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Qualitative properties and construction of solutions to some semilinear elliptic PDEs.

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Summary

This thesis, that recollects the results obtained during my PhD ([30, 70, 69]), is devoted to the study of some semilinear elliptic equations. On the one hand, we study some qualitative properties, such as symmetry of solutions, on the other hand we explicitly construct some solutions vanishing near some fixed manifold. In this summary we just give a brief account of the main issues treated in this work. Since we dealt with two different kinds of problems, the manuscript is divided in two parts. The first one is devoted to the study of qualitative properties of some given solutions to some equations, the second one is about the construction of solutions. Every part will be endowed with its own introduction, in which we explain the main ideas and the techniques used to solve our problems in a more detailed way.

- In Chapter 1, we study the equation

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N, \tag{1}$$

where f is a sufficiently smooth function satisfying

$$f^{'}(s) \leq 0 \quad \text{for } s \in (0, \varepsilon), \text{ for some } \varepsilon > 0, \text{ and } f(0) = 0. \tag{2}$$

A well-known particular case is $f(u) = u|u|^{p-1} - u$, that is the nonlinearity appearing in the Schrödinger equation.

We consider solutions that decay in some variables and we wonder whether they are symmetric in those variables. We will see that this is true under some additional hypothesis, such as, for instance, the periodicity in the other variables or the decay of the gradient. Considering solutions decaying in all the variables, we get the famous theorem by Gidas, Ni and Nirenberg, of which we give an alternative proof. The proofs basically exploit the moving planes technique, that is based on the maximum principles, and some integral techniques.

- In Chapter 2 we consider the Otha-Kawasaki functional

$$\mathcal{E}_\varepsilon(u) := \frac{1}{2}\int_{T^3} |\nabla u| dx + \varepsilon \int_{T^3}\int_{T^3} G(x,y)(u(x) - m)(u(y) - m) dx dy, \tag{3}$$

where u is a bounded variation function on the Torus T^3 with values in $\{\pm 1\}$, G is the Green function of $-\Delta$ in Ω and $\varepsilon > 0$ is small. This functional arises from the diblock copolymers theory. It is well known that some critical points are given by those functions u whose interface has a periodic structure corresponding to spheres, cylinders, gyroids and lamellae. Moreover, the

functional $\mathcal{E}_\varepsilon$ is known to be translation invariant [1, 18], thus any translated of a critical point is still critical (since the functions that we are considering take values in $\{\pm 1\}$, the critical points can be identified to sets). Now, we add a small linear perturbation to our functional of the form

$$\varepsilon \int_{T^3} f(x)u(x)dx,$$

that breaks the translation invariance and we find at least four critical points in a neighbourhood of a suitable translated of a fixed set, that is the Schwarz P surface studied by M. Ross [72], who shows that this surface is volume preserving stable, that is stable with respect to volume preserving transformations. Furthermore, it is possible to prove that the kernel of the Jacobi operator is exactly given by the subspace generated by translations [1, 18, 42]. The properties of this kernel force the solutions to fulfill a volume constraint.

- In Chapter 3 we deal with the phase transition theory, in particular we construct some entire solutions to the Cahn-Hilliard equation

$$-\varepsilon^2\Delta(-\varepsilon^2\Delta u + W'(u)) + W''(u)(-\varepsilon^2\Delta u + W'(u)) = 0 \quad \text{in } \mathbb{R}^3, \tag{4}$$

where $W(u) := \frac{1}{4}(1-u^2)^2$, whose nodal set is close to the Clifford Torus

$$\Sigma := \Big\{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \Big(\sqrt{2} - \sqrt{x_1^2 + x_2^2}\Big)^2 + x_3^2 - 1 = 0\Big\}. \tag{5}$$

The Cahn-Hilliard equation (4) can be seen as a fourth-order generalization of the Allen-Cahn equation

$$-\varepsilon^2\Delta u = u - u^3, \tag{6}$$

arising from the phase transitions. If we consider (6) in a bounded domain $\Omega \subset \mathbb{R}^N$, for example, we know that it is the Euler equation of the energy

$$E_\varepsilon(u) = \int_\Omega \big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2\big)dx.$$

Modica and Mortola [57] prove that this energy Γ-converges to the perimeter

$$E(u) = \begin{cases} cPer_\Omega(\{u = 1\}) & \text{if } u = \pm 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

as $\varepsilon \to 0$. The Euler equation of the perimeter is $H = 0$, where H is defined to be the mean curvature of the manifold $\partial\{x \in \Omega : u(x) = 1\}$, that is $H := k_1 + \cdots + k_{N-1}$, where the k_i are the scalar curvatures. Since the Γ-convergence implies the convergence of minimizers, if we consider a family u_ε of minimizers of the energy (7), the zero-level set tends to a minimal hypersuface, that a hypersurface satisfying $H = 0$. Pacard and Ritoré prove a sort of viceversa, in the sense that they start from a given minimal hypersurface Σ in a compact manifold M and they construct a family of solutions u_ε whose nodal set approaches Σ as $\varepsilon \to 0$, provided Σ is nondegenerate, in the sense that the Jacobi operator

$$L_0\phi = -\Delta_\Sigma\phi - |A|^2\phi - Ric_g(\nu_\Sigma, \nu_\Sigma) \tag{7}$$

which represents the second variation of the perimeter, given by

$$E''(\Sigma)(\phi, \psi) = \int_\Sigma \tilde{L}_0\phi\psi d\sigma, \tag{8}$$

has to be invertible. Here, $|A|^2 := k_1^2 + \cdots + k_{N-1}^2$ is the squared norm of the second fundamental form, Ric_g is the Ricci tensor of M and ν_Σ is a choice of unit normal to Σ.

In the thesis, we have some similar results for the Cahn-Hilliard equation. It is possible to verify that (4) is the Euler equation of the energy

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{2\varepsilon}\int_\Omega \left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^2 dx & \text{if } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

There are some Γ-convergence results that relate this energy to the Willmore functional

$$\mathcal{W}(\Sigma) = c\int_\Sigma H_\Sigma^2(y)d\sigma,$$

where H_Σ is the mean curvature of Σ (for instance [8, 71, 61]). This functional is studied, for example in the context of general relativity, being related to the Hawking mass

$$m_H(\Sigma) = \sqrt{\frac{Area(\Sigma)}{16\pi}}(1 - \frac{1}{16\pi}\mathcal{W}(\Sigma)).$$

We call *Willmore surfaces* the critical points of this functional, which are characterized by the Euler equation,

$$-\Delta_\Sigma H + \frac{1}{2}H(H^2 - 2|A|^2) = 0.$$

Therefore, it looks reasonable to wonder whether it is possible to construct a family of solutions u_ε of the Cahn-Hilliard equation (4) whose nodal set is close to some prescribed Willmore surface. In our case, we start from the Clifford torus (5), of which we exploit the nondegeneracy properties. In fact, the proof is based on the fact that the kernel of the self-adjoint operator

$$\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2 L_0\phi - H(\nabla_\Sigma\phi, \nabla_\Sigma H) + 2(A\nabla_\Sigma\phi, \nabla_\Sigma H) + \\ 2H < A, \nabla^2\phi > + \phi(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2 + 2H\mathrm{tr}A^3) \quad (9)$$

appearing in the second variation of the Willmore functional exactly coincides with the subspace generated by conformal transformations, that is isometries, dilations and inversions with respect to spheres. Due to the dilation invariance of the Willmore functional, we impose the solutions to satisfy the volume constraint

$$\int_{\mathbb{R}^3} (1 - u_\varepsilon)dx = 4\sqrt{2}\pi^2.$$

Due to the other invariances, we impose the solutions to respect the symmetries of the torus. The proof is based on the Lyapunov-Schmidt reduction.

Résumé

Cette thèse, qui réunit les résultats obtenus pendant mon doctorat [30, 70, 69], est consacrée à l'étude de certaines équations elliptiques semi-linéaires. D'un côté, nous étudions certaines propriétés qualitatives, comme la symétrie des solutions, de l'autre nous construisons explicitement des solutions qui s'annulent près d'une variété donnée. Ce résumé contient seulement un bref compte rendu du travail. Ayant traité deux types de problèmes différents, l'ouvrage est divisé en deux parties. La première est dévouée à l'étude des propriétés qualitatives de solutions données de certaines équations, la deuxième concernant la construction de solutions. Chaque partie sera dotée de sa propre introduction, où nous expliquons de façon plus détaillée les idées fondamentales et les techniques utilisées pour résoudre nos problèmes.

- Dans le Chapitre 1, nous étudions l'équation

$$-\Delta u = f(u) \quad \text{en } \mathbb{R}^N, \tag{10}$$

où f est une fonction assez régulière satisfaisant

$$f^{'}(s) \leq 0 \quad \text{pour } s \in (0, \varepsilon), \text{ pour quelques } \varepsilon > 0, \text{ et } f(0) = 0. \tag{11}$$

Il est intéressant de voir comment les propriétés de décroissance influencent les symétries des solutions. Il y a un résultat très général de Gidas, Ni et Nirenberg qui dit que toute solution positive de (10) décroissante à l'infini est à symétrie radiale.

Théorème 1. *[37] Soit $u > 0$ une solution de l'équation (10), avec f satisfaisant la condition (11). Supposons en plus que*

$$u(x) \to 0 \quad \textit{pour } |x| \to \infty. \tag{12}$$

Alors, à une translation près, u est à symétrie radiale, c'est-à-dire $u = u(|x|)$, et $\partial u/\partial r(x) < 0$, pour tout $x \neq 0$.

Dans ma thèse, nous considérons des hypothèses plus faibles. Plus précisément, nous étudions des solutions bornées et décroissantes seulement dans certaines variables, c'est à dire nous supposons que

$$u(y, z) \to 0 \quad \text{pour } |z| \to \infty, \text{ uniformément en } y, \tag{13}$$

où nous avons posé $x = (y, z)$, avec $y \in \mathbb{R}^M$, $z \in \mathbb{R}^{N-M}$, et nous nous demandons si elles sont symétriques par rapport à ces variables. Nous verrons

que cela est vrai sous des hypothèses supplémentaires, comme par exemple la périodicité dans les autres variables. Nous disons que une fonction est périodique en y de période $T = (T_1, \ldots, T_M)$ si, pour tout $(y, z) \in \mathbb{R}^N$,

$$u(y + T_j e_j, z) = u(y, z) \quad \text{pour } 1 \leq j \leq M$$

où $\{e_1, \ldots, e_M\}$ désigne la base standard en $\mathbb{R}^M$.

Théorème 2. *Soit $u > 0$ une solution bornée de l'équation (10), avec f satisfaisant (11). Nous écrivons $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$, et nous supposons que*

(i) u est périodique en y.

(ii) $u(y, z) \to 0$ pour $|z| \to \infty$, uniformément en y.

Alors u est à symétrie radiale en z, c'est-à-dire $u = u(y, |z - z_0|)$, et $u_j(y, |z - z_0|) < 0$ pour $z^j > z_0^j$, $1 \leq j \leq N - M$, pour quelques $z_0 \in \mathbb{R}^{N-M}$.

Remarque 3. *En particulier, dans le cas $M = 0$, ce résultat coïncide avec le Théorème 1 de Gidas, Ni et Nirenberg, dont nous fournissons une démonstration alternative.*

Un cas particulier très étudié est l'équation de Schrödinger

$$-\Delta u + u = |u|^{p-1} u \tag{14}$$

issue de la mécanique quantique mais aussi de la biologie, par exemple dans l'étude des systèmes de réaction-diffusion proposés par Gierer et Meinhardt en 1972 (voir [38, 53]).

En [22], Dancer a démontré que, pour T assez grand, il existe une solution u_T de (60) satisfaisant les propriétés suivantes

- $u_T(x)$ pair et périodique en y avec période T,
- $u_T(x)$ est à symétrie radiale en z,
- $u_T(y, z) \to 0$ exponentiellement pour $|z| \to \infty$, uniformément en y,

où nous avons posé $x = (y, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$.

Le Théorème 2, dans le cas $M = 1$, montre que toute solution qui est pair et périodique en y et décroissante dans les autres variables doit nécessairement être symétrique en z, comme la solution de Dancer.

Après, nous considérons les solutions satisfaisant (13) et

$$\text{pour tout } x_N,\ \nabla_{x'} u(x', x_N) \to 0 \quad \text{pour } |x'| \to \infty. \tag{15}$$

Pour comprendre le comportement de ce type de solutions, il est utile d'étudier le problème (avec $f(0) = 0$)

$$\begin{cases} -v'' = f(v) & \text{on } \mathbb{R} \\ v \geq 0 \\ v(t) \to 0 & \text{as } |t| \to \infty. \end{cases} \tag{16}$$

Pour le Théorème d'unicité de Cauchy, $v > 0$ ou $v \equiv 0$. Nous démontrons que, s'il existe une solution positive de (16), alors elle est unique. Nous verrons qu'il est fondamental de distinguer les cas où cette solution positive existe ou non. Si elle n'existe pas, nous avons un résultat général assez simple, qui assure la symétrie radiale.

Théorème 4. *Soit f une fonction satisfaisant la condition (11) telle que le problème (16) n'admet pas de solution positive. Si $u > 0$ est une solution bornée de*

$$\begin{cases} -\Delta u = f(u) & \text{en } \mathbb{R}^N \\ u(x', x_N) \to 0 & \text{pour } |x_N| \to \infty, \text{ uniformément en } x' \\ \nabla_{x'} u(x', x_N) \to 0 & \text{pour } |x'| \to \infty, \text{ pour tout } x_N, \end{cases} \tag{17}$$

alors u est à symétrie radiale, c'est-à-dire il existe $y \in \mathbb{R}^N$ tel que $u = u(|x - y|)$.

Nous observons que, si $f(t) = 0$ pour tout $0 < t < \delta$, alors le problème (16) n'admet pas de solution positive. En effet, pour t assez grand, toute solution positive qui tend vers zéro doit être affine, c'est-à-dire $v(t) = at + b$, mais $v(t) \to 0$ pour $t \to \infty$, donc $a = b = 0$; pour le Théorème d'unicité de Cauchy, $v \equiv 0$. Donc, dans ce cas, le Théorème 23 est vrai. En outre, pour ce type de nonlinéarités, en dimension $N = 2$, nous avons un résultat de non-existence.

Corollaire 5. *Soit $N = 2$. Soit f une fonction $C^1(\mathbb{R})$ telle que $f(t) = 0$ pour tout $0 < t < \delta$, pour quelques $\delta > 0$. Alors la seule solution bornée $u \geq 0$ de (63) c'est $u \equiv 0$.*

Un exemple bien connu de nonlinéarité de ce type est $f(u) = ((u-\beta)^+)^p$, avec $p > 1$. Dans ce cas, quand $N \geq 3$ et $1 < p < \frac{N+2}{N-2}$, Dupaigne et Farina en [28] ont démontré que la solution à symétrie radiale est unique et ils ont trouvé l'expression explicite

$$u(x) = \begin{cases} \phi_R(|x|) + \beta & \text{pour } |x| \leq R \\ \alpha|x|^{2-N} & \text{pour } |x| \geq R \end{cases}$$

où

$$R = \left(\frac{1}{\beta(N-2)} \int_0^1 \phi_1^p(r) r^{N-1} dr \right)^{(p-1)/2},$$

$\alpha = \beta R^{N-2}$ et ϕ_R est la seule solution à symétrie radiale et décroissante du problème

$$\begin{cases} -\Delta\phi_R = \phi_R^p & \text{pour } |x| \leq R \\ \phi_R = 0 & \text{si } |x| = R \\ \phi_R > 0 & \text{si } |x| < R \\ \frac{\partial \phi_R}{\partial r} < 0 & \text{pour } 0 < |x| \leq R \end{cases}$$

Cet exemple montre que, en dimension $N \geq 3$, le Corollaire 5 n'est pas vrai.

Avec des techniques similaires, nous obtenons une borne inférieure pour la norme L^∞ des solutions de l'équation (14), décroissantes dans une variable et satisfaisant (15). Dans [50], Kwong a démontré qu'il existe une unique solution positive (à une translation près) à symétrie radiale de l'équation (14), que sera dénotée par U. Nous observons que

$$\max U > \left(\frac{p+1}{2} \right)^{\frac{1}{p-1}}.$$

En effet, à une translation près, nous pouvons supposer que $\max U = U(0)$, c'est-à-dire $U(x) = v(|x|)$, où v est une solution de l'équation ordinaire

$$-v'' - \frac{N-1}{r} v'(r) = f(v(r))$$

avec $f(t) = t^p - t$. Multipliant l'équation par v' et intégrant,

$$\frac{d}{dr}\left(\frac{1}{2}(v'(r))^2 + F(v(r)) \right) = -\frac{N-1}{r}(v')^2 < 0$$

pour $r > 0$, avec $F(t) = \frac{1}{p+1}t^{p+1} - \frac{1}{2}t^2$. Donc l'énergie $E(r) = \frac{1}{2}(v^{'}(r))^2 + F(v(r))$ est strictement décroissante et $E(r) \to 0$ pour $r \to \infty$. Donc $E(r) > 0$ pour tout r, en particulier $E(0) = F(U(0)) > 0$, donc $\max U > (p + 1/2)^{1/p-1}$.

Cette observation sera utile pour démontrer la proposition suivante.

Proposition 6. *Soit $u > 0$ une solution bornée de l'équation (14) satisfaisant la condition (13) avec $z = x_N$. Supposons que $\nabla_{x'} u(x^{'}, x_N) \to 0$ pour $|x^{'}| \to \infty$, pour tout x_N. Alors*

$$||u||_\infty \geq \left(\frac{p+1}{2}\right)^{1/p-1}.$$

En tout cas, il y a des nonlinéarités pour lesquelles le problème (16) admet une solution positive, par exemple $f(u) = |u|^{p-1}u - u$. Pour traiter ce cas, nous considérons la fonctionnelle

$$H(u, x^{'}) = \int_{-\infty}^{\infty} \frac{1}{2}\big(u_N^2 - |\nabla_{x'} u|^2\big) - F(u)dx_N$$

et, pour tout $\lambda \in \mathbb{R}$, le moment

$$E_\lambda(u, x^{'}) = \int_{-\infty}^{\infty} (x_N - \lambda)\left(\frac{1}{2}\big(u_N^2 - |\nabla_{x'} u|^2\big) - F(u)\right)dx_N.$$

Dans ces définitions, nous avons posé $F(u) = \int_0^u f(t)dt$, c'est-à-dire la primitive de f qui s'annule en zéro.

Remarque 7. *Si $f^{'}(0) < 0$, la condition (13) avec $z = x_N$ est suffisante pour que l'énergie et le moment soient bien définis, parce que u et ∇u décroissent de façon exponentielle dans x_N,*

$$u(x), |\nabla u(x)| \leq Ce^{-\gamma|x_N|} \quad \textit{pour } |x_N| \geq B, \tag{18}$$

pour quelques constantes $B > 0$, $\gamma > 0$.

Si $f^{'}(0) = 0$, nous avons besoin d'une hypothèse plus forte sur u pour que ces définitions soient bien posées, c'est-à-dire $|H(u, x^{'})|, |E_\lambda(u, x^{'})| < \infty$. Ici, nous imposons que

$$u(x) \leq C|x_N|^{-(1+\sigma)} \quad \text{for } |x_N| > B \tag{19}$$

pour quelques constantes $B > 0$, $\sigma > 0$.

Théorème 8. *Soit $u > 0$ une solution bornée de l'équation (56) satisfaisant la condition (64), avec $f \in C^2(\mathbb{R})$ satisfaisant (57). Supposons que*

(a) Il existe $\overline{x_N} \in \mathbb{R}$ et $\delta > 0$ tels que $u(x', \overline{x_N}) \geq \delta > 0$, pour tout $x' \in \mathbb{R}^{N-1}$.

(b) $\nabla_{x'} u(x', x_N) \to 0$ pour $|x'| \to \infty$, pour tout x_N.

Alors u est symétrique par rapport à x_N, c'est-à-dire $u = u(x', |x_N - \lambda|)$, pour quelques $\lambda \in \mathbb{R}$, et $u_N(x', x_N) > 0$ si $x_N < \lambda$.

Remarque 9. *Dans le Théorème 28, nous pouvons supposer qu'il existe une solution positive du Problème (16), autrement, pour le Théorème 4, il n'y a aucune solution u satisfaisant les hypothèses du Théorème 8.*

Dans le Théorème 8, il serait intéressant d'enlever l'hypothèse (a). Jusqu'à présent, nous avons pu le faire en dimension $N = 2$.

Théorème 10. *Soit $N = 2$. Soit $u > 0$ une solution bornée de l'équation (10) satisfaisant la condition (19), avec $f \in C^2(\mathbb{R})$ satisfaisant (11). Supposons en outre que $u_1(x_1, x_2) \to 0$ pour $|x_1| \to \infty$, pour tout x_2.*

Alors u est symétrique en x_2, c'est-à-dire $u = u(x_1, |x_2 - \lambda|)$, pour quelques $\lambda \in \mathbb{R}$.

Remarque 11. *(i) Si $f'(0) < 0$, (18), les Théorèmes 8 et 10 sont valables même si nous remplaçons la condition (19) par l'hypothèse plus faible (13).*

(ii) En dimension $N = 2$, le Théorème 10 est une extension du Théorème 1.1 de [14] à des nonlinéarités plus générales, parce que nous n'avons pas besoin de prendre $f(u) = u + g(u)$, avec g satisfaisant les hypothèses $(f1)$, $(f2)$ et $(f3)$. De l'autre côté, nous avons besoin d'un peu plus de régularité, nous supposons que $f \in C^2$ au lieu de $C^{1,\beta}$.

Après, nous nous sommes concentrés sur les solutions décroissantes dans $N - 1$ variables

Théorème 12. *Soit $N \geq 5$. Soit $u > 0$ une solution bornée de l'équation (10), avec $f \in C^2(\mathbb{R})$ satisfaisant (57). Supposons que*

$$u(x', x_N) \to 0 \quad \text{pour } |x'| \to \infty, \text{ uniformément en } x_N \tag{20}$$

et

$$\text{pour quelques } x_0^{'},\ u(x_0^{'}, x_N) \to 0 \quad \text{pour } x_N \to \infty\ . \tag{21}$$

Alors u est à symétrie radiale.

Remarque 13. *(i) Nous observons que, si nous supposons $f \in C^1$ avec $f^{'}(0) < 0$, alors, grâce à la décroissance exponentielle, le Théorème 12 est valable en toute dimension $N \geq 2$.*

(ii) Le Théorème 12 est une généralisation du Théorème 1 de Gidas, Ni et Nirenberg.

Les preuves reposent surtout sur la technique des moving planes, qui se base sur les principes de maximum, et sur des techniques intégrales. Nous avons souvent besoin d'une version non-standard du principe du maximum, valable pour des domaines non-bornés aussi, due à Berestycki, Caffarelli et Nirenberg.

Lemme 14 (Principe de Maximum pour des domaines éventuellement non-bornés, [9])**.** *Soit D un domaine (un ouvert connexe) en $\mathbb{R}^N$, éventuellement non-borné. Supposons que $\overline{D}$ est disjoint de la fermeture d'un cône ouvert connecté Σ. Supposons qu'il existe une fonction z en $C(\overline{D})$ bornée au-dessus et satisfaisant*

$$\begin{aligned} \Delta z + c(x)z \geq 0 \quad & \text{en } D \text{ avec } c(x) \leq 0, \\ z \leq 0 \quad & \text{sur } \partial D, \end{aligned}$$

où c est une fonction continue. Alors $z \leq 0$ en D.

Pour une preuve, voir [9], Lemme $2,1$.

- Dans le Chapitre 2 nous considérons la fonctionnelle de Otha-Kawasaki

$$\mathcal{E}_\varepsilon(u) := \frac{1}{2}\int_{T^3} |\nabla u| dx + \varepsilon \int_{T^3}\int_{T^3} G(x,y)(u(x)-m)(u(y)-m)dxdy, \quad (22)$$

u étant une fonction à variation bornée sur le tore T^3 à valeurs dans $\{\pm 1\}$, G étant la fonction de Green de $-\Delta$ sur Ω, c'est-à-dire elle est la solution distributionnelle de

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x) - \frac{1}{|\Omega|} & \text{in } \Omega \\ \partial_{\nu(x)} G(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

On peut montrer que G est la somme de la fonction de Green de $-\Delta$ sur $\mathbb{R}^3$ et une partie régulière,

$$G(x,y) = \frac{c}{|x-y|} + R(x,y),$$

(voir [64]). $\varepsilon \geq 0$ est un paramètre dépendant du matériel, ici nous supposons qu'il est petit.

Cette fonctionnelle est issue de la théorie des copolymères, des molécules complexes où des chaînes de deux types différents de monomères sont greffés ensemble. L'expérience montre que, au-dessus d'une certaine température, ces mélanges se comportent comme des fluides, c'est-à-dire les monomères sont mélangées de façon désordonnée, au contraire, au-dessous de cette température critique on observe une séparation de phase. Des structures périodiques assez communes sont les sphères, les cylindres, les gyroids et les lamellae, qui correspondent à des points critiques de cette fonctionnelle.

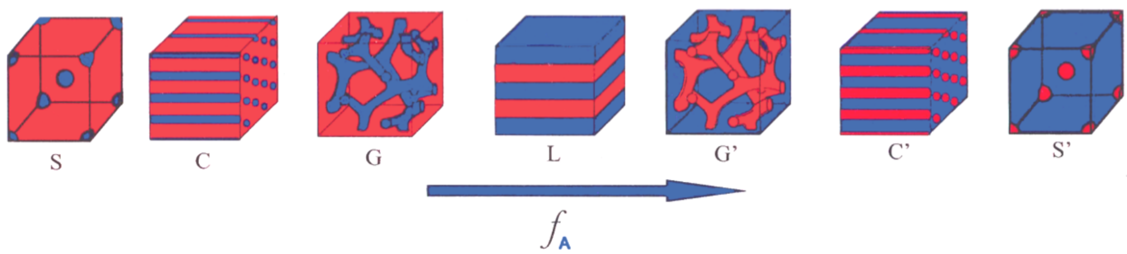

Figure 1: Les structures périodiques plus communes sont sphères, cylindres, gyroids et lamellae

Cette énergie apparaît comme étant la Γ-limite pour $\varepsilon \to 0$ des fonctionnelles approximantes

$$\mathcal{E}_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\varepsilon}\int_\Omega \frac{(1-u^2)^2}{4} dx$$
$$+\frac{16\gamma}{3}\int_\Omega\int_\Omega G(x,y)(u(x)-m)(u(y)-m)dxdy,$$

introduites par Otha et Kawasaki (see [1, 15, 16, 17]).

De façon plus géométrique, notre fonctionnelle est donnée par

$$J_\gamma(E) := P_\Omega(E) + \gamma\int_\Omega\int_\Omega G(x,y)(u_E(x)-m)(u_E(y)-m)dxdy \qquad (23)$$

où

$$E := \{x \in \Omega : u(x) = 1\},$$

de telle façon que $u_E = \chi_E - \chi_{\Omega\setminus E}$. La variation première $J_\gamma^{'}$ est donnée par

$$J_\gamma^{'}(E)[\varphi] = \int_\Sigma (H_\Sigma(x) + 4\gamma v_E(x))\varphi(x)d\sigma(x), \qquad (24)$$

tandis que la variation seconde est

$$J_\gamma^{''}(E)[\varphi] = \int_\Sigma L\varphi(x)\varphi(x)d\sigma(x), \qquad (25)$$

où

$$L\varphi = -\Delta_\Sigma\varphi - |A|^2\varphi + 8\gamma\int_\Sigma G(\cdot\,,y)\varphi(y)d\sigma(y) + 4\gamma\partial_\nu v\varphi. \qquad (26)$$

Ici φ est dans l'éspace

$$W := \left\{w \in H^1(\Sigma) : \int_\Sigma w(x)\nu_i(x)d\sigma(x) = 0, \quad 1 \le i \le 3\right\}, \qquad (27)$$

$\Sigma := \partial E$ et

$$v_E(x) := \int_{T^3} G(x,y)(u_E(y)-m)dy \qquad (28)$$

est la seule solution du problème

$$\begin{cases} -\Delta v_E = u_E - m & \text{dans } T^3 \\ \int_{T^3} v_E dx = 0. \end{cases} \qquad (29)$$

Pour le calcul explicite de la première et de la deuxième variation, voir par exemple [18]. Il est bien connu que la fonctionnelle $\mathcal{E}_\varepsilon$ est invariante par translations [1, 18], c'est-à-dire $J_\gamma(E+\xi) = J_\gamma(E)$, pour tout $\xi \in \Omega = T^3$, donc tout translaté d'un point critique est encore critique (étant donné que les fonctions que nous considérons sont à valeurs dans $\{\pm 1\}$, il est possible d'identifier les points critiques à des ensembles).

Il y a plein de résultats sur le points critiques de cette fonctionnelle. Par exemple, il est intéressant de comprendre si tous les minimiseurs locaux sont périodiques, comme dans les cas précédents (sphères, cylindres, gyroids et lamellae). On sait que cela est vrai en dimension 1 (voir [60]), mais le problème est encore ouvert en dimension plus élevée. D'autres auteurs, comme Ren et Wei [64, 65, 66, 67, 68], ont construit des exemples explicites de minimiseurs périodiques stables, c'est-à-dire avec variation seconde positive. En outre, Acerbi Fusco and Morini [1] ont prouvé que chaque point critique stable est un minimiseur par rapport à des perturbations L^1.

Maintenant, nous ajoutons à notre fonctionnelle une petite perturbation linéaire de la forme

$$\varepsilon \int_{T^3} f(x)u(x)dx,$$

qui brise l'invariance par translation, et nous trouvons au moins quatre points critiques dans un voisinage d'un translaté d'un ensemble fixé, c'est-à-dire la surface P de Schwarz étudiée par M. Ross [72], qui démontre qu'elle est stable par rapport aux variations normales qui préservent le volume. En outre, il est possible de démontrer que le noyau de l'opérateur de Jacobi coïncide exactement avec le sous-espace engendré par les translations.

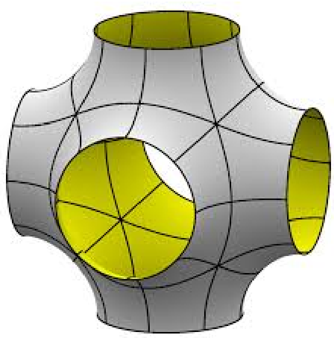

Figure 2: Schwarz' P surface

En effet, si $\nu(x)$ dénote la normale extérieure unitaire à Σ dans x. Puisque I_0 est invariant par translation, $\nu_i(x) := (\nu(x), e_i)$ sont des champs de Jacobi de Σ, c'est-à-dire ils satisfont

$$-\Delta_\Sigma \nu_i - |A|^2 \nu_i = 0 \quad \text{in } \Sigma, \tag{30}$$

(voir [1],[18]). En outre, Grosse-Brauckmann et Wohlgemuth ont montré dans [42] que Σ est nondégénérée à une translation près. Autrement dit

$$Ker(I_0''(E)) = span\{\nu_i\}_{1\leq i\leq 3}. \tag{31}$$

Remark 15. *Nous observons que, a cause de la géométrie de Σ, ν_i sont linéairement indépendants.*

Les ν_i ont moyenne nulle, parce que

$$\int_\Sigma \nu_i(x) d\sigma(x) = \int_{T^3} \text{div} e_i = 0. \tag{32}$$

En plus, nous décomposons $H^1(\Sigma)$ dans la somme directe

$$H^1(\Sigma) = span\{\nu_i\}_{1\leq i\leq 3} \oplus W, \tag{33}$$

(voir (2.23) pour la définition de W), et nous posons

$$W^0 := \left\{ w \in W : \int_\Sigma w(x) d\sigma(x) = 0 \right\}. \tag{34}$$

Cette discussion peut être résumée en disant que

$$\int_\Sigma |\nabla_\Sigma w|^2 - |A|^2 w^2 d\sigma \geq c||w||^2_{H^1(\Sigma)} \quad \text{for any } w \in W^0. \tag{35}$$

Les propriétés du noyau forcent les solutions à satisfaire une contrainte de volume.

Théorème 16. *Soit I_γ la fonctionnelle définie dans (2.8) et $\nu(x)$ la normale extérieure unitaire à la surface P de Schwarz Σ. Alors il existe $\gamma_0 > 0$ tel que, pour tout $0 < \gamma < \gamma_0$, il existe $\xi_j \in T^3$, $1 \leq j \leq 4$, et $w_{\gamma,j} \in C^{2,\alpha}(\overline{\Sigma})$, avec*

$$||w_{\gamma,j}||_{C^{2,\alpha}(\Sigma)} \leq c\gamma, \tag{36}$$

tels que les ensembles F_j définis comme étant l'intérieur de

$$\Gamma_j := \{x + \xi_j + \nu(x) w_{\gamma,j}(x) : x \in \Sigma\} \tag{37}$$

sont des points critiques de I_γ sous la contrainte de volume

$$\mathcal{L}_3(F_j) = \mathcal{L}_3(E). \tag{38}$$

Remarque 17. *(i) On peut choisir* $w_{\gamma,j}$ *de telle façon qu'elles satisfassent les symétries de* Σ*.*

(ii) Si nous prenons $f \equiv 0$*, nous trouvons un seul point critique* F*, c'est-à-dire l'intérieur de*

$$\Gamma := \{x + \nu(x) w_\gamma(x) : x \in \partial E\}, \tag{39}$$

où w_γ *est une petite correction, dans le sens que* $||w_\gamma||_{C^{2,\alpha}(\Sigma)} \leq c\gamma$*, trouvée grâce au théorème de la fonction implicite. Pour l'invariance de la fonctionnelle, tout translaté est encore un point critique. Un résultat pareil a été prouvé par Cristoferi (voir [21], Théorème* 4.18*), qui a construit un point critique de* J_γ *proche de chaque surface à courbure moyenne constante, périodique, régulière et strictement stable.*

(iii) Nous avons énoncé le théorème dans le cas de I_γ *pour la simplicité. La même preuve devrait permettre d'obtenir des résultats de multiplicité même dans le cas de perturbations nonlinéaires régulières et des coefficients différents dans le terme nonlocal.*

- Dans le Chapitre 3 nous nous occupons de la théorie des transitions de phase, en particulier de la construction de solutions entières de l'équation de Cahn-Hilliard

$$-\varepsilon^2\Delta(-\varepsilon^2\Delta u+W^{'}(u))+W^{''}(u)(-\varepsilon^2\Delta u+W^{'}(u))=0 \quad \text{en } \mathbb{R}^3, \tag{40}$$

où $W(u):=\frac{1}{4}(1-u^2)^2$, dont l'ensemble nodal soit proche du tore de Clifford

$$\Sigma:=\Big\{x=(x_1,x_2,x_3)\in\mathbb{R}^3:\Big(\sqrt{2}-\sqrt{x_1^2+x_2^2}\Big)^2+x_3^2-1=0\Big\}. \tag{41}$$

L'équation de Cahn-Hilliard (40) peut être vue comme étant une généralisation à l'ordre quatre de l'équation de Allen-Cahn

$$-\varepsilon^2\Delta u=u-u^3, \tag{42}$$

issue de la théorie des transitions de phase, par exemple l'étude des configurations stables de deux liquides différents mélangés dans un récipient borné Ω. Si $u(x)$ est la densité d'un des deux liquides au point $x\in\Omega$ et l'énergie pour unité de volume est donnée par une fonction W de u, il semble raisonnable d'obtenir les configurations stables en minimisant la fonctionnelle énergie

$$E(u)=\int_\Omega W(u)dx$$

entre toutes les distributions satisfaisant la contrainte de volume

$$\int_\Omega udx=m. \tag{43}$$

Si, par exemple, $W(u)=(1-u^2)^2$, toute fonction affine par morceaux prenant seulement les valeurs ±1 et satisfaisant (43) est un minimiseur, indépendamment de la forme de l'interface. Donc ce modèle n'est pas satisfaisant, parce qu'il est très loin de l'expérience physique qui montre que les interfaces sont des minimiseurs du périmètre, donc nous remplaçons l'énergie par

$$E_\varepsilon(u)=\int_\Omega\big(\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{4\varepsilon}(1-u^2)^2\big)dx.$$

Nous pouvons voir qu'il y a une compétition entre le potentiel qui force la solution à être proche de ±1 et le terme de gradient qui pénalise la transition de phase. Minimisant cette fonctionnelle, nous cherchons les interfaces physiques où la transition de phase se manifeste.

Les minimiseurs u_ε de E_ε sont des solutions de l'équation d'Euler, c'est-à-dire (42). Pour voir si les interfaces sont vraiment des surfaces minimales, il est intéressant d'étudier le comportement asymptotique des ensembles de niveau $\{u_\varepsilon = c\}$ quand le paramètre $\varepsilon \to 0$. Il est utile d'exploiter la nature variationnelle du problème. Modica and Mortola [57] démontrent que cette énergie Γ-converge au périmètre

$$E(u) = \begin{cases} cPer_\Omega(\{u = 1\}) & \text{si } u = \pm 1 \text{ p.p. en } \Omega \\ +\infty & \text{autrement en } L^1(\Omega) \end{cases}$$

pour $\varepsilon \to 0$.

En outre, Modica a démontré que, si u_ε sont minimiseurs de E_ε sous la contrainte de volume

$$\int_\Omega u_\varepsilon dx = m,$$

où $m \in (-1, 1)$ est une constante arbitraire, alors il existe une suite $\varepsilon_k \to 0$ telle que u_{ε_k} converge à une fonction u dans $L^1(\Omega)$ (voir proposition 3 de [56]). En outre, le Théorème 1 de [56] dit que $u = \pm 1$ p. p. dans Ω, et l'ensemble

$$E = \{x \in \Omega : u(x) = 1\}$$

est réellement un minimiseur du périmètre entre parmi tous les ensembles $F \subset \Omega$ satisfaisant la contrainte de volume

$$|F| = \frac{|\Omega| + m}{2}.$$

Il est possible de trouver des résultats similaires sur la rélation entre les minimiseurs de E_ε et les minimiseurs du périmètre dans [56] et dans [19], où Choksi et Sternberg décrivent la relation entre la théorie des transitions de phase et l'étude d'un certain type de polymères.

Au contraire, il est intéressant de se demander si chaque hypersurface minimale peut être obtenue comme étant la limite d'ensembles nodaux de points critiques de l'énergie de Ginzburg-Landau E_ε.

Le premier résultat dans cette direction est du à Kohn et Sternberg (see [48]). Ils considèrent un domaine assez régulier et borné $\Omega \subset \mathbb{R}^2$ et une union disjointe de segments l_i qui coupent le bord $\partial\Omega$ orthogonalement comme étant l'interface. Ils définissent u_0 comme étant constante par morceaux sur $\Omega \setminus \cup_i l_i$, prenant les valeurs ± 1, et ils construisent une suite de minimiseurs u_ε qui converge a u_0 dans $L^1(\Omega)$.

Dans [63], Pacard et Ritoré démontrent un résultat plus général, qui est valide pour une classe plus grande d'interfaces. Ils partent d'une hypersurface minimale Σ dans une variété Riemanienne compacte M et, sous des hypothèses supplémentaires, ils montent qu'elle peut être vue comme étant la limite pour $\varepsilon \to 0$ d'ensembles nodaux de solutions u_ε de l'équation de Allen-Cahn (42). Ces solutions u_ε ont été construites avec des techniques non-variationnelles, comme des théorèmes de point fixé et la réduction de Lyapunov-Schmidt, et elles ne sont pas nécessairement des minimiseurs.

Pour ce qui regarde l'hypersurface Σ, ils imposent certaines restrictions. Avant tout, elle doit être *admissible*, c'est-à-dire l'ensemble nodal d'une fonction régulière $f : M \to \mathbb{R}$, qui divise M en deux régions

$$M^+(\Sigma) = \{p \in M : f(p) > 0\} \quad \text{et } M^-(\Sigma) = \{p \in M : f(p) < 0\}.$$

En outre, Σ doit être *non-dégénérée.* Pour expliquer la notion de non-dégénéréscence, nous donnons la caractérisation variationnelle de hypersurface minimale. Une hypersurface Σ dans une variété Riemannienne M est définie comme étant *minimale* si elle est un point critique de la fonctionnelle périmètre, c'est-à-dire elle satisfait l'équation de courbure moyenne nulle $H = 0$, où

$$H = k_1 + \cdots + k_{N-1},$$

dénote la courbure moyenne, et les k_j sont les courbures principales.

La variation seconde du périmètre est donnée par

$$A''(\Sigma)[\phi, \psi] = \int_\Sigma L_0\phi(y)\psi(y)d\sigma(y),$$

où l'opérateur auto-adjoint

$$L_0\phi = -\Delta_\Sigma\phi - |A|^2\phi$$

est appelé l'*opérateur de Jacobi* de Σ et

$$|A|^2 = k_1^2 + \cdots + k_{N-1}^2$$

est la norme au carré de sa seconde forme fondamentale. Une hypersurface minimale Σ est définie comme étant non-dégénérée si son opérateur de Jacobi

$$L_0 : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$$

est un isomorphisme. Pour une introduction à ces sujets, on peut voir [26] aussi.

En outre, les résultats de [63] sont valides même si le potentiel $W(t) = (1-t^2)^2/4$ est remplacé par un potentiel a double puis plus général, c'est-à-dire une fonction régulière W telle que

$$\begin{cases} W(t) \geq 0 & \text{pour tout } t, \\ W(t) = 0 & \text{si et seulement si } t = \pm 1, \\ W''(\pm 1) > 0. \end{cases} \tag{44}$$

Pour résumer, ils prouvent le Théorème suivant

Theorem 18 ([63]). *Soit W comme dans (3.3). Soit Σ une hypersurface minimale non-dégénérée dans une variété compacte M. Alors il existe ε_0 tel que, pour tout $0 < \varepsilon < \varepsilon_0$, il existe une solution u_ε de l'équation de Allen-Cahn*

$$-\varepsilon^2 \Delta u_\varepsilon + W'(u_\varepsilon) = 0$$

telle que $u_\varepsilon \to \pm 1$ sur les compactes de $M^\pm(\Sigma)$.

En tout cas, malgré beaucoup de résultats montrent une certaine similarité entre les ensembles nodaux des solutions de l'équation de Allen-Cahn et les surfaces minimales, il y a aussi des solutions dont l'ensemble nodal est loin d'être minimal. Par exemple, Agudelo, Del Pino and Wei ont construit des solutions à symétrie axiale in $\mathbb{R}^3$, c'est-à-dire des solutions de la forme $u = u(|x'|, x_3)$, dont les composantes de l'ensemble nodal ressemblent à une caténoïde pour $|x'|$ assez grand.

La réduction de Lyapunov-Schmidt a été exploitée dans le cas non-compact aussi, pour construire des solutions entières de l'équation de Allen-Cahn dans $\mathbb{R}^9$ qui sont monotones dans une variable mais non monodimensionnelles, parce que l'ensemble nodal est proche du graphe de Bombieri-De Giorgi-Giusti, qui est un graph minimal sur $\mathbb{R}^8$ qui n'est pas affine (voir [11, 24]). Ces solutions sont liées à une célèbre conjecture de De Giorgi, qui dit que, au moins pour $n \leq 8$, chaque solution $|u| < 1$ de l'équation de Allen-Cahn

$$-\Delta u = u - u^3$$

satisfaisant $\partial_N u > 0$ dans $\mathbb{R}^N$ doit être monodimensionnelle, c'est-à-dire elle doit dépendre d'une seule variable euclidienne, ou, autrement dit, $u(x) = u(< a, x >)$, où $a \in S^{N-1}$ est une direction fixée. Le résultat de Del Pino, Kowalczyk et Wei montre que la borne sur la dimension dans la conjecture de Giorgi est optimale. Jusqu'à présent, on sait que la conjecture est vraie en dimension $N = 2$ (voir [36],[31]) et $N = 3$ (voir [7],[31]). La conjecture est encore ouverte en dimension $4 \leq N \leq 8$. Un résultat très intéressant a été obtenu par Savin, (voir [73]), qui a prouvé que la conjecture est vraie en dimension $4 \leq N \leq 8$, à condition que

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1, \quad \forall x' \in \mathbb{R}^{N-1},$$

qui implique que ces solutions sont des minimiseurs locaux de l'énergie

$$\int_{\mathbb{R}^3} \Big(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\Big)dx.$$

Les résultats les plus généraux sur la validité de la conjecture peuvent être trouvés dans [32, 33].

Dans la thèse, nous démontrons des résultats similaires pour l'équation de Cahn-Hilliard. Il est possible de vérifier que (40) est l'équation de Euler de l'énergie

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{2\varepsilon}\int_\Omega \big(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\big)^2 dx & \text{si } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{autrement en } L^1(\Omega). \end{cases}$$

Il y a des résultats de Γ-convergence qui relient cette énergie à la fonctionnelle de Willmore

$$\mathcal{W}(\Sigma) = c\int_\Sigma H_\Sigma^2(y)d\sigma,$$

où H_Σ est la courbure moyenne de Σ. Dans [8] Bellettini et Paolini ont démontré que l'inégalité $\Gamma - \limsup$ pour les surfaces de Willmore régulières, tandis que l'inégalité $\Gamma - \liminf$ est beaucoup plus difficile à prouver. Jusqu'à présent elle a été démontrée par Röger et Schätzle en [71], et, indépendamment, en dimension $N = 2$, par Nagase et Tonegawa dans [61]. Le problème est encore ouvert en dimension plus élevée, par contre il est connu que l'approximation n'est pas valide pour les ensembles non-réguliers, même en dimension $N = 2$.

Cette fonctionnelle est étudiée, par exemple, dans le contexte de la relativité générale, étant liée à la masse de Hawking

$$m_H(\Sigma) = \sqrt{\frac{Area(\Sigma)}{16\pi}}\Big(1 - \frac{1}{16\pi}\mathcal{W}(\Sigma)\Big).$$

Ici Σ peut être interprétée comme la surface d'un objet dont la masse doit être mesurée. En outre, cette fonctionnelle apparaît en biologie aussi, sous le nom de *énergie de Helfrich*, et elle est utilisée pour décrire le comportement des membranes des cellulaires. Pour plus d'informations et de références, nous conseillons de voir [51, 44, 45].

Nous appelons *Surfaces de Willmore* les points critiques de cette fonctionnelle, qui sont caractérisés par l'équation d'Euler

$$-\Delta_\Sigma H + \frac{1}{2} H(H^2 - 2|A|^2) = 0.$$

Vus les résultats de Γ-convergence, il est raisonnable de se demander si on peut construire une famille de solutions u_ε de l'équation de Cahn-Hilliard (40) dont l'ensemble nodal soit proche d'une surface de Willmore prescrite. Dans notre cas, on part du tore de Clifford (41), dont nous exploitons les propriétés de non-dégénérescence. La variation seconde de la fonctionnelle de Willmore est donnée par

$$\mathcal{W}''(\Sigma)[\phi,\psi] = \int_\Sigma \tilde{L}_0 \phi(y)\psi(y) d\sigma(y)$$

où $\tilde{L}_0$ est l'opérateur auto-adjoint

$$\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2 L_0\phi - H(\nabla_\Sigma\phi, \nabla_\Sigma H) + 2(A\nabla_\Sigma\phi, \nabla_\Sigma H) + 2H < A, \nabla^2\phi > + \phi(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2 + 2H \mathrm{tr} A^3). \qquad (45)$$

La démonstration repose sur le fait que le noyau de cet opérateur coïncide exactement avec le sous-espace engendré par les transformations conformes, c'est-à-dire les isométries, les dilatations et les inversions par rapport à des sphères. En effet, d'un côté, pour un résultat de White [78], la fonctionnelle de Willmore est invariante par transformations conformes, de l'autre, pour le Corollaire 2, page 34, de Weiner [77], on sait que la variation seconde est positive définie sur l'orthogonal à ces transformations.

