Scuola Internazionale Superiore di Studi Avanzati

Area of Mathematics

Symplectic isotopy conjecture for elliptic ruled surfaces

Ph.D. Thesis

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Contents

0. Introduction

Abstract:

Three problems are studied in this thesis; the first problem is about four-dimensional symplectic manifolds. It was formulated by McDuff and Salamon in the very latest edition of their famous book [McD-Sa-1]. This problem is to prove that the Torelli part of the symplectic mapping class group of a geometrically ruled surface is trivial. In *Section 0* a partial solution for this problem is given, see *Theorem 0.2*. The proof can be found in my recent joint work with Vsevolod Shevchishin [S-S], though the proof there is a bit sketchy. The goal of this thesis is to provide the reader with more details.

The second problem is to compute the symplectic mapping class group of the one-point blow-up of $S^2 \times T^2$, the direct product of the 2-sphere S^2 and the 2-torus T^2. A partial solution to this problem is given in *Section 3*, see also my joint work with Shevchishin [S-S]. Namely, it is proved that the abelianization of the corresponding symplectic mapping class group is $\mathbb{Z}^2$, see *Theorem 3.3*.

The third problem has nothing to do with symplectic geometry, it is purely topological. This problem studies necessary and sufficient conditions for the existence of Lorentzian cobordisms between closed smooth manifolds of arbitrary dimension such that the structure group of the cobordism is $Spin(1,n)_0$, see *Theorem 4.2*. This extends a result of Gibbons-Hawking on $\boldsymbol{Sl}(2,\mathbb{C})$-Lorentzian cobordisms between 3-manifolds and results of Reinhart and Sorkin on the existence of Lorentzian cobordisms. We compute the kernel of the inclusion forgetful homomorphism from $Spin(1,n)_0$-Lorentzian cobordism ring to the spin cobordism ring, see *Corollary 4.9*. The proof is explained very carefully in my recent joint work with Rafael Torres [S-T]. Here the explanation tends to be briefly, see [S-T] for more details.

This thesis is devoted to study four-dimensional symplectic manifolds, focusing on their symplectomorphisms and symplectic submanifolds. It is intended to bring us one step closer to a complete description of the homotopy type of the symplectomorphism groups of some 4-manifolds, with a particular regard to rational and ruled surfaces.

Given a compact closed symplectic manifold (X,ω), the symplectomorphism group $\mathscr{S}ymp(X,\omega)$ consists of all diffeomorphisms that preserve the symplectic form ω. The group is equipped naturally with the C^∞-topology. By McDuff-Salamon, see [McD-Sa-4], Remark 9.5.6, this group has the homotopy type of a countable CW-complex. It is an interesting and important problem in symplectic topology to understand the homotopy type of this group in detail.

In the case of 2-dimensional manifolds it follows from classical arguments that $\mathscr{S}ymp(X,\omega)$ is homotopy equivalent to the group $\mathscr{D}iff_+(X)$ of orientation-preserving diffeomorphisms of X, and the latter group is very well understood. However, in dimension 4, there is no general description for $\mathscr{S}ymp(X,\omega)$ (as well as for $\mathscr{D}iff_+(X)$), though there are 4-manifolds for which the homotopy type of $\mathscr{S}ymp(X,\omega)$ can be described. The study of topological properties of $\mathscr{S}ymp(X,\omega)$ goes back to the seminal paper of Gromov [Gro], where it was proved that the group $\mathscr{S}ymp_c(\mathbb{R}^4,\omega)$ of compactly supported symplectomorphisms of $\mathbb{R}^4$ is contractible, while the group $\mathscr{S}ymp(\mathbb{CP}^2,\omega)$ is homotopy equivalent to the projective unitary group $\boldsymbol{PU}(3)$.

Since the foundational paper of Gromov [Gro], symplectomorphism groups of 4-manifolds have been extensively studied by many mathematicians, see [Ab-McD, AG, AL, McD-B, Bu, Eli, H-Iv, Kh, LLW, McD-3, Sei1, Sh-4, T] and references therein. Probably one of the major developments in symplectic topology is Gromov's technique of pseudoholomorphic curves. By using pseudoholomorphic curves, Abreu and McDuff developed a framework that helps to attack the problem of computing the symplectomorphism groups for rational and ruled surfaces, and then obtained a number of results toward to this problem, see [Ab-McD, McD-B]. Although a tremendous amount of work has been done in this direction, this problem is largely open and it keeps being the subject of vigorous ongoing activities in symplectic geometry. So far there is no one-size-fits-all method that would work for every ruled surface. It appears that the more complicated the topology of a rational or ruled surface gets, the more deeply one should be involved in complex algebraic geometry [Ab-McD], geometric topology [Lal-Pin, Pin-1], group theory [Li-1], Seiberg-Witten invariants [Kh], Gromov-Witten invariants and quantum cohomology [AL] in order to compute the corresponding symplectomorphism groups. The use of tools of all these branches of mathematics, on the one hand, makes the theory of symplectomorphism groups of rational and ruled surfaces pretty technical, but on the other hand, truly interdisciplinary and challenging.

