathrm{D}_{\boldsymbol{\pi}}(V_1 - V_2) = \mathrm{D}_{\boldsymbol{\pi}} V_1 - \mathrm{D}_{\boldsymbol{\pi}} V_2 = 0$, we have that $V_1 - V_2$ is a parallel transport and by assumption we know that $|V_{1,t_0} - V_{2,t_0}| = 0$ in the $\boldsymbol{\pi}$-a.e. sense. Thus Proposition 6.27 grants that for every $t \in [0,1]$ it $\boldsymbol{\pi}$-a.e. holds that $|V_{1,t} - V_{2,t}| = 0$, i.e. $V_{1,t} = V_{2,t}$. □

Remark 6.29 We emphasise that the norm preservation property is a consequence of the Leibniz formula in Proposition 6.24. We do not know if such formula holds for $V, W \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$ and this is why we defined the parallel transport as an element of $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ with null convective derivative, as opposed to an element of $\mathscr{W}^{1,2}(\boldsymbol{\pi})$ with the same property. ■

We now assume existence of parallel transport along some/all Lipschitz test plans and see what can be derived from such assumption. First of all, we show that a parallel transport sends bases into bases:

Proposition 6.30 *Let $\boldsymbol{\pi}$ be a Lipschitz test plan on* X *with this property: given any $t \in [0,1]$ and $\bar{V}_t \in \mathrm{e}_t^* L^2(T\mathrm{X})$, there exists a (unique) parallel transport V along $\boldsymbol{\pi}$ such that $V_t = \bar{V}_t$. Call $(E_n)_{n \in \mathbb{N} \cup \{\infty\}}$ the dimensional decomposition of $\mathrm{e}_0^* L^2(T\mathrm{X})$ and choose any orthonormal basis $(\bar{V}_n)_{n \in \mathbb{N}}$ of $\mathrm{e}_0^* L^2(T\mathrm{X})$, i.e. $\bar{V}_1, \ldots, \bar{V}_n$ is an orthonormal basis for $\mathrm{e}_0^* L^2(T\mathrm{X})$ on E_n for any $n \in \mathbb{N}$. Denote by $t \mapsto V_{n,t}$ the parallel transport of $\bar{V}_n$ along $\boldsymbol{\pi}$. Then for every $t \in [0,1]$ it holds that the partition $(E_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is also the dimensional decomposition of $\mathrm{e}_t^* L^2(T\mathrm{X})$ and that the set $(V_{n,t})_{n \in \mathbb{N}}$ is an orthonormal basis of $\mathrm{e}_t^* L^2(T\mathrm{X})$.*

Proof. For every $t, s \in [0,1]$, let us consider the map sending $\bar{V} \in \mathrm{e}_t^* L^2(T\mathrm{X})$ to $V_s \in \mathrm{e}_s^* L^2(T\mathrm{X})$, where $V \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ is the parallel transport along $\boldsymbol{\pi}$ such that $V_t = \bar{V}$. Proposition 6.27 ensures that this map preserves the pointwise norm. Since it is clearly linear, it is easily verified that it is an isomorphism of $\mathrm{e}_t^* L^2(T\mathrm{X})$ and $\mathrm{e}_s^* L^2(T\mathrm{X})$. The conclusions follow. □

We shall apply the previous result to show that – under the same assumptions – we have the equality $\mathscr{W}^{1,2}(\boldsymbol{\pi}) = \mathscr{H}^{1,2}(\boldsymbol{\pi})$:

Proposition 6.31 $(\mathscr{H} = \mathscr{W})$ *Let $\boldsymbol{\pi}$ be a Lipschitz test plan on* X *with the following property: given any $t \in [0,1]$ and $\bar{V}_t \in \mathrm{e}_t^* L^2(T\mathrm{X})$, there exists a (unique) parallel transport V along $\boldsymbol{\pi}$ such that $V_t = \bar{V}_t$. Then $\mathscr{H}^{1,2}(\boldsymbol{\pi}) = \mathscr{W}^{1,2}(\boldsymbol{\pi})$.*

Proof. Fix $V \in \mathscr{W}^{1,2}(\boldsymbol{\pi})$. Choose an orthonormal basis $(\bar{V}_i)_{i \in \mathbb{N}} \subseteq \mathrm{e}_0^* L^2(T\mathrm{X})$ of $\mathrm{e}_0^* L^2(T\mathrm{X})$ and call $t \mapsto V_{i,t}$ the parallel transport of $\bar{V}_i$ along $\boldsymbol{\pi}$. Then by Proposition 6.30 we see that

$$V_t = \sum_{i \in \mathbb{N}} a_{i,t} V_{i,t} \qquad \text{where we set } a_{i,t} \doteq \langle V_t, V_{i,t} \rangle \text{ for a.e. } t \in [0,1], \tag{6.41}$$

being intended that the series absolutely converges in $\mathrm{e}_t^* L^2(T\mathrm{X})$ for almost every $t \in [0,1]$. By Proposition 6.24, we see that $t \mapsto a_{i,t}$ is in $W^{1,1}\big([0,1], L^1(\boldsymbol{\pi})\big)$, with derivative given by

$$a'_{i,t} = \langle \mathrm{D}_{\boldsymbol{\pi}} V_t, V_{i,t} \rangle. \tag{6.42}$$

In particular, since $|V_{i,t}| \leq 1$ we see that $a_{i,t}, a'_{i,t} \in L^2\big([0,1], L^2(\boldsymbol{\pi})\big)$ and in turn this implies – by Proposition C.3 – that the mapping $t \mapsto a_{i,t}$ belongs to $W^{1,2}\big([0,1], L^2(\boldsymbol{\pi})\big)$. This fact and item ii) of Proposition 6.25 give that $\big(t \mapsto a_{i,t}V_{i,t}\big) \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ for every $i \in \mathbb{N}$, thus accordingly we have $\big(t \mapsto \sum_{i=0}^{n} a_{i,t}V_{i,t}\big) \in \mathscr{H}^{1,2}(\boldsymbol{\pi})$ for any $n \in \mathbb{N}$.

Hence to conclude it is sufficient to show that these partial sums form a $\mathscr{W}^{1,2}(\boldsymbol{\pi})$-Cauchy sequence, as then it is clear from (6.41) that the limit coincides with V. From (6.41) and (6.42) we have that

$$\sum_{i\in\mathbb{N}} \iint_0^1 |a_{i,t}|^2 + |a'_{i,t}|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi} = \|V\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} + \|\mathrm{D}_{\boldsymbol{\pi}}V\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} < +\infty.$$

Since we also have the identity

$$\bigg\|\sum_{i=n}^{m} a_i\,V_i\bigg\|^2_{\mathscr{W}^{1,2}(\boldsymbol{\pi})} = \bigg\|\sum_{i=n}^{m} a_i\,V_i\bigg\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} + \bigg\|\sum_{i=n}^{m} a'_i\,V_i\bigg\|^2_{\mathscr{L}^2(\boldsymbol{\pi})} = \sum_{i=n}^{m} \iint_0^1 |a_{i,t}|^2 + |a'_{i,t}|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi},$$

the conclusion follows. □

We shall now prove that if the parallel transport exists along all Lipschitz test plans, then the dimension of X – intended as the dimension of the tangent module – must be constant.

Theorem 6.32 (From parallel transport to constant dimension) *Suppose that, given any Lipschitz test plan $\boldsymbol{\pi}$ on X, any $t \in [0,1]$ and any $\bar{V} \in \mathrm{e}_t^* L^2(T\mathrm{X})$, there exists the parallel transport V along $\boldsymbol{\pi}$ with $V_t = \bar{V}$. Then the tangent module $L^2(T\mathrm{X})$ has constant dimension, i.e. in its dimensional decomposition $(E_i)_{i\in\mathbb{N}\cup\{\infty\}}$ one of the E_i's has full measure.*

Proof. We argue by contradiction: assume to have $\mathfrak{m}(E_i), \mathfrak{m}(E_j) > 0$ for some $i, j \in \mathbb{N}\cup\{\infty\}$ with $i \neq j$. Let $F_0 \subseteq E_i$, $F_1 \subseteq E_j$ be bounded Borel sets of positive (finite) measure. Consider

$$\mu_0 \doteq \mathfrak{m}(F_0)^{-1}\,\mathfrak{m}_{|F_0},$$
$$\mu_1 \doteq \mathfrak{m}(F_1)^{-1}\,\mathfrak{m}_{|F_1}.$$

Let $\boldsymbol{\pi}$ be the unique optimal geodesic plan connecting them and recall that it is a test plan (cf. Remark 4.42). Since $\boldsymbol{\pi}\big(\mathrm{e}_0^{-1}(F_0)\big) = \mu_0(F_0) = 1$ and the dimension of $L^2(T\mathrm{X})$ on F_0 is i, by Theorem 3.38 we see that for the dimensional decomposition $(\tilde{E}^0_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of $\mathrm{e}_0^* L^2(T\mathrm{X})$ we have that $\boldsymbol{\pi}(\tilde{E}^0_i) = 1$ and $\boldsymbol{\pi}(\tilde{E}^0_k) = 0$ for every $k \neq i$. Similarly, for the dimensional decomposition $(\tilde{E}^1_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of $\mathrm{e}_1^* L^2(T\mathrm{X})$ we have $\boldsymbol{\pi}(\tilde{E}^1_j) = 1$ and $\boldsymbol{\pi}(\tilde{E}^1_k) = 0$ for every $k \neq j$. In particular, it holds that

$$\boldsymbol{\pi}\big(\tilde{E}^0_i \,\Delta\, \tilde{E}^1_i\big) = \boldsymbol{\pi}(\tilde{E}^0_i) = 1 > 0. \tag{6.43}$$

Now notice that – from basic considerations about optimal transport – we have that $\boldsymbol{\pi}$ is concentrated on geodesics starting from F_0 and ending in F_1. The constant speed of any such geodesic is bounded from above by $\sup_{x\in F_0,\, y\in F_1} \mathsf{d}(x,y) < \infty$, so that $\boldsymbol{\pi}$ is a Lipschitz test plan. Therefore Proposition 6.30 grants that the dimensional decomposition of $\mathrm{e}_t^* L^2(T\mathrm{X})$ does not depend on t. This contradicts (6.43), whence the proof is achieved. □

Remark 6.33 It has been recently proved in [BS18a] that any finite-dimensional RCD space has constant dimension, in the sense of Theorem 6.32. As already mentioned in the Introduction,this represented one of the most important open problems in the theory of RCD spaces. The proof relies upon some regularity estimates for the regular Lagrangian flows, that have been obtained in [BS18b]. ■

6.2.2 Existence of the parallel transport in a special case

It is unclear whether on general RCD spaces the parallel transport exists. Aim of the present subsection is to show at least that the theory we propose is not empty, i.e. that – under suitable assumptions on the space – the parallel transport exists. We will not insist in trying to make such assumptions as general as possible (for instance, the 'good base' defined below could consist of different vector fields on different open sets covering our space), as our main concern is just to show that in some circumstances our notion of parallel transport can be shown to exist.

We shall work with spaces admitting the following sort of basis for the tangent module:

Definition 6.34 (Good basis) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a given* $\mathsf{RCD}(K,N)$ *space, for some* $K\in\mathbb{R}$ *and* $N\in(1,\infty)$*. Let us denote by* $(A_k)_{k=1}^n$ *the dimensional decomposition of* X*. Then a family* $\mathcal{W}=\{w_1,\dots,w_n\}\subseteq H^{1,2}_C(T\mathrm{X})$ *of Sobolev vector fields on* X *is said to be a* good basis *for* $L^2(T\mathrm{X})$ *provided there exists* $M>0$ *such that the following properties are satisfied:*

i) *For any* $k=1,\dots,n$*, we have that* $w_1,\dots,w_k$ *constitute a basis for* $L^2(T\mathrm{X})$ *on* A_k *and*

$$\text{(6.44)}\qquad \begin{cases} |w_i|\in(M^{-1},M),\\ \big|\langle w_i,w_j\rangle\big|<\frac{1}{M^2k}\end{cases}\quad \mathfrak{m}\text{-a.e. in } A_k\qquad \text{for every } i,j=1,\dots,k \text{ with } i\neq j.$$

ii) *It holds that*

$$\text{(6.45)}\qquad |\nabla w_i|_{\mathsf{HS}}\leq M\quad \mathfrak{m}\text{-a.e. in X}\qquad \text{for every } i=1,\dots,n.$$

Let us notice that the 'hard' assumption here is given by point ii) – perhaps coupled with the lower bound in i) – which imposes an L^∞ bound on the covariant derivative, when in our setting the L^2 ones are more natural (compare with Theorem 6.39 below). Let us mention, in particular, that for spaces admitting a good basis it is not hard to prove – regardless of parallel transport – that the dimension is constant, as we shall see in Proposition 6.36.

Let us start the technical work with the following simple lemma:

Lemma 6.35 *Let* $\mathscr{H}$ *be a Hilbert* $L^2(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module. Fix a Borel subset* A *of* X *and a constant* $M>1$*. Let* $k\in\mathbb{N}$*. Suppose that there exist* $w_1,\dots,w_k\in\mathscr{H}$ *such that*

$$\text{(6.46)}\qquad \begin{cases} |w_i|\in(M^{-1},M)\\ \big|\langle w_i,w_j\rangle\big|\leq\frac{1}{M^2k}\end{cases}\quad \text{hold } \mathfrak{m}\text{-a.e. in } A,\qquad \text{for every } i,j=1,\dots,k \text{ with } i\neq j.$$

Given any $h_1,\dots,h_k\in L^0\big(\mathfrak{m}|_A\big)$*, let us define* $w\doteq\sum_{i=1}^k h_i\,w_i\in\mathscr{H}^0$*, where* $\mathscr{H}^0$ *denotes the* L^0*-completion of* $\mathscr{H}$*. Then it holds that*

$$\text{(6.47)}\qquad \frac{1}{M^2\,k}\sum_{i=1}^k|h_i|^2\leq|w|^2\leq M^2\,k\sum_{i=1}^k|h_i|^2\qquad \mathfrak{m}\text{-a.e. in } A,$$

thus in particular $w\in\mathscr{H}|_A$ *if and only if* $h_i\in L^2\big(\mathfrak{m}|_A\big)$ *for every* $i=1,\dots,k$*.*

Proof. For the second inequality in (6.47), note that $\langle w_i, w_j\rangle \le M^2$ $\mathfrak{m}$-a.e. on A for every i, j, thus accordingly

$$|w|^2 = \Big|\sum_{i=1}^k h_i\, w_i\Big|^2 = \sum_{i,j=1}^k h_i\, h_j\, \langle w_i, w_j\rangle \le M^2 \sum_{i,j=1}^k \frac{1}{2}\,|h_i|^2 + \frac{1}{2}\,|h_j|^2 = M^2\, k \sum_{i=1}^k |h_i|^2.$$

For the first inequality in (6.47), we recall that $|w_i| > M^{-1}$ and $\langle w_i, w_j\rangle \ge -\frac{1}{M^2\,k}$ hold $\mathfrak{m}$-a.e. on A for $i \neq j$ to deduce that

$$\begin{aligned}|w|^2 &= \Big|\sum_{i=1}^k h_i\, w_i\Big|^2 = \sum_{i=1}^k |h_i|^2\, |w_i|^2 + \sum_{i\neq j} h_i\, h_j\, \langle w_i, w_j\rangle \ge \frac{1}{M^2}\sum_{i=1}^k |h_i|^2 - \frac{1}{M^2\, k}\sum_{i\neq j}|h_i\, h_j| \\ &\ge \frac{1}{M^2}\sum_{i=1}^k |h_i|^2 - \frac{1}{M^2\, k}\sum_{i\neq j}\frac{1}{2}|h_i|^2 + \frac{1}{2}|h_j|^2 = \frac{1}{M^2 k}\sum_{i=1}^k |h_i|^2.\end{aligned}$$

Therefore the statement is achieved. □

The constant dimension now easily follows – without using [BS18a] – from Lemma 6.35 and the fact that if a good basis exists, then there is another one for which the functions $\langle w_i, w_j\rangle$ are Lipschitz, as shown in the proof of the following proposition.

Proposition 6.36 *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space admitting a good basis for* $L^2(T\mathrm{X})$*. Then the tangent module has constant dimension, meaning that in its dimensional decomposition* $(E_k)_{k=1}^n$ *one of the* E_k*'s has full measure.*

Proof. Let $k \in \mathbb{N}$ be the maximal index such that $\mathfrak{m}(E_k) > 0$; its existence follows from the finiteness results in [Han18] and [GP16b]. To conclude, it is enough to show that on a neighbourhood of E_k the tangent module has dimension $\ge k$. Let $(w_i)_{i=1}^k$ be a good basis and let $f \in C^\infty_{\mathrm{c}}(\mathbb{R})$ be such that $f(z) = z$ for every $z \in [0, M]$. Let us consider the vector fields $\tilde{w}_i \doteq f\big(|w_i|^2\big)\, w_i$. Note that $f\big(|w_i|^2\big) \in W^{1,2}(\mathrm{X})$ with

$$\nabla f\big(|w_i|^2\big) = 2\, f'\big(|w_i|^2\big)\, \nabla w_i(\cdot, w_i),$$

hence by (6.45) and the choice of f we see that $f\big(|w_i|^2\big)$ is bounded with bounded gradient. It follows that $\tilde{w}_i \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ with

$$\nabla \tilde{w}_i = \nabla f\big(|w_i|^2\big) \otimes w_i + f\big(|w_i|^2\big)\, \nabla w_i,$$

so that from the expression of $\nabla f(|w_i|^2)$ we deduce that $\tilde{w}_i$ is bounded with bounded covariant derivative. Hence each $g_{i,j} \doteq \langle \tilde{w}_i, \tilde{w}_j\rangle$ belongs to $W^{1,2}(\mathrm{X})$ and is bounded with bounded gradient as well. By the Sobolev-to-Lipschitz property (recall item iii) of Definition 4.40) we deduce that $g_{i,j}$ has a Lipschitz – in particular continuous – representative. By construction, the bounds (6.44) hold on E_k for the $\tilde{w}_i$'s, hence the continuity of $g_{i,j}$ grants that they hold also on some neighbourhood of E_k. By Lemma 6.35, this is sufficient to conclude that the vector fields $\tilde{w}_i$ are independent – by the first in (6.47) – on such neighbourhood, thus concluding the proof of the statement. □

We now prove existence of the parallel transport for the class of those RCD spaces that admit a good basis for their tangent module. In the proof we shall use, for simplicity, the fact just proved that the dimension must be constant, but actually the same argument works even without knowing a priori this fact (this remark is perhaps irrelevant, since constant dimension follows so directly from the existence of a good basis).

Theorem 6.37 (Existence of the parallel transport) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any* $\mathsf{RCD}(K,N)$ *space, with* $K\in\mathbb{R}$ *and* $N\in(1,\infty)$*, that admits a good basis. Let* $\boldsymbol{\pi}$ *be a Lipschitz test plan on* X *and fix* $\bar{V}\in\mathrm{e}_0^*L^2(T\mathrm{X})$*. Then there exists the parallel transport* $V\in\mathscr{H}^{1,2}(\boldsymbol{\pi})$ *along* $\boldsymbol{\pi}$ *such that* $V_0=\bar{V}$*.*

Proof. We know by Proposition 6.36 that the dimension of the tangent module must be constant. From this fact and Definition 6.34, we see that there exist $w_1,\ldots,w_n\in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$, for some $n\in\mathbb{N}^+$, such that (6.44) and (6.45) hold on X. Put $W_{i,t}\doteq\mathrm{e}_t^*w_i$ for all $t\in[0,1]$. By item i) of Proposition 6.25 we have that $W_i\in\mathscr{H}^{1,2}(\boldsymbol{\pi})$ with

$$\mathrm{D}_{\boldsymbol{\pi}}W_{i,t}=\mathrm{Cov}_t(w_i),$$

therefore from (6.23), (6.45) and the assumption that $\boldsymbol{\pi}$ is Lipschitz we get

$$|\mathrm{D}_{\boldsymbol{\pi}}W_{i,t}|\leq M\,\mathsf{L}(\boldsymbol{\pi}). \tag{6.48}$$

Moreover, by the defining property of the pullback map and from (6.44) we know that for every $t\in[0,1]$ it holds

$$\begin{cases}|W_{i,t}|\in(M^{-1},M),\\ \big|\langle W_{i,t},W_{j,t}\rangle\big|\leq(M^2\,k)^{-1}\end{cases}\quad\boldsymbol{\pi}\text{-a.e.}\qquad\text{for every } i,j=1,\ldots,n \text{ with } i\neq j,$$

thus accordingly Lemma 6.35 grants the existence of suitable functions $\bar{g}_1,\ldots,\bar{g}_n\in L^2(\boldsymbol{\pi})$ that satisfy $\bar{V}=\sum_{i=1}^n\bar{g}_i\,\mathrm{e}_0^*w_i$. A similar argument applied to the pullback of the map e (recall Proposition 6.9 and the definition (6.13)) – and based on the bound (6.48) – shows that there are functions $H_{i,j}\in L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)$ such that

$$\mathrm{D}_{\boldsymbol{\pi}}W_{i,t}=\sum_j H_{i,j,t}\,W_{j,t}\qquad\text{for a.e. } t\in[0,1]. \tag{6.49}$$

It will be technically convenient to fix once and for all Borel representatives of these functions – still denoted by $H_{i,j}$ – such that

$$\sup_{\gamma,t}\big|H_{i,j,t}(\gamma)\big|=\|H_{i,j}\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)}\qquad\text{for every } i=1,\ldots,n. \tag{6.50}$$

We shall look for a parallel transport of the form $V\doteq\sum_i g_i\,W_i$ with $g_i\in AC^2\big([0,1],L^2(\boldsymbol{\pi})\big)$. Notice that Lemma 6.35 grants that any such V belongs to $\mathscr{H}^{1,2}(\boldsymbol{\pi})$ with

$$\begin{aligned}\mathrm{D}_{\boldsymbol{\pi}}V_t&\overset{(6.38)}{=}\sum_{i=1}^n g'_{i,t}\,W_{i,t}+\sum_{i=1}^n g_{i,t}\,\mathrm{D}_{\boldsymbol{\pi}}W_{i,t}\overset{(6.49)}{=}\sum_{i=1}^n g'_{i,t}\,W_{i,t}+\sum_{i,j=1}^n g_{i,t}\,H_{i,j,t}\,W_{j,t}\\&=\sum_{i=1}^n\bigg(g'_{i,t}+\sum_{j=1}^n H_{j,i,t}\,g_{j,t}\bigg)W_{i,t}\qquad\text{for a.e. } t\in[0,1].\end{aligned}$$

Hence our V is the desired parallel transport if and only if $g_1,\ldots,g_n$ solve the system

$$\begin{cases}g_{i,0}=\bar{g}_i,\\ g'_{i,t}+\sum_{j=1}^n H_{j,i,t}\,g_{j,t}=0\quad\text{for a.e. } t\end{cases}\qquad\text{for every } i=1,\ldots,n. \tag{6.51}$$

To solve the previous system, we shall apply Theorem C.6 to the Banach (in fact, Hilbert) space $\mathbb{B}\doteq\big[L^2(\boldsymbol{\pi})\big]^n$ equipped with the norm

$$\|f\|_{\mathbb{B}}\doteq\bigg(\sum_{i=1}^n\int|f_i|^2\,\mathrm{d}\boldsymbol{\pi}\bigg)^{1/2}\qquad\text{for every } f=(f_1,\ldots,f_n)\in\mathbb{B}.$$

For every $t \in [0,1]$, let us define $\lambda_t \in \mathrm{End}(\mathbb{B})$ as

$$(\lambda_t f)_i \doteq -\sum_{j=1}^n H_{j,i,t}\, f_j \qquad \text{for every } i = 1,\dots,n \text{ and } f = (f_1,\dots,f_n) \in \mathbb{B},$$

so that the system (6.51) can be equivalently rewritten in the compact form

$$\begin{cases} g_0 = \bar{g}, \\ g'_t = \lambda_t g_t \end{cases} \qquad \text{for a.e. } t \in [0,1],$$

where $\bar{g} \doteq (\bar{g}_1,\dots,\bar{g}_n)$. Theorem C.6 grants that a solution in $\mathrm{LIP}\big([0,1],\mathbb{B}\big) \subseteq AC^2\big([0,1],\mathbb{B}\big)$ exists provided the λ_t's are equibounded and $t \mapsto \lambda_t f$ is strongly measurable for every $f \in \mathbb{B}$. The former follows from

$$\begin{aligned}\big\|\lambda_t f\big\|_{\mathbb{B}}^2 &= \sum_{i=1}^n \Big\| \sum_{j=1}^n H_{j,i,t}\, f_j \Big\|_{L^2(\boldsymbol{\pi})}^2 \leq n \sum_{i,j=1}^n \big\|H_{j,i,t}\, f_j\big\|_{L^2(\boldsymbol{\pi})}^2 \\ &\leq n \max_{i,j,t} \|H_{j,i,t}\|_{L^\infty(\boldsymbol{\pi})}^2 \sum_{i,j=1}^n \|f_j\|_{L^2(\boldsymbol{\pi})}^2 \overset{(6.50)}{\leq} n^2 \max_{i,j} \|H_{i,j}\|_{L^\infty(\boldsymbol{\pi}\times\mathcal{L}_1)}^2 \|f\|_{\mathbb{B}}^2.\end{aligned}$$

For the latter, notice that since $\mathbb{B}$ is separable it is sufficient to prove that for any $f \in \mathbb{B}$ the map $t \mapsto \lambda_t f \in \mathbb{B}$ is weakly measurable. Given that $\mathbb{B}$ is also Hilbert, we just need to show that for any $f, g \in \mathbb{B}$ the function $t \mapsto \langle \lambda_t f, g\rangle_{\mathbb{B}} \in \mathbb{R}$ is measurable. Since we have that

$$\langle \lambda_t f, g\rangle_{\mathbb{B}} = -\sum_{i,j=1}^n \int H_{j,i,t}\, f_j\, g_i \,\mathrm{d}\boldsymbol{\pi},$$

the conclusion follows from Fubini theorem. □

6.3 Sobolev basis of the tangent module

In this conclusive section we show that one can always build a basis of the tangent module of an RCD space which has Sobolev regularity, as opposed to just L^2 regularity. The basic idea used in the construction is based on the observation that 'being a basis' is a non-linear requirement. Technically speaking, the crucial argument is contained in the following lemma:

Lemma 6.38 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space, for some $K \in \mathbb{R}$. Let $w \in L^2(T\mathrm{X})$ be given. Then there exists $v \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ such that $\langle v, w\rangle \neq 0$ holds $\mathfrak{m}$-a.e. on $\big\{|w| \neq 0\big\}$.*

Proof. We can assume $w \neq 0$ or otherwise there is nothing to prove; then replacing if necessary w with $\big(\chi_{\{|w|\leq 1\}} + \chi_{\{|w|>1\}}|w|^{-1}\big)w$, we can assume that $|w| \leq 1$ holds $\mathfrak{m}$-a.e. in X. Let $(w_n)_n \subseteq \mathrm{TestV}(\mathrm{X})$ be $L^2(T\mathrm{X})$-converging to w and $\tilde{\mathfrak{m}}$ a Borel probability measure on X such that $\mathfrak{m} \ll \tilde{\mathfrak{m}} \leq C\mathfrak{m}$ for some $C > 0$. Then $\langle w_n, w\rangle \to |w|^2$ in $L^2(\tilde{\mathfrak{m}})$, thus accordingly

$$m_n \doteq \tilde{\mathfrak{m}}\big(\big\{|\langle w_n, w\rangle| > 0\big\}\big) \longrightarrow m_\infty \doteq \tilde{\mathfrak{m}}\big(\big\{|w| > 0\big\}\big). \tag{6.52}$$

We now observe that:

$$\begin{aligned}&\text{For every } v, \tilde{w} \in L^2(T\mathrm{X}) \text{ and } a > 0 \text{ there exists } b \in (0,a) \\ &\text{such that } \tilde{\mathfrak{m}}\Big(\big\{|\langle \tilde{w}, w\rangle| > 0\big\} \cap \big\{\langle v + b\tilde{w}, w\rangle = 0\big\}\Big) = 0.\end{aligned} \tag{6.53}$$

Indeed – setting for brevity $E_b \doteq \{|\langle\tilde w,w\rangle|>0\}\cap\{\langle v+b\tilde w,w\rangle=0\}$ – we have that

$$\tilde{\mathfrak m}\big(E_b\cap E_{b'}\big)\le\tilde{\mathfrak m}\Big(\{|\langle\tilde w,w\rangle|>0\}\cap\{(b-b')\langle\tilde w,w\rangle=0\}\Big)=0\qquad\text{whenever } b\neq b',$$

so that the claim follows from the finiteness of $\tilde{\mathfrak m}$ and the fact that the set $(0,a)$ is uncountable.