Vu que la fonctionnelle de Willmore est invariante par dilatations, nous cherchons des solutions satisfaisant la contrainte de volume

$$\int_{\mathbb{R}^3} (1 - u_\varepsilon) dx = 4\sqrt{2}\pi^2.$$

Pour les autres invariances, nous imposons les autres symétries. La démonstration repose sur la réduction de Lyapunov-Schmidt et sur des expansions géométriques du laplacien.

Théorème 19. *Soit W un potentiel double puis satisfaisant (44). Soit Σ le Tore de Clifford. Alors il existe une solution u_ε de (40) satisfaisant la contrainte de volume*

$$\int_{\mathbb{R}^3} (1 - u_\varepsilon) dx = 4\sqrt{2}\pi^2, \tag{46}$$

avec $u_\varepsilon \to \pm 1$ et $\partial_k u_\varepsilon \to 0$ uniformement sur les compacts de $\Sigma^\pm$, pour $1 \le k \le 4$. En plus, $u_\varepsilon(x_1, x_2, x_3) = u_\varepsilon(x_1, x_2, -x_3)$ et $u_\varepsilon(x) = u_\varepsilon(Rx)$, pour tout $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ et pour tout rotation $R \in SO(3)$ telle que $R(0, 0, 1) = (0, 0, 1)$.

où dans le théorème ci-dessus, nous avons posé

$$\Sigma^+ = \{x \in \mathbb{R}^3 : f(x) > 0\} \quad \text{and } \Sigma^- = \{x \in \mathbb{R}^3 : f(x) < 0\}.$$

Introduzione

Questa tesi, che unisce i risultati ottenuti durante il mio dottorato [30, 70, 69], è dedicata allo studio di alcune equazioni alle derivate parziali semilineari. Da una parte studiamo alcune proprietà qualitative come la simmetria delle soluzioni, dall'altra costruiamo esplicitamente delle soluzioni che si annullano vicino a una varietà data. Questo riassunto contiene solo un breve resoconto del lavoro. Avendo trattato due tipi di problemi diversi, l'opera è suddivisa in due parti. La prima è dedicata allo studio di proprietà qualitative di soluzioni date di certe equazioni, la seconda riguarda la costruzione di soluzioni. Ogni parte sarà dotato della propria introduzione, in cui spieghiamo in modo più dettagliato le idee fondamentali e le tecniche usate per risolvere i nostri problemi.

- Nel capitolo 1, studiamo l'equazione

  $$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N, \tag{47}$$

  dove f è una funzione abbastanza regolare che soddisfa

  $$f'(s) \leq 0 \quad \text{per } s \in (0, \varepsilon), \text{ per qualche } \varepsilon > 0, \text{ e } f(0) = 0. \tag{48}$$

  Un caso particulare ampiamente studiato è $f(u) = u|u|^{p-1} - u$, cioè la nonlinearità che compare nell'equazione di Schrödinger.

  Consideriamo soluzioni che decadono in certe variabili e ci chiediamo se siano simmetriche ripetto a certe variabili. Vedremo che questo è vero sotto ulteriori ipotesi, come ad esempio la periodicità nelle altre variabili o il decadimento del gradiente. Considerando soluzioni che decadono in tutte le variabili, otteniamo il famoso teorema di Gidas, Ni e Nirenberg, di cui diamo una dimostrazione alternativa. Le dimostrazioni usano soprattutto la tecnica dei moving planes, che si basa sui principi di massimo, e su tecniche integrali.

- Nel capitolo 2 consideriamo il funzionale di Otha-Kawasaki

  $$\mathcal{E}_\varepsilon(u) := \frac{1}{2}\int_{T^3} |\nabla u| dx + \varepsilon \int_{T^3}\int_{T^3} G(x,y)(u(x)-m)(u(y)-m)dxdy, \tag{49}$$

  dove u è una funzione a variazione limitata sul toro T^3 a valori in $\{\pm 1\}$, G è la funzione di Green di $-\Delta$ su Ω e $\varepsilon > 0$ è piccolo. Questo funzionale è legato alla teoria dei copolimeri. Si sa che alcuni punti critici di questo funzionale sono le funzioni u la cui interfaccia ha una struttura periodica data da sfere, cilindri, giroidi e lamelle. Inoltre è noto che il funzionale $\mathcal{E}_\varepsilon$ è invariante per traslazioni [1, 18], quindi ogni traslato di un punto critico è ancora critico (poiché le funzioni che consideriamo sono a valori in $\{\pm 1\}$, i punti critici si possono identificare a insiemi). Ora aggiungiamo al nostro funzionale una piccola perturbatione lineare della forma

$$\varepsilon \int_{T^3} f(x)u(x)dx,$$

che rompe l'invarianza per traslazione e troviamo almeno quattro punti critici in un intorno di un traslato di un insieme fissato, cioè la superficie P di Schwarz, studiata da M. Ross [72], che dimostra che è stabile rispetto alle variazioni normali che preservano il volume. Inoltre, si può dimostrare che il nucleo dell'operatore di Jacobi coincide esattamente con il sottospazio generato dalle traslazioni [1, 18, 42]. In virtù delle proprietà del nucleo, le soluzioni devono soddisfare un vincolo di volume.

- Nel capitolo 3 ci occupiamo della teoria delle transizioni di fase, in particolare della costruzione di soluzioni intere dell'equazione di Cahn-Hilliard

$$-\varepsilon^2\Delta(-\varepsilon^2\Delta u + W^{'}(u)) + W^{''}(u)(-\varepsilon^2\Delta u + W^{'}(u)) = 0 \quad \text{in } \mathbb{R}^3, \tag{50}$$

dove $W(u) := \frac{1}{4}(1-u^2)^2$, il cui insieme nodale sia vicino al toro di Clifford

$$\Sigma := \Big\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \Big(\sqrt{2} - \sqrt{x_1^2 + x_2^2}\Big)^2 + x_3^2 - 1 = 0 \Big\}. \tag{51}$$

L'equazione di Cahn-Hilliard (50) può essere vista come una generalizzazione all'ordine quattro dell'equazione di Allen-Cahn

$$-\varepsilon^2\Delta u = u - u^3, \tag{52}$$

legata alla teoria delle transizioni di fase. Se consideriamo (52) su un dominio limitato $\Omega \subset \mathbb{R}^N$, ad esempio, si sa che è l'equazione di Eulero dell'energia

$$E_\varepsilon(u) = \int_\Omega \big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2\big)dx.$$

Modica and Mortola [57] dimostrano che questa energia Γ-converge al perimetro

$$E(u) = \begin{cases} cPer_\Omega(\{u=1\}) & \text{se } u = \pm 1 \text{ q.o. in } \Omega \\ +\infty & \text{altrimenti in } L^1(\Omega) \end{cases}$$

per $\varepsilon \to 0$. L'equazione di Eulero del perimetro è $H = 0$, dove H è definita come la curvatura media della varietà $\partial\{x \in \Omega : u(x) = 1\}$, cioè $H := k_1 + \cdots + k_{N-1}$, e le k_i sono le curvature scalari. Poiché la Γ-convergenza implica la convergenza dei minimizzatori, se consideriamo una famiglia u_ε di minimizzatori dell'energia (53), l'insieme di livello zero si avvicina a una superficie minima Σ, cioè una superficie che soddisfa $H = 0$. Pacard e Ritoré provano una sorta di viceversa, cioè partono da una superficie minima Σ fissata in una varietà Riemanniana M e costruiscono una famiglia di soluzioni u_ε il cui insieme nodale si avvicina a Σ quando $\varepsilon \to 0$, purché Σ sia nondegenere, nel senso che l'operatore di Jacobi

$$L_0\phi = -\Delta_\Sigma\phi - |A|^2\phi - Ric_g(\nu_\Sigma, \nu_\Sigma) \tag{53}$$

che rappresenta la variazione seconda del perimetro, data da

$$E''(\Sigma)(\phi, \psi) = \int_\Sigma \tilde{L}_0\phi\psi d\sigma, \tag{54}$$

deve essere invertibile. Sopra, $|A|^2 := k_1^2 + \cdots + k_{N-1}^2$ è la norma al quadrato della seconda forma fondamentale, Ric_g è il tensore di Ricci di M e ν_Σ è una normale unitaria a Σ.

Nella tesi, ci sono risultati simili per l'equazione di Cahn-Hilliard. Si può vedere che (50) è l'equazione di Eulero dell'energia

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{2\varepsilon}\int_\Omega \left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^2 dx & \text{se } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{altrimenti in } L^1(\Omega). \end{cases}$$

Ci sono risultati di Γ-convergenza che mettono in relazione questa energia al funzionale di Willmore

$$\mathcal{W}(\Sigma) = c\int_\Sigma H_\Sigma^2(y) d\sigma,$$

dove H_Σ è la curvatura media di Σ (per esempio [8, 71, 61]). Questo funzionale è studiato, per esempio, nel contesto della relatività generale, essendo legata alla massa di Hawking

$$m_H(\Sigma) = \sqrt{\frac{Area(\Sigma)}{16\pi}}(1 - \frac{1}{16\pi}\mathcal{W}(\Sigma)).$$

Chiamiamo *Superfici di Willmore* i punti critici di questo funzionale, che sono caratterizzati dall'equazione di Eulero

$$-\Delta_\Sigma H + \frac{1}{2}H(H^2 - 2|A|^2) = 0.$$

Di conseguenza, è ragionevole chiedersi se possiamo costruire una famiglia di soluzioni u_ε dell'equazione di Cahn-Hilliard (50) ilcui insieme nodale sia vicino a una superficie di Willmore fissata. Nel nostro caso, partiamo dal Toro di Clifford (51), di cui sfruttiamo le proprietà di nondegenerazione. Infatti le dimostrazione si basa sul fatto che il nucleo dell'operatore autoaggiunto

$$\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2L_0\phi - H(\nabla_\Sigma\phi, \nabla_\Sigma H) + 2(A\nabla_\Sigma\phi, \nabla_\Sigma H) + \\ 2H < A, \nabla^2\phi > +\phi(2 < A, \nabla^2 H > +|\nabla_\Sigma H|^2 + 2H\mathrm{tr}A^3) \tag{55}$$

che compare nella variazione seconda del funzionale di Willmore coincide esattamente con il sottospazio generato dalle trasformazioni conformi, cioè le isometrie, le dilatazioni e le inversioni rispetto a sfere. Per l'invarianza per dilatazioni del funzionale di Willmore, imponiamo che le soluzioni soddisfino il vincolo di volume

$$\int_{\mathbb{R}^3}(1 - u_\varepsilon)dx = 4\sqrt{2}\pi^2.$$

In virtù delle altre invarianze, imponiamo che le soluzioni rispettare le simmetrie del toro. La dimostrazione si basa sulla riduzione di Lyapunov-Schmidt.

Part I

Qualitative properties of solutions to some PDEs

Introduction and main results

In this part of my thesis we consider positive bounded solutions to the equation

$$-\Delta u = f(u) \tag{56}$$

on $\mathbb{R}^N$. The nonlinearity will always be a C^1 function decreasing in a right neighborhood of the origin, that is

$$f^{'}(s) \leq 0 \quad \text{for } s \in (0, \varepsilon), \text{ for some } \varepsilon > 0, \text{ and } f(0) = 0. \tag{57}$$

The aim is to establish some symmetry results. In [37] Gidas, Ni and Nirenberg proved the following theorem.

Theorem 20. *[37] Let $u > 0$ be a solution to equation (56), with f satisfying condition (57). Assume furthermore that*

$$u(x) \to 0 \quad as\ |x| \to \infty. \tag{58}$$

Then, up to a translation, u is radially symmetric and decreasing to 0, that is $u = u(|x|)$, with $\partial u/\partial r(x) < 0$, for any $x \neq 0$.

The main problem we are concerned with is the following: if we replace the decay hypothesis (58) by the weaker assumptions that u is bounded and satisfies

$$u(y, z) \to 0 \quad \text{as } |z| \to \infty, \text{ uniformly in } y, \tag{59}$$

where we have set $x = (y, z)$, with $y \in \mathbb{R}^M$, $z \in \mathbb{R}^{N-M}$, is it true that u is radially symmetric in z, that is $u = u(y, |z - z^0|)$, for some z^0, with $u_j(y, z) < 0$ for $z_j > z_j^0$, $1 \leq j \leq N - M$, where we have set $u_j = \partial u/\partial z_j$?

In the sequel, we will give some sufficient conditions for this to be true. An example of sufficient condition to get symmetry is periodicity in the y variables.

We say that a function $u : \mathbb{R}^N \to \mathbb{R}$ is periodic in y of period $T = (T_1, \dots, T_M)$ if, for any $(y, z) \in \mathbb{R}^N$,

$$u(y + T_j e_j, z) = u(y, z) \quad \text{for } 1 \leq j \leq M$$

where $\{e_1, \dots, e_M\}$ denotes the standard basis in $\mathbb{R}^M$.

Theorem 21. *Let $u > 0$ be a bounded solution to equation (56), with f satisfying (57). Let us write $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$, and assume that*

(i) u is periodic in y

(ii) $u(y, z) \to 0$ as $|z| \to \infty$, uniformly in y.

Then u is radially symmetric and decreasing with respect to z, that is $u = u(y, |z - z_0|)$, and $u_j(y, |z - z_0|) < 0$ for $z^j > z_0^j$, $1 \leq j \leq N - M$, for some $z_0 \in \mathbb{R}^{N-M}$.

Remark 22. *In particular, in the case $M = 0$, this result reduces to Theorem 20 by Gidas, Ni and Nirenberg, of which we give an alternative proof.*

An interesting case is represented by the semilinear equation

$$-\Delta u + u = u^p \tag{60}$$

with $1 < p < \frac{N+1}{N-3}$ if $N > 3$ and $p > 1$ if $2 \leq N \leq 3$. This equation arises naturally in several scientific contexts, such as, for example the nonlinear-Schrodinger equation in quantum mechanics but also biology, for instance in the study of the reaction-diffusion system proposed by Gierer and Meinhardt in 1972. For further information, we refer to the papers [38, 53].

Dancer in [22] showed that, for sufficiently large T, there exists a solution u_T to (60) fulfilling the following properties:

- $u_T(x)$ is even and periodic in y with period T,
- $u_T(x)$ is radially symmetric in z,
- $u_T(y, z) \to 0$ exponentially fast as $|z| \to \infty$, uniformly in y,

where we have set $x = (y, z) \in \mathbb{R} \times \mathbb{R}^{N-1}$.

Theorem 21, in the case $M = 1$, shows that any solution which is even and periodic in y and decays in the other variables has to be symmetric in z, like Dancer' s solution. These results with periodicity will be proved in Sections 1.1 and 1.2.

After that, we will consider solutions fulfilling (59) and

$$\text{for any } x_N,\ \nabla_{x'} u(x', x_N) \to 0 \quad \text{as } |x'| \to \infty. \tag{61}$$

In order to investigate the behaviour of this kind of solutions, it is useful to study the problem (with $f(0) = 0$)

$$\begin{cases} -v'' = f(v) & \text{on } \mathbb{R} \\ v \geq 0 & \\ v(t) \to 0 & \text{as } |t| \to \infty. \end{cases} \tag{62}$$

By the Cauchy uniqueness Theorem, either $v > 0$ or $v \equiv 0$. We will show that, if there exists a positive solution to (62), then it is unique. It turns out that it is worth distinguishing the cases in which such a positive solution exists or not.

Theorem 23. *Let f be a function fulfilling condition (57) such that problem (62) admits no positive solution. If $u > 0$ is a bounded solution to*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \\ u(x', x_N) \to 0 & \text{as } |x_N| \to \infty, \text{ uniformly in } x' \\ \nabla_{x'} u(x', x_N) \to 0 & \text{as } |x'| \to \infty, \text{ for any } x_N, \end{cases} \tag{63}$$

then u is radially symmetric, that is $u = u(|x-y|)$, for an appropriate $y \in \mathbb{R}^N$.

We observe that, if $f(t) = 0$ for any $0 < t < \delta$, for some $\delta > 0$, then problem (62) has no positive solution. In fact, for t large enough, v has to be affine, that is $v(t) = at + b$, but $v(t) \to 0$ as $t \to \infty$, hence $a = b = 0$; by the Cauchy uniqueness theorem, $v \equiv 0$. As a consequence, in this case, Theorem 23 holds true. For this kind of nonlinearities, in dimension $N = 2$, we can get a non-existence result.

Corollary 24. *Let $N = 2$. Let f be a $C^1(\mathbb{R})$ function such that $f(t) = 0$ for any $0 < t < \delta$, for a suitable $\delta > 0$. Then the only bounded solution $u \geq 0$ to (63) is $u \equiv 0$.*

A relevant example of nonlinearity of this type is $f(u) = ((u - \beta)^+)^p$, with $p > 1$. In this case, when $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$, L. Dupaigne and A. Farina in [28] showed that the radially symmetric solution is unique and found the explicit expression

$$u(x) = \begin{cases} \phi_R(|x|) + \beta & \text{for } |x| \leq R \\ \alpha |x|^{2-N} & \text{for } |x| \geq R \end{cases}$$

where

$$R = \left(\frac{1}{\beta(N-2)} \int_0^1 \phi_1^p(r) r^{N-1} dr \right)^{(p-1)/2},$$

$\alpha = \beta R^{N-2}$ and ϕ_R is the unique radially symmetric and radially decreasing solution to the problem

$$\begin{cases} -\Delta \phi_R = \phi_R^p & \text{for } |x| \leq R \\ \phi_R = 0 & \text{on } |x| = R \\ \phi_R > 0 & \text{in } |x| < R \\ \frac{\partial \phi_R}{\partial r} < 0 & \text{for } 0 < |x| \leq R \end{cases}$$

This example shows that, in dimension $N \geq 3$, Corollary 24 is not true.

With similar techniques, we obtain a lower bound for the L^∞-norm of nontrivial solutions to equation (60), decaying in one variable and fulfilling (61). In [50], Kwong showed that there exists a unique (up to a translation) positive radially symmetric solution to equation (60), that we will denote by U. We observe that

$$\max U > \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}.$$

In fact, up to a translation, we can assume that $\max U = U(0)$, that is $U(x) = v(|x|)$, where v is a solution to the ODE

$$-v'' - \frac{N-1}{r}v'(r) = f(v(r))$$

with $f(t) = t^p - t$. Multiplying the equation by v' and integrating, we get

$$\frac{d}{dr}\left(\frac{1}{2}(v'(r))^2 + F(v(r))\right) = -\frac{N-1}{r}(v')^2 < 0$$

for $r > 0$, with $F(t) = \frac{1}{p+1}t^{p+1} - \frac{1}{2}t^2$. So the energy $E(r) = \frac{1}{2}(v'(r))^2 + F(v(r))$ is strictly decreasing and $E(r) \to 0$ as $r \to \infty$. Therefore $E(r) > 0$ for any r, in particular $E(0) = F(U(0)) > 0$, hence $\max U > (p+1/2)^{1/p-1}$.

This remark will be useful to prove the following proposition.

Proposition 25. *Let $u > 0$ be a bounded solution to equation (60) satisfying condition (59) with $z = x_N$. Assume that $\nabla_{x'} u(x', x_N) \to 0$ as $|x'| \to \infty$, for any x_N. Then*

$$||u||_\infty \geq \left(\frac{p+1}{2}\right)^{1/p-1}.$$

Anyway, there are examples of nonlinearities for which problem (62) admits a positive solution, such as $f(u) = |u|^{p-1}u - u$. In order to deal with this case, we consider the energy-like functional

$$H(u, x') = \int_{-\infty}^{\infty} \frac{1}{2}\left(u_N^2 - |\nabla_{x'} u|^2\right) - F(u) dx_N$$

and, for any $\lambda \in \mathbb{R}$, the momentum

$$E_\lambda(u, x') = \int_{-\infty}^{\infty} (x_N - \lambda)\left(\frac{1}{2}\left(u_N^2 - |\nabla_{x'} u|^2\right) - F(u)\right) dx_N.$$

In the above definitions, we have denoted by $F(u) = \int_0^u f(t)dt$, the primitive of f vanishing at the origin.

Remark 26. *In view of Lemma 42, condition (59) is sufficient for the energy and the momentum to be well defined and finite if* $f'(0) < 0$.

In the proof of Lemma 42, we used a non-standard version of the maximum principle for possibly unbounded domains due to Berestycki, Caffarelli and Nirenberg, that will be exploited many times troughout this chapter, thus we think it is worth to recall it.

Lemma 27 (Maximum principle for possibly unbounded domains, [9])**.** *Let D be a domain (open connected set) in $\mathbb{R}^N$, possibly unbounded. Assume that $\overline{D}$ is disjoint from the closure of an infinite open connected cone Σ. Suppose there is a function z in $C(\overline{D})$ that is bounded above and satisfies for some continuous function $c(x)$*

$$\begin{aligned} \Delta z + c(x)z \geq 0 \quad & \text{in } D \text{ with } c(x) \leq 0, \\ z \leq 0 \quad & \text{on } \partial D, \end{aligned}$$

Then $z \leq 0$ in D.

For the proof, see [9], Lemma 2.1.

If $f^{'}(0) = 0$, we need some further assumptions about u in order for these definitions to be well posed, that is $|H(u, x^{'})|, |E_\lambda(u, x^{'})| < \infty$. In this context, we require

$$u(x) \leq C|x_N|^{-(1+\sigma)} \quad \text{for } |x_N| > B \tag{64}$$

for suitable constants $B > 0$, $\sigma > 0$. We will show in Section 1.3 that this condition is sufficient for $H(u, x^{'})$ and $E_\lambda(u, x^{'})$ to be well defined and finite, provided $f \in C^2(\mathbb{R}^N)$.

Theorem 28. *Let $u > 0$ be a bounded solution to equation (56) satisfying condition (64), with $f \in C^2(\mathbb{R})$ satisfying (57). Assume furthermore that*

(a) There exists $\overline{x_N} \in \mathbb{R}$ and $\delta > 0$ such that $u(x^{'}, \overline{x_N}) \geq \delta > 0$, for any $x^{'} \in \mathbb{R}^{N-1}$.

(b) $\nabla_{x^{'}} u(x^{'}, x_N) \to 0$ as $|x^{'}| \to \infty$, for any x_N.

Then u is symmetric in x_N, that is $u = u(x^{'}, |x_N - \lambda|)$, for some $\lambda \in \mathbb{R}$, and $u_N(x^{'}, x_N) > 0$ if $x_N < \lambda$.

Remark 29. *In Theorem 28, we can assume that there exists a positive solution to Problem (62), otherwise, by Theorem 23, there are no solutions u fulfilling hypothesis of Theorem 28.*

Section 1.3 will be devoted to the proof of this theorem, that holds true in any dimension $N \geq 2$. In Theorem 28, we would like to be able to drop assumption (a). Up to now, we have been able to do so only in dimension $N = 2$.

Theorem 30. *Let $N = 2$. Let $u > 0$ be a bounded solution to equation (56) satisfying condition (64), with $f \in C^2(\mathbb{R})$ satisfying (57). Assume furthermore that $u_1(x_1, x_2) \to 0$ as $|x_1| \to \infty$, for any x_2.*

Then u is symmetric in x_2, that is $u = u(x_1, |x_2 - \lambda|)$, for some $\lambda \in \mathbb{R}$.

Remark 31. *If $f^{'}(0) < 0$, thanks to Lemma 42, Theorems 28 and 30 hold true even if we replace condition (64) with the weaker assumption (59).*

Remark 32. *In dimension $N = 2$, Theorem 30 is a extension to Theorem 1.1 of [14] to more general nonlinearities, since we do not need to take $f(u) = u + g(u)$, with g satisfying their assumptions $(f1)$, $(f2)$ and $(f3)$. On the other hand, we need some more regularity, we take $f \in C^2$ instead of $C^{1,\beta}$.*

Unfortunately, if f is flat near the origin, condition (59) does not necessarily imply (64), at least in dimension $N \geq 3$. In fact, the solution constructed by L. Dupaigne and A. Farina in [28] in dimension $N = 3$ decays as $|x|^{-1}$ (see the above discussion for the explicit expression). This function, seen as a solution in higher dimension, is a counter-example in dimension $N \geq 4$ too.

In Section 1.5, we consider solutions to (56) decaying in $N - 1$ variables, and we prove an extension of Theorem 20.

Theorem 33. *Let $N \geq 5$. Let $u > 0$ be a bounded solution to equation (56), with $f \in C^2(\mathbb{R})$ satisfying (57). Assume that*

$$u(x^{'}, x_N) \to 0 \quad \text{as } |x^{'}| \to \infty, \text{ uniformly in } x_N \tag{65}$$

and

$$\text{for some } x_0^{'},\ u(x_0^{'}, x_N) \to 0 \quad \text{as } x_N \to \infty\ . \tag{66}$$

Then u is radially symmetric.

Remark 34. *We observe that, if we assume $f \in C^1$ with $f^{'}(0) < 0$, then, thanks to the exponential decay proved in Lemma 42, Theorem 33 holds true in any dimension $N \geq 2$.*

Remark 35. *In dimension $2 \leq N \leq 4$, Theorem 33 holds true under the assumption*

$$u(x), |\nabla u(x)| \leq C|x^{'}|^{-\frac{N-1+\sigma}{2}} \quad \text{for } |x^{'}| \geq B \tag{67}$$

for suitable constants $B > 0$, $\sigma > 0$ and $f \in C^1$.

In order to deal with the case $f'(0) = 0$, we study the decay rate at infinity of functions fulfilling (65). This will be carried out in section 1.5.

Chapter 1

Decaying solutions to some PDEs

1.1 Starting the moving plane procedure

First we define, for $\lambda \in \mathbb{R}$, $u_\lambda(x) = u(x', 2\lambda - x_N)$, $\Sigma_\lambda = \{x_N < \lambda\}$. In the following proposition, we prove that the moving plane procedure can be started. In order to do so, it is enough to replace condition (59) with the weaker assumption

$$u(x', x_N) \leq \varepsilon \quad \text{in the subspace } \{x_N > \lambda_0\} \tag{1.1}$$

for a suitable $\lambda_0 \in \mathbb{R}$, if f is nonincreasing in the interval $(0, \varepsilon)$.

Proposition 36 (Initiation)**.** *Let $u > 0$ be a bounded solution to equation (56) fulfilling (1.1). Assume that f satisfies (57). Then $u - u_\lambda \geq 0$ in Σ_λ, for any $\lambda \geq \lambda_0$.*

Remark 37. *In particular, this proposition holds true if we assume that*

$$u(x', x_N) \to 0 \quad \textit{as } x_N \to \infty \textit{ uniformly in } x'.$$

Proof. We assume by contradiction that it is possible to find $\lambda \geq \lambda_0$ such that the open set $\Omega_\lambda = \{u - u_\lambda < 0\} \cap \Sigma_\lambda$ is not empty. By the monotonicity of f near the origin, we get that, for any nonempty connected component ω of Ω_λ,

$$\begin{cases} -\Delta(u - u_\lambda) = f(u) - f(u_\lambda) \geq 0 & \text{in } \omega \\ u - u_\lambda = 0 & \text{on } \partial\omega. \end{cases}$$

Hence, by the maximum principle for possibly unbounded domains (see Lemma 27 and [9], Lemma 2, 1), we conclude that $u - u_\lambda \geq 0$ in ω, a contradiction. □

In view of this proposition, we can define (continuation)

$$\overline{\lambda} = \inf\{\lambda_0 : u - u_\lambda \geq 0 \text{ in } \Sigma_\lambda, \forall \lambda \geq \lambda_0\}. \tag{1.2}$$

By construction, we see that $\overline{\lambda} < \infty$.

Lemma 38. *Let $u \geq 0$ be a bounded solution to equation (56) fulfilling (1.1). Assume that f satisfies (57).*

(i) If $\overline{\lambda} = -\infty$, then $u_N \equiv 0$ or $u_N(x) < 0$ for any $x \in \mathbb{R}^N$.

(ii) If u satisfies condition (59) and $\overline{\lambda} = -\infty$, then $u \equiv 0$.

(iii) If u satisfies condition (59) and $f'(t) \leq 0$ for $t > 0$, then $u \equiv 0$.

Proof. (i) If $\overline{\lambda} = -\infty$, that is the moving plane method does not stop, then $u_N \leq 0$. Since u_N verifies the linearized equation $-\Delta u_N = f'(u)u_N$, by the strong maximum principle, we get that $u_N \equiv 0$ or $u_N < 0$ in the whole $\mathbb{R}^N$.

(ii) If $\overline{\lambda} = -\infty$, the monotonicity, together with condition (59), yields that $u \equiv 0$.

(iii) If $f'(t) \leq 0$ for any $t > 0$, then $\overline{\lambda} = -\infty$, hence, by statement ($ii$), $u \equiv 0$. □

Proposition 39. *Let $u > 0$ be a bounded solution to equation (56) fulfilling (1.1). Assume that f satisfies (57). Assume, in addition, that $\overline{\lambda} > -\infty$.*

(i) For any positive integer k, there exists $\overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda}$ and a point $x^k \in \Sigma_{\lambda_k}$, with $\{x_N^k\}$ bounded, such that

$$u(x^k) < u_{\lambda_k}(x^k) \tag{1.3}$$

(ii) If, in addition, u is periodic in x_N, then the sequence x^k can be chosen to be bounded.

Proof. (i) It follows from the definition of $\overline{\lambda}$ that we can choose a sequence $\overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda}$ and a point $x^k \in \Sigma_{\lambda_k}$ such that $u(x^k) < u_{\lambda_k}(x^k)$. By construction, we have that $x_N^k < \lambda_k < \overline{\lambda}$; what remains to prove is that we can choose these sequences in such a way that x_N^k is bounded from below. We define

$$\Lambda = \{\big((\lambda_k)_k, (x^k)_k\big) : \overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda},\, x^k \in \Sigma_{\lambda_k} \text{ and } u(x^k) < u_{\lambda_k}(x^k)\}$$

and we argue by contradiction. We assume that for any couple of sequences $(\tilde{\lambda}, \tilde{x}) = \big((\lambda_k)_k, (x^k)_k\big) \in \Lambda$, we have $x_N^k \to -\infty$. Hence, once we fix $B > 0$ and such a couple $(\tilde{\lambda}, \tilde{x})$, we can find $\overline{k}$ such that $x_N^k < -B$, for $k \geq \overline{k}$. Now, if we set

$$k_0(\tilde{\lambda}, \tilde{x}) = \min\{\overline{k} : x_N^k < -B, \text{ for } k \geq \overline{k}\},$$

we have that $x_N^k < -B$ for $k \geq k_0(\tilde{\lambda}, \tilde{x})$, while $x^{k_0(\tilde{\lambda}, \tilde{x})-1} \geq -B$.

After that we set

$$k_0 = \sup\{k_0(\tilde{\lambda}, \tilde{x}) : (\tilde{\lambda}, \tilde{x}) \in \Lambda\};$$

if $k_0 = \infty$, the family $\{k_0(\tilde{\lambda}, \tilde{x}) : (\tilde{\lambda}, \tilde{x}) \in \Lambda\}$ would be a diverging sequence k_j of positive integers, that we can assume to be increasing and such that $k_j > j$. For any j, we set $i = k_j - 1$ and consider the corresponding couple $(\tilde{\lambda}, \tilde{x})$: we set $\mu_i = \lambda_i$ and $s^i = x^i$. The couple (μ, s) still belongs to Λ and $s^i_N \geq -B$, a contradiction.

Therefore, we have that $k_0 < \infty$ and, for any $k \geq k_0$, $u - u_{\lambda_k} \geq 0$ in $\{-B < x_N < \lambda_k\}$. Now, if we choose B so large that $u(x) < \varepsilon$ for $x_N > 2(\overline{\lambda} - 1) - B$, we have, for $k \geq k_0$

$$\begin{cases} -\Delta(u - u_{\lambda_k}) = f(u) - f(u_{\lambda_k}) \geq 0 & \text{in } \omega \\ u - u_{\lambda_k} = 0 & \text{on } \partial\omega, \end{cases}$$

where ω is any connected component of the set $\Omega_k = \{x_N < -B\} \cap \{u - u_{\lambda_k} < 0\}$. Therefore, by the maximum principle for possibly unbounded domains (see Lemma 27 and [9], Lemma 2,1), we get that $u - u_{\lambda_k} \geq 0$ in ω, hence $\Omega_k = \emptyset$, that is $u - u_{\lambda_k} \geq 0$ in Σ_{λ_k}, for $k \geq k_0$.

The same is true for any $\lambda > \lambda_{k_0+1}$. Otherwise, we would be able to find a couple (λ, x^λ) such that $u(x^\lambda) < u_\lambda(x^\lambda)$, with $x^\lambda \in \Sigma_\lambda$ and $\lambda > \lambda_{k_0+1}$. As a consequence, $\lambda = \tilde{\lambda}_{k_0+1}$, for an appropriate $\tilde{\lambda}$, so $u - u_\lambda \geq 0$ in Σ_λ, which is not possible.

(ii) It follows from the periodicity that we can redefine x^k in order for $(x')^k$ to be bounded. □

1.2 Results with periodicity

Now we can proceed with the proof of Theorem 21 in the case $M = N - 1$.

Proof. At first we note that, by statement *(ii)* of Lemma 38, $\overline{\lambda} > -\infty$, otherwise $u \equiv 0$. Since $u - u_{\overline{\lambda}} \geq 0$ in $\Sigma_{\overline{\lambda}}$, the strong maximum principle yields that either $u \equiv u_{\overline{\lambda}}$ or $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$. Now we argue by contradiction and assume that the second possibility holds true. We take a sequence of real numbers λ_k and a sequence of points $x^k \in \Sigma_{\lambda_k}$ as in Proposition 39. By the boundedness of x^k, we have that, up to a subsequence, $x^k \to x^\infty$, so, by (1.3), we get that $u(x^\infty) \leq u_{\overline{\lambda}}(x^\infty)$. Since we are assuming that $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, we have that $x^\infty_N = \overline{\lambda}$. By the Hopf Lemma, we obtain that $u_N(x', \overline{\lambda}) < 0$, but the mean value theorem yields that

$$0 < u_{\lambda_k}(x^k) - u(x^k) = 2(\lambda_k - x^k_N) u_N((x')^k, \xi^k)$$

with $x_N^k < \xi^k < 2\lambda^k - x_N^k$. Letting $k \to \infty$, we conclude that $u_N(x^\infty) \geq 0$, a contradiction. Hence we have $u = u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$. □

Now let us consider the general case. In the next proposition, hypothesis (*ii*) of Theorem 21 can be replaced by the weaker assumptions

$$\begin{cases} u(y, z', z_N) \to 0 & \text{as } |z'| \to \infty, \text{ uniformly in the other variables} \\ u(y, z', z_N) \to 0 & \text{as } z_N \to \infty, \text{ uniformly in the other variables.} \end{cases} \tag{1.4}$$

Under these hypotheses, it is possible to define $\overline{\lambda} < \infty$ as before.

Proposition 40. *Let $u > 0$ be a bounded solution to equation (56) satisfying condition (1.4). Assume that f satisfies (57). Assume furthermore that $\overline{\lambda} > -\infty$.*

(i) Then, for any positive integer k, there exists $\overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda}$ and a point $x^k = (y^k, z^k) \in \Sigma_{\lambda_k}$, with $\{z^k\}$ bounded, such that

$$u(x^k) < u_{\lambda_k}(x^k)$$

(ii) If, in addition, u is periodic in y, then the sequence x^k can be taken in such a way that it is bounded.

This is a generalisation of Proposition 39, for which we have nevertheless presented an independent proof.

Proof. As in the proof of Proposition 39, by definition of $\overline{\lambda}$, we can find a sequence of real numbers $\overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda}$ and a sequence of points $x^k = (y^k, z^k) \in \Sigma_{\lambda_k}$ such that (1.3) holds. The difference is that now we want to prove that this sequence can be chosen in such a way that z^k is bounded. In order to do so we will argue by contradiction. By construction, we know that $z_N^k \leq \overline{\lambda}$. In the notation of Proposition 39, we define, for $R > 0$ and $(\tilde{\lambda}, \tilde{x}) \in \Lambda$, the number

$$k_0(R, \tilde{\lambda}, \tilde{x}) = \inf\{k_0 : z_N^k \leq -R, |(z')^k| \geq R \ \forall k \geq k_0\}.$$

Now we put

$$k_0(R) = \sup\{k_0(R, \tilde{\lambda}, \tilde{x})\};$$

exactly as in Proposition 39, we get that $k_0(R) < \infty$, for any $R > 0$ and $u - u_{\lambda_k} \geq 0$ in $\Sigma_{\lambda_k} \cap Q_R$ for any $k \geq k_0$, where we have set $Q_R = \{|z'| \leq R, \ z_N \geq -R\}$.

By the decay assumptions, if R is large enough, we have that $u(y, z) < \varepsilon$ for $|z'| > R$ and $u_{\lambda_k}(y, z) < \varepsilon$ for $z_N < -R$ and for any k. Hence, if we set $\Omega_k = \{u - u_{\lambda_k} < 0\} \cap \Sigma_{\lambda_k}$, we get that, for any connected component ω of Ω_k,

$$\begin{cases} -\Delta(u - u_{\lambda_k}) = f(u) - f(u_{\lambda_k}) \geq 0 & \text{in } \omega \\ u - u_{\lambda_k} = 0 & \text{on } \partial\omega, \end{cases}$$

hence, by the maximum principle for possibly unbounded domains (see Lemma 27 and [9], Lemma 2.1), $\omega = \emptyset$, as desired. □

The conclusion of the proof of Theorem 21 is similar to what we have done in the case $M = N-1$. As first we observe that, by the behaviour of u for $z_N \to -\infty$, applying Lemma 38, we get $\overline{\lambda} > -\infty$. Then we take a sequence $x^k = (y^k, z^k)$ as in Proposition 40; up to a subsequence, we can assume that $x^k \to x^\infty = (y^\infty, z^\infty)$. Passing to the limit in (1.3), we can see that $u(y^\infty, z^\infty) \leq u_{\overline{\lambda}}(y^\infty, z^\infty)$. If $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, we get that $(y^\infty, z^\infty) \in \partial\Sigma_{\overline{\lambda}}$, but this contradicts the Hopf Lemma, as we have seen above.

1.3 Results without periodicity

First we state two general Lemmas, that enable us to define the energy and the momentum.

Lemma 41. *Let $u > 0$ be a bounded solution to (56), with $f \in C^1$ satisfying (57). Assume furthermore that (59) holds. Then*

$$\nabla u(y,z) \to 0 \quad \textit{as } |z| \to \infty, \textit{ uniformly in } y.$$

Proof. Assume, by contradiction, that it is possible to find a $\delta > 0$ and a sequence $|z^k| \to \infty$ such that

$$\sup_y |\nabla u(y, z^k)| \geq 2\delta.$$

So we can take a sequence $y^k \in \mathbb{R}^M$ such that $|\nabla u(y^k, z^k)| \geq \delta$ and define $u^k(x) = u(x + x^k)$. Up to a subsequence, $u^k \to v$ in $C^2_{loc}(\mathbb{R}^N)$, and $v \geq 0$ is still a solution to equation (56). Now we observe that, on the one hand

$$u^k(0) = u(x^k) \to 0 = v(0),$$

hence, by the strong maximum principle, $v \equiv 0$. On the other hand,

$$\delta \leq |\nabla u(y^k, z^k)| = |\nabla u^k(0)| \to |\nabla v(0)| = 0,$$

a contradiction. □

Lemma 42. *If $f'(0) < 0$, then u and ∇u actually decay exponentially in z, that is, $\forall \gamma < \sqrt{-f'(0)}$*

$$u(x), |\nabla u(x)| \leq Ce^{-\gamma|z|} \quad \textit{for } |z| \geq B, \tag{1.5}$$

for some constant $B > 0$.

Proof. The idea is to use the function $e^{-\gamma|z|}$ as a barrier. First we note that, for any $0<\gamma<\sqrt{-f'(0)}$, there exists $B>0$ such that $f(u)<-\gamma^2 u$ in the region $\Omega:=\{x=(y,z)\in\mathbb{R}^M\times\mathbb{R}^{N-M}:|z|>B\}$, thus

$$(-\Delta+\gamma^2)(u-(\lambda e^{-\gamma|z|}+\sigma))\leq f(u)+\gamma^2 u+\lambda(\Delta-\gamma^2)e^{-\gamma|z|}-\sigma\gamma^2<0$$

in Ω. Moreover, if $\lambda>||u||_\infty e^{\gamma B}$, then

$$u(x)<\lambda e^{-\gamma|z|}<\lambda e^{-\gamma|z|}+\sigma,\quad \text{if } |z|=B,$$

for any $\sigma>0$. Now we fix a point $x_0=(y_0,z_0)\in\Omega_1$ and $R>|z_0|$ such that $u(y,z)<\sigma$ if $|z|>R$. In particular, we have

$$u(x)<\sigma<\lambda e^{-\gamma|z|}+\sigma,\quad \text{if } |z|=R,$$

therefore, by the maximum principle for possibly unbounded domains (see [9]) applied to the region $\{x=(y,z)\in\mathbb{R}^M\times\mathbb{R}^{N-M}:B<|z|<R\}$, we have $u(x_0)\leq\lambda e^{-\gamma|z_0|}+\sigma$. Letting $\sigma\to 0$, we prove the exponential decay of u.

As regards the decay of the gradient, by Lemma 41, we have that $\nabla u(y,z)\to 0$ as $|z|\to\infty$, uniformly in y. Moreover, the partial derivatives satisfy

$$-\Delta u_j=f'(u)u_j<-\gamma^2 u \quad \text{in } \Omega,\ j=1,\ldots,N,$$

provided B is large enough. As a consequence, the exponential decay is proven as above. □

Now we observe that condition (61) enables us to relate the study of equation (56) to the study of one dimensional problem (62). The results concerning this one-dimensional problem are probably known, for sake of completeness we present the proofs.

Before giving these proofs, let us fix some terminology. If u is a bounded solution to (56), then for any sequence $|x^k|\to\infty$, it is possible to find a subsequence such that $u^k(x)=u(x+x^k)\to u^\infty(x)$ in the $C^2_{loc}(\mathbb{R}^N)$ sense, and u^∞ is still a solution. In the sequel, this kind of solutions, obtained as a limit of sequences constructed as above, will be referred to as *profiles*. In the sequel, we will say that a profile is one-dimensional if it is a function depending just on the x_N-variable.

Lemma 43. *Let u be a bounded solution to equation (56) satisfying (59) with $z=x_N$, and with f fulfilling (57). Then any profile is one-dimensional if and only if (61) holds.*

Proof. If any profile is one-dimensional, for any $|(x')^k|\to\infty$, there is a subsequence such that $u^k(x)=u(x'+(x')^k,x_N)\to v(x)$ in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$, with $v_j\equiv 0$, for $1 \leq j \leq N-1$. This implies, in particular, that $u_j^k \to 0$ pointwise, therefore $u_j((x')^k, x_N) \to 0$ for any $x_N \in \mathbb{R}$. Since the sequence $(x')^k$ is arbitrary, we conclude that $u_j(x', x_N) \to 0$ as $|x'| \to \infty$, for any x_N.

The converse is true because C^2_{loc} convergence implies pointwise convergence. □

Now we are going to study Problem (62). For solutions satisfying

$$v(t) \leq C|t|^{-(1+\sigma)} \quad \text{for any } |t| \geq M \tag{1.6}$$

for suitable constants $M > 0$, $\gamma > 0$, we define

$$H(v) = \int_{-\infty}^{\infty} \frac{1}{2}(v')^2 - F(v)dt$$

and, for any $\lambda \in \mathbb{R}$,

$$E_\lambda(v) = \int_{-\infty}^{\infty} (t-\lambda)\Big(\frac{1}{2}(v')^2 - F(v)\Big)dt.$$

In order to show that $H(v)$ and $E(v)$ are well defined and finite for such solutions, we prove the following lemma.

Lemma 44. *Let $v > 0$ be a solution to Problem (62). Then*

(i) v is symmetric with respect to λ, for some $\lambda \in \mathbb{R}$, and $v'(t) > 0$ for any $t < \lambda$.

(ii) For any $t \in \mathbb{R}$, we have $\frac{1}{2}(v'(t))^2 + F(v(t)) = 0$.

(iii) If we assume, in addition, that $v(t) \leq C|t|^{-(1+\sigma)}$, for some $\sigma > 0$, then $|v'(t)| \leq C|t|^{-(1+\sigma)}$, for any $|t| \geq B$.

Proof. (i) Since $v(t) \to 0$ as $t \to \infty$, the solution must have a maximum point at $t = \lambda$, for some $\lambda \in \mathbb{R}$. In particular it satisfies the Cauchy problem

$$\begin{cases} -v'' = f(v) & \text{on } \mathbb{R} \\ v(\lambda) = v_{max} \\ v'(\lambda) = 0. \end{cases}$$

A computation shows that $v_\lambda(t) = v(2\lambda - t)$ satisfies the same Cauchy problem, hence $v_\lambda = v$. If v had another critical point $\mu \neq \lambda$, it would also be symmetric with respect to μ, and hence periodic, but this is not possible because it tends to 0 at infinity.

(ii) Multiplying the ODE by v' and integrating we obtain the relation

$$0 \leq \frac{1}{2}(v')^2 = -F(v) + C,$$

where C is a suitable constant. Letting $t \to \infty$, we get that $C \geq 0$. If we had $C > 0$, we would get that $(v')^2 \to 2C > 0$ as $t \to \infty$, which is not possible because $v \to 0$ as $t \to \infty$. Finally we get that $C = 0$ and $v' \to 0$ as $t \to \infty$.

(iii) If $f'(0) < 0$, the claim follows from the exponential decay of the derivative, so we can assume that $f'(0) = 0$. We assume by contradiction that for any positive integer k, we can find $|t_k| > k$ such that $|v'(t_k)| > k|t_k|^{-(1+\sigma)}$. Now we set

$$v^k(t) = |t_k|^\sigma v(|t_k|t)$$

A computation shows that, for k large enough and for any $\frac{1}{2} < |t| < 2$, we have

$$(v^k)'(t) = |t_k|^{1+\sigma}\sqrt{-2F(v(|t_k|t))} \leq |t_k|^{1+\sigma}\sqrt{2C|t_k|^{-2(1+\sigma)}|t|^{-2(1+\sigma)}} \leq C.$$

However, we can see that

$$(v^k)'(t_k/|t_k|) \geq |t_k|^{1+\sigma}k|t_k|^{-(1+\sigma)} = k, \tag{1.7}$$

a contradiction. □

Proposition 45. *If there exists a nontrivial solution to Problem (62), then it is unique up to a translation.*

It follows from the Cauchy uniqueness theorem that any nontrivial solution to Problem (62) is strictly positive. Nevertheless, we point out that a nontrivial solution does not always exist, for instance if $f(u) = ((u - \beta)^+)^p$ with $\beta > 0$, as we will see later.

Proof. Let us assume that there are two solutions $v > 0$ and $w > 0$, that are not one the translated of the other. Up to a translation, we can assume that the symmetry axes are the same, that is there exists $\lambda \in \mathbb{R}$ such that $v = v(|t - \lambda|)$ and $w = w(|t - \lambda|)$.

If $v(\lambda) = w(\lambda)$, then we also have $v'(\lambda) = w'(\lambda) = 0$, since λ is a maximum point for both v and w; therefore, by the Cauchy uniqueness theorem, we get that $v \equiv w$.

Now, assume, for instance, that $w(\lambda) > v(\lambda)$. By continuity, there exists $t_0 > \lambda$ such that $w(t_0) = v(\lambda)$. As a consequence, we conclude that

$$0 > w'(t_0) = \sqrt{-2F(w(t_0))} = \sqrt{-2F(v(\lambda))} = v'(\lambda),$$

a contradiction. □

Now, let us prove a quite general Lemma, in which we do not need to assume that u is a solution to some PDE.