Although I have made an effort to illustrate all main ideas of symplectic topology of ruled surfaces here, the reader will benefit immensely from the following works on symplectomorphism groups: [Bu, Ev, Sh-4], which are very closely related to the topic of the thesis. The reader also might find it useful to take a look at McDuff-Salamon books on symplectic geometry [McD-Sa-1, McD-Sa-3].

0.1. **Symplectic isotopy conjecture.** Recall that a symplectic manifold is a pair (X,ω), where X is a smooth manifold and ω is a closed non-degenerate 2-form on X. Here we assume that X is four-dimensional and closed.

A symplectomorphism of (X,ω) is a diffeomorphism $f\colon X \to X$ such that $f^*\omega = \omega$. Given a symplectic manifold (X,ω), the set of its symplectomorphisms is naturally equipped with the C^∞-topology and forms a topological group $\mathscr{Symp}(X,\omega)$. Every symplectic manifold possesses a great amount of symplectomorphisms. Indeed, every smooth function $f \in C^\infty(X)$ gives rise to a vector field $\mathsf{sgrad} f$ as follows:

$$\mathrm{d}\, f(\boldsymbol{\xi}) = \omega(\boldsymbol{\xi}, \mathsf{sgrad}\, f) \quad \text{for arbitrary vector filed } \boldsymbol{\xi}.$$

The flow generated by $\mathsf{sgrad} f$ preserves the symplectic structure. Therefore every smooth function $f \in C^\infty(X,\omega)$ corresponds to a certain one-dimensional group of symplectomorphisms of (X,ω).

Although the group $\mathscr{Symp}(X,\omega)$ is infinite-dimensional, it has foreseeable homotopy groups. The thesis focuses on the symplectic mapping class group $\pi_0\,\mathscr{Symp}(X,\omega)$. A fundamental problem in symplectic topology is to understand the kernel and the image of the following homomorphism

$$\pi_0\,\mathscr{Symp}(X,\omega) \to \pi_0\,\mathscr{Diff}(X). \tag{0.1}$$

The kernel of this homomorphism will be called the *(reduced) symplectic mapping class group* of X. To say this group is trivial means that every symplectomorphism that is smoothly isotopic to the identity is isotopic to the identity within the symplectomorphism group. There are 4-manifolds for which this kernel is expected to be trivial. It was conjectured by McDuff and Salamon that all geometrically ruled surfaces have this injectivity property, see Problem 14 in [McD-Sa-1].

Problem 0.1 (Symplectic isotopy conjecture for ruled surfaces). Every symplectomorphism of a geometrically ruled surface is symplectically isotopic to the identity if and only if it is smoothly isotopic to the identity.

This conjecture is known to hold for $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. This follows essentially from the explanation given by Gromov in his seminal paper [Gro]. However, a much more comprehensive treatment of these cases was worked out by Abreu [AbTh] and Abreu-McDuff [Ab-McD]. As far as I know, this conjecture is open for the case of irrational ruled surfaces, though it was partially proved by McDuff under certain constraints on symplectic forms, see [McD-B]. Here I will give a proof for elliptic ruled surfaces. We denote by $S^2 \tilde{\times} T^2$ the total space of the non-trivial S^2-bundle over T^2.

Theorem 0.2. *The Symplectic isotopy conjecture holds for $S^2 \times T^2$ and $S^2 \tilde{\times} T^2$.*

Note that the conjecture was proved by McDuff for $S^2 \times T^2$, see *Proposition 1.5* in [McD-B].