Now let us set $\alpha_n \doteq \big\||w_n|\big\|_{L^\infty(\mathrm X)}+\|w_n\|_{W^{1,2}_{\mathrm C}(T\mathrm X)}$. We can recursively define decreasing sequences $(\beta_n)_n,(\gamma_n)_n\subseteq(0,\infty)$ with $\beta_1=1$ and such that for any $n\in\mathbb N$ we have

$$\begin{cases}3\beta_{n+1}\le\gamma_{n+1}\le\beta_n,\\ \tilde{\mathfrak m}(E_n)\ge m_n/\big(1+\frac1n\big),\end{cases}\qquad\text{where we set } E_n\doteq\Big\{\big|\big\langle\textstyle\sum_{i=1}^n\frac{\beta_i}{\alpha_i}\,w_i,w\big\rangle\big|\ge\gamma_{n+1}\Big\}.$$

To see that this is possible, let $\beta_1\doteq1$ and notice that trivially

$$\Big\{\big|\big\langle\tfrac{\beta_1}{\alpha_1}\,w_1,w\big\rangle\big|>0\Big\}=\big\{|\langle w_1,w\rangle|>0\big\},$$

so that for $\gamma_2\in(0,\beta_1)$ sufficiently small the above holds. Now assume that β_{n-1} and γ_n have been already chosen. We use property (6.53) for $v\doteq\sum_{i=1}^{n-1}\frac{\beta_i}{\alpha_i}\,w_i$, $\tilde w\doteq w_n$ and $a\doteq\gamma_n/3$ to find $\beta_n\doteq b<\gamma_n/3$ such that $\tilde{\mathfrak m}\big(\{|\langle\sum_{i=1}^n\frac{\beta_i}{\alpha_i}\,w_i,w\rangle|>0\}\big)\ge\tilde{\mathfrak m}\big(\{|\langle w_n,w\rangle|>0\}\big)=m_n$. Hence for $\gamma_{n+1}\in(0,\beta_n)$ sufficiently small the above claim holds.

We claim that the vector $v\doteq\sum_{i\ge1}\frac{\beta_i}{\alpha_i}\,w_i$ satisfies the conclusion of the statement and start by observing that $\beta_i\le3^{-i}$, thus accordingly

$$\big\|\tfrac{\beta_i}{\alpha_i}\,w_i\big\|_{W^{1,2}_{\mathrm C}(T\mathrm X)}\le3^{-i}\,\|{\alpha_i}^{-1}\,w_i\|_{W^{1,2}_{\mathrm C}(T\mathrm X)}\le3^{-i}$$

by definition of α_i. Hence the series converges in $W^{1,2}_{\mathrm C}(T\mathrm X)$, so that v is well-defined and belongs to $H^{1,2}_{\mathrm C}(T\mathrm X)$. Now notice that by construction and (6.52) we have that $\tilde{\mathfrak m}(E_n)\to m_\infty$ and $\tilde{\mathfrak m}\big(E_n\setminus\{|w|>0\}\big)=0$, so that $\tilde{\mathfrak m}\big(\{|w|>0\}\setminus\bigcup_n E_n\big)=0$. Hence to conclude it is sufficient to show that for every $n\ge1$ one has that $\langle v,w\rangle\neq0$ holds $\tilde{\mathfrak m}$-a.e. on E_n. Fix $n\ge1$, let $m>n$ and observe that – by definition of the α_i's and the β_i's – we have that

$$\big|\big\langle\tfrac{\beta_m}{\alpha_m}\,w_m,w\big\rangle\big|\le3^{n-m+1}\,\beta_{n+1}\,\big|\big\langle\alpha_m^{-1}\,w_m,w\big\rangle\big|\le3^{n-m+1}\,\beta_{n+1}\qquad\text{holds }\tilde{\mathfrak m}\text{-a.e. in X},$$

so that $\big|\sum_{m>n}\big\langle\frac{\beta_m}{\alpha_m}\,w_m,w\big\rangle\big|\le\frac32\,\beta_{n+1}\le\frac12\,\gamma_{n+1}$. On the other hand, we have by construction that $\big|\sum_{i=1}^n\big\langle\frac{\beta_i}{\alpha_i}\,w_i,w\big\rangle\big|\ge\gamma_{n+1}$ holds $\tilde{\mathfrak m}$-a.e. on E_n. Finally, this grants that $|\langle v,w\rangle|\ge\frac12\,\gamma_{n+1}$ is verified $\tilde{\mathfrak m}$-a.e. on E_n. Therefore the proof is achieved. □

By repeatedly applying Lemma 6.38, we can find a family of $H^{1,2}_{\mathrm C}(T\mathrm X)$-Sobolev generators of the tangent module on any $\mathsf{RCD}(K,\infty)$ space X, as follows:

Theorem 6.39 (Sobolev base of the tangent module) *Let* $(\mathrm X,\mathsf d,\mathfrak m)$ *be an* $\mathsf{RCD}(K,\infty)$ *space, for some constant* $K\in\mathbb R$. *Suppose that the dimensional decomposition of* X *is given by* $(A_n)_{n\in\mathbb N}$. *Then there exists a sequence of vector fields* $(v_n)_{n\ge1}\subseteq H^{1,2}_{\mathrm C}(T\mathrm X)$ *such that*

(6.54) $\qquad v_1,\ldots,v_n$ *is a local basis for* $L^2(T\mathrm X)$ *on* A_n $\qquad$ *for every* $n\in\mathbb N^+$.

Proof. The statement can be equivalently rewritten in the following way:

(6.55) $\qquad v_1,\ldots,v_n$ are independent on $\bigcup_{k\ge n}A_k$ $\qquad$ for every $n\in\mathbb N^+$.

We build the sequence $(v_n)_n$ by means of a recursive argument. First of all, choose a vector field $w \in L^2(T\mathrm{X})$ such that $0 < |w| \le 1$ $\mathfrak{m}$-a.e. in $\bigcup_{k\ge 1} A_k$, then pick $v_1 \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ such that $\langle v_1, w\rangle \neq 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k\ge 1} A_k$, whose existence is granted by Lemma 6.38. Thus in particular we have $|v_1| > 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k\ge 1} A_k$, proving (6.55) for $n = 1$. Now suppose to have already found $v_1, \dots, v_n$ satisfying the required property. It can be easily seen that there exists $w \in L^2(T\mathrm{X})$ such that $\langle v_1, w\rangle = \dots = \langle v_n, w\rangle = 0$ and $0 < |w| \le 1$ hold $\mathfrak{m}$-a.e. in the set $\bigcup_{k>n} A_k$. Hence Lemma 6.38 ensures the existence of a vector field $v_{n+1} \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ such that $\langle v_{n+1}, w\rangle \neq 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$.

Now take any $f_1, \dots, f_{n+1} \in L^\infty(\mathfrak{m})$ such that $\sum_{i=1}^{n+1} f_i\, v_i = 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$, thus one has $f_{n+1}\langle v_{n+1}, w\rangle = \sum_{i=1}^{n+1} f_i\langle v_i, w\rangle = 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$, from which we can deduce that $f_{n+1} = 0$ holds $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$. Therefore $\sum_{i=1}^{n} f_i\, v_i = 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$ and accordingly also $f_1 = \dots = f_n = 0$ $\mathfrak{m}$-a.e. in $\bigcup_{k>n} A_k$, as a consequence of the independence of $v_1, \dots, v_n$. This grants that the vector fields $v_1, \dots, v_{n+1}$ are independent on $\bigcup_{k>n} A_k$, proving (6.55) for $n + 1$. The statement is then achieved. □

7

Quasi-continuous vector fields on RCD spaces

Contents

The purpose of this chapter is to explain in which sense any Sobolev vector field on an $\mathsf{RCD}(K,\infty)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ – more precisely, any element of the space $H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ – admits a (unique) quasi-continuous representative. The whole discussion is taken from [DGP18].

Consider a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and the variational 2-capacity associated to the Sobolev space $W^{1,2}(\mathrm{X})$, which will be shortly called *capacity* and denoted by Cap (cf. Subsection 7.1.1). We recall that a function $f:\mathrm{X}\to\mathbb{R}$ is *quasi-continuous* if for any $\varepsilon>0$ we can find an open subset Ω of X with $\mathrm{Cap}(\Omega)<\varepsilon$ such that f is continuous when restricted to $\mathrm{X}\setminus\Omega$. Then it is well-known that – provided continuous functions are dense in $W^{1,2}(\mathrm{X})$ – any Sobolev function has a (unique up to Cap-a.e. equality) quasi-continuous representative. This means that an element of $W^{1,2}(\mathrm{X})$ is 'more regular' than a generic element of $L^2(\mathfrak{m})$, since it is defined not only $\mathfrak{m}$-a.e. but also Cap-a.e. (it is worth to point out that $\mathfrak{m}\ll\mathrm{Cap}$).

A natural question arises: given an $\mathsf{RCD}(K,\infty)$ space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and some Sobolev vector field v over X, is it possible to speak about 'quasi-continuous representative' of v? As we are going to see, the answer is positive. However, it is still unclear whether this theory could shed new light on the 'fine' properties of RCD spaces.

We now briefly describe some technical details of our construction. We denote by $L^0(\mathrm{Cap})$ the space of all equivalence classes (up to Cap-a.e. equality) of Borel functions on X. It turns out that $L^0(\mathrm{Cap})$ has a natural structure of topological ring (cf. Definition 7.6), whence we can give a meaningful definition of $L^0(\mathrm{Cap})$-normed $L^0(\mathrm{Cap})$-module (or *normed* Cap-*module* for brevity); cf. Definition 7.21. The axiomatisation of normed Cap-module mimics that of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module, but with the capacity in place of the reference measure $\mathfrak{m}$, so that – morally speaking – the elements of a normed Cap-module are defined Cap-a.e..

The only normed Cap-module we shall actually exhibit is the *tangent* Cap-*module*, which is introduced in Theorem 7.25 and denoted by $L^0_{\mathrm{Cap}}(T\mathrm{X})$. The idea is the following: given any test function $f \in \mathrm{TestF}(\mathrm{X})$, we already know that its minimal weak upper gradient $|Df|$ belongs to the Sobolev space $W^{1,2}(\mathrm{X})$ (cf. Subsection 4.2.2); this tells us that, heuristically, the 'gradient of f' is defined Cap-a.e., thus also the limits of $L^0(\mathrm{Cap})$-linear combinations of 'gradients' of test functions (that constitute the tangent module) are defined Cap-a.e..

Given a test function $f \in \mathrm{TestF}(\mathrm{X})$, we denote by $\bar{\nabla} f$ the gradient of f as an element of $L^0_{\mathrm{Cap}}(T\mathrm{X})$. Then we can give the definition of quasi-continuous vector field over X in the following way: we say that an element $v \in L^0_{\mathrm{Cap}}(T\mathrm{X})$ is *quasi-continuous* provided the function $|v - \bar{\nabla} f|$ is quasi-continuous for every $f \in \mathrm{TestF}(\mathrm{X})$; cf. Definition 7.29.

Therefore the main result of the chapter states the following:

Any element of $H^{1,2}_{C}(T\mathrm{X})$ has a quasi-continuous representative in $L^0_{\mathrm{Cap}}(T\mathrm{X})$.

We refer to Theorem 7.34 for the precise formulation and the proof of such result.

7.1 Capacity on metric measure spaces

7.1.1 Definition and main properties of capacity

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a fixed metric measure space. We briefly recall the definition of capacity in this context and its main properties; the forthcoming discussion is mainly taken from [BH91].

Definition 7.1 (Capacity) *Let E be a given subset of* X. *Let us denote*

$$\mathcal{F}_E \doteq \left\{ f \in W^{1,2}(\mathrm{X}) \;\middle|\; f \geq 1 \;\; \mathfrak{m}\text{-a.e. on some open neighbourhood of } E \right\}. \tag{7.1}$$

Then the capacity *of the set E is defined as the quantity* $\mathrm{Cap}(E) \in [0, +\infty]$, *given by*

$$\mathrm{Cap}(E) \doteq \inf_{f \in \mathcal{F}_E} \|f\|^2_{W^{1,2}(\mathrm{X})}, \tag{7.2}$$

with the convention that $\mathrm{Cap}(E) \doteq +\infty$ *whenever the family $\mathcal{F}_E$ is empty.*

In the following result we collect the main properties of the capacity; cf. Appendix A for the language of outer measures that will be used.

Proposition 7.2 *The capacity* Cap *is a submodular outer measure on the space* X. *Moreover, it satisfies the following properties:*

i) $\mathfrak{m}(E) \leq \mathrm{Cap}(E)$ *for every Borel subset E of* X.

ii) Cap *is continuous from below, i.e.*

$$\mathrm{Cap}\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) = \lim_{n\to\infty} \mathrm{Cap}(E_n) \tag{7.3}$$

for any increasing sequence $(E_n)_n$ *of subsets of* X.

iii) *Given any* $x \in \mathrm{X}$ *and* $0 < r < R$, *it holds that*

$$\mathrm{Cap}\big(B_r(x)\big) \le \left(1 + \frac{1}{(R-r)^2}\right)\mathfrak{m}\big(B_R(x)\big). \tag{7.4}$$

Proof. The only statement that is not proven in [BH91, Proposition 8.1.3] is item iii). To show its validity, fix $x \in \mathrm{X}$ and $0 < r < R$. Given any $\lambda \in (0, R-r)$, define $f_\lambda \in \mathcal{F}_{B_r(x)}$ as

$$f_\lambda \doteq \left(1 - \frac{\mathsf{d}\big(\cdot\,, B_r(x)\big)}{\lambda}\right)^+.$$

Notice that $f_\lambda \le 1$, $|Df_\lambda| \le 1/\lambda$ hold $\mathfrak{m}$-a.e. on X and $\mathrm{spt}(f_\lambda) \subseteq B_R(x)$, whence accordingly

$$\mathrm{Cap}\big(B_r(x)\big) \le \|f_\lambda\|^2_{W^{1,2}(\mathrm{X})} = \|f\|^2_{L^2(\mathfrak{m})} + \big\||Df|\big\|^2_{L^2(\mathfrak{m})} \le \mathfrak{m}\big(B_R(x)\big) + \frac{\mathfrak{m}\big(B_R(x)\big)}{\lambda^2}.$$

By letting $\lambda \nearrow R - r$ in the previous inequality, we thus obtain (7.4). □

Remark 7.3 (The capacity is σ-finite) It follows from item iii) of Proposition 7.2 that there exists a partition $(A_k)_{k\in\mathbb{N}}$ of X into bounded Borel sets satisfying $0 < \mathrm{Cap}(A_k) < +\infty$ for every $k \in \mathbb{N}$ and with the following property: there exists a point $\bar{x} \in \mathrm{X}$ such that the ball $B_k(\bar{x})$ is contained in $A_1 \cup \ldots \cup A_k$ for every $k \in \mathbb{N}$. ■

Example 7.4 Consider the open segment $S \doteq (0,1)$ in $\mathbb{R}$ (equipped with Euclidean distance and Lebesgue measure). Given any $n \ge 2$, denote by $P_n \subseteq S$ the singleton $\{1/n\}$. One can check that $0 < \mathrm{Cap}(P_n) < \mathrm{Cap}(S)$, but that $\mathrm{Cap}(S \setminus P_n) = \mathrm{Cap}(S)$. In other words, we have that $\int \chi_{S\setminus P_n}\,\mathrm{dCap} = \int \chi_S\,\mathrm{dCap}$ and $\mathrm{Cap}\big(\{\chi_{S\setminus P_n} \ne \chi_S\}\big) > 0$. This shows that for $\mu \doteq \mathrm{Cap}$ and $f \le g$ the converse of item iv) of Proposition A.1 fails. ■

Remark 7.5 In this context, the monotone convergence theorem can be easily shown to hold. Namely, given functions $f, f_n : \mathrm{X} \to [0, +\infty]$, $n \in \mathbb{N}$ we have that

$$f_n(x) \nearrow f(x) \;\text{ for Cap-a.e. } x \in \mathrm{X} \quad \Longrightarrow \quad \int f\,\mathrm{dCap} = \lim_{n\to\infty} \int f_n\,\mathrm{dCap}. \tag{7.5}$$

In order to prove it, call F the set of points $x \in \mathrm{X}$ with $f_n(x) \nearrow f(x)$, thus $\mathrm{Cap}(\mathrm{X}\setminus F) = 0$. For any fixed $t \ge 0$, we have that the sequence of sets $\{\chi_F f_n > t\}$ is increasing with respect to n and satisfies $\bigcup_n \{\chi_F f_n > t\} = \{\chi_F f > t\}$. Hence by applying the monotone convergence theorem (for the Lebesgue measure) and item iv) of Proposition A.1 we conclude that

$$\begin{aligned}\int f\,\mathrm{dCap} &= \int_F f\,\mathrm{dCap} = \int_0^{+\infty} \mathrm{Cap}\big(\{\chi_F f > t\}\big)\,\mathrm{d}t = \lim_{n\to\infty}\int_0^{+\infty} \mathrm{Cap}\big(\{\chi_F f_n > t\}\big)\,\mathrm{d}t\\ &= \lim_{n\to\infty}\int_F f_n\,\mathrm{dCap} = \lim_{n\to\infty}\int f_n\,\mathrm{dCap},\end{aligned}$$

thus proving that the claim (7.5) is verified.

On the other hand, an analogue of the dominated convergence theorem cannot hold, as shown by the following counterexample. For any $n \geq 2$, let us consider the point P_n in $\mathbb{R}$ defined in Example 7.4. Since the capacity in the space $\mathbb{R}$ is translation-invariant, one has that $\mathrm{Cap}(P_n) = \mathrm{Cap}(P_2) > 0$ for all $n \geq 2$. Moreover, we have $\lim_n \chi_{P_n}(x) = 0$ for all $x \in \mathbb{R}$ and $\chi_{P_n} \leq \chi_{(0,1)}$ for all $n \geq 2$, with $\int \chi_{(0,1)} \,\mathrm{dCap} = \mathrm{Cap}\big((0,1)\big) < +\infty$. Nevertheless, it holds that $\int \chi_{P_n} \,\mathrm{dCap} \equiv \mathrm{Cap}(P_2)$ does not converge to 0 as $n \to \infty$, thus proving the failure of the dominated convergence theorem. In order to provide such a counterexample, we exploited the fact that the capacity is not σ-additive; indeed, we built a sequence of pairwise disjoint sets, all having the same positive capacity, which are contained in a fixed set of finite capacity. The lack of a result such as the dominated convergence theorem explains the technical difficulties we will find in the proofs of Proposition 7.7 and Theorem 7.10. ■

7.1.2 The space $L^0(\mathrm{Cap})$

It makes sense to consider the integral associated to Cap and that such integral is subadditive, by Proposition 7.2 and Theorem A.3. In light of this, the following definition is meaningful:

Definition 7.6 (The space $L^0(\mathrm{Cap})$) *Given any two functions $f, g : \mathrm{X} \to \mathbb{R}$, we will say that* $f = g$ in the Cap-a.e. sense *provided* $\mathrm{Cap}\big(\{f \neq g\}\big) = 0$. *We define $L^0(\mathrm{Cap})$ as the space of all the equivalence classes – up to Cap-a.e. equality – of Borel functions on X. It turns out that $L^0(\mathrm{Cap})$ is both a topological vector space and a topological ring when endowed with the usual pointwise operations and the distance*

$$\mathsf{d}_{\mathrm{Cap}}(f,g) \doteq \sum_{k\in\mathbb{N}} \frac{1}{2^k\big(\mathrm{Cap}(A_k)\vee 1\big)} \int_{A_k} |f-g| \wedge 1 \,\mathrm{dCap} \qquad \text{for every } f,g \in L^0(\mathrm{Cap}), \tag{7.6}$$

where $(A_k)_k$ is any fixed Borel partition of X as in Remark 7.3.

Note that the integral $\int_{A_k} |f-g| \wedge 1 \,\mathrm{dCap}$ is well-defined, since its value does not depend on the particular representatives of f and g, as granted by item iv) of Proposition A.1. Moreover, we point out that the fact that $\mathsf{d}_{\mathrm{Cap}}$ satisfies the triangle inequality is a consequence of the subadditivity of the integral associated with the capacity.

The next result shows that, even if the choice of the particular sequence $(A_k)_k$ might affect the distance $\mathsf{d}_{\mathrm{Cap}}$, its induced topology remains unaltered.

Proposition 7.7 *Let $(f_n)_n \subseteq L^0(\mathrm{Cap})$ be given. Then the following are equivalent:*

i) $\lim_{n,m} \mathsf{d}_{\mathrm{Cap}}(f_n, f_m) = 0$,

ii) $\lim_{n,m} \mathrm{Cap}\big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\big) = 0$ *for any $\varepsilon > 0$ and any bounded set $E \subseteq \mathrm{X}$.*

Proof. We separately prove the two implications:
i) $\Longrightarrow$ ii) Fix any $0 < \varepsilon < 1$ and a bounded set $E \subseteq \mathrm{X}$. Choose $k \in \mathbb{N}$ such that $E \subseteq B_k(\bar{x})$, so that $E \subseteq A_1 \cup \ldots \cup A_k$. Since $\mathsf{d}_{\mathrm{Cap}}(f_n, f_m) \overset{n,m}{\to} 0$, we have $\lim_{n,m} \int_{A_i} |f_n - f_m| \wedge 1 \,\mathrm{dCap} = 0$ for all $i = 1, \ldots, k$. Therefore we conclude that

$$\begin{aligned}\varlimsup_{n,m\to\infty} \varepsilon \,\mathrm{Cap}\big(E \cap \big\{|f_n - f_m| > \varepsilon\big\}\big) &\leq \varlimsup_{n,m\to\infty} \varepsilon \sum_{i=1}^k \mathrm{Cap}\big(A_i \cap \big\{|f_n - f_m| > \varepsilon\big\}\big) \\ &\leq \sum_{i=1}^k \lim_{n,m\to\infty} \int_{A_i} |f_n - f_m| \wedge 1 \,\mathrm{dCap} = 0.\end{aligned}$$

ii) $\Longrightarrow$ i) Let $\varepsilon > 0$ be fixed. Choose $k \in \mathbb{N}$ with $2^{-k} \leq \varepsilon$. By our hypothesis, there is $\bar{n} \in \mathbb{N}$ such that $\mathrm{Cap}\big(A_i \cap \{|f_n - f_m| > \varepsilon\}\big) \leq \varepsilon\,\mathrm{Cap}(A_i)$ for every $n, m \geq \bar{n}$ and $i = 1, \ldots, k$. Let us call $B_i^{nm} \doteq A_i \cap \{|f_n - f_m| > \varepsilon\}$ and $C_i^{nm} \doteq A_i \setminus B_i^{nm}$. Therefore for any $n, m \geq \bar{n}$ it holds that

$$\begin{aligned}
\mathsf{d}_{\mathrm{Cap}}(f_n, f_m) &\leq \sum_{i=1}^{k} \frac{1}{2^i\,\mathrm{Cap}(A_i)} \int_{A_i} |f_n - f_m| \wedge 1 \,\mathrm{dCap} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \\
&\leq \sum_{i=1}^{k} \frac{1}{2^i\,\mathrm{Cap}(A_i)} \left[\int_{B_i^{nm}} |f_n - f_m| \wedge 1 \,\mathrm{dCap} + \int_{C_i^{nm}} |f_n - f_m| \wedge 1 \,\mathrm{dCap}\right] + \varepsilon \\
&\leq \sum_{i=1}^{k} \frac{1}{2^i\,\mathrm{Cap}(A_i)} \big[\mathrm{Cap}(B_i^{nm}) + \varepsilon\,\mathrm{Cap}(A_i)\big] + \varepsilon \leq 3\,\varepsilon,
\end{aligned}$$

proving that $\lim_{n,m} \mathsf{d}_{\mathrm{Cap}}(f_n, f_m) = 0$, as required. □

We omit the proof of the following result, since it is analogous to that of Proposition 7.7.

Proposition 7.8 *Let $f \in L^0(\mathrm{Cap})$ and $(f_n)_n \subseteq L^0(\mathrm{Cap})$. Then the following are equivalent:*

i) $\lim_n \mathsf{d}_{\mathrm{Cap}}(f_n, f) = 0$,

ii) $\lim_n \mathrm{Cap}\big(E \cap \{|f_n - f| > \varepsilon\}\big) = 0$ *for any $\varepsilon > 0$ and any bounded set $E \subseteq \mathrm{X}$.*

Remark 7.9 It can be readily checked that any converging sequence in $L^0(\mathrm{Cap})$ admits a pointwise Cap-a.e. converging subsequence. Namely, if $\mathsf{d}_{\mathrm{Cap}}(f_n, f) \overset{n}{\to} 0$ for some $f \in L^0(\mathrm{Cap})$ and $(f_n)_n \subseteq L^0(\mathrm{Cap})$, then there exists a subsequence $(f_{n_i})_i$ of $(f_n)_n$ such that $f_{n_i}(x) \to f(x)$ is verified as $i \to \infty$ for Cap-a.e. $x \in \mathrm{X}$.

On the other hand, the converse implication is in general false, as shown by the following counterexample. Consider P_n as in Example 7.4 for any $n \geq 2$. We have that $f_n \doteq \chi_{P_n}$ pointwise converges to 0 as $n \to \infty$. However, it holds that

$$\mathrm{Cap}\big((0,1) \cap \{|f_n| > 1/2\}\big) = \mathrm{Cap}(P_n) \equiv \mathrm{Cap}(P_2) > 0$$

does not converge to 0, thus we do not have $\lim_n \mathsf{d}_{\mathrm{Cap}}(f_n, 0) = 0$ by Proposition 7.8. ■

We now prove that $\big(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}}\big)$ is complete. Observe that Propositions 7.7 and 7.8 ensure that such completeness does not depend on the particular choice of $(A_k)_k$.