Lemma 46. *Let us denote* $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$. *Let* $u : \mathbb{R}^N \to \mathbb{R}$ *be a continuous function such that*

$$u(y, z) \to 0 \quad \text{as } |z| \to \infty, \text{ uniformly in } y.$$

(i) Assume that for any sequence $|y^k| \to \infty$ *it is possible to find a subsequence such that* $u^k(x) = u(y^k, z) \to 0$ *in the* C^0_{loc} *sense. Then* $u(x) \to 0$ *as* $|x| \to \infty$.

(ii) Let $M = 1$. *Assume that for any sequence* $y^k \to \infty$ *it is possible to find a subsequence such that* $u^k(x) = u(y + y^k, z) \to 0$ *in the* C^0_{loc} *sense. Then*

$$u(y, z) \to 0 \quad \text{as } y \to \infty, \text{ uniformly in } z. \tag{1.8}$$

Proof. (i) By the decay in z, we have that, for any $\varepsilon > 0$, there exists $B > 0$ such that $|u(y, z)| < \varepsilon$ for $|z| \geq B$. Since $u^k \to 0$ in the C^0_{loc} sense, the convergence is uniform in the compact set $K = \{|z| \leq B, y = 0\}$. Hence for any sequence $|y^k| \to \infty$, there is a subsequence such that

$$\sup_K |u^k(x)| = \sup_{|z| \leq B} |u(y^k, z)| \to 0,$$

therefore $u(y, z) \to 0$ as $|y| \to \infty$, uniformly in z, so we have the statement.

(ii) We essentially repeat the same proof, with the only difference that we consider only sequences $y^k \to \infty$. □

Now we are going to prove Theorem 23.

Proof. For any sequence $|(x')^k| \to \infty$, by Lemma 43, any corresponding profile v is one-dimensional and satisfies (62), so, by our assumption about f, $v \equiv 0$. Since this is true for any profile, Lemma 46 yields that $u(x) \to 0$ as $|x| \to \infty$, hence, by the result by Gidas, Ni, Nirenberg in [37], u is radially symmetric and radially decreasing.

□

Now we can prove Corollary 24

Proof. Assume by contradiction that such a solution exists. Then by Theorem 23 it is radially symmetric, that is, up to a translation, $u(x) = v(|x|)$, and harmonic outside a ball, so $u(x) = a \log(|x|) + b$, for $|x|$ large enough. If $a = 0$, by (59), we get $b = 0$, so $u \equiv 0$. Otherwise, $a \neq 0$ and $b \in \mathbb{R}$, but this contradicts condition (59). □

Now we can prove Proposition 25.

Proof. In the proof, we set $F(u) = \frac{1}{p+1}|u|^{p+1} - \frac{1}{2}u^2$ and $f(u) = |u|^{p-1}u - u$.

In order to prove the proposition, we will assume by contradiction that $0 < ||u||_\infty < (p+1/2)^{1/p-1}$ and we will see that this yields that for any $|(x^{'})^k| \to \infty$, the corresponding profile is identically 0, hence, by Lemma 46, up to a translation, $u(x) = U(|x|)$, but, by our assumption, we have that $||u||_\infty < (p+1/2)^{1/p-1} < \max U$, a contradiction.

As before, by Lemma 43, we get that any profile is one dimensional and satisfies (62), therefore, by point (i) of Lemma 44, we know that $v = v(|t-\lambda|)$, for an appropriate $\lambda \in \mathbb{R}$. By symmetry, we get that $v^{'}(\lambda) = 0$, hence, by point (ii) of Lemma 44, $F(v(\lambda)) = 0$. Anyway, we have that $v(\lambda) = ||v||_\infty \leq ||u||_\infty < (p+1/2)^{1/p-1}$, so we conclude that $||v||_\infty = 0$.

□

1.4 Proof of Theorems 9 and 11.

In this section we are going to deal with the cases in which problem (62) has a positive solution, so we can have a positive profile when we translate in the $x^{'}$-directions. In order to deal with this case, we need to consider the energy $H(u, x^{'})$ and the momentum $E_\lambda(u, x^{'})$ of a solution, hence we need some further assumptions about the decay rate of u in x_N. In the next lemma, we see that it is enough to prescribe the decay rate of u, we do not need any further assumption about the gradient.

Lemma 47. *Let $u > 0$ be a bounded solution to (56) with $f \in C^2(\mathbb{R})$ satisfying (57) and $f^{'}(0) = 0$. Assume furthermore that $u(x) \leq C|x_N|^{-\alpha}$ for $|x_N| \geq B$, for some constants $B > 0$ and $\alpha \geq 1$. Then*

(i) *the gradient satisfies*

$$|\nabla u(x)| \leq C|x_N|^{-\alpha} \quad \text{for } |x_N| \geq B. \tag{1.9}$$

(ii) *If $\alpha \geq 2$, then*

$$|\nabla u(x)| \leq C|x_N|^{-(1+\alpha)} \quad \text{for } |x_N| \geq B. \tag{1.10}$$

Proof. (i) Assume by contradiction that (1.9) fails. Then it is possible to find a sequence of points $x^k \in \mathbb{R}^N$, with $|x_N^k| \geq k$, such that

$$|\nabla u((x^{'})^k, x_N^k)| \geq k|x_N^k|^{-\alpha}.$$

Now we define

$$v^k(x', x_N) = |x_N^k|^{\alpha-1} u\Big(|x_N^k|\big(x' + \frac{(x')^k}{|x_N^k|}\big), |x_N^k| x_N\Big)$$

and

$$\Omega = \Big\{|x'| < 1, \frac{1}{2} < |x_N| < 2\Big\}.$$

By the decay rate of u in x_N and the fact that $|x_N^k| \to \infty$, we have

$$|v^k(x)| \leq C|x_N^k|^{-1}|x_N|^{-\alpha} \leq C.$$

for any $x \in \Omega$ and for k large enough. Since $f \in C^2$ and $f'(0) = 0$, we deduce that $|f(u)|/u^2$ is bounded in a neighbourhood of the origin, so

$$0 \leq |\Delta v^k(x)| = |x_N^k|^{\alpha+1}\Big|f\Big(u\Big(|x_N^k|\big(x' + \frac{(x')^k}{|x_N^k|}\big), |x_N^k| x_N\Big)\Big)\Big| \leq$$
$$C|x_N^k|^{\alpha+1} u^2\Big(|x_N^k|\big(x' + \frac{(x')^k}{|x_N^k|}\big), |x_N^k| x_N\Big) \leq C|x_N^k|^{1-\alpha}|x_N|^{-2\alpha} \leq C$$

for any $x \in \Omega$ and for k large enough .

By elliptic estimates we have that, for any ball $B \subset\subset \Omega$, for any $p > 1$ and for any k,

$$||v^k||_{W^{2,p}(B)} \leq C(||v^k||_{L^\infty(\Omega)} + ||\Delta v^k||_{L^\infty(\Omega)}) \leq C.$$

Now we take $p > N$ and we conclude, by the Sobolev embedding $C^{1,\alpha}(B) \subset W^{2,p}(B)$ and since the ball is arbitrary, we have that $||\nabla v^k||_{L^\infty(\Omega)}$ is uniformly bounded with respect to k.

On the other hand, an explicit computation gives that

$$\Big|\nabla v^k\Big(0, \frac{x_N^k}{|x_N^k|}\Big)\Big| = |x_N^k|^\alpha |\nabla u((x')^k, x_N^k)| \geq k \to \infty,$$

a contradiction.

(ii) The proof is the same as before, with the only difference that now we set

$$v^k(x', x_N) = |x_N^k|^{\alpha} u\Big(|x_N^k|\big(x' + \frac{(x')^k}{|x_N^k|}\big), |x_N^k| x_N\Big).$$

The only point where we use that $\alpha \geq 2$ is to say that $||\Delta v^k||_{L^\infty(\Omega)}$ is uniformly bounded with respect to k. □

By this lemma we see that, if u fulfills (64) then the gradient satisfies

$$|\nabla u(x)| \le C|x_N|^{-(1+\sigma)} \quad \text{for } |x_N| \ge M, \tag{1.11}$$

for suitable constants $M > 0$, $\sigma > 0$, so it is possible to define the energy and the momentum, even if $f'(0) = 0$.

Now we recall that, under condition (59), it is possible to start the moving plane procedure from the positive x_N direction (see Proposition 36) and define $\overline{\lambda}$ as in (1.2).

It is possible to show that, under assumption (a) of Theorem 28, any profile is positive and we can find a profile v that is symmetric with respect to $\overline{\lambda}$.

Proposition 48. *If $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, then there exists a positive solution v which is symmetric about the hyperplane $\{x_N = \overline{\lambda}\}$.*

Proof. We take a sequence x^k as in Proposition 39 and we define

$$u^k(x) = u(x' + (x')^k, x_N).$$

By the Ascoli-Arzelá theorem, up to a subsequence, u^k converges to a non-negative solution v to equation (56).

Now we want to prove that $v > 0$. We point out that $|(x')^k| \to \infty$. If not, by the boundedness of x_N^k, it would be possible to find a subsequence $x^k \to x^\infty$. Hence, passing to the limit in (1.3), we would get that $u(x^\infty) \le u_{\overline{\lambda}}(x^\infty)$; since $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, we get that $x^\infty \in \partial\Sigma_{\overline{\lambda}}$, which contradicts the Hopf lemma. In fact, still by (1.3) and by Lagrange theorem, we have that

$$0 < u_{\lambda_k}(x^k) - u(x^k) = 2(\lambda_k - x^k)u_N^k(0, \xi^k),$$

for an appropriate $x_N^k < \xi^k < 2\lambda_k - x_N^k$. Therefore, passing to the limit, we get that $u_N(0, \overline{\lambda}) \ge 0$, which contradicts the Hopf Lemma.

We are now in position to show that $v > 0$. In fact, $H(u^k, x') = H(u, x' + (x')^k) \to H(v, x')$, so $|H(v, x')| > \gamma > 0$, hence $v > 0$.

It remains to prove that such a profile is symmetric. Since the translation is orthogonal to the x_N direction, we have that $u^k \ge u^k_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, hence $v \ge v_\lambda$ in $\Sigma_{\overline{\lambda}}$. By the strong maximum principle, we can see that $v > v_\lambda$ or $v \equiv v_\lambda$ in $\Sigma_{\overline{\lambda}}$; we want to exclude the first possibility. In order to do so, we take a subsequence such that $x_N^k \to x_N^\infty$ and pass to the limit in (1.3), and we obtain that $v(0, x^\infty) \le v(0, x^\infty)$. Now we observe that, if $v > v_{\overline{\lambda}}$, we get that $x_N^\infty = \overline{\lambda}$, which contradicts the Hopf Lemma, exactly as above. □

In view of condition (64), the decay in x_N holds both for $x_N \to \infty$ and for $x_N \to -\infty$, therefore we can also start the moving plane procedure from the left, and define

$$\underline{\lambda} = \sup\{\lambda_0 : u - u_\lambda \ge 0 \text{ in } \tilde{\Sigma}_\lambda, \forall \lambda \le \lambda_0\},$$

where $\tilde{\Sigma}_\lambda = \{x_N > \lambda\}$.

As above, by construction, we get $\underline{\lambda} > -\infty$. Furthermore, we can prove that $\underline{\lambda} \leq \overline{\lambda}$. If not, we would have $u_N \geq 0$ in $\{x_N < \underline{\lambda}\}$ and $u_N \leq 0$ in $\{x_N > \overline{\lambda}\}$, so $u_N = 0$ in $\{\overline{\lambda} < x_N < \underline{\lambda}\}$. By the strong maximum principle we get, for instance, that $u_N \equiv 0$ in $\Sigma_{\underline{\lambda}}$, hence $u \equiv 0$.

Remark 49. *If $\underline{\lambda} = \overline{\lambda}$, then u is symmetric with respect to x_N, that is $u = u(x^{'}, |x_N - \overline{\lambda}|)$.*

To conclude the proof of Theorem 28, we have to rule out the possibility $\underline{\lambda} < \overline{\lambda}$. In order to do so, we prove the following proposition.

Proposition 50. *Let $u > 0$ be a bounded positive solution to equation (56) satisfying (64), with f as in (57). Assume furthermore that*

(i) $|H(u, x^{'})| \geq \gamma > 0$ for $|x|^{'}$ large enough

(ii) there exists μ such that $E_\mu(u, x^{'}) \to 0$ for $x^{'} \to \infty$.

Then $u(x) = u(x^{'}, |x_N - \lambda|)$, for a suitable $\lambda \in \mathbb{R}$ (that is, u is symmetric in x_N).

Proof. We divide the proof in two steps.

(*i*) If $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, then $\overline{\lambda} = \mu$.

We define

$$\tilde{u}(x) = u(x^{'}, x_N + \overline{\lambda} - \mu)$$

and

$$\tilde{u}^k(x) = \tilde{u}(x^{'} + (x^{'})^k, x_N).$$

It is worth to remark that the profile of the translated solution $\tilde{u}$ coincides with the translation of the profile $\tilde{v}$, that is $\tilde{u}_k \to \tilde{v}$, up to a subsequence. Since v is symmetric about the hyperplane $\{x_N = \overline{\lambda}\}$, $\tilde{v}$ is symmetric about the hyperplane $\{x_N = \mu\}$, therefore, if we set

$$g(x) = \frac{1}{2}\left(u_N^2 - |\nabla_{x^{'}} u|^2\right) - F(u),$$

then we have

$$\begin{aligned}
0 = E_\mu(\tilde{v}, x^{'}) = \lim_{k\to\infty} E_\mu(\tilde{u}^k, x^{'}) = \lim_{k\to\infty} E_\mu(\tilde{u}, x^{'} + (x^{'})^k) = \\
\lim_{k\to\infty} \int_{-\infty}^{\infty} (x_N - \mu) g(x^{'} + (x^{'})^k, x^N + \overline{\lambda} - \mu) dx_N = \\
\lim_{k\to\infty} \left\{ \int_{-\infty}^{\infty} (z_N - \mu) g(x^{'} + (x^{'})^k, z_N) dz_N - \int_{-\infty}^{\infty} (\overline{\lambda} - \mu) g(x^{'} + (x^{'})^k, z_N) dz_N \right\} = \\
\lim_{k\to\infty} \left\{ E_\mu(u, x^{'} + (x^{'})^k) - (\overline{\lambda} - \mu) H(u, x^{'} + (x^{'})^k) \right\} = -(\overline{\lambda} - \mu) H(v, x^{'}).
\end{aligned}$$

Since $H(v,x^{'}) \neq 0$, we have $\overline{\lambda} = \mu$.

(ii) $\underline{\lambda} = \mu = \overline{\lambda}$.

In order to prove the statement, we start the reflection from the left and obtain that either u is symmetric about the hyperplane $\{x_N = \underline{\lambda}\}$ or $u > u_{\underline{\lambda}}$ in $\tilde{\Sigma}_{\underline{\lambda}}$; in the second case, exactly as in Proposition 39, we are able to construct a sequence $\underline{\lambda} < \lambda_k < \underline{\lambda} + 1/k$ and a sequence of points $s^k \in \tilde{\Sigma}_{\lambda_k}$ such that $u(s^k) < u_{\lambda_k}(s_k)$, with $|(s^{'})^k| \to \infty$ and $\{s_N^k\}$ bounded. Passing to the limit, we get a profile w which is symmetric about the hyperplane $\{x_N = \underline{\lambda}\}$. Since $\overline{\lambda} = \mu$, we have

$$0 = \lim_{k\to\infty} E_{\overline{\lambda}}(u, x^{'} + (s^{'})^k) = \lim_{k\to\infty} \int_{-\infty}^{\infty} (x_N - \overline{\lambda}) g(x^{'} + (s^{'})^k, x_N) dx_N =$$

$$\lim_{k\to\infty} \left\{ \int_{-\infty}^{\infty} (x_N - \underline{\lambda}) g(x^{'} + (s^{'})^k, x_N) dx_N - \int_{-\infty}^{\infty} (\overline{\lambda} - \underline{\lambda}) g(x^{'} + (s^{'})^k, x_N) dx_N \right\} =$$

$$E_{\underline{\lambda}}(w, x^{'}) - (\underline{\lambda} - \overline{\lambda}) H(w, x^{'}) = -(\underline{\lambda} - \overline{\lambda}) H(w, x^{'}).$$

Since $H(w,x^{'}) \neq 0$, then $\underline{\lambda} = \overline{\lambda}$. □

Now we can recollect our results to conclude the proof of Theorem 28.

Proof. The idea is to apply Proposition 50. Therefore, we have to check that hypothesis (i) and (ii) are satisfied. As first, we will prove that $H(u,x^{'})$ tends to a finite positive limit as $|x^{'}| \to \infty$. In order to do so, we take an arbitrary sequence $|(x^{'})^k| \to \infty$ and we prove that, up to a subsequence, $H(u,(x^{'})^k)$ converges to a positive limit which is indipendent of the chosen sequence.

By the Arzelá-Ascoli theorem, for any sequence $|(x^{'})^k| \to \infty$, we can find a subsequence such that $u^k(x) = u(x^{'} + (x^{'})^k, x_N)$ converges to a nonnegative profile v, which still verifies $-\Delta v = f(v)$. By hypothesis (a), we have that $v > 0$; by Lemma 43, we get that v is one-dimensional, that is $v = v(x_N)$. Moreover, by condition (64) we get that $v(x_N) \leq C|x_N|^{-(1+\sigma)}$ for $|x_N| \geq B$.

As a consequence, v is a solution to problem (62) for which the energy $H(v)$ and the momentum $E_\lambda(v)$ are well defined and finite. Moreover,

$$H(u,(x^{'})^k) = H(u^k, 0) \to H(v, 0) = H(v) = \int_{-\infty}^{\infty} (v^{'})^2 > 0.$$

By the uniqueness of the positive solution to (62), proven in Proposition 45, we get that the limit does not depend on the particular choice of the sequence $|(x^{'})^k| \to \infty$, hence

$$H(u,x^{'}) \to H(v) > 0 \quad \text{as } |(x^{'})| \to \infty.$$

In the same way as above, it is possible to prove that $E_0(u,x^{'}) \to E_0(v)$ as $|(x^{'})| \to \infty$. Therefore

$$E_\mu(u,x^{'}) = E_0(u,x^{'}) - \mu H(u,x^{'}) \to E_0(v) - \mu H(v),$$

so it is enough to take $\mu = E_0(v)/H(v)$. This concludes the proof of Theorem 28. □

Now we prove Theorem 30. In the proof, we will use a result by Malchiodi, Gui and Xu (see [43], Proposition 2). If $N=2$, they show that $H(u, x_1)$ is actually independent of x_1, hence it may be referred to as $H(u)$. If $H(u) \neq 0$, we can apply Proposition 50 with $\mu = E_0(u)/H(u)$, and the proof is finished.

It remains to deal with the case $H(u) = 0$. We claim that in this case u is radially symmetric, that is, up to a translation, $u = u(|x|)$, where $x = (x_1, x_2) \in \mathbb{R}^2$.

Proposition 51. *In the hypothesis of Theorem 30, if $H(u) = 0$, then u is radially symmetric.*

Proof. In view of Lemma 46, it is enough to show that any profile is identically 0 and apply the result by Gidas, Ni and Nirenberg in [37].

Assume, by contradiction, that one can find a sequence $|x_1^k| \to \infty$ whose correspondent profile v is stricly positive. By Lemma 43, this profile is one-dimensional, therefore it is the unique (up to a translation) solution to Problem (62), hence we already know that $H(v) = \int_{-\infty}^{\infty} (v')^2 > 0$. On the other hand, by the dominated convergence theorem, we have that $H(v) = H(u) = 0$, a contradiction. □

1.5 Solutions decaying in $N-1$ variables

Now we are considering solutions to equation (56) fulfilling (65). The nonlinearity will always satisfy (57), sometimes it will be required to be of class C^2, sometimes C^1 will be enough.

For such solutions, we define the energy-like functional

$$\mathcal{H}(u, x_N) = \int_{\mathbb{R}^{N-1}} \frac{1}{2}\left(|\nabla_{x'} u|^2 - u_N^2\right) - F(u)dx'.$$

We point out that, in order for such a functional to be well defined and finite, we need some further information about the decay rate of u, for example it is enough to consider solutions u fulfilling (67).

Remark 52. *If $f'(0) < 0$, any solution satisfying (65) actually decays exponentially in x', and the same is true for the gradient, that is*

$$u(x), |\nabla u(x)| \leq Ce^{-\gamma|x'|} \quad \text{for } |x'| \geq B,$$

for some $B > 0$, $\gamma > 0$, and this is true in any dimension $N \geq 2$, hence there are no problems to define $\mathcal{H}(u, x_N)$.

It is interesting to understand what happens in the case $f'(0) = 0$. It turns out that, at least in dimension $N \geq 5$, if $f \in C^2$, any solution fulfilling (65) actually decays fast enough in x', so it is still possible to define $\mathcal{H}(u, x_N)$. In dimension $2 \leq N \leq 4$, it is possible to do the same under hypothesis (67).

Moreover, we recall that in [43] Malchiodi, Gui and Xu showed that $\mathcal{H}(u, x_N)$ actually depends only on u, hence it will be referred to simply as $\mathcal{H}(u)$.

Lemma 53. *Let us denote $x = (y, z) \in \mathbb{R}^M \times \mathbb{R}^{N-M}$. Assume that $N - M \geq 3$. Let $u > 0$ be a bounded $C^2(\mathbb{R}^N)$ function such that $-\Delta u \leq 0$ for $|z| \geq r$, for some $r > 0$. Assume furthermore that*

$$u(y,z) \to 0 \quad \text{as } |z| \to \infty, \text{ uniformly in } y \tag{1.12}$$

Then

$$u(x) \leq C|z|^{2-(N-M)} \quad \text{for } |z| \geq R \tag{1.13}$$

for a suitable constant $R > 0$.

Proof. We will give an estimate of u by dominating it with a barrier. In this construction we use the function $v(y,z) = |z|^{2-(N-M)}$, because we know that $v > 0$ and $\Delta v = 0$ on $\mathbb{R}^N$, for $N - M \geq 3$. We observe that, for any $\sigma > 0$ and $\lambda \in \mathbb{R}$,

$$-\Delta\big(u - (\sigma + \lambda v)\big) \leq 0 \quad \text{for } |z| \geq r$$

By the decay in $|z|$, we deduce that, for any $\varepsilon > 0$, we can find $\rho = \rho(\varepsilon) > 0$ such that $u(y,z) < \varepsilon$ for $|z| \geq \rho$. Now we set $R = \max\{\rho, r\}$. We fix $0 < \sigma < \varepsilon$, $x_0 = (y_0, z_0)$ such that $|z_0| > R$ and we take $A > |z_0|$ so large that $u < \sigma$ for $|z| \geq A$. Hence we have

$$\begin{cases} u < \sigma < \sigma + \lambda R^{2-(N-M)} & \text{for } |z| = A \\ u < \varepsilon < \lambda R^{2-(N-M)} < \sigma + \lambda R^{2-(N-M)} & \text{for } |z| = R \end{cases}$$

if we choose $\lambda > \varepsilon R^{N-M-2}$. Therefore, by the maximum principle for possibly unbounded domains (see [9], Lemma 2.1) applied to the region $C = \{x \in \mathbb{R}^N : R < |z| < A\}$, we get $u \leq \sigma + \lambda v$ on C, in particular $u(x_0) \leq \sigma + \lambda v(x_0)$. Letting $\sigma \to 0$, we have the statement. □

Corollary 54. *Let $u > 0$ be a bounded solution to (56), with f satisfying (57).*

(i) If $N \geq 4$ and u satisfies (65), then

$$u(x', x_N) \leq C|x'|^{3-N} \quad \text{for } |x'| \geq B \tag{1.14}$$

for a suitable constant $B > 0$.

(ii) If $N \geq 3$ and $u(x) \to 0$ as $|x| \to \infty$, then

$$u(x) \leq C|x|^{2-N} \quad \text{for } |x| \geq B \tag{1.15}$$

Proof. It is enough to apply Lemma 53 with $M = 1$ in case (i) and with $M = 0$ in case (ii). □

Lemma 55. *Let $u > 0$ be a bounded solution to (56), with $f \in C^2(\mathbb{R})$ satisfying (57).*

(i) If $N \geq 5$ and u satisfies (65), then

$$|\nabla u(x', x_N)| \leq C|x'|^{2-N} \quad \textit{for } |x'| \geq B \tag{1.16}$$

for a suitable constant $B > 0$.

(ii) If $N \geq 4$ and $u(x) \to 0$ as $|x| \to \infty$, then

$$|\nabla u(x)| \leq C|x|^{1-N} \quad \textit{for } |x| \geq B \tag{1.17}$$

for a suitable constant $B > 0$.

Proof. It is enough to apply statement (ii) of Lemma 47 with $\alpha = N - 3$ in case (i) and $\alpha = N - 2$ in case (ii). □

Lemma 56. *Let $u > 0$ be a bounded solution to the (56), with $f \in C^2(\mathbb{R})$ satisfying the (57). Assume furthermore that (65) holds.*

(i) Let $N \geq 5$. Then $\mathcal{H}(u)$ is well defined and finite.

(ii) If $2 \leq N \leq 4$, the same is true under condition (67).

Proof. As above, we can assume that $f'(0) = 0$, otherwise the result follows from the exponential decay.

(i) Applying Lemma 55, we get that

$$\int_{|x'| \geq M} u_j^2 dx' \leq C \int_M^\infty r^{2(2-N)} r^{N-2} dr$$

that is finite because $N \geq 5$.

By the assumption $f'(0) = 0$ and $f \in C^2$, we get that $F(u)/u^3$ is bounded in a neighbourhood of the origin. If $N \geq 5$, this yields that

$$\Big| \int_{|x'| \geq R} F(u) dx' \Big| \leq C \int_R^\infty r^{3(3-N)} r^{N-2} dr < \infty$$

(ii) If $2 \leq N \leq 4$, condition (67) yields that

$$\int_{|x'| \geq M} u_j^2 dx' \leq C \int_M^\infty r^{-(N-1+\sigma)} r^{N-2} dr < \infty$$

and

$$\Big| \int_{|x'| \geq R} F(u) dx' \Big| \leq C \int_R^\infty r^{-3\frac{N-1+\sigma}{2}} r^{N-2} dr < \infty$$

□

Let $N \geq 4$. For a solution $u > 0$ to (56) such that $u(x) \to 0$ as $|x| \to \infty$, we define

$$J(u) = \int_{\mathbb{R}^N} \frac{1}{2}|\nabla u|^2 - F(u)dx.$$

We point out that, if $f^{'}(0) < 0$, any positive solution decaying to 0 decays exponentially, so the restirction on the dimension is not necessary, we can define $J(u)$ for any $N \geq 1$.

Anyway by Corollary 54 and Lemma 55, in dimension $N \geq 4$, even if $f^{'}(0) = 0$, the fact that $u \to 0$ as $|x| \to \infty$ is sufficient to guarantee that $J(u)$ is well defined and finite. In fact

$$\int_{\mathbb{R}^N} |\nabla u|^2 \leq C \int_0^\infty r^{2(1-N)} r^{N-1} dr < \infty$$

and

$$\left| \int_{\mathbb{R}^N} F(u) \right| \leq C \int_0^\infty r^{3(2-N)} r^{N-1} dr < \infty.$$

In dimension $1 \leq N \leq 3$, the decay to 0 is not sufficient to define $J(u)$, at least if $f^{'}(0) = 0$. In order to do so, we have to assume some further conditions about the decay of u, for instance

$$u(x), |\nabla u(x)| \leq C|x|^{-\frac{N+\sigma}{2}} \quad \text{for } |x| \geq B \tag{1.18}$$

for appropriate constants $B > 0$, $\sigma > 0$. In the next lemma, we will compute explicitly $J(u)$, and we will see that $J(u) > 0$.

Lemma 57. *Let $u > 0$ be a solution to the problem*

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with $f \in C^2(\mathbb{R})$ satisfying (57).

(i) If $N \geq 4$, then

$$J(u) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 > 0 \tag{1.19}$$

(ii) If $1 \leq N \leq 3$, the same formula holds if u fulfills condition (1.18) and $f \in C^1$.

Remark 58. *If $f \in C^1(\mathbb{R})$ with $f^{'}(0) < 0$, thanks to the exponential decay (1.12), formula (1.19) holds true in any dimension $N \geq 1$.*

Proof. If $N = 1$, condition (1.18) guarantees that $J(u)$ is well defined and finite. By statement (ii) of Lemma 44, $\frac{1}{2}(u^{'})^2 + F(u) = 0$, therefore $J(u) = \int_{-\infty}^{\infty}(u^{'})^2 > 0$, unless $u \equiv 0$.

Now we observe that, in any dimension $N \geq 2$ and for any nonlinearity f fulfilling the (57), any solution to (1.19) is radially symmetric, that is, up to a translation, $u(x) = v(|x|)$, where v satisfies that ODE

$$-v^{''} - \frac{N-1}{r}v^{'} = f(v)$$

We multilpy the ODE by $v^{'}r^N$ and integrate to obtain

$$-\int_0^{\infty} v^{''}v^{'}r^N dr - (N-1)\int_0^{\infty}(v^{'})^2 r^{N-1}dr = \int_0^{\infty} f(v)v^{'}r^N dr$$

Integrating by parts we get

$$\int_0^{\infty} f(v)v^{'}r^N dr = \left[F(v)r^N\right]_0^{\infty} - N\int_0^{\infty} F(v)r^{N-1}dr$$

and

$$2\int_0^{\infty} v^{''}v^{'}r^N dr = \left[(v^{'})^2 r^N\right]_0^{\infty} - N\int_0^{\infty}(v^{'})^2 r^{N-1}dr$$

If $f^{'}(0) < 0$, thanks to the exponential decay, all integrals are well defined and finite and all boundary terms vanish. Finally, we get

$$\frac{N-2}{2N}\int_0^{\infty}(v^{'})^2 r^{N-1}dr = \int_0^{\infty} F(v)r^{N-1}dr \tag{1.20}$$

If $N = 2$, we already see that $\int_0^{\infty} F(v)r^{N-1}dr = 0$, hence $J(u) = \frac{1}{2}\int_0^{\infty}(v^{'})^2 r dr > 0$. In higher dimension, a computation show that $J(u) = \frac{1}{N}\int_0^{\infty}(v^{'})^2 r^{N-1}dr > 0$.

If $f^{'}(0) = 0$, we have no exponential decay, so it is harder to verify that all the integrals are well defined and finite and that the boundary terms vanish. In order to do so, in dimension $1 \leq N \leq 3$ we use condition (1.18), while in higher dimension, by Corollary 54 and 55, the decay at infinity is enough to guarantee (1.15) and (1.17), hence all the integrals are well defined and finite and the boundary terms vanish. □

Now we prove Theorem 33.

Proof. As first, we point out that, in dimension $N \geq 5$, by Lemma 56, condition (65) is enough to guarantee suitable decay to define $H(u)$. For any sequence $x_N^k \to \infty$, it is possible to find a subsequence such that $u^k(x) = u(x', x_N + x_N^k)$ converges to a profile $\to u^\infty$ in the $C^{2,\alpha}_{loc}$ sense. By hypothesis (66),

$$u^\infty(x_0', 0) = \lim_{k \to \infty} u^k(x_0', 0) = \lim_{k \to \infty} u(x_0', x_N^k) = 0$$

hence $u^\infty \equiv 0$. Since the sequence is arbitrary, by Lemma 46, $u(x', x_N) \to 0$ as $x_N \to \infty$, uniformly in x', so we can apply Proposition 36 to begin the moving plane procedure (see Remark 4). Now, since we do not know the behaviour of u for $x_N \to -\infty$, we have to be careful to exclude the case $\overline{\lambda} = -\infty$. Assume, by contradiction, that $\overline{\lambda} = -\infty$. Then we get $u_N \leq 0$ and therefore, since u_N satisfies $-\Delta u_N = f'(u)u_N$, by the strong maximum principle we have $u_N < 0$, hence it is possible to define, for any $x' \in \mathbb{R}^{N-1}$,

$$\underline{u}(x') = \lim_{x_N \to -\infty} u(x', x_N).$$

By the Arzelà-Ascoli theorem, it is possible to check that the convergence holds in C^2_{loc}, hence the profile $\underline{u}$ satisfies

$$\begin{cases} -\Delta \underline{u} = f(\underline{u}) & \text{in } \mathbb{R}^{N-1} \\ \underline{u} > 0 & \\ \underline{u}(x') \to 0 & \text{as } |x'| \to \infty \end{cases}$$

Therefore, applying Lemma 57 to $\underline{u}$, we get that $J(\underline{u}) > 0$.

However, by relation (1.18), $J(\underline{u})$ is well defined and finite and

$$J(\underline{u}) = \int_{\mathbb{R}^{N-1}} \frac{1}{2}|\nabla \underline{u}|^2 - F(\underline{u})dx' =$$
$$\lim_{x_N} \mathcal{H}(u, x_N) = \mathcal{H}(u) = \lim_{x_N \to \infty} \mathcal{H}(u, x_N) = 0,$$

a contradiction.

As a consequence, we get that $\overline{\lambda} \in \mathbb{R}$ and $u - u_{\overline{\lambda}} \geq 0$ in $\Sigma_{\overline{\lambda}}$. By the strong maximum principle, we have that $u > u_{\overline{\lambda}}$ or $u \equiv u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$. To conclude the proof of the theorem we have to exclude the first possibility.

Assume, by contradiction, that $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$. By Proposition 40, applied to the case $M = 0$, we can find a sequence of real numbers $\overline{\lambda} - 1/k \leq \lambda_k < \overline{\lambda}$ and a bounded sequence of points $x^k \in \Sigma_{\lambda_k}$, such that

$$u(x^k) < u_{\lambda_k}(x^k).$$

Up to a subsequence, $x^k \to x^\infty$, therefore $u(x^\infty) \leq u_{\overline{\lambda}}(x^\infty)$. Since we are assuming that $u > u_{\overline{\lambda}}$ in $\Sigma_{\overline{\lambda}}$, we get that $x_N^\infty = \overline{\lambda}$, but this is a contradiction to the Hopf lemma, as above. To conclude, we observe that the symmetry in the x_N variable yields that

$$u(x', x_N) \to 0 \quad \text{as } x_N \to -\infty, \text{ uniformly in } x',$$

hence, by the result by Gidas, Ni and Nirenberg in [37], we get the radial symmetry.

□

Part II

Construction of solutions to some semilinear PDEs

Introduction: the Lyapunov-Schmidt reduction

In this section we will give the outlines of some type of Lyapunov-Schmidt reduction. This technique is useful, for instance, to solve differential equations, both in the case of ODEs and PDEs. In some sense, it is a generalization of the implicit function theorem to the case of a noninjective operator.

Let Y_1, Y_2 be Banach subspaces with respect to the norms $||\cdot||_1$ and $||\cdot||_2$ respectively, and

$$F : Y_1 \to Y_2 \tag{1.21}$$

a possibly nonlinear operator. Suppose we want to find a family u_ε of solutions to the equation

$$F(u_\varepsilon) = 0, \quad u_\varepsilon \in Y_1, \tag{1.22}$$

for $\varepsilon > 0$ small enough. If F is a $C^{2,0}(Y_1, Y_2)$, we can Taylor-expand (1.22) and see that (1.22) is equivalent to an equation of the form

$$0 = F(v + w) = F(v) + F'(v)w + Q_v(w), \tag{1.23}$$

where

$$Q_v(w) := \int_0^1 dt \int_0^t F''(v + sw)[w, w]ds \tag{1.24}$$

satisfies

$$||Q_v(w)||_2 \le c||w||_1^2 \tag{1.25}$$

$$||Q_v(w_1) - Q_v(w_2)||_2 \le c(||w_1||_1 + ||w_2||_1)||w_1 - w_2||_1, \tag{1.26}$$

for any $w, w_1, w_2 \in Y_1$ such that $||w||_1, ||w_1||_1, ||w_2||_1 < 1$, for some constant $c > 0$ independent of v.

Proposition 59. *Assume that, for any $\varepsilon > 0$, there exists $v_\varepsilon \in Y_1$ such that $||F(v_\varepsilon)||_2 \le c_1\varepsilon$. Moreover, we assume that $F'(v_\varepsilon) : Y_1 \to Y_2$ satisfies*

$$||w||_1 \le c_2||F'(v_\varepsilon)w||_2, \quad \forall w \in Y_1, \tag{1.27}$$

for some constant $c_2 > 0$ independent of ε and it is surjective. Then, for ε small enough, there exists a unique $w = w_\varepsilon \in Y_1$ such that $u_\varepsilon := v_\varepsilon + w$ solves (1.22).

Proof. Our hypotheses yield that $F'(v_\varepsilon)$ is invertible. Expanding F in Taylor series and composing with the inverse of $F'(v_\varepsilon)$, we can see that (1.22) is equivalent to

$$w = Tw := F'(v_\varepsilon)^{-1}(-F(v_\varepsilon) - Q_{v_\varepsilon}(w)).$$

By (1.27) and (1.25), we can see that

$$||Tw||_1 \leq c_2(c_1\varepsilon + cC^2\varepsilon^2) < C\varepsilon \tag{1.28}$$

if $||w||_1 \leq C\varepsilon$, provided $C > c_1c_2$. In other words, T maps the ball $\{w \in Y_1 : ||w||_1 \leq C\varepsilon\}$ into itself.

Moreover, T is Lipschitz continuous with constant of order ε, that is it fulfills

$$||Tw_1 - Tw_2||_1 \leq \tilde{c}\varepsilon||w_1 - w_2||_1,$$

by (1.26). Hence it is a contraction, thus it admits a unique fixed point $w = w_\varepsilon$. □

Remark 60. *The same arguments work even if*

$$F'(v_\varepsilon) = \mathcal{L}_\varepsilon + \varepsilon\overline{L}_\varepsilon,$$

where $\mathcal{L}_\varepsilon$ satisfies (1.27) and is surjective and $\overline{L}_\varepsilon : Y_1 \to Y_2$ is bounded uniformly in ε.

In other words, up to now we have shown that, if we can find an appropriate approximate solution v_ε and $F'(v_\varepsilon)$ is invertible, with inverse bounded uniformly in ε, then our original equation $F(u_\varepsilon) = 0$ is solvable thanks to the implicit function theorem.

The situation described above is quite simple, but it is often far from what actually happens in most of the applications, where $F'(v_\varepsilon)$ usually has a nontrivial kernel. In order to overcome this difficulty, one introduces the *Lyapunov Schmidt reduction.* We consider families $\{Y_{1,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ and $\{Y_{2,\varepsilon}\}_{0<\varepsilon<\varepsilon_0}$ of Banach spaces and a family $\{H_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of Hilbert spaces, endowed with the scalar product $<\cdot\,,\cdot>_\varepsilon$, such that $Y_{1,\varepsilon} \subset Y_{2,\varepsilon} \subset H_\varepsilon$ with dense inclusions. $Y_{1,\varepsilon}$ and $Y_{2,\varepsilon}$ are endowed with the norms $||\cdot||_1$ and $||\cdot||_2$ (we omit the subscript ε in the norms in order to simplify the notation). We introduce a space Ω of parameters, that is a subset of some Banach space X, endowed with the norm $|\cdot|_X$, and a family $\{\mathcal{L}_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of operators

$$\mathcal{L}_\varepsilon : \Omega \to L(Y_{1,\varepsilon}, Y_{2,\varepsilon}),$$

where $L(Y_{1,\varepsilon}, Y_{2,\varepsilon})$ is the space of linear continuous operators from $Y_{1,\varepsilon}$ to $Y_{2,\varepsilon}$, with respect to the norms $||\cdot||_1$ and $||\cdot||_2$, and a family $\{\mathcal{F}_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of functionals

$$\mathcal{F}_\varepsilon : X \times Y_{1,\varepsilon} \to Y_{2,\varepsilon},$$

possibly nonlinear in both variables. If X is finite-dimensional, we speak of *finite-dimensional* Lyapunov-Schmidt reduction, otherwise we speak of *infinite-dimensional* Lyapunov-Schmidt reduction. The aim is to find a family of solutions $(\phi, w) = (\phi, w)_\varepsilon \in \Omega \times Y_{1,\varepsilon}$ to

$$\mathcal{L}_\varepsilon(\phi)[w] = \mathcal{F}_\varepsilon(\phi, w). \tag{1.29}$$

We suppose that there exists a subspace $T_{\varepsilon,\phi} \subset H_\varepsilon$ such that

$(a1)$ $\forall g \in Y_{2,\varepsilon} \cap T_{\varepsilon,\phi}$, there exists a unique $w := \Phi_{\varepsilon,\phi}(g) \in Y_{1,\varepsilon} \cap T_{\varepsilon,\phi}$ such that $\mathcal{L}_\varepsilon(\phi)[w] = g$, and $||w||_1 \leq c||g||_2$, for some constant $c > 0$ independent of ε and ϕ.

$(a2)$ $\mathcal{L}_\varepsilon(\phi)(Y_{1,\varepsilon} \cap T_{\varepsilon,\phi}) \subset T_{\varepsilon,\phi}$, $\forall \phi \in \Omega$, $\forall 0 < \varepsilon < \varepsilon_0$.

Requirement $(a2)$ is equivalent to the existence of a right inverse of $\mathcal{L}_\varepsilon(\phi)$, independent of ε and ϕ. In the applications, it is often convenient to show this property by means of the Fredholm alternative or, more frequently, by variational techniques, such as the direct method of calculus of variations.

We define, for any $0 < \varepsilon < \varepsilon_0$ and for any $\phi \in \Omega$, the projection $\Pi_{\varepsilon,\phi} : H_\varepsilon \to T_{\varepsilon,\phi}$. We assume that

$(a3)$ $\Pi_{\varepsilon,\phi}(Y_{i,\varepsilon}) \subset Y_{i,\varepsilon}$ and $||\Pi_{\varepsilon,\phi} w||_i \leq c||w||_i$, for $i = 1, 2$, for some constant $c > 0$ independent of ε and ϕ.

By $(a2)$, composing (1.29) with the projection $\Pi_{\varepsilon,\phi}$ and with $Id - \Pi_{\varepsilon,\phi}$, where $Id = Id_{Y_{2,\varepsilon}}$, we get the system

$$\mathcal{L}_\varepsilon(\phi)[w] = \Pi_{\varepsilon,\phi}\mathcal{F}_\varepsilon(\phi, w), \quad w \in W_{\varepsilon,\phi} \tag{1.30}$$

$$(Id - \Pi_{\varepsilon,\phi})\mathcal{F}_\varepsilon(\phi, w) = 0, \tag{1.31}$$

where we have set $W_{\varepsilon,\phi} := T_{\varepsilon,\phi} \cap Y_{1,\varepsilon}$. We note that our system is well defined thanks to $(a3)$. First we fix $0 < \varepsilon < \varepsilon_0$ and $\phi \in \Omega$ and we find a solution $w = w_\varepsilon(\phi) \in Y_{1,\varepsilon}$ to the projected equation (1.30), known as the *auxiliary equation*, and then we look for a solution to the reduced problem (1.31), known as the *bifurcation equation*. In this way we actually have two unknows, ϕ and w, instead of one, and we need to solve a system. The auxiliary equation is the easier to solve, since either variational techniques or the Fredholm alternative usually work in this kind of situation, while it is usually harder to prove existence of a solution to the bifurcation equation. However, we will see that this decomposition is convenient in many situations, especially in case we have some further information about the solvability of the bifurcation equation, due to the particular problem from which our equation comes.

In most of the applications, it happens that

$$T_{\varepsilon,\phi} := \{w \in H_\varepsilon :< w, v >_\varepsilon = 0, \quad \forall v \in V_{\varepsilon,\phi}\}, \tag{1.32}$$

where

$$V_{\varepsilon,\phi} := \{n \in Y_{1,\varepsilon} : \mathcal{L}_\varepsilon(\phi)[n] = 0\} \tag{1.33}$$

is the kernel of $\mathcal{L}_\varepsilon(\phi)$. In this particular framework, it follows that, if there exists a solution $w \in W_{\varepsilon,\phi} = T_{\varepsilon,\phi} \cap Y_{1,\varepsilon}$ to $\mathcal{L}_\varepsilon(\phi)[w] = g$, then it is automatically unique. Moreover, if $\mathcal{L}_\varepsilon(\phi)$ is self-adjoint with respect to the scalar product $<\cdot,\cdot>_\varepsilon$, then condition $(a2)$ above is automatically satisfied and $T_{\varepsilon,\phi} \cap Y_{2,\varepsilon} = Range(\mathcal{L}_\varepsilon(\phi))$. However, sometimes it happens that $T_{\varepsilon,\phi}$ is just *contained* in the orthogonal complement of the kernel, and not exactly equal (see Chapter 3 or [24, 63]).

As regards $\mathcal{F}_\varepsilon$, we require

$(\mathcal{F}1)$ $||\mathcal{F}_\varepsilon(\phi, 0)||_2 \leq c\varepsilon^\beta$, for some $\beta > 0$.

$(\mathcal{F}2)$ $||\partial_w \mathcal{F}_\varepsilon(\phi, 0)[h]||_2 \leq c\varepsilon||h||_1$, for any $0 < \varepsilon < \varepsilon_0$, $\phi \in \Omega$ and $h \in Y_{1,\varepsilon}$, with $c > 0$ independent of ε and ϕ.

$(\mathcal{F}3)$ $||\partial_w^2 \mathcal{F}_\varepsilon(\phi, w)[h, k]||_2 \leq c||h||_1||k||_1$, for any $0 < \varepsilon < \varepsilon_0$, $\phi \in \Omega$, $w \in Y_{1,\varepsilon}$ and $h, k \in Y_{1,\varepsilon}$, with $c > 0$ independent of ε and ϕ.