Surprisingly, the proof for $S^2 \tilde{\times} T^2$ is much more complicated than for $S^2 \times T^2$. This is because a symplectic $S^2 \tilde{\times} T^2$ may contain an embedded symplectic (-1)-torus. In [S-S] it is shown that if a symplectic 4-manifold (X, ω_0) contains a symplectic torus T of self-intersection number (-1), then one can construct a family of symplectic forms ω_t on X such that

$$\int_T \omega_t < 0 \quad \text{for } t \text{ sufficiently large.}$$

and an ω_t-symplectomorphism $E_C \colon X \to X$ called the *elliptic twist along* T. This symplectomorphism E_T is constructed to be smoothly isotopic to the identity. However, it is possible in principle that such a symplectomorphism E_T is not symplectically isotopic to the identity. While elliptic twists appear to exist for $S^2 \tilde{\times} T^2$, we will prove them to be symplectically trivial, see *Section 2.7*.

With the absence of elliptic twists for $S^2 \times T^2$, it becomes easy to prove the conjecture for $S^2 \times T^2$. An argument similar to what is presented in *Section 2.7* works, so few details are given here.

The methods to be used to prove *Theorem 0.2* were introduced by Abreu-McDuff [Ab-McD] and then extended further by McDuff [McD-B].

0.2. **The Diagram.** One can attack the problem of computing a symplectic mapping class group $\pi_0 \mathscr{S}ymp(X,\omega)$ by using the fibration first introduced by Kronheimer [Kh]. This fibration is as follows:

$$\mathscr{S}ymp(X,\omega) \cap \mathscr{D}iff_0(X) \to \mathscr{D}iff_0(X) \to \Omega(X,\omega), \tag{0.2}$$

where $\mathscr{D}iff_0(X)$ is the identity component of the group of diffeomorphisms of X and $\Omega(X,\omega)$ is the space of all symplectic forms on X in the class of $[\omega]$. Here the last arrow stands for the map

$$\mathscr{D}iff_0(X) \xrightarrow{\varphi} \Omega(X,\omega) \colon f \to f_*\omega. \tag{0.3}$$

To shorten notation, we write $\mathscr{S}ymp^*(X,\omega)$ instead of $\mathscr{S}ymp(X,\omega) \cap \mathscr{D}iff_0(X)$. Clearly, we have this homotopy fibration sequence

$$\ldots \to \pi_1(\mathscr{D}iff_0(X)) \xrightarrow{\varphi_*} \pi_1(\Omega(X,\omega)) \xrightarrow{\partial} \pi_0(\mathscr{S}ymp^*(X,\omega)) \to 0, \tag{0.4}$$

where by ∂ we denote the boundary homomorphism.

Let us define and denote by $\mathcal{J}(X,\omega)$ the space of those J for which there exists a taming symplectic form $\omega_J \in \Omega(X,\omega)$.

Lemma 0.3 (McDuff, see [McD-B]). *The space $\mathcal{J}(X,\omega)$ is canonically homotopy equivalent to $\Omega(X,\omega)$.*

Proof. This proof is taken from [McD-B]. Let us consider the space of pairs

$$(\Omega,\mathcal{J}) = \{\ (\omega,J) \in \Omega(X,\omega) \times \mathcal{J}(X,\omega) \mid \omega \text{ tames } J\ \}.$$

This space is naturally equipped with two projections

$$\begin{tikzcd} (\Omega,\mathcal{J}) \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{J}(X,\omega) \\ \Omega(X,\omega) \arrow[ur, "\psi"] & \end{tikzcd} \tag{0.5}$$

Both p_1 and p_2 are projections with contractible fibers. The reader is invited to check that both p_1 and p_2 do satisfy the homotopy lifting property.

Given $\omega \in \Omega(X,\omega)$, then $p_1^{-1}(\omega)$ is the space of ω-tamed almost-complex structures. It is Gromov's observation that the latter space is contractible.

On the other hand, given $J \in \mathcal{J}(X,\omega)$, then $p_2^{-1}(J)$ is nothing but the space of symplectic forms which tame J. The latter space is convex and hence contractible.

Since p_1 and p_2 are homotopy equivalences, there is a unique homotopy equivalence

$$\psi\colon \Omega(X,\omega) \to \mathcal{J}(X,\omega). \tag{0.6}$$

such that (0.5) becomes a homotopy commutative diagram. □

Further, let us consider one more homotopy commutative diagram

$$\begin{tikzcd} \mathscr{D}\!\mathit{iff}_0(X) \arrow[r, "\varphi"] \arrow[dr, "\nu"] & \Omega(X,\omega) \arrow[d, "\psi"] \\ & \mathcal{J}(X,\omega), \end{tikzcd} \tag{0.7}$$

where the diagonal arrow is for the map

$$\nu : \mathscr{D}\!\mathit{iff}_0(X) \to \mathcal{J}(X,\omega) \qquad \text{with} \qquad \nu : f \mapsto f_* J, \tag{0.8}$$