Theorem 7.10 *The metric space $\big(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}}\big)$ is complete.*

Proof. Let $(f_n)_n$ be a $\mathsf{d}_{\mathrm{Cap}}$-Cauchy sequence of Borel functions $f_n : \mathrm{X} \to \mathbb{R}$. Fix any $k \in \mathbb{N}$. Let $(f_{n_i})_i$ be an arbitrary subsequence of $(f_n)_n$. Up to passing to a further (not relabeled) subsequence, it holds that

$$\mathrm{Cap}\big(A_k \cap \{|f_{n_i} - f_{n_{i+1}}| > 2^{-i}\}\big) \leq 2^{-i} \qquad \text{for every } i \in \mathbb{N}. \tag{7.7}$$

Let us call $F_i \doteq A_k \cap \{|f_{n_i} - f_{n_{i+1}}| > 2^{-i}\}$ for every $i \in \mathbb{N}$ and $F \doteq \bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} F_j$. Given that we have $\sum_{i \in \mathbb{N}} \mathrm{Cap}(F_i) < +\infty$ by (7.7), we deduce from Lemma A.4 that $\mathrm{Cap}(F) = 0$. Note that if $x \in A_k \setminus F = \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} A_k \setminus F_j$, then there is $i \in \mathbb{N}$ such that $\big|f_{n_j}(x) - f_{n_{j+1}}(x)\big| \leq 2^{-j}$

for all $j \geq i$, which grants that $\big(f_{n_i}(x)\big)_i \subseteq \mathbb{R}$ is a Cauchy sequence for every $x \in A_k \setminus F$. Therefore we define the Borel function $g^k : A_k \to \mathbb{R}$ as

$$g^k(x) \doteq \begin{cases} \lim_i f_{n_i}(x) & \text{if } x \in A_k \setminus F, \\ 0 & \text{if } x \in F. \end{cases}$$

Now fix any $\varepsilon > 0$. Choose $\bar{i} \in \mathbb{N}$ such that $\sum_{i \geq \bar{i}} 2^{-i} \leq \varepsilon$. If $i \geq \bar{i}$ and $x \in \bigcap_{j \geq i} A_k \setminus F_j$ (thus in particular $x \notin F$), hence one has $\big|f_{n_i}(x) - g^k(x)\big| \leq \sum_{j \geq i} \big|f_{n_j}(x) - f_{n_{j+1}}(x)\big| \leq \sum_{j \geq i} 2^{-j} \leq \varepsilon$. This implies that

$$A_k \cap \big\{|f_{n_i} - g^k| > \varepsilon\big\} \subseteq \bigcup_{j \geq i} F_j \qquad \text{for every } i \geq \bar{i}. \tag{7.8}$$

Then $\mathrm{Cap}\big(A_k \cap \big\{|f_{n_i} - g^k| > \varepsilon\big\}\big) \leq \sum_{j \geq i} \mathrm{Cap}(F_j) \leq \sum_{j \geq i} 2^{-j}$ holds for every $i \geq \bar{i}$, thus accordingly

$$\lim_{i \to \infty} \mathrm{Cap}\big(A_k \cap \big\{|f_{n_i} - g^k| > \varepsilon\big\}\big) = 0 \qquad \text{for every } \varepsilon > 0. \tag{7.9}$$

We proved this property for some subsequence of a given subsequence $(f_{n_i})_i$ of $(f_n)_n$, hence this shows that

$$\lim_{n \to \infty} \mathrm{Cap}\big(A_k \cap \big\{|f_n - g^k| > \varepsilon\big\}\big) = 0 \qquad \text{for every } \varepsilon > 0. \tag{7.10}$$

Now let us define the Borel function $f : \mathrm{X} \to \mathbb{R}$ as $f := \sum_{k \in \mathbb{N}} \chi_{A_k} g^k$. Notice that the equality $A_k \cap \big\{|f_n - f| > \varepsilon\big\} = A_k \cap \big\{|f_n - g^k| > \varepsilon\big\}$ and property (7.10) yield

$$\lim_{n \to \infty} \mathrm{Cap}\big(A_k \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0 \qquad \text{for every } k \in \mathbb{N} \text{ and } \varepsilon > 0.$$

Since any bounded subset of X is contained in the union of finitely many A_k's, we immediately deduce that $\lim_n \mathrm{Cap}\big(E \cap \big\{|f_n - f| > \varepsilon\big\}\big) = 0$ whenever $\varepsilon > 0$ and $E \subseteq \mathrm{X}$ is bounded. This grants that $\lim_n \mathsf{d}_{\mathrm{Cap}}(f_n, f) = 0$ by Proposition 7.8, thus proving that $\big(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}}\big)$ is a complete metric space and accordingly the statement. □

In the next section, we shall need the density result we are now going to present.

Proposition 7.11 *The space* $\mathsf{Sf}(\mathrm{X})$ *of simple functions, which for this chapter is defined as*

$$\mathsf{Sf}(\mathrm{X}) \doteq \left\{ \sum_{n=1}^{\infty} \alpha_n \chi_{E_n} \;\middle|\; \begin{array}{l} (\alpha_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \text{ and } (E_n)_{n \in \mathbb{N}} \\ \text{is a Borel partition of } \mathrm{X} \end{array} \right\}, \tag{7.11}$$

is dense in $\big(L^0(\mathrm{Cap}), \mathsf{d}_{\mathrm{Cap}}\big)$.

Proof. Fix $f \in L^0(\mathrm{Cap})$ and $\varepsilon > 0$. Choose a Borel representative $\bar{f} : \mathrm{X} \to \mathbb{R}$ of f. For any integer $i \in \mathbb{Z}$, let us define $E_i \doteq \bar{f}^{-1}\big([i\,\varepsilon, (i+1)\,\varepsilon)\big)$. Then $(E_i)_{i \in \mathbb{Z}}$ constitutes a partition of X into Borel sets, so that $\bar{g} \doteq \sum_{i \in \mathbb{Z}} i\,\varepsilon\,\chi_{E_i}$ is a well-defined Borel function that belongs to $\mathsf{Sf}(\mathrm{X})$. Finally, it holds that $\big|\bar{f}(x) - \bar{g}(x)\big| < \varepsilon$ for every $x \in \mathrm{X}$, which grants that $\mathsf{d}_{\mathrm{Cap}}(f, g) \leq \varepsilon$, where $g \in L^0(\mathrm{Cap})$ denotes the equivalence class of $\bar{g}$. Hence the statement follows. □

7.1.3 Quasi-continuous representative of a Sobolev function

We conclude the present section by investigating the notion of quasi-continuous function and its main properties. We refer to [BH91] for a more detailed discussion about this topic.

Definition 7.12 (Quasi-continuity for functions) *We say that a function $f : \mathrm{X} \to \mathbb{R}$ is* quasi-continuous *provided for every $\varepsilon > 0$ there exists an open set $\Omega \subseteq \mathrm{X}$ with $\mathrm{Cap}(\Omega) < \varepsilon$ such that $f|_{\mathrm{X}\setminus\Omega} : \mathrm{X} \setminus \Omega \to \mathbb{R}$ is continuous. Moreover, we denote by $\mathcal{C}^{qc}(\mathrm{X}) \subseteq L^0(\mathrm{Cap})$ the set of all equivalence classes – up to Cap-a.e. equality – of Borel quasi-continuous functions.*

The well-posedness of the definition of $\mathcal{C}^{qc}(\mathrm{X})$ stems from the following result:

Lemma 7.13 *Let $f, \tilde{f} : \mathrm{X} \to \mathbb{R}$ be two functions that coincide Cap-a.e. on X. Then f is quasi-continuous if and only if $\tilde{f}$ is quasi-continuous.*

It can be readily checked that sums and products of two quasi-continuous functions are quasi-continuous functions as well. More generally, given any $n \in \mathbb{N}$ and any continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}$, it holds that the function $\mathrm{X} \ni x \mapsto \Phi\big(f_1(x), \ldots, f_n(x)\big)$ is quasi-continuous whenever $f_1, \ldots, f_n : \mathrm{X} \to \mathbb{R}$ are quasi-continuous functions.

Remark 7.14 Let $f : \mathrm{X} \to \mathbb{R}$ be a given quasi-continuous function. The very definition of quasi-continuity grants the existence of an increasing sequence $(C_n)_n$ of closed subsets of X with $\lim_n \mathrm{Cap}(\mathrm{X} \setminus C_n) = 0$ such that f is continuous on each C_n. Then $N \doteq \bigcap_n \mathrm{X} \setminus C_n$ is a Borel set with null capacity – in particular, we have $\mathfrak{m}(N) = 0$ by item i) of Proposition 7.2 – and f is Borel on $\mathrm{X} \setminus N$. This proves that any quasi-continuous function is $\mathfrak{m}$-measurable and Cap-a.e. equivalent to a Borel function. ■

Since $\mathfrak{m}$ is absolutely continuous with respect to Cap, there is a natural projection map

$$\Pi : L^0(\mathrm{Cap}) \longrightarrow L^0(\mathfrak{m}). \tag{7.12}$$

Namely, we define $\Pi\big([f]_{\mathrm{Cap}}\big) \doteq [f]_{\mathfrak{m}}$ for every Borel function $f : \mathrm{X} \to \mathbb{R}$, where by $[f]_{\mathrm{Cap}}$ (resp. $[f]_{\mathfrak{m}}$) we intend the equivalence class up to Cap-a.e. (resp. $\mathfrak{m}$-a.e.) equality of f.

Proposition 7.15 (Uniqueness of the quasi-continuous representative) *Fix any two quasi-continuous functions f, g on X. Then $f = g$ $\mathfrak{m}$-a.e. on X implies $f = g$ Cap-a.e. on X. In other words,*

$$\Pi|_{\mathcal{C}^{qc}(\mathrm{X})} : \mathcal{C}^{qc}(\mathrm{X}) \longrightarrow L^0(\mathfrak{m}) \qquad \text{is an injective map.} \tag{7.13}$$

Theorem 7.16 (Quasi-continuous representative of Sobolev functions) *We assume that the space $C(\mathrm{X}) \cap W^{1,2}(\mathrm{X})$ is dense in $W^{1,2}(\mathrm{X})$. Then there exists a unique map*

$$T : W^{1,2}(\mathrm{X}) \longrightarrow \mathcal{C}^{qc}(\mathrm{X}) \tag{7.14}$$

such that the composition $\Pi \circ T : W^{1,2}(\mathrm{X}) \to L^0(\mathfrak{m})$ is the inclusion map $W^{1,2}(\mathrm{X}) \subseteq L^0(\mathfrak{m})$. Moreover, T is linear and $\big|T(f)\big| = T\big(|f|\big)$ holds for every $f \in W^{1,2}(\mathrm{X})$.

We conclude the section by recalling the notion of quasi-uniform convergence and two important results concerning such concept.

Definition 7.17 (Quasi-uniform convergence) *Let $f_n : \mathrm{X} \to \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$ be any functions. Then we say that f_n* quasi-uniformly converges *to f_∞ as $n \to \infty$ if for any $\varepsilon > 0$ there exists an open set $\Omega \subseteq \mathrm{X}$ with $\mathrm{Cap}(\Omega) < \varepsilon$ such that $f_n \to f_\infty$ uniformly on $\mathrm{X} \setminus \Omega$.*

Remark 7.18 Note that the quasi-uniform convergence is invariant under Cap-a.e. modifications. Namely, given any functions $f_n, g_n : \mathrm{X} \to \mathbb{R}$, $n \in \mathbb{N} \cup \{\infty\}$ such that $f_n = g_n$ in the Cap-a.e. sense for all $n \in \mathbb{N} \cup \{\infty\}$, it can be readily checked that $f_n \to f_\infty$ quasi-uniformly if and only if $g_n \to g_\infty$ quasi-uniformly. In particular, this grants that – as in the next result – it makes sense to speak about quasi-uniform convergence for elements of $L^0(\mathrm{Cap})$. ■

Proposition 7.19 *Let $(f_n)_n \subseteq W^{1,2}(\mathrm{X})$ be a sequence that $W^{1,2}(\mathrm{X})$-converges to some limit function $f \in W^{1,2}(\mathrm{X})$. Then there exists a subsequence $(f_{n_i})_i$ such that $T(f_{n_i})$ quasi-uniformly converges to $T(f)$ as $i \to \infty$.*

Lemma 7.20 *Let $(f_n)_n \subseteq L^0(\mathrm{Cap})$ be a sequence such that $f_n \to f$ quasi-uniformly for some limit function $f \in L^0(\mathrm{Cap})$. Then $\lim_n \mathsf{d}_{\mathrm{Cap}}(f_n, f) \to 0$.*

Proof. Let f_n, f be fixed representatives. Pick any $\varepsilon > 0$. Let us choose any open set $\Omega \subseteq \mathrm{X}$ with $\mathrm{Cap}(\Omega) < \varepsilon$ such that $f_n \to f$ uniformly on the set $\mathrm{X} \setminus \Omega$. Then there exists $\bar{n} \in \mathbb{N}$ such that $\sup_{\mathrm{X}\setminus\Omega} |f_n - f| \leq \varepsilon$ holds for every $n \geq \bar{n}$. Given $\mathsf{d}_{\mathrm{Cap}}$ as in (7.6), we thus have that

$$\mathsf{d}_{\mathrm{Cap}}(f_n, f) \leq \sum_{k\in\mathbb{N}} \frac{\mathrm{Cap}(A_k \cap \Omega)}{2^k} + \sum_{k\in\mathbb{N}} \frac{1}{2^k\,\mathrm{Cap}(A_k)} \int_{A_k\setminus\Omega} \varepsilon \,\mathrm{dCap} \leq 2\,\varepsilon$$

holds for every $n \geq \bar{n}$. Therefore $\lim_n \mathsf{d}_{\mathrm{Cap}}(f_n, f) = 0$, as required. □

7.2 Quasi-continuous vector fields on RCD spaces

7.2.1 Normed Cap-modules

In the present section, we shall use the term *normed* $\mathfrak{m}$*-module* in place of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and we will typically denote by $\mathscr{M}_\mathfrak{m}$ any such object. Here we introduce a new notion of normed module – called *normed* Cap*-module* – in which the measure under consideration is the capacity Cap instead of the reference measure $\mathfrak{m}$.

Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $(A_k)_k$ a partition of X as in Remark 7.3.

Definition 7.21 (Normed Cap-module) *We say that a quadruple $\big(\mathscr{M}_{\mathrm{Cap}}, \tau, \cdot, |\cdot|\big)$ is a* normed Cap-module *over $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ provided the following hold:*

i) *$(\mathscr{M}_{\mathrm{Cap}}, \tau)$ is a topological vector space.*

ii) *The bilinear map $\cdot : L^0(\mathrm{Cap}) \times \mathscr{M}_{\mathrm{Cap}} \to \mathscr{M}_{\mathrm{Cap}}$ satisfies $f \cdot (g \cdot v) = (fg) \cdot v$ and $1 \cdot v = v$ for every $f, g \in L^0(\mathrm{Cap})$ and $v \in \mathscr{M}_{\mathrm{Cap}}$.*

iii) *The map $|\cdot| : \mathscr{M}_{\mathrm{Cap}} \to L^0(\mathrm{Cap})$, called* pointwise norm, *satisfies*

$$\begin{aligned} |v| &\geq 0 && \text{for every } v \in \mathscr{M}_{\mathrm{Cap}}, \text{ with equality if and only if } v = 0, \\ |v + w| &\leq |v| + |w| && \text{for every } v, w \in \mathscr{M}_{\mathrm{Cap}}, \\ |f \cdot v| &= |f||v| && \text{for every } v \in \mathscr{M}_{\mathrm{Cap}} \text{ and } f \in L^0(\mathrm{Cap}), \end{aligned} \tag{7.15}$$

where all equalities and inequalities are intended in the Cap*-a.e. sense.*

iv) *The distance* $\mathsf{d}_{\mathscr{M}_{\rm Cap}}$ *on* $\mathscr{M}_{\rm Cap}$*, given by*

$$\mathsf{d}_{\mathscr{M}_{\rm Cap}}(v,w) \doteq \sum_{k\in\mathbb{N}} \frac{1}{2^k\big(\mathrm{Cap}(A_k)\vee 1\big)} \int_{A_k} |v-w|\wedge 1\,\mathrm{dCap} \tag{7.16}$$

for every $v,w\in\mathscr{M}_{\rm Cap}$*, is complete and induces the topology* τ*.*

The relation with the normed $\mathfrak{m}$-modules is explained by the following result:

Proposition 7.22 *Let* $\mathscr{M}_{\rm Cap}$ *be a normed* Cap*-module over* $(\mathrm{X},\mathsf{d},\mathfrak{m})$*. We define an equivalence relation* $\sim_{\mathfrak{m}}$ *on* $\mathscr{M}_{\rm Cap}$ *as follows: given any* $v,w\in\mathscr{M}_{\rm Cap}$*, we declare that*

$$v\sim_{\mathfrak{m}} w \qquad\Longleftrightarrow\qquad |v-w|=0\ \ \mathfrak{m}\text{-a.e. in } \mathrm{X}. \tag{7.17}$$

Then the quotient $\mathscr{M}_{\mathfrak{m}}\doteq\mathscr{M}_{\rm Cap}/\sim_{\mathfrak{m}}$ *inherits a natural structure of normed* $\mathfrak{m}$*-module.*

Proof. Let us denote by $[v]_{\mathfrak{m}}\in\mathscr{M}_{\mathfrak{m}}$ the equivalence class of $v\in\mathscr{M}_{\rm Cap}$. Given $[v]_{\mathfrak{m}},[w]_{\mathfrak{m}}\in\mathscr{M}_{\mathfrak{m}}$ and $[f]_{\mathfrak{m}}\in L^0(\mathfrak{m})$, we define

$$\begin{aligned}
[v]_{\mathfrak{m}}+[w]_{\mathfrak{m}} &\doteq [v+w]_{\mathfrak{m}}\in\mathscr{M}_{\mathfrak{m}},\\
[f]_{\mathfrak{m}}\cdot[v]_{\mathfrak{m}} &\doteq \big[[f]_{\rm Cap}\cdot v\big]_{\mathfrak{m}}\in\mathscr{M}_{\mathfrak{m}},\\
\big|[v]_{\mathfrak{m}}\big| &\doteq \Pi\big(|v|\big)\in L^0(\mathfrak{m}).
\end{aligned}$$

Hence standard verifications show that the above operations are well-posed and endow $\mathscr{M}_{\mathfrak{m}}$ with a normed $\mathfrak{m}$-module structure. □

Remark 7.23 In analogy with the case of normed $\mathfrak{m}$-modules, one could be tempted to define the dual of a normed Cap-module $\mathscr{M}_{\rm Cap}$ as the space of all $L^0(\mathrm{Cap})$-linear continuous operators $L:\mathscr{M}_{\rm Cap}\to L^0(\mathrm{Cap})$ and to declare that the pointwise norm $|L|$ of any such L is the minimal element of $L^0(\mathrm{Cap})$ (where minimality is intended in the Cap-a.e. sense) such that the inequality $|L|\geq L(v)$ holds Cap-a.e. for any $v\in\mathscr{M}_{\rm Cap}$ that Cap-a.e. satisfies $|v|\leq 1$.

Technically speaking, for normed $\mathfrak{m}$-modules this can be achieved by using the notion of essential supremum of a family of Borel functions. Nevertheless, it seems that this tool cannot be adapted to the situation in which we want to consider the capacity instead of the reference measure, as suggested by Example 7.4. ■

Definition 7.24 *Let* $\mathscr{H}_{\rm Cap}$ *be a normed* Cap*-module over* $(\mathrm{X},\mathsf{d},\mathfrak{m})$*. Then we say that* $\mathscr{H}_{\rm Cap}$ *is a* Hilbert module *provided*

$$|v+w|^2+|v-w|^2=2\,|v|^2+2\,|w|^2 \qquad \text{holds Cap-a.e. in X} \tag{7.18}$$

for every $v,w\in\mathscr{H}_{\rm Cap}$*.*

By polarisation, we define a pointwise scalar product $\langle\cdot,\cdot\rangle:\mathscr{H}_{\rm Cap}\times\mathscr{H}_{\rm Cap}\to L^0(\mathrm{Cap})$ as

$$\langle v,w\rangle \doteq \frac{|v+w|^2-|v|^2-|w|^2}{2} \qquad \text{Cap-a.e. in X.} \tag{7.19}$$

Then the operator $\langle\cdot,\cdot\rangle$ is $L^0(\mathrm{Cap})$-bilinear and satisfies

$$\begin{aligned}
\big|\langle v,w\rangle\big| &\leq |v||w|\\
\langle v,v\rangle &= |v|^2
\end{aligned} \qquad \text{Cap-a.e. for every } v,w\in\mathscr{H}_{\rm Cap}. \tag{7.20}$$

7.2.2 The tangent Cap-module

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a given $\mathsf{RCD}(K,\infty)$ space, for some constant $K\in\mathbb{R}$. We point out that we are in a position to apply Theorem 7.16 above, since Lipschitz functions with bounded support are dense in $W^{1,2}(\mathrm{X})$ by Theorem 2.27.

We use the notation $L^0_{\mathfrak{m}}(T\mathrm{X})$ to indicate the tangent $\mathfrak{m}$-module over X. We now introduce the so-called *tangent* Cap*-module* $L^0_{\mathrm{Cap}}(T\mathrm{X})$ over X, which is a normed Cap-module in the sense of Definition 7.21.

Theorem 7.25 (Tangent Cap-module) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be an $\mathsf{RCD}(K,\infty)$ space. Then there exists a unique couple $\big(L^0_{\mathrm{Cap}}(T\mathrm{X}),\bar\nabla\big)$, where $L^0_{\mathrm{Cap}}(T\mathrm{X})$ is a normed* Cap*-module over* X *and the operator $\bar\nabla:\mathrm{TestF}(\mathrm{X})\to L^0_{\mathrm{Cap}}(T\mathrm{X})$ is linear, such that the following properties hold:*

i) *For any $f\in\mathrm{TestF}(\mathrm{X})$ we have that the equality $|\bar\nabla f|=T\big(|Df|\big)$ holds* Cap*-a.e. on* X *(note that $|Df|\in W^{1,2}(\mathrm{X})$ as a consequence of Lemma 4.54).*

ii) *The space of $\sum_{n\in\mathbb{N}}\chi_{E_n}\bar\nabla f_n$, with $(f_n)_n\subseteq\mathrm{TestF}(\mathrm{X})$ and $(E_n)_n$ Borel partition of* X*, is dense in $L^0_{\mathrm{Cap}}(T\mathrm{X})$.*

Uniqueness is intended up to unique isomorphism: given another couple $(\mathscr{M}_{\mathrm{Cap}},\bar\nabla')$ with the same properties, there exists a unique isomorphism $\Phi:L^0_{\mathrm{Cap}}(T\mathrm{X})\to\mathscr{M}_{\mathrm{Cap}}$ with $\Phi\circ\bar\nabla=\bar\nabla'$.

The space $L^0_{\mathrm{Cap}}(T\mathrm{X})$ is called tangent Cap-module *associated to $(\mathrm{X},\mathsf{d},\mathfrak{m})$, while its elements are said to be* Cap-vector fields *on* X*. Moreover, the operator $\bar\nabla$ is called* gradient.

Proof. We first prove the uniqueness part of the statement and then the existence one.
UNIQUENESS. Consider any simple vector field $v\in L^0_{\mathrm{Cap}}(T\mathrm{X})$, i.e. $v=\sum_{n\in\mathbb{N}}\chi_{E_n}\bar\nabla f_n$ for some $(f_n)_n\subseteq\mathrm{TestF}(\mathrm{X})$ and $(E_n)_n$ Borel partition of X. We are thus forced to set

$$\Phi(v)\doteq\sum_{n\in\mathbb{N}}\chi_{E_n}\bar\nabla' f_n\in\mathscr{M}_{\mathrm{Cap}}. \tag{7.21}$$

Such definition is well-posed, as granted by the Cap-a.e. equalities

$$\bigg|\sum_{n\in\mathbb{N}}\chi_{E_n}\bar\nabla' f_n\bigg|=\sum_{n\in\mathbb{N}}\chi_{E_n}|\bar\nabla' f_n|=\sum_{n\in\mathbb{N}}\chi_{E_n}|Df_n|=\sum_{n\in\mathbb{N}}\chi_{E_n}|\bar\nabla f_n|=|v|,$$

which also show that Φ preserves the pointwise norm of simple vector fields. In particular, the map Φ is linear and continuous, whence it can be uniquely extended to a linear and continuous operator $\Phi:L^0_{\mathrm{Cap}}(T\mathrm{X})\to\mathscr{M}_{\mathrm{Cap}}$ by density of simple vector fields in $L^0_{\mathrm{Cap}}(T\mathrm{X})$. It follows from Remark 7.9 that Φ preserves the pointwise norm. Moreover, we know from the definition (7.21) that $\Phi(fv)=f\,\Phi(v)$ is satisfied for any simple f and v, whence for all $f\in L^0(\mathrm{Cap})$ and $v\in L^0_{\mathrm{Cap}}(T\mathrm{X})$ by Proposition 7.11. To conclude, just notice that the image of Φ is dense in $\mathscr{M}_{\mathrm{Cap}}$ by density of simple vector fields in $\mathscr{M}_{\mathrm{Cap}}$, thus accordingly Φ is surjective (as its image is closed, being Φ an isometry). Therefore we proved that there exists a unique module isomorphism $\Phi:L^0_{\mathrm{Cap}}(T\mathrm{X})\to\mathscr{M}_{\mathrm{Cap}}$ such that $\Phi\circ\bar\nabla=\bar\nabla'$, as required.
EXISTENCE. We define the 'pre-tangent module' Ptm as the set of all sequences $(E_n,f_n)_n$, where $(f_n)_n\subseteq\mathrm{TestF}(\mathrm{X})$ and $(E_n)_n$ is a Borel partition of X. We now define an equivalence relation $\sim$ on Ptm: we declare that $(E_n,f_n)_n\sim(F_m,g_m)_m$ provided

$$T\big(|D(f_n-g_m)|\big)=0\quad\text{holds Cap-a.e. on } E_n\cap F_m\qquad\text{for every } n,m\in\mathbb{N}.$$

The equivalence class of $(E_n, f_n)_n$ will be denoted by $[E_n, f_n]_n$. Moreover, let us define

$$\alpha\,[E_n, f_n]_n + \beta\,[F_m, g_m]_m \doteq [E_n \cap F_m, \alpha\, f_n + \beta\, g_m]_{n,m}$$

for every $\alpha, \beta \in \mathbb{R}$ and $[E_n, f_n]_n, [F_m, g_m]_m \in \mathsf{Ptm}/\sim$, so that $\mathsf{Ptm}/\sim$ inherits a vector space structure; well-posedness of these operations is granted by the locality property of minimal weak upper gradients and by Theorem 7.16. We define the pointwise norm of any given element $[E_n, f_n]_n \in \mathsf{Ptm}/\sim$ as

$$\big|[E_n, f_n]_n\big| \doteq \sum_{n\in\mathbb{N}} \chi_{E_n} T\big(|Df_n|\big) \in L^0(\mathrm{Cap}). \tag{7.22}$$

Then we define $L^0_{\mathrm{Cap}}(T\mathrm{X})$ as the completion of the metric space $\big(\mathsf{Ptm}/\sim, \mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathrm{X})}\big)$, where

$$\mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathrm{X})}(v,w) \doteq \sum_{k\in\mathbb{N}} \frac{1}{2^k\big(\mathrm{Cap}(A_k)\vee 1\big)} \int_{A_k} |v-w| \wedge 1 \,\mathrm{dCap} \quad \text{for all } v,w \in \mathsf{Ptm}/\sim, \tag{7.23}$$

while we set $\bar\nabla f \doteq [\mathrm{X}, f] \in L^0_{\mathrm{Cap}}(T\mathrm{X})$ for every test function $f \in \mathrm{TestF}(\mathrm{X})$, thus obtaining a linear operator $\bar\nabla : \mathrm{TestF}(\mathrm{X}) \to L^0_{\mathrm{Cap}}(T\mathrm{X})$. Item i) of the statement is thus clearly satisfied. Observe that $[E_n, f_n]_n = \sum_{n\in\mathbb{N}} \chi_{E_n} \bar\nabla f_n$ for every $[E_n, f_n]_n \in \mathsf{Ptm}/\sim$, so that also item ii) is verified, as a consequence of the density of $\mathsf{Ptm}/\sim$ in $L^0_{\mathrm{Cap}}(T\mathrm{X})$. Now let us define the multiplication operator $\,\cdot : \mathsf{Sf}(\mathrm{X}) \times (\mathsf{Ptm}/\sim) \to \mathsf{Ptm}/\sim$ as follows:

$$\Big(\sum_{m\in\mathbb{N}} \alpha_m\, \chi_{F_m}\Big) \cdot [E_n, f_n]_n \doteq [E_n \cap F_m, \alpha_m\, f_n]_{n,m} \in \mathsf{Ptm}/\sim . \tag{7.24}$$

Therefore the maps that are defined in (3.41) and (7.24) can be uniquely extended by continuity to a pointwise norm operator $|\cdot| : L^0_{\mathrm{Cap}}(T\mathrm{X}) \to L^0(\mathrm{Cap})$ and a multiplication by $L^0(\mathrm{Cap})$-functions $\,\cdot : L^0(\mathrm{Cap}) \times L^0_{\mathrm{Cap}}(T\mathrm{X}) \to L^0_{\mathrm{Cap}}(T\mathrm{X})$, respectively. It also turns out that the distance $\mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathrm{X})}$ is expressed by the formula in (7.23) for any $v, w \in L^0_{\mathrm{Cap}}(T\mathrm{X})$, as one can readily deduce from Remark 7.9. Finally, standard verifications show that $L^0_{\mathrm{Cap}}(T\mathrm{X})$ is a normed Cap-module over $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, thus concluding the proof. □

Remark 7.26 An analogous construction has been carried out in Theorem 4.1 to define the cotangent $\mathfrak{m}$-module $L^0_{\mathfrak{m}}(T^*\mathrm{X})$, while the tangent $\mathfrak{m}$-module $L^0_{\mathfrak{m}}(T\mathrm{X})$ was obtained in Definition 4.7 from the cotangent one by duality. However, since we cannot consider duals of normed Cap-modules (as pointed out in Remark 7.23), we opted for a different axiomatisation.