Proposition 61. *Assume that $Y_{1,\varepsilon} \subset Y_{2,\varepsilon} \subset H_\varepsilon$ are dense subspaces of the Hilbert space H_ε, Banach with respect to the norms $||\cdot||_1$ and $||\cdot||_2$ respectively. Assume furthermore that $\mathcal{F}$ satisfies $(\mathcal{F}1)$, $(\mathcal{F}2)$ and $(\mathcal{F}3)$ and $(a1)$, $(a2)$, $(a3)$ are fulfilled.*

Then equation (1.30) admits a unique solution $w = w_{\varepsilon,\phi} \in W_{\varepsilon,\phi}$ satisfying

$$||w_{\varepsilon,\phi}||_1 \leq C\varepsilon^\beta. \tag{1.34}$$

Remark 62. *Sometimes it is convenient to look for a solution in a smaller closed subspace $\tilde{Y}_{1,\varepsilon} \subset Y_{1,\varepsilon}$. In this situation, it is quite natural to assume that there exists a closed subspace $\tilde{Y}_{2,\varepsilon} \subset Y_{2,\varepsilon}$ such that $\mathcal{F}_\varepsilon(\phi, w) \in \tilde{Y}_{2,\varepsilon}$, for any $w \in \tilde{Y}_{1,\varepsilon}$, and, that, if $g \in \tilde{Y}_{2,\varepsilon}$, then the solution $w = \Phi_{\varepsilon,\phi}(g)$ found thanks to $(\mathcal{L}2)$ belongs to $Y_{2,\varepsilon}$. These hypothesis are particularly useful in case we want our solutions to fulfill* *some symmetry properties, since in general the spaces of functions that respect some symmetries are not dense in* L^2*, and hence not convenient to speak about self-adjointness. However, these hypothesis are quite frequently verified thanks to uniqueness.*

Proof. The existence and estimate (1.34) follow from a fixed point argument, like in the proof of Proposition (59). In fact, equation (1.30) is equivalent to

$$w = Tw := \Phi_{\varepsilon,\phi}\Pi_{\varepsilon,\phi}\mathcal{F}_\varepsilon(\phi, w),$$

where T is a contraction on the ball

$$B := \{w \in Y_1 : ||w||_1 \leq C\varepsilon^\beta\}.$$

In fact, expanding $\mathcal{F}_\varepsilon$ in Taylor series with respect to w, we can see that

$$\mathcal{F}_\varepsilon(\phi, w) = \mathcal{F}_\varepsilon(\phi, 0) + \partial_w \mathcal{F}_\varepsilon(\phi, 0)[w] + Q_{\varepsilon,\phi}(w),$$

where

$$Q_{\varepsilon,\phi}(w) := \int_0^1 dt \int_0^t \partial_w^2 \mathcal{F}_\varepsilon(\phi, sw)[w, w]ds,$$

so in particular, by $(\mathcal{F}1)$,

$$||\mathcal{F}(\varepsilon, \phi, w)||_2 \leq c_1\varepsilon^\beta + c_2\varepsilon||w||_1 + c_3||w||_1^2, \quad \forall w \in W, \forall \phi \in \Omega,$$

thus

$$||\Phi_{\varepsilon,\phi}\Pi_{\varepsilon,\phi}\mathcal{F}_\varepsilon(\phi, w)||_1 \leq c(c_1\varepsilon^\beta + c_2 C\varepsilon^{\beta+1} + c_3 C^2\varepsilon^{2\beta}) \leq C\varepsilon^\beta, \tag{1.35}$$

if $w \in B$ and $C > cc_1$. Moreover, once again by the Taylor expansion and by $(\mathcal{F}2)$,

$$||\mathcal{F}(\varepsilon, \phi, w_1) - \mathcal{F}(\varepsilon, \phi, w_2)||_2 \leq c(\varepsilon + ||w_1||_1 + ||w_2||_1)||w_1 - w_2||_1$$

for any $w_1, w_2 \in Y_{1,\varepsilon}$ with $||w_1||_1, ||w_2||_1 < 1$, for any $\phi \in \Omega$, so that

$$||Tw_1 - Tw_2||_1 \leq 2c\varepsilon||w_1 - w_2||_1, \quad \forall w_1, w_2 \in B.$$

We stress that, in the hypothesis of Remark 62, T maps $\tilde{Y}_{1,\varepsilon}$ into itself, thus it possible to find a solution in $\tilde{B} := B \cap \tilde{Y}_{1,\varepsilon}$. □

Since it is useful to have a solution $w_\varepsilon(\phi)$ depending on the datum ϕ in a Lipschitz way, we introduce some further hypothesis about the Lipschitz dependence of $\mathcal{L}_\varepsilon$ and $\mathcal{F}_\varepsilon$ on the parameter ϕ. More precisely, we require

$$||\mathcal{F}_\varepsilon(\phi_1, w) - \mathcal{F}_\varepsilon(\phi_2, w)||_2 \leq c(\varepsilon^\beta + ||w||_1)|\phi_1 - \phi_2|_X, \tag{1.36}$$
$$\forall \phi_1, \phi_2 \in \Omega, |\phi_1|_X, |\phi_2|_X \leq c\varepsilon, c > 0, \forall w \in Y_{1,\varepsilon}, ||w||_1 < 1$$

and

$$||(\mathcal{L}_\varepsilon(\phi_1) - \mathcal{L}_\varepsilon(\phi_2))w||_2 \leq c|\phi_1 - \phi_2|_X ||w||_1, \quad \forall \phi_1, \phi_2 \in \Omega, \forall w \in Y_{1,\varepsilon}. \tag{1.37}$$
$$||(\Pi_{\varepsilon,\phi_1} - \Pi_{\varepsilon,\phi_2})g||_2 \leq c|\phi_1 - \phi_2|_X ||g||_2, \quad \forall \phi_1, \phi_2 \in \Omega, \forall g \in Y_{2,\varepsilon}. \tag{1.38}$$

Lemma 63. *If we define $T_{\varepsilon,\phi}$ and $V_{\varepsilon,\phi}$ as in (1.32) and (1.33), then condition (1.38) is automatically fulfilled.*

Proof. First we observe that, if $n_2 \in V_2$, then

$$(\mathcal{L}_1 - \mathcal{L}_2)n_2 = \mathcal{L}_1 n_2 = \mathcal{L}_1 \Pi_1 n_2,$$

thus $(\mathcal{L}_1 - \mathcal{L}_2)n_2 \in R_1 = Y_{\varepsilon,2} \cap W_1$. Since, by construction, $\Pi_2 n_2 = 0$ and by uniqueness, we have

$$||(\Pi_1 - \Pi_2)n_2||_2 = ||\Pi_1 n_2||_2 \leq c||(\mathcal{L}_1 - \mathcal{L}_2)n_2||_2 \leq c||n_2||_2|\phi_1 - \phi_2|_X, \quad \forall n_2 \in N_2.$$

Now we show that the same is true if $w \in W_2$. In fact, $\mathcal{L}_1(Id - \Pi_1)w = 0$, therefore

$$(\mathcal{L}_1 - \mathcal{L}_2)(Id - \Pi_1)w = -\mathcal{L}_2(Id - \Pi_1)w,$$

which yields that $(\mathcal{L}_1 - \mathcal{L}_2)(Id - \Pi_1)w \in R_2 = Y_{\varepsilon,2} \cap T_2$. Once again by uniqueness and by the fact that, by construction, $w = \Pi_2 w$, we get

$$||(\Pi_2 - \Pi_1)w||_2 = ||(Id - \Pi_1)w||_2 \leq$$
$$c||(\mathcal{L}_1 - \mathcal{L}_2)(Id - \Pi_1)w||_2 \leq c||w||_2|\phi_1 - \phi_2|_1, \quad \forall w \in W_2.$$

□

In conclusion, we state the existence result for the solution to (1.30) in the particular case in which $(a1)$, $(a2)$, $(a3)$ are fulfilled, which also yields the Lipschitz dependence on the datum ϕ.

Proposition 64. *Assume the same hypothesis as in Proposition 61. Assume furthermore that (1.36), (1.37) and (1.39) are fulfilled too. Then there exists a unique solution $w_{\varepsilon,\phi} \in W_{\varepsilon,\phi}$ to (1.30) and*

$$||w_{\varepsilon,\phi_1} - w_{\varepsilon,\phi_2}||_1 \leq C\varepsilon^\beta |\phi_1 - \phi_2|_X, \tag{1.39}$$

for any $\phi_1, \phi_2 \in \Omega$.

Proof. Existence follows from Proposition 61 .

In order to prove the Lipschitz dependence on the datum ϕ, we set, for the sake of simplicity, $w_i := w_{\varepsilon,\phi_i}$, $\mathcal{L}_i := \mathcal{L}_\varepsilon(\phi_i)$ and so on, $i = 1, 2$. In the sequel of the proof, we will use the notations of $(a1)$, $(a2)$ and $(a3)$. We observe that

$$\mathcal{L}_1(w_1 - w_2) = \Pi_1(\mathcal{F}_\varepsilon(\phi_1, w_1) - \mathcal{F}_\varepsilon(\phi_2, w_1)) \\ +\Pi_1(\mathcal{F}_\varepsilon(\phi_2, w_1) - \mathcal{F}_\varepsilon(\phi_2, w_2)) + (\Pi_1 - \Pi_2)\mathcal{F}_\varepsilon(\phi_2, w_2).$$

Moreover, since $w_2 \in W_2$, that is $w_2 = \Pi_2 w_2$, we have

$$w_1 - w_2 = w_1 - (\Pi_1 w_2 + (Id - \Pi_1)w_2) = w_1 - \Pi_1 w_2 - (\Pi_2 - \Pi_1)w_2.$$

Since, by $(a3)$,

$$\Pi_{\varepsilon,\phi}(Y_{1,\varepsilon}) \subset Y_{1,\varepsilon}, \tag{1.40}$$

then $w_1 - \Pi_1 w_2$ and $(\Pi_2 - \Pi_1)w_2$ belong to $Y_{1,\varepsilon}$ and, by (1.38),

$$||(\Pi_2 - \Pi_1)w_2||_1 \le c||w_2||_1|\phi_1 - \phi_2|_X \le c\varepsilon^\beta|\phi_1 - \phi_2|_X.$$

It remains to estimate $||w_1 - \Pi_1 w_2||_1$. In order to do so, we compute

$$\mathcal{L}_1(w_1 - \Pi_1 w_2) = \mathcal{L}_1(\Pi_2 - \Pi_1)w_2 + \Pi_1(\mathcal{F}_\varepsilon(\phi_1, w_1) - \mathcal{F}_\varepsilon(\phi_2, w_1)) \\ +\Pi_1(\mathcal{F}_\varepsilon(\phi_2, w_1) - \mathcal{F}_\varepsilon(\phi_2, w_2)) + (\Pi_1 - \Pi_2)\mathcal{F}_\varepsilon(\phi_2, w_2).$$

We observe that, by $(\mathcal{F}2)$ and $(\mathcal{F}3)$, we have

$$||\mathcal{F}_\varepsilon(\phi, v_1) - \mathcal{F}_\varepsilon(\phi, v_2)||_1 \le ||\partial_w \mathcal{F}_\varepsilon(\phi, 0)[v_1 - v_2]||_1 \\ +||Q_{\varepsilon,\phi}(v_1) - Q_{\varepsilon,\phi}(v_2)||_1 \le c(\varepsilon + ||v_1||_1 + ||v_2||_1)||v_1 - v_2||_1,$$

for any $\phi \in X$ and for any v_1, v_2 with $||v_1||_1, ||v_2||_1 < 1$. Since $w_1 - \Pi_1 w_2 \in W_1$, by uniqueness, (1.37) and (1.38), we can see that

$$||w_1 - \Pi_1 w_2||_1 \le c||(\Pi_2 - \Pi_1)w_2||_1 + c(\varepsilon^\beta + ||w_1||_1)||w_1 - \Pi_1 w_2||_1 \\ +c(\varepsilon + ||w_1||_1 + ||w_2||_1)||w_1 - w_2||_1 + c|\phi_1 - \phi_2|_X||\mathcal{F}_\varepsilon(\phi_2, w_2)||_2 \le \\ c(\varepsilon^\beta|\phi_1 - \phi_2|_X + \varepsilon||w_1 - \Pi_1 w_2||_1),$$

thus

$$\frac{1}{2}||w_1 - \Pi_1 w_2||_1 \le (1 - c\varepsilon)||w_1 - \Pi_1 w_2||_1 \le c\varepsilon^\beta|\phi_1 - \phi_2|_X.$$

□

Now the auxiliary equation is solved, and the solution depends on ϕ in a Lipschitz way. It remains to consider the bifurcation equation. Here it is hard to give the outlines of a general theory, since the way of solving it really depends on the problem. However, in many situation this equation is solved by means of geometric expansions (see, for instance, Section 3.1, [24, 63]) and fixed point arguments, based on the Lipschitz dependence of w on the datum ϕ, that we proved above (see 1.39). Moreover, let us stress that, up to now, there is no difference between the finite-dimensional case and the infinite-dimensional one, apart from the fact that the absolute value of ϕ has to be replaced by some suitable norm in an appropriate Banach space. These difference only affects the bifurcation equation, that is either a $m \times m$ system in the finite-dimensional case, or something more complicated, typically a differential equation, in the infinite-dimensional one.

Chapter 2

Critical points of the Otha-Kawasaki functional

2.1 The Otha-Kawasaki functional

A diblock copolymer is a complex molecule where chains of two different kinds of monomers, say A and B, are grafted togheter. Diblock copolymer melts are large collections of diblock copolymers. The experiments show that, above a certain temperature, these melts behave like fluids, that is the monomers are mixed in a disordered way, while below this critical temperature phase separation is observed. Some common periodic structures observed in experiments are spheres, cylinders, gyroids and lamellae (see figure 2.1). These patterns can be found by

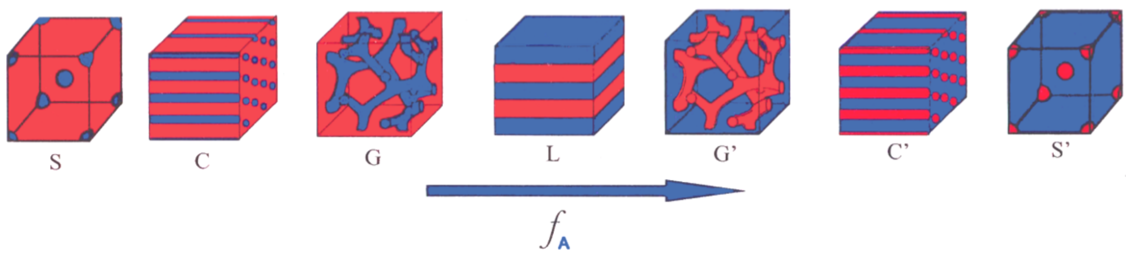

Figure 2.1: The most commonly observed periodic structures are spheres, cylinders, gyroids and lamellae

minimizing some energy. It looks reasonable to describe the phenomenon through an energy given by the sum of the perimeter, that forces the separation surfaces to be minimal, plus some nonlocal term that keeps trace of the long-range interactions between monomers. More explicitly, one can take the functional

$$\mathcal{E}_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u| dx + \varepsilon \int_\Omega \int_\Omega G(x,y)(u(x)-m)(u(y)-m)dxdy \tag{2.1}$$

as an energy. Here Ω is a bounded domain of $\mathbb{R}^3$, that can be seen as the container where the diblock copolymer melt is confined, u is a bounded variation function in Ω with values in $\{\pm 1\}$ (for instance, we can assume that $u(x) = 1$ if there are only monomers of type A at x, $u(x) = -1$ if there are only monomers of type B at x), $m := \frac{1}{|\Omega|}\int_\Omega u dx$, $\int_\Omega |\nabla u| dx$ is its total variation, or equivalently the perimeter of the set $\{x \in \Omega : u(x) = 1\}$, G is the Green's function of $-\Delta$ on Ω, that is the distributional solution to

$$\begin{cases} -\Delta_x G(x,y) = \delta_y(x) - \frac{1}{|\Omega|} & \text{in } \Omega \\ \partial_{\nu(x)} G(x,y) = 0 & \text{on } \partial\Omega. \end{cases}$$

G turns out to be the sum of the Green's function of $-\Delta$ over $\mathbb{R}^3$ and a regular part $R(x,y)$, namely

$$G(x,y) = \frac{c}{|x-y|} + R(x,y),$$

(see [64]). $\varepsilon \geq 0$ is a parameter depending on the material, that we will assume to be small.

This energy appears as the Γ-limit as $\gamma \to 0$ of the approximating functionals

$$\mathcal{E}_{\varepsilon,\gamma}(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\gamma}\int_\Omega \frac{(1-u^2)^2}{4} dx$$
$$+ \frac{16\varepsilon}{3}\int_\Omega \int_\Omega G(x,y)(u(x)-m)(u(y)-m)dxdy,$$

introduced by Otha and Kawasaki (see [1, 15, 16, 17]).

In a more geometric way our functional is given by

$$J_\varepsilon(E) := P_\Omega(E) + \varepsilon \int_\Omega \int_\Omega G(x,y)(u_E(x)-m)(u_E(y)-m)dxdy \tag{2.2}$$

where

$$E := \{x \in \Omega : u(x) = 1\},$$

so that $u_E = \chi_E - \chi_{\Omega\setminus E}$. The first variation of J_ε is given by

$$J_\varepsilon'(E)[\varphi] = \int_\Sigma (H_\Sigma(x) + 4\varepsilon v_E(x))\varphi(x)d\sigma(x), \tag{2.3}$$

while its second variation is given by

$$J_\varepsilon''(E)[\varphi] = \int_\Sigma L\varphi(x)\varphi(x)d\sigma(x), \tag{2.4}$$

where

$$L\varphi = -\Delta_\Sigma\varphi - |A|^2\varphi + 8\varepsilon \int_\Sigma G(\cdot\,, y)\varphi(y)d\sigma(y) + 4\varepsilon\partial_\nu v\varphi. \tag{2.5}$$

Here $\varphi \in H^1(\Sigma)$, $\Sigma := \partial E$ and

$$v_E(x) := \int_{T^3} G(x,y)(u_E(y) - m)dy \tag{2.6}$$

is the unique solution to the problem

$$\begin{cases} -\Delta v_E = u_E - m & \text{in } T^3 \\ \int_{T^3} v_E dx = 0. \end{cases} \tag{2.7}$$

For an explicit computation of the first and the second variation, see for instance [18]. In the sequel, Ω will always be the 3-dimensional torus T^3, that is the quotient of the cube $[0,1]^3$ by the equivalence relation that identifies the opposite faces. It is known that J_ε is translation invariant, that is $J_\varepsilon(E+\phi) = J_\varepsilon(E)$, for any $\phi \in T^3$ (see [1],[18]), thus, once we find a critical point of it, any translation in T^3 is still critical.

There are several results in the literature about critical points of this functional. For instance, an interesting problem is to understand whether all global minimizers are periodic, like the patterns described above (spheres, cylinders, gyroids and lamellae, see Figure 2.1). This is known to be true in dimension one (see [59]), but the problem is still open in higher dimension. We refer to [3, 74] for further results. Some other authors, such as Ren and Wei [64, 65, 66, 67, 68], constructed explicit examples of stable periodic local minimizers, that is with positive second variation. Moreover, Acerbi, Fusco and Morini [1] showed that any stable critical point is actually a local minimizer with respect to small L^1 perturbations.

Here we add a small linear perturbation that corresponds to an external force f applied to the system, that can be taken to be $C^{0,1}_{loc}(\mathbb{R}^3)$ and periodic, with triple period 1. The energy becomes

$$I_\varepsilon(E) := J_\varepsilon(E) + \varepsilon \int_\Omega f(x)u_E(x)dx. \tag{2.8}$$

The additional linear term breakes the translation invariance. We will construct at least four critical points F_j of I_ε, $1 \leq j \leq 4$, for ε small enough, that are close to suitable translations of the Schwarz P surface Σ (see figure 2.2), under the volume constraint

$$|F_j|_3 = |E|_3, \tag{2.9}$$

where E is the interior of Σ.

Remark 65. *The Schwartz P surface can be seen as a periodic surface in* $\mathbb{R}^3$*, with triple period* 1*. Moreover, it divides the Torus into two components, an interior and an exterior. In the sequel, E will denote the interior part.*

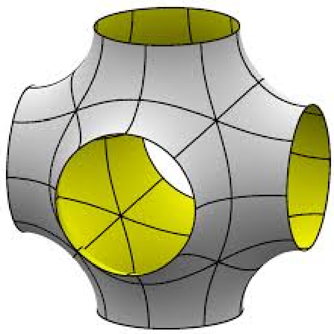

Figure 2.2: Schwarz' P surface

We will use a technique based on a finite dimensional Lyapunov-Schmidt reduction (see [6], Chapter 2.2), and on the Lusternik-Schnirelman theory (see [5], Chapter 9) for the multiplicity.

For $0 < \alpha < 1$ and for any integer $k = 0, 1$, we introduce the Hölder spaces

$$C_s^{k,\alpha}(\overline{\Sigma}) := \{w \in C^{k,\alpha}(\overline{\Sigma}) : w(x) = w(T_j x), 1 \leq j \leq 3\}, \tag{2.10}$$

where T_j are the reflections defined by

$$T_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \quad T_2(x_1, x_2, x_3) = (x_1, -x_2, x_3), \quad T_3(x_1, x_2, x_3) = (x_1, x_2, -x_3).$$

Here it is understood that we have put the origin in the centre of the cube (see Figure 2.2), in such a way that these spaces consist of functions that respect the symmetries of Σ, that is the symmetries with respect to the coordinate planes $\{x_j = 0\}$, $1 \leq j \leq 3$. Moreover, we set

$$C_s^{2,\alpha}(\overline{\Sigma}) := \{w \in C^{k,\alpha}(\overline{\Sigma}) : w(x) = w(T_j x), 1 \leq j \leq 3, \partial_n w = 0 \quad \text{on } \partial\Sigma\}, \tag{2.11}$$

$\partial_n w := (\nabla w, n)$, and n is the outward pointing unit normal to $\partial\Sigma$ in Σ. We endow these spaces with the norm

$$||w||_{C^{k,\alpha}(\Sigma)} = \sum_{j=0}^{k} ||\nabla^j w||_{L^\infty(\Sigma)} + \sup_{x\neq y} \sup_{|\beta|=k} \frac{|\partial_\beta w(x) - \partial_\beta w(y)|}{d(x,y)^\alpha}, \tag{2.12}$$

where d is the geodesics distance on Σ.

Theorem 66. *Let I_ε be defined as in (2.8) and $\nu(x)$ be the outward-pointing unit normal to the Schwarz P surface Σ. Then there exists $\varepsilon_0 > 0$ such that, for any $0 < \varepsilon < \varepsilon_0$, there exist $\phi_j \in T^3$, $1 \leq j \leq 4$, and $w_{\varepsilon,j} \in C^{2,\alpha}_s(\overline{\Sigma})$, with*

$$||w_{\varepsilon,j}||_{C^{2,\alpha}(\Sigma)} \leq c\varepsilon, \tag{2.13}$$

such that the sets F_j defined as the interior of

$$\Gamma_j := \{x + \phi_j + \nu(x) w_{\varepsilon,j}(x) : x \in \Sigma\} \tag{2.14}$$

are critical points of I_ε under the volume constraint

$$|F_j|_3 = |E|_3. \tag{2.15}$$

Remark 67. *(i) We stress that $w_{\varepsilon,\phi_j} \in C^{2,\alpha}_s(\overline{\Sigma})$, thus they satisfy the symmetries Σ.*

(ii) If we take $f \equiv 0$, we find a unique critical point F, that is the interior of

$$\Gamma := \{x + \nu(x) w_\varepsilon(x) : x \in \partial E\}, \tag{2.16}$$

where w_ε is a small correction, namely $||w_\varepsilon||_{C^{2,\alpha}(\Sigma)} \leq c\varepsilon$, found by means of the implicit function Theorem (see Remark 71). Then any translation $F + \phi$ is still a critical point of J_ε. A similar result was proved by Cristoferi (see [21], Theorem 4.18*), who constructed a critical point of J_ε close to any smooth periodic strictly stable constant mean curvature surface.*

(iii) We stated the theorem in the case of I_ε for simplicity. The same proof should yield existence and multiplicity results also for regular nonlinear perturbations and different coefficients in the nonlocal and forcing terms.

A similar result was obtained by Bonacini and Cristoferi [12], who studied a nonlocal version of the isoperimetric problem, that is they considered a small nonlocal perturbation of the perimeter and showed that the unique minimizers F under the volume constraint $|F|_N = m$ are the balls, provided m is small enough.

The critical points we construct here are not necessarily stable, since we apply the Lusternik-Schnirelmann theory (see [5], chapter 9).

A crucial tool in the proof is nondegeneracy up to translations of the Jacobi operator of the Schwarz P surface. In [72], Ross showed that the Schwarz P surface is a critical point of the area and it is *volume preserving stable*, that is it the second variation of the area is non-negative on any normal variation with zero average. More precisely, setting $I_0 := P_\Omega$, we have

$$I_0''(E)(\varphi,\varphi) = \int_\Sigma |\nabla_\Sigma \varphi|^2 - |A|^2\varphi^2 d\sigma \geq 0 \tag{2.17}$$

for any $\varphi \in H^1(\Sigma)$ satisfying

$$\int_\Sigma \varphi d\sigma = 0, \tag{2.18}$$

(see Theorem 1 of [72]). Let $\nu(x)$ denote the exterior unit normal to Σ at x. Since I_0 is translation invariant, then $\nu_i(x) := (\nu(x), e_i)$ are Jacobi fields of Σ, that is they satisfy

$$-\Delta_\Sigma \nu_i - |A|^2 \nu_i = 0 \quad \text{in } \Sigma, \tag{2.19}$$

(see [1],[18]). Moreover, Grosse-Brauckmann and Wohlgemuth showed in [42] that Σ is nondegenerate up to translations, that is there are no other nontrivial Jabobi fields. In other words

$$Ker(I_0''(E)) = span\{\nu_i\}_{1\leq i\leq 3}. \tag{2.20}$$

Remark 68. *Let us observe that the ν_i's are linearly independent. In fact, if not, there would exist a constant vector $b = (b_1, b_2, b_3) \neq 0$ such that $0 = (b, \nu(x))$ for any $x \in \Sigma$, but this contradicts the geometry of Σ.*

We note that the ν_i's have zero average, since

$$\int_\Sigma \nu_i(x) d\sigma(x) = \int_{T^3} \mathrm{div} e_i = 0. \tag{2.21}$$

In addition, we decompose $H^1(\Sigma)$ into the orthogonal sum

$$H^1(\Sigma) = span\{\nu_i\}_{1\leq i\leq 3} \oplus W, \tag{2.22}$$

where

$$W := \left\{ w \in H^1(\Sigma) : \int_\Sigma w(x)\nu_i(x) d\sigma(x) = 0, \quad 1 \leq i \leq 3 \right\}, \tag{2.23}$$

and we define

$$W^0 := \left\{ w \in W : \int_\Sigma w(x) d\sigma(x) = 0 \right\}. \tag{2.24}$$

The above discussion can be rephrased by saying that

$$\int_\Sigma |\nabla_\Sigma w|^2 - |A|^2 w^2 d\sigma \geq c||w||^2_{H^1(\Sigma)} \quad \text{for any } w \in W^0. \tag{2.25}$$

2.2 Proof of the main result

In this section, we give the outlines of the proof of Theorem 66. We need to find at least four sets F of the form (2.14) and a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$H_{\partial F}(y) + 4\varepsilon v_F(y) + \varepsilon f(y) = \lambda \quad \forall y \in \partial F, \tag{2.26}$$

or equivalently

$$I_\varepsilon'(F) = \lambda. \tag{2.27}$$

Exploiting the variational nature of the problem and the fact that $H_\Sigma = 0$, equation (2.26) is equivalent to

$$\lambda = 4\varepsilon v_E(x) + Lw(x) + Q(w)(x) + \varepsilon f(y), \quad \forall x \in \Sigma, \tag{2.28}$$

where y is seen as a function of x depending on the parameter ξ, namely $y = x + \phi + w(x)\nu(x)$, and

$$Q(w) := J_\varepsilon'(F) - J_\varepsilon'(E) - J_\varepsilon''(E)w. \tag{2.29}$$

Writing

$$Lw = -\Delta_\Sigma w - |A|^2 w + \varepsilon \tilde{L} w, \tag{2.30}$$

where

$$\tilde{L}w = 8 \int_\Sigma G(\cdot\,, \zeta) w(\zeta) d\sigma(\zeta) + 4\partial_\nu v_E w, \tag{2.31}$$

we can see that (2.28) is equivalent to

$$-\Delta_\Sigma w - |A|^2 w = \lambda + \mathcal{F}_\varepsilon^1(\phi, w), \tag{2.32}$$

where the nonlinear functional $\mathcal{F}_\varepsilon^1$ is given by

$$\mathcal{F}_\varepsilon^1(\phi, w)(x) = -4\varepsilon v_E(x) - \varepsilon \tilde{L} w(x) - Q(w)(x) - \varepsilon f(y), \quad \forall x \in \Sigma. \tag{2.33}$$

The unknowns are the function w, the vector $\phi \in T^3$ and $\lambda \in \mathbb{R}$.

2.2.1 The volume constraint

Now we will consider the relation between the volume of F and w. In order to do so, we point out that there exists a global parametrization

$$\xi : Y \to \Sigma, \tag{2.34}$$

defined on an open set $Y \in \mathbb{R}^2$ (see [35], section 3), that induces a change of coordinates on a neighbourhood of Σ given by

$$X(\mathrm{y}_1, \mathrm{y}_2, z) := \xi(\mathrm{y}_1, \mathrm{y}_2) + z\nu(\mathrm{y}_1, \mathrm{y}_2), \tag{2.35}$$

where, with an abuse of notation, $\nu(\mathrm{y}_1, \mathrm{y}_2)$ is the outward-pointing unit normal to Σ at $\xi(\mathrm{y}_1, \mathrm{y}_2)$. The volume of F is given by

$$|F|_3 = |E|_3 + \int_Y d\mathrm{y} \int_0^{w(\mathrm{y})} \det JX(\mathrm{y}, z) dz,$$

where JX is the Jacobian of X. We expand

$$\det JX(\mathrm{y}, z) = \det JX(\mathrm{y}, 0) + zA(\mathrm{y}) + z^2 B(\mathrm{y}),$$

thus we get

$$|F|_3 = |E|_3 + \int_Y d\mathrm{y} \int_0^{w(\mathrm{y})} \Big(\det JX(\mathrm{y}, 0) + zA(\mathrm{y}) + z^2 B(\mathrm{y}) \Big) dz$$
$$= |E|_3 + \int_Y \det JX(\mathrm{y}, 0) w(\mathrm{y}) d\mathrm{y} + \int_Y \left(\frac{1}{2} w^2(\mathrm{y}) A(\mathrm{y}) + \frac{1}{3} w^3(\mathrm{y}) B(\mathrm{y}) \right) d\mathrm{y}.$$

Since $\det JX(\mathrm{y}, 0) = (\nu(\mathrm{y}), \partial_{\mathrm{y}_1}\phi \times \partial_{\mathrm{y}_2}\phi) \neq 0$ for any $\mathrm{y} \in Y$,

$$|F|_3 = |E|_3 + \int_\Sigma w(x) d\sigma(x) + \int_\Sigma \tilde{Q}(x, w(x)) d\sigma(x), \tag{2.36}$$

where

$$\tilde{Q}(x, w) = \frac{1}{\det JX(x)} \left(\frac{1}{2} w^2(x) A(x) + \frac{1}{3} w^3(x) B(x) \right). \tag{2.37}$$

Therefore the volume constraint is equivalent to an equation of the form

$$\int_\Sigma w(x) d\sigma(x) = - \int_\Sigma \tilde{Q}(x, w(x)) d\sigma(x). \tag{2.38}$$

2.2.2 The Lyapunov-Schmidt reduction

To sum up, we need to solve the problem

$$\begin{cases} -\Delta_\Sigma w - |A|^2 w = \lambda + \mathcal{F}^1_\varepsilon(\phi, w) & \text{in } \Sigma, \\ \int_\Sigma w(x)d\sigma(x) = -\int_\Sigma \tilde{Q}(x, w(x))d\sigma(x), & \\ \partial_n w = 0 & \text{on } \partial\Sigma \end{cases} \tag{2.39}$$

We want to solve it by means of the finite-dimensional Lyapunov-Schmidt reduction. In the notation of the introduction, we set $\Omega := T^3$ and

$$H := L^2(\Sigma) \times \mathbb{R}, \quad Y_1 := C^{2,\alpha}_0(\overline{\Sigma}) \times \mathbb{R}, \quad Y_2 := C^{0,\alpha}(\overline{\Sigma}) \times \mathbb{R}. \tag{2.40}$$

Here, H is endowed with the scalar product

$$< (w_1, \lambda_1), (w_2, \lambda_2) >:= \int_\Sigma w_1 w_2 d\sigma + \lambda_1\lambda_2$$

and

$$C^{2,\alpha}_0(\overline{\Sigma}) := \{w \in C^{2,\alpha}(\overline{\Sigma}) : \partial_n w = 0 \quad \text{on } \partial\Sigma\}, \tag{2.41}$$

Y_1 is endowed with the norm

$$||(w, \lambda)||_1 := ||w||_{C^{2,\alpha}(\Sigma)} + |\lambda|$$

and Y_2 is endowed with the norm

$$||(\varphi, a)||_2 := ||\varphi||_{C^{0,\alpha}(\Sigma)} + |a|.$$

Equation (2.39) can be written in the form

$$\mathcal{L}[w, \lambda] = \mathcal{F}_\varepsilon(\phi, (w, \lambda)), \tag{2.42}$$

where $\mathcal{L}$ is given by

$$\mathcal{L}[w, \lambda] := \Big(-\Delta_\Sigma w - |A|^2 w - \lambda, \int_\Sigma w d\sigma \Big), \quad \forall (w, \lambda) \in Y_1 \times \mathbb{R}, \tag{2.43}$$

and

$$\mathcal{F}_\varepsilon(\phi, (w, \lambda)) := \Big(\mathcal{F}^1_\varepsilon(\phi, w), -\int_\Sigma \tilde{Q}(x, w(x))d\sigma(x) \Big) \tag{2.44}$$

is actually independent of λ, hence it will be denoted by $\mathcal{F}_\varepsilon(\phi, w)$. It is possible to see that the Kernel of $\mathcal{L}$ is given by $N = span\{\nu_i\}_{i=1,2,3} \times \{0\}$ and

$$\begin{aligned} N^\perp &:= \{(w, \lambda) \in H \times \mathbb{R} :< (w, \lambda), (n, \mu) >= 0, \forall (n, \mu) \in N\} \\ &= \Big\{ \varphi \in L^2(\Sigma) : \int_\Sigma \varphi(x)\nu_i(x)d\sigma(x) = 0, \quad 1 \le i \le 3 \Big\} \times \mathbb{R}. \end{aligned} \tag{2.45}$$

Since here $\mathcal{L}$ is actually independent of ε and ϕ, we do not need the notation $\mathcal{L}_\varepsilon(\phi)$, and the same is true for N and $N^\perp$.

Remark 69. *We note that $\mathcal{L}$ is self adjoint on Y_1 with respect to the scalar product (2.41), thanks to the boundary conditions, since*

$$\int_\Sigma (-\Delta_\Sigma w_1 - |A|^2 w_1) w_2 d\sigma + \lambda_1 \lambda_2 =$$
$$\int_\Sigma (-\Delta_\Sigma w_2 - |A|^2 w_2) w_1 d\sigma + \int_{\partial\Sigma} (\partial_n w_1) w_2 d\sigma_{\partial\Sigma} + \int_{\partial\Sigma} (\partial_n w_1) w_2 d\sigma_{\partial\Sigma} + \lambda_1\lambda_2 =$$
$$\int_\Sigma (-\Delta_\Sigma w_2 - |A|^2 w_2) w_1 d\sigma + \lambda_1 \lambda_2.$$

As a consequence, condition $(a2)$ is fulfilled with $T_{\varepsilon,\phi} := N^\perp$, for any $\phi \in \Omega$ and $0 < \varepsilon < \varepsilon_0$. Composing with the projection $\Pi : H \to N^\perp$ onto $N^\perp$, we split (2.42) into the system

$$\mathcal{L}[w,\lambda] = \Pi \mathcal{F}_\varepsilon(\phi, (w,\lambda)) \tag{2.46}$$
$$(Id - \Pi)\mathcal{F}_\varepsilon(\phi,(w,\lambda)) = 0. \tag{2.47}$$

According to the terminology introduced above, (2.46) is the *auxialiary equation* and (2.47) is the *bifurcation equation.*

2.2.3 The auxiliary equation

The auxiliary equation (2.46) is equivalent to

$$\begin{cases} -\Delta_\Sigma w - |A|^2 w = \lambda + \Pi_1 \Big(\mathcal{F}^1_\varepsilon(\phi, w), -\int_\Sigma \tilde{Q}(x, w(x)) d\sigma(x) \Big) & \text{in } \Sigma \\ \partial_n w = 0 & \text{on } \partial\Sigma, \\ \int_\Sigma w d\sigma = -\int_\Sigma \tilde{Q}(x, w(x)) d\sigma(x), \end{cases} \tag{2.48}$$

where $\Pi_1 : H \to L^2(\Sigma)$ is the first component of the projection Π onto $N^\perp$, and Π_2 is just the projection on the second component.

This problem will be solved by a fixed point argument in the following Proposition, proved in section 2.3.

Proposition 70. *For any $\phi \in T^3$ and for any ε sufficiently small, there exists a unique solution $(w_{\varepsilon,\phi}, \lambda_{\varepsilon,\phi}) \in C^{2,\alpha}_s(\overline{\Sigma}) \times \mathbb{R}$ to problem (2.48) satisfying*

$$||w_{\varepsilon,\phi}||_{C^{2,\alpha}(\Sigma)} + |\lambda_{\varepsilon,\phi}| \le C\varepsilon, \tag{2.49}$$
$$\int_\Sigma w(x)\nu_i(x) d\sigma(x) = 0, \quad 1 \le i \le 3, \tag{2.50}$$

for some constant $C > 0$. Moreover, the solution is Lipschitz continuous with respect to the parameter ξ, that is

$$||w_{\varepsilon,\phi_1} - w_{\varepsilon,\phi_2}||_{C^{2,\alpha}(\Sigma)} + |\lambda_{\varepsilon,\phi_1} - \lambda_{\varepsilon,\phi_1}| \leq C\varepsilon|\phi_1 - \phi_2|, \quad \forall \phi_1, \phi_2 \in T^3. \quad (2.51)$$

Remark 71. *If we take $f \equiv 0$, in order to get the right correction w, we just solve (2.48) for $\phi = 0$, due to the translation invariance of J_ε (see Remark 67). We do not need the Lyapunov-Schmidt reduction.*

2.2.4 The bifurcation equation

In order to conclude the proof of Theorem 66, we have to find at least four points $\phi \in T^3$ such that $(Id - \Pi)\mathcal{F}_\varepsilon(\phi, w_{\varepsilon,\phi})(x) = 0$. By (2.48), $w := w_{\varepsilon,\phi}$ solves

$$\Pi(\mathcal{L}[w, \lambda] - \mathcal{F}_\varepsilon(\phi, w)) = 0.$$

Since, as we will see in section 2.3.1, $R = N^\perp \cap Y_2$, this is equivalent to say that $\mathcal{L}[w, \lambda] - \mathcal{F}_\varepsilon(\phi, w) \in N \cap Y_2$, or

$$\mathcal{L}[w, \lambda] - \mathcal{F}_\varepsilon(\phi, w)$$

$$= \Big(- \Delta_\Sigma w - |A|^2 w - \lambda, \int_\Sigma w(x) d\sigma(x) \Big) - \Big(\mathcal{F}_\varepsilon^1(\phi, w), - \int_\Sigma \tilde{Q}(w(x)) d\sigma(x) \Big)$$

$$= (\sum_{i=1}^{3} A_{i,\varepsilon,\phi} \nu_i, 0)$$

for some coefficients $A_{i,\varepsilon,\phi}$ $i = 1, 2, 3$. This is equivalent to

$$-\Delta_\Sigma w - |A|^2 w - \lambda - \mathcal{F}_\varepsilon^1(\phi, w) = \sum_{i=1}^{3} A_{i,\varepsilon,\phi} \nu_i \quad \text{in } \Sigma. \quad (2.52)$$

As a consequence, the bifurcation equation

$$-(Id - \Pi)\mathcal{F}_\varepsilon(\phi, w) = (Id - \Pi)(\mathcal{L}[w, \lambda] - \mathcal{F}_\varepsilon(\phi, w)) = 0$$

is equivalent to

$$A_{i,\varepsilon,\phi} = 0 \quad \text{for } i = 1, 2, 3. \quad (2.53)$$

Equation (2.53) is solvable thanks to the Lusternik-Schnirelmann theory and the compactness of the Torus. We recall that the Torus T^3 has category 4 (see [5], example 9.4, *(iii)*).

Proposition 72. *Equation (2.53) is satisfied if ϕ is a critical point of the function $\Psi_\varepsilon : T^3 \to \mathbb{R}$ defined by*

$$\Psi_\varepsilon(\phi) := I_\varepsilon(F), \tag{2.54}$$

where F is the interior of

$$\Gamma := \{x + \phi + w_{\varepsilon,\phi}(x)\nu(x) : x \in \Sigma\}.$$

The proof of Proposition 72 will be carried out in Section 2.4. It is possible to see that Ψ_ε actually admits at least 4 critical points, due to Theorem 9.10 of [5] applied to I_ε, with $M = T^3$. The compactness of the torus T^3 is crucial, since it guarantees that I_ε is bounded from below on M and the Palais-Smale condition is satisfied.

2.3 Solving the auxiliary equation

The aim of this section is to prove Proposition 70. The idea is to apply Proposition 61. First, in Section 2.3.1, we will treat the corresponding linear problem, in order to show that $(a1)$ and $(a3)$ are satisfied. Then, in Section 2.3.2, we will show that $\mathcal{F}$ satisfies the hypothesis of Proposition 61, that is $(\mathcal{F}1)$, $(\mathcal{F}2)$, $(\mathcal{F}3)$ and condition (1.36). We stress that conditions (1.37) and (1.38) are automatically verified since both $\mathcal{L}$ and Π are independent of ϕ.

2.3.1 The linear problem

Proposition 73. *Let $a \in \mathbb{R}$ and $\varphi \in C^{0,\alpha}(\overline{\Sigma})$ be such that*

$$\int_\Sigma \varphi \nu_i d\sigma = 0 \quad \text{for } i = 1, 2, 3. \tag{2.55}$$

Then there exists a unique solution $(w, \lambda) = \Phi(\varphi, a) \in C_0^{2,\alpha}(\overline{\Sigma}) \times \mathbb{R}$ to the problem

$$\begin{cases} -\Delta_\Sigma w - |A|^2 w = \lambda + \varphi & \text{in } \Sigma \\ \partial_n w = 0 & \text{on } \partial\Sigma \\ \int_\Sigma w\nu_i d\sigma = 0 & \text{for } 1 \le i \le 3, \\ \int_\Sigma w d\sigma = a. & \end{cases} \tag{2.56}$$

Moreover, we have the estimate

$$||w||_{C^{2,\alpha}(\Sigma)} + |\lambda| \le c(||\varphi||_{C^{0,\alpha}(\Sigma)} + |a|). \tag{2.57}$$

If, in addition, $\varphi \in C_s^{0,\alpha}(\overline{\Sigma})$, then $w \in C_s^{2,\alpha}(\overline{\Sigma})$.

Remark 74. *(i) Since the ν_i's are linearly independent (see Remark 68), then the matrix*

$$L_{ki} := \int_\Sigma \nu_k \nu_i d\sigma \tag{2.58}$$

is invertible (for a detailed proof, see the appendix of Chapter 2).

(ii) Proposition 73 shows that $(a1)$ and $(a3)$ hold.

Proof. Step (i): existence and uniqueness.
First we look for a weak solution $w \in W$. We write any $w \in W$ as

$$w = w_0 + \frac{1}{|\Sigma|} \int_\Sigma w d\sigma,$$

with $w_0 \in W^0$. The linear problem can be rephrased as follows

$$\begin{cases} -\Delta_\Sigma w_0 - |A|^2 w_0 = \lambda + \varphi + |A|^2 \frac{a}{|\Sigma|} & \text{in } \Sigma \\ \int_\Sigma w_0 = 0. \end{cases} \tag{2.59}$$

We note that the right-hand side of (2.59) is orthogonal to ν_i, for $i = 1, 2, 3$, due to the fact that

$$\int_\Sigma |A|^2 \frac{a}{|\Sigma|} \nu_i(x) d\sigma = \int_\Sigma \Big(\Delta_\Sigma \nu_i + |A|^2 \nu_i \Big)(x) \frac{a}{|\Sigma|} d\sigma = 0, \tag{2.60}$$

since $\partial_n \nu_i = 0$ on $\partial\Sigma$, and

$$\int_\Sigma \nu_i(x) d\sigma(x) = \int_E \text{div}(e_1) dx = 0. \tag{2.61}$$

In addition, the norm defined by

$$||w|| = \int_\Sigma |\nabla_\Sigma w|^2 - |A|^2 w^2 \tag{2.62}$$

is equivalent to the $H^1(\Sigma)$-norm on W^0, thus the functional

$$I(w) = \int_\Sigma |\nabla_\Sigma w|^2 - |A|^2 w^2 d\sigma - \int_\Sigma \left(\varphi + |A|^2 \frac{a}{|\Sigma|} \right) w d\sigma$$

is bounded from below by

$$I(w) \geq c||w||^2_{H^1(\Sigma)} - ||\varphi||_{L^2(\Sigma)} ||w||_{H^1(\Sigma)}, \tag{2.63}$$

on W^0, hence it is coercive on it. Moreover, this functional is also w.l.s.c. and strictly convex on W^0, therefore any minimizing sequence $w_k \in W^0$ weakly converges, up to subsequence, to the unique minimizer $w_0 \in W^0$, which satisfies the Euler-Lagrange equation

$$\int_{\partial\Sigma} \partial_n w v d\sigma_{\partial\Sigma} + \int_\Sigma (-\Delta_\Sigma w_0 - |A|^2 w_0) v d\sigma = \\ \lambda \int_\Sigma v d\sigma + \sum_{i=1}^3 \beta_i \int_\Sigma \nu_i v d\sigma + \int_\Sigma \varphi v d\sigma + \int_\Sigma |A|^2 \frac{a}{|\Sigma|} v d\sigma, \tag{2.64}$$

for any $v \in H^1(\Sigma)$, for some Lagrange multipliers $\lambda, \beta_i \in \mathbb{R}$. Since $\varphi \in C^{0,\alpha}(\overline{\Sigma})$, then $w \in C^{2,\alpha}(\overline{\Sigma})$ (see for instance [62]). Taking the test functions $v \in C^1_c(\Sigma)$, we can see that w satsfies

$$-\Delta_\Sigma w_0 - |A|^2 w_0 = \lambda + \sum_{i=1}^3 \beta_i \nu_i + \varphi + |A|^2 \frac{a}{|\Sigma|} \quad \text{in } \Sigma,$$

in the classical sense. Taking now $v \in C^1(\overline{\Sigma})$, we can see that the Neumann boundary condition is satisfied in the classical sense too. Moreover, if $\varphi \in C^{0,\alpha}_s(\Sigma)$, then w respects the same simmetries because of the symmetries of the laplacian and uniqueness, that is $w \in C^{2,\alpha}_s(\Sigma)$. Taking ν_j as a test function in (2.64), using (2.61), (2.55), (2.60) the Neumann boundary condition and the fact that $\partial_n \nu_i = 0$ on $\partial\Sigma$, we get

$$\sum_{i=1}^3 \beta_i \int_\Sigma \nu_i \nu_j d\sigma = 0,$$

therefore by Remark 74, $\beta_i = 0$.

Step (ii): Regularity estimates.
Multiplying (2.59) by w_0, integrating by parts and using (2.25), the Neumann boundary conditions and Hölder's inequality, we can see that

$$c||w_0||^2_{H^1(\Sigma)} \le \int_\Sigma |\nabla_\Sigma w_0|^2 - |A|^2 w_0^2 d\sigma = \int_\Sigma \varphi w_0 d\sigma + \frac{a}{|\Sigma|} \int_\Sigma |A|^2 w_0 d\sigma \le \\ ||w_0||_{L^2(\Sigma)}(||\varphi||_{L^2(\Sigma)} + \tilde{c}|a|) \le ||w_0||_{H^1(\Sigma)}(||\varphi||_{L^2(\Sigma)} + \tilde{c}|a|).$$

Since $||w||^2_{H^1(\Sigma)} = ||w_0||^2_{H^1(\Sigma)} + a^2$, then

$$||w||_{H^1(\Sigma)} \le c(||\varphi||_{L^2(\Sigma)} + |a|).$$

In order to estimate λ, we integrate (2.56) and we get

$$\lambda|\Sigma| + \int_\Sigma \varphi d\sigma = -\int_\Sigma |A|^2 w d\sigma,$$

since, by the Neumann boundary conditions,

$$\int_\Sigma \Delta_\Sigma w d\sigma = \int_{\partial\Sigma} \partial_n w d\sigma_{\partial\Sigma} = 0, \tag{2.65}$$

thus

$$|\lambda| \leq c(||\varphi||_{L^2(\Sigma)} + ||w||_{L^2(\Sigma)}).$$

To sum up, we have the estimate

$$|\lambda| + ||w||_{H^1(\Sigma)} \leq c(||\varphi||_{L^2(\Sigma)} + |a|), \tag{2.66}$$

In order to get the estimate with respect to the norms we are interested in, we point out that, by the Sobolev embeddings

$$||w||_{L^\infty(B_\delta(x))} \leq c||w||_{W^{2,2}(B_\delta(x))} \leq c(||w||_{L^2(B_{2\delta}(x))} + ||\varphi + \lambda||_{L^2(B_{2\delta}(x))} + |a|) \leq c(||\varphi||_{L^2(\Sigma)} + |a|),$$

for any $\delta > 0$ small but fixed and $x \in \Sigma$ such that $d(x, \partial\Sigma) > \delta$ (here, $B_\delta(x)$ is the geodesic ball of radius δ centered at x in Σ). In particular,

$$||w||_{L^\infty(\Sigma)} \leq c(||\varphi||_{L^\infty(\Sigma)} + |a|).$$

By the Hölder's regularity estimates, we conclude that,

$$||w||_{C^{2,\alpha}(\Sigma)} \leq c(||w||_{L^\infty(\Sigma)} + ||\varphi + \lambda||_{C^{0,\alpha}(\Sigma)}) \leq c(||\varphi||_{C^{0,\alpha}(\Sigma)} + |a|),$$

(see [39], Chapter 6, Theorem 6.30). Since the same is true for $|\lambda|$, the proof is over. □

2.3.2 Proof of Proposition 70: a fixed point argument

In order to show existence, uniqueness and Lipschitz continuity with respect to ϕ of the solution (w, λ) to (2.48), it is enough to show that the hypothesis of Proposition 61 are satisfied.