for an arbitrarily chosen ω-tamed almost-complex structure J. Following the fundamental idea of Gromov's theory [Gro] we study the space $\mathcal{J}(X,\omega)$ rather than $\Omega(X,\omega)$. We see from the following diagram

$$\begin{tikzcd} \ldots \arrow[r] & \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \arrow[r, "\varphi_*"] \arrow[d, "\mathsf{id}"'] & \pi_1(\Omega(X,\omega)) \arrow[r, "\partial"] \arrow[d, "\psi_*"'] & \pi_0(\mathscr{S}\!\mathit{ymp}^*(X,\omega)) \arrow[r] & 0 \\ \ldots \arrow[r] & \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \arrow[r, "\nu_*"] & \pi_1(\mathcal{J}(X,\omega)), \end{tikzcd} \tag{0.9}$$

that each loop in $\mathcal{J}(X,\omega)$ contributes to the symplectic mapping class group of X, provided this loop does not come from $\mathscr{D}\!\mathit{iff}_0(X)$. We will refer to (0.9) as a *fundamental diagram*. It will become our main tool to prove *Theorem 0.2*. The reader is referred to [McD-B] for more extensive discussion of the topic.

In what follows we work with a slightly bigger space $\mathcal{J}^k(X,\omega)$ of C^k-smooth almost-complex structures. The reason to do this is that the space $\mathcal{J}^k(X,\omega)$ is a Banach manifold while the space of C^∞-smooth structures $\mathcal{J}(X,\omega)$ is not of that kind. What we prove for $\pi_i(\mathcal{J}^k(X,\omega))$ works perfectly for $\pi_i(\mathcal{J}(X,\omega))$ because the inclusion $\mathcal{J}(X,\omega) \hookrightarrow \mathcal{J}^k(X,\omega)$ induces the weak homotopy equivalence $\pi_i(\mathcal{J}(X,\omega)) \to \pi_i(\mathcal{J}^k(X,\omega))$.

1. Short intro to Gromov's theory

Let us review some basic statements on pseudoholomorphic curves. The reader is referred to [Iv-Sh-1] for a comprehensive introduction to Gromov's theory of pseudoholomorphic curves, where the Gromov compactness theorem and a number of other important statements were generalized as much as reasonable.

1.1. **Basic.** Recall that an *almost-complex structure* J on a manifold X is an endomorphism of $J\colon TX \to TX$ such that $J^2 = -\mathsf{id}$. An almost-complex structure J on a symplectic manifold (X,ω) is called ω-tamed if

$$\omega(J\boldsymbol{\xi},\boldsymbol{\xi}) > 0 \quad \text{for arbitrary non-zero vector field } \boldsymbol{\xi}.$$

It is very well-known that the space $\mathcal{J}_\omega$ of ω-tamed almost-complex structures is non-empty and contractible, see e.g. [Iv-Sh-1]. In particular, every two ω-tamed almost-complex structures J_0 and J_1 can be connected by a homotopy (by path) J_t, $t \in [0,1]$, inside J_ω. A relative analogue of this states that, for a given compact $K \subset X$ and an ω-tamed almost-complex structure J defined over K, one can always extend J to be an ω-tamed almost-complex structure over the whole X. Moreover, the space $\mathcal{J}_\omega^K$ of such extensions is contractible. In particular, every two extensions $J_0, J_1 \in \mathcal{J}_\omega^K$ can be connected by a homotopy (by path) $J_t \in \mathcal{J}_\omega^K$.

A *parametrized J-holomorphic curve* in X is a C^1-map $u\colon S \to X$ from a (connected) Riemann surface S with a complex structure j on S to (X,J) that satisfy the Cauchy-Riemann equation:

$$\partial_J u = \frac{1}{2}(\mathrm{d}\,u \circ j - J \circ \mathrm{d}\,u) = 0. \tag{1.1}$$

Through out this work we mainly consider (but not restrict ourselves to) the case of embedded curves, i.e. we assume that $u\colon S \to X$ is an embedding.

In what follows we need an appropriate version of the elliptic regularity property for pseudoholomorphic curves.