We just point out the fact that – since RCD spaces are infinitesimally Hilbertian – the modules $L^0_{\mathfrak{m}}(T^*\mathrm{X})$ and $L^0_{\mathfrak{m}}(T\mathrm{X})$ can be canonically identified via the Riesz isomorphism. ■

Proposition 7.27 *The tangent* Cap*-module* $L^0_{\mathrm{Cap}}(T\mathrm{X})$ *is a Hilbert module.*

Proof. Given any $f, g \in \mathrm{TestF}(\mathrm{X})$, we deduce from item i) of Theorem 7.25 and the last statement of Theorem 7.16 that

$$\begin{aligned}|\bar\nabla f + \bar\nabla g|^2 + |\bar\nabla f - \bar\nabla g|^2 &= T\big(|D(f+g)|^2 + |D(f-g)|^2\big) = T\big(2\,|Df|^2 + 2\,|Dg|^2\big)\\ &= 2\,|\bar\nabla f|^2 + 2\,|\bar\nabla g|^2.\end{aligned}$$

This grants that the pointwise parallelogram identity (7.18) is satisfied whenever v, w are $L^0(\mathrm{Cap})$-linear combinations of elements of $\big\{\bar\nabla f \,:\, f \in \mathrm{TestF}(\mathrm{X})\big\}$, whence also for any two Cap-vector fields $v, w \in L^0_{\mathrm{Cap}}(T\mathrm{X})$ by approximation. This proves that $L^0_{\mathrm{Cap}}(T\mathrm{X})$ is a Hilbert module, as required. □

The next result illustrates the relation that subsists between tangent Cap-module and tangent $\mathfrak{m}$-module. The projection $\bar{\Pi}$ we are going to describe represents a generalisation of the projection Π introduced in (7.12).

Proposition 7.28 *There is a unique linear continuous operator* $\bar{\Pi} : L^0_{\rm Cap}(T{\rm X}) \to L^0_{\mathfrak{m}}(T{\rm X})$ *that satisfies the following properties:*

i) $\bar{\Pi}(\bar{\nabla} f) = \nabla f$ *for every* $f \in {\rm TestF}({\rm X})$.

ii) $\bar{\Pi}(gv) = \Pi(g)\,\bar{\Pi}(v)$ *for every* $g \in L^0({\rm Cap})$ *and* $v \in L^0_{\rm Cap}(T{\rm X})$.

Moreover, the operator $\bar{\Pi}$ *satisfies the equality*

$$\big|\bar{\Pi}(v)\big| = \Pi\big(|v|\big) \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } v \in L^0_{\rm Cap}(T{\rm X}). \tag{7.25}$$

Proof. Given a Borel partition $(E_n)_{n\in\mathbb{N}}$ of X and $(v_n)_{n\in\mathbb{N}} \subseteq L^0_{\rm Cap}(T{\rm X})$, we are forced to set

$$\bar{\Pi}\bigg(\sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\rm Cap}\bar{\nabla} f_n\bigg) \doteq \sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\mathfrak{m}}\nabla f_n. \tag{7.26}$$

The well-posedness of such definition stems from the following $\mathfrak{m}$-a.e. equalities:

$$\begin{aligned}\bigg|\sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\mathfrak{m}}\nabla f_n\bigg| &= \sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\mathfrak{m}}\,|Df_n| = \sum_{n\in\mathbb{N}}\Pi\big([\chi_{E_n}]_{\rm Cap}\big)\,\Pi\big(T\big(|Df_n|\big)\big)\\ &= \Pi\bigg(\sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\rm Cap}\,T\big(|Df_n|\big)\bigg) = \Pi\bigg(\sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\rm Cap}\,|\bar{\nabla} f_n|\bigg)\\ &= \Pi\bigg(\bigg|\sum_{n\in\mathbb{N}}[\chi_{E_n}]_{\rm Cap}\bar{\nabla} f_n\bigg|\bigg).\end{aligned} \tag{7.27}$$

Moreover, we also infer that such map $\bar{\Pi}$ – which is linear by construction – is also continuous, whence it admits a unique linear and continuous extension $\bar{\Pi} : L^0_{\rm Cap}(T{\rm X}) \to L^0_{\mathfrak{m}}(T{\rm X})$. Property i) is clearly satisfied by (7.26). From the linearity of ∇ and $\bar{\nabla}$, we deduce that property ii) holds for any simple function $g \in L^0({\rm Cap})$, thus also for any $g \in L^0({\rm Cap})$ by approximation. Finally, again by approximation we see that (7.25) follows from (7.27). □

7.2.3 Quasi-continuity of Sobolev vector fields on RCD spaces

Let $({\rm X},{\sf d},\mathfrak{m})$ be an ${\sf RCD}(K,\infty)$ space, for some $K \in \mathbb{R}$. The aim of this conclusive subsection is to prove that any element of the space $H^{1,2}_{\rm C}(T{\rm X})$ admits a quasi-continuous representative, in a suitable sense. We begin with the definition of quasi-continuous vector field on X:

Definition 7.29 (Quasi-continuity for vector fields) *Let* $v \in L^0_{\rm Cap}(T{\rm X})$ *be given. Then we say that* v *is* quasi-continuous *provided*

$$|v - \bar{\nabla} f| \in \mathcal{C}^{qc}({\rm X}) \qquad \text{for every } f \in {\rm TestF}({\rm X}). \tag{7.28}$$

We denote by $\mathcal{C}^{qc}(T{\rm X}) \subseteq L^0_{\rm Cap}(T{\rm X})$ *the set of all quasi-continuous* Cap-*vector fields on* X.

The well-posedness of the previous definition immediately follows from Lemma 7.13.

Remark 7.30 It is well-known that a vector field $v : \mathbb{R}^n \to \mathbb{R}^n$ in the Euclidean space is quasi-continuous if and only if $\mathbb{R}^n \ni x \mapsto \big|v(x) - \nabla f(x)\big|$ is quasi-continuous for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$. This served as a motivation for our definition and shows its consistency with the classical notion of quasi-continuous vector field in the smooth setting. ■

Lemma 7.31 *The family* $\mathcal{C}^{qc}(T\mathrm{X}) \subseteq L^0_{\mathrm{Cap}}(T\mathrm{X})$ *contains the set*

$$(7.29) \qquad \mathrm{Test}\bar{\mathrm{V}}(\mathrm{X}) \doteq \left\{ \sum_{i=1}^{n} T(g_i)\bar{\nabla} f_i \;\middle|\; n \in \mathbb{N},\, (f_i)_{i=1}^n, (g_i)_{i=1}^n \subseteq \mathrm{TestF}(\mathrm{X}) \right\}.$$

Moreover, it holds that $|v| \in \mathcal{C}^{qc}(\mathrm{X})$ *for every* $v \in \mathcal{C}^{qc}(T\mathrm{X})$.

Proof. In order to prove the first statement, let us fix $v = \sum_{i=1}^n T(g_i)\bar{\nabla} f_i \in \mathrm{Test}\bar{\mathrm{V}}(\mathrm{X})$. Given any $f_{n+1} \in \mathrm{TestF}(\mathrm{X})$ and called $g_{n+1} \equiv -1$, it clearly holds that

$$\begin{aligned}
|v - \bar{\nabla} f_{n+1}|^2 &= \sum_{i,j=1}^{n+1} T(g_i)\,T(g_j)\,\langle \bar{\nabla} f_i, \bar{\nabla} f_j \rangle \\
&= \sum_{i,j=1}^{n+1} T(g_i)\,T(g_j)\,\frac{\big|\bar{\nabla}(f_i + f_j)\big|^2 - |\bar{\nabla} f_i|^2 - |\bar{\nabla} f_j|^2}{2} \\
&= \sum_{i,j=1}^{n+1} T(g_i)\,T(g_j)\,\frac{T\big(|D(f_i + f_j)|^2\big) - T\big(|Df_i|^2\big) - T\big(|Df_j|^2\big)}{2} \in \mathcal{C}^{qc}(\mathrm{X}).
\end{aligned}$$

Therefore $|v - \bar{\nabla} f_{n+1}| \in \mathcal{C}^{qc}(\mathrm{X})$ for every $f_{n+1} \in \mathrm{TestF}(\mathrm{X})$ and accordingly $v \in \mathcal{C}^{qc}(T\mathrm{X})$. The last statement is trivial: since the identically null function 0 belongs to $\mathrm{TestF}(\mathrm{X})$, one has that $|v| = |v - \bar{\nabla} 0| \in \mathcal{C}^{qc}(\mathrm{X})$ for all $v \in \mathcal{C}^{qc}(T\mathrm{X})$. □

Remark 7.32 It is not clear whether in general $\mathcal{C}^{qc}(T\mathrm{X})$ is a vector space. ■

Proposition 7.33 *Let* $\mathcal{V} \subseteq \mathcal{C}^{qc}(T\mathrm{X})$ *be any given vector subspace of* $L^0_{\mathrm{Cap}}(T\mathrm{X})$. *Then the map* $\bar{\Pi}|_{\mathcal{V}} : \mathcal{V} \to L^0_{\mathfrak{m}}(T\mathrm{X})$ *is injective.*

Proof. Let $v, w \in \mathcal{V}$ be such that $\bar{\Pi}(v) = \bar{\Pi}(w)$. In other words, we have that

$$\Pi\big(|v - w|\big) \overset{(7.25)}{=} \big|\bar{\Pi}(v - w)\big| = 0 \qquad \text{holds } \mathfrak{m}\text{-a.e. in X},$$

whence Proposition 7.15 grants that $|v - w| = 0$ holds Cap-a.e. in X. This shows that $v = w$, thus proving the claim. □

Finally, we are ready to state and prove the main result of the chapter: any element of the space $H^{1,2}_{\mathrm{C}}(T\mathrm{X})$ admits a quasi-continuous representative in $\mathcal{C}^{qc}(T\mathrm{X})$, in the sense described above. This is a generalisation of Theorem 7.16 to vector fields over an RCD space.

Theorem 7.34 (Quasi-continuous representative of Sobolev vector field) *Let us fix an* $\mathsf{RCD}(K, \infty)$ *space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, *for some* $K \in \mathbb{R}$. *Then there exists a unique map*

$$(7.30) \qquad \bar{T} : H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \longrightarrow \mathcal{C}^{qc}(T\mathrm{X})$$

such that $\bar{\Pi} \circ \bar{T} : H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \to L^0_{\mathfrak{m}}(T\mathrm{X})$ *coincides with the inclusion* $H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \subseteq L^0_{\mathfrak{m}}(T\mathrm{X})$. *Moreover,* $\bar{T}$ *is linear and* $\big|\bar{T}(v)\big| = T\big(|v|\big)$ *holds for every* $v \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$.

Proof. Fix $v \in H^{1,2}_{\mathrm{C}}(T\mathrm{X})$. Pick $(\bar{v}_n)_n \subseteq \mathrm{Test}\bar{\mathrm{V}}(\mathrm{X})$ such that $v_n \doteq \bar{\Pi}(\bar{v}_n) \to v$ in $W^{1,2}_{\mathrm{C}}(T\mathrm{X})$. We know from Lemma 4.54 that $|v_n - v| \in W^{1,2}(\mathrm{X})$ and $\big|D|v_n - v|\big| \leq \big|\nabla(v_n - v)\big|_{\mathsf{HS}}$ $\mathfrak{m}$-a.e. for all $n \in \mathbb{N}$, thus accordingly $|v_n - v| \to 0$ in $W^{1,2}(\mathrm{X})$ as $n \to \infty$. Proposition 7.19 grants that – up to a (not relabeled) subsequence – we have that $T\big(|v_n - v|\big) \to 0$ quasi-uniformly as $n \to \infty$, whence $T\big(|v_n - v_m|\big) \to 0$ quasi-uniformly as $n, m \to \infty$. Thus Lemma 7.20 yields

$$\mathsf{d}_{L^0_{\mathrm{Cap}}(T\mathrm{X})}(\bar{v}_n, \bar{v}_m) = \mathsf{d}_{\mathrm{Cap}}\big(T\big(|v_n - v_m|\big), 0\big) \longrightarrow 0 \qquad \text{as } n, m \to \infty.$$

This shows that $(\bar{v}_n)_n \subseteq L^0_{\mathrm{Cap}}(T\mathrm{X})$ is Cauchy, thus it converges to some $\bar{v} \in L^0_{\mathrm{Cap}}(T\mathrm{X})$. Hence one has $\bar{\Pi}(\bar{v}) = \bar{\Pi}\big(\lim_n \bar{v}_n\big) = \lim_n \bar{\Pi}(\bar{v}_n) = \lim_n v_n = v$, so that we define $\bar{T}(v) \doteq \bar{v}$. Proposition 7.33 grants that the map $\bar{T} : H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \to \mathcal{C}^{qc}(T\mathrm{X})$ is well-defined and is the unique map such that $\bar{\Pi} \circ \bar{T}$ coincides with the inclusion $H^{1,2}_{\mathrm{C}}(T\mathrm{X}) \subseteq L^0_{\mathfrak{m}}(T\mathrm{X})$. Finally, the last two statements follow from linearity of $\bar{\Pi}$, Theorem 7.16 and Proposition 7.33. □

Remark 7.35 We point out that $\bar{T}\big(\mathrm{TestV}(\mathrm{X})\big) = \mathrm{Test}\bar{\mathrm{V}}(\mathrm{X})$. ■

8

Differential of metric-valued Sobolev maps

Contents

Let $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m})$ be a metric measure space and let $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ be a complete and separable metric space. We shall also fix $p = 2$ for simplicity. There are several possible definitions of the concept of Sobolev map from X to Y. Here we work with the one based on *post-composition* (see [HKST15] for historical remarks): we say that f belongs to $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$ provided there exists $G \in L^2(\mathfrak{m})$ such that for any Lipschitz function $\varphi : \mathrm{Y} \to \mathbb{R}$ we have $\varphi \circ f \in \mathrm{S}^2(\mathrm{X})$ and

$$\big|D(\varphi \circ f)\big| \leq \mathrm{Lip}(\varphi)\, G \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.}$$

The least such G is denoted by $|Df|$ and called *minimal weak upper gradient* of the map f. Since Y has no linear structure, the set $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$ is not a vector space in general. We refer to Section 8.1 below for the relative discussion.

The question we address in this chapter – entirely taken from [GPS18] – is the following: in analogy with the fact that 'behind' the minimal weak upper gradient $|Df|$ of a real-valued Sobolev function there is an abstract differential $\mathrm{d}f$, does there exist a notion of differential for metric-valued Sobolev maps?

Let us now motivate our interest in such problem. In the celebrated paper [ES64], J. Eells and J. H. Sampson proved the Lipschitz regularity of harmonic maps between Riemannian manifolds, when the target manifold N has non-positive curvature and is simply connected; the Lipschitz estimate is given in terms of a lower Ricci curvature bound and an upper dimension bound on the source manifold M. A key point in their proof is the establishment of the so-called *Bochner-Eells-Sampson formula* for maps $f: M \to N$, which can be informally stated in the following way:

$$\Delta \frac{|\mathrm{d}f|^2}{2} \geq \nabla f(\Delta f) + K\,|\mathrm{d}f|^2, \tag{8.1}$$

where $|\mathrm{d}f|$ is the Hilbert-Schmidt norm of the differential of f and $K \in \mathbb{R}$ is a lower bound for the Ricci curvature of M; let us remain vague about the meaning of the term $\nabla f(\Delta f)$. A direct consequence of (8.1) is that if f is harmonic, then

$$\Delta \frac{|\mathrm{d}f|^2}{2} \geq K\,|\mathrm{d}f|^2. \tag{8.2}$$

This bound and Moser's iteration technique show that $|\mathrm{d}f|$ is locally bounded from above in the domain of definition of f, thus showing the local Lipschitz regularity of f (the upper dimension bound for M enters into play in the constants appearing in Moser's argument).

Since the Lipschitz regularity of harmonic maps does not depend on the smoothness of M and N, but only on the stated curvature bounds, it is natural to ask whether the same results hold by only assuming the appropriate curvature bounds on the source space and the target space – without any reference to smoothness. Efforts in this direction have been made by Gromov-Schoen in [GS92], by Korevaar-Shoen in [KS93] and by Zhang-Zhu in [ZZ18]. The most general result is in [ZZ18], where the authors handle the case of source spaces that are finite-dimensional Alexandrov spaces with (sectional) curvature bounded from below and targets that are $\mathsf{CAT}(0)$ spaces (i.e. metric spaces with non-positive sectional curvature). Still, given Eells-Sampson's result, the natural synthetic setting appears to be that of maps from an $\mathsf{RCD}(K,N)$ space to a $\mathsf{CAT}(0)$ space. The content of this chapter aims at being a first step in the direction of obtaining (8.1) for maps from $\mathsf{RCD}(K,N)$ spaces to $\mathsf{CAT}(0)$ spaces (cf. also [GT18]). If successful, this would easily imply the desired Lipschitz regularity for harmonic maps and at the same time improve the understanding of the subject even in previously studied non-smooth settings. The very first step to tackle in order to write down (8.1) is to understand what the differential '$\mathrm{d}f$' is, which is what we are going to do.

Let $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ be a Sobolev map. As we will see in Subsection 8.1, a special measure on Y associated to u is given by $\mu \doteq u_*\big(|Du|^2\,\mathfrak{m}\big)$. The importance of this measure is due to the fact that it has nice composition properties (cf. Proposition 8.3), which will allow us to define the differential

$$\mathrm{d}u : L^0(T\mathrm{X}) \to \big(u^* L^0_\mu(T^*\mathrm{Y})\big)^*$$

of u as an appropriate adjoint of the well-posed map $\mathrm{d}f \mapsto \mathrm{d}(f \circ u)$ (cf. Definition 8.4).

Once this definition is given, we verify that it is compatible with some previously known notions of differentials in the non-smooth setting:

- The differential of a real-valued Sobolev function on X, as an element of the cotangent module $L^0(T^*\mathrm{X})$; see Subsection 8.2.1.
- The differential of a map of bounded deformation between metric measure spaces, in the sense of Theorem 4.34; see Subsection 8.2.2.

- The 'metric differential' for metric-valued Lipschitz maps defined on the Euclidean space, which has been studied in [Kir94] and corresponds to the pointwise magnitude of the differential of a map (rather than a true differential); see Subsection 8.2.3.

Finally, in Section 8.3 we shall also consider the case of locally Sobolev maps $u \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X};\mathrm{Y})$. More precisely, we will discuss how to define the differential in this situation. The main complication here is that the weighted pushforward $\mu = u_*\big(|Du|^2\,\mathfrak{m}\big)$ no longer defines a σ-finite measure. We shall see that this difficulty can be handled by proving a sort of sheaf property for pullback of differentials, so that the differential of u can be defined via an appropriate inverse limit construction.

Although we formulate our results for maps in $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$, one could use precisely the same arguments to treat Sobolev maps with general exponent $p \in (1,\infty)$. In this case, one has to keep in mind that the constructions using minimal weak upper gradients – in particular the notion of differential of a Sobolev map – will depend on p. Nevertheless, the exponent $p = 2$ is sufficient for the applications we have in mind.

8.1 Metric-valued Sobolev maps and their differential

Let $(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m})$ be a metric measure space and $(\mathrm{Y},\mathsf{d}_\mathrm{Y})$ be a complete separable metric space.

Definition 8.1 (Metric valued Sobolev map) *The class $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$ is defined as the collection of all Borel maps $u : \mathrm{X} \to \mathrm{Y}$ for which there exists a function $G \in L^2(\mathfrak{m})^+$ such that for any $f \in \mathrm{LIP}(\mathrm{Y})$ it holds that $f \circ u \in \mathrm{S}^2(\mathrm{X})$ and*

$$\big|\mathrm{d}(f\circ u)\big| \leq \mathrm{Lip}(f)\,G \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{8.3}$$

The minimal function G (in the $\mathfrak{m}$-a.e. sense) for which the above holds is denoted by $|Du|$.

Given any $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$, the class of $G \in L^2(\mathfrak{m})$ for which (8.3) holds is a closed lattice, hence an $\mathfrak{m}$-a.e. minimal element exists and accordingly the definition of $|Du|$ is well-posed. Our study of the maps in $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$ begins with the following basic lemma:

Lemma 8.2 *Let $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ and $f \in \mathrm{LIP}(\mathrm{Y})$ be given. Then*

$$\big|\mathrm{d}(f\circ u)\big| \leq \mathrm{lip}_a(f)\circ u\,|Du| \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{8.4}$$

Proof. Let $(y_n)_n \subseteq \mathrm{Y}$ be a countable dense set. For any $r \in \mathbb{Q}^+$, let $f_{r,n} \in \mathrm{LIP}(\mathrm{Y})$ be any McShane extension of $f|_{B_r(y_n)}$, i.e. any Lipschitz function defined on the whole Y that coincides with f on $B_r(y_n)$ and such that $\mathrm{Lip}(f_{r,n}) = \mathrm{Lip}\big(f;B_r(y_n)\big)$; recall property (1.21). Then from (8.3) and the locality of the differential we see that

$$\big|\mathrm{d}(f\circ u)\big| \leq \mathrm{Lip}\big(f;B_r(y_n)\big)\,|Du| \qquad \mathfrak{m}\text{-a.e. on } u^{-1}\big(B_r(y_n)\big).$$

Since for every $y \in \mathrm{Y}$ we have that $\mathrm{lip}_a(f)(y) = \inf \mathrm{Lip}\big(f;B_r(y_n)\big)$ – where the infimum is taken among all n and r such that $y \in B_r(y_n)$ – the conclusion follows. □

Let us fix $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ and equip the target space Y with the finite Radon measure

$$\mu \doteq u_*\big(|Du|^2\mathfrak{m}\big). \tag{8.5}$$

Notice that for any $f \in L^0(\mu)$ the function $f \circ u$ is not well-defined up to $\mathfrak{m}$-a.e. equality, in the sense that if $f = \tilde{f}$ holds μ-a.e. then not necessarily $f \circ u = \tilde{f} \circ u$ in the $\mathfrak{m}$-a.e. sense. Still, we certainly have $f \circ u = \tilde{f} \circ u$ $\mathfrak{m}$-a.e. on $\{|Du| > 0\}$. For this reason, we have that the equality $f \circ u\,|Du| = \tilde{f} \circ u\,|Du|$ holds $\mathfrak{m}$-a.e., or in other words the map $f \mapsto f \circ u\,|Du|$ is well-defined from $L^0(\mu)$ to $L^0(\mathfrak{m})$. Then the trivial identity $\int \big|f \circ u\,|Du|\big|^2\,\mathrm{d}\mathfrak{m} = \int |f|^2\,\mathrm{d}\mu$ shows that the operator

$$L^2(\mu) \ni f \longmapsto f \circ u\,|Du| \in L^2(\mathfrak{m}) \qquad \text{is linear and continuous.} \tag{8.6}$$

Moreover, we define the space $\mathrm{LIP_{bd}(Y)}$ as follows:

$$\mathrm{LIP_{bd}(Y)} \doteq \Big\{ f : \mathrm{Y} \to \mathbb{R} \;\Big|\; f_{|B} \text{ is Lipschitz for any } B \subseteq \mathrm{Y} \text{ bounded} \Big\}. \tag{8.7}$$

In particular, any element of $\mathrm{LIP_{bd}(Y)}$ is continuous. It is then easy to check that:

$$\begin{aligned} &\text{For any } f \in \mathrm{S}^2(\mathrm{Y},\mu) \text{ there is a sequence } (f_n)_n \subseteq \mathrm{LIP_{bd}(Y)} \\ &\mu\text{-a.e. converging to } f \text{ such that } \mathrm{lip}_a(f_n) \to |Df| \text{ in } L^2(\mu). \end{aligned} \tag{8.8}$$

We now turn to our key basic result about pullback of Sobolev functions:

Proposition 8.3 *Let $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ be given. Put $\mu \doteq u_*\big(|Du|^2\mathfrak{m}\big)$ and let $f \in \mathrm{S}^2(\mathrm{Y},\mu)$. Then there exists $g \in \mathrm{S}^2(\mathrm{X})$ such that $g = f \circ u$ holds $\mathfrak{m}$-a.e. on $\{|Du| > 0\}$ and*

$$|\mathrm{d}g| \leq |\mathrm{d}_\mu f| \circ u\,|Du| \qquad \textit{in the } \mathfrak{m}\textit{-a.e. sense.} \tag{8.9}$$

More precisely, there exist $g \in \mathrm{S}^2(\mathrm{X})$ and a sequence $(f_n)_n \subseteq \mathrm{LIP_{bd}(Y)}$ such that

$$\begin{aligned} f_n &\longrightarrow f && \textit{in the } \mu\textit{-a.e. sense,} \\ \mathrm{lip}_a(f_n) &\longrightarrow |\mathrm{d}_\mu f| && \textit{in } L^2(\mu), \\ f_n \circ u &\longrightarrow g && \textit{in the } \mathfrak{m}\textit{-a.e. sense,} \\ \mathrm{lip}_a(f_n) \circ u\,|Du| &\longrightarrow |\mathrm{d}_\mu f| \circ u\,|Du| && \textit{in } L^2(\mathfrak{m}). \end{aligned} \tag{8.10}$$

Proof. Up to a truncation and diagonalisation argument, we can assume that $f \in L^\infty(\mu)$. Then let $(f_n)_n \subseteq \mathrm{LIP_{bd}(Y)}$ be as in (8.8). Since f is bounded, by truncation we can assume the f_n's to be uniformly bounded. Thus the first two claims in (8.10) hold and – by taking (8.6) into account – we see that also the last claim in (8.10) holds. Now observe that if we could prove that the sequence $(f_n \circ u)_n$ has a pointwise $\mathfrak{m}$-a.e. limit g, then (8.9) would follow from Lemma 8.2, property (8.6) and the closure of the differential. Let $B \subseteq \mathrm{X}$ be bounded and Borel. The functions $f_n \circ u$ are equibounded and $\mathfrak{m}(B) < \infty$, hence $(f_n \circ u)_n$ is bounded in $L^2\big(\mathfrak{m}_{|B}\big)$. Thus by passing to an appropriate (not relabeled) sequence of convex combinations – which do not affect the already proven convergences in (8.10) – we obtain that $(f_n \circ u)_n$ has a strong limit in $L^2\big(\mathfrak{m}_{|B}\big)$. Thus some subsequence converges $\mathfrak{m}$-a.e. on B. Therefore considering a sequence $(B_k)_k$ of bounded sets such that $\mathrm{X} = \bigcup_k B_k$, we conclude by a diagonalisation argument. □

Let us notice that, since μ is a finite measure on Y, we have $\mathrm{LIP(Y)} \subseteq \mathrm{S}^2(\mathrm{Y},\mu)$. Also:

$$\text{For } f \in \mathrm{LIP(Y)} \text{ and } g \in \mathrm{S}^2(\mathrm{X}) \text{ as in Proposition 8.3, we have } \mathrm{d}(f \circ u) = \mathrm{d}g. \tag{8.11}$$

Indeed, the locality of the differential gives $\mathrm{d}(f \circ u) = \mathrm{d}g$ on $\{|Du| > 0\}$, while the bounds (8.4) and (8.9) grant that $\big|\mathrm{d}(f \circ u)\big| = |\mathrm{d}g| = 0$ holds $\mathfrak{m}$-a.e. on $\{|Du| = 0\}$.