First we can see that

$$||\mathcal{F}^1_\varepsilon(\phi, 0)||_{C^{0,\alpha}(\Sigma)} \leq \varepsilon(4||v_E||_{C^{2,\alpha}(\Sigma)} + ||f||_{C^{0,\alpha}(\Sigma)})$$

In addition, the second component fulfills

$$\int_\Sigma \tilde{Q}(x,0)d\sigma = 0,$$

thus $\mathcal{F}$ fulfills ($\mathcal{F}1$).

Furthermore,

$$\partial_w \mathcal{F}_\varepsilon(\phi, 0)[h] = (\varepsilon \tilde{L}[h], 0), \forall h \in Y_1, \tag{2.67}$$

therefore ($\mathcal{F}2$) is satisfied. ($\mathcal{F}3$) follows from the explicit expressions of $\tilde{Q}(w)$ and $\mathcal{F}^1_\varepsilon$.

In conclusion, we point out that $\tilde{Q}$ is independent of ϕ and

$$||\mathcal{F}^1_\varepsilon(\phi_1, w) - \mathcal{F}^1_\varepsilon(\phi_2, w)||_{C^{0,\alpha}(\Sigma)} =$$
$$\varepsilon||f(x+\phi_1+\nu w(x)) - f(x+\phi_2+\nu w(x))||_{C^{0,\alpha}(\Sigma)} \leq c\varepsilon|\phi_1-\phi_2|,$$

which yields that (1.36) is satisfied too.

It remains to show that we can find our solution $w_{\varepsilon,\phi} \in C^{2,\alpha}_s(\overline{\Sigma})$. In order to do so, we point out that $\mathcal{L}$ maps the subspace

$$\tilde{Y}_1 := C^{2,\alpha}_s(\overline{\Sigma}) \times \mathbb{R}$$

into

$$\tilde{Y}_2 := C^{0,\alpha}_s(\overline{\Sigma}) \times \mathbb{R},$$

and, by Proposition 73, Φ maps $\tilde{Y}_2$ in $\tilde{Y}_1$, therefore our claim follows from Remark 62, point *(ii)*.

2.4 Solving the bifurcation equation.

The parametrization $\xi : Y \to \Sigma$ of Σ introduced in (2.34) induces a parmetrization $\beta : Y \to \Gamma := \partial F$ given by

$$\beta(\mathrm{y}_1, \mathrm{y}_2) := \xi(\mathrm{y}_1, \mathrm{y}_2) + \phi + w_{\varepsilon,\phi}(\mathrm{y}_1, \mathrm{y}_2)\nu(\mathrm{y}_1, \mathrm{y}_2). \tag{2.68}$$

The volume element can be expressed in terms of ϕ in this way

$$|\beta_{\mathrm{y}_1} \times \beta_{\mathrm{y}_2}| = |\xi_{\mathrm{y}_1} \times \xi_{\mathrm{y}_2}| + L^1_\phi w_\phi + Q^1_\phi w_\phi,$$

where L^1_ϕ depends linearly on w_ϕ and on its gradient and Q^1_ϕ is quadratic in the same quantites. More precisely, they satisfy the estimates

$$\begin{cases} |L^1_\phi w| \leq c||w||_{C^{2,\alpha}(\Sigma)} \\ |Q^1_\phi(w)| \leq c||w||^2_{C^{2,\alpha}(\Sigma)}. \end{cases} \tag{2.69}$$

Using the Taylor expansion of the function $\frac{1}{1+s}$, we can show that the outward-pointing unit normal to Γ is

$$\nu_\Gamma = \frac{\beta_{y_1} \times \beta_{y_2}}{|\beta_{y_1} \times \beta_{y_2}|} = \frac{\xi_{y_1} \times \xi_{y_2}}{|\xi_{y_1} \times \xi_{y_2}|} + \tilde{L}^1_\phi w_\phi + \tilde{Q}^1_\phi w_\phi = \nu + \tilde{L}^1_\phi w_\phi + \tilde{Q}^1_\phi w_\phi, \tag{2.70}$$

with $\tilde{L}^1_\phi$ and $\tilde{Q}^1_\phi$ satisfying (2.69).

Now we point out that, if ϕ is a critical point of Ψ_ε, then

$$\partial_{\phi_i} \Psi_\varepsilon(\phi) = 0. \tag{2.71}$$

We will rephrase this fact in a more convenient way, that will be more suitable for the forthcoming computations. We define the one-parameter family of diffeomorphisms

$$y_t : Y \to \mathbb{R}^3$$

by

$$y_t(y_1, y_2) := \xi(y_1, y_2) + \phi + te_i + w_{\varepsilon,\phi+te_i}(y_1, y_2)\nu(y_1, y_2), \tag{2.72}$$

for $i = 1, 2, 3$; $\Gamma_t := y_t(Y)$ is the image of y_t. By construction, Γ_t is actually a submanifold of T^3 and $\Gamma_0 = \Gamma$. In terms of Γ_t, condition (2.71) is equivalent to

$$\frac{d}{dt} I_\varepsilon(\Gamma_t)|_{t=0} = 0. \tag{2.73}$$

By a result of Fall and Mahmoudi (see [29]),

$$0 = \frac{d}{dt} I_\varepsilon(\Gamma_t)|_{t=0} = \int_\Gamma (H_\Gamma + 4\varepsilon v_F + \varepsilon f)(\zeta, \nu_\Gamma) d\sigma_\Gamma + \frac{1}{|\partial\Gamma|} \int_{\partial\Gamma} (\zeta, \nu^\Gamma_{\partial\Gamma}) ds, \tag{2.74}$$

where

$$\zeta = \frac{d}{dt} y_t(x)|_{t=0} = e_i + \partial_{\phi_i} w_\phi \nu. \tag{2.75}$$

and $\nu^{\Gamma}_{\partial\Gamma}$ is the unit normal to $\partial\Gamma$ in Γ. The boundary term vanishes by periodicity and by the symmetries of the problem. Using the parametrization β of Γ and expansions (2.71) and (2.69), the latter relation becomes

$$\int_Y \Big\{ (H_\Gamma + 4\varepsilon v_F + \varepsilon f)(\beta(\mathrm{y}_1, \mathrm{y}_2)) \\ (e_i + \partial_{\phi_i} w_\phi \nu, \nu + \tilde{L}^1_\phi w_\phi + \tilde{Q}^1_\phi w_\phi) \\ (|\xi_x \times \xi_y| + L^1_\phi w_\phi + Q^1_\phi w_\phi) \Big\} d\mathrm{y}_1 d\mathrm{y}_2 = 0.$$

By the auxiliary equation, we know that

$$(H_\Gamma + 4\varepsilon v_F + \varepsilon f)(\beta(\mathrm{y}_1, \mathrm{y}_2)) = \sum_{k=1}^{3} A_{k,\varepsilon,\phi} \nu_k(\mathrm{y}_1, \mathrm{y}_2) + \lambda, \tag{2.76}$$

thus

$$\sum_{k=1}^{3} A_{k,\varepsilon,\phi} \Big(\int_\Sigma \nu_k \nu_i d\sigma + b_{ki} \Big) + \lambda \int_\Gamma (\zeta, \nu_\Gamma) d\sigma_\Gamma = 0, \quad \text{for } i = 1, 2, 3, \tag{2.77}$$

with $b_{ki} = O(\varepsilon)$. Moreover, once again by [29], we know that

$$\frac{d}{dt} |F_t|_3 = \int_\Gamma (\zeta, \nu_F) d\sigma_\Gamma,$$

hence, by the volume constraint,

$$\int_\Gamma (\zeta, \nu_F) d\sigma_\Gamma = 0,$$

thus we get

$$\sum_{k=1}^{3} A_{k,\varepsilon,\phi} \Big(\int_\Sigma \nu_k \nu_i d\sigma + b_{ki} \Big) = 0, \quad \text{for } i = 1, 2, 3. \tag{2.78}$$

Since the matrix L_{ki} is invertible (see Remark 74) and the coefficients b_{ki} are small, the matrix $L_{ki} + b_{ki}$ is invertible too, therefore $A_{k,\varepsilon,\phi} = 0$ for $k = 1, 2, 3$.

2.5 Appendix of Chapter 2

Proof of Remark 74

We argue by contradiction. If the statement were not true, there would exist a vector $c = (c_1, c_2, c_3) \neq 0$ such that $Lc = 0$, or equivalently

$$\sum_{j=1}^{3} \left(\int_{\Sigma} \nu_i(x)\nu_j(x)d\sigma(x) \right) c_j = 0. \tag{2.79}$$

Furthermore, writing ν_i as a linear combination of an orthonormal basis $\{e_i\}_{1\leq i\leq 3}$ of $span\{\nu_i\}_{1\leq i\leq 3}$, namely

$$\nu_i(x) = \sum_{k=1}^{3} \nu_{ik} e_k(x),$$

where

$$\nu_{ik} := \int_{\Sigma} \nu_i(x)e_k(x)d\sigma(x),$$

we can see that, setting $a_k := \sum_{j=1}^{3} \nu_{jk}c_j$, (2.79) is equivalent to

$$0 = \sum_{k=1}^{3} \left(\int_{\Sigma} \nu_i(x)e_k(x)d\sigma(x) \right) a_k = \int_{\Sigma} \nu_i(x)a(x)d\sigma(x)$$

with $a(x) := \sum_{k=1}^{3} a_k e_k(x) \in span\{e_i\}_{1\leq i\leq 3} = span\{\nu_i\}_{1\leq i\leq 3}$, so in particular $a \equiv 0$. On the other hand, $a_k = 0$ for any k is equivalent to

$$\int_{\Sigma} c(x)e_k(x)d\sigma(x) = 0 \quad \forall 1 \leq k \leq 3,$$

with $c(x) = \sum_{j=1}^{3} c_j\nu_j(x)$. Thus $c \equiv 0$, that is $c_j = 0$ for any $1 \leq j \leq 3$, a contradiction.

Chapter 3

Clifford Tori and the Cahn-Hilliard equation

3.1 The Cahn-Hilliard equation and Willmore surfaces

The Allen-Cahn equation

$$-\varepsilon^2 \Delta u = u - u^3, \tag{3.1}$$

arises in several physical contexts, such as the study of the stable configurations of two different fluids confined in a bounded container Ω. If $u(x)$ is the density of one of the two fluids at a point $x \in \Omega$ and the energy per unit volume is given by a function W of u, it looks reasonable to obtain stable configurations by minimizing the energy functional

$$E(u) = \int_\Omega W(u)dx$$

among all distributions fulfilling the volume constraint

$$\int_\Omega u dx = m. \tag{3.2}$$

If, for instance, $W(u) = (1-u^2)^2$, and $m \in (-1, 1)$, any piecewise constant function taking only the values ± 1 and satisfying (3.2) is a minimizer, irrespectively of the shape of the interface. Therefore this model is unsatisfactory, since it is very far from the reasonable physical assumption that the interfaces are area minimizers, so one replaces the energy by

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{(1-u^2)^2}{4\varepsilon} \right) dx.$$

We can see that there is a competition between the potential energy, that forces u to be close to ± 1, and the gradient term that penalizes the phase transition. By minimizing this functional, we are looking for the physical interfaces in which the phase transition can occur.

The minimizers u_ε of E_ε are solutions to the Euler Lagrange equation, that is (3.1). In order to see if the interfaces are actually minimal surfaces, it is interesting to study the asymptotic behaviour of the level sets $\{u_\varepsilon = c\}$ as the parameter $\varepsilon \to 0$. It is useful to exploit the variational structure of the problem. It was shown by Modica and Mortola that the energy E_ε, seen as a functional on $L^1(\Omega)$ and extended to be $+\infty$ when the integrand is not an L^1 function, $\Gamma-$converges to the functional

$$E(u) = \begin{cases} cPer_\Omega(\{u=1\}) & \text{if } u = \pm 1 \text{ a.e. in } \Omega \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

in the strong topology of $L^1(\Omega)$ (see [57]), where $c > 0$ is a suitable constant.

Moreover, Modica showed that, if u_ε are minimizers of F_ε under the volume constraint

$$\int_\Omega u_\varepsilon dx = m,$$

for some $m \in (-1, 1)$, then there exists a sequence $\varepsilon_k \to 0$ such that u_{ε_k} converges to some function u in $L^1(\Omega)$ (see proposition 3 of [56]). Furthermore, Theorem 1 of [56] asserts that $u = \pm 1$ a. e. in Ω, and the set

$$E = \{x \in \Omega : u(x) = 1\}$$

is actually a perimeter minimizer between all the subsets $F \subset \Omega$ satisfying the volume constraint

$$|F| = \frac{|\Omega| + m}{2}.$$

Further results about the relation between the minimizers of E_ε and the minimizers of the perimeter can be found in [56] and in [19], where Choksi and Sternberg also described the relation between phase transition theory and the study of a certain kind of polymers.

Conversely, it is an interesting problem to understand if any minimal hypersurface can be achieved as the limit of nodal sets of minimizers of the Ginzburg-Landau energy E_ε.

The first result in this direction is due to Kohn and Sternberg (see [48]). They considered a smooth bounded domain $\Omega \subset \mathbb{R}^2$ and, as an interface, a disjoint union of segments l_i meeting the boundary $\partial\Omega$ orthogonally. They defined u_0 to be locally constant on $\Omega\setminus\cup_i l_i$, taking the values ± 1, and constructed a sequence of minimizers u_ε converging to u_0 in $L^1(\Omega)$.

In [63], Pacard and Ritoré proved a more general result, that holds true for a larger class of interfaces. They started from a minimal hypersurface Σ in a compact Riemannian manifold M and, under suitable assumptions, they showed that it can be achieved as the limit as $\varepsilon \to 0$ of nodal sets (that is 0-level sets) of solutions u_ε of the rescaled Allen-Cahn equation (3.1). These solutions u_ε were constructed with techniques such as fixed point theorems and the Lyapunov-Schmidt reduction, and are not necessarily minimizers.

As regards the hypersurface Σ, they imposed some restrictions. They required it to be *admissible*, that is the nodal set of a smooth function $f : M \to \mathbb{R}$. In the sequel, we will set

$$M^+(\Sigma) = \{p \in M : f(p) > 0\} \quad \text{and } M^-(\Sigma) = \{p \in M : f(p) < 0\}.$$

Moreover, Σ has to be *non-degenerate*. In order to explain the notion of non-degeneracy, let us give the variational characterization of minimal hypersurfaces. A hypersurface Σ in a compact Riemannian manifold M is said to be minimal if it is a minimizer for the area functional, whose critical points are characterized by the Euler equation $H = 0$, where H denotes the mean curvature of Σ. In the sequel, the mean curvature H of a hypersurface Σ embedded in $\mathbb{R}^N$ will always be

$$H = k_1 + \cdots + k_{N-1},$$

where the k_j's are the principal curvatures.

The second variation of the area functional is given by

$$A''(\Sigma)[\phi, \psi] = \int_\Sigma L_0\phi(y)\psi(y)d\sigma(y),$$

where the self-adjoint operator

$$L_0\phi = -\Delta_\Sigma\phi - |A|^2\phi$$

is called the Jacobi operator of Σ and

$$|A|^2 = k_1^2 + \cdots + k_{N-1}^2$$

is the squared norm of its second fundamental form. By definition, a minimal hypersurface Σ is said to be non-degenerate if its Jacobi operator

$$L_0 : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$$

is an isomorphism. For an introduction to these topics, see also [26].

Moreover, the results in [63] hold even if the potential $W(t) = (1-t^2)^2/4$ is replaced by a more general double-well potential, that is a smooth function W such that

$$\begin{cases} W(t) \geq 0 & \text{for any } t, \\ W(t) = 0 & \text{if and only if } t = \pm 1, \\ W''(\pm 1) > 0. \end{cases} \tag{3.3}$$

To sum up, they proved the following Theorem.

Theorem 75 ([63]). *Let W be as in (3.3). Let Σ be an admissible non-degenerate minimal hypersurface in a compact Riemannian manifold M. Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ there exists a solution u_ε to the rescaled Allen-Cahn equation*

$$-\varepsilon^2 \Delta u_\varepsilon + W'(u_\varepsilon) = 0$$

such that $u_\varepsilon \to \pm 1$ on compact subsets of $M^{\pm}(\Sigma)$.

Anyway, despite several results lead to think that, in some sense, the nodal sets of the solutions to the Allen-Cahn equation resemble minimal surfaces, there are also solutions for which the nodal set is far from being minimal. For instance, Agudelo, Del Pino and Wei constructed axially symmetric solutions $u = u(|x'|, x_3)$ in $\mathbb{R}^3$ such that the components of the nodal set, for $|x'|$ large enough, look like a catenoid (see [2]).

The Lyapunov-Schmidt reduction was also applied to the non compact case, to construct entire solutions to the Allen-Cahn equation in $\mathbb{R}^9$ that are monotone in one variable but not one-dimensional, since their nodal set resembles the Bombieri-De Giorgi-Giusti graph, that is a minimal graph over $\mathbb{R}^8$ that is not affine (see [11],[24]). This solutions are related to a famous conjecture of De Giorgi, that asserts that, at least for $N \leq 8$, any entire bounded solution $|u| < 1$ to the Allen-Cahn equation

$$-\Delta u = u - u^3$$

satisfying $\partial_N u > 0$ in the whole $\mathbb{R}^N$ must be one-dimensional, that is it must depend just on one euclidean variable, in other words $u(x) = u(< a, x >)$, for some unit vector $a \in S^{N-1}$. The result by Del Pino, Kowalczyk and Wei shows that de Giorgi's conjecture is sharp about the upper bound on the dimension. Up to now it is known that the conjecture is true in dimension $N = 2$ (see [36],[31]) and $N = 3$ (see [7],[31]). The conjecture is still open in dimension $4 \leq N \leq 8$, although notable progress was made by Savin (see [73]), that proved that the conjecture is true in dimension $4 \leq N \leq 8$ under the reasonable assumption that, for any $x' \in \mathbb{R}^{N-1}$,

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1,$$

that yields that these solutions are minimizers of the energy

$$E(u) = \int_{\mathbb{R}^3} \big(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}(1-u^2)^2\big)dx.$$

We are interested here in analogues of these results for the Cahn-Hilliard equation

$$-\varepsilon^2 \Delta(-\varepsilon^2 \Delta u + W'(u)) + W''(u)(-\varepsilon^2 \Delta u + W'(u)) = 0, \tag{3.4}$$

with W satisfying (3.3). Note that, as in the case of Allen-Cahn, we rescale the equation in order to treat Γ-convergence. If, for instance, we study the equation in a bounded domain $\Omega \subset \mathbb{R}^N$, it is possible to see that it is the Euler equation of the functional

$$\mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{2\varepsilon}\int_\Omega \big(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\big)^2 dx & \text{if } u \in L^1(\Omega) \cap H^2(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

As in the case of the functionals E_ε related to the Allen-Cahn equation, some $\Gamma-$convergence results are known about $\mathcal{W}_\varepsilon$. More precisely, the asymptotic behaviour of $\mathcal{W}_\varepsilon$ as $\varepsilon \to 0$ is related to the Willmore functional

$$\mathcal{W}(u) = c\int_{\partial E \cap \Omega} H^2_{\partial E}(y) d\mathcal{H}^{N-1},$$

where $E = \{u = 1\}$, if $u = \pm 1$ a. e., defined when the interface ∂E is smooth enough. The nodal sets of the critical points u of $\mathcal{W}$ are called *Willmore hypersurfaces*. The Euler equation satisfied by this kind of hypersurfaces is

$$-\Delta_\Sigma H = \frac{1}{2}H^3 - 2HK,$$

where H is the mean curvature and K is the Gauss curvature of $\Sigma = \partial E$. In the sequel, the Gauss curvature K of hypersurface Σ embedded in $\mathbb{R}^N$ will always be

$$K = k_1 \ldots k_{N-1}.$$

An equivalent form of the Willmore equation is

$$-\Delta_\Sigma H + \frac{1}{2}H(H^2 - 2|A|^2) = 0. \tag{3.5}$$

The Willmore functional arises naturally in general relativity, since it is related to the Hawking mass, that is

$$m_H(\Sigma) = \sqrt{\frac{Area(\Sigma)}{16\pi}}(1 - \frac{1}{16\pi}\mathcal{W}(\Sigma)).$$

Here Σ can be interpreted as the surface of a body whose mass has to be measured. Furthermore, this functional is also appears in biology, under the name of *Helfrich energy*, and it is used to describe the behaviour of some lipid bilayer cell membranes. For further details and references, we suggest to see [51, 44, 45].

In [8] Bellettini and Paolini proved the $\Gamma - \limsup$ inequality for smooth Willmore hypersurfaces, while the $\Gamma - \liminf$ inequality is much harder to prove. Up to now it has been proved in dimension $N = 2, 3$ by Röger and Schätzle in [71], and, independently, in dimension $N = 2$, by Nagase and Tonegawa in [61]. The problem is still open in higher dimension, while it is known that the approximation does not hold, in general, for non smooth sets, even in dimension $N = 2$.

In view of these $\Gamma-$convergence results that establish a link between the Cahn-Hilliard functional and the Willmore functional, it is interesting to see if also the above counter-part is true. In other words, we try to answer the following question: given a Willmore hypersurface Σ, is it possible to construct a sequence of solutions u_ε of the Cahn-Hilliard equation (3.4) whose nodal sets approach Σ as $\varepsilon \to 0$? In the paper, we show that this result holds true up to a Lagrange multiplier if, for instance, Σ is the standard Clifford Torus, that is the zero level set of the function

$$f(x) = \left(\sqrt{2} - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 - 1. \tag{3.6}$$

It has been recently proved in [55] that the Clifford Torus is the unique minimizer of the Willmore energy (up to conformal transformations) among surfaces of genus greater or equal than 1. The Lagrange multiplier is due to a volume constaint, as it will be clear from the statement of Theorem 75 and Remark 77.

It is interesting to see that it is possible to construct these solutions in such a way that they respect the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to any rotation that fixes the x_3-axis. A Lagrange multiplier λ_ε appears due to this volume constraint.

Theorem 76. *Let W be an even double-well potential satisfying (3.3). Let Σ be the Clifford Torus. Then there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$ there exists a solution $(u_\varepsilon, \lambda_\varepsilon) \in C^{4,\alpha}(\mathbb{R}^3) \times \mathbb{R}$ to*

$$-\varepsilon^2\Delta(-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon)) + W''(u_\varepsilon)(-\varepsilon^2\Delta u_\varepsilon + W'(u_\varepsilon)) = \varepsilon^4\lambda_\varepsilon(1 - u_\varepsilon), \tag{3.7}$$

where $\lambda_\varepsilon = O(\varepsilon)$ is a Lagrange multiplier, under the volume constraint

$$\int_{\mathbb{R}^3} (1-u_\varepsilon)^2 dx = 8\sqrt{2}\pi^2 c_\varepsilon, \quad c_\varepsilon \to 1 \ \text{as } \varepsilon \to 0. \tag{3.8}$$

Moreover, $u_\varepsilon \to \pm 1$ and $\partial_k u_\varepsilon \to 0$ uniformly on compact subsets of $\Sigma^\pm$, for $1 \le k \le 4$, and u respects the symmetries of the Torus, that is $u_\varepsilon(x_1,x_2,x_3) = u_\varepsilon(x_1,x_2,-x_3)$ and $u_\varepsilon(x) = u_\varepsilon(Rx)$, for any $x=(x_1,x_2,x_3) \in \mathbb{R}^3$ and for any rotation $R \in SO(3)$ such that $R(0,0,1) = (0,0,1)$.

In the statement of the Theorem, we denoted

$$\Sigma^+ := \{x \in \mathbb{R}^3 : f(x) > 0\} \quad \text{and } \Sigma^- := \{x \in \mathbb{R}^3 : f(x) < 0\}.$$

and

$$c_\varepsilon := 1 + \varepsilon \int_0^\infty (v_\star^2 - 1) dt.$$

This result is a fourth order analogue of Theorem 75 by Pacard and Ritoré (see [63]). The proof is based on the Lyapunov-Schmidt reduction, that is we split equation (3.4) into a system of two equations. The auxiliary equation will be solved by using the spectral decomposition of the linearized Allen-Cahn operator and the bifurcation equation will be solved thanks to the *nondegeneracy* of the Clifford Torus, up to conformal maps. For a more detailed introduction to the techniques developed in the proof, see section 2.

In order to explain what we mean by nondegeneracy, we go back to the variational definition of Willmore hypersurface and we consider the second variation of the Willmore functional, that is

$$\mathcal{W}''(\Sigma)[\phi,\psi] = \int_\Sigma \tilde{L}_0 \phi \psi d\sigma,$$

where $\tilde{L}_0$ is the self-adjoint operator given by

$$\begin{aligned}\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2 L_0\phi - H(\nabla_\Sigma\phi, \nabla_\Sigma H) + 2(A\nabla_\Sigma\phi, \nabla_\Sigma H) + \\ 2H < A, \nabla^2\phi > +\phi(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2 + 2H \mathrm{tr} A^3).\end{aligned} \tag{3.9}$$

Here we have denoted by $(\cdot\,,\cdot)$ the scalar product induced by the metric g on Σ, indeed, for instance $(\nabla\phi, \nabla H) = g^{ij}H_i\phi_j$, and by $<\cdot\,,\cdot>$ the trace of the product of two matrices, so for instance $< A, \nabla^2\phi >= A^{ij}\nabla^2_{ij}\phi$, and $A^{ij} = g^{ik}g^{jl}A_{kl}$. It is possible to find the explicit computation of the first and the second variation of the Willmore functional $\mathcal{W}$ in [51], section 3. This is the analogue of the Jacobi operator in the case of minimal hypersurfaces. In view of a result by White [78], the Willmore functional is invariant under conformal transformations of the Euclidean space, that is homotheties, isometries and Möbius transformations, i.e. inversions with respect to spheres. On the other hand, by Corollary 2, page 34, of [77], we know that its second variation is positive definite on the orthogonal complement of the space of conformal transformations, hence the kernel of $\tilde{L}_0$ exactly consists of these transformations.

Remark 77. *In view of the above discussion, $\tilde{L}_0$ is injective if restricted to the space of functions with zero average and fulfilling the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to all rotations of $\mathbb{R}^3$ that fix the x_3 axis. In fact, the variation of internal volume under a normal variation ϕ is given by $\int_\Sigma \phi$. Mantaining constant volume corresponds to variations ϕ with zero average. Working in this class we then exclude non-trivial homotheties. When considering sharp interfaces, this constraint is equivalent to prescribe the integral of $(1-u_\varepsilon)^2$, that is to impose*

$$\begin{aligned}\int_{\mathbb{R}^3}(1-u_\varepsilon)^2dx &= 8\sqrt{2}\pi^2 c_\varepsilon\\ &= 4|\Sigma^+|_3\left(1+\varepsilon\int_0^\infty (v_\star^2-1)dt\right),\end{aligned}$$

where $|\Sigma^+|_3 = 2\sqrt{2}\pi^2$ is the volume of the interior of the Clifford Torus, that is its 3-dimensional Lebesgue measure.

3.2 Some useful facts in differential geometry

For $0<\varepsilon\le 1$, we define the rescaled Clifford Torus as $\Sigma_\varepsilon := \{\varepsilon^{-1}\zeta : \zeta\in\Sigma\}$. In other words, $\Sigma_\varepsilon = \{y\in\mathbb{R}^3 : f_\varepsilon(y)=0\}$, where $f_\varepsilon(y) := \varepsilon^{-2}f(\varepsilon y)$ and f is defined in (3.6).

For $0<\tau<\sqrt{2}-1$ and $0<\varepsilon\le 1$, we define the tubular neighbourhood of width τ/ε of Σ_ε as

$$V_{\tau/\varepsilon} = \{x\in\mathbb{R}^3 : dist(x,\Sigma_\varepsilon) < \tau/\varepsilon\}.$$

On this neighbourhood of Σ_ε, we introduce a new system of coordinates, known as Fermi coordinates. First we define

$$Z_\varepsilon : \Sigma_\varepsilon \times (-\tau/\varepsilon, \tau/\varepsilon) \to V_{\tau/\varepsilon}$$

by the relation

$$Z_\varepsilon(y,z) = \exp_y(z\nu(\varepsilon y)) = y + z\nu(\varepsilon y), \tag{3.10}$$

where $\nu(\varepsilon y)$ is the outward-pointing unit normal to the original Torus Σ at εy, that coincides with the the outward-pointing unit normal to Σ_ε at y, and $\exp_y$ is the exponential map of $\mathbb{R}^3$ at y seen as a point of $\mathbb{R}^3$. If τ is small enough, that is $0 < \tau < \sqrt{2} - 1$ in the case of the Clifford Torus, Z_ε is a diffeomorphism. In other words, Z_ε is a change of coordinates on $V_{\tau/\varepsilon}$, and the coordinates $(y, z) = Z_\varepsilon^{-1}(x)$ are known as Fermi coordinates of the rescaled torus Σ_ε, or stretched Fermi coordinates of the Torus.

Remark 78. *Any function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ can be seen as a function of (y, z). More precisely, we can consider the composition $u^\star(y, z) = u(Z_\varepsilon(y, z))$. In the sequel, with a slight abuse of notation, we will write $u = u(y, z)$.*

Let us fix a point $\zeta_0 \in \Sigma$ and a parametrization onto a neighbourhood $V \subset \Sigma$ of ζ_0, that is a smooth function

$$Y : U \to V$$

on an open set $U \subset \mathbb{R}^2$ such that $Y(\xi_0) = \zeta_0$, for some $\xi_0 \in U$. Then, setting $U_\varepsilon = \varepsilon^{-1} U$ and $V_\varepsilon = \varepsilon^{-1} V$, the function

$$Y_\varepsilon : U_\varepsilon \to V_\varepsilon$$

given by $Y_\varepsilon(\mathrm{y}) := \varepsilon^{-1} Y(\varepsilon \mathrm{y})$ is a parametrization of Σ_ε. In the sequel, we will denote by y the points in U_ε and by $y = Y_\varepsilon(\mathrm{y})$ the points in V_ε. For any $|z| < \tau/\varepsilon$, we consider the surface

$$\Sigma_{\varepsilon,z} := \{y + z\nu(\varepsilon y), y \in \Sigma_\varepsilon\}. \tag{3.11}$$

On this surface, we consider the parametrization

$$X_\varepsilon(\mathrm{y}, z) := Y_\varepsilon(\mathrm{y}) + z\nu(\varepsilon Y_\varepsilon(\mathrm{y})). \tag{3.12}$$

In particular, $X := X_1$ is a parametrization of $\Sigma_z := \Sigma_{1,z}$, the homothetic surface to Σ at distance z. It is known that the tangent vectors $\{\partial_i X_\varepsilon(\mathrm{y}, z)\}_{i=1,2}$ constitute a basis of the tangent space $T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z}$, that will be referred to as the standard basis. We define the coefficients of the metric of $\Sigma_{\varepsilon,z}$ at $y + z\nu(\varepsilon y)$ as follows

$$\tilde{g}_{\varepsilon,ij}(y, z) :=< \partial_i X_\varepsilon(\mathrm{y}), \partial_j X_\varepsilon(\mathrm{y}) >= \tilde{g}_{ij}(\varepsilon y, \varepsilon z), \tag{3.13}$$

where $<\cdot\,,\cdot>$ denotes the scalar product of $\mathbb{R}^3$ and $i, j = 1, 2$. The Laplacian on $\Sigma_{\varepsilon,z}$ is given by

$$\Delta_{\Sigma_{\varepsilon,z}} = \frac{1}{\sqrt{\det \tilde{g}_\varepsilon(y,z)}} \partial_j \big(\sqrt{\det \tilde{g}_\varepsilon(y,z)}\, \tilde{g}_\varepsilon^{ij}(y,z) \partial_i \big) = \tilde{g}_\varepsilon^{ij}(y,z)\partial_{ij} + \tilde{b}_\varepsilon^i(y,z)\partial_i, \tag{3.14}$$

where

$$\tilde{b}^i_\varepsilon(y,z) := \partial_j \tilde{g}^{ij}_\varepsilon(y,z) + \frac{1}{2}\partial_j\big(\log\det\tilde{g}_\varepsilon(y,z)\big)\tilde{g}^{ij}_\varepsilon(y,z) \tag{3.15}$$

and $\tilde{g}^{ij}_\varepsilon := (\tilde{g}^{-1}_\varepsilon)_{ij}$ are the elements of the inverse of the metric. These quantities are related to the ones of Σ_z through the relations

$$\begin{aligned}\tilde{g}^{ij}_\varepsilon(y,z) &= \tilde{g}^{ij}(\varepsilon y, \varepsilon z),\\ \tilde{b}^i_\varepsilon(y,z) &= \varepsilon\tilde{b}^i(\varepsilon y, \varepsilon z),\end{aligned}$$

with $\tilde{g}^{ij} := \tilde{g}^{ij}_1$ and $\tilde{b}^i := \tilde{b}^i_1$. We define the second fundamental form at $y+z\nu(\varepsilon y) \in \Sigma_{\varepsilon,z}$ to be the linear application of the tangent space $T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z}$ into itself that, in the standard basis $\{\partial_i X_\varepsilon(\mathrm{y},z)\}_{i=1,2}$, is represented by the matrix

$$\tilde{A}_{\varepsilon,ij}(y,z) := -<\partial_i\nu(\varepsilon y), \partial_j X_\varepsilon(\mathrm{y},z)> . \tag{3.16}$$

We introduce the mean curvature $\tilde{H}_\varepsilon(y,z)$ of $\Sigma_{\varepsilon,z}$ at $y+z\nu(\varepsilon y)$ as follows

$$\tilde{H}_\varepsilon(y,z) := (\tilde{A}_\varepsilon)^i_i(y,z) = \tilde{g}^{ij}_\varepsilon(y,z)\tilde{A}_{\varepsilon,ij}(y,z).$$

In other words

$$\tilde{H}_\varepsilon(y,z) = \tilde{k}_{\varepsilon,1}(y,z) + \tilde{k}_{\varepsilon,2}(y,z),$$

where $\tilde{k}_{\varepsilon,i}(y,z)$ are the *principal curvatures* of $\Sigma_{\varepsilon,z}$, that is eigenvalues of the matrix $\tilde{g}^{-1}_\varepsilon(y,z)\tilde{A}_\varepsilon(y,z)$. Therefore we can see that the metric $\tilde{g}_{\varepsilon,ij}(y,z)$ and the matrix representing the second fundamental $\tilde{A}_{\varepsilon,ij}(y,z)$ form depend on the parametrization, while this is not the case for $\tilde{H}_\varepsilon(y,z)$. Setting, as above $\tilde{A}_{ij} := \tilde{A}_{1,ij}$ and $\tilde{H} := \tilde{H}_1$, we have $\tilde{A}_{\varepsilon,ij}(y,z) = \varepsilon\tilde{A}_{ij}(\varepsilon y, \varepsilon z)$ and $\tilde{H}_\varepsilon(y,z) = \varepsilon\tilde{H}(\varepsilon y,\varepsilon z)$.

Lemma 79. *For a function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ of class C^2, the Laplacian in Fermi coordinates is given by*

$$\Delta u(y,z) = \Delta_{\Sigma_{\varepsilon,z}} u(y,z) - \varepsilon\tilde{H}(\varepsilon y,\varepsilon z)\partial_z u(y,z) + \partial_{zz}u(y,z). \tag{3.17}$$

For the notation, see Remark 78.

Proof. For any $y \in \Sigma_\varepsilon$ and $|z| < \tau/\varepsilon$, $\mathbb{R}^3$ splits into the direct sum of the tangent space to $\Sigma_{\varepsilon,z}$ and the one dimensional subspace generated by the unit normal $\nu(\varepsilon y)$, that is $\mathbb{R}^3 = T_{y+z\nu(\varepsilon y)}\Sigma_{\varepsilon,z} + \mathbb{R}$. The vectors $\{\partial_i X_\varepsilon(y,z), \nu(\varepsilon y)\}_{i=1,2}$ constitute a basis of $\mathbb{R}^3 = T_{y+z\nu(\varepsilon y)}\mathbb{R}^3$. The metric in this basis is given by

$$G_\varepsilon(y,z) = \begin{bmatrix}\tilde{g}_\varepsilon(y,z) & 0\\ 0 & 1\end{bmatrix}. \tag{3.18}$$

The inverse is

$$G_\varepsilon^{-1}(y,z) = \begin{bmatrix} \tilde{g}_\varepsilon^{-1}(y,z) & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.19}$$

Here $1 \le I, J \le 3$ and $1 \le i, j \le 2$. The laplacian on $\mathbb{R}^3$ in the metric G_ε is given by

$$\Delta u = \frac{1}{\sqrt{\det G_\varepsilon(y,z)}} \partial_J \big(\sqrt{\det G_\varepsilon(y,z)}\, G_\varepsilon^{IJ}(y,z)\partial_I\big) =$$

$$G_\varepsilon^{IJ}(y,z)\partial_{IJ}u(y,z) + \partial_J G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) + \frac{1}{2}\partial_J(\log\det G_\varepsilon(y,z)) G_\varepsilon^{IJ}(y,z)\partial_I u(y,z).$$

Thus

$$\begin{aligned} G_\varepsilon^{IJ}(y,z)\partial_{IJ}u(y,z) &= \tilde{g}_\varepsilon^{ij}(y,z)\partial_{ij}u(y,z) + \partial_{zz}u(y,z) \\ \partial_J G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) &= \partial_j \tilde{g}_\varepsilon^{ij}(y,z)\partial_i u(y,z) \\ \frac{1}{2}\partial_J(\log\det G_\varepsilon(y,z)) G_\varepsilon^{IJ}(y,z)\partial_I u(y,z) &= \end{aligned}$$

$$\frac{1}{2}\partial_j(\log\det \tilde{g}_\varepsilon(y,z))\tilde{g}_\varepsilon^{ij}(y,z)\partial_i u(y,z) + \frac{1}{2}\partial_z(\log\det \tilde{g}_\varepsilon(y,z))\partial_z u(y,z).$$

To conclude, we point out that

$$\frac{1}{2}\partial_z(\log\det \tilde{g}_\varepsilon(y,z)) = -\tilde{H}_\varepsilon(y,z) = -\varepsilon \tilde{H}(\varepsilon y, \varepsilon z).$$

□

Exploiting the Taylor expansion of $\tilde{H}$ of the mean curvature of a given hypersurface provided by Del Pino, Kowalczyk and Wei (see [24]), we have that

$$\tilde{H}(\varepsilon y, \varepsilon z) = \sum_{i=1}^{2} \frac{k_i(\varepsilon y)}{1 - \varepsilon z k_i(\varepsilon y)} = \sum_{j \ge 1} (\varepsilon z)^{j-1} H_j(\varepsilon y), \quad H_j(\varepsilon y) := \textstyle\sum_{i=1}^2 k_i^j(\varepsilon y) \tag{3.20}$$

Here $k_i(\varepsilon y) := \tilde{k}_{\varepsilon,i}(y,0)$ are the principal curvatures of the Clifford Torus Σ at εy. Therefore the Taylor expansions of the first and the second derivatives of $\tilde{H}$ are

$$\begin{cases} \tilde{H}_z(\varepsilon y, \varepsilon z) = \sum_{j\ge 1} j(\varepsilon z)^{j-1} H_{j+1}(\varepsilon y), \\ \tilde{H}_{zz}(\varepsilon y, \varepsilon z) = \sum_{j \ge 1} j(j+1)(\varepsilon z)^{j-1} H_{j+2}(\varepsilon y). \end{cases} \tag{3.21}$$

In the sequel, we will set $H(\varepsilon y) := H_1(\varepsilon y)$, $|A(\varepsilon y)|^2 := H_2(\varepsilon y)$ and $\mathrm{tr} A^3(y) := H_3(\varepsilon y)$.

Now we need the Taylor expansion in εz of $\Delta_{\Sigma_{\varepsilon,z}}$. For our purposes, it is enough to know the terms of order zero and one, while we also need the term of order two in the expansion of $\tilde{H}$. For this reason, we prefer not to expand the full Laplacian on $\mathbb{R}^3$. In fact, an expansion up to order one would not be enough, because we cannot neglect the terms involving $\mathrm{tr}A^3$, while an expansion up to order two would be a useless effort, in fact it would involve the terms of order two of $\Delta_{\Sigma_{\varepsilon,z}}$, that will always simplify in our forthcoming calculations. Before stating next Lemma, we recall that

$$\Delta_{\Sigma_\varepsilon} = \frac{1}{\sqrt{\det g_\varepsilon(y)}}\partial_j\big(\sqrt{\det g_\varepsilon(y)}g_\varepsilon^{ij}(y)\partial_i\big) = g_\varepsilon^{ij}(y)\partial_{ij} + b_\varepsilon^i(y)\partial_i, \tag{3.22}$$

where

$$\begin{aligned} g_\varepsilon^{ij}(y) &:= \tilde{g}_\varepsilon^{ij}(y,0) = \tilde{g}^{ij}(\varepsilon y,0) = g^{ij}(\varepsilon y) \\ b_\varepsilon^i(y) &:= \tilde{b}_\varepsilon^i(y,0) = \varepsilon\tilde{b}^i(\varepsilon y,0) = \varepsilon b^i(\varepsilon y). \end{aligned} \tag{3.23}$$

It is possible to find similar computations in [52], where Mahmoudi, Sànchez and Yao treat the more general case of a k-dimensional submanifold in an N-dimensional manifold.

Lemma 80. *For a function $u : V_{\tau/\varepsilon} \to \mathbb{R}$ of class C^2, for any $y \in \Sigma_\varepsilon$, for any $|z| \le \tau/\varepsilon$,*

$$\begin{aligned} \Delta_{\Sigma_{\varepsilon,z}}u = \Delta_{\Sigma_\varepsilon}u + \varepsilon z(a_1^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_1^i(\varepsilon y)\partial_i) \\ +(\varepsilon z)^2(a_2^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_2^i(\varepsilon y)\partial_i) + \overline{a}^{ij}(\varepsilon y,\varepsilon z)\partial_{ij} + \varepsilon\overline{b}_i(\varepsilon y,\varepsilon z)\partial_i, \end{aligned}$$

where

$$\begin{aligned} a_1^{ij} &:= 2A^{ij}, b_1^i := 2\partial_j A^{ij} + 2\Gamma_{kj}^k A^{ij} - g^{ij}H_j, \\ a_2^{ij} &:= \frac{1}{2}\partial_{zz}\tilde{g}^{ij}(\varepsilon y,0), b_2^i := \frac{1}{2}\partial_{zz}\tilde{b}^i(\varepsilon y,0), \end{aligned}$$

everything evaluated at εy, and the remainders satisfy $|\overline{a}^{ij}(\varepsilon y,\varepsilon z)|, |\overline{b}^i(\varepsilon y,\varepsilon z)| \le c\varepsilon^3|z|^3$, for some constant $c > 0$ depending on Σ.

Remark 81. *This expansion was already known up to the order zero in ε (see, for instance, [24]). What is really new here is the computation of a_1^{ij} and b_1^i, necessary to study our fourth order problem.*

Let $\phi, \psi : \Sigma \to \mathbb{R}$ be C^2 functions. Let us set $\phi_i := \partial_i\phi$. We recall that, by the properties of the covariant derivative,

$$\begin{aligned} \nabla_k A^{ij} &= \partial_k A^{ij} + \Gamma_{kl}^i A^{lj} + \Gamma_{kl}^j A^{li}, \\ \nabla_{ij}^2\phi &= \phi_{ij} - \Gamma_{ij}^k\phi_k, \end{aligned}$$

where everything is evaluated at εy. Moreover, by Codazzi's equation, $\nabla_j A^{ij} = g^{ik}\nabla_k A^j_j$, so in particular,

$$\begin{aligned} a_1^{ij}\phi_i\psi_j &= 2(A\nabla\phi, \nabla\psi) \\ a_1^{ij}\psi_{ij} + b_1^i\psi_i &= 2A^{ij}\psi_{ij} - 2\Gamma^k_{ji}A^{ij}\psi_k + 2\nabla_j A^{ij}\psi_i \\ -(\nabla_\Sigma H, \nabla_\Sigma\psi) &= 2 < A, \nabla^2\psi > +(\nabla_\Sigma\psi, \nabla_\Sigma H), \end{aligned} \tag{3.24}$$

where we have set

$$< A, \nabla^2\psi >:= A^{ij}\nabla^2_{ij}\psi = A^{ij}\psi_{ij} + \Gamma^k_{ij}\psi_k.$$

Proof. By (3.12) and (3.13), we can see that

$$\tilde{g}_{\varepsilon,ij}(y,z) = g_{ij} + \varepsilon z(< \partial_i Y, \partial_j\nu > + < \partial_j Y, \partial_i\nu >) + (\varepsilon z)^2 < \partial_i\nu, \partial_j\nu > .$$

In the proof, it is understood that the geometric quantities of Σ are evaluated at εy. In view of (3.16) with $z = 0$, we have

$$\partial_i\nu = -A^k_i\partial_k Y$$

therefore

$$\begin{aligned} \tilde{g}_{\varepsilon,ij}(y,z) = g_{ij} - \varepsilon z(g_{ik}A^k_j + g_{jk}A^k_i) + (\varepsilon z)^2 < \partial_i\nu, \partial_j\nu >= \\ g_{ij} - 2\varepsilon z A_{ij} + (\varepsilon z)^2 < \partial_i\nu, \partial_j\nu > . \end{aligned} \tag{3.25}$$

In order to expand the Laplacian, we need the expansion of the inverse of the metric. It is useful to write it as

$$\tilde{g}_\varepsilon = L + M,$$

with $L_{ij} = g_{ij}$ and $M = -2\varepsilon z A_{ij} + (\varepsilon z)^2 < \partial_i\nu, \partial_j\nu >$. Equivalently, $\tilde{g}_\varepsilon = L(I + L^{-1}M)$, hence

$$\tilde{g}_\varepsilon^{-1} = (I + L^{-1}M)^{-1}L^{-1} = (I - L^{-1}M + O((\varepsilon z)^2))L^{-1} = L^{-1} - L^{-1}ML^{-1} + O((\varepsilon z)^2),$$

thus

$$\tilde{g}^{ij}_\varepsilon(y,z) = g^{ij} + 2\varepsilon z A^{ij} + O((\varepsilon z)^2).$$

where $A^{ij} = g^{ik}g^{jl}A_{kl}$. Moreover

$$\log\det\tilde{g}_\varepsilon(y,z) = \log\det g_\varepsilon(y) + \mathrm{tr}(L^{-1}M) + O((\varepsilon z)^2) = \log\det g_\varepsilon - 2\varepsilon z H + O((\varepsilon z)^2),$$

so, since $\frac{1}{2}\partial_j(\log\det g)A^{ij} = \Gamma^k_{kj}A^{ij}$,

$$\begin{aligned}\Delta_{\Sigma_{\varepsilon,z}} &= (g^{ij} + 2\varepsilon z A^{ij})\partial_{ij} + \varepsilon(\partial_j g^{ij} + 2\varepsilon z\partial_j A^{ij})\partial_i \\ &+\varepsilon\Big(\frac{1}{2}\partial_j(\log\det g) - \varepsilon z H_j\Big)(g^{ij} + 2\varepsilon z A^{ij})\partial_i + O((\varepsilon z)^2) = \\ \Delta_{\Sigma_\varepsilon} &+ \varepsilon z\Big\{2A^{ij}\partial_{ij} + \varepsilon(2\partial_j A^{ij} + 2\Gamma^k_{kj}A^{ij} - g^{ij}H_j)\partial_i\Big\} + O((\varepsilon z)^2).\end{aligned}$$

□

As a consequence, we have the following expansion of the Laplacian

$$\begin{aligned}\Delta &= \partial_{zz} - \varepsilon\tilde{H}(\varepsilon y, \varepsilon z)\partial_z + \Delta_{\Sigma_\varepsilon} + \varepsilon z(a_1^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_1^i(\varepsilon y)\partial_i) \\ &+(\varepsilon z)^2(a_2^{ij}(\varepsilon y)\partial_{ij} + \varepsilon b_2^i(\varepsilon y)\partial_i) + \overline{a}^{ij}(\varepsilon y, \varepsilon z)\partial_{ij} + \varepsilon\overline{b}^i(y,z)\partial_i.\end{aligned} \tag{3.26}$$

Although (3.26) looks nice, we prefer to look for the expression of the Laplacian in a slightly different system of coordinates. We fix a C^2 function $\phi : \Sigma \to \mathbb{R}$ whose $L^\infty(\Sigma)$ is less than $1/4$ and we introduce a new change of variables, that is we put

$$t = z - \phi(\varepsilon y). \tag{3.27}$$

The expression of the Laplacian will be more complicated than (3.26), but more appropriate for our purposes. The reason is that we know the kernel of the operator $-(\Delta_{\Sigma_\varepsilon} + \partial_{tt}) + W''(v_\star(t))$, that is the one dimensional space generated by $v_\star'(t)$, while we do not know exactly the kernel (if any) of $-(\Delta_{\Sigma_\varepsilon}+\partial_{zz})+W''(v_\star(z-\phi(\varepsilon y)))$.