Take α that satisfy $0 < \alpha < 1$. Let $\Delta \subset \mathbb{C}$ be a unit disk, and let $C^{k,\alpha}(\Delta,\mathbb{C})$ be the space of C^k-functions from Δ to $\mathbb{C}$ with k-th derivative Hölder continuous, exponent α. One puts

$$\|f\|_{k,\alpha} := \|f\|_{C^k} + \mathsf{sup}_{x\neq y}\frac{\left\|D^k f(x) - D^k f(y)\right\|}{|x-y|^\alpha},$$

so $C^{k,\alpha}(\Delta,\mathbb{C})$ becomes a Banach space. Here $D^k f$ denotes the vector of derivatives of f of order k. In a similar way, one can define the space $C^{k,\alpha}(S,X)$ of $C^{k,\alpha}$-smooth maps from S to X, as well as the space of $C^{k,\alpha}$-smooth almost-complex structures on X.

Theorem 1.1 (Elliptic regularity, see e.g. [Iv-Sh-1], Corollary 3.2.2)**.** *If J is $C^{k,\alpha}$-smooth, then every J-holomorphic curve is $C^{k+1,\alpha}$-smooth.*

Since we work with smooth almost-complex structure, we can restrict attention to smooth curves.

Although a definition was given for parametrized curves, what we are really interested in is something different. Let us take S to be $\mathbb{CP}^1$, and let $C := u(S)$. We say that C is a non-parametrized J-curve and denote by $\boldsymbol{C} := [C]$ the homology class given by u. From the topological point of view it becomes much more convenient to work with non-parametrized curves, simply because the space of parametrized curves is too big. Indeed, consider the space $\mathcal{M}_J(C)$ of J-holomorphic curves $u\colon S \to X$ such that $[u(S)] = \boldsymbol{C}$. If $u \in \mathcal{M}_J(\boldsymbol{C})$, then $u \circ g \in \mathcal{M}_J(\boldsymbol{C})$, where $g \in \boldsymbol{PSL}(2,\mathbb{C})$. Therefore the group $G = \boldsymbol{PSL}(2,\mathbb{C})$ acts freely on $\mathcal{M}_J(C)$ by reparametrization; the quotient $\mathscr{M}_J(\boldsymbol{C}) := \mathcal{M}_J(\boldsymbol{C})/G$ is nothing but the space of non-parametrized J-curves, and it appears to be not that big, see *§1.5*.

The situation becomes more subtle when S is of genus greater than zero. This is because non-zero genus surfaces have many complex structures. However, when it goes to non-parametrized curves of non-zero genus it does not really matter what complex structure is chosen on S. We count them as if they were biholomorphic. A little technical work is needed to define the moduli space of non-parametrized curves properly, see [Iv-Sh-1].

It follows very easily from the definition of the tameness condition that every J-holomorphic emdedding $u\colon S \to X$ with $J \in J_\omega$ is symplectic. To some extent, the converse is also true: every C^{k+1}-smooth symplectic embedding is J-holomorphic for an appropriate C^k-smooth ω-tamed almost-complex structure, though for immersions the situation is a bit more complicated.

Lemma 1.2 (see e.g. [Iv-Sh-1], Lemma 1.4.2)**.** *Let $u\colon S \to X$ be an ω-symplectic C^1-smooth immersion such that $u(S)$ has only simple transversal positive self-intersections, then there exists an ω-tamed almost-complex structure J on X and a complex structure j on S such that $u\colon (S,j) \to (X,J)$ becomes J-holomorphic.*

1.2. **Positivity of Intersections.** If $x \in X$ is a self-intersection point of a J-curve u, $x = u(z_1) = u(x_2)$, $z_1, z_2 \in S$, $z_1 \neq z_2$, or an intersection point of two J-curves u_1 and u_2, $z_1 \in S_1$, $z_2 \in S_2$, such that the tangent planes $\mathrm{d}\,u(T_{x_i}) \subset T_xX$ are transversal to each other and complex w.r.t. J_x in T_xX, then the intersection number of planes $\mathrm{d}\,u(T_{z_i})$ at x is positive. This positivity intersection property will be constantly used in the sequel, without any particular reference. Note, however, that it is possible for two symplectic planes in $\mathbb{R}^4$ to intersect each other negatively.

1.3. **Adjunction formula.** If $u\colon S \to X$ is J-holomorphic, and $C := u(S)$, define the virtual genus of C to be

$$\mathsf{g}(C) = 1 + \frac{1}{2}([C]^2 - K_X^*(C)), \tag{1.2}$$

where K_X^* is for the anti-canonical class of (X, J). The so-called *adjunction inequality* holds

$$\mathsf{g}(C) \geqslant g(S) = \text{genus of } S \tag{1.3}$$

with the equality iff u is an embedding. In the latter case this equality is called *the adjunction formula*.