Observe that for $\nu \doteq \mathfrak{m}|_{\{|Du|>0\}}$ we have $u_*\nu \ll \mu$, thus $u^*L^0_\mu(T^*\mathrm{Y})$ is a well-defined $L^0(\nu)$-normed $L^0(\nu)$-module. Recalling the 'extension' functor introduced in Remark 3.11, our definition of the differential $\mathrm{d}u$ is the following:

Definition 8.4 (Differential of metric-valued Sobolev maps) *The* differential

$$\mathrm{d}u : L^0(T\mathrm{X}) \longrightarrow \mathrm{Ext}\big(\big(u^*L^0_\mu(T^*\mathrm{Y})\big)^*\big) \tag{8.12}$$

of a map $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ is the operator defined in the following way. For any $v \in L^0(T\mathrm{X})$, the object $\mathrm{d}u(v) \in \mathrm{Ext}\big(\big(u^*L^0_\mu(T^*\mathrm{Y})\big)^*\big)$ is characterised by this property: for any $f \in \mathrm{S}^2(\mathrm{Y},\mu)$ and $g \in \mathrm{S}^2(\mathrm{X})$ as in Proposition 8.3, we have that

$$\mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big) = \mathrm{d}g(v) \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{8.13}$$

We now verify that this is a good definition and check its very basic properties:

Proposition 8.5 (Well-posedness of $\mathrm{d}u$) *The definition of $\mathrm{d}u(v)$ is well-posed and the map $\mathrm{d}u : L^0(T\mathrm{X}) \to \mathrm{Ext}\big(\big(u^*L^0_\mu(T^*\mathrm{Y})\big)^*\big)$ is $L^0(\mathfrak{m})$-linear continuous. Moreover, it holds*

$$|\mathrm{d}u| = |Du| \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{8.14}$$

Proof. Let $f \in \mathrm{S}^2(\mathrm{Y},\mu)$ be given. Observe that if $g, g' \in \mathrm{S}^2(\mathrm{X})$ satisfy the properties listed in Proposition 8.3, then the locality of the differential and the bound (8.9) show that $\mathrm{d}g = \mathrm{d}g'$. Hence the right hand side of (8.13) depends only on f, u and v. Then notice that again the bound (8.9) gives

$$\Big|\mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big)\Big| \overset{(8.13)}{=} \big|\mathrm{d}g(v)\big| \leq |\mathrm{d}g|\,|v| \overset{(8.9)}{\leq} |\mathrm{d}_\mu f| \circ u\,|Du|\,|v| = \big|\mathrm{ext}(u^*\mathrm{d}_\mu f)\big|\,|Du|\,|v|.$$

Thus the arbitrariness of $f \in \mathrm{S}^2(\mathrm{Y},\mu)$, the universal property of the pullback and (3.10) ensure that $\mathrm{d}u(v)$ is a well-defined element of $\big(\mathrm{Ext}\big(u^*L^0_\mu(T^*\mathrm{Y})\big)\big)^* \sim \mathrm{Ext}\big(\big(u^*L^0_\mu(T^*\mathrm{Y})\big)^*\big)$, as desired, with

$$\big|\mathrm{d}u(v)\big| \leq |Du|\,|v| \qquad \text{in the } \mathfrak{m}\text{-a.e. sense.} \tag{8.15}$$

The fact that $\mathrm{d}u(v)$ is $L^0(\mathfrak{m})$-linear in v is trivial, while the bound (8.15) gives both continuity and $\boxed{\leq}$ in (8.14). To get $\boxed{\geq}$, let $f : \mathrm{Y} \to \mathbb{R}$ be 1-Lipschitz and notice that since $\mu(\mathrm{Y}) < \infty$ we also have that $f \in \mathrm{S}^2(\mathrm{Y},\mu)$. Notice that since $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ we have that $f \circ u \in \mathrm{S}^2(\mathrm{X})$, thus we can find an element $v \in L^0(T\mathrm{X})$ such that

$$|v| = 1 \;\text{ and }\; \mathrm{d}(f \circ u)(v) = \big|\mathrm{d}(f \circ u)\big| \qquad \text{in the } \mathfrak{m}\text{-a.e. sense} \tag{8.16}$$

(the existence of such v follows by Banach-Alaoglu theorem, see [Gig17b, Corollary 1.2.16]). Moreover, pick any $g \in \mathrm{S}^2(\mathrm{X})$ as in Proposition 8.3 and observe that

$$\begin{aligned}\big|\mathrm{d}(f \circ u)\big| &\overset{(8.16)}{=} \big|\mathrm{d}(f \circ u)(v)\big| \overset{(8.11)}{=} \big|\mathrm{d}g(v)\big| \overset{(8.13)}{=} \big|\mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big)\big| \leq \big|\mathrm{ext}(u^*\mathrm{d}_\mu f)\big|\,|\mathrm{d}u|\,|v| \\ &\overset{(8.16)}{=} |\mathrm{d}_\mu f| \circ u\,|\mathrm{d}u| \leq |\mathrm{d}u|,\end{aligned}$$

having used the fact that f is 1-Lipschitz in the last step. By arbitrariness of f and the very definition of $|Du|$ given in Definition 8.1, this establishes $\boxed{\geq}$ in (8.14). □

8.2 Consistency with previously known notions

8.2.1 The case $\mathrm{Y} = \mathbb{R}$

In this subsection we assume that $\mathrm{Y} = \mathbb{R}$ and prove that – once few natural identifications are taken into account – the newly defined differential $\mathrm{d}u$ is ‘the same’ as the one as defined by Theorem 4.1, which for the moment we shall denote by $\underline{\mathrm{d}}u \in L^0(T^*\mathrm{X})$.

To start with, let us observe that directly from the definitions and the chain rule

$$\mathrm{d}(f \circ u) = f' \circ u \, \underline{\mathrm{d}}u \qquad \text{for every } u \in \mathrm{S}^2(\mathrm{X}) \text{ and } f \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R}) \tag{8.17}$$

(recall Proposition 4.3), we have that the class $\mathrm{S}^2(\mathrm{X};\mathrm{Y})$ coincides with $\mathrm{S}^2(\mathrm{X})$ as soon as $\mathrm{Y} = \mathbb{R}$ and that the two notions of minimal weak upper gradient coincide. Now fix any $u \in \mathrm{S}^2(\mathrm{X})$, define $\mu \doteq u_*\big(|\mathrm{d}u|^2\mathfrak{m}\big)$ and consider the $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathrm{Ext}\big(u^*L^0_\mu(T^*\mathbb{R})\big)$. From the separability of $L^0_\mu(T\mathbb{R})$ provided by Corollary 4.29, the characterisation of the dual of the pullback described in Theorem 3.34 and property (3.10), we see that

$$\mathrm{Ext}\big(u^*L^0_\mu(T\mathbb{R})\big) \sim \mathrm{Ext}\big(u^*L^0_\mu(T^*\mathbb{R})\big)^*$$

via the coupling $\mathrm{ext}(u^*L)\big(\mathrm{ext}(u^*v)\big) \doteq \mathrm{ext}\big(L(v) \circ u\big)$. Hence in our situation we shall think of $\mathrm{d}u$ as a map from $L^0(T\mathrm{X})$ to $\mathrm{Ext}\big(u^*L^0_\mu(T\mathbb{R})\big)$.

Let us consider the maps $P : L^0(\mathbb{R},\mathbb{R}^*;\mu) \to L^0_\mu(T^*\mathbb{R})$ and $\iota : L^0_\mu(T\mathbb{R}) \to L^0(\mu)$ as in Theorem 4.27. Put $\nu \doteq \chi_{\{|Du|>0\}}\mathfrak{m}$ and consider the $L^0(\nu)$-linear continuous operators

$$u^*P : u^*L^0(\mathbb{R},\mathbb{R}^*;\mu) \longrightarrow u^*L^0_\mu(T^*\mathbb{R}), \qquad u^*\iota : u^*L^0_\mu(T\mathbb{R}) \longrightarrow u^*L^0(\mu) \sim L^0(\nu)$$

defined via the universal property of the pullback module. It is then clear that $u^*\iota$ is the adjoint of u^*P, thus we deduce that

$$(u^*Df)\big((u^*\iota)(V)\big) = (u^*\mathrm{d}_\mu f)(V) \quad \nu\text{-a.e.} \qquad \text{for all } V \in u^*L^0_\mu(T\mathbb{R}) \text{ and } f \in C^1_c(\mathbb{R}). \tag{8.18}$$

Finally, noticing that $\mathrm{ext} : u^*L^0_\mu(T\mathbb{R}) \to \mathrm{Ext}\big(u^*L^0_\mu(T\mathbb{R})\big)$ is invertible, we define

$$\mathcal{I} : \mathrm{Ext}\big(u^*L^0_\mu(T\mathbb{R})\big) \longrightarrow L^0(\mathfrak{m}) \qquad \text{as } \mathcal{I} \doteq \mathrm{ext} \circ (u^*\iota) \circ \mathrm{ext}^{-1}. \tag{8.19}$$

Then we can prove the following result:

Theorem 8.6 *With the above notation and assumptions, we have* $|\mathrm{d}u| = |\underline{\mathrm{d}}u|$ $\mathfrak{m}$*-a.e. and*

$$\mathcal{I}\big(\mathrm{d}u(v)\big) = \underline{\mathrm{d}}u(v) \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } v \in L^0(T\mathrm{X}). \tag{8.20}$$

Proof. The identity $|\mathrm{d}u| = |\underline{\mathrm{d}}u|$ follows from (8.14) and the fact that for $u \in \mathrm{S}^2(\mathrm{X}) = \mathrm{S}^2(\mathrm{X};\mathbb{R})$ the two notions of minimal weak upper gradient underlying the two spaces coincide. We turn to (8.20). For $f \in C^1_c(\mathbb{R})$, let us denote by $Df : \mathbb{R} \to \mathbb{R}^*$ its differential and by $f' : \mathbb{R} \to \mathbb{R}$ its derivative. Clearly – up to identifying $\mathbb{R}$ and $\mathbb{R}^*$ via the Riesz isomorphism – these two objects coincide and thus checking first the case $h = u^*g$ we easily get that

$$f' \circ u \, h = \mathrm{ext}(u^*Df)(h) \qquad \text{holds } \mathfrak{m}\text{-a.e. in X} \tag{8.21}$$

for every $h \in \mathrm{Ext}\big(u^*L^0(\mu)\big) \sim \mathrm{Ext}\big(L^0(\nu)\big) \subseteq L^0(\mathfrak{m})$. Then for g as in Proposition 8.3 we have

$$\begin{aligned}
f' \circ u \, \mathcal{I}\big(\mathrm{d}u(v)\big) &\overset{(8.21)}{=} \mathrm{ext}(u^*Df)\,\mathcal{I}\big(\mathrm{d}u(v)\big) \overset{(8.19),(3.10)}{=} \mathrm{ext}\Big((u^*Df)\big((u^*\iota)\big(\mathrm{ext}^{-1}(\mathrm{d}u(v))\big)\big)\Big) \\
&\overset{(8.18)}{=} \mathrm{ext}\Big((u^*\mathrm{d}_\mu f)\big(\mathrm{ext}^{-1}(\mathrm{d}u(v))\big)\Big) \overset{(3.10)}{=} \mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big) \overset{(8.13)}{=} \mathrm{d}g(v) \\
&\overset{(8.11)}{=} \mathrm{d}(f \circ u)(v) \overset{(8.17)}{=} f' \circ u \, \underline{\mathrm{d}}u(v).
\end{aligned}$$

Since $\big\{f' \circ u \,:\, f \in C^1_c(\mathbb{R})\big\}$ generates $L^0(\mathfrak{m})$, this is sufficient to establish (8.20). □

8.2.2 The case u of bounded deformation

In this subsection we also assume that $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ carries a non-negative Radon measure $\mathfrak{m}_\mathrm{Y}$ giving finite mass to bounded sets and study the differential of a map $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y})$ that is also of bounded deformation. Let us call $\mathfrak{m}_\mathrm{X} \doteq \mathfrak{m}$. We can consider the notion of differential

$$\hat{\mathrm{d}}u : L^2(T\mathrm{X}) \longrightarrow \big(u^* L^2_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y})\big)^*$$

defined in Theorem 4.34. We now study the relation between $\hat{\mathrm{d}}u$ and $\mathrm{d}u$. We start by noticing that the definition of $|Du|$ trivially gives $|Du| \leq \mathrm{Lip}(u)$ in the $\mathfrak{m}_\mathrm{X}$-a.e. sense, so we have

$$\mu = u_*\big(|Du|^2 \mathfrak{m}_\mathrm{X}\big) \leq \mathrm{Lip}(u)^2\, u_*\mathfrak{m}_\mathrm{X} \leq C\,\mathrm{Lip}(u)^2\,\mathfrak{m}_\mathrm{Y}. \tag{8.22}$$

Furthermore, let us prove the following general statement:

Lemma 8.7 *Let $\mu_1, \mu_2 \geq 0$ be two non-zero Radon measures on the complete space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ with $\mu_1 \leq \mu_2$. Then $\mathrm{S}^2(\mathrm{Y},\mu_2) \subseteq \mathrm{S}^2(\mathrm{Y},\mu_1)$ and there exists a unique $L^0(\mu_2)$-linear continuous operator $P : L^0_{\mu_2}(T^*\mathrm{Y}) \to \mathrm{Ext}\big(L^0_{\mu_1}(T^*\mathrm{Y})\big)$ such that*

$$P(\mathrm{d}_{\mu_2} f) = \mathrm{ext}(\mathrm{d}_{\mu_1} f) \qquad \textit{for every } f \in \mathrm{S}^2(\mathrm{Y},\mu_2). \tag{8.23}$$

Moreover, it holds that $\big|P(\omega)\big| \leq |\omega|$ in the μ_2-a.e. sense for every $\omega \in L^0_{\mu_2}(T^*\mathrm{Y})$.

Proof. The assumption $\mu_1 \leq \mu_2$ ensures that the topologies of $L^2(\mu_2)$ and $L^0(\mu_2)$ are stronger than those of $L^2(\mu_1)$ and $L^0(\mu_1)$, respectively. Thus both the inclusion $\mathrm{S}^2(\mathrm{Y},\mu_2) \subseteq \mathrm{S}^2(\mathrm{Y},\mu_1)$ and the μ_2-a.e. bound $\mathrm{ext}\big(|\mathrm{d}_{\mu_1} f|\big) \leq |\mathrm{d}_{\mu_2} f|$ for $f \in \mathrm{S}^2(\mathrm{Y},\mu_2)$ follow from the definition of Sobolev class. In order to conclude, just apply Proposition 3.31 with φ the identity operator and $T(\mathrm{d}_{\mu_2} f) \doteq \mathrm{ext}(\mathrm{d}_{\mu_1} f) \in \mathrm{Ext}\big(L^0_{\mu_1}(T^*\mathrm{Y})\big)$. □

By applying this lemma to the case under consideration, we get the next result:

Proposition 8.8 *There exists a unique $L^0(\mathfrak{m}_\mathrm{Y})$-linear and continuous operator*

$$\pi : L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y}) \longrightarrow \mathrm{Ext}\big(L^0_{\mu}(T^*\mathrm{Y})\big) \tag{8.24}$$

such that $\pi(\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f) = \mathrm{ext}(\mathrm{d}_\mu f)$ for every $f \in \mathrm{S}^2(\mathrm{Y},\mathfrak{m}_\mathrm{Y})$, which also satisfies $\big|\pi(\omega)\big| \leq |\omega|$ in the $\mathfrak{m}_\mathrm{Y}$-a.e. sense for all $\omega \in L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y})$. Moreover, for any $f \in \mathrm{S}^2(\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ and $g \in \mathrm{S}^2(\mathrm{X})$ as in Proposition 8.3 we have that

$$\mathrm{d}g = \mathrm{d}(f \circ u). \tag{8.25}$$

Proof. The first part of the statement follows from Lemma 8.7 and (8.22). To prove (8.25), notice that thanks to the locality of the differential we know that (8.25) holds $\mathfrak{m}_\mathrm{X}$-a.e. on the set $\big\{|Du| > 0\big\}$, while (8.9) shows that $\mathrm{d}g = 0$ $\mathfrak{m}_\mathrm{X}$-a.e. on $\big\{|Du| = 0\big\}$. Hence in order to conclude it is sufficient to prove that $\big|\mathrm{d}(f \circ u)\big| = 0$ $\mathfrak{m}_\mathrm{X}$-a.e. on $\big\{|Du| = 0\big\}$. To see this, pick a sequence $(f_n)_n \subseteq \mathrm{LIP}_{\mathrm{bd}}(\mathrm{Y})$ such that $f_n \to f$ in the $\mathfrak{m}_\mathrm{Y}$-a.e. sense and $\mathrm{lip}_a(f_n) \to |\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f|$ in $L^2(\mathfrak{m}_\mathrm{Y})$. Then the assumption $u_*\mathfrak{m}_\mathrm{X} \leq C\mathfrak{m}_\mathrm{Y}$ grants that $f_n \circ u \to f \circ u$ in the $\mathfrak{m}_\mathrm{X}$-a.e. sense and $\mathrm{lip}_a(f_n) \circ u \to |\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f| \circ u$ in $L^2(\mathfrak{m}_\mathrm{X})$. Therefore by passing to the limit in (8.4), we conclude that $\big|\mathrm{d}(f \circ u)\big| = 0$ is satisfied $\mathfrak{m}_\mathrm{X}$-a.e. on $\big\{|Du| = 0\big\}$, as desired. □

It can be readily verified that the map sending $u^*\mathrm{ext}(\omega)$ to $\mathrm{ext}(u^*\omega)$ is an isomorphism between $u^*\mathrm{Ext}\big(L^0_\mu(T^*\mathrm{Y})\big)$ and $\mathrm{Ext}\big(u^*L^0_\mu(T^*\mathrm{Y})\big)$, hence from Proposition 8.8 above and the universal property of the pullback we see that there is a unique $L^0(\mathfrak{m}_\mathrm{X})$-linear and continuous map $u^*\pi : u^*L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y}) \to \mathrm{Ext}\big(u^*L^0_\mu(T^*\mathrm{Y})\big)$ such that

$$(u^*\pi)(u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}}f) = \mathrm{ext}(u^*\mathrm{d}_\mu f) \qquad \text{for every } f \in \mathrm{S}^2(\mathrm{Y},\mathfrak{m}_\mathrm{Y}) \tag{8.26}$$

and such map satisfies the inequality

$$\big|(u^*\pi)(\omega)\big| \le |\omega| \quad \mathfrak{m}_\mathrm{X}\text{-a.e.} \qquad \text{for every } \omega \in u^*L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y}). \tag{8.27}$$

Denoting by $(u^*\pi)^* : \big(\mathrm{Ext}\big(u^*L^0_\mu(T^*\mathrm{Y})\big)\big)^* \to \big(u^*L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y})\big)^*$ the adjoint of $u^*\pi$, we have:

Theorem 8.9 *With the above notation and assumptions, it holds that*

$$\hat{\mathrm{d}}u(v) = (u^*\pi)^*\big(\mathrm{d}u(v)\big) \qquad \textit{for every } v \in L^0(T\mathrm{X}). \tag{8.28}$$

Moreover, it holds that

$$\big|\hat{\mathrm{d}}u(v)\big| \le \big|\mathrm{d}u(v)\big| \quad \mathfrak{m}_\mathrm{X}\textit{-a.e.} \qquad \textit{for every } v \in L^0(T\mathrm{X}). \tag{8.29}$$

Proof. Let $f \in \mathrm{S}^2(\mathrm{Y},\mathfrak{m}_\mathrm{Y})$ and notice that

$$\begin{aligned}(u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}}f)\big(\hat{\mathrm{d}}u(v)\big) &\overset{(4.47)}{=} \mathrm{d}(f\circ u)(v) \overset{(8.25)}{=} \mathrm{d}g(v) \overset{(8.13)}{=} \mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big)\\ &\overset{(8.26)}{=} (u^*\pi)(u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}}f)\big(\mathrm{d}u(v)\big).\end{aligned}$$

Since elements of the form $u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}}f$ generate $u^*L^0_{\mathfrak{m}_\mathrm{Y}}(T^*\mathrm{Y})$, this is sufficient to prove (8.28). Now observe that (8.27) yields $\big|(u^*\pi)^*(V)\big| \le |V|$ $\mathfrak{m}_\mathrm{X}$-a.e. for every $V \in \big(\mathrm{Ext}\big(u^*L^2_\mu(T^*\mathrm{Y})\big)\big)^*$ by duality, hence (8.29) follows from (8.28). □

Equality in (8.29) can be obtained under appropriate assumptions on either X or Y:

Proposition 8.10 *Suppose that either $W^{1,2}(\mathrm{X},\mathfrak{m}_\mathrm{X})$ or $W^{1,2}(\mathrm{Y},\mu)$ is reflexive. Then*

$$\big|\hat{\mathrm{d}}u(v)\big| = \big|\mathrm{d}u(v)\big| \quad \mathfrak{m}_\mathrm{X}\textit{-a.e.} \qquad \textit{for every } v \in L^0(T\mathrm{X}). \tag{8.30}$$

Proof. We separately consider the two cases:
$W^{1,2}(\mathrm{X},\mathfrak{m}_\mathrm{X})$ IS REFLEXIVE. By inequality (8.29) and a density argument, to conclude is sufficient to show that for any $f \in L^\infty(\mu)\cap \mathrm{S}^2(\mathrm{Y},\mu)$, $g \in \mathrm{S}^2(\mathrm{X},\mathfrak{m}_\mathrm{X})$ as in Proposition 8.3 and for any $v \in L^\infty(T\mathrm{X})$ with bounded support it holds

$$\mathrm{d}g(v) \le |\mathrm{d}_\mu f|\circ u\,\big|\hat{\mathrm{d}}u(v)\big| \qquad \text{in the } \mathfrak{m}_\mathrm{X}\text{-a.e. sense.} \tag{8.31}$$

Let us observe that (8.29) and the very definition of $|\mathrm{d}u|$ give $\big|\hat{\mathrm{d}}u(v)\big| \le \big|\mathrm{d}u(v)\big| \le |\mathrm{d}u|\,|v|$ in the $\mathfrak{m}$-a.e. sense, hence the $\mathfrak{m}_\mathrm{X}$-a.e. value of $G\circ u\,\big|\hat{\mathrm{d}}u(v)\big|$ is independent of the μ-a.e. representative of $G \in L^2(\mu)$, thus the right hand side of (8.31) is well-defined $\mathfrak{m}_\mathrm{X}$-a.e. and vanishes $\mathfrak{m}_\mathrm{X}$-a.e. on the set $\big\{|\mathrm{d}u| = 0\big\}$. Also, the trivial bound

$$\int |G|^2\circ u\,\big|\hat{\mathrm{d}}u(v)\big|^2\,\mathrm{d}\mathfrak{m}_\mathrm{X} \le \int |G|^2\circ u\,|\mathrm{d}u|^2\,|v|^2\,\mathrm{d}\mathfrak{m}_\mathrm{X} \le \big\||v|\big\|^2_{L^\infty(\mathfrak{m}_\mathrm{X})}\int |G|^2\,\mathrm{d}u_*\big(|\mathrm{d}u|^2\mathfrak{m}\big)$$

shows that

$$L^2(\mu) \ni G \longmapsto G\circ u\,\big|\hat{\mathrm{d}}u(v)\big| \in L^2(\mathfrak{m}_\mathrm{X}) \qquad \text{is linear and continuous.} \tag{8.32}$$

Now fix f, v as in (8.31). Let $\eta \in \mathrm{LIP_{bs}}(\mathrm{X})$ be identically 1 on the support of v and take a sequence $(f_n)_n \subseteq \mathrm{LIP_{bd}}(\mathrm{Y})$ as in (8.8) for the space $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mu)$. Since we assumed f to be bounded, up to a truncation argument we can assume the f_n's to be equibounded. Thus the functions $f_n \circ u$ are equibounded as well and (by taking into account the Leibniz rule) we see that $\eta\, f_n \circ u \in W^{1,2}(\mathrm{X}, \mathfrak{m}_\mathrm{X})$ with equibounded norm. Since we assumed such space to be reflexive, up to passing to a non-relabeled subsequence we can assume that $(\eta\, f_n \circ u)_n$ has a $W^{1,2}(\mathrm{X}, \mathfrak{m}_\mathrm{X})$-weak limit and it is then clear that such limit is $\eta\, g$. Thus we have that the sequence $\big(\mathrm{d}(\eta f_n \circ u)\big)_n$ converges to $\mathrm{d}(\eta\, g)$ weakly in $L^2(T^*\mathrm{X})$ and – by the choices of v, η – this implies that $\big(\mathrm{d}(f_n \circ u)(v)\big)_n$ weakly converges to $\mathrm{d}g(v)$ in $L^2(\mathfrak{m}_\mathrm{X})$. Now notice that

$$\mathrm{d}(f_n \circ u)(v) = (u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f_n)\big(\hat{\mathrm{d}}u(v)\big) \le |\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f_n| \circ u\, \big|\hat{\mathrm{d}}u(v)\big| \le \mathrm{lip}_a(f_n) \circ u\, \big|\hat{\mathrm{d}}u(v)\big|.$$

Property (8.32) and the choice of $(f_n)_n$ give that the rightmost side of the above converges to the right hand side of (8.31) in $L^2(\mathfrak{m}_\mathrm{X})$, whence we conclude.