Given a function

$$f : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$$

of class C^2, it is possible to define

$$\mathrm{f} : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$$

by setting $\mathrm{f}(y,t) := f(y, z - \phi(\varepsilon y))$. A computation shows that

$$\begin{aligned}\mathrm{f}_t(y,t) &= f_z(y, z-\phi) \\ \mathrm{f}_i(y,t) &= f_i(y, z-\phi) - \varepsilon\phi_i f_z(y, z-\phi) \\ \mathrm{f}_{ij}(y,t) &= f_{ij}(y, z-\phi) - \varepsilon\phi_i f_{zj}(y, z-\phi) - \varepsilon\phi_i f_{zj}(y, z-\phi) \\ &+\varepsilon^2\phi_{ij} f_z(y, z-\phi) + \varepsilon\phi_i\phi_j f_{zz}(y, z-\phi),\end{aligned}$$

where ϕ and its derivatives are evaluated at εy, thus, in these coordinates, the expression of the Laplacian of a function u defined in $V_{\tau/\varepsilon}$ of class C^2 is given by

$$\Delta = \partial_{tt} + g^{ij}\partial_{ij} + \varepsilon b^i\partial_i + \mathrm{D} = \partial_{tt} + \Delta_{\Sigma_\varepsilon} + \mathrm{D}, \tag{3.28}$$

where the operator D is given by

$$\begin{aligned} \mathrm{D} := & -\varepsilon\hat{H}(\varepsilon y, \varepsilon(t+\phi))\partial_t - \varepsilon^2\Delta_\Sigma\phi\partial_t - 2\varepsilon g^{ij}\phi_i\partial_{tj} + \varepsilon^2|\nabla_\Sigma\phi|^2\partial_{tt} \\ & +\varepsilon(t+\phi)\big\{a_1^{ij}\partial_{ij} + \varepsilon b_1^i\partial_i - \varepsilon^2(a_1^{ij}\phi_{ij} + b_1^i\phi_i)\partial_t - 2\varepsilon a_1^{ij}\phi_i\partial_{tj} + \varepsilon^2 a_1^{ij}\phi_i\phi_j\partial_{tt}\big\} \\ & +\varepsilon^2(t+\phi)^2\big\{a_2^{ij}\partial_{ij} + \varepsilon b_2^i\partial_i - \varepsilon^2(a_2^{ij}\phi_{ij} + b_2^i\phi_i)\partial_t - 2\varepsilon a_2^{ij}\phi_i\partial_{tj} + \varepsilon^2 a_2^{ij}\phi_i\phi_j\partial_{tt}\big\} \\ & +\hat{a}^{ij}\partial_{ij} + \varepsilon\hat{b}^i\partial_i - \varepsilon^2(\hat{a}^{ij}\phi_{ij} + \hat{b}^i\phi_i)\partial_t - 2\varepsilon\hat{a}^{ij}\phi_i\partial_{tj} + \varepsilon^2\hat{a}^{ij}\phi_i\phi_j\partial_{tt}. \end{aligned} \tag{3.29}$$

Here we have set $\hat{H}(\varepsilon y, \varepsilon(t+\phi)) := \tilde{H}(\varepsilon y, \varepsilon z)$, $\hat{a}^{ij}(\varepsilon y, \varepsilon(t+\phi)) = \overline{a}^{ij}(\varepsilon y, \varepsilon z)$, $\hat{b}^i(\varepsilon y, \varepsilon(t+\phi)) = \overline{b}^i(\varepsilon y, \varepsilon z)$ and all the geometric quantities of Σ are evaluated at εy.

3.3 Functional setting

3.3.1 Functions on Σ

As first we define, for $0 < \alpha < 1$, the space $C^{k,\alpha}(\Sigma)$ as the set of functions $\phi : \Sigma \to \mathbb{R}$ that are k times differentiable and whose $k-$th partial derivatives are Hölder continuous with exponent α. We endow these spaces with the norms

$$|\phi|_{C^{k,\alpha}(\Sigma)} := \sum_{j=0}^{k} ||\nabla^j\phi||_\infty + \sup_{p\neq q}\frac{|\nabla^k\phi(p) - \nabla^k\phi(q)|}{d(p,q)^\alpha}.$$

In order to treat $\tilde{L}_0$, we define the spaces of functions that respect the symmetries of the Torus, that is the symmetry with respect to the x_1x_2-plane and with respect to any rotation that keeps the x_3-axis fixed. To be precise, we set $T(x_1, x_2, x_3) := (x_1, x_2, -x_3)$ and

$$SO_{x_3}(3) := \{R \in SO(3) : Re_3 = e_3\},$$

where $e_3 = (0, 0, 1)$, and we define

$$\begin{aligned} C^{k,\alpha}(\Sigma)_s := \{\phi \in C^{k,\alpha}(\Sigma) : \phi(\zeta) = \phi(T\zeta) \text{ for any } \zeta \in \Sigma, \\ \phi(\zeta) = \phi(R\zeta) \text{ for any } R \in SO_{x_3}(3)\}. \end{aligned}$$

We note that $SO_{x_3}(2) \simeq SO(2)$, in the sense that any matrix $R \in SO_{x_3}(3)$ has the form

$$R = \begin{bmatrix} \tilde{R} & 0 \\ 0 & 1 \end{bmatrix},$$

for some rotation of the x_1x_2-plane $\tilde{R} \in SO(2)$.

Equivalently, we can see the Torus as the quotient of the square $[0,2\pi]^2$ by the equivalence relation that identifies the opposite sides. In this way, our spaces will become

$$C^{k,\alpha}(\Sigma)_s := \{\phi \in C^{k,\alpha}([0,2\pi]) : \phi(\vartheta_1) = \phi(2\pi - \vartheta_1) \text{ for any } \vartheta_1 \in [0,\pi]\}. \quad (3.30)$$

In other words, functions respecting these symmetries are actually periodic functions of one real variable, symmetric with respect to $\vartheta_1 = \pi$. In the sequel, we will also be interested in the spaces

$$\begin{aligned} C^{k,\alpha}(\Sigma)_{s,0} := \{\phi \in C^{k,\alpha}([0,2\pi]) : \phi(\vartheta_1) = \phi(2\pi - \vartheta_1) \text{ for any } \vartheta_1 \in [0,\pi], \quad (3.31) \\ \phi'(0) = \phi'(2\pi) = \phi^{(3)}(0) = \phi^{(3)}(2\pi) = 0\}. \end{aligned}$$

By the symmetries of the Laplacian, the gradient and the geometric quantities of Σ, one can show that $\tilde{L}_0$ preserves the symmetries of functions $\phi \in C^{4,\alpha}(\Sigma)_s$, that is it maps $C^{4,\alpha}(\Sigma)_s$ into $C^{0,\alpha}(\Sigma)_s$.

Let us introduce the operator

$$\mathcal{L} : C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R} \to C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$$

defined by

$$\mathcal{L}(\phi,\lambda) := \left(\tilde{L}_0\phi + \lambda, \int_\Sigma \phi(\zeta) d\sigma(\zeta)\right).$$

When we solve the equation $\mathcal{L}(\phi,\lambda) = (h,a)$, with h symmetric with respect to y_1, the solution ϕ will satisfy the same symmetry property, thus also its second and fourth derivative will do, while the first and third will be antisymmetric. In order to extend it by periodicity, we need $\phi'(0) = \phi'(2\pi)$ and $\phi^{(3)}(0) = \phi^{(3)}(2\pi)$, hence we need zero Neumann boundary conditions.

It is known that $\tilde{L}_0$ is self-adjoint with respect to the $L^2(\Sigma)$-scalar product (see [51], section 3), thus, $\mathcal{L}$ is self adjoint with respect to the scalar product

$$< (\phi,\lambda),(\psi,\mu) > := \int_\Sigma \phi(\zeta)\psi(\zeta) d\sigma(\zeta) + \lambda\mu.$$

Remark 77 can be rephrased by saying that the operator $\mathcal{L}$ is injective. In fact, if $\mathcal{L}(\phi,\lambda) = 0$, multiplying by ϕ and integrating over Σ we get

$$\int_\Sigma \tilde{L}_0\phi(\zeta)\phi(\zeta) d\sigma(\zeta) = -\lambda \int_\Sigma \phi(\zeta) d\sigma(\zeta) = 0.$$

By the result of Weiner ([77], Corollary 2, page 34), the elements of the Kernel of $\tilde{L}_0$ are the normal components of the vector fiels generated by conformal transformations. The only ones that preserve the symmetries of the Torus are dilations, that are excluded by the volume constraint. This is equivalent to say that $X \cap N = 0$, where

$$N := \{\phi \in C^4(\Sigma) : \tilde{L}_0 \phi = 0\}, X := \left\{\phi \in C^{4,\alpha}(\Sigma)_{s,0} : \int_\Sigma \phi(\zeta) d\zeta = 0\right\}. \quad (3.32)$$

Moreover, once again by [77], Corollary 2, page 34, $\tilde{L_0}$ is positive definite on

$$N^\perp := \left\{\phi \in C^{4,\alpha}(\Sigma) : \int_\Sigma \phi n d\sigma(\zeta) = 0, \forall n \in N\right\} \quad (3.33)$$

thus we conclude that $\phi = 0$, so $\lambda = 0$.

In order to show the solvability of the linear problem

$$\begin{cases} \tilde{L}_0 \phi(\zeta) + \lambda = h(\zeta) & \forall \zeta \in \Sigma, \\ \int_\Sigma \phi = a. \end{cases} \quad (3.34)$$

also surjectivity is needed. For this purpose, we will use the Fredholm theory. First we note that, if $\phi \in C^{4,\alpha}(\Sigma)_{s,0}$, then

$$\tilde{L}_0 \phi = \phi^{(4)} + L_1 \phi,$$

for some linear operator of order 3, that will be denoted by L_1, that can be computed explicitely exploiting the parametrization (3.83) of Σ, $\phi^{(4)}$ denotes the fourth derivative with respect to ϑ_1 and

$$\int_\Sigma \phi d\sigma(\zeta) = \int_0^{2\pi} \phi(\vartheta_1)(\sqrt{2} + \cos \vartheta_1) d\vartheta_1,$$

therefore, since $h \in C^{0,\alpha}(\Sigma)_s$ (3.34) becomes

$$\begin{cases} \phi^{(4)} + L_1 \phi + \lambda = h & \forall \zeta \in \Sigma, \\ \int_0^{2\pi} \phi(\vartheta_1) d\vartheta_1 = \frac{a}{\sqrt{2}} - \frac{1}{\sqrt{2}} \int_0^{2\pi} \phi(\vartheta_1) \cos \vartheta_1 d\vartheta_1, \end{cases} \quad (3.35)$$

or equivalently

$$(\mathcal{L}_1 + \mathcal{L}_2)(\phi, \lambda) = (h, a/\sqrt{2}), \quad (3.36)$$

where

$$\mathcal{L}_1(\phi, \lambda) := \left(\phi^{(4)} + \lambda, \int_0^{2\pi} \phi(\vartheta_1) d\vartheta_1\right) \quad (3.37)$$

and

$$\mathcal{L}_2(\phi, \lambda) := \Big(L_1\phi, \frac{1}{\sqrt{2}} \int_0^{2\pi} \phi(\vartheta_1) \cos\vartheta_1 d\vartheta_1 \Big). \tag{3.38}$$

In the next Proposition, we will prove the invertibility of $\mathcal{L}_1$, whose inverse will enable us to apply the Fredholm theory.

Proposition 82. *For any $(h, a) \in C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$, there exists a unique solution $(\phi, \lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ to the problem*

$$\begin{cases} \phi^{(4)} + \lambda = h & \forall \vartheta_1 \in [0, 2\pi] \\ \int_0^{2\pi} \phi(\vartheta_1) d\vartheta_1 = a, \end{cases} \tag{3.39}$$

satisfying the estimate

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \leq c(|h|_{C^{0,\alpha}(\Sigma)} + |a|), \tag{3.40}$$

for some constant $c > 0$.

Proof. First we write $\phi = \phi^{||} + \phi^{\perp}$, where $\phi^{||}$ is a constant and $\phi^{\perp}$ has zero-average. Rephrasing (3.39) in this way, we get $\phi^{||} = a/2\pi$ and $\phi^{\perp}$ has to satisfy

$$(\phi^{\perp})^{(4)} + \lambda = h \quad \forall \vartheta_1 \in [0, 2\pi]$$

$$\int_0^{2\pi} \phi^{\perp} = 0, \tag{3.41}$$

$$(\phi^{\perp})'(0) = (\phi^{\perp})'(2\pi) = (\phi^{\perp})^{(3)}(0) = (\phi^{\perp})^{(3)}(2\pi) = 0. \tag{3.42}$$

As a consequence, our problem is solved if we set

$$\lambda = \frac{1}{2\pi} \int_0^{2\pi} h, \phi^{\perp}(\vartheta_1) = \int_0^1 G(\vartheta_1, s)(h(s) - \lambda) ds + \phi_0,$$

where G is the Grenn function of $\frac{d^4}{d\vartheta_1^4}$ with double Neumann boundary conditions and $\phi_0 \in \mathbb{R}$ is chosen in such a way that $\phi^{\perp}$ has zero average. Since h is symmetric with respect to $\vartheta_1 = \pi$, then, by uniqueness, the same is true for $\phi^{\perp}$. The regularity of $\phi^{\perp}$ and the estimates of the norms follow by construction. □

Proposition 83. *Problem (3.34) is uniquely solvable for any $(h, a) \in C^{0,\alpha}(\Sigma)_s \times \mathbb{R}$, and the solution $(\phi, \lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ fulfills the estimate*

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \leq c(|h|_{C^{0,\alpha}(\Sigma)} + |a|), \tag{3.43}$$

for some constant $c > 0$.

Proof. It is enough to write (3.34) in the form

$$(\phi, \lambda) + \mathcal{L}_1^{-1}\mathcal{L}_2(\phi, \lambda) = \mathcal{L}_1^{-1}(h, a)$$

and to apply the Fredholm alternative theorem. Notice that $\mathcal{L}_1^{-1}\mathcal{L}_2$ is compact and $I + \mathcal{L}_1^{-1}\mathcal{L}_2$ is injective on the space $C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$, since $\mathcal{L}$ is, as we observed above. □

In the sequel, we will often use the notation

$$B_k(1/4) := \{\phi \in C^{k,\alpha}(\Sigma)_{s,0} : |\phi|_{C^{k,\alpha}(\Sigma)} \leq 1/4\}.$$

In constructing approximate solutions, we will generate error terms with high powers of ε but with high derivatives of ϕ. To deal with these terms, we introduce some smoothing operators, as in [63]. Let us recall that Alinhac and Gérard (see [4]) constructed a family of smoothing operators $\{R_\theta\}_{\theta \geq 1}$ such that

$$|R_\theta \phi|_{C^{k,\alpha}(\Sigma)} \leq c|\phi|_{C^{k',\alpha'}(\Sigma)} \quad \text{if } k + \alpha \leq k' + \alpha' \tag{3.44}$$

$$|R_\theta \phi|_{C^{k,\alpha}(\Sigma)} \leq c\theta^{k+\alpha-k'-\alpha'}|\phi|_{C^{k',\alpha'}(\Sigma)} \quad \text{if } k + \alpha \geq k' + \alpha' \tag{3.45}$$

$$|\phi - R_\theta \phi|_{C^{k,\alpha}(\Sigma)} \leq c\theta^{k+\alpha-k'-\alpha'}|\phi|_{C^{k',\alpha'}(\Sigma)} \quad \text{if } k + \alpha \leq k' + \alpha'. \tag{3.46}$$

We point out that, since the construction of the R_θ's relies on cutting-off high Fourier modes, we have

$$\int_\Sigma \phi = \int_\Sigma R_\theta \phi, \quad \forall \phi \in C^{4,\alpha}(\Sigma)_s.$$

It is possible to find further details in [23], where the periodic (compact) case is specifically treated.

These operators are useful to gain regularity. For instance, we will start our construction of the approximate solution from a function $\phi \in C^{4,\alpha}(\Sigma)$, since the operator $\tilde{L}_0$ is a fourth-order one, but some higher order derivatives will appear in the construction. As a consequence, we replace ϕ by $\phi_\star := R_{1/\varepsilon}\phi$ in the expansion of the Laplacian (3.29), that is we set $t = z - \phi_\star(\varepsilon y)$, for any $y \in \Sigma_\varepsilon$. We need to estimate the derivatives of $\phi_\star$ up to order six, which will be done by means of the properties of the operators R_θ.

3.3.2 Exponentially decaying functions on $\mathbb{R}^3$

For any $\delta > 0$ and for any $x \in \mathbb{R}^N$, we define

$$\varphi_\delta(x) := \zeta(|x|) + (1 - \zeta(|x|))e^{\delta|x|},$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is a C^∞ cutoff function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } t < 1 \\ 0 & \text{for } t > 2. \end{cases}$$

Moreover, we introduce the weighted spaces

$$C^{k,\alpha}_\delta(\mathbb{R}^3) := \{u \in C^{k,\alpha}(\mathbb{R}^3) : ||\tilde{u}_\delta||_{C^{k,\alpha}(\mathbb{R}^3)} < \infty\},$$

where $\tilde{u}_\delta := u\varphi_\delta$ and $C^{k,\alpha}(\mathbb{R}^3)$ is the space of $C^k(\mathbb{R}^3)$ functions whose fourth derivatives are Hölder continuous with exponent α. We point out that functions belonging $C^{k,\alpha}_\delta(\mathbb{R}^3)$ decay exponentially with rate δ, and the same is true for their derivatives.

This spaces are endowed with the norm

$$||u||_{C^{k,\alpha}(\mathbb{R}^3)} := \sum_{j=0}^{k} ||\nabla^j u||_\infty + [\nabla^k u]_\alpha.$$

In order to construct solutions to (3.4) that respect the symmetry of the Torus, we need to introduce the spaces of functions fulfilling these symmetries, that is

$$C^{k,\alpha}_{\delta,s}(\mathbb{R}^3) := \{u \in C^{k,\alpha}_\delta(\mathbb{R}^3) : u(Tx) = u(x) \quad , u(Rx) = u(x) \quad \text{for any } R \in SO_{x_3}(3)\}.$$

Remark 84. *We note that, for instance, if $u \in C^{2,\alpha}_{\delta,s}(\mathbb{R}^3)$, then $\Delta u \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$. In fact, by definition, any $u \in C^{2,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfies $u(x) = u_T(x)$, where $u_T(x) := u(Tx)$. Taking the Laplacian, we can see that $\Delta u(x) = \Delta u_T(x) = \Delta u(Tx)$, and similarly, if $R \in SO_{x_3}(3)$ and we set $u_R(x) = u(Rx)$, then $\Delta u(x) = \Delta u_R(x) = \Delta u(Rx)$.*

3.3.3 Functions on $\Sigma_\varepsilon \times \mathbb{R}$

First we will show existence and uniqueness of the heteroclinic solution to the ODE $-v_\star'' + W'(v_\star) = 0$. The result is known, but since the proof is quite short, we report it for completeness.

Lemma 85. *Let W be an even double well potential satisfying (3.3). Then there exists a unique solution $v_\star$ to the problem*

$$\begin{cases} -v_\star'' + W'(v_\star) = 0 \\ v_\star(0) = 0 \\ v_\star \to \pm 1 & \text{as } t \to \pm\infty. \end{cases} \tag{3.47}$$

and this solution is odd.

It is known that, if $W(t) = \frac{1}{4}(1-t^2)^2$ is the classical double-well potential, then $v_\star(t) = \tanh(t/\sqrt{2})$.

Proof. Let $v_\star$ be the unique solution to the Cauchy Problem

$$\begin{cases} -v_\star'' + W'(v_\star) = 0 \\ v_\star(0) = 0 \\ v_\star'(0) = \sqrt{2W(0)}. \end{cases}$$

Let (a,b) be its maximal interval of definition, with $a < 0 < b$. Since the function $w(t) = -v_\star(-t)$ is still a solution to the same Cauchy Problem, $v_\star$ is an odd function, so it is enough to study $v_\star$ in the positive half line and $a = -b$. Multiplying the ODE by $v_\star'$ and integrating we have

$$\frac{1}{2}(v_\star')^2 = W(v_\star) + c. \tag{3.48}$$

Evaluating at $t = 0$, it is possible to see that $c = 0$. As a consequence, $v_\star' > 0$ in $(0,b)$. In fact, if we assume by contradiction that there exists a first t_0 such that $v_\star'(t_0) = 0$, then $W(v_\star(t_0)) = 0$, so in particular $v_\star(t_0) = 1$, but, by the uniqueness Cauchy Theorem, this implies that $v_\star \equiv 1$ in a neighbourhood of t_0, a contradiction. As a consequence, it is possible to define

$$l := \lim_{t\to b} v_\star(t).$$

By monotonicity, we know that $l > 0$. Now we want to rule out the case $l = \infty$. Indeed, if this were true, we would have $v_\star'' < 0$ near 0 and $v_\star'' > 0$ near b, so there should exist $t_1 > 0$ such that $v_\star''(t_1) = 0$. Therefore, using the equation and (3.48), we can see that $v_\star(t_1) = 1$ and $v_\star'(t_1) = 0$, which is not possible.

Since $l < \infty$, we have $b = \infty$. Now, always by (3.48), we get that $v_\star' \to \sqrt{2W(l)}$ as $t \to \infty$. Since u is bounded, $W(l) = 0$, hence $l = 1$.

Uniqueness follows from the Cauchy Theorem. □

It is known that $v_\star$ converges exponentially to ± 1 as $t \to \pm\infty$ at a rate which is given by $\sqrt{W''(1)} = \sqrt{W''(-1)}$, since W is even. More precisely, for any $k \in \mathbb{N}$, there exists a constant c_k such that

$$|\partial_t^k(v_\star - 1)| \le c_k e^{-t\sqrt{W''(1)}} \quad \text{for any } t \ge 0 \tag{3.49}$$

and

$$|\partial_t^k(v_\star + 1)| \le c_k e^{t\sqrt{W''(1)}} \quad \text{for any } t \le 0. \tag{3.50}$$

For instance, in the classical case $W(t) = \frac{1}{4}(1-t^2)^2$, we have $\sqrt{W''(\pm 1)} = \sqrt{2}$.

For $0 < \delta < \sqrt{W''(1)}$, we define the function

$$\psi_\delta(t) = (1+e^t)^\delta (1+e^{-t})^\delta.$$

For $0 < \varepsilon \leq 1$ and $0 < \alpha < 1$, we define the space $C^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ as the set of functions $U : \Sigma_\varepsilon \times \mathbb{R} \to \mathbb{R}$ that are k times differentiable and whose $k-$th partial derivatives are Hölder continuous with exponent α. This space is endowed with the norm

$$||U||_{C^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} = ||U\psi_\delta||_{C^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})},$$

where

$$||U||_{C^{k,\alpha}(\Sigma_\varepsilon \times \mathbb{R})} = \sum_{j=0}^{k} ||\nabla^j U||_{L^\infty(\Sigma_\varepsilon \times \mathbb{R})} + \sup_{x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha}.$$

Given the heteroclinic solution $v_\star$, we can define the spaces

$$\mathcal{E}^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) := \left\{ U \in C^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) : \int_{-\infty}^{\infty} U(y,t) v_\star'(t) dt = 0 \text{ for any } y \in \Sigma_\varepsilon \right\}$$

of functions that orthogonal, for any $y \in \Sigma_\varepsilon$, to $v_\star'$.

Moreover, as above, we will be interested in the spaces of functions that respect the symmetries of the Torus, thus we define

$$C^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}) := \{U \in C^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) : U_T = U, \quad U_R = U \quad \text{for any } R \in SO_{x_3}(3)\},$$

where we have set $U_T(y,z) := U(Ty,z)$ and $U_R(y,z) := U(Ry,z)$. Furthermore, we set $\mathcal{E}^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}) := \mathcal{E}^{k,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) \cap C^{k,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. These spaces consist of functions that are both symmetric and orthogonal to $v_\star'$.

3.4 Sketch of proof

By a rescaling argument, it is enough to construct solutions to

$$-\Delta(-\Delta u + W'(u)) + W''(u)(-\Delta u + W'(u)) = \varepsilon^4 \lambda (1-u),$$

whose nodal set is close to Σ_ε, since we can obtain the required solutions to (3.4) by setting $\tilde{u}(x) := u(x/\varepsilon)$. Thus we set

$$F(u) = -\Delta(-\Delta u + W'(u)) + W''(u)(-\Delta u + W'(u)). \tag{3.51}$$

A computation shows that

$$F'(u)[v] = -\Delta(-\Delta v + W''(u)v) + W''(u)(-\Delta v + W''(u)v) + W'''(u)(-\Delta u + W'(u))v \tag{3.52}$$

and

$$F''(u)[v, w] = -\Delta(W'''(u)vw) + (W'''(u)W''(u) + W^{(4)}(u)(-\Delta u + W'(u)))vw + W'''(u)[w(-\Delta v + W''(u)v) + v(-\Delta w + W''(u)w)]. \tag{3.53}$$

In order to produce the required solutions we fix $\varepsilon > 0$ small and a small function $\phi \in C^{4,\alpha}(\Sigma)_{s,0}$, in the sense that $|\phi|_{C^{4,\alpha}(\Sigma)} < 1/4$, and we define the approximate solution $v_{\varepsilon,\phi}$ in such a way that its nodal set is close to

$$\Sigma_{\varepsilon,\phi} := \{y + \phi(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\},$$

for ε small enough and $v_{\varepsilon,\phi} \equiv \pm 1$ outside a sufficiently small neighbourhood of $\Sigma_{\varepsilon,\phi}$. More precisely, we set $\phi_\star := R_{1/\varepsilon}\phi$,

$$\Sigma_{\varepsilon,\phi_\star} := \{y + \phi_\star(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\},$$

$$\mathbb{H}(x) := \begin{cases} 1 & \text{if } f_\varepsilon(x) > 0 \\ 0 & \text{if } f_\varepsilon(x) = 0 \\ -1 & \text{if } f_\varepsilon(x) < 0 \end{cases}$$

and, for any $\varepsilon > 0$ and for any integer $m > 0$,

$$\chi_m(x) := \begin{cases} \zeta(|t| - \frac{\tau}{2\varepsilon} - m) & \text{if } x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) \text{ and } d(x, \Sigma_{\varepsilon,\phi_\star}) < \tau/\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We look for an approximate solution of the form

$$v_{\varepsilon,\phi}(x) = \chi_5(x)\tilde{v}_{\varepsilon,\phi}(y,t) + (1 - \chi_5(x))\mathbb{H}(x), \tag{3.54}$$

where $t = z - \phi_\star(\varepsilon y)$ and $v_{\varepsilon,\phi}$ is understood to coincide with $\mathbb{H}$ outside the support of χ_5. From now on, we will consider the expansion of the Laplacian (3.29) with ϕ replaced by $\phi_\star$. This is more convenient for the forthcoming computations. Moreover $v_{\varepsilon,\phi}$ will vanish close to $\Sigma_{\varepsilon,\phi}$ and it will respect the symmetries of the Torus. We stress that these cutoff functions actually depend on ϕ, but we prefer not to put the subscript ϕ to simplify the notation. However, we will see that the error $F(v_{\varepsilon,\phi})$ is small, but not zero, therefore we have to add a correction $w = w_{\varepsilon,\phi,\lambda}$ depending on ε, ϕ and λ in order to obtain a real solution, that is $F(v_{\varepsilon,\phi} + w) = 0$.

Rephrasing our problem in this way, the unknowns are ϕ and w, for any $\varepsilon > 0$ small but fixed. Expanding F in Taylor series, our equation becomes

$$F(v_{\varepsilon,\phi}) + F^{'}(v_{\varepsilon,\phi})[w] + Q_{\varepsilon,\phi}(w) = 0, \tag{3.55}$$

where

$$Q_{\varepsilon,\phi}(w) = \int_0^1 dt \int_0^t F^{''}(v_{\varepsilon,\phi} + sw)[w,w]ds, \tag{3.56}$$

However, we are not able to solve it directly, because of the lack of coercivity of $F^{'}(v_{\varepsilon,\phi})$.

3.4.1 A gluing procedure

We look for a solution of the form

$$w(x) = \chi_2(x)U(y,t) + V(x), \tag{3.57}$$

where V is defined in the whole $\mathbb{R}^3$, U is defined in the entire $\Sigma_\varepsilon \times \mathbb{R}$. Since we want our solutions u_ε to respect the symmetries of the Torus, we look for solutions U and V such that

$$\begin{aligned} U(y,t) = U(Ty,t), \quad U(y,t) = U(Ry,t), \quad &\text{for any } R \in SO_{x_3}(3) \text{ and } (y,t) \in \Sigma_\varepsilon \times \mathbb{R} \\ V(x) = V(Tx), \quad V(x) = V(Rx), \quad &\text{for any } R \in SO_{x_3}(3) \text{ and } x \in \mathbb{R}^3. \end{aligned}$$

Now we observe that the potential

$$\Gamma_{\varepsilon,\phi}(x) := (1 - \chi_1(x))W^{''}(v_{\varepsilon,\phi}) + \chi_1(x)W^{''}(1) \tag{3.58}$$

is positive and bounded away from 0 in the whole $\mathbb{R}^3$, that is, for any $0 < \delta < \sqrt{W''(1)}$, $0 < \delta^2 < \Gamma_{\varepsilon,\phi}(x) < W''(1)$ provided ε is small enough, the estimate is uniform in ϕ. Moreover, using that $\chi_2\chi_1 = \chi_1$, (3.51) becomes

$$\begin{aligned} 0 = \chi_2 \Big\{ & F(\tilde{v}_{\varepsilon,\phi}) + F^{'}(\tilde{v}_{\varepsilon,\phi})[U] + \chi_1 Q_{\varepsilon,\phi}(U+V) + \chi_1 \mathrm{M}_{\varepsilon,\phi}(V) \\ & - \varepsilon^4 \chi_1 \lambda(1 - \tilde{v}_{\varepsilon,\phi} - V) + \varepsilon^4 \lambda U \Big\} \\ & + (-\Delta + \Gamma_{\varepsilon,\phi})^2 V + (1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V) \\ & + \mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) - \varepsilon^4\lambda(1-\chi_1)(1 - v_{\varepsilon,\phi} - V), \end{aligned} \tag{3.59}$$

where

$$\begin{aligned} \mathrm{M}_{\varepsilon,\phi}(V) &:= (W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))(-\Delta V + \Gamma_{\varepsilon,\phi}V) \\ &+(-\Delta + W''(\tilde{v}_{\varepsilon,\phi}))\big[(W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V\big] \end{aligned} \tag{3.60}$$

$$\begin{aligned} \mathrm{N}_{\varepsilon,\phi}(U) &:= -2 < \nabla\chi_2, \nabla(-\Delta U + W''(\tilde{v}_{\varepsilon,\phi})U) > -\Delta\chi_2(-\Delta U + W''(\tilde{v}_{\varepsilon,\phi})U) \\ &+(-\Delta + W''(\tilde{v}_{\varepsilon,\phi}))(-2 < \nabla\chi_2, \nabla U > -\Delta\chi_2 U) \end{aligned} \tag{3.61}$$

$$\begin{aligned} \mathrm{P}_{\varepsilon,\phi}(V) &:= -2 < \nabla\chi_1, \nabla((W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V) > -\Delta\chi_1(W''(\tilde{v}_{\varepsilon,\phi}) - W''(1))V \\ &+W'''(v_{\varepsilon,\phi})(-\Delta v_{\varepsilon,\phi} + W'(v_{\varepsilon,\phi}))V. \end{aligned} \tag{3.62}$$

Hence we have reduced our problem to finding a solution (V, U) to the system

$$\begin{aligned} &(-\Delta + \Gamma_{\varepsilon,\phi})^2 V + (1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V) \\ &\quad +\mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) = \varepsilon^4\lambda(1-\chi_1)(1 - v_{\varepsilon,\phi} - V) \qquad \text{in } \mathbb{R}^3 \end{aligned} \tag{3.63}$$

$$\begin{aligned} &F(\tilde{v}_{\varepsilon,\phi}) + F'(\tilde{v}_{\varepsilon,\phi})[U] + \chi_1 Q_{\varepsilon,\phi}(U+V) \\ &\quad +\chi_1\mathrm{M}_{\varepsilon,\phi}(V) = \varepsilon^4\lambda\chi_1(1 - \tilde{v}_{\varepsilon,\phi} - V) - \varepsilon^4\lambda U \quad \text{for } |t| \le \tau/2\varepsilon + 3 \end{aligned} \tag{3.64}$$

The system of equations (3.63) and (3.64) is known as *auxiliary equation*. First we solve equation (3.63) for any fixed U, thanks to coercivity, due to the fact that $\Gamma_{\varepsilon,\phi}$ is bounded away from 0 uniformly in ε and ϕ. We will see that our solution also depends on the data U and ϕ in a Lipschitz way.

Proposition 86. *For any $\varepsilon > 0$ small enough, for any $U \in C^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ satisfying $||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le 1$, for any $\phi \in B_4(1/4)$ and for any $|\lambda| < 1$, equation (3.63) admits a unique solution $V_{\varepsilon,\phi,\lambda,U} \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfying*

$$\begin{cases} ||V_{\varepsilon,\phi,\lambda,U}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon} \\ ||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon}||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \\ ||V_{\varepsilon,\phi_1,\lambda_1,U} - V_{\varepsilon,\phi_2,\lambda_2,U}||_{C^{4,\alpha}(\mathbb{R}^3)} \le C_1 e^{-a/\varepsilon}(|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|) \end{cases} \tag{3.65}$$

$a := \delta\tau/2$, for any U_1, U_2 satisfying $||U_1||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}, ||U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le 1$, for any $\phi_1, \phi_2 \in B_4(1/4)$, for any $|\lambda_1|, |\lambda_2| < 1$, for some constants $a, C_1 > 0$ independent of U, ε and ϕ.

The proof of Proposition 86 is based on a fixed point argument (see section 3.6).

3.4.2 An infinite-dimensional Lyapunov-Schmidt reduction

Now we consider equation (3.64), extended to the whole $\Sigma_\varepsilon \times \mathbb{R}$. We solve it by means of the infinite-dimensional Lyapunov-Schmidt reduction. In the notations of Section 2.1.1, we set

$$H_\varepsilon := L^2(\Sigma_\varepsilon \times \mathbb{R}), \quad Y_{1,\varepsilon} := C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}), \quad Y_{2,\varepsilon} := C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}), \tag{3.66}$$

$0 < \delta < \sqrt{2}$. These spaces are endowed with the norms

$$||U||_1 := ||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \tag{3.67}$$

and

$$||f||_2 := ||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}. \tag{3.68}$$

Moreover, we nothe that

$$F^{'}(v_{\varepsilon,\phi})[U] = \mathcal{L}_\varepsilon(\phi)[U] + \varepsilon\overline{L}_{\varepsilon,\phi}[U],$$

where $\mathcal{L}_\varepsilon(\phi) = \mathcal{M}_\varepsilon(\phi)^2$ and

$$\mathcal{M}_\varepsilon(\phi)[U] := -(\Delta_{\Sigma_\varepsilon} + \partial_{tt})U(y,t) + W^{''}(v_\star(t))U(y,t),$$

defined for any $U \in C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})$. The dependence on ϕ is hidden in $v_\star(t)$, where $t = z - \phi(\varepsilon y)$ actually depends on ϕ. However, thanks to the smoothness of $v_\star$ and W, (1.37) is satisfied. In addition, we have set

$$\begin{aligned}\overline{L}_{\varepsilon,\phi}[U] := \varepsilon^{-1}\mathcal{M}_\varepsilon(\phi)(\mathrm{D} + W^{''}(\tilde{v}_{\varepsilon,\phi}) - W^{''}(v_\star))[U] + \varepsilon^{-1}(\mathrm{D} + W^{''}(\tilde{v}_{\varepsilon,\phi}) - W^{''}(v_\star))\mathcal{M}_\varepsilon(\phi)[U]\\ +\varepsilon^{-1}(\mathrm{D} + W^{''}(\tilde{v}_{\varepsilon,\phi}) - W^{''}(v_\star))^2[U] + \varepsilon^{-1}W^{'''}(\tilde{v}_{\varepsilon,\phi})(-\Delta\tilde{v}_{\varepsilon,\phi} + W^{'}(\tilde{v}_{\varepsilon,\phi}))[U],\end{aligned}$$

D is defined in (3.29). Therefore we reduced ourselves to consider the equation

$$\mathcal{L}_\varepsilon(\phi)[U] = \mathcal{F}_\varepsilon(\phi, U), \tag{3.69}$$

in the entire $\Sigma_\varepsilon \times \mathbb{R}$, where

$$\begin{aligned}\mathcal{F}_\varepsilon(\phi,U) := -F(\tilde{v}_{\varepsilon,\phi}) - \chi_1 Q_{\varepsilon,\phi}(U+V) - \varepsilon\overline{L}_{\varepsilon,\phi}[U]\\ -\chi_1\mathrm{M}_{\varepsilon,\phi}(V) + \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda(\chi_1 U + V).\end{aligned} \tag{3.70}$$

We note that $\mathcal{L}_\varepsilon(\phi)$ is self-adjoint on $Y_{1,\varepsilon}$ with respect to the $L^2(\Sigma_\varepsilon\times\mathbb{R})$-scalar product and $(a2)$ is satisfied, because

$$\begin{aligned}\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{M}_\varepsilon(\phi)[\tilde{U}](y,t)v_\star^{'}(t)dt =\\ -\Delta_{\Sigma_\varepsilon}\int_{-\infty}^{\infty}\tilde{U}(y,t)v_\star^{'}(t)dt + \int_{-\infty}^{\infty}\tilde{U}(y,t)(-v_\star^{'''} + W^{''}(v_\star)v_\star^{'})(t)dt =\\ -\Delta_{\Sigma_\varepsilon}\int_{-\infty}^{\infty}\tilde{U}(y,t)v_\star^{'}(t)dt, \quad \forall\tilde{U}\in C^{2,\alpha}(\Sigma_\varepsilon\times\mathbb{R}),\end{aligned}$$

since $-v_\star''' + W''(v_\star)v_\star' = 0$. In particular, taking $\tilde{U} := \mathcal{M}_\varepsilon(\phi)[U]$, with $U \in W_{\varepsilon,\phi}$, we get

$$\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{L}_\varepsilon(\phi)[\tilde{U}](y,t)v_\star'(t)dt = -\Delta_{\Sigma_\varepsilon}\int_{-\infty}^{\infty}\mathcal{M}_\varepsilon(\phi)[U](y,t)v_\star'(t)dt$$
$$= -\Delta_{\Sigma_\varepsilon}\Big(-\Delta_{\Sigma_\varepsilon}\int_{-\infty}^{\infty}U(y,t)v_\star'(t)dt + \int_{-\infty}^{\infty}U(y,t)(-v_\star''' + W''(v_\star)v_\star')(t)dt\Big) = 0,$$

as required. Moreover we set

$$T_{\varepsilon,\phi} := \Big\{U \in H_\varepsilon : \int_{-\infty}^{\infty}U(y,t)v_\star'(t)dt = 0, \quad \text{a. e. } y \in \Sigma_\varepsilon\Big\}, \tag{3.71}$$

and $W_{\varepsilon,\phi} = T_{\varepsilon,\phi} \cap Y_{1,\varepsilon}$. We define the projection $\Pi_{\varepsilon,\phi} : H_\varepsilon \to T_{\varepsilon,\phi}$ as follows

$$\Pi_{\varepsilon,\phi}U(y,t) := U(y,t) - \frac{\int_\mathbb{R} U(y,s)v_\star'(s)ds}{\int_\mathbb{R}(v_\star'(s))^2ds}v_\star'(t). \tag{3.72}$$

It follows from the definition that $\Pi_{\varepsilon,\phi}$ satisfies $(a3)$. Once again, we stress that the dependence on ϕ is hidden in $t = z - \phi(\varepsilon y)$. We also notice that

$$N_{\varepsilon,\phi} = \{U \in Y_{1,\varepsilon} : \mathcal{L}_\varepsilon(\phi)[U] = 0\} = span\{v_\star'\}.$$

In fact, if $\mathcal{L}_\varepsilon(\phi)[U] = 0$, then, by Lemma 6, 1 of [24], $\mathcal{M}_\varepsilon(\phi)[U] = cv_\star'(t)$, thus, integrating over $\Sigma_\varepsilon \times \mathbb{R}$, we have

$$0 = \int_{\Sigma_\varepsilon\times\mathbb{R}}\mathcal{M}_\varepsilon(\phi)[U](y,t)v_\star'(t)dt = 4\pi^2\sqrt{2}c\varepsilon^{-2}\int_\mathbb{R}(v_\star'(t))^2dt,$$

therefore $c = 0$, that is $\mathcal{M}_\varepsilon(\phi)[U] = 0$, thus $U = \tilde{c}v_\star'$. It is worth to stress that in this case $T_{\varepsilon,\phi}$ is contained in

$$N_{\varepsilon,\phi}^\perp = \Big\{U \in H_\varepsilon : \int_{\Sigma_\varepsilon\times\mathbb{R}}U(y,t)v_\star'(t)d\sigma dt = 0\Big\}, \tag{3.73}$$

but the opposite inclusion may be false.

We split equation (3.69) into the system

$$\mathcal{L}_\varepsilon(\phi)[U] = \Pi_{\varepsilon,\phi}\mathcal{F}_\varepsilon(\phi,U), \quad w \in W_{\varepsilon,\phi}, \tag{3.74}$$
$$(Id - \Pi_{\varepsilon,\phi})\mathcal{F}_\varepsilon(\phi,U) = 0. \tag{3.75}$$

Before stating the next proposition, let us observe that any function $U : \Sigma_\varepsilon\times\mathbb{R} \to \mathbb{R}$ can be written as the sum of an even part and an odd part, the even part being $U_e(y,t) := \frac{1}{2}(U(y,t) + U(y,-t))$ and the odd part being $U_o(y,t) := \frac{1}{2}(U(y,t) + U(y,-t))$.

Proposition 87. *For any $\varepsilon > 0$ small enough and for any $\phi \in B_4(1)$, we can find a solution $U_{\varepsilon,\phi} \in \mathcal{E}^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ to equation (3.74) satisfying*

$$\begin{cases} ||U_{\varepsilon,\phi}||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^3 \\ ||(U_{\varepsilon,\phi})_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^4 \\ ||U_{\varepsilon,\phi_1} - U_{\varepsilon,\phi_2}||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq C_2\varepsilon^3|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}, \end{cases} \tag{3.76}$$

for any $\phi_1, \phi_2 \in B_4(1)$, for some constant $C_2 > 0$ independent of ε.

The proof of Proposition 87 will be given in section 3.6.

3.4.3 The bifurcation equation

In conclusion, we will show that it is possible to find ϕ that solves

$$\int_{-\infty}^{\infty} \big(F(\tilde{v}_{\varepsilon,\phi}) - \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}) + \mathrm{T}(U, V_{\varepsilon,\phi,U},\phi)\big)(y,t)v_\star'(t)dt = 0 \tag{3.77}$$

for any $y \in \Sigma_\varepsilon$, where

$$\mathrm{T}(U,V,\phi) := \chi_1 Q_{\varepsilon,\phi}(U+V) + \mathrm{R}_{\varepsilon,\phi}(U) + \chi_1\mathrm{M}_{\varepsilon,\phi}(V) + \varepsilon^4\lambda(\chi_1 V + U) \tag{3.78}$$

Moreover, we want the real solution $u_\varepsilon(x) := v_{\varepsilon,\phi}(x/\varepsilon) + w_{\varepsilon,\phi,\lambda}(x/\varepsilon)$ to satisfy the volume constraint (3.8). We will show in Proposition (92) and in the proof of Proposition (87) in Section 6.4 that equation (3.77) is equivalent to

$$\tilde{L}_0\phi(y) + \frac{2}{c_\star}\lambda = \varepsilon q^1_\varepsilon(\phi,\lambda)(y) + \varepsilon^2 q^2_\varepsilon(\phi,\lambda)(y), \tag{3.79}$$

where q^1_ε satisfies

$$\begin{cases} |q^1_\varepsilon(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \leq c \\ |q^1_\varepsilon(\phi_1,\lambda_1) - q^1_\varepsilon(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \leq c(|\phi_1-\phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases} \tag{3.80}$$

for any $(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ fulfilling $|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < 1$, and q^2_ε satisfies

$$\begin{cases} |q^2_\varepsilon(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \leq C \\ |q^2_\varepsilon(\phi_1,\lambda_1) - q^2_\varepsilon(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \leq C(|\phi_1-\phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1-\lambda_2|), \end{cases} \tag{3.81}$$

for any $(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0} \times \mathbb{R}$ fulfilling $|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < \tilde{c}\varepsilon$, for some $\tilde{c} > 0$. We stress that the constant C appearing in (3.81) may depend on $\tilde{c}$. As regards the volume constraint, we note that, by the change of variables $x' = x/\varepsilon$,

$$8\pi^2\sqrt{2}c_\varepsilon = \int_{\mathbb{R}^3}(1-u_\varepsilon(x))^2dx = \varepsilon^3\int_{\mathbb{R}^3}1-(v_{\varepsilon,\phi}(x)+w_{\varepsilon,\phi,\lambda}(x))^2dx,$$
$$c_\varepsilon := 1+\varepsilon\int_0^\infty(v_\star^2-1)dt.$$

The latter integral can be calculated exploiting the natural change of variables

$$\begin{cases} x_1 = \varepsilon^{-1}\cos(\varepsilon \mathrm{y}_2)\big((z+\varepsilon^{-1})\cos(\varepsilon \mathrm{y}_1)+\varepsilon^{-1}\sqrt{2}\big),\\ x_2 = \varepsilon^{-1}\sin(\varepsilon \mathrm{y}_2)\big((z+\varepsilon^{-1})\cos(\varepsilon \mathrm{y}_1)+\varepsilon^{-1}\sqrt{2}\big),\\ x_3 = \varepsilon^{-1}(z+\varepsilon^{-1})\sin(\varepsilon \mathrm{y}_1).\end{cases} \tag{3.82}$$

on $V_{\tau/\varepsilon}$, induced by the parametrization $Y_\varepsilon(\mathrm{y}) = \varepsilon^{-1}Y(\varepsilon\mathrm{y})$, where

$$Y(\vartheta_1,\vartheta_2) := (\cos\vartheta_2(\cos\vartheta_1+\sqrt{2}),\sin\vartheta_2(\cos\vartheta_1+\sqrt{2}),\sin\vartheta_2) \tag{3.83}$$

and $(\vartheta_1,\vartheta_2) = \varepsilon(\mathrm{y}_1,\mathrm{y}_2)\in[0,2\pi)^2$.