Example: let $u\colon S \to X$ be J-holomorphic, and take S to be $\mathbb{CP}^1$. If $[C]^2 - K_X^*(C) = -2$, then u should be an embedding! We will use the adjunction formula literally in every section, often without mentioning it specifically.

1.4. **Sard's Lemma.** The purpose of this subsection is to recall Sard's lemma for infinite-dimensional manifolds. The lemma was proved by Smale in his seminal paper [Sm], and it is known to be an important tool in studying elliptic equations and, in particular, the Cauchy-Riemann equation. Here we present a very short summary of the results of [Sm], no details are given here.

Recall that a *Fredholm* operator is a continuous linear map $L\colon E_1 \to E_2$ from one Banach space to another with the properties:

(1) $\mathsf{dim}\,\mathsf{Ker}\, L < \infty$

(2) $\mathsf{Im}\, L$ is closed

(3) $\mathsf{dim}\,\mathsf{Coker}\, L < \infty$

If L is Fredholm, then its index is defined to be equal to

$$\mathsf{ind}\, L := \mathsf{dim}\,\mathsf{Ker}\, L - \mathsf{dim}\,\mathsf{Coker}\, L.$$

The set $\mathcal{F}(E_1, E_2)$ of Fredholm operators is open in the space of all continuous operators $\mathcal{L}(E_1, E_2)$ in the norm topology. Furthermore the index is a locally constant function on $\mathcal{F}(E_1, E_2)$. See e.g. [McD-Sa-4].

We now consider differentiable Banach manifolds X and Y. We will assume our manifolds to be connected and to have countable base.

A *Fredholm map* is a C^1-map $f\colon X \to Y$ such that at each $x \in M$, the derivative $f_*(x)\colon T_xX \to T_{f(x)}Y$ is a Fredholm operator. The index of f is defined to be the index of $f_*(x)$ for some $x \in X$. Since X is connected, the definition does not depend on x.

Let $f\colon X \to Y$ be a C^1-map. A point $x \in X$ is called a *regular point* of f if $f_*(x)$ is an epimorphism and is *singular* if not regular. The images of singular points under f are called *singular values* and their complement the *regular values*.

Theorem 1.3 (Smale, [Sm], Theorem 1.3)**.** *Let $f\colon X \to Y$ be a C^q-Fredholm map with $q > \mathsf{max}(\mathsf{ind} f, 0)$. Then the regular values of f are almost all of Y.*

This theorem implies

Theorem 1.4 (Smale, [Sm], Corollary 1.5)**.** *If $f\colon X \to Y$ is a C^q-Fredholm map with $q > \mathsf{max}(\mathsf{ind} f, 0)$, then for almost all $y \in Y$, $f^{-1}(y)$ is a submanifold of X whose dimension is equal to index of f or it is empty.*

If fact, a more general statement holds true. Let $f\colon X \to Y$ be a C^1-map and $g\colon Z \to Y$ be a C^1-embedding. We shall say that f is *transversal* to g if for each $y \in g(Z)$ and $x \in X$, $f(x) = y$ the spaces $\mathsf{Im}\, f_*(x)$, $T_y g(Z)$, span the tangent space T_yY.

Theorem 1.5 (Smale, [Sm], Theorems 3.1 & 3.3)**.** *Let $f\colon X \to Y$ be a C^q-Fredholm map and $g\colon Z \to Y$ be a C^1-embedding of a finite dimensional manifold Z with*

$$q > \mathsf{max}(\mathsf{ind} f + \mathsf{dim}\, Z, 0).$$

i) Then there exists a C^1-approximation g' of g such that f is transversal to g'. Furthermore if f is transversal to the restriction of g to a closed subset A of Y, then g' may be chosen so that $g' = g$ on A.

ii) If f is transversal to g, then $f^{-1}(g(Z))$ is a submanifold of X of dimension equal to $\mathsf{ind} f + \mathsf{dim}\, Z$.

In his paper [Sm], Smale describes a notion of a generalized degree for a proper Fredholm map. Here we give a short summary of his construction for the case $\mathsf{ind} f = 0$. Let $f\colon X \to Y$ be a proper C^2-Fredholm map, $\mathsf{ind} f = 0$. Given a generic $y \in Y$, then $f^{-1}(y)$ is discrete, and if f is proper, is a finite number of points. We define a *generalized degree* of f to be the number of point of $f^{-1}(y)$ modulo 2

$$\mathsf{deg}\,(f) = \# f^{-1}(y) \bmod 2.$$

To see that $\operatorname{deg}(f)$ is independent of y let y_1, be another regular value of f and suppose $g\colon I \to Y$ is an embedded path with $g(0) = y$, $g(1) = y_1$. By *Theorem 1.5* we suppose f is transversal to g, so $f^{-1}(g(I))$ is a one-dimensional manifold. This gives us the invariance of $\operatorname{deg}(f)$.