$W^{1,2}(\mathrm{Y}, \mu)$ IS REFLEXIVE. As already observed, given any $f \in W^{1,2}(\mathrm{Y}, \mu)$ we can find a sequence $(f_n)_n \subseteq \mathrm{LIP_{bs}}(\mathrm{Y}) \subseteq W^{1,2}(\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ converging to f in $W^{1,2}(\mathrm{Y}, \mu)$ and such that we have $\mathrm{lip}_a(f_n) \to |\mathrm{d}_\mu f|$ in $L^2(\mu)$. Notice that the definitions of $\mathrm{d}u$ and $\hat{\mathrm{d}}u$ give

$$\begin{aligned}\mathrm{ext}(u^*\mathrm{d}_\mu f_n)\big(\mathrm{d}u(v)\big) &\overset{(8.11)}{=} \mathrm{d}(f_n \circ u)(v) = (u^*\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f_n)\big(\hat{\mathrm{d}}u(v)\big) \le \big|\hat{\mathrm{d}}u(v)\big|\, |\mathrm{d}_{\mathfrak{m}_\mathrm{Y}} f_n| \circ u \\ &\le \big|\hat{\mathrm{d}}u(v)\big|\, \mathrm{lip}_a(f_n) \circ u.\end{aligned}$$

Since the construction also ensures that $u^*\mathrm{d}_\mu f_n \to u^*\mathrm{d}_\mu f$ as $n \to \infty$, by passing to the limit in the above we get that

$$\mathrm{ext}(u^*\mathrm{d}_\mu f)\big(\mathrm{d}u(v)\big) \le \big|\hat{\mathrm{d}}u(v)\big|\, |\mathrm{d}_\mu f| \circ u = \big|\hat{\mathrm{d}}u(v)\big|\, \big|\mathrm{ext}(u^*\mathrm{d}_\mu f)\big| \qquad \text{in the } \mathfrak{m}_\mathrm{X}\text{-a.e. sense.}$$

By arbitrariness of $f \in W^{1,2}(\mathrm{Y}, \mu)$, this is sufficient to conclude. □

8.2.3 The case $\mathrm{X} = \mathbb{R}^d$ and u Lipschitz

In this subsection we assume that the source space X is the Euclidean space $(\mathbb{R}^d, \mathsf{d}_{\mathrm{Eucl}}, \mathcal{L}^d)$ and that the map $u \in \mathrm{S}^2(\mathbb{R}^d; \mathrm{Y})$ is also Lipschitz. In this case, B. Kirchheim proved in [Kir94] that for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ there is a seminorm $\mathrm{md}(u, x)$ on $\mathbb{R}^d$ – called *metric differential* – such that the following property holds:

$$\text{(8.33)} \qquad \text{For } \mathcal{L}^d\text{-a.e. } x \in \mathbb{R}^d \qquad \mathrm{md}(u,x)(v) = \lim_{t \searrow 0} \frac{\mathsf{d}_\mathrm{Y}\big(u(x+tv), u(x)\big)}{t} \quad \text{for every } v \in \mathbb{R}^d,$$

where it is part of the claim the fact that the limit in the right hand side exists.

We now show that such concept is fully compatible with our notion of differential:

Theorem 8.11 *Let $u : \mathbb{R}^d \to \mathrm{Y}$ be a Lipschitz map in $\mathrm{S}^2(\mathbb{R}^d; \mathrm{Y})$ and let $v \in \mathbb{R}^d \sim T\mathbb{R}^d$. Denote by $\bar{v} \in L^0(T\mathbb{R}^d)$ the vector field constantly equal to v. Then*

$$\text{(8.34)} \qquad \big|\mathrm{d}u(\bar{v})\big|(x) = \mathrm{md}(u,x)(v) \qquad \textit{for } \mathcal{L}^d\textit{-a.e. } x \in \mathbb{R}^d.$$

Proof. We separately prove the two inequalities:

$\boxed{\ge}$ Let $(y_n)_n$ be countable and dense in $u(\mathbb{R}^d) \subseteq \mathrm{Y}$. For any $n \in \mathbb{N}$, put $f_n(\cdot) \doteq \mathsf{d}_\mathrm{Y}(\cdot, y_n)$.

From the compatibility of the abstract differential with the classical distributional notion in the case $X = \mathbb{R}^d$ and Rademacher theorem, we see that

$$\mathrm{d}(f_n \circ u)(\bar v) = \lim_{h\to 0} \frac{(f_n \circ u)(\cdot + hv) - (f_n\circ u)(\cdot)}{h} \qquad \text{holds } \mathcal{L}^d\text{-a.e. in } \mathbb{R}^d. \tag{8.35}$$

For $x \in \mathbb{R}^d$, let $\gamma^x : [0,1] \to \mathrm{Y}$ be the Lipschitz curve defined by $\gamma^x_t \doteq u(x+tv)$ and let us put $g^x_{n,t} \doteq f_n \circ \gamma^x_t$. By [AGS08, Theorem 1.1.2] and its proof we know that for the metric speed $|\dot\gamma^x_t|$ it holds that $|\dot\gamma^x_t| = \sup_n \frac{\mathrm{d}}{\mathrm{d}t}\, g^x_{n,t}$ for every $x \in \mathbb{R}^d$ and a.e. t, so that taking (8.35) into account we obtain

$$\mathrm{md}(u, x+tv)(v) = |\dot\gamma^x_t| = \sup_n \frac{\mathrm{d}}{\mathrm{d}t}\, g^x_{n,t} = \sup_n \mathrm{d}(f_n \circ u)(\bar v)(x+tv) \qquad \text{for a.e. } x \in \mathbb{R}^d \text{ and a.e. } t.$$

Therefore Fubini theorem yields

$$\mathrm{md}(u,\cdot)(v) = \sup_n \mathrm{d}(f_n\circ u)(\bar v) \overset{(8.11)}{=} \sup_n \mathrm{ext}(u^*\mathrm{d}_\mu f_n)\big(\mathrm{d}u(\bar v)\big) \le \big|\mathrm{d}u(\bar v)\big| \qquad \mathcal{L}^d\text{-a.e. in } \mathbb{R}^d,$$

having used the trivial μ-a.e. bound $|\mathrm{d}_\mu f_n| \le 1$ in the last step.
$\boxed{\le}$ Let $f \in \mathrm{S}^2(\mathrm{Y},\mu)$ be arbitrary and let $g \in \mathrm{S}^2(\mathrm{X})$ as in Proposition 8.3. We will show that

$$\mathrm{d}g(v) \le |\mathrm{d}_\mu f| \circ u \,\mathrm{md}(u,\cdot)(v) \qquad \text{holds } \mathcal{L}^d\text{-a.e.,} \tag{8.36}$$

which is sufficient to conclude. The already proven bound $\boxed{\ge}$ in (8.34) and the same arguments used in studying (8.31) show that the right hand side of (8.36) is well-defined $\mathcal{L}^d$-a.e. and that

$$L^2(\mu) \ni G \longmapsto G\circ u\,\mathrm{md}(u,\cdot)(v) \in L^2(\mathbb{R}^d) \qquad \text{is linear and continuous.} \tag{8.37}$$

Now let $(f_n)_n \subseteq \mathrm{LIP}_{\mathrm{bd}}(\mathrm{Y})$ be as in Proposition 8.3. Observe that for every $n \in \mathbb{N}$ the identity (8.35) yields for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ the following inequalities:

$$\big|\mathrm{d}(f_n\circ u)(\bar v)\big|(x) \le \mathrm{lip}_a(f_n)\big(u(x)\big) \lim_{h\to 0} \frac{\mathsf{d}_{\mathrm{Y}}\big(u(x+hv), u(x)\big)}{|h|} = \big(\mathrm{lip}_a(f_n)\circ u\big)(x)\,\mathrm{md}(u,x)(v).$$

By (8.37) and the choice of $(f_n)_n$, we see that the rightmost side of the above converges to the right hand side of (8.36) in $L^2(\mathbb{R}^d)$. By following again the arguments in the first part of the proof of Proposition 8.10 – that are applicable as $W^{1,2}(\mathbb{R}^d)$ is reflexive – we see that the sequence $\big(\mathrm{d}(f_n\circ u)(\bar v)\big)_n$ converges to $\mathrm{d}g(v)$ in the weak topology of $L^2(\mathbb{R}^d)$. Hence the inequality (8.36) is obtained. □

8.3 Differential of locally Sobolev maps between metric spaces

8.3.1 Inverse limits of modules

Here we briefly discuss some properties of inverse limits in the category of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules, where morphisms are $L^0(\mathfrak{m})$-linear *contractions*, i.e. maps $T : \mathscr{M} \to \mathscr{N}$ such that $\big|T(v)\big| \le |v|$ holds $\mathfrak{m}$-a.e. in X. We start with:

Proposition 8.12 *Let* $\big(\{\mathscr{M}_i\}_{i\in I}, \{P^i_j\}_{i\leq j}\big)$ *be an inverse system of* $L^0(\mathfrak{m})$*-normed modules. Then the inverse limit* $\big(\mathscr{M}, \{P^i\}_{i\in I}\big)$ *exists and for every family* $I \ni i \mapsto v^i \in \mathscr{M}_i$ *such that*

$$P^i_j(v^j) = v^i \qquad \text{and} \qquad \operatorname*{ess\,sup}_{i\in I} |v^i| \in L^0(\mathfrak{m}) \tag{8.38}$$

there is a unique $v \in \mathscr{M}$ *such that* $v^i = P^i(v)$ *for all* $i \in I$*. Moreover,* $|v| = \operatorname{ess\,sup}_{i\in I} |v^i|$.

Proof. The system $\big(\{\mathscr{M}_i\}_{i\in I}, \{P^i_j\}_{i\leq j}\big)$ is also an inverse system in the category of algebraic modules over the ring $L^0(\mathfrak{m})$, in the sense of [Lan84, Chapter III.§10]. Hence – according to [Lan84, Chapter III, Theorem 10.2] and its proof – the algebraic inverse limit $(\mathscr{M}_{\rm Alg}, P^i_{\rm Alg})$ exists and for every family $i \mapsto v^i \in \mathscr{M}_i$ there is a unique $v \in \mathscr{M}_{\rm Alg}$ such that $P^i_{\rm Alg}(v) = v^i$ for every $i \in I$. Now define $|v|$ for any $v \in \mathscr{M}_{\rm Alg}$ as

$$|v| \doteq \operatorname*{ess\,sup}_{i\in I} \big|P^i_{\rm Alg}(v)\big|, \tag{8.39}$$

so that $|v| : \mathrm{X} \to [0, +\infty]$ is the equivalence class of a Borel function up to $\mathfrak{m}$-a.e. equality. Furthermore, let us set

$$\mathscr{M} \doteq \big\{v \in \mathscr{M}_{\rm Alg} \;:\; |v| \in L^0(\mathfrak{m})\big\} = \big\{v \in \mathscr{M}_{\rm Alg} \;:\; |v| < +\infty \ \mathfrak{m}\text{-a.e.}\big\}$$

and $P^i \doteq P^i_{\rm Alg}|_{\mathscr{M}}$. We claim that $(\mathscr{M}, P^i)$ is the desired inverse limit and start by noticing that (8.39) ensures that $\big|P^i(v)\big| \leq |v|$ $\mathfrak{m}$-a.e., i.e. the P^i's are contractions, as required. Let us now check that $\mathscr{M}$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module: the only non-trivial thing to verify is that it is complete, i.e. that if $(v^n)_n$ is Cauchy in $\mathscr{M}$ then it has a limit. Since the P^i's are contractions, we see that $n \mapsto P^i(v_n)$ is Cauchy in $\mathscr{M}_i$ and thus has a limit v^i for every $i \in I$. Passing to the limit in the identity $P^i(v^n) = P^i_j\big(P^j(v^n)\big)$ – valid for every $i \leq j$ – and using the continuity of P^i_j, we deduce that $v^i = P^i_j(v^j)$, whence $v \doteq \{v^i\}_{i\in I} \in \mathscr{M}_{\rm Alg}$. Since $(v_n)_n$ is Cauchy and the pointwise norm in $\mathscr{M}$ trivially satisfies the triangle inequality, we see that the sequence $\big(|v_n|\big)_n$ has a limit f in $L^0(\mathfrak{m})$. Then the bound $|v^i| = \lim_n \big|P^i(v_n)\big| \leq \lim_n |v_n| \doteq f$, valid for every $i \in I$, grants that $|v| \leq f$ and thus $v \in \mathscr{M}$. Similarly, from

$$\big|v^i - P^i(v_n)\big| = \lim_{m\to\infty} \big|P^i(v_m) - P^i(v_n)\big| \leq \lim_{m\to\infty} |v_m - v_n|$$

we deduce that $|v - v_n| \leq \lim_m |v_m - v_n|$, whence by passing to the $L^0(\mathfrak{m})$-limit in n and by using that $(v_n)_n$ is $\mathscr{M}$-Cauchy we conclude that $v_n \to v$ in $\mathscr{M}$, showing completeness. The fact that for v^i's as in (8.38) there exists a unique $v \in \mathscr{M}$ projecting on them is consequence of the construction and from this the universality property of $(\mathscr{M}, P^i)$ follows. □

It is now easy to check that there exists the inverse limit of a compatible family of maps:

Proposition 8.13 *Let* $\big(\{\mathscr{M}^i\}_{i\in I}, \{P^i_j\}_{i\leq j}\big)$, $\big(\{\mathscr{N}^i\}_{i\in I}, \{Q^i_j\}_{i\leq j}\big)$ *be two inverse systems of* $L^0(\mathfrak{m})$*-normed* $L^0(\mathfrak{m})$*-modules and call* $(\mathscr{M}, P^i)$, $(\mathscr{N}, Q^i)$ *their inverse limits. For any* $i \in I$*, let* $T^i : \mathscr{M}^i \to \mathscr{N}^i$ *be an* $L^0(\mathfrak{m})$*-linear continuous operator such that the diagram*

$$\begin{array}{ccc} \mathscr{M}^j & \xrightarrow{\;T^j\;} & \mathscr{N}^j \\ {\scriptstyle P^i_j}\big\downarrow & & \big\downarrow{\scriptstyle Q^i_j} \\ \mathscr{M}^i & \xrightarrow[\;T^i\;]{} & \mathscr{N}^i \end{array} \tag{8.40}$$

commutes for all $i, j \in I$ with $i \leq j$ and so that for some $\ell \in L^0(\mathfrak{m})$ we have

$$|T^i(v^i)| \leq \ell\,|v^i| \quad \mathfrak{m}\text{-a.e.} \qquad \text{for every } i \in I \text{ and } v^i \in \mathscr{M}^i. \tag{8.41}$$

Then there exists a unique $L^0(\mathfrak{m})$-linear continuous operator $T : \mathscr{M} \to \mathscr{N}$ such that

$$\begin{array}{ccc} \mathscr{M}^i & \xrightarrow{T^i} & \mathscr{N}^i \\ {\scriptstyle P^i}\uparrow & & \uparrow{\scriptstyle Q^i} \\ \mathscr{M} & \xrightarrow[T]{} & \mathscr{N} \end{array} \tag{8.42}$$

is a commutative diagram for every $i \in I$. Moreover, the inequality $|T(v)| \leq \ell\,|v|$ is satisfied $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$.

Proof. Let $v \in \mathscr{M}$, put $w^i \doteq T^i\big(P^i(v)\big) \in \mathscr{N}^i$ and notice that (8.40) yields $Q^i_j(w^j) = w^i$ and (8.41) that $|w^i| \leq \ell\,|v|$ $\mathfrak{m}$-a.e. for every $i \leq j$. Thus Proposition 8.12 above ensures that there is a unique $T(v) \in \mathscr{N}$ such that $Q^i(T(v)) = w^i$ for every $i \in I$ and it satisfies $|T(v)| \leq \ell\,|v|$ in the $\mathfrak{m}$-a.e. sense. Since the assignment $v \mapsto T(v)$ is $L^0(\mathfrak{m})$-linear, the proof is done. □

8.3.2 Locally Sobolev maps and their differential

In this subsection we come back to the case of general $(\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ as in Section 8.1 and study the case of $u \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}; \mathrm{Y})$, which is the collection of all maps u such that every point $x \in \mathrm{X}$ has a neighbourhood U_x where u coincides $\mathfrak{m}$-a.e. with some $u_x \in \mathrm{S}^2(\mathrm{X}; \mathrm{Y})$. Then for $u \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}; \mathrm{Y})$ the locality of the differential ensures that the formula

$$|Du| \doteq |Du_x| \quad \mathfrak{m}\text{-a.e. on } U_x \qquad \text{for every } x \in \mathrm{X} \tag{8.43}$$

well-defines a function $|Du| \in L^2_{\mathrm{loc}}(\mathrm{X})$. For this kind of map u, the measure $u_*\big(|Du|^2\mathfrak{m}\big)$ is – in general – not anymore σ-finite, hence to define the differential $\mathrm{d}u$ we need to suitably adapt the definition previously given. This is the scope of the current subsection.

Let $u \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}; \mathrm{Y})$ be fixed. By $\mathcal{F}(u)$ we denote the collection of all open sets $\Omega \subseteq \mathrm{X}$ such that $\int_\Omega |Du|^2 \,\mathrm{d}\mathfrak{m} < \infty$. Since $u \in \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X}; \mathrm{Y})$, we see that the family $\mathcal{F}(u)$ is a cover of X. We shall now build two inverse limits of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules indexed over $\mathcal{F}(u)$, directed by inclusion. For the first one, define for $\Omega \in \mathcal{F}(u)$ the measure μ_Ω on Y as

$$\mu_\Omega \doteq u_*\big(|Du|^2 \mathfrak{m}_{|\Omega}\big). \tag{8.44}$$

Thus μ_Ω is a Radon measure. We can consider the cotangent module $L^0_{\mu_\Omega}(T^*\mathrm{Y})$ of $(\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mu_\Omega)$ and its pullback $u^*L^0_{\mu_\Omega}(T^*\mathrm{Y})$, which is an $L^0\big(\mathfrak{m}_{|\Omega\cap\{|Du|>0\}}\big)$-normed module. Then we put

$$u^*L^0_\Omega(T^*\mathrm{Y}) \doteq \mathrm{Ext}\big(u^*L^0_{\mu_\Omega}(T^*\mathrm{Y})\big). \tag{8.45}$$

Observe that for $\Omega, \Omega' \in \mathcal{F}(u)$, $\Omega' \subseteq \Omega$ we have $\mu_{\Omega'} \leq \mu_\Omega$ and thus Lemma 8.7 provides a canonical 'projection' map $P^{\Omega'}_\Omega : L^0_{\mu_\Omega}(T^*\mathrm{Y}) \to \mathrm{Ext}\big(L^0_{\mu_{\Omega'}}(T^*\mathrm{Y})\big)$. Then we can consider the (extended) pullback map $u^*P^{\Omega'}_\Omega : u^*L^0_\Omega(T^*\mathrm{Y}) \to u^*L^0_{\Omega'}(T^*\mathrm{Y})$ and notice that – since one has that $P^{\Omega_1}_{\Omega_2} \circ P^{\Omega_2}_{\Omega_3} = P^{\Omega_1}_{\Omega_3}$ for every $\Omega_1, \Omega_2, \Omega_3 \in \mathcal{F}(u)$, $\Omega_3 \subseteq \Omega_2 \subseteq \Omega_1$ – the functoriality of the pullback grants that

$$\Big(\big\{u^*L^0_\Omega(T^*\mathrm{Y})\big\}_{\Omega\in\mathcal{F}(u)}, \big\{u^*P^{\Omega'}_\Omega\big\}_{\Omega\subseteq\Omega'}\Big)$$

is an inverse system of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules. We call $\big(u^*L^0_u(T^*\mathrm{Y}), \{P^\Omega\}_{\Omega\in\mathcal{F}(u)}\big)$ its inverse limit (recall Proposition 8.12).

Remark 8.14 For every $f:\mathrm{Y}\to\mathbb{R}$ Lipschitz with bounded support we have $f\in\mathrm{S}^2(\mathrm{Y},\mu)$ for any finite Radon measure μ and obviously $|\mathrm{d}_\mu f|\leq \mathrm{Lip}(f)$ μ-a.e.. Hence there is an element $\omega\in u^*L^0_u(T^*\mathrm{Y})$ such that $P^\Omega(\omega)=\mathrm{ext}(u^*\mathrm{d}_{\mu_\Omega}f)$ for every $\Omega\in\mathcal{F}(u)$. ■

For the second, consider for any open set $\Omega\subseteq\mathrm{X}$ the module

$$L^0_\Omega(T^*\mathrm{X})\doteq\mathrm{Ext}\big(L^0_{\mathfrak{m}|_\Omega}(T^*\mathrm{X})\big),$$

which is $L^0(\mathfrak{m})$-normed. Since trivially for $\Omega'\subseteq\Omega$ we have that $\mathfrak{m}|_{\Omega'}\leq\mathfrak{m}|_\Omega$, Lemma 8.7 grants the existence of canonical (extended) 'projection' maps $Q^{\Omega'}_\Omega: L^0_\Omega(T^*\mathrm{X})\to L^0_{\Omega'}(T^*\mathrm{X})$. By construction, it is also clear that $\big(\{L^0_\Omega(T^*\mathrm{X})\}_{\Omega\in\mathcal{F}(u)},\{Q^{\Omega'}_\Omega\}_{\Omega'\subseteq\Omega}\big)$ is an inverse system of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules. We then have the following non-obvious result:

Lemma 8.15 *The inverse limit of the inverse system* $\big(\{L^0_\Omega(T^*\mathrm{X})\}_{\Omega\in\mathcal{F}(u)},\{Q^{\Omega'}_\Omega\}_{\Omega'\subseteq\Omega}\big)$ *is given by* $\big(L^0(T^*\mathrm{X}),\{Q^\Omega_\mathrm{X}\}_{\Omega\in\mathcal{F}(u)}\big)$.

Proof. The fact that $Q^{\Omega'}_\Omega\circ Q^\Omega_\mathrm{X}=Q^{\Omega'}_\mathrm{X}$ for $\Omega'\subseteq\Omega\subseteq\mathrm{X}$ open is a direct consequence of the definition of the Q's. For the universality, we recall [AGS14b, Theorem 4.19] and its proof (the assumption $\mathfrak{m}(\mathrm{X})=1$ plays no role) to get that $\big|Q^{\Omega'}_\Omega(\mathrm{d}_{\mathfrak{m}|_\Omega}f)\big|=\big|\mathrm{d}_{\mathfrak{m}|_{\Omega'}}f\big|$ $\mathfrak{m}$-a.e. on Ω'. Moreover, if $f\in\mathrm{S}^2(\mathrm{X},\mathfrak{m}|_{\Omega'})$ has support at positive distance from $\mathrm{X}\setminus\Omega'$, then $f\in\mathrm{S}^2(\mathrm{X},\mathfrak{m}|_\Omega)$ as well. It easily follows that $Q^{\Omega'}_\Omega: L^0_\Omega(T^*\mathrm{X})\to L^0_{\Omega'}(T^*\mathrm{X})$ has a unique norm-preserving right inverse; call it $P^{\Omega'}_\Omega$. Hence if $\mathcal{F}(u)\ni\Omega\mapsto\omega^\Omega\in L^0_\Omega(T^*\mathrm{X})$ satisfies $Q^{\Omega'}_\Omega(\omega^\Omega)=\omega^{\Omega'}$ for every choice of $\Omega,\Omega'\in\mathcal{F}(u)$ with $\Omega'\subseteq\Omega$, then it is clear that there exists a unique $\omega\in L^0(T^*\mathrm{X})$ such that $\chi_\Omega\,\omega=P^\Omega_\mathrm{X}(\omega^\Omega)$ for every $\Omega\in\mathcal{F}(u)$. This is sufficient to conclude the proof of the statement. □

Let $\Omega\in\mathcal{F}(u)$ and define $S_\Omega:\big\{\mathrm{d}_{\mu_\Omega}f\ :\ f\in\mathrm{S}^2(\mathrm{Y},\mu_\Omega)\big\}\to L^0_\Omega(T^*\mathrm{X})$ by putting

$$S_\Omega(\mathrm{d}_{\mu_\Omega}f)\doteq\mathrm{ext}\big(\mathrm{d}_{\mathfrak{m}|_\Omega}g\big),\tag{8.46}$$

where g is related to f as in Proposition 8.3, in this case applied to the space $\big(\mathrm{X},\mathsf{d}_\mathrm{X},\mathfrak{m}|_\Omega\big)$. In particular, the bound (8.9) yields

$$\big|S_\Omega(\mathrm{d}_{\mu_\Omega}f)\big|\leq\chi_\Omega\,|\mathrm{d}_{\mu_\Omega}f|\circ u\,|Du|,\tag{8.47}$$

which is easily seen to ensure that the map S_Ω is well-posed (i.e. the value of S_Ω depends only on $\mathrm{d}_{\mu_\Omega}f$ and not on f). Thus by the universal property of the pullback we see that there exists a unique $L^0(\mathfrak{m})$-linear continuous map $T_\Omega: u^*L^0_\Omega(T^*\mathrm{Y})\to L^0_\Omega(T^*\mathrm{X})$ such that

$$T_\Omega\big(\mathrm{ext}(u^*\mathrm{d}_{\mu_\Omega}f)\big)=S_\Omega(\mathrm{d}_{\mu_\Omega}f)\qquad\text{for every } f\in\mathrm{S}^2(\mathrm{Y},\mu_\Omega)$$

and from (8.47) we deduce that such T_Ω satisfies

$$\big|T_\Omega(\omega)\big|\leq|Du|\,|\omega|\quad\mathfrak{m}\text{-a.e.}\qquad\text{for every }\omega\in u^*L^0_\Omega(T^*\mathrm{Y}).\tag{8.48}$$

It is now only a matter of keeping track of the various definitions introduced so far, to check that for every $\Omega,\Omega'\in\mathcal{F}(u)$ with $\Omega'\subseteq\Omega$ it holds that

$$T_{\Omega'}\big(u^*P^{\Omega'}_\Omega(\omega)\big)=Q^{\Omega'}_\Omega\big(T_\Omega(\omega)\big)\tag{8.49}$$

for every $\omega \in u^* L^0_\Omega(T^*\mathrm{Y})$ of the form $\omega = \mathrm{ext}\big(u^*\mathrm{d}_{\mathfrak{m}|_\Omega} f\big)$, for some $f \in \mathrm{S}^2(\mathrm{Y}, \mu_\Omega)$. Then by $L^0(\mathfrak{m})$-linearity and continuity we see that (8.49) holds for every $\omega \in u^* L^0_\Omega(T^*\mathrm{Y})$, so that (keeping (8.48) into account) we infer from Proposition 8.13 and Lemma 8.15 that there is a unique $L^0(\mathfrak{m})$-linear and continuous map $T : u^* L^0_u(T^*\mathrm{Y}) \to L^0(T^*\mathrm{X})$ such that

$$\begin{array}{ccc} L^0(T^*\mathrm{X}) & \xrightarrow{Q^\Omega_\mathrm{X}} & L^0_\Omega(T^*\mathrm{X}) \\ {\scriptstyle T}\uparrow & & \uparrow{\scriptstyle T_\Omega} \\ u^* L^0_u(T^*\mathrm{Y}) & \xrightarrow[P^\Omega]{} & u^* L^0_\Omega(T^*\mathrm{Y}) \end{array} \tag{8.50}$$

is a commutative diagram for every $\Omega \in \mathcal{F}(u)$.