Proposition 88. *For any $\varepsilon>0$ small enough, for any $\phi\in C^{4,\alpha}(\Sigma)_{s,0}$, we have*

$$\int_{\mathbb{R}^3}1-(v_{\varepsilon,\phi}(x)+w_{\varepsilon,\phi,\lambda}(x))^2dx = \varepsilon^{-3}8\pi^2\sqrt{2}c_\varepsilon+4\varepsilon^{-2}\int_\Sigma\phi(\zeta)d\sigma(\zeta)$$
$$+16\sqrt{2}\pi^2\varepsilon^{-1}\int_0^\infty t(1-v_\star(t))dt+4G_\varepsilon(\phi,\lambda),$$

with G_ε fulfilling

$$\begin{cases}|G_\varepsilon(\phi,\lambda)|\le c,\\ |G_\varepsilon(\phi_1,\lambda_1)-G_\varepsilon(\phi_2,\lambda_2)|\le c(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}+|\lambda_1-\lambda_2|)\end{cases}$$

if $\phi,\phi_1,\phi_2\in C^{4,\alpha}(\Sigma)_s$ satisfy $|\phi|_{C^{4,\alpha}(\Sigma)},|\phi_1|_{C^{4,\alpha}(\Sigma)},|\phi_2|_{C^{4,\alpha}(\Sigma)}\le\tilde{c}\varepsilon$, for some $\tilde{c}>0$, for any $\lambda,\lambda_1,\lambda_2$ such that $|\lambda|,|\lambda_1|,|\lambda_2|<1$.

The proof of this Proposition will be given in Section 7. Therefore, in terms of ϕ and λ, equation (3.8) is equivalent to equation

$$\int_\Sigma\phi(\zeta)d\sigma(\zeta) = -4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1-v_\star(t))dt-\varepsilon^2G_\varepsilon(\phi,\lambda). \tag{3.84}$$

In order to solve the bifurcation equation (3.79), we need to couple it with (3.84), due to the properties of $\tilde{L}_0$ (see Section 3.4.1), and we solve the system by a fixed point argument, that will be explained in this Proposition, whose proof will be carried out in Section 7.

Proposition 89. *For any $\varepsilon > 0$ small enough, the bifurcation equation*

$$\tilde{L}_0\phi + \frac{2}{c_\star}\lambda = \varepsilon q^1_\varepsilon(\phi,\lambda) + \varepsilon^2 q^2_\varepsilon(\phi,\lambda) \tag{3.85}$$

$$\int_\Sigma \phi(\zeta)d\sigma(\zeta) = -4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1-v_\star(t))dt - \varepsilon^2 G_\varepsilon(\phi,\lambda), \tag{3.86}$$

admits a solution $(\phi,\lambda) \in C^{4,\alpha}(\Sigma)_{s,0}\times\mathbb{R}$ satisfying

$$|\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| \le C_3\varepsilon, \tag{3.87}$$

for some constant $C_3 = C_3(W,\tau) > 0$.

Remark 90. *(i) In the statement of Proposition 89, we use the term bifurcation equation for a system. This is a slight abuse of language, however it is convenient since it is consistent with the bifurcation theory.*

(ii) As we will see in the proof of Proposition 92 below, the Willmore equation will appear at order ε^3, while the linearized operator

$$\begin{aligned}\tilde{L}_0\phi = L_0^2\phi + \frac{3}{2}H^2L_0\phi - H(\nabla_\Sigma\phi,\nabla_\Sigma H) + 2(A\nabla_\Sigma\phi,\nabla_\Sigma H) + \\ 2H < A,\nabla^2\phi > +\phi(2 < A,\nabla^2 H > +|\nabla_\Sigma H|^2 + 2HtrA^3).\end{aligned} \tag{3.88}$$

will appear at order ε^4, thus it is crucial for the remainder to be smaller in order to apply a contraction mapping principle. This is actually the case thanks to the fact that the odd part of $U_{\varepsilon,\phi,\lambda}$ is of order ε^4.

3.5 The approximate solution

3.5.1 Construction

First one can try to take $v_\star(z-\phi(\varepsilon y))$ as an approximate solution, where $\phi \in B_4(1/4) = \{\phi \in C^{4,\alpha}(\Sigma)_{s,0} : |\phi|_{C^{4,\alpha}} < 1/4\}$ is some small function that respects the symmetries of Σ and (y,z) are the Fermi coordinates of Σ. We will see that these symmetries will be inherited by our approximate solution (see Remark 91 below). In this way, our approximate solution vanishes exactly on

$$\Sigma_{\varepsilon,\phi} := \{y + \phi(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\} \tag{3.89}$$

and goes from -1 to 1 in a monotone way. The speed of the phase transition is independent of ε, but it becomes faster and faster when ε becomes small if compared to the scale of Σ_ε, that increases as $1/\varepsilon$.

With the change of variables $t = z - \phi_\star(\varepsilon y)$, we have

$$v_\star(z - \phi(\varepsilon y)) = v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)).$$

We recall that $\phi_\star := R_{1/\varepsilon}\phi$, where the smoothing operators $R_{1/\varepsilon}$ are introduced in Section 3.1 and satisfy (3.44),(3.45),(3.46). We write

$$v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)) = v_\star(t) + v_{1,\varepsilon,\phi}(y,t),$$

where

$$v_{1,\varepsilon,\phi}(y,t) := v_\star(t + \phi_\star(\varepsilon y) - \phi(\varepsilon y)) - v_\star(t).$$

Since the Fermi coordinates are just defined in a neighbourhood of the Torus, our approximate solution is not defined everywhere. For our purposes, it is enough to consider it in the set

$$B = \{x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) \in \mathbb{R}^3 : |t| < \tau/2\varepsilon + 7\}, \tag{3.90}$$

that is a tubular neighbourhood of

$$\Sigma_{\varepsilon,\phi_\star} = \{y + \phi_\star(\varepsilon y)\nu(\varepsilon y) : y \in \Sigma_\varepsilon\}$$

of width $\tau/2\varepsilon + 7$. Then it will be extended to the whole $\mathbb{R}^3$ with the aid of a cutoff function.

Now let us first compute $F(v_\star(t))$. In the sequel, by a Taylor expansion, we will also consider the contribution of $v_{1,\varepsilon,\phi}$. In the forthcoming computations, $v_\star$ and its derivatives will always be evaluated at t, the geometric quantities, ϕ and its derivatives will always be evaluated at εy. By (3.26),

$$\begin{aligned}
-\Delta v_\star + W'(v_\star) = -v_\star'' + W'(v_\star) + \varepsilon \hat{H}(\varepsilon y, \varepsilon(t+\phi_\star))v_\star' \quad (3.91)\\
+\varepsilon^2 \Delta_\Sigma \phi_\star v_\star' - \varepsilon^2 |\nabla \phi_\star|^2 v_\star'' + \varepsilon^3 (t+\phi_\star)(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)v_\star' - \varepsilon^3(t+\phi_\star)a_1^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''\\
+\varepsilon^4(t+\phi_\star)^2(a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)v_\star' - \varepsilon^4(t+\phi_\star)^2 a_2^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''\\
\varepsilon^2(\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2 \overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''.
\end{aligned}$$

The term of order 0 in ε vanishes since $v_\star$ satisfies the ODE $-v_\star'' + W'(v_\star) = 0$. Thus, in order to compute $F(v_\star)$, we need to apply the linear operator $-\Delta + W''(v_\star)$ to the remaining terms. We will write down all terms of order less or equal than 4, the other ones being lower order terms, in some sense that will be clear soon. Let us set, for any function $v \in C^2(\mathbb{R})$, $L_\star v := -v'' + W''(v_\star)v$. Differentiating the ODE satisfied by $v_\star$, we get $L_\star v_\star' = 0$, thus using the Taylor expansion of $\tilde{H}$, the first term of (3.91) gives

$$T^1_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon\hat{H}(\varepsilon y, \varepsilon(t+\phi_\star))v'_\star) = \varepsilon^2(H^2 - 2|A|^2)v''_\star \tag{3.92}$$
$$+\varepsilon^3\Big\{(2H|A|^2 - 4\mathrm{tr}A^3)(t+\phi_\star)v''_\star + (H|A|^2 - 2\mathrm{tr}A^3)v'_\star - \Delta_\Sigma H v'_\star$$
$$+2(\nabla_\Sigma H, \nabla_\Sigma\phi_\star)v''_\star - H|\nabla_\Sigma\phi_\star|^2 v'''_\star + H\Delta_\Sigma\phi_\star v''_\star\Big\}$$
$$+\varepsilon^4\Big\{(|A|^4 - 6H_4 + 2H\mathrm{tr}A^3)((t+\phi_\star)^2 v''_\star + (t+\phi_\star)v'_\star) - \Delta_\Sigma|A|^2(t+\phi_\star)v'_\star$$
$$+2(\nabla_\Sigma|A|^2, \nabla_\Sigma\phi_\star)(t+\phi_\star)v''_\star - |A|^2|\nabla_\Sigma\phi_\star|^2(t+\phi_\star)v'''_\star + \Delta_\Sigma\phi_\star|A|^2(t+\phi_\star)v''_\star$$
$$-(a_1^{ij}H_{ij} + b_1^i H_i)(t+\phi_\star)v'_\star + 2a_1^{ij}H_i(\phi_\star)_j(t+\phi_\star)v''_\star$$
$$+H(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(t+\phi_\star)v''_\star - Ha_1^{ij}(\phi_\star)_i(\phi_\star)_j v'''_\star\Big\} + \varepsilon^5 F^1_{\varepsilon,\phi}(y,t),$$

with $F^1_{\varepsilon,\phi}$ is small and Lipschitzian in ϕ, in the sense that, in view of (3.44),

$$\begin{cases} |S_{\varepsilon,\phi}F^1_{\varepsilon,\phi}|_{C^{0,\alpha}(\Sigma)} \leq c \\ |S_{\varepsilon,\phi_1}F^1_{\varepsilon,\phi_1} - S_{\varepsilon,\phi_2}F^1_{\varepsilon,\psi}|_{C^{0,\alpha}(\Sigma)} \leq c|\phi_1 - \phi_2|_{C^{4,\alpha}(\Sigma)}, \end{cases} \tag{3.93}$$

for any $\phi, \phi_1, \phi_2 \in B_4(1/4)$, for some constant $c = c(W,\tau) > 0$ independent of ε and ϕ.

Similarly, the second term of (3.91) gives

$$T^2_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon^2\Delta_\Sigma\phi_\star v'_\star) = \varepsilon^3 H\Delta_\Sigma\phi_\star v''_\star \tag{3.94}$$
$$+\varepsilon^4\Big\{-(\Delta_\Sigma)^2\phi_\star v'_\star + |A|^2\Delta_\Sigma\phi_\star(t+\phi_\star)v''_\star + 2(\nabla_\Sigma\Delta_\Sigma\phi_\star, \nabla_\Sigma\phi_\star)v''_\star$$
$$+(\Delta_\Sigma\phi_\star)^2 v''_\star - |\nabla_\Sigma\phi_\star|^2\Delta_\Sigma\phi_\star v'''_\star\Big\} + \varepsilon^5 F^2_{\varepsilon,\phi}(y,t),$$

with $F^2_{\varepsilon,\phi}$ fulfilling (3.93).

The third term of (3.91) is already quadratic in ϕ, but, for the sake of completeness, we prefer to write it down.

$$T^3_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(-\varepsilon^2|\nabla_\Sigma\phi_\star|^2 v''_\star) = \varepsilon^2|\nabla_\Sigma\phi_\star|^2(v^{(4)}_\star - W''(v_\star)v''_\star) \tag{3.95}$$
$$-\varepsilon^3 H|\nabla_\Sigma\phi_\star|^2 v'''_\star + \varepsilon^4\Big\{-|A|^2|\nabla_\Sigma\phi_\star|^2(t+\phi_\star)v'''_\star + \Delta_\Sigma|\nabla_\Sigma\phi_\star|^2 v''_\star$$
$$-2(\nabla_\Sigma|\nabla_\Sigma\phi_\star|^2, \nabla_\Sigma\phi_\star)v'''_\star + |\nabla_\Sigma\phi_\star|^4 v^{(4)}_\star - |\nabla_\Sigma\phi_\star|^2\Delta_\Sigma\phi_\star v'''_\star\Big\} + \varepsilon^5 F^3_{\varepsilon,\phi}(y,t)$$

The fouth term of (3.91) gives

$$T^4_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)\big(\varepsilon^3(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(t+\phi_\star)v_\star'\big) = \tag{3.96}$$
$$-2\varepsilon^3(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)v_\star'' + \varepsilon^4 H(a_1^{ij}(\phi_\star)_{ij} + b_1^i(\phi_\star)_i)(v_\star' + (t+\phi_\star)v_\star'') + \varepsilon^5 F^4_{\varepsilon,\phi}(y,t).$$

The fifth term of (3.91) gives

$$\begin{aligned} T^5_{\varepsilon,\phi}(y,t) &= \big(-\Delta + W''(v_\star)\big)(-\varepsilon^3 a_1^{ij}(\phi_\star)_i(\phi_\star)_j(t+\phi_\star)v_\star'') = \\ &\varepsilon^3 a_1^{ij}(\phi_\star)_i(\phi_\star)_j((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star''') \\ &\quad -\varepsilon^4 H a_1^{ij}(\phi_\star)_i(\phi_\star)_j(v_\star'' + (t+\phi_\star)v_\star''') + \varepsilon^5 F^5_{\varepsilon,\phi}(y,t), \end{aligned}$$

with $F^3_{\varepsilon,\phi}, F^4_{\varepsilon,\phi}, F^5_{\varepsilon,\phi}$ fulfilling (3.93).

Now we consider the terms involving a_2^{ij} and b_2^i. We will see that all the contributions of order ε^4 coming from these terms will simplify, therefore we do not need to know the explicit expression of a_2^{ij} and b_2^i.

$$\begin{aligned} T^6_{\varepsilon,\phi}(y,t) = \Big\{ &\big(-\Delta + W''(v_\star)\big)\big(\varepsilon^4(a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)(t+\phi_\star)^2 v_\star' - \varepsilon^4 a_2^{ij}(\phi_\star)_i(\phi_\star)_j(t+\phi_\star)^2 v_\star''\big) \\ &+\varepsilon^2(\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2\overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''\Big\} \\ &= -\varepsilon^4(a_2^{ij}(\phi_\star)_{ij} + b_2^i(\phi_\star)_i)(2v_\star' + 4(t+\phi_\star)v_\star'') \\ &-\varepsilon^4 a_2^{ij}(\phi_\star)_i(\phi_\star)_j(2v_\star'' + 4(t+\phi_\star)v_\star''' + (t+\phi_\star)^2 v_\star^{(4)} + W''(v_\star)v_\star'') \\ &+\big(-\Delta + W''(v_\star)\big)\Big\{\varepsilon^2(\overline{a}^{ij}(\phi_\star)_{ij} + \overline{b}^i(\phi_\star)_i)v_\star' - \varepsilon^2\overline{a}^{ij}(\phi_\star)_i(\phi_\star)_j v_\star''\Big\} + \varepsilon^5 F^6_{\varepsilon,\phi}(y,t), \end{aligned} \tag{3.97}$$

with $F^6_{\varepsilon,\phi}$ fulfilling (3.93).

It turns out that, in the expansion of $F(v_\star(t))$, the term of order ε^2 is $(H^2 - 2|A|^2)v_\star'' + |\nabla\phi_\star|^2(v_\star^{(4)} - W''(v_\star)v_\star'')$. Since it is too large for our purposes, we add a correction to the approximate solution in order to cancel it. Moreover, there is also a quadratic term appearing at order ε^3, that is $-2H|\nabla\phi_\star|^2 v_\star'''$. Althogh it is smaller if, for instance, $|\phi|_{C^{4,\alpha}(\Sigma)} \leq c\varepsilon$, for some constant $c > 0$, we would like to get rid of it in order to improve the approximation.

In order to do so, we set

$$\begin{aligned} \eta(t) &:= -v_\star'(t)\int_0^t (v_\star'(s))^{-2}ds \int_{-\infty}^s \frac{\tau(v_\star'(\tau))^2}{2}d\tau, \\ \tilde\eta(t) &:= -v_\star'(t)\int_0^t (v_\star'(s))^{-2}ds \int_{-\infty}^s v_\star'(\tau)v_\star''(\tau)d\tau. \end{aligned}$$

This functions are exponentially decaying, odd and solve

$$L_\star\eta(t) = -\eta''(t) + W''(v_\star(t))\eta(t) = \frac{1}{2}tv_\star'(t), \quad L_\star\tilde{\eta}(t) = v_\star''(t),$$
$$\int_{-\infty}^{\infty}\eta(t)v_\star'(t)dt = \int_{-\infty}^{\infty}\tilde{\eta}(t)v_\star'(t)dt = 0.$$

Differentiating this relation once more, it is possible to see that $L_\star^2\eta(t) = -v_\star''(t)$ and $L_\star^2\tilde{\eta} = -v_\star^{(4)} + W''(v_\star)v_\star''$. Our new approximate solution will be

$$\tilde{v}_{\varepsilon,\phi}(y,t) := v_\star(t) + v_{1,\varepsilon,\phi}(y,t) + v_{2,\varepsilon,\phi}(y,t),$$

where

$$v_{2,\varepsilon,\phi}(y,t) = \varepsilon^2(\psi(\varepsilon y) + \varepsilon L\phi_\star(\varepsilon y))\eta(t) + \varepsilon^2|\nabla\phi_\star(\varepsilon y)|^2\tilde{\eta}(t), \tag{3.98}$$

with $\psi : \Sigma \to \mathbb{R}$ and L linear in ϕ to be determined later. In the sequel, $\eta, \tilde{\eta}$ and its derivatives are evaluated at t, the geometric quantities, ϕ and its derivatives will be evaluated at εy.

Taking the Taylor expansion of F,

$$\begin{aligned} F(\tilde{v}_{\varepsilon,\phi})(y,t) = {} & F(v_\star) + F'(v_\star)[v_{1,\varepsilon,\phi}] + F'(v_\star)[v_{2,\varepsilon,\phi}] \\ & + \frac{1}{2}F''(v_\star)[v_{1,\varepsilon,\phi}, v_{1,\varepsilon,\phi}] + F''(v_\star)[v_{1,\varepsilon,\phi}, v_{2,\varepsilon,\phi}] \\ & + \frac{1}{2}F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}] + C_{\varepsilon,\phi}[v_{1,\varepsilon,\phi} + v_{2,\varepsilon,\phi}], \end{aligned}$$

where

$$C_{\varepsilon,\phi}[w] = \int_0^1 dt \int_0^t ds \int_0^s F'''\big(v_\star + \tau w\big)[w,w,w]d\tau.$$

It follows from the expansion of the laplacian and the properties of the smoothing operators (3.44,3.45,3.46) that

$$F'(v_\star)[v_{1,\varepsilon,\phi}](y,t) = \varepsilon^4\Delta_\Sigma^2(\phi_\star - \phi) + \varepsilon^5 F^{12}_{\varepsilon,\phi}(y,t), \tag{3.99}$$

with $F^{12}_{\varepsilon,\phi}$ satisfying (3.93). We point out that this extra term does not give rise to terms of order ε^2 and ε^3.

Now we compute $F'(v_\star)[v_{2,\varepsilon,\phi}]$. As first we note that

$$T^7_{\varepsilon,\phi}(y,z) = W'''(v_\star)(-\Delta v_\star + W'(v_\star))(\varepsilon^2\psi\eta + \varepsilon^3 L\phi_\star\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta})$$
$$= \varepsilon^3 H\psi W'''(v_\star)\eta v_\star' + \varepsilon^3 H|\nabla\phi_\star|^2 W'''(v_\star)\tilde{\eta}v_\star'$$
$$+\varepsilon^4\Big\{(\psi\Delta_\Sigma\phi_\star + HL\phi_\star + (t+\phi_\star)\psi|A|^2)W'''(v_\star)\eta v_\star' - |\nabla\phi_\star|^2\psi W'''(v_\star)\eta v_\star''$$
$$+|\nabla\phi_\star|^2 W'''(v_\star)\tilde{\eta}(|A|^2(t+\phi_\star)v_\star' + \varepsilon^2\Delta_\Sigma\phi_\star v_\star' - |\nabla\phi_\star|^2 v_\star'')\Big\}$$
$$+\varepsilon^5 F^7_{\varepsilon,\phi}(y,t),$$

with $F^7_{\varepsilon,\phi}$ fulfilling (3.93).

After that, we have to compute $\big(-\Delta + W''(v_\star)\big)^2(\varepsilon^2(\psi+\varepsilon L\phi_\star)\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta})$. We obtain

$$\big(-\Delta + W''(v_\star)\big)(\varepsilon^2(\psi+\varepsilon L\phi_\star)\eta + \varepsilon^2|\nabla\phi_\star|^2\tilde{\eta}) =$$
$$\varepsilon^2(\psi L_\star\eta + |\nabla\phi_\star|^2 L_\star\tilde{\eta}) + \varepsilon^3(H\psi\eta' + L\phi_\star L_\star\eta + H|\nabla\phi_\star|^2\tilde{\eta}')$$
$$+\varepsilon^4\Big\{-\Delta_\Sigma\psi\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star + 2(\nabla_\Sigma\psi,\nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)\eta'$$
$$-\psi|\nabla_\Sigma\phi_\star|^2\eta'' - \Delta|\nabla\phi_\star|^2\tilde{\eta} - |\nabla\phi_\star|^4\tilde{\eta}''$$
$$\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star,\nabla|\nabla\phi_\star|^2)\big)\tilde{\eta}'\Big\} + \varepsilon^5\tilde{F}_{\varepsilon,\phi}(y,t),$$

with $\tilde{F}_{\varepsilon,\phi}$ satisfying (3.93).

Applying the operator once more, we obtain

$$T^8_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon^2(\psi+\varepsilon L\phi_\star)L_\star\eta + \varepsilon^2|\nabla\phi_\star|^2 L_\star\tilde{\eta}) \; (3.100)$$
$$\varepsilon^2(\psi L_\star^2\eta + |\nabla\phi_\star|^2 L_\star^2\tilde{\eta}) + \varepsilon^3\Big\{L\phi_\star L_\star^2\eta + H\psi(L_\star\eta)' + H|\nabla\phi_\star|^2(L_\star\tilde{\eta})'\Big\}$$
$$+\varepsilon^4\Big\{-\Delta_\Sigma\psi L_\star\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star + 2(\nabla_\Sigma\psi,\nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)(L_\star\eta)'$$
$$-\psi|\nabla_\Sigma\phi_\star|^2(L_\star\eta)'' - \Delta|\nabla\phi_\star|^2 L_\star\tilde{\eta} - |\nabla\phi_\star|^4(L_\star\tilde{\eta})''$$
$$\big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star,\nabla|\nabla\phi_\star|^2)\big)(L_\star\tilde{\eta})'\Big\} + \varepsilon^5 F^8_{\varepsilon,\phi}(y,t),$$

with $F^8_{\varepsilon,\phi}$ satisfying (3.93).

Moreover,

$$T^9_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)(\varepsilon^3 H\psi\eta' + \varepsilon^3 H|\nabla\phi_\star|^2\tilde{\eta}') = \; (3.101)$$
$$\varepsilon^3(\psi HL_\star(\eta') + H|\nabla\phi_\star|^2 L_\star(\tilde{\eta}')) + \varepsilon^4(H^2\psi\eta'' + H^2|\nabla\phi_\star|^2\tilde{\eta}'') + \varepsilon^5 F^9_{\varepsilon,\phi}(y,t),$$

with $F^9_{\varepsilon,\phi}$ satisfying (3.93).

As regards the term of order ε^4 of (3.100), we note that

$$\begin{aligned} T^{10}_{\varepsilon,\phi}(y,t) &= \varepsilon^4\big(-\Delta + W''(v_\star)\big)\Big\{-\Delta_\Sigma\psi\eta + \big(|A|^2\psi(t+\phi_\star) + HL\phi_\star + 2(\nabla_\Sigma\psi,\nabla_\Sigma\phi_\star) \\ &\quad +\psi\Delta_\Sigma\phi_\star\big)\eta' - \psi|\nabla_\Sigma\phi_\star|^2\eta'' - \Delta|\nabla\phi_\star|^2\tilde\eta - |\nabla\phi_\star|^4\tilde\eta'' \\ &\quad \big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star,\nabla|\nabla\phi_\star|^2)\big)\tilde\eta'\Big\} \\ &= \varepsilon^4\Big\{-\Delta_\Sigma\psi L_\star\eta + \big(HL\phi_\star + 2(\nabla_\Sigma\psi,\nabla_\Sigma\phi_\star) + \psi\Delta_\Sigma\phi_\star\big)L_\star(\eta') \\ &\quad +|A|^2\psi L_\star((t+\phi_\star)\eta') - \psi|\nabla_\Sigma\phi_\star|^2 L_\star(\eta'') - \Delta|\nabla\phi_\star|^2\tilde\eta - |\nabla\phi_\star|^4(L_\star\tilde\eta)'' \\ &\quad \big(|A|^2(t+\phi_\star)|\nabla\phi_\star|^2 + \Delta_\Sigma\phi_\star|\nabla\phi_\star|^2 + 2(\nabla\phi_\star,\nabla|\nabla\phi_\star|^2)\big)(L_\star\tilde\eta)'\Big\} + \varepsilon^5 F^{10}_{\varepsilon,\phi}(y,t), \end{aligned} \tag{3.102}$$

with $F^{10}_{\varepsilon,\phi}$ satisfying (3.93). To conclude, also

$$F^{11}_{\varepsilon,\phi}(y,t) = \big(-\Delta + W''(v_\star)\big)\tilde F_{\varepsilon,\phi}(y,t)$$

is negligible, that is it satisfies (3.93), since $\tilde F_{\varepsilon,\phi}$ does.

Now we have to considered the contribution of $F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}]$, since it gives rise to a term of order ε^4. However, we will see that this contribution will cancel after projection, since the term of order ε^4 is orthogonal to $v'_\star$, indeed

$$\begin{aligned} &F''(v_\star)[v_{2,\varepsilon,\phi}, v_{2,\varepsilon,\phi}] \\ &= \varepsilon^4\psi^2\big(W'''(v_\star)W''(v_\star)\eta^2 - (W'''(v_\star)\eta^2)'' + 2W'''(v_\star)\eta L_\star\eta\big) + \varepsilon^5 F^{13}_{\varepsilon,\phi}(y,t), \end{aligned} \tag{3.103}$$

with $F^{13}_{\varepsilon,\phi}$ satisfying (3.93). By the properties of the smoothing operators R_θ (see (3.44,3.45,3.46)), we can see that

$$||F''(v_\star)[v_{1,\varepsilon,\phi}, v_{2,\varepsilon,\phi}]||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \le c\varepsilon^6|\phi|_{C^{4,\alpha}(\Sigma)} \tag{3.104}$$

and the same is true for $F''(v_\star)[v_{1,\varepsilon,\phi}, v_{1,\varepsilon,\phi}]$ and $C_{\varepsilon,\phi}[v_{1,\varepsilon,\phi} + v_{2,\varepsilon,\phi}]$.

We point out that, with this choice of approximate solution, the term of order ε^2 of $F(\tilde v_{\varepsilon,\phi})$ is precisely

$$(H^2 - 2|A|^2)v''_\star + \psi L^2_\star\eta + |\nabla\phi_\star|^2(v^{(4)}_\star - W''(v_\star)v''_\star + L^2_\star\tilde\eta),$$

which vanishes if we set $\psi := H^2 - 2|A|^2$, since $L^2_\star\eta = -v''_\star$ and $L^2_\star\tilde\eta = -v^{(4)}_\star + W''(v_\star)v''_\star$. Moreover, if we set $L\phi = \phi(2H|A|^2 - 4\mathrm{tr}A^3) + 2H\Delta_\Sigma\phi - 4(A,\nabla^2\phi)$, the term of order ε^3 is

$$\begin{aligned}(H|A|^2-2\mathrm{tr}A^3)(2tv_\star''+v_\star)+|\nabla\phi_\star|^2(-2v_\star'''+2(L_\star\tilde{\eta})')-\Delta_\Sigma Hv_\star'\\+(\phi_\star(2H|A|^2-4\mathrm{tr}A^3)+2H\Delta_\Sigma\phi_\star-4(A,\nabla^2\phi_\star))v_\star''+L\phi L_\star^2\eta\\2H(H^2-2|A|^2)(L_\star\eta)'+2(A\nabla\phi_\star,\nabla\phi_\star)\big((t+\phi_\star)v_\star^{(4)}-(t+\phi_\star)W''(v_\star)v_\star''+2v_\star'''\big)=\\(H|A|^2-2\mathrm{tr}A^3)(2tv_\star''+v_\star')-\Delta_\Sigma Hv_\star'+2H(H^2-2|A|^2)(L_\star\eta)'\\+2(A\nabla\phi_\star,\nabla\phi_\star)\big((t+\phi_\star)v_\star^{(4)}-(t+\phi_\star)W''(v_\star)v_\star''+2v_\star'''\big),\end{aligned}$$

since $(L_\star\tilde{\eta})'=v_\star'''$ and $L_\star^2\eta=-v_\star''$.

We recall that $\tilde{v}_{\varepsilon,\phi}$ is just defined in B, while our global approximate solution is $v_{\varepsilon,\phi}(x)=\chi_5(x)\tilde{v}_{\varepsilon,\phi}(y,t)+(1-\chi_5(x))\mathbb{H}(x)$ (see (3.54)).

Remark 91. *It follows from the construction that our approximate solution respects the symmetries of the Torus, that is $v_{\varepsilon,\phi}(x)=v_{\varepsilon,\phi}(Tx)$ and $v_{\varepsilon,\phi}(x)=v_{\varepsilon,\phi}(Rx)$, for any $R\in SO_{x_3}(3)$.*

3.5.2 Projection

As we noticed in section 4, 2, we need to consider the projection of the error $F(\tilde{v}_{\varepsilon,\phi})$.

Proposition 92. *Let us set, for any $\phi\in B_4(1/4)=\{\phi\in C^{4,\alpha}(\Sigma)_{s,0}:|\phi|_{C^{4,\alpha}(\Sigma)}<1/4\}$,*

$$L\phi:=-4<A,\nabla^2\phi>+2H\Delta_\Sigma\phi+\phi(2H|A|^2-4trA^3), \tag{3.105}$$

where $c_\star:=\int_{-\infty}^{\infty}(v_\star'(t))^2dt$. Then, for any $y\in\Sigma_\varepsilon$, the projection of $F_\varepsilon(\tilde{v}_{\varepsilon,\phi})$ satisfies

$$\int_{-\infty}^{\infty}(F(\tilde{v}_{\varepsilon,\phi})-\varepsilon^4\chi_1\lambda(1-\tilde{v}_{\varepsilon,\phi}))(y,t)v_\star'(t)dt=-\varepsilon^4c_\star\tilde{L}_0\phi-2\varepsilon^4\lambda+\varepsilon^5\mathcal{F}_{\varepsilon,\phi,\lambda}(\varepsilon y), \tag{3.106}$$

with $\mathcal{F}_{\varepsilon,\phi}$ uniformly bounded and Lipschitzian in $\phi\in B_4(1/4)$ and in ε, that is there exists a constant $c=c(W,\tau)>0$ such that

$$\begin{cases}|\mathcal{F}_{\varepsilon,\phi,\lambda}|_{C^{0,\alpha}(\Sigma)}\le c,\\|\mathcal{F}_{\varepsilon,\phi_1,\lambda_1}-\mathcal{F}_{\varepsilon,\phi_2,\lambda_2}|_{C^{0,\alpha}(\Sigma)}\le c(|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}+|\lambda_1-\lambda_2|),\end{cases} \tag{3.107}$$

for any $\phi,\phi_1,\phi_2\in C^{4,\alpha}(\Sigma)_s$ with $|\phi|_{C^{4,\alpha}(\Sigma)},|\phi_1|_{C^{4,\alpha}(\Sigma)},|\phi_2|_{C^{4,\alpha}(\Sigma)}\le\tilde{c}\varepsilon$, for any $|\lambda|,|\lambda_1|,|\lambda_2|<\tilde{c}\varepsilon$, for some $C>0$, and for any $\varepsilon>0$ small enough.

Remark 93. *It follows from the computations that* $\mathcal{F}_{\varepsilon,\phi,\lambda}(y) = \mathcal{H}(y) + \varepsilon\mathcal{G}_{\varepsilon,\phi,\lambda}(y)$, *where* $|\mathcal{H}|_{C^{0,\alpha}(\Sigma)} < \infty$ *is bounded and*

$$|\mathcal{G}_{\varepsilon,\phi,\lambda}|_{C^{0,\alpha}(\Sigma)} \leq C, \quad \text{if } |\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < \tilde{c}\varepsilon,$$

where $\tilde{c} > 0$ *is arbitrary and* C *may depend on* $\tilde{c}$, *but not on* ε.

Proof. Above we computed $F(\tilde{v}_{\varepsilon,\phi})$ using (3.28), now we just project it term by term.

Integrating by parts we can show that

$$\int_{-\infty}^{\infty} tv_\star''(t)v_\star'(t)dt = -\frac{1}{2}c_\star \tag{3.108}$$

$$\int_{-\infty}^{\infty} (L_\star\eta(t))'v_\star'(t)dt = \frac{1}{4}c_\star \tag{3.109}$$

$$\int_{-\infty}^{\infty} L_\star(\eta'(t))v_\star'(t)dt = 0, \tag{3.110}$$

so in particular

$$\int_{-\infty}^{\infty} W'''(v_\star(t))\eta(t)(v_\star'(t))^2dt = \int_{-\infty}^{\infty} \left\{L_\star\eta(t) - L_\star(\eta'(t))\right\}v_\star'(t)dt = \frac{1}{4}c_\star.$$

Moreover, setting $b_\star := \int_{-\infty}^{\infty}(v_\star''(t))^2dt = -\int_{-\infty}^{\infty} v_\star'''(t)v_\star'(t)dt$, we can see that

$$\int_{-\infty}^{\infty} \left\{tv_\star^{(4)}(t) - tW''(v_\star(t))v_\star''(t) + 2v_\star'''(t)\right\}v_\star'(t)dt = \tag{3.111}$$

$$-\int_{-\infty}^{\infty} tL_\star(v_\star''(t))v_\star'(t)dt - 2b_\star = -\int_{-\infty}^{\infty} tv_\star''(t)L_\star(v_\star'(t))dt + 2b_\star - 2b_\star = 0$$

because $L_\star(v_\star') = 0$.

First we show that the term of order ε^3 vanishes after projection. In fact, by (3.108,3.109,3.111) and the Willmore equation,

$$\begin{aligned}(H|A|^2 - 2\mathrm{tr}A^3)\int_{-\infty}^{\infty}(2tv_\star'' + v_\star)v_\star'dt - \Delta_\Sigma H\int_{-\infty}^{\infty}(v_\star')^2dt + 2H(H^2-2|A|^2)\int_{-\infty}^{\infty}(L_\star\eta)'v_\star'dt \\ +2(A\nabla\phi_\star,\nabla\phi_\star)\int_{-\infty}^{\infty}\left((t+\phi_\star)v_\star^{(4)} - (t+\phi_\star)W''(v_\star)v_\star'' + 2v_\star'''\right)v_\star'dt \\ = c_\star(-\Delta_\Sigma H + \frac{1}{2}H(H^2-2|A|^2)) = 0\end{aligned}$$

In the forthcoming calculations, we will only consider the terms of order at least ε^4. For notational convenience, we define $\tilde{T}^k_{\varepsilon,\phi}$ to be $\tilde{T}^k_{\varepsilon,\phi}$ minus the terms of order

ε^2 and ε^3. For instance

$$\tilde{T}^1_{\varepsilon,\phi}(y,t) = T^1_{\varepsilon,\phi} - \varepsilon^2(H^2 - 2|A|^2)v_\star''$$
$$-\varepsilon^3\Big\{(2H|A|^2 - 4\mathrm{tr}A^3)(t+\phi_\star)v_\star'' + (H|A|^2 - 2\mathrm{tr}A^3)v_\star' - \Delta_\Sigma H v_\star'$$
$$+2(\nabla_\Sigma H, \nabla_\Sigma \phi_\star)v_\star'' - H|\nabla_\Sigma\phi_\star|^2 v_\star''' + H\Delta_\Sigma\phi_\star v_\star''\Big\}.$$

Moreover, the right-hand side will always be evaluated at εy. By (3.108) and (3.24),

$$\int_{-\infty}^{\infty} \tilde{T}^1_{\varepsilon,\phi}(y,t)v_\star'(t)dt$$
$$= \varepsilon^4 c_\star\Big\{ - \phi_\star\Delta_\Sigma|A|^2 - (\nabla_\Sigma|A|^2, \nabla_\Sigma\phi_\star) - \frac{1}{2}|A|^2\Delta_\Sigma\phi_\star - \phi_\star(2 < A, \nabla^2 H > + |\nabla_\Sigma H|^2)$$
$$-2(A\nabla_\Sigma H, \nabla_\Sigma\phi_\star) - \frac{1}{2}H(2 < A, \nabla^2\phi_\star > + (\nabla_\Sigma H, \nabla_\Sigma\phi_\star))\Big\} + \varepsilon^5\mathcal{F}^1_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty} \tilde{T}^2_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^4 c_\star\Big\{ - (\Delta_\Sigma)^2\phi_\star - \frac{1}{2}|A|^2\Delta_\Sigma\phi_\star\Big\} + \varepsilon^5\mathcal{F}^2_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty} \tilde{T}^3_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5\mathcal{F}^3_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty} \tilde{T}^4_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^4\frac{1}{2}c_\star H(2 < A, \nabla^2\phi_\star > + (\nabla_\Sigma H, \nabla_\Sigma\phi_\star)) + \varepsilon^5\mathcal{F}^4_{\varepsilon,\phi},$$

with $\mathcal{F}^1_{\varepsilon,\phi}, \mathcal{F}^2_{\varepsilon,\phi}, \mathcal{F}^3_{\varepsilon,\phi}, \mathcal{F}^4_{\varepsilon,\phi}$ satisfying (3.107).

By (3.111),

$$\int_{-\infty}^{\infty} \tilde{T}^5_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5\mathcal{F}^5_{\varepsilon,\phi},$$

with $\mathcal{F}^5_{\varepsilon,\phi}$ satisfying (3.107). Once again by (3.108), we can see that

$$\int_{-\infty}^{\infty} \tilde{T}^6_{\varepsilon,\phi}(y,t)v_\star'(t)dt = \varepsilon^5\mathcal{F}^6_{\varepsilon,\phi},$$

with $\mathcal{F}^6_{\varepsilon,\phi}$ satisfying (3.107).

Now let us consider the terms coming from the correction.

$$\int_{-\infty}^{\infty} \tilde{T}^7_{\varepsilon,\phi}(y,t)v_\star^{'}(t)dt$$
$$= \varepsilon^4 c_\star \frac{1}{4}\Big\{(H^2-2|A|^2)\Delta_\Sigma\phi_\star + HL\phi_\star + (H^2-2|A|^2)|A|^2\phi_\star\Big\} + \varepsilon^5\mathcal{F}^7_{\varepsilon,\phi},$$

$$\int_{-\infty}^{\infty} \tilde{T}^8_{\varepsilon,\phi}(y,t)v_\star^{'}(t)dt$$
$$= \varepsilon^4 c_\star\Big\{\frac{1}{4}(H^2-2|A|^2)|A|^2\phi_\star + \frac{1}{4}HL\phi_\star + H(\nabla_\Sigma H, \nabla_\Sigma\phi_\star)$$
$$-(\nabla_\Sigma|A|^2, \nabla_\Sigma\phi_\star) + \frac{1}{4}(H^2-2|A|^2)\Delta_\Sigma\phi_\star\Big\} + \varepsilon^5\mathcal{F}^8_{\varepsilon,\phi},$$

with $\mathcal{F}^7_{\varepsilon,\phi}, \mathcal{F}^8_{\varepsilon,\phi}$ satisfying (3.107). To conclude, also

$$\mathcal{F}^9_{\varepsilon,\phi} = \int_{-\infty}^{\infty}\{\tilde{T}^9_{\varepsilon,\phi}(y,t) + \tilde{T}^{10}_{\varepsilon,\phi}(y,t)\}v_\star^{'}(t)dt \tag{3.112}$$

fulfills (3.107). The contribution of the term $F'(v_\star)[v_{2,\varepsilon,\phi}]$ is given by

$$\int_{-\infty}^{\infty} F'(v_\star)[v_{2,\varepsilon,\phi}](y,t)v_\star^{'}(t)dt = -\varepsilon^4 c_\star\Delta^2_\Sigma(\phi-\phi_\star) + \mathcal{F}^{10}_{\varepsilon,\phi}, \tag{3.113}$$

with $\mathcal{F}^{10}_{\varepsilon,\phi}$ satisfying (3.107). In conclusion, by the choice of L (see 3.105) we obtain exactly $\tilde{L}_0 R_{1/\varepsilon} + \Delta^2(Id - R_{1/\varepsilon})$ as a linear term at order ε^4. Moreover, by (3.104), the terms involving $F^{''}(v_\star)$ and $C_{\varepsilon,\phi}$ do not give rise to terms of order ε^4 in the projection. In conclusion, we have

$$\int_{-\infty}^{\infty} F(\tilde{v}_{\varepsilon,\phi})(y,t)v_\star^{'}(t)dt = -\varepsilon^4 c_\star(\tilde{L_0}\phi_\star(\varepsilon y) + \Delta^2_\Sigma(\phi-\phi_\star)(\varepsilon y)) + \varepsilon^5\mathcal{F}^{11}_{\varepsilon,\phi}(\varepsilon y),$$

where $\mathcal{F}^{11}_{\varepsilon,\phi}$ satisfies (3.107). We can see that

$$\tilde{L_0}\phi_\star + \Delta^2_\Sigma(\phi-\phi_\star) = \tilde{L}_0\phi + (\tilde{L}_0 - \Delta^2_\Sigma)(\phi_\star - \phi),$$

and $\varepsilon^{-1}(\tilde{L}_0 - \Delta^2_\Sigma)(\phi_\star - \phi)$ satisfies (3.107), by the property (3.46), because $\tilde{L}_0 - \Delta^2_\Sigma$ is a second order operator, thus, for instance

$$|\varepsilon^{-1}(\tilde{L}_0 - \Delta^2_\Sigma)(\phi_\star - \phi)|_{C^{0,\alpha}(\Sigma)} \le c\varepsilon^{-1}|\phi_\star - \phi|_{C^{2,\alpha}(\Sigma)} \le c\varepsilon|\phi|_{C^{2,\alpha}(\Sigma)}.$$

Using the oddness of $\tilde{v}_{\varepsilon,\phi}$ in t, we directly compute

$$-\int_{\mathbb{R}} \varepsilon^4\lambda\chi_1(1-\tilde{v}_{\varepsilon,\phi}(y,t))v_\star^{'}(t)dt = -2\varepsilon^4\lambda + \varepsilon^4\lambda\mathcal{F}^{12}_{\varepsilon,\phi}(y),$$

with $\mathcal{F}^{12}_{\varepsilon,\phi}$ satisfying (3.107) □

3.6 Solvability far from Σ_ε

This Section will be devoted to the proofs of Propositions 86. First we study the associated linear problem, then we will conclude the proof by a fixed point argument.

3.6.1 Solvabilty far away from Σ_ε: the linear problem

We will prove the following Proposition.

Proposition 94. *Let $0 < \delta < \sqrt{W''(1)}$. Then, for any $\varepsilon > 0$ small enough, for any $\phi \in B_4(\tau/4)$, and for any $f \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$, the equation*

$$(-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f \tag{3.114}$$

admits a unique solution $V = \Psi_{\varepsilon,\phi}(f)$ in $C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$ satisfying $||V||_{C^{4,\alpha}(\mathbb{R}^3)} \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$, for some constant $c > 0$ independent of ε and ϕ.

Remark 95. *(i) The symmetries of the solution follow for free from the symmetries of the laplacian and of $\Gamma_{\varepsilon,\phi}$. In fact, if $f \in C^{0,\alpha}_{\delta,s}(\mathbb{R}^3)$, and V is a solution to $(-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f$, then also $u_T(x) := u(Tx)$ is a solution, thus, by uniqueness, $u = u_T$. The same argument also shows that $u = u_R$, for any $R \in SO_{x_3}(3)$, hence $u \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3)$.*

(ii) In particular, if $|f| \leq ce^{-\sqrt{W''(1)}|x|}$, then the absolute value of the solution is bounded by $e^{-\delta|x|}$, for any $0 < \delta < \sqrt{W''(1)}$.

We split the proof into some lemmas and a proposition, with the aid of some remarks. First we reduce ourselves to consider a second order PDE, then, by a bootstrap argument, we will solve our fourth order equation.

Proposition 96. *Let $0 < \delta < \sqrt{W''(1)}$. Then, for any $\varepsilon > 0$ small enough, for any $\phi \in B_4(\tau/4)$, and for any $f \in C^{0,\alpha}_{\delta}(\mathbb{R}^3)$, the equation*

$$-\Delta u + \Gamma_{\varepsilon,\phi} u = f \tag{3.115}$$

admits a unique solution $u = \tilde{\Psi}_{\varepsilon,\phi}(f)$ in $C^{2,\alpha}_{\delta}(\mathbb{R}^3)$ satisfying $||u||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$, for some constant $c > 0$ independent of ε and ϕ.

Proof. Step (i): existence, uniqueness and local Hölder regularity.

Existence and uniqueness of the weak solution follow from the Riesz representation theorem. Since $f \in C^{0,\alpha}_{loc}(\mathbb{R}^3)$, then $u \in C^{2,\alpha}_{loc}(\mathbb{R}^3)$.

Step (ii): Decay of the solution: $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$

We will use the function $e^{-\delta|x|}$ as a barrier. More precisely, we fix $\rho > 0$ and $|z| > \rho$. Then we fix $\sigma > 0$ and $R > |z|$ so large that $u(x) < \sigma$ for $|x| \geq R$. Therefore u fulfills

$$\begin{cases} u < \max_{\partial B_\rho} u < \lambda e^{-\delta\rho} < \lambda e^{-\delta\rho} + \sigma & \text{for } |x| = \rho \\ u < \sigma < \lambda e^{-\delta R} + \sigma & \text{for } |x| = R \\ (-\Delta + \Gamma_{\varepsilon,\phi})(u - (\lambda e^{-\delta|x|} + \sigma)) \leq \left(c - \lambda\delta\frac{N-1}{R}\right)e^{-\delta r} \leq 0 & \text{for } \rho < |x| < R, \end{cases}$$

provided $\lambda \geq \lambda_0$, with λ_0 independent of σ. By the maximum principle we get that $u(z) < \lambda e^{-\delta|z|} + \sigma$, for any $|z| \geq \rho$ and for any $\sigma > 0$. In the same way, one can prove that $u(z) > -\lambda e^{-\delta|z|} - \sigma$ Letting $\sigma \to 0$, we get that $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$.