1.5. **Compactness.** In general, the moduli space of non-parametrized curves $\mathscr{M}_J(\boldsymbol{C})$ fails to be compact. However, Gromov's compactness theorem asserts that it can be compactified in some nice way. It states that if a sequence J_i of $C^{k,\alpha}$-structures converges uniformly to an ω-tamed almost-complex structure J_∞, and if $u_i\colon S \to X$, $C_i := u_i(S)$ are (non-parametrized, and, for simplicity, embedded) J_i-holomorphic curves, then there exists a subsequence C_j that converges to a J_∞-holomorphic curve or a cusp-curve C_∞. To say that C_∞ is a cusp-curve means that it may be non-smooth, reducible, or having multiple components.

Here the concept of convergence is understood with respect to the Gromov topology on the space of stable maps. For a precise definition of this convergence with estimates in neighbourhoods of vanishing circles the reader is referred to [Iv-Sh-1]. Note, however, that it is fine for our purpose to use the ***cycle topology***. A sequence C_i of pseudoholomorphic curves is said to converge to C_∞ with respect to the cycle topology if

$$\lim_{i\to\infty} \int_{C_i} \varphi = \int_{C_\infty} \varphi$$

for every 2-form φ on X. In fact, even a less subtle topology, the Hausdorff topology, would do. We may equip X with some Riemannian metric, so it becomes a complete metric space. Then a sequence C_i of pseudoholomorphic curves is said to converge to C_∞ with respect to the Hausdorff topology if

$$\lim_{i\to\infty} \operatorname{dist}(C_i, C_\infty) = 0.$$

Although the limit C_∞ need not be embedded, the elliptic regularity ensures that C_∞ has reasonable smoothness, provided J_∞ is smooth.

Importantly, if all C_i are of homology class $\boldsymbol{C} \in \mathsf{H}_2(X;\mathbb{Z})$, then so is C_∞. Thus, if we take $J_i = J$, then the theorem ensures some compactification for $\mathscr{M}_J(\boldsymbol{C})$, by cusp-curves.

Note that if every J-curve of class $\boldsymbol{C} \in \mathsf{H}_2(X;\mathbb{Z})$ is embedded, then the moduli space $\mathscr{M}_J(\boldsymbol{C})$ is automatically compact. An important example of this phenomenon is given by irrational geometrically ruled surfaces, which are orientable S^2-bundles over compact Riemann surfaces of non-zero genus, see *Theorem 2.8*.

1.6. **Universal moduli space.** Here and below "smoothness" means some $C^{k,\alpha}$-smoothness with $0<\alpha<1$ and k natural sufficiently large.

Let $\mathcal{J}$ be an open subset in the Banach manifold of all $C^{k,\alpha}$-smooth almost-complex structures on X for k sufficiently large. The interesting examples are given by the set $\mathcal{J}_\omega$ of ω-tamed almost-complex structures, or, even more useful for us, by the $\mathcal{J}(X,\omega)$ of those structures which are tamed by some form in $\Omega(X,\omega)$; in the latter example the symplectic forms need not be the same.

Fix also a homology class $\boldsymbol{C}\in H_2(X;\mathbb{Z})$. Denote by $\mathcal{M}(\boldsymbol{C})$, or by $\mathcal{M}$ for short, the space of pairs (J,C), where $J\in\mathcal{J}$, C is a non-parametrized immersed irreducible non-multiple J-holomorphic curve of genus g and of class $\boldsymbol{C}$. The space $\mathcal{M}(\boldsymbol{C})$ is a smooth Banach manifold and the natural projection $pr\colon \mathcal{M}(\boldsymbol{C})\to\mathcal{J}$ is a smooth Fredholm map of $\mathbb{R}$-index

$$\mathsf{ind}(\mathsf{pr}) = 2\,(K_X^*(\boldsymbol{C})+g-1)\,, \tag{1.4}$$

where as usual K_X^* is for the anti-canonical class of $J\in\mathcal{J}$; it does not actually depend on $J\in\mathcal{J}$, provided that $\mathcal{J}$ is connected.