We can now give the main definition of this section:

Definition 8.16 *The differential* $\mathrm{d}u : L^0(T\mathrm{X}) \to \big(u^* L^0_u(T^*\mathrm{Y})\big)^*$ *is the adjoint of* T*.*

Observe that from (8.48) it follows that $\big|T(\omega)\big| \leq |Du|\,|\omega|$ for every $\omega \in u^* L^0_u(T^*\mathrm{Y})$. Hence by duality we also get that $\big|\mathrm{d}u(v)\big| \leq |Du|\,|v|$ $\mathfrak{m}$-a.e. for every $v \in L^0(T\mathrm{X})$, i.e. that the inequality $|\mathrm{d}u| \leq |Du|$ holds $\mathfrak{m}$-a.e. in X. Then – arguing as in Proposition 8.5 – we can prove that actually $|\mathrm{d}u| = |Du|$ in the $\mathfrak{m}$-a.e. sense. Analogously, natural variants of the properties stated in Subsections 8.2.1, 8.2.2 and 8.2.3 hold for this more general notion of differential. We omit the details.

We conclude by noticing that if $u \in \mathrm{S}^2(\mathrm{X};\mathrm{Y}) \subseteq \mathrm{S}^2_{\mathrm{loc}}(\mathrm{X};\mathrm{Y})$ then $\mathrm{X} \in \mathcal{F}(u)$, i.e. the directed family $\mathcal{F}(u)$ has a maximum. It is therefore clear that the differential $\mathrm{d}u$ in the sense of Definition 8.16 canonically coincides with the one given by Definition 8.4.

A
Integration of outer measures

Given any set X, let us denote by 2^{X} its *power set*, i.e. the collection of all the subsets of X. By *outer measure* on X we intend any set-function $\mu : 2^{\mathrm{X}} \to [0,+\infty]$ with $\mu(\emptyset) = 0$ that is monotone and σ-subadditive, i.e. such that

$$\mu(E_\infty) \leq \sum_{n\in\mathbb{N}} \mu(E_n) \qquad \text{for every } (E_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq 2^{\mathrm{X}} \text{ with } E_\infty \subseteq \bigcup_{n\in\mathbb{N}} E_n. \tag{A.1}$$

We say that an outer measure μ on X is *continuous from below* provided

$$\mu\bigg(\bigcup_{n\in\mathbb{N}} E_n\bigg) = \lim_{n\to\infty} \mu\bigg(\bigcup_{i=0}^{n} E_i\bigg) \qquad \text{for every sequence } (E_n)_{n\in\mathbb{N}} \subseteq 2^{\mathrm{X}}. \tag{A.2}$$

Observe that the limit in the right hand side of formula (A.2) always exists, because the sequence $n \mapsto \mu\big(\bigcup_{i\leq n} E_i\big)$ is non-decreasing by monotonicity of μ.

Let us fix a set X and an outer measure μ on X. Then the integral $\int f \,\mathrm{d}\mu$ of any given function $f : \mathrm{X} \to [0,+\infty]$ can be defined as

$$\int f \,\mathrm{d}\mu \doteq \int_0^{+\infty} \mu\big(\{f > t\}\big)\,\mathrm{d}t. \tag{A.3}$$

The definition (A.3) of $\int f \,\mathrm{d}\mu$ is well-posed, since the function $t \mapsto \mu\big(\{f > t\}\big) \in [0,+\infty]$ is non-increasing, thus in particular it is Lebesgue measurable. Given any set $E \subseteq \mathrm{X}$, we define

$$\int_E f \,\mathrm{d}\mu \doteq \int \chi_E \, f \,\mathrm{d}\mu. \tag{A.4}$$

In the next result we collect the basic properties of the above-defined integral:

Proposition A.1 *Let $f, g : \mathrm{X} \to [0,+\infty]$ be fixed. Then the following holds:*

i) $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$ *provided* $f \leq g$.

ii) $\int \lambda f \,\mathrm{d}\mu = \lambda \int f \,\mathrm{d}\mu$ *for every* $\lambda > 0$.

iii) *If* $\mu(\{f \neq 0\}) = 0$ *then* $\int f \,\mathrm{d}\mu = 0$, *while the converse implication holds provided* μ *is continuous from below.*

iv) $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$ *provided* $\mu(\{f \neq g\}) = 0$.

Proof. We divide the proof into several steps:
i) Given any $t > 0$ it clearly holds that $\{f > t\} \subseteq \{g > t\}$, whence $\mu(\{f > t\}) \leq \mu(\{g > t\})$ for all $t > 0$ and accordingly i) follows.
ii) Observe that

$$\int \lambda f \, d\mu = \int_0^{+\infty} \mu(\{\lambda f > t\}) \,\mathrm{d}t = \int_0^{+\infty} \mu(\{f > t/\lambda\}) \,\mathrm{d}t = \lambda \int_0^{+\infty} \mu(\{f > s\}) \,\mathrm{d}s$$
$$= \lambda \int f \,\mathrm{d}\mu,$$

thus proving the validity of ii).
iii) Suppose that $\mu(\{f \neq 0\}) = 0$. Then $\mu(\{f > t\}) = 0$ as well for all $t > 0$ by monotonicity of μ, whence accordingly $\int f \,\mathrm{d}\mu = \int_0^{+\infty} \mu(\{f > t\}) \,\mathrm{d}t = 0$. Conversely, suppose that μ is continuous from below and $\int f \,\mathrm{d}\mu = \int_0^{+\infty} \mu(\{f > t\}) \,\mathrm{d}t = 0$. This grants that $\mu(\{f > t\}) = 0$ for a.e. $t > 0$, thus also for every $t > 0$ by monotonicity of μ. Given that $\{f > 1/n\} \nearrow \{f \neq 0\}$ as $n \to \infty$, we conclude that $\mu(\{f \neq 0\}) = \lim_n \mu(\{f > 1/n\}) = 0$ by continuity from below of μ. Therefore property iii) is proved.
iv) For any $t > 0$ it holds that $\{f > t\} \subseteq \{g > t\} \cup \{f \neq g\}$, whence

$$\mu(\{f > t\}) \leq \mu(\{g > t\}) + \mu(\{f \neq g\}) = \mu(\{g > t\}) \qquad \text{for all } t > 0,$$

so that $\int f \,\mathrm{d}\mu \leq \int g \,\mathrm{d}\mu$. By interchanging the roles of f and g, we obtain that $\int g \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu$, thus $\int f \,\mathrm{d}\mu = \int g \,\mathrm{d}\mu$. This shows the validity of iv). □

Nevertheless, this notion of integral for outer measures is – in general – not additive. Even to get just its subadditivity, we need to require the following strong assumption on μ:

Definition A.2 (Submodularity) *We say that an outer measure* μ *is* submodular *provided*

$$\mu(E \cup F) + \mu(E \cap F) \leq \mu(E) + \mu(F) \quad \textit{for every } E, F \subseteq \mathrm{X}. \tag{A.5}$$

Then a key result is the following, for whose proof we refer to [Den10, Theorem 6.3]:

Theorem A.3 (Subadditivity theorem) *Let* μ *be an outer measure over* X. *Then* μ *is submodular if and only if the integral associated to* μ *is subadditive, i.e.*

$$\int (f+g) \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu + \int g \,\mathrm{d}\mu \qquad \textit{for every } f, g : \mathrm{X} \to [0, +\infty). \tag{A.6}$$

We shall also make use of the following simple property of outer measures:

Lemma A.4 (Borel-Cantelli) *Let* $(E_n)_{n\in\mathbb{N}}$ *be subsets of* X *satisfying* $\sum_n \mu(E_n) < +\infty$. *Then* $\mu\big(\bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} E_m\big) = 0$.

Proof. Call $E \doteq \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} E_m$. Then $\mu(E) \leq \mu\big(\bigcup_{m\geq n} E_m\big)$ for all $n \in \mathbb{N}$, so that

$$\mu(E) \leq \varliminf_{n\to\infty} \mu\left(\bigcup_{m\geq n} E_m\right) \leq \lim_{n\to\infty} \sum_{m\geq n} \mu(E_m) = 0,$$

which proves the statement. □

B
Hausdorff measures

Given any real number $k \in [0, +\infty)$, let us define

$$\omega_k \doteq \frac{\pi^{k/2}}{\Gamma(1+k/2)}, \qquad \text{where } \Gamma(t) \doteq \int_0^{+\infty} x^{t-1}\, \mathrm{e}^{-x}\, \mathrm{d}x. \tag{B.1}$$

The function Γ is the so-called *gamma function*. Recall that $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$. In the case $k \in \mathbb{N}$, the quantity ω_k coincides with the $\mathcal{L}^k$-measure of the unit ball in $\mathbb{R}^k$.

Let (X, d) be a metric space. Given any real number $\delta \in (0, +\infty)$ and any subset $E \subseteq \mathrm{X}$, we define the quantity $\mathcal{H}^k_\delta(E) \in [0, +\infty]$ as

$$\mathcal{H}^k_\delta(E) \doteq \inf \left\{ \frac{\omega_k}{2^k} \sum_{n=0}^{\infty} \mathrm{diam}(E_n)^k \;\middle|\; E \subseteq \bigcup_{n=0}^{\infty} E_n,\ \mathrm{diam}(E_n) < \delta \text{ for all } n \in \mathbb{N} \right\}. \tag{B.2}$$

More precisely, in the case $k = 0$ we make use of the following convention: $\mathrm{diam}(\{x\})^0 \doteq 1$ for every $x \in \mathrm{X}$ and $\mathrm{diam}(\emptyset)^0 \doteq 0$. It holds that each function $\mathcal{H}^k_\delta$ is an outer measure on X. Let us now associate to any set $E \subseteq \mathrm{X}$ the quantity $\mathcal{H}^k(E) \in [0, \infty]$, which is given by

$$\mathcal{H}^k(E) \doteq \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(E). \tag{B.3}$$

Given that $\mathcal{H}^k_\delta(E)$ is non-increasing with respect to δ, in the above definition $\lim_{\delta \searrow 0}$ can be replaced by $\sup_{\delta > 0}$. The function $\mathcal{H}^k : 2^{\mathrm{X}} \to [0, +\infty]$, which turns out to be an outer measure on X, is referred to as the *k-dimensional Hausdorff measure* on the space (X, d).

An important property of $\mathcal{H}^k$ is the following: given $k \in (0, +\infty)$ and $E \subseteq \mathrm{X}$, one has

$$\mathcal{H}^k(E) > 0 \quad \Longrightarrow \quad \mathcal{H}^{k'}(E) = +\infty \text{ for every } k' \in [0, k). \tag{B.4}$$

Given any subset E of X, we define the *Hausdorff dimension* of E as

$$\dim_{\mathcal{H}}(E) \doteq \inf \big\{ k \in [0, +\infty) \;\big|\; \mathcal{H}^k(E) = 0 \big\}. \tag{B.5}$$

A consequence of (B.4) is that $\mathcal{H}^k(E) = +\infty$ if $k < \dim_{\mathcal{H}}(E)$ and $\mathcal{H}^k(E) = 0$ if $k > \dim_{\mathcal{H}}(E)$, while in general nothing can be said about the value of $\mathcal{H}^{\dim_{\mathcal{H}}(E)}(E)$.

Given two metric spaces $(\mathrm{X}, \mathsf{d}_\mathrm{X})$, $(\mathrm{Y}, \mathsf{d}_\mathrm{Y})$ and any $k \in (0, +\infty)$, it holds that

$$\mathcal{H}^k\big(f(A)\big) \leq \mathrm{Lip}(f)^k\, \mathcal{H}^k(A) \qquad \text{for every } f \in \mathrm{LIP}(\mathrm{X}, \mathrm{Y}) \text{ and } A \subseteq \mathrm{X}, \tag{B.6}$$

whence in particular $\dim_{\mathcal{H}}\big(f(A)\big) \leq \dim_{\mathcal{H}}(A)$. Another important property of Hausdorff measures is the following, for whose proof we refer to [AT04, Theorem 2.4.3]:

Proposition B.1 *Let $(\mathrm{X}, \mathsf{d}, \mu)$ be a metric measure space and $k \in \mathbb{N}$. Let $A \subseteq \mathrm{X}$ be a Borel set and $\lambda \in (0, +\infty)$. Then*

$$\varlimsup_{r \searrow 0} \frac{\mu\big(B_r(x)\big)}{\omega_k\, r^k} \geq \lambda \quad \textit{for every } x \in A \qquad \Longrightarrow \qquad \lambda\, \mathcal{H}^k(A) \leq \mu(A). \tag{B.7}$$

Given a metric space (X, d), we say that a Borel set $A \subseteq \mathrm{X}$ is *countably $\mathcal{H}^k$-rectifiable* provided there exist a sequence of Borel subsets $(B_n)_n$ of $\mathbb{R}^k$ and Lipschitz maps $f_n : B_n \to \mathrm{X}$ such that $\mathcal{H}^k\big(A \setminus \bigcup_n f_n(B_n)\big) = 0$. A fundamental property of countably $\mathcal{H}^k$-rectifiable sets is:

Theorem B.2 (Spherical density) *Let (X, d) be a metric space and $k \in \mathbb{N}$. Let $A \subseteq \mathrm{X}$ be a countably $\mathcal{H}^k$-rectifiable set and $\theta : A \to (0, +\infty)$ be a Borel map. Define $\mu \doteq \theta\, \mathcal{H}^k|_A$ and suppose that μ is a finite measure. Then*

$$\lim_{r \searrow 0} \frac{\mu\big(B_r(x)\big)}{\omega_k\, r^k} = \theta(x) \qquad \textit{holds for } \mathcal{H}^k\textit{-a.e. } x \in A. \tag{B.8}$$

The previous result is proven, for instance, in [AK00, Theorem 5.4].

C
Bochner integral

We recall some basic results about measurability and integration of Banach-valued maps of a single variable $t \in [0,1]$. A detailed discussion about this topic can be found e.g. in [DU77].

For the sake of brevity, we shall use the notation $\mathcal{L}_1$ to indicate the 1-dimensional Lebesgue measure restricted to $[0,1]$, namely

$$\mathcal{L}_1 \doteq \mathcal{L}^1|_{[0,1]}. \tag{C.1}$$

Let $\mathbb{B}$ be a fixed Banach space. We denote by $\mathbb{B}'$ its dual space. A *simple map* is any $\mathbb{B}$-valued mapping $y : [0,1] \to \mathbb{B}$ that can be written in the form

$$y = \sum_{i=1}^{k} \chi_{E_i} \, v_i, \qquad \text{for some } E_1, \ldots, E_k \subseteq [0,1] \text{ Borel and } v_1, \ldots, v_k \in \mathbb{B}.$$

A map $y : [0,1] \to \mathbb{B}$ is said to be *strongly measurable* provided there exists a sequence $(y_n)_n$ of simple maps $y_n : [0,1] \to \mathbb{B}$ such that $\lim_n \big\| y_n(t) - y(t) \big\|_{\mathbb{B}} = 0$ for $\mathcal{L}_1$-a.e. $t \in [0,1]$, while it is said to be *weakly measurable* provided $[0,1] \ni t \mapsto \omega\big(y(t)\big) \in \mathbb{R}$ is a Borel function for every $\omega \in \mathbb{B}'$. It directly follows from the very definition that linear combinations of strongly (resp. weakly) measurable maps are strongly (resp. weakly) measurable. Moreover, if some map $y : [0,1] \to \mathbb{B}$ is strongly measurable, then $[0,1] \ni t \mapsto \big\| y(t) \big\|_{\mathbb{B}} \in [0,+\infty)$ is Borel.

The relation between the strongly measurable maps and the weakly measurable ones is fully described by a theorem of Pettis, which states that a map $y : [0,1] \to \mathbb{B}$ is strongly measurable if and only if it is weakly measurable and there exists a Borel set $N \subseteq [0,1]$ of null $\mathcal{L}_1$-measure such that $y\big([0,1] \setminus N\big)$ is a separable subset of $\mathbb{B}$. The latter property is often referred to as *essential separably-valuedness*.

We now describe how to define $\mathbb{B}$-valued integrals, the so-called *Bochner integrals*. First of all, given any simple map $y : [0,1] \to \mathbb{B}$ – written in the form $y = \sum_{i=1}^{k} \chi_{E_i} \, v_i$ – we define

$$\int_0^1 y(t) \, \mathrm{d}t \doteq \sum_{i=1}^{k} \mathcal{L}_1(E_i) \, v_i \in \mathbb{B}. \tag{C.2}$$

It can be readily checked that this definition does not depend on the particular way of expressing the map y. Further, we say that any strongly measurable function $y : [0,1] \to \mathbb{B}$ is *Bochner integrable* provided there exists a sequence $(y_n)_n$ of simple maps $y_n : [0,1] \to \mathbb{B}$ such that $\lim_n \int_0^1 \|y_n(t) - y(t)\|_{\mathbb{B}} \, \mathrm{d}t = 0$. In particular, the sequence $\big(\int_0^1 y_n(t) \, \mathrm{d}t\big)_n \subseteq \mathbb{B}$ is Cauchy, so that it makes sense to define

$$\int_0^1 y(t) \, \mathrm{d}t \doteq \lim_{n\to\infty} \int_0^1 y_n(t) \, \mathrm{d}t \in \mathbb{B}. \tag{C.3}$$

It turns out that the value of $\int_0^1 y(t) \, \mathrm{d}t$ just defined is independent of the approximating simple maps $(y_n)_n$ and that it satisfies the fundamental inequality

$$\left\| \int_0^1 y(t) \, \mathrm{d}t \right\|_{\mathbb{B}} \leq \int_0^1 \|y(t)\|_{\mathbb{B}} \, \mathrm{d}t. \tag{C.4}$$

An alternative characterisation of the $\mathbb{B}$-valued integrable maps is given by the following theorem, which is due to Bochner: we have that a strongly measurable map $y : [0,1] \to \mathbb{B}$ is Bochner integrable if and only if it satisfies $\int_0^1 \|y(t)\|_{\mathbb{B}} \, \mathrm{d}t < +\infty$.

The previous result naturally leads to the notion of $\mathbb{B}$-valued L^p space: given $p \in [1,\infty]$, we define $L^p\big([0,1],\mathbb{B}\big)$ as the space of all (equivalence classes of) those strongly measurable maps $y : [0,1] \to \mathbb{B}$ for which the quantity $\|y\|_{L^p([0,1],\mathbb{B})}$ is finite, where

$$\|y\|_{L^p([0,1],\mathbb{B})} \doteq \begin{cases} \left(\displaystyle\int_0^1 \|y(t)\|_{\mathbb{B}}^p \, \mathrm{d}t \right)^{1/p} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{t\in[0,1]} \|y(t)\|_{\mathbb{B}} & \text{if } p = \infty. \end{cases} \tag{C.5}$$

Therefore $L^p\big([0,1],\mathbb{B}\big)$ itself is a Banach space for any $p \in [1,\infty]$.

Definition C.1 (Vector-valued Sobolev/AC maps) *Let $p \in [1,\infty]$ be a given exponent. Then the space $W^{1,p}\big([0,1],\mathbb{B}\big)$ consists of all those curves $y \in L^p\big([0,1],\mathbb{B}\big)$ for which there exists a map $y' \in L^p\big([0,1],\mathbb{B}\big)$ such that*

$$\int_0^1 \varphi'(t) \, y(t) \, \mathrm{d}t = -\int_0^1 \varphi(t) \, y'(t) \, \mathrm{d}t \qquad \textit{for every } \varphi \in C_c^\infty(0,1). \tag{C.6}$$

We endow the Sobolev space $W^{1,p}\big([0,1],\mathbb{B}\big)$ with the norm

$$\|y\|_{W^{1,p}([0,1],\mathbb{B})} \doteq \begin{cases} \left(\|y\|_{L^p([0,1],\mathbb{B})}^p + \|y'\|_{L^p([0,1],\mathbb{B})}^p \right)^{1/p} & \textit{if } p < \infty, \\ \|y\|_{L^\infty([0,1],\mathbb{B})} + \|y'\|_{L^\infty([0,1],\mathbb{B})} & \textit{if } p = \infty. \end{cases} \tag{C.7}$$

Moreover, we define $AC^p\big([0,1],\mathbb{B}\big)$ as the space of all curves $y : [0,1] \to \mathbb{B}$ for which there exists $f \in L^p(0,1)$ such that

$$\|y(t) - y(s)\|_{\mathbb{B}} \leq \int_s^t f(r) \, \mathrm{d}r \qquad \textit{for every } t,s \in [0,1] \textit{ with } s < t. \tag{C.8}$$

We point out that the above definition of $AC^p\big([0,1],\mathbb{B}\big)$ is consistent with Definition 1.1.

Proposition C.2 (Absolutely continuous representative) *Let $y \in W^{1,p}\big([0,1],\mathbb{B}\big)$ be given. Then there exists $\tilde{y} \in AC^p\big([0,1],\mathbb{B}\big)$ such that $y(t) = \tilde{y}(t)$ for a.e. $t \in [0,1]$. Moreover, such curve $\tilde{y}$ satisfies the identity*

$$\tilde{y}(t) - \tilde{y}(s) = \int_s^t y'(r)\,\mathrm{d}r \qquad \text{for every } t,s \in [0,1] \text{ with } s < t. \tag{C.9}$$

Proof. The curve $t \mapsto z(t) \doteq \int_0^t y'(s)\,\mathrm{d}s$ belongs to $AC^p\big([0,1],\mathbb{B}\big) \cap W^{1,p}\big([0,1],\mathbb{B}\big)$, thus in particular the curve $t \mapsto y(t) - z(t)$ belongs to $W^{1,p}\big([0,1],\mathbb{B}\big)$ and has derivative almost everywhere equal to 0. Hence to conclude it is sufficient to show that any such curve is a.e. constant. This follows by noticing that for any $\ell \in \mathbb{B}'$ the function $t \mapsto \ell\big(y(t) - z(t)\big)$ belongs to $W^{1,p}(0,1)$ (by direct verification) and has derivative a.e. equal to 0. □

Proposition C.3 (Characterization of Sobolev curves) *Let $y,z \in L^p\big([0,1],\mathbb{B}\big)$. Then the following conditions are equivalent:*

i) *It holds that $y \in W^{1,p}\big([0,1],\mathbb{B}\big)$ and $z = y'$.*

ii) *For some $D \subseteq \mathbb{B}'$ dense, we have $\ell \circ y \in W^{1,1}(0,1)$ with $(\ell \circ y)' = \ell \circ z$ for any $\ell \in D$.*

In particular, if $\mathbb{B} = L^q(\mu)$ for some Radon measure μ and some exponent $q \in [1,\infty)$, then we have $y \in W^{1,p}\big([0,1],L^q(\mu)\big)$ and $z = y'$ if and only if for every Borel set E it holds that the function $t \mapsto \int_E y(t)\,\mathrm{d}\mu$ belongs to $W^{1,1}(0,1)$ with derivative $t \mapsto \int_E z(t)\,\mathrm{d}\mu$.

Proof. By assumption and by using the fact that the Bochner integral commutes with the application of ℓ, we see that

$$\ell\bigg(\int_0^1 \varphi'(t)\,y(t)\,\mathrm{d}t\bigg) = \ell\bigg(-\int_0^1 \varphi(t)\,z(t)\,\mathrm{d}t\bigg) \qquad \text{for every } \varphi \in C_c^\infty(0,1),$$

for every $\ell \in D$. Then the conclusion follows by density of D in $\mathbb{B}'$.

For the second claim, just observe that the linear span of the set of all characteristic functions of Borel sets is dense in $L^{q/(q-1)}(\mu) \cong L^q(\mu)'$. □

It is important to underline that in general absolute continuity does not imply almost everywhere differentiability: this has to do with the so-called Radon-Nikodým property of the target Banach space (we refer to [BL98] for a thorough discussion about this topic).

A sufficient condition for this implication to hold is given by the next theorem:

Theorem C.4 *Let $\mathbb{B}$ be a reflexive Banach space. Let $p \in [1,\infty]$ and $y \in AC^p\big([0,1],\mathbb{B}\big)$. Then for a.e. $t \in [0,1]$ it holds that the limit of $\big(y(t+h) - y(t)\big)/h$ as $h \to 0$ exists in $\mathbb{B}$.*

In particular, we have $AC^p\big([0,1],\mathbb{B}\big) \cong W^{1,p}\big([0,1],\mathbb{B}\big)$, i.e. every absolutely continuous curve is the (only) continuous representative of a curve in $W^{1,p}\big([0,1],\mathbb{B}\big)$.

Given any Banach space $\mathbb{B}$, we denote by $\mathrm{End}(\mathbb{B})$ the space of all linear and continuous maps of $\mathbb{B}$ to itself, which is a Banach space if endowed with the operator norm.

The space $\Gamma(\mathbb{B}) = C\big([0,1],\mathbb{B}\big)$ is a Banach space with respect to the norm $\|\cdot\|_{\Gamma(\mathbb{B})}$, given by $\|y\|_{\Gamma(\mathbb{B})} \doteq \max\big\{\|y_t\|_{\mathbb{B}} : t \in [0,1]\big\}$ for every $y \in \Gamma(\mathbb{B})$.