Step (iii): Estimate of the L^∞-norm of the solution.
Since $u\varphi_\delta \in L^\infty(\mathbb{R}^3)$, then it exists a point $y \in \mathbb{R}^3$ such that $|u(y)| = ||u||_\infty$. If $u(y) > 0$, then y is a maximum point, thus

$$\delta^2 u(y) \leq -\Delta u(y) + \Gamma_{\varepsilon,\phi}(y)u(y) = f(y) \leq ||f||_\infty,$$

that is

$$||u||_{L^\infty(\mathbb{R}^3)} \leq c||f||_{L^\infty(\mathbb{R}^3)}.$$

A similar argument shows that the same estimate is true if $u(y) < 0$ (a minimum).

Step (iv): Continuity of the right inverse.
By [39] (chapter 6.1, Corollary 6.3), we have, for any $x \in \mathbb{R}^3$,

$$||u||_{C^{2,\alpha}(B_1(x))} \leq c(||f||_{C^{0,\alpha}(B_2(x))} + ||u||_{L^\infty(B_2(x))}). \tag{3.116}$$

Since x is arbitrary, we conclude that

$$||u||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c(||u||_{L^\infty(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}. \tag{3.117}$$

Step (v): Decay of the derivatives.

By the decay of u, we already know that $\tilde{u}_\delta \in L^\infty(\mathbb{R}^3)$. Moreover, $\tilde{u}_\delta$ satisfies the equation

$$\begin{aligned} -\Delta\tilde{u}_\delta + \Gamma_{\varepsilon,\phi}\tilde{u}_\delta = \tilde{f}_\delta - 2 < \nabla u, \nabla\varphi_\delta > -u\Delta\varphi_\delta = \\ \tilde{f}_\delta - 2\varphi_{-\delta} < \nabla\tilde{u}_\delta, \nabla\varphi_\delta > +\tilde{u}_\delta(2(\varphi_{-\delta})^2|\nabla\varphi_\delta|^2 - \varphi_{-\delta}\Delta\varphi_\delta), \end{aligned} \tag{3.118}$$

thus, once again by [39] (chapter 6.1, Corollary 6.3),

$$||\tilde{u}_\delta||_{C^{2,\alpha}(B_1(x))} \leq c(||\tilde{u}_\delta||_{L^\infty(\mathbb{R}^3)} + ||\tilde{f}_\delta||_{C^{0,\alpha}(\mathbb{R}^3)}) < \infty, \tag{3.119}$$

for any $x \in \mathbb{R}^3$, thus $u \in C^{2,\alpha}_\delta(\mathbb{R}^3)$. □

Now we can conclude the proof of Proposition 94.

Proof. Given $f \in C^{0,\alpha}_\delta(\mathbb{R}^3)$, we have to find $V \in C^{4,\alpha}_\delta(\mathbb{R}^3)$ fulfilling

$$\begin{cases} (-\Delta + \Gamma_{\varepsilon,\phi})^2 V = f \\ ||V||_{C^{4,\alpha}(\mathbb{R}^3)} \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}. \end{cases}$$

In order to do so, we use proposition 96 twice to find $u \in C^{2,\alpha}_\delta(\mathbb{R}^3)$ and $V \in C^{2,\alpha}_\delta(\mathbb{R}^3)$, such that

$$\begin{cases} (-\Delta + \Gamma_{\varepsilon,\phi})u = f \\ (-\Delta + \Gamma_{\varepsilon,\phi})V = u, \end{cases}$$

and

$$\begin{cases} ||u||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)} \\ ||V||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c||u||_{C^{0,\alpha}(\mathbb{R}^3)}. \end{cases}$$

Now it remains to estimate the higher order derivatives of u. For this purpose, we differentiate the equation satisfied by u and we get

$$(-\Delta + \Gamma_{\varepsilon,\phi})V_j = u_j - (\Gamma_{\varepsilon,\phi})_j V \tag{3.120}$$

for $j = 1, \ldots, 3$, hence, applying the regularity estimates for $(-\Delta + \Gamma_{\varepsilon,\phi})$,

$$||V_j||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c(||u_j||_{C^{0,\alpha}(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}$$

and, since the right-hand side of (3.120) behaves like $e^{\delta|x|}$, then, arguinig as in the proof of Proposition 96, step (ii), we can see that $V_j \in C^{2,\alpha}_\delta(\mathbb{R}^3)$, that is $V \in C^{3,\alpha}_\delta(\mathbb{R}^3)$.

Similarly, differentiating the equation once again, we see that

$$(-\Delta + \Gamma_{\varepsilon,\phi})V_{ij} = u_{ij} - (\Gamma_{\varepsilon,\phi})_i V_j - (\Gamma_{\varepsilon,\phi})_j V_i - (\Gamma_{\varepsilon,\phi})_{ij} V,$$

for $i, j = 1, \ldots, 3$, so in particular

$$||V_{ij}||_{C^{2,\alpha}(\mathbb{R}^3)} \leq c(||u_{ij}||_{C^{0,\alpha}(\mathbb{R}^3)} + ||f||_{C^{0,\alpha}(\mathbb{R}^3)}) \leq c||f||_{C^{0,\alpha}(\mathbb{R}^3)}.$$

and $V \in C^{4,\alpha}_\delta(\mathbb{R}^3)$. □

3.6.2 Proof of Proposition 86: solving equation (3.63) by a fixed point argument

Equation (3.63) is equivalent to the fixed point problem

$$V = T_1(V) := -\Psi_{\varepsilon,\phi}\Big\{(1-\chi_2)F(v_{\varepsilon,\phi}) + (1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V)$$
$$+\mathrm{N}_{\varepsilon,\phi}(U) + \mathrm{P}_{\varepsilon,\phi}(V) - \varepsilon^4\lambda(1-\chi_1)(1-v_{\varepsilon,\phi}-V)\Big\},$$

that we will solve by showing that T_1 is a contraction on the ball

$$\Lambda_1 := \{V \in C^{4,\alpha}_{\delta,s}(\mathbb{R}^3) : ||V||_{C^{4,\alpha}(\mathbb{R}^3)} \leq C_1 e^{-a/\varepsilon}\},$$

provided the constant C_1 is large enough. In fact, by the exponential decay of U far from Σ_ε, we get that

$$||\mathrm{N}_{\varepsilon,\phi}(U)||_{C^{4,\alpha}(\mathbb{R}^3)} \leq \tilde{c}e^{-a/\varepsilon},$$

$a := \delta\tau/2$, for some constant $\tilde{c} > 0$ independent of ε and ϕ. By (3.49) and (3.50), the same is true for $(1-\chi_2)F(v_{\varepsilon,\phi})$. Moreover, by (3.91), (3.49) and (3.50),

$$||\mathrm{P}_{\varepsilon,\phi}(V)||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \leq c\varepsilon||V||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \leq c\varepsilon e^{-a/\varepsilon},$$

with $c > 0$ depending on W, τ, δ but not of ε and ϕ. Moreover, using that

$$||(1-\chi_1)V||_{C^{4,\alpha}(\mathbb{R}^3)} \leq c||V||_{C^{4,\alpha}(\mathbb{R}^3)}$$

and

$$||(1-\chi_1)\chi_2 U||_{C^{4,\alpha}(\mathbb{R}^3)} \leq ce^{-a/\varepsilon},$$

where $(1-\chi_1)\chi_2 U$ is understood to be 0 outside the support of χ_2, and the definition of $Q_{\varepsilon,\phi}$ (see (3.56)), we get

$$||(1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U + V)||_{C^{4,\alpha}(\mathbb{R}^3)} \leq ce^{-2a/\varepsilon}.$$

Furthermore, we can see that

$$||\lambda(1-\chi_1)(1-\tilde{v}_{\varepsilon,\phi}-V)||_{C^{0,\alpha}(\mathbb{R}^3)} \leq c|\lambda|(e^{-a/\varepsilon} + ||V||_{C^{4,\alpha}(\mathbb{R}^3)}). \qquad (3.121)$$

Up to now, we have just proved that T_1 maps Λ_1 in itself, provided C_1 is large enough. In order to show that it is actually a contraction, we need to estimate its Lipschitz constant. The terms depending on V are $P_{\varepsilon,\phi}$, that fulfills

$$||P_{\varepsilon,\phi}(V_1) - P_{\varepsilon,\phi}(V_2)||_{C^{4,\alpha}(\mathbb{R}^3)} \leq c\varepsilon||V_1 - V_2||_{C^{4,\alpha}(\mathbb{R}^3)}$$

for some constant $c > 0$ independent of ε and ϕ, $(1-\chi_1)Q_{\varepsilon,\phi}(\chi_2 U+V)$, that fulfills

$$||(1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U+V) - Q_{\varepsilon,\phi}(\chi_2 U+V))||_{C^{4,\alpha}_\delta(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||V_1 - V_2||_{C^{4,\alpha}(\mathbb{R}^3)}.$$

and $\varepsilon^4\lambda(1-\chi_1)V$, that satisfies a similar estimate.

Lipschitz dependence on U, ϕ and λ.

Given $\phi \in B_4(1/4)$ and $U_1, U_2 \in C^{4,\alpha}(\Sigma_\varepsilon \times \mathbb{R})$, the difference between the solutions V_{ε,ϕ,U_1} and $V_{\varepsilon,\phi,\lambda,U_1}$ fulfills

$$\begin{aligned}(-\Delta+\Gamma_{\varepsilon,\phi})^2(V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}) = (1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U_2 + V_{\varepsilon,\phi,\lambda,U_2}) - Q_{\varepsilon,\phi}(\chi_2 U_1 + V_{\varepsilon,\phi,\lambda,U_1}))\\ +\mathrm{N}_{\varepsilon,\phi}(U_2) - \mathrm{N}_{\varepsilon,\phi}(U_1) + \mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_2}) - \mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_1}).\end{aligned}$$

By (3.61), the terms involving $\mathrm{N}_{\varepsilon,\phi}$ satisfy

$$||\mathrm{N}_{\varepsilon,\phi}(U_1) - \mathrm{N}_{\varepsilon,\phi}(U_2)||_{C^{0,\alpha}(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||U_2 - U_1||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.$$

By (3.62), the terms involving $\mathrm{N}_{\varepsilon,\phi}$ can be estimated with the difference between the solutions, that is

$$||\mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_1}) - \mathrm{P}_{\varepsilon,\phi}(V_{\varepsilon,\phi,\lambda,U_2})||_{C^{0,\alpha}(\mathbb{R}^3)} \leq ce^{-a/\varepsilon}||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \tag{3.122}$$

and

$$\begin{aligned}||(1-\chi_1)(Q_{\varepsilon,\phi}(\chi_2 U_1 + V_{\varepsilon,\phi,\lambda,U_1}) - Q_{\varepsilon,\phi}(\chi_2 U_2 + V_{\varepsilon,\phi,\lambda,U_2}))||_{C^{0,\alpha}(\mathbb{R}^3)} \leq\\ ce^{-a/\varepsilon}(||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} + ||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}).\end{aligned}$$

Therefore, applying $\Psi_{\varepsilon,\phi}$ to the right-hand side of (3.122), we obtain

$$\begin{aligned}||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} \leq\\ ce^{-a/\varepsilon}(||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} + ||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}),\end{aligned}$$

thus, reabsorbing the norm of the difference between the solutions,

$$\begin{aligned}\frac{1}{2}||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)} &\leq (1-ce^{-a/\varepsilon})||V_{\varepsilon,\phi,\lambda,U_1} - V_{\varepsilon,\phi,\lambda,U_2}||_{C^{4,\alpha}(\mathbb{R}^3)}\\ &\leq ce^{-a/\varepsilon}||U_1 - U_2||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}.\end{aligned}$$

The Lipschitz dependence on ϕ and λ can be treated with a similar argument. It is worth to point out that also the potential $\Gamma_{\varepsilon,\phi}$ actually depends on ϕ, through the approximate solution and the cutoff function. However, this dependence is mild enough for our purposes, in fact the difference of the potentials $\Gamma_{\varepsilon,\phi_1} - \Gamma_{\varepsilon,\phi_2}$ is exponentially small in ε.

3.7 Solving the auxiliary equation

This section is devoted to the proof of Proposition (87). We already know that $(a2)$ is satisfied (see Section 3.5.2), therefore, in order to solve the auxiliary equation, we need to show that $(a1)$ is fulfilled too. We will show that this is true in section 3.7.1. Then, in Section 3.7.2 we will prove that $\mathcal{F}$ defined in (3.71) satisfies $(\mathcal{F}1)$, $(\mathcal{F}2)$, $(\mathcal{F}3)$ and (1.36), and we will see that (1.37) and (1.38) are verified, in order to apply Proposition 61, which enables us to find a solution $U_{\varepsilon,\phi}$ to (3.64).

3.7.1 The linear problem

Now we look for a solution to equation (3.64) respecting the symmetries of the Torus. First we study the linear operator $\mathcal{M}_\varepsilon(\phi)$.

Proposition 97. *Let $0 < \delta < \sqrt{W''(\pm 1)}$ and $\phi \in B_4(1)$. For any $f \in \mathcal{E}^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$, there exists a unique solution $U = \Phi_{\varepsilon,\phi}(f)$ in $\mathcal{E}^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ to $\mathcal{L}_\varepsilon(\phi)[U] = f$ such that*

$$||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})},$$

for some constant $C > 0$ which is independent of ε.

In the notation of section 3.1, we have

$$\mathcal{E}^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) = Y_{1,\varepsilon} \cap T_{\varepsilon,\phi} = W_{\varepsilon,\phi}, \quad \mathcal{E}^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R}) = Y_{2,\varepsilon} \cap T_{\varepsilon,\phi},$$

and since we want our solution $U_{\varepsilon,\phi}$ to respect the symmetries of the Torus, we set

$$\tilde{Y}_{1,\varepsilon} := C^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}), \quad \tilde{Y}_{2,\varepsilon} := C^{0,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R}).$$

If f respects the symmetries of the Torus, that is $f \in \tilde{Y}_{2,\varepsilon} \cap W_{\varepsilon,\phi}$ then also the solution $U = \Phi_{\varepsilon,\phi}(f)$ does, that is it belongs to $\tilde{Y}_{2,\varepsilon} \cap W_{\varepsilon,\phi}$. In other words, $\Phi_{\varepsilon,\phi}$ maps $\mathcal{E}^{4,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$ into $\mathcal{E}^{0,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. This fact follows from uniqueness.

It is useful to see that we can control the odd part of the solution with the odd part (in t) of f and the same is true for the even parts.

Lemma 98. *Let $0 < \delta < \sqrt{W''(\pm 1)}$ and $f \in C^{0,\alpha}_{\delta,s}(\Sigma_\varepsilon \times \mathbb{R})$. Let $U \in C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})$ be the solution to $\mathcal{L}_\varepsilon(\phi)[U] = f$. Then*

$$\begin{cases} ||U_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \leq c||f_o||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \\ ||U_e||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})} \leq c||f_e||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon \times \mathbb{R})}, \end{cases}$$

where c is the constant found in Proposition 97.

Proof. We set, for any $(y,t) \in \Sigma_\varepsilon \times \mathbb{R}$, $\tilde{U}(y,t) := U(y,-t)$ and $\tilde{f}(y,t) := f(y,-t)$. Using that W'' is even and $v_\star$ is odd, we can see that $\mathcal{L}^2_\varepsilon \tilde{U} = \tilde{f}$. Therefore, subtracting and multiplying by $1/2$, we get

$$\mathcal{L}_\varepsilon(\phi)\left[\frac{U(y,t) - \tilde{U}(y,t)}{2}\right] = \frac{f(y,t) - \tilde{f}(y,t)}{2},$$

that is $\mathcal{L}_\varepsilon(\phi)U_o(y,t) = f_o$. In addition,

$$\int_{-\infty}^{\infty} U_o(y,t)v_\star'(t)dt = \int_{-\infty}^{\infty} f_o(y,t)v_\star'(t)dt = 0,$$

for any $y \in \Sigma_\varepsilon$, hence $U_o = G_{\varepsilon,\phi}(f_o)$, so in particular the first estimate holds true. The second one can be proved by a similar argument. □

Now we prove Proposition 97, with the aid of some Lemmas and Remarks. First we consider the spectral decomposition of $\mathcal{M}_\varepsilon(\phi)$. We will denote by $(\lambda_j, \phi_j)_{j\geq 0}$ the eingendata of $-\Delta_\Sigma$. We observe that $\lambda_0 = 0$, $\lambda_j \geq \lambda_1 > 0$, ϕ_0 is constant and, without loss of generality, we can assume that $||\phi_j||_{L^2(\Sigma)} = 1$ (see [63]). Similarly, we will denote by $\{\mu_k\}_{k\geq 0}$ the eigenvalues of $L_\star = -\partial_{tt} + W''(v_\star(t))$. In [59], Müller proved that $\mu_0 = 0$, and the corresponding eigenspace, that is the Kernel, is generated by $v_\star'(t)$, while $\mu_k \geq \mu_1 > 0$ (see also [54]).

Remark 99. *The eigenvalues of $\mathcal{M}_\varepsilon(\phi)$ are $\{\mu_k + \varepsilon^2\lambda_j\}_{j,k\geq 0}$, thus all non-zero eigenvalues are positive and bounded away from 0, indeed $\mu_k + \varepsilon^2\lambda_j \geq \varepsilon^2\lambda_1 > 0$.*

Lemma 100. *Let*

$$\mathcal{M}_\varepsilon(\phi) : H^1(\Sigma_\varepsilon \times \mathbb{R}) \to H^{-1}(\Sigma_\varepsilon \times \mathbb{R})$$

be defined by the duality relation

$$\left\langle \mathcal{M}_\varepsilon(\phi)[U_1], U_2 \right\rangle = \int_{\Sigma_\varepsilon \times \mathbb{R}} \left\{ (\nabla_{\Sigma_\varepsilon} U_1, \nabla_{\Sigma_\varepsilon} U_2) + \partial_t U_1 \partial_t U_2 + W''(v_\star(t))U_1 U_2 \right\} d\sigma(y)dt,$$

for any $U_1, U_2 \in H^1(\Sigma_\varepsilon \times \mathbb{R})$. Then

$$Ker(\mathcal{M}_\varepsilon(\phi)) = span(v_\star'(t)).$$

In the sequel, we will use the notation

$$\int_{\Sigma_\varepsilon \times \mathbb{R}} \mathcal{M}_\varepsilon(\phi)[U_1]U_2 d\sigma(y)dt := \left\langle \mathcal{M}_\varepsilon(\phi)[U_1], U_2 \right\rangle, \quad \forall U_1, U_2 \in H^1(\Sigma_\varepsilon \times \mathbb{R}) \tag{3.123}$$

Proof. It is possible to see that $(\lambda_{\varepsilon,j}, \phi_{\varepsilon,j})_{j\geq 0} := (\varepsilon^2\lambda_j, \varepsilon^2\phi_j(\varepsilon y))_{j\geq 0}$ are eigendata of Σ_ε and $\phi_{\varepsilon,j}$ are orthonormal in $L^2(\Sigma_\varepsilon)$. Any function $w \in H^1(\Sigma_\varepsilon \times \mathbb{R})$ can be expanded in Fourier series as follows

$$U(y,t) = \sum_{j\geq 0} U_j(t)\phi_{\varepsilon,j}(y)$$

where

$$U_j(t) = \int_{\Sigma_\varepsilon} U(y,t)\phi_{\varepsilon,j}(y)d\sigma(y).$$

If $\mathcal{M}_\varepsilon(\phi)[w] = 0$, applying the operator to each term in the series, we get

$$-\partial_{tt}U_j(t) + \lambda_{\varepsilon,j}U_j(t) + W''\big(v_\star(t))U_j(t) = 0$$

for any $j \geq 0$, so $U_0(t) = cv_\star'(t)$ and $w_j = 0$ for $j \geq 1$. $\square$

Let

$$\mathcal{O} := \left\{U \in H^1(\Sigma_\varepsilon \times \mathbb{R}) : \int_{\Sigma_\varepsilon\times\mathbb{R}} U(y,t)v_\star'(t)d\sigma(y)dt = 0\right\}.$$

be the orthogonal to $v_\star'(t)$ in $H^1(\Sigma_\varepsilon \times \mathbb{R})$.

Lemma 101. *For any $f \in L^2(\Sigma_\varepsilon \times \mathbb{R})$ satisfying*

$$\int_{-\infty}^{\infty} f(y,t)v_\star'(t)dt = 0 \quad \text{for any } y \in \Sigma_\varepsilon,$$

there exists a unique $U \in H^1(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\begin{cases} \mathcal{M}_\varepsilon(\phi)[U] = f \\ \int_{-\infty}^{\infty} U(y,t)v_\star'(t)dt = 0 \quad \text{for any } y \in \Sigma_\varepsilon. \end{cases}$$

Proof. At first we observe that

$$||U|| = \int_{\Sigma_\varepsilon\times\mathbb{R}} |\nabla_{\Sigma_\varepsilon}U(y,z)|^2 + (\partial_{tt}U(y,t))^2 + W''(v_\star'(z))U^2(y,z)d\sigma(y)dt \quad (3.124)$$

is an equivalent norm on $\mathcal{O}$, that is, for any $U \in X$, we have

$$c_{\varepsilon,1}||U||_{H^1(\Sigma_\varepsilon\times\mathbb{R})} \leq ||U|| \leq c_{\varepsilon,2}||U||_{H^1(\Sigma_\varepsilon\times\mathbb{R})},$$

for some constants $c_{\varepsilon,1}, c_{\varepsilon,2} > 0$. In fact, by the spectral decomposition of $\mathcal{M}_\varepsilon(\phi)$, (see Remark 99),

$$\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{M}_\varepsilon(\phi)[U]U d\sigma(y)dt \geq \varepsilon^2\lambda_1 \int_{\Sigma_\varepsilon\times\mathbb{R}} U^2 d\sigma(y)dt.$$

Since $W''(v_\star(t))$ is bounded, a pointwise estimate yields that

$$\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{M}_\varepsilon(\phi)[U]U d\sigma(y)dt \geq \int_{\Sigma_\varepsilon\times\mathbb{R}} |\nabla_{\Sigma_\varepsilon} U|^2 + (\partial_{tt}U)^2 d\sigma(y)dt - c\int_{\Sigma_\varepsilon\times\mathbb{R}} U^2 d\sigma(y)dt, \tag{3.125}$$

for some constant $c > 0$. Now we point out that, for any $0 < \lambda < 1$, we have

$$\begin{aligned}\int_{\Sigma_\varepsilon\times\mathbb{R}} &\mathcal{M}_\varepsilon(\phi)[U]U d\sigma(y)dt \\ = \lambda\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{M}_\varepsilon(\phi)[U]U d\sigma(y)dt &+ (1-\lambda)\int_{\Sigma_\varepsilon\times\mathbb{R}} \mathcal{M}_\varepsilon(\phi)[U]U d\sigma(y)dt \geq \\ \lambda\Big(\int_{\Sigma_\varepsilon\times\mathbb{R}} |\nabla_{\Sigma_\varepsilon} U|^2 &+ (\partial_{tt}U)^2 d\sigma(y)dt - c\int_{\Sigma_\varepsilon\times\mathbb{R}} U^2 d\sigma(y)dt\Big) \\ &+(1-\lambda)\varepsilon^2\lambda_1\int_{\Sigma_\varepsilon\times\mathbb{R}} U^2 d\sigma(y)dt,\end{aligned}$$

so, in order to prove the lower bound, it is enough to choose $\lambda < \varepsilon^2\lambda_1/(c+\varepsilon^2\lambda_1)$. As a consequence, by the Riesz representation theorem, for any $f \in L^2(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\int_{\Sigma_\varepsilon\times\mathbb{R}} f(y,t)v_\star'(t) d\sigma(y)dt = 0, \tag{3.126}$$

the equation $\mathcal{M}_\varepsilon(\phi)[U] = f$ admits a unique solution $U \in \mathcal{O}$. We observe that orthogonality condition (3.126) is necessary for solvability, since

$$\begin{aligned}\int_{\Sigma_\varepsilon\times\mathbb{R}} f(y,t)v_\star'(t) d\sigma(y)dt = \int_{\Sigma_\varepsilon\times\mathbb{R}} &\mathcal{M}_\varepsilon(\phi)[U(y,t)]v_\star'(t) d\sigma(y)dt = \\ \int_{\Sigma_\varepsilon\times\mathbb{R}} &U(y,t)\mathcal{M}_\varepsilon(\phi)[v_\star'(t)] d\sigma(y)dt = 0.\end{aligned}$$

If in particular f satisfies (3.124), then, by proposition 8, 4 of [63], also w satisfies (3.124). □

Now we are ready to conclude the proof of Proposition 97.

Proof. There are two more steps. As first we need some regularity theory to estimate the $C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$ norm of the solution U if $f \in \mathcal{E}^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$, then we have to iterate the estimates to deal with the operator $\mathcal{L}_\varepsilon(\phi)$. For the first step, see Proposition 8, 3 of [63]. As regards the second one, we argue as follows.

If $f \in \mathcal{E}^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$, the above discussion yields that we can find $\tilde{U} \in \mathcal{E}^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$ such that

$$\begin{cases} \mathcal{M}_\varepsilon(\phi)[\tilde{U}] = f \\ ||\tilde{U}||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}, \end{cases} \tag{3.127}$$

for some constant $C > 0$ independent of ε. Now, by the same argument, we can find $U \in \mathcal{E}^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$ satisfying

$$\begin{cases} \mathcal{M}_\varepsilon(\phi)[U] = \tilde{U} \\ ||U||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||\tilde{U}||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}, \end{cases} \tag{3.128}$$

for some constant $C > 0$ independent of ε. To conclude the proof, we have to show that $U \in C^{4,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$ and

$$||U||_{C^{4,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}. \tag{3.129}$$

In order to do so we apply a bootstrap argument. We differentiate (3.128) with respect to y_j and we get

$$\mathcal{M}_\varepsilon(\phi)[U_j] = \tilde{U}_j.$$

By (3.127), we get that $U_j \in C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})$ and

$$||U_j||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||\tilde{U}_j||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||\tilde{U}||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}.$$

In the same way, taking the derivative with respect to t, we get

$$\mathcal{M}_\varepsilon(\phi)[U_t] = \tilde{U}_t - \frac{1}{\varepsilon}W'''(v_\star(t))v_\star'(t)U.$$

Exactly as before, we have

$$||U_t||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C(||\tilde{U}_t||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} + ||W'''(v_\star(z))v_\star'(t)U||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}) \leq$$
$$C(||\tilde{U}||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} + ||U||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}) \leq C(||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} + ||\tilde{U}||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}) \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}.$$

Therefore we have

$$||\nabla^3(U\psi_\delta)||_\infty \leq C||\nabla U||_{C^{2,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})} \leq C||f||_{C^{0,\alpha}_{\varepsilon,\delta}(\Sigma \times \mathbb{R})}.$$

Differentiating the equation once again, we get

$$||\nabla^4(U\psi_\delta)||_\infty + [\nabla^4(U\psi_\delta)]_\alpha \leq C||f||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon \times \mathbb{R})}.$$

In conclusion, we have (3.129). □

3.7.2 Proof of Proposition 87

It remains to prove that $\mathcal{F}$ satisfies $(\mathcal{F}1)$, $(\mathcal{F}2)$ and $(\mathcal{F}3)$.

We note that

$$\begin{aligned}||\mathcal{F}_\varepsilon(\phi,0)||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} = ||F(\tilde{v}_{\varepsilon,\phi})||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + ||\chi_1(Q(U+V_{\varepsilon,\phi,U}) + M_{\varepsilon,\phi(V)})||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \\ +\varepsilon^4||\lambda(1-\tilde{v}_{\varepsilon,\phi}) - \lambda V||_{C^{0,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c(\varepsilon^3 + e^{-a/\varepsilon}) < 2c\varepsilon^3,\end{aligned}$$

for some constant $c>0$ depending just on W,τ and the geometric quantities of Σ, therefore $(\mathcal{F}1)$ is verified with $\beta = 3$.

Furthermore,

$$\partial_U\mathcal{F}_\varepsilon(\phi,0)[h] = -\varepsilon\overline{L}_{\varepsilon,\phi}[h] - \varepsilon^4\lambda\chi_1 h, \quad \forall h\in Y_{2,\varepsilon},$$

hence $(\mathcal{F}2)$ is fulfilled.

Condition $(\mathcal{F}3)$ is satisfied since $\partial^2_U\mathcal{F}_\varepsilon(\phi,U)[h,k] = \partial^2_U(\chi_1 Q(U+V_{\varepsilon,\phi,U}))$.
Lipschitz dependence on ϕ.

Let us fix $\phi_1,\phi_2\in B_4(1)$ with $|\phi_1|_{C^{4,\alpha}(\Sigma)}, |\phi_2|_{C^{4,\alpha}(\Sigma)} \leq c\varepsilon$. In this proof, ε will always be small but fixed, and we will be interested in the dependence on ϕ.
First we note that

$$||F(\tilde{v}_{\varepsilon,\phi_1}) - F(\tilde{v}_{\varepsilon,\phi_2})||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^3|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)},$$

Using the Lipschitz dependence of V on the data proved in Proposition 86 and the definitions of $\mathrm{M}_{\varepsilon,\phi}$, $Q_{\varepsilon,\phi}$ and $\overline{L}_{\varepsilon,\phi}$, it is possible to see that

$$\begin{aligned}||\mathrm{M}_1(V_1) - \mathrm{M}_2(V_2)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq ce^{-a/\varepsilon}|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}, \\ ||\chi_1^1 Q_1(U+V_1) - \chi_1^2 Q_2(U+V_2)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c(||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} + e^{-a/\varepsilon})|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}, \\ ||\varepsilon\overline{L}_1 U - \varepsilon\overline{L}_2 U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon||U||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})}|\phi_1-\phi_2|_{C^{4,\alpha}(\Sigma)}.\end{aligned}$$

We recall that the notations χ_1^1 and χ_1^2 are due to the fact that the cutoff functions actually depend on ϕ, through t. In conclusion, (1.36) is fulfilled too.

As regards (1.37) and (1.38), they hold true since,

$$|v_\star'(z-\phi_1(\varepsilon y)) - v_\star'(z-\phi_2(\varepsilon y))| \leq c|\phi_1(\varepsilon y) - \phi_2(\varepsilon y)| \tag{3.130}$$

and therefore, for instance

$$|W''(v_\star(z-\phi_1(\varepsilon y))) - W''(v_\star(z-\phi_2(\varepsilon y)))| \leq c|\phi_1(\varepsilon y) - \phi_2(\varepsilon y)|.$$

Estimate of the odd part of the solution $U_{\varepsilon,\phi}$.

Up to now we have proved the existence of a solution $U_{\varepsilon,\phi}$ to equation (3.64) satisfying $||U_{\varepsilon,\phi}||_{C^{4,\alpha}_{\delta}(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^3$. However, we point out that the only terms of order ε^3 in the right-hand side come from $\chi_4 F(\tilde{v}_{\varepsilon,\phi})$. In fact, as we observed above, $||\mathrm{T}(U, V_{\varepsilon,\phi,U}, \phi)||_{C^{0,\alpha}_{\delta}(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^4$, so in particular the same is true for

$$\frac{1}{c_\star}\left(\int_{-\infty}^{\infty} \mathrm{T}(U, V_{\varepsilon,\phi,U}, \phi) v_\star'(t)dt\right) v_\star'(t).$$

Moreover, by Proposition 92,

$$\int_{-\infty}^{\infty} F(\tilde{v}_{\varepsilon,\phi})(y,t)v_\star'(t)dt$$

is of order ε^4, since the second term is exponentially small. Going back to Section 3.5.1, it is possible to see that the only terms of order ε^3 in $F(\tilde{v}_{\varepsilon,\phi})$ are even in t, thus the odd part of the right-hand side is of order ε^4, and therefore, by Lemma 98, the same is true for $U_{\varepsilon,\phi}$, namely $||(U_{\varepsilon,\phi})_o||_{C^{4,\alpha}_{\delta}(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^4$.

3.8 Solving the bifurcation equation

3.8.1 The proof of Proposition 88

First we note that, setting $w := w_{\varepsilon,\phi,\lambda}$,

$$\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi} - w)^2dx = \int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi})^2dx - 2\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi})wdx + \int_{\mathbb{R}^3} w^2dx.$$

Now let us fix some notation. For any $\phi \in C^{4,\alpha}(\Sigma)_s$ and $0 < \varepsilon \leq 1$, $|\Sigma_{\varepsilon,\phi_\star}|_3$ will be the volume of the interior of $\Sigma_{\varepsilon,\phi_\star}$, that is its 3-Lebesgue measure. Moreover, we set

$$B_1 := \{x = Z_\varepsilon(y, t + \phi_\star(\varepsilon y)) : -7 - \tau/2\varepsilon < t < 0\}$$
$$B_2 := \{x = Z_\varepsilon(y, z) : 0 < t < 7 + \tau/2\varepsilon\},$$

V_i will be the volume of B_i, for $i = 1, 2$, and $A := \mathbb{R}^3\backslash B$. Now we note that

$$\int_{\mathbb{R}^3}(1 - v_{\varepsilon,\phi}(x))^2dx = \int_A(1 - v_{\varepsilon,\phi}(x))^2dx + \int_B(1 - v_{\varepsilon,\phi}(x))^2dx$$

and

$$\int_A (1 - v_{\varepsilon,\phi}(x))^2 dx = 4(|\Sigma_{\varepsilon,\phi_\star}|_3 - V_1). \tag{3.131}$$

where the expression of the volume of $\Sigma_{\varepsilon,\phi_\star}$ is

$$|\Sigma_{\varepsilon,\phi_\star}|_3 = \varepsilon^{-3} 2\pi^2\sqrt{2} + \varepsilon^{-2}\int_\Sigma \phi_\star(\zeta) d\sigma(\zeta) \tag{3.132}$$
$$+2\pi\varepsilon^{-1}\int_0^{2\pi} \phi_\star^2(\vartheta)(\cos\vartheta + \sqrt{2}/2) d\vartheta + \frac{2\pi}{3}\int_0^{2\pi} \phi_\star^3(\vartheta)\cos(\vartheta) d\vartheta.$$

In order to compute this integral, we used the change of variables induced by the parametrization (3.83) of the Torus. Moreover,

$$\int_B (1-v)^2 = \int_B 1 - 2\int_B v + \int_B v^2.$$

We can see that

$$\int_B 1 = V_1 + V_2 \tag{3.133}$$

and

$$-2\int_B v = -8\pi \int_0^{2\pi} (2\phi_\star(\vartheta)\cos\vartheta + \varepsilon^{-1}\sqrt{2}) d\vartheta \int_0^{7+\tau/2\varepsilon} t v_\star(t) dt + G_\varepsilon^1(\phi) \tag{3.134}$$

$$\int_B v^2 = 4\pi \int_0^{2\pi} (\phi_\star^2(\vartheta)\cos\vartheta + \varepsilon^{-1}\phi_\star(\vartheta)(2\cos\vartheta + \sqrt{2}) + \varepsilon^{-2}) d\vartheta \int_0^{6+\tau/2\varepsilon} v_\star^2(t) dt \tag{3.135}$$
$$+G_\varepsilon^2(\phi).$$

Moreover, using that

$$\int_\Sigma (\phi_\star - \phi)(\zeta) d\zeta = 0,$$

we can see that the term $v_\star(t+\phi_\star-\phi)-v_\star(t)$ does not contribute to the main term of our integral. Taking the sum of (3.131,3.133,3.134,3.135),

$$\begin{aligned}\int_{\mathbb{R}^3}(1-v)^2 &= \varepsilon^{-3}8\pi^2\sqrt{2}\left(1+\varepsilon\int_0^\infty (v_\star^2-1)dt\right)-8\pi^2\sqrt{2}\int_{t>6+\tau/\sqrt{2}}(v_\star^2-1)dt \qquad (3.136)\\ &+4\int_\Sigma \phi_\star+8\pi\varepsilon^{-1}\int_0^{2\pi}\phi_\star^2(\vartheta)(\cos\vartheta+\sqrt{2}/2)d\vartheta+\frac{8\pi}{3}\int_0^{2\pi}\phi_\star^3(\vartheta)\cos(\vartheta)d\vartheta\\ &+8\pi\int_0^{2\pi}(2\phi(\vartheta)\cos\vartheta+\varepsilon^{-1}\sqrt{2})\int_0^{6+\tau/2\varepsilon}t(1-v_\star(t))dt\\ &+4\pi\int_0^{2\pi}(\phi^2(\vartheta)\cos\vartheta+\varepsilon^{-1}\phi(\vartheta)(2\cos\vartheta+\sqrt{2}))d\vartheta\int_0^{6+\tau/2\varepsilon}(v_\star^2-1)dt\\ &+2\pi\int_0^{2\pi}(\phi^2(\vartheta)\cos\vartheta+\varepsilon^{-1}\phi(\vartheta)(2\cos\vartheta+\sqrt{2}))(\phi_\star-\phi)(\vartheta)d\vartheta\int_{-6-\tau/2\varepsilon}^{6+\tau/2\varepsilon}v_\star'(t)dt\\ &+2\pi\int_0^{2\pi}\cos\vartheta(\phi_\star-\phi)(\vartheta)d\vartheta\int_{-6-\tau/2\varepsilon}^{6+\tau/2\varepsilon}v_\star'(t)t^2dt+G_\varepsilon^3(\phi)=\\ &\varepsilon^{-3}8\pi^2\sqrt{2}\left(1+\varepsilon\int_0^\infty(v_\star^2-1)dt\right)+4\int_\Sigma\phi_\star(\zeta)d\zeta\\ &+16\pi^2\sqrt{2}\varepsilon^{-1}\int_0^\infty t(1-v_\star(t))dt+G_\varepsilon^4(\phi)\\ &=\varepsilon^{-3}8\pi^2\sqrt{2}c_\varepsilon+4\int_\Sigma\phi(\zeta)d\zeta+16\pi^2\sqrt{2}\varepsilon^{-1}\int_0^\infty t(1-v_\star(t))dt+G_\varepsilon^4(\phi),\end{aligned}$$

where G_ε^i satisfy (3.84), $i=1,\dots,4$.

It remains to deal with the terms involving w. First we note that, if $f\in C_\delta^{0,\alpha}(\mathbb{R}^3)$, then the unique solution u of

$$(-\Delta+2)^2u=f \quad \text{in } \mathbb{R}^3$$

is given by the convolution between f and some suitable green function $G:\mathbb{R}^3\backslash\{0\}\to\mathbb{R}$, namely

$$u(x)=\int_{\mathbb{R}^3}G(x-y)f(y)dy.$$

Since w satisfies

$$(-\Delta+2)^2w=-F(v_{\varepsilon,\phi})-Q_{\varepsilon,\phi}(w)+((-\Delta+2)^2-F'(v_{\varepsilon,\phi}))[w], \qquad (3.137)$$

then it can be written as

$$w(x)=\int_{\mathbb{R}^3}G(x-y)(-F(v_{\varepsilon,\phi})-Q_{\varepsilon,\phi}(w)+((-\Delta+2)^2-F'(v_{\varepsilon,\phi}))[w])(y)dy.$$

Here we are interested in showing the estimate

$$\int_{\mathbb{R}^3} |w(x)|dx \le c, \tag{3.138}$$

for some constant $c > 0$. We already know by construction that $w \in C^{4,\alpha}_\delta(\mathbb{R}^3)$ and $||w||_{C^{4,\alpha}(\mathbb{R}^3)} \le c\varepsilon^3$. We note that, by Fubini's theorem,

$$\int_{\mathbb{R}^3} |w(x)|dx = \int_{\mathbb{R}^3} dx \left| \int_{\mathbb{R}^3} G(x-y)(-F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w])(y)dy \right| \le$$
$$\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} G(x-y)| - F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w]|(y)dy =$$
$$\int_{\mathbb{R}^3} | - F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w]|(y)dy \int_{\mathbb{R}^3} G(x-y)dx.$$

With a change of variables, we get that, for any $y \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3} G(x-y)dx = \int_{\mathbb{R}^3} G(z)dz$$

is a positive constant, thus it remains to estimate the other integral. We can see that, by construction

$$F(v_{\varepsilon,\phi}) = \chi_5 F(\tilde{v}_{\varepsilon,\phi}) - 2(\nabla\chi_5, \nabla\tilde{v}_{\varepsilon,\phi}) - \Delta\chi_5\tilde{v}_{\varepsilon,\phi}$$
$$+(-\Delta + W''(v_{\varepsilon,\phi}))\big(W'(v + (1-\chi_5)(\mathbb{H} - v_{\varepsilon,\phi})) - \chi_5 W'(v_{\varepsilon,\phi})\big),$$

therefore it is supported in a ball of radius c/ε, for some constant $c > 0$, and it is bounded by a constant times ε^3, hence

$$\int_{\mathbb{R}^3} |F(v_{\varepsilon,\phi})|dy = \int_{B_{c/\varepsilon}(0)} |F(v_{\varepsilon,\phi})|dy \le c|B_{c/\varepsilon}(0)|\varepsilon^3 \le c.$$

A similar estimate holds for the linear term, since

$$((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))w = 2(W''(v) - 2)\Delta w + 2(\nabla(W''(v) - 2), \nabla w) + \Delta(W''(v) - 2)w$$
$$+(4 - W''(v)^2)w - W^{(3)}(v)(-\Delta v + W'(v))w,$$

is supported in the same ball and it is bounded, in $L^\infty(\mathbb{R}^3)$ norm, by a constant times ε^3. The term involving $Q_{\varepsilon,\phi}(w)$ is the most tricky. It is explicitly given by

$$\int_{\mathbb{R}^3} |Q_{\varepsilon,\phi}(w)(y)|dy \le \int_{\mathbb{R}^3}\int_0^1 dt \int_0^t |\Delta(W''(v_{\varepsilon,\phi} + sw)w^2)$$
$$+2W'''(v_{\varepsilon,\phi} + sw)(-\Delta w + W''(v_{\varepsilon,\phi} + sw)w) +$$
$$+(W''(v_{\varepsilon,\phi} + sw)W'''(v_{\varepsilon,\phi} + sw) + W^{(4)}(v_{\varepsilon,\phi} + sw)(-\Delta(v_{\varepsilon,\phi} + sw) + W'(v_{\varepsilon,\phi} + sw)))w^2|ds \le$$
$$c\varepsilon^3\bigg(\int_{\mathbb{R}^3}(|w(y)| + |\nabla w(y)|)dy\bigg).$$

In conclusion,

$$\int_{\mathbb{R}^3} |w(x)|dx \leq c\Big(1 + \varepsilon^3 \int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx\Big) \tag{3.139}$$

Similarly, the first derivatives of w satisfy

$$\partial_j w(x) = \int_{\mathbb{R}^3} \partial_j G(x-y)(-F(v_{\varepsilon,\phi}) - Q_{\varepsilon,\phi}(w) + ((-\Delta+2)^2 - F'(v_{\varepsilon,\phi}))[w])(y)dy,$$

hence the gradient satisfies an estimate like (3.139), thus

$$\int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx \leq c\Big(1 + \varepsilon^3 \int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx\Big)$$

that is

$$\int_{\mathbb{R}^3} (|w(x)| + |\nabla w(x)|)dx \leq c. \tag{3.140}$$

In particular, (3.140) yields that

$$\left|\int_{\mathbb{R}^3} (1 - v_{\varepsilon,\phi})w dx\right| \leq c \int_{\mathbb{R}^3} |w|dx \leq c$$
$$\left|\int_{\mathbb{R}^3} w^2 dx\right| \leq c\varepsilon^3 \int_{\mathbb{R}^3} |w|dx \leq c\varepsilon^3.$$

The Lipschitz dependence on the data follows from similar arguments.

3.8.2 The proof of Proposition 89

Before giving the proof, we state a technical Lemma, in which we prove that the term $\tilde{p}_4$ is small enough.

Lemma 102. *For any $\varepsilon > 0$ small enough, for any ϕ, ϕ_1, ϕ_2 satisfying $|\phi|_{C^{4,\alpha}(\Sigma)}, |\phi_1|_{C^{4,\alpha}(\Sigma)}, |\phi_2|_{C^{4,\alpha}(\Sigma)} < 1$, for any $|\lambda|, |\lambda_1|, |\lambda_2| < 1$, we have*

$$\begin{cases} |\tilde{p}_4(\phi,\lambda)|_{C^{0,\alpha}(\Sigma)} \leq c\varepsilon^5 \\ |\tilde{p}_4(\phi_1,\lambda_1) - \tilde{p}_4(\phi_2,\lambda_2)|_{C^{0,\alpha}(\Sigma)} \leq c\varepsilon^5(|\phi_1 - \phi_2|_{C^{0,\alpha}(\Sigma)} + |\lambda_1 - \lambda_2|), \end{cases}$$

for some constant $c > 0$.

Proof. We write $U = U_o + U_e$, where we have set, for the sake of simplicity, $U := U_{\varepsilon,\phi,\lambda}$. By Proposition 87, we know that $||U_o||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^4$, therefore $||\mathrm{R}_{\varepsilon,\phi}(U_o)||_{C^{4,\alpha}_\delta(\Sigma_\varepsilon\times\mathbb{R})} \leq c\varepsilon^5$, since all the coefficients of $\mathrm{R}_{\varepsilon,\phi}$ are at least of order ε. It remains to deal with the even part U_e. We will see that all the terms of order ε^4 in the expression of $\mathrm{R}_{\varepsilon,\phi}(U_e)$ will vanish after projection. This can be seen by a direct computation

$$\begin{aligned}\mathrm{R}_{\varepsilon,\phi}(U_e) = \varepsilon\big\{&HW'''(v_\star)v_\star' U_e - \Delta_{\Sigma_\varepsilon}(\partial_t U_e + a_1^{ij}\partial_{ij}U_e t)\\ &+H\partial_{ttt}U_e + W''(v_\star)(H\partial_t U_e + a_1^{ij}\partial_{ij}U_e t)\\ &+(H\partial_t + t a_1^{ij}\partial_{ij})\mathcal{L}_\varepsilon U_e + \tilde{R}_{\varepsilon,\phi}(U_e)\big\},\end{aligned}$$

where $\tilde{R}_{\varepsilon,\phi}(U_e)$ is some linear operator with coefficients of order at least ε^2. All the terms of order ε are odd, thus they vanish when we multiply by $v_\star'$ and integrate, the other ones give rise to terms of order ε^5, being $U_{\varepsilon,\phi}$ of order ε^3. □

Remark 103. *Before proving Proposition 89, we point out that*

$$q^1_\varepsilon(\phi,\lambda)(y) = (\frac{1}{c_\star}\mathcal{H} + \varepsilon^{-5}\tilde{p}_4(\phi,\lambda))(y)$$

$$q^2_\varepsilon(\phi,\lambda)(y) = \left(\frac{1}{c_\star}\mathcal{G}_{\varepsilon,\phi,\lambda} + \varepsilon^{-6}(\tilde{p}_1(\phi,\lambda) + \tilde{p}_2(\phi,\lambda) + \tilde{p}_3(\phi,\lambda))\right)(y)$$

actually satisfy (3.80) and (3.81). For the notations, see Remark 93 and Proposition 92.

Now we are ready to prove Proposition 89.

Proof. In view of Proposition 92, the system of equations (3.79) and (3.84) is equivalent to the fixed point problem

$$\begin{aligned}(\phi,\lambda) = T_3(\phi,\lambda) := \mathcal{L}^{-1}\Big(&\varepsilon q^1_\varepsilon(\phi, c_\star\lambda/2)(y) + \varepsilon^2 q^2_\varepsilon(\phi, c_\star\lambda/2)(y),\\ &4\sqrt{2}\pi^2\varepsilon\int_0^\infty t(1-v_\star(t))dt + \varepsilon^2 G_\varepsilon(\phi, c_\star\lambda/2)\Big),\end{aligned}$$

Using the properties (3.80), (3.81), Proposition 88 and (3.84), we we can show show that T_3 is a contraction on the ball

$$\Lambda_3 := \{(\phi,\lambda)\in C^{4,\alpha}(\Sigma)_{s,0}\times\mathbb{R} : |\phi|_{C^{4,\alpha}(\Sigma)} + |\lambda| < C_3\varepsilon\},$$

provided C_3 is large enough.

□

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