Theorem C.5 (Integral solutions to vector-valued linear ODEs) *Let $\mathbb{B}$ be a Banach space. Let $z \in \Gamma(\mathbb{B})$. Let $\lambda : [0,1] \to \mathrm{End}(\mathbb{B})$ be a bounded function, i.e. there exists $c > 0$ such that $\|\lambda(t)\|_{\mathrm{End}(\mathbb{B})} \leq c$ for every $t \in [0,1]$. Assume that $[0,1] \ni t \mapsto \lambda(t)v \in \mathbb{B}$ is strongly measurable for every $v \in \mathbb{B}$. Then there exists a unique curve $y \in \Gamma(\mathbb{B})$ such that*

$$y(t) = z(t) + \int_0^t \lambda(s)y(s)\,\mathrm{d}s \qquad \text{for every } t \in [0,1]. \tag{C.10}$$

Moreover, the solution y satisfies $\|y\|_{\Gamma(\mathbb{B})} \leq e^c \|z\|_{\Gamma(\mathbb{B})}$.

Proof. Given any simple mapping $t \mapsto y_t = \sum_{i=1}^k \chi_{A_i}(t)\,v_i$, with $A_1,\dots,A_k \in \mathscr{B}([0,1])$ and $v_1,\dots,v_k \in \mathbb{B}$, we have that $t \mapsto \lambda(t)y_t = \sum_{i=1}^k \chi_{A_i}(t)\,\lambda(t)v_i$ is strongly measurable by hypothesis on λ. Now fix $y \in \Gamma(\mathbb{B})$. In particular, $y : [0,1] \to \mathbb{B}$ is strongly measurable, hence there exists a sequence $(y^k)_k$ of simple maps $y^k : [0,1] \to \mathbb{B}$ such that $\lim_k \|y_t^k - y_t\|_{\mathbb{B}} = 0$ holds for $\mathcal{L}_1$-a.e. $t \in [0,1]$. This grants that $\|\lambda(t)y_t^k - \lambda(t)y_t\|_{\mathbb{B}} \leq c\,\|y_t^k - y_t\|_{\mathbb{B}} \overset{k}{\to} 0$ is satisfied for $\mathcal{L}_1$-a.e. $t \in [0,1]$, thus accordingly the map $t \mapsto \lambda(t)y_t$ is strongly measurable as pointwise limit of strongly measurable functions. Moreover, since $\|\lambda(t)y_t\|_{\mathbb{B}} \leq c\,\|y\|_{\Gamma(\mathbb{B})}$ for all $t \in [0,1]$, one has that $t \mapsto \lambda(t)y_t$ actually belongs to $L^\infty([0,1],\mathbb{B})$. Therefore it makes sense to define the function $\Lambda y : [0,1] \to \mathbb{B}$ as $(\Lambda y)(t) \doteq \int_0^t \lambda(s)y_s\,\mathrm{d}s$ for every $t \in [0,1]$. Note that

$$\|\Lambda y(t_1) - \Lambda y(t_0)\|_{\mathbb{B}} \leq c\,\|y\|_{\Gamma(\mathbb{B})}(t_1 - t_0) \qquad \text{for every } t_0, t_1 \in [0,1] \text{ with } t_0 < t_1. \tag{C.11}$$

Therefore Λy is Lipschitz with $\mathrm{Lip}(\Lambda y) \leq c\,\|y\|_{\Gamma(\mathbb{B})}$, so that in particular $\Lambda y \in \Gamma(\mathbb{B})$. By plugging $t_1 = t$ and $t_0 = 0$ into (C.11), we deduce that $\|\Lambda y(t)\|_{\mathbb{B}} \leq c\,\|y\|_{\Gamma(\mathbb{B})} t$ for all $t \in [0,1]$ and accordingly that $\|\Lambda y\|_{\Gamma(\mathbb{B})} \leq c\,\|y\|_{\Gamma(\mathbb{B})}$. This guarantees that the map $\Lambda : \Gamma(\mathbb{B}) \to \Gamma(\mathbb{B})$ is linear and continuous, with $\|\Lambda\|_{\mathrm{End}(\Gamma(\mathbb{B}))} \leq c$. Now observe that

$$y \in \Gamma(\mathbb{B}) \text{ satisfies (C.10)} \qquad \Longleftrightarrow \qquad (\mathrm{id}_{\Gamma(\mathbb{B})} - \Lambda)(y) = z. \tag{C.12}$$

For any $n \in \mathbb{N}^+$, the iterated operator $\Lambda^n = \Lambda \circ \ldots \circ \Lambda$ satisfies

$$\begin{aligned}
\|\Lambda^n y(t)\|_{\mathbb{B}} &\leq c \int_0^t \|\Lambda^{n-1} y(t_n)\|_{\mathbb{B}}\,\mathrm{d}t_n \\
&\leq c^2 \int_0^t \int_0^{t_n} \|\Lambda^{n-2} y(t_{n-1})\|_{\mathbb{B}}\,\mathrm{d}t_{n-1}\,\mathrm{d}t_n \\
&\leq \ldots \\
&\leq c^n \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} \|y(t_1)\|_{\mathbb{B}}\,\mathrm{d}t_1 \ldots \mathrm{d}t_{n-1}\,\mathrm{d}t_n \\
&\leq c^n \|y\|_{\Gamma(\mathbb{B})} \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} \mathrm{d}t_1 \ldots \mathrm{d}t_{n-1}\,\mathrm{d}t_n \\
&= c^n \|y\|_{\Gamma(\mathbb{B})} \frac{t^n}{n!}
\end{aligned}$$

for every $y \in \Gamma(\mathbb{B})$ and $t \in [0,1]$, whence $\|\Lambda^n\|_{\mathrm{End}(\Gamma(\mathbb{B}))} \leq c^n/n!$. Hence $\mathrm{id}_{\Gamma(\mathbb{B})} - \Lambda$ is invertible and the operator norm of its inverse $(\mathrm{id}_{\Gamma(\mathbb{B})} - \Lambda)^{-1} = \sum_{n=0}^\infty \Lambda^n$ is bounded above by e^c. In light of (C.12), we finally conclude that there exists a unique curve $y \in \Gamma(\mathbb{B})$ fulfilling (C.10), namely $y \doteq (\mathrm{id}_{\Gamma(\mathbb{B})} - \Lambda)^{-1}(z)$, which also satisfies $\|y\|_{\Gamma(\mathbb{B})} \leq e^c \|z\|_{\Gamma(\mathbb{B})}$. □

We will actually make use of the following consequence of Theorem C.5:

Corollary C.6 (Differential solutions to vector-valued linear ODEs) *Fix a reflexive Banach space* $\mathbb{B}$*. Let* $\overline{y} \in \mathbb{B}$ *be given. Let* $\lambda : [0,1] \to \mathrm{End}(\mathbb{B})$ *be a bounded function. Suppose that the map* $[0,1] \ni t \mapsto \lambda(t)v \in \mathbb{B}$ *is strongly measurable for every* $v \in \mathbb{B}$*. Then there exists a unique curve* $y \in \mathrm{LIP}\big([0,1], \mathbb{B}\big)$ *such that*

$$\begin{cases} y'(t) = \lambda(t)y(t) & \text{for } \mathcal{L}_1\text{-a.e. } t \in [0,1], \\ y(0) = \bar{y}. \end{cases} \tag{C.13}$$

Moreover, the solution y *satisfies* $\|y\|_{\Gamma(\mathbb{B})} \leq e^c \, \|\overline{y}\|_{\mathbb{B}}$*, where* $c \doteq \max_{t\in[0,1]} \big\|\lambda(t)\big\|_{\mathrm{End}(\mathbb{B})}$*.*

Proof. Define $z(t) \doteq \overline{y}$ for all $t \in [0,1]$. Consider the curve $y \in \Gamma(\mathbb{B})$ given by Theorem C.5. For every $t, s \in [0,1]$ with $s < t$ we have that

$$\big\|y(t) - y(s)\big\|_{\mathbb{B}} = \bigg\| \int_s^t \lambda(r)y(r)\,\mathrm{d}r \bigg\|_{\mathbb{B}} \overset{\text{(C.4)}}{\leq} \int_s^t \big\|\lambda(r)y(r)\big\|_{\mathbb{B}}\,\mathrm{d}r \leq c \int_s^t \big\|y(r)\big\|_{\mathbb{B}}\,\mathrm{d}r.$$

Since $\big\|y(\cdot)\big\|_{\mathbb{B}} \in L^\infty(0,1)$, we deduce that the map y is Lipschitz, so that Theorem C.4 grants that y is a.e. differentiable. Then (C.13) trivially follows from (C.10).

Conversely, let $y \in \mathrm{LIP}\big([0,1], \mathbb{B}\big)$ be any curve such that (C.13) holds true. By integration we conclude that y satisfies also property (C.10), thus proving uniqueness. □

Bibliography

[ACDM12] Luigi Ambrosio, Maria Colombo and Simone Di Marino. *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope.* Adv. St. in Pure Math., 67:1–58, (2012).

[AG13] Luigi Ambrosio and Nicola Gigli. *A user's guide to optimal transport.* In Modelling and Optimisation of Flows on Networks, Lecture Notes in Mathematics, pages 1–155. Springer Berlin Heidelberg, (2013).

[AGMR15] Luigi Ambrosio, Nicola Gigli, Andrea Mondino and Tapio Rajala. *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure.* Trans. Amer. Math. Soc., 367(7):4661–4701, (2015).

[AGS08] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, (2008).

[AGS13] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces.* Rev. Mat. Iberoam., 29(3):969–996, (2013).

[AGS14a] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below.* Invent. Math., 195(2):289–391, (2014).

[AGS14b] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Metric measure spaces with Riemannian Ricci curvature bounded from below.* Duke Math. J., 163(7):1405–1490, (2014).

[AGS15] Luigi Ambrosio, Nicola Gigli and Giuseppe Savaré. *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds.* The Annals of Probability, 43(1):339–404, (2015).

[AK00] Luigi Ambrosio and Bernd Kirchheim. *Rectifiable sets in metric and Banach spaces.* Math. Annalen., 318:527–555, (2000).

[Amb18] Luigi Ambrosio. *Calculus, heat flow and curvature-dimension bounds in metric measure spaces.* Proceedings of the ICM 2018, (2018).

[AMS15] Luigi Ambrosio, Andrea Mondino and Giuseppe Savaré. *Nonlinear diffusion equations and curvature conditions in metric measure spaces.* Accepted at Memoirs Amer. Math. Soc., (2015).

[AT04] Luigi Ambrosio and Paolo Tilli. *Topics on Analysis in Metric Spaces.* Oxford Lecture Series in Mathematics and Its Applications, (2004).

[Awo06] Steve Awodey. *Category Theory.* Oxford Logic Guides, (2006).

[Bak85] Dominique Bakry. *Transformations de Riesz pour les semi-groupes symetriques. II. Étude sous la condition $\Gamma_2 \geq 0$.* In Séminaire de probabilités, XIX, 1983/84, volume 1123 of Lecture Notes in Math., pages 145–174. Springer, Berlin, (1985).

[BBI01] Dmitri Burago, Yuri Burago and Sergei Ivanov. *A course in metric geometry.* volume 33 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, (2001).

[BÉ85] Dominique Bakry and Michel Émery. *Diffusions hypercontractives.* In Séminaire de probabilités, XIX, 1983/84, volume 1123 of Lecture Notes in Math., pages 177–206. Springer, Berlin, (1985).

[BGL12] Dominique Bakry, Ivan Gentil and Michel Ledoux. *On Harnack inequalities and optimal transportation.* Annali della Scuola Normale Superiore di Pisa, Classe di scienze, 14, 10, (2012).

[BGP92] Yuri Burago, Michail L. Gromov and Grigori Y. Perel'man. *A.D. Alexandrov spaces with curvature bounded below.* Russian Mathematical Surveys, 47(2):1, (1992).

[BH91] Nicolas Bouleau and Francis Hirsch. *Dirichlet forms and analysis on Wiener space.* De Gruyter studies in mathematics. W. de Gruyter, (1991).

[BL98] Yoav Benyamini and Joram Lindenstrauss. *Geometric Nonlinear Functional Analysis.* Number pt. 1 in American Mathematical Society colloquium publications. American Mathematical Soc., (1998).

[Bog07] Vladimir I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, (2007).

[BS10] Kathrin Bacher and Karl-Theodor Sturm. *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces.* J. Funct. Anal., 259(1):28–56, (2010).

[BS18a] Elia Brué and Daniele Semola. *Constancy of the dimension for* $\mathsf{RCD}(K,N)$ *spaces via regularity of Lagrangian flows.* Accepted at Comm. Pure and Applied Math., (2018).

[BS18b] Elia Brué and Daniele Semola. *Regularity of Lagrangian flows over* $\mathsf{RCD}^*(K,N)$ *spaces.* Preprint, arXiv:1803.04387, (2018).

[CC96] Jeff Cheeger and Tobias H. Colding. *Lower bounds on Ricci curvature and the almost rigidity of warped products.* Ann. of Math. (2), 144(1):189–237, (1996).

[CC97] Jeff Cheeger and Tobias H. Colding. *On the structure of spaces with Ricci curvature bounded below. I.* J. Differential Geom., 46(3):406–480, (1997).

[CC00a] Jeff Cheeger and Tobias H. Colding. *On the structure of spaces with Ricci curvature bounded below. II.* J. Differential Geom., 54(1):13–35, (2000).

[CC00b] Jeff Cheeger and Tobias H. Colding. *On the structure of spaces with Ricci curvature bounded below. III.* J. Differential Geom., 54(1):37–74, (2000).

[CEMS01] Dario Cordero-Erausquin, Robert J. McCann and Michael Schmuckenschläger. *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb.* Inventiones mathematicae, 146(2):219–257, (2001).

[Che99] Jeff Cheeger. *Differentiability of Lipschitz functions on metric measure spaces.* Geom. Funct. Anal., 9(3):428–517, (1999).

[CM17] Fabio Cavalletti and Andrea Mondino. *Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds.* Inventiones mathematicae, 208(3):803–849, (2017).

[CM16] Fabio Cavalletti and Emanuel Milman. *The globalization theorem for the curvature dimension condition.* Preprint, arXiv:1612.07623, (2016).

[CN12] Tobias H. Colding and Aaron Naber. *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications.* Ann. of Math. (2), 176(2):1173–1229, (2012).

[Den10] Dieter Denneberg. *Non-Additive Measure and Integral.* Theory and Decision Library B. Springer Netherlands, (2010).

[DGP18] Clément Debin, Nicola Gigli and Enrico Pasqualetto. *Quasi-continuous vector fields on* RCD *spaces.* In progress, (2018).

[DM14] Simone Di Marino. *Sobolev and BV spaces on metric measure spaces via derivations and integration by parts.* Submitted, arXiv:1409.5620, (2014).

[DMRV06] Oleksiy Dovgoshey, Olli Martio, Vladimir Ryazanov and Matti Vuorinen. *The Cantor function.* Expositiones Mathematicae, 24(1):1–37, (2006).

[DMS15] Simone Di Marino and Gareth Speight. *The p-weak gradient depends on p.* Proc. Amer. Math. Soc. 143:5239-5252, (2015).

[DPG16] Guido De Philippis and Nicola Gigli. *From volume cone to metric cone in the nonsmooth setting.* Geometric and Functional Analysis, 26(6):1526–1587, (2016).

[DPG18] Guido De Philippis and Nicola Gigli. *Non-collapsed spaces with Ricci curvature bounded from below.* Journal de l'Ecole Polytechnique, 5:613–650, (2018).

[DPMR17] Guido De Philippis, Andrea Marchese and Filip Rindler. *On a conjecture of Cheeger.* Measure Theory in Non-Smooth Spaces, edited by N. Gigli. De Gruyter, (2017) [DPR16] Guido De Philippis and Filip Rindler. *On the structure of $\mathscr{A}$-free measures and applications.* Annals of Math., Volume 184, 1017-1039, (2016).

[DU77] Joseph Diestel and John J. Uhl, Jr. *Vector measures.* American Mathematical Society, Providence, R.I.. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15, (1977).

[EKS14] Matthias Erbar, Kazumasa Kuwada and Karl-Theodor Sturm. *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces.* Inventiones mathematicae, 201(3):1–79, (2014).

[ES64] James Eells, Jr. and J.H. Sampson. *Harmonic mappings of Riemannian manifolds.* Amer. J. Math., 86:109–160, (1964).

[Fed69] Herbert Federer. *Geometric measure theory.* Grundlehren der mathematischen Wissenschaften. Springer, (1969).

[Fug57] Bent Fuglede. *Extremal length and functional completion.* Acta Math., 98:171–219, (1957).

[GH16] Nicola Gigli and Bangxian Han. *Independence on p of weak upper gradients on* RCD *spaces.* Journal of Functional Analysis, volume 271, 1-11, (2016).

[Gig13] Nicola Gigli. *The splitting theorem in non-smooth context.* Submitted, arXiv:1302.5555, (2013).

[Gig14b] Nicola Gigli. *An overview of the proof of the splitting theorem in spaces with non-negative Ricci curvature.* Analysis and Geometry in Metric Spaces, 2:169–213, (2014).

[Gig15] Nicola Gigli. *On the differential structure of metric measure spaces and applications.* Mem. Amer. Math. Soc., 236(1113):vi+91, (2015).

[Gig17a] Nicola Gigli. *Lecture notes on differential calculus on* RCD *spaces.* Accepted for publication by RIMS lecture notes, arXiv:1703.06829, (2017).

[Gig17b] Nicola Gigli. *Nonsmooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below.* Mem. Amer. Math. Soc., 251(1196), (2017).

[GM13] Nicola Gigli and Andrea Mondino. *A PDE approach to nonlinear potential theory in metric measure spaces.* J. Math. Pures Appl. (9), 100(4):505–534, (2013).

[GM14] Nicola Gigli and Sunra Mosconi. *The Abresch-Gromoll inequality in a non-smooth setting.* Discrete Contin. Dyn. Syst., 34(4):1481–1509, (2014).

[GMR15] Nicola Gigli, Andrea Mondino and Tapio Rajala. *Euclidean spaces as weak tangents of infinitesimally hilbertian metric spaces with Ricci curvature bounded below.* Journal für die reine und angewandte Mathematik (Crelles Journal), 2015(705):33-244, (2015).

[GMS15] Nicola Gigli, Andrea Mondino and Giuseppe Savaré. *Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows.* Proceedings of the London Mathematical Society, 111(5):1071–1129, (2015).

[GP16a] Nicola Gigli and Enrico Pasqualetto. *Behaviour of the reference measure on* RCD *spaces under charts.* Accepted at Communications in Analysis and Geometry, (2016).

[GP16b] Nicola Gigli and Enrico Pasqualetto. *Equivalence of two different notions of tangent bundle on rectifiable metric measure spaces.* Submitted, arXiv:1611.09645, (2016).

[GP17] Nicola Gigli and Enrico Pasqualetto. *On the notion of parallel transport on* RCD *spaces.* Submitted, arXiv:1803.05374, (2017).

[GP18] Nicola Gigli and Enrico Pasqualetto. *Differential structure associated to axiomatic Sobolev spaces.* Submitted, arXiv:1807.05417, (2018).

[GPS18] Nicola Gigli, Enrico Pasqualetto and Elefterios Soultanis. *Differential of metric valued Sobolev maps.* Submitted, arXiv:1807.10063, (2018).

[GR17] Nicola Gigli and Chiara Rigoni. *Recognizing the flat torus among* $\mathsf{RCD}^*(0,N)$ *spaces via the study of the first cohomology group.* Calculus of Variation and Partial Differential Equations, 57(4), (2017).

[Gro87] Michail L. Gromov. *Hyperbolic Groups.* Pages 75–263. Springer New York, New York, NY, (1987).

[GRS16] Nicola Gigli, Tapio Rajala and Karl-Theodor Sturm. *Optimal Maps and Exponentiation on Finite-Dimensional Spaces with Ricci Curvature Bounded from Below.* J. Geom. Anal., 26(4):2914–2929, (2016).

[GS92] Michail L. Gromov and Richard Schoen. *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one.* Inst. Hautes Études Sci. Publ. Math., (76):165–246, (1992).

[GT01] Vladimir Gol'dshtein and Marc Troyanov. *Axiomatic theory of Sobolev spaces.* Expositiones Mathematicae, 19:289–336, 12, (2001).

[GT18] Nicola Gigli and Alexander Tyulenev. *Korevaar-Schoen's directional energy and Ambrosio's regular flows.* In progress, (2018).

[Haj96] Piotr Hajłasz. *Sobolev spaces on an arbitrary metric space.* Potential Analysis, 5(4):403–415, (1996).

[Han18] Bangxian Han. *Ricci tensor on* $\mathsf{RCD}^*(K,N)$ *spaces.* The Journal of Geometric Analysis, 28(2):1295-1314, (2018).

[Hei01] Juha Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, (2001).

[HKST15] Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam and Jeremy T. Tyson. *Sobolev spaces on metric measure spaces. An approach based on upper gradients.* Volume 27 of New Mathematical Monographs. Cambridge University Press, Cambridge, (2015).

[Ket15a] Christian Ketterer. *Cones over metric measure spaces and the maximal diameter theorem.* J. Math. Pures Appl. (9), 103(5):1228–1275, (2015).

[Ket15b] Christian Ketterer. *Obata's rigidity theorem for metric measure spaces.* Analysis and Geometry in Metric Spaces, 3(1)278–295, (2015).

[Kir94] Bernd Kirchheim. *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure.* Proceedings of the American Mathematical Society, 121(1):113–123, (1994).

[Kit18] Yu Kitabeppu. *A sufficient condition to a regular set of positive measure on* RCD *spaces.* Potential Analysis, pp 1–18, (2018).

[KM18] Martin Kell and Andrea Mondino. *On the volume measure of non-smooth spaces with Ricci curvature bounded below.* Annali SNS, XVIII(2):593–610, (2018).

[KM98] Pekka Koskela and Paul MacManus. *Quasiconformal mappings and Sobolev spaces.* Studia Math., 131(1):1–17, (1998).

[KS93] Nicholas J. Korevaar and Richard Schoen. *Sobolev spaces and harmonic maps for metric space targets.* Comm. Anal. Geom., 1(3-4):561–659, (1993).

[Kuw10] Kazumasa Kuwada. *Duality on gradient estimates and Wasserstein controls.* J. Funct. Anal., 258(11):3758–3774, (2010).

[Lan84] Serge Lang. *Algebra.* Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, second edition, (1984).

[Lev06] Beppo Levi. *Sul principio di Dirichlet.* Rend. Circ. Mat. Palermo, (1906).

[Lis07] Stefano Lisini. *Characterization of absolutely continuous curves in Wasserstein spaces.* Calc. Var. Partial Differential Equations, 28(1):85–120, (2007).

[LP18] Danka Lučić and Enrico Pasqualetto. *The Serre-Swan theorem for normed modules.* Rend. Circ. Mat. Palermo, (2018).

[LV07] John Lott and Cédric Villani. *Weak curvature conditions and functional inequalities.* J. Funct. Anal., 245(1):311–333, (2007).

[McC97] Robert J. McCann. *A convexity principle for interacting gases.* Advances in Mathematics, 128(1):153–179, (1997).

[MN14] Andrea Mondino and Aaron Naber. *Structure Theory of Metric-Measure Spaces with Lower Ricci Curvature Bounds.* Accepted at Journ. European Math Soc. (JEMS), (2014).

[Nes03] JET NESTRUEV. *Smooth Manifolds and Observables.* Graduate Texts in Mathematics. Springer, (2003).

[OV00] FELIX OTTO AND CÉDRIC VILLANI. *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality.* Journal of Functional Analysis, 173(2):361–400, (2000).

[Pas18] ENRICO PASQUALETTO. *Lecture notes on nonsmooth differential geometry.* Work in progress, available at: http://www.math.sissa.it/course/phd-course/nonsmooth-differential-geometry, (2018).

[Pet98] ANTON PETRUNIN. *Parallel transportation for Alexandrov space with curvature bounded below.* Geom. Funct. Anal., 8(1):123–148, (1998).

[Pet10] ANTON PETRUNIN. *Alexandrov meets Lott-Villani-Sturm.* Münster J. Math., volume 4, (2010).

[Raj12] TAPIO RAJALA. *Local Poincaré inequalities from stable curvature conditions on metric spaces.* Calc. Var. Partial Differential Equations, 44(3-4):477–494, (2012).

[Roy88] HALSEY L. ROYDEN. *Real analysis.* Macmillan Publishing Company, New York, third edition, (1988).

[Sau89] JEAN-LUC SAUVAGEOT. *Tangent bimodule and locality for dissipative operators on C^*-algebras.* In Quantum probability and applications, IV (Rome, 1987), volume 1396 of Lecture Notes in Math., pages 322–338. Springer, Berlin, (1989).

[Sau90] JEAN-LUC SAUVAGEOT. *Quantum Dirichlet forms, differential calculus and semigroups.* In Quantum probability and applications, V (Heidelberg, 1988), volume 1442 of Lecture Notes in Math., pages 334–346. Springer, Berlin, (1990).

[Sav14] GIUSEPPE SAVARÉ. *Self-improvement of the Bakry-Émery condition and Wasserstein contraction of the heat flow in $\mathsf{RCD}(K,\infty)$ metric measure spaces.* Discrete Contin. Dyn. Syst., 34(4):1641–1661, (2014).

[Sha00] NAGESWARI SHANMUGALINGAM. *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces.* Rev. Mat. Iberoamericana, 16(2):243–279, (2000).

[Sha09] NAGESWARI SHANMUGALINGAM. *A Universality Property of Sobolev Spaces in Metric Measure Spaces.* Sobolev Spaces in Mathematics I: Sobolev Type Inequalities, pages 345–359. Springer New York, New York, NY, (2009).

[Stu06a] KARL-THEODOR STURM. *On the geometry of metric measure spaces. I.* Acta Math., 196(1):65–131, (2006).

[Stu06b] KARL-THEODOR STURM. *On the geometry of metric measure spaces. II.* Acta Math., 196(1):133–177, (2006).

[Tak79] MASAMICHI TAKESAKI. *Theory of operator algebras. I.* Springer-Verlag, New York-Heidelberg, (1979).

[Tim06] Sergey Timoshin. *Regularity in metric spaces.* PhD thesis, Lausanne, available at: https://infoscience.epfl.ch/record/85799/files/EPFL TH3571.pdf, (2006).

[Vil09] Cédric Villani. *Optimal transport. Old and new.* Volume 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, (2009).

[Vil16] Cédric Villani. *Synthetic theory of Ricci curvature bounds.* Japanese Journal of Mathematics, 11(2):219–263, (2016).

[Vil17] Cédric Villani. *Inégalités isopérimétriques dans les espaces métriques mesurés [d'après F. Cavalletti & A. Mondino].* Séminaire Bourbaki, available at: http://www.bourbaki.ens.fr/TEXTES/1127.pdf, (2017).

[vRS09] Max-K. von Renesse and Karl-Theodor Sturm. *Entropic measure and Wasserstein diffusion.* Ann. Probab., 37(3):1114–1191, (2009).

[Wea99] Nik Weaver. *Lipschitz algebras.* World Scientific Publishing Co., Inc., River Edge, NJ, (1999).

[Wea00] Nik Weaver. *Lipschitz algebras and derivations. II. Exterior differentiation.* J. Funct. Anal., 178(1):64–112, (2000).

[ZZ18] Hui-Chun Zhang and Xi-Ping Zhu. *Lipschitz continuity of harmonic maps between Alexandrov spaces.* Invent. Math., 211(3):863–934, (2018).