We normally denote $\mathcal{M}_J(\boldsymbol{C}) := \mathsf{pr}^{-1}(J)$, or by $\mathcal{M}_J$ for short. In the case of a path $h\colon I\to\mathcal{J}$ with $J_t:=h(t)$, $t\in I$ we denote

$$\mathcal{M}_h := \{(C,J_t,t)\mid t\in I,\ J_t=h(t),\ (C,J_t)\in\mathcal{M}_{J_t}\}\,.$$

Lemma 1.6 (see [Sh-4], Lemma 1.1)**.** *i) In the case* $\mathsf{ind}(\mathsf{pr})\leqslant 0$ *the map* $\mathsf{pr}\colon\mathcal{M}\to\mathcal{J}$ *is a smooth immersion of codimension* $-\mathsf{ind}(\mathsf{pr})$*, and an embedding in the case when pseudoholomorphic curves* C *in* $\mathcal{M}$ *are embedded and* $\boldsymbol{C}^2<0$*; the image* $\mathsf{pr}(\mathcal{M})$ *can be naturally cooriented.*

ii) In the case $\mathsf{ind}(\mathsf{pr})\geqslant 0$ *and* $\mathcal{M}$ *is non-empty, there is a subset* $\mathcal{J}_{\mathrm{reg}}\subset\mathcal{J}$ *of second Baire category such that for every* $J\in\mathcal{J}_{\mathrm{reg}}$ *the set* $\mathcal{M}_J$ *is a naturally oriented smooth manifold of dimension* $\mathsf{ind}(\mathsf{pr})$*, for a generic path* $h\colon I\to\mathcal{J}$ *the space* $\mathcal{M}_h$ *is a smooth manifold of dimension* $\mathsf{ind}(\mathsf{pr})+1$*, and so on.*

iii) If, in addition to ii) , C *is* $\mathbb{CP}^1$*, then* $\mathcal{J}_{\mathrm{reg}}=\mathcal{J}$*.*

1.7. **Counting tori.** Given a class $\boldsymbol{C}\in H_2(X;\mathbb{Z})$, we study the space $\mathcal{M}_J$, or briefly $\mathcal{M}$, of non-parametrized J-curves representing the class $\boldsymbol{C}$ in X. It follows from *Lemma 1.6* that, for a sufficiently generic almost-complex structure J on X, the space $\mathcal{M}$ is in fact a compact (strictly speaking, it can be compactified) manifold of dimension $\mathsf{dim}\,\mathscr{M} = K_X^*(\boldsymbol{C}) + \boldsymbol{C}^2$.

Assume that the class $\boldsymbol{C}$ can be represented by a smooth symplectic torus of self-intersection number 0, then, for generic J, the moduli space $\mathscr{M}$ is an oriented 0-dimensional manifold, and hence its points can be counted, with signs to be determined by the orientation of $\mathscr{M}$; see [Tb] where it is explained how this orientation is chosen. This counting is especially easy to do when every J-holomorphic representative of $\boldsymbol{C}$ is irreducible. Fortunately for us, we only will deal with an irreducible case, see section *§2.4*. The result of this counting is the Gromov invariant

$$\mathsf{Gr}(\boldsymbol{C}) = \#\mathscr{M}.$$

Importantly, if J is integrable, all points of $\mathscr{M}$ have positive signs, so a good strategy for counting curves is to work with J integrable. This been said, keep in mind that "generic J" might not be "the integrable choice of J". Indeed, let us consider $X \cong \mathbb{CP}^1 \times E$, where E is some smooth elliptic curve; being fibered by elliptic curves, X has infinitely many holomorphic tori homologous to each other, and these tori are of self-intersection number zero. Be that as it may, the general theory promises that we should have finitely many J-tori for J generically chosen.

To achieve transversality one has to find a complex structure J such that every J-torus of class $\boldsymbol{C}$ has the holomorphically non-trivial normal bundle; this also ensures that (X,J) contains only finitely many holomorphic tori of class $\boldsymbol{C}$, see [McD-E] where the tori counting technique discussed in details and it is explained how to deal with multiple tori. A toy example of computation for Gr is given in *§2.2*.

1.8. **Symplectic economics.** Here we give a brief description of the inflation technique developed by Lalonde-McDuff [La-McD] and McDuff [McD-B], and a generalization of this procedure given by Buşe, see [Bu].

Theorem 1.7 (Inflation)**.** *Let J be an ω_0-tamed almost complex structure on a symplectic 4-manifold (X,ω_0) that admits an embedded J-holomorphic curve C with $[C]\cdot[C] \geqslant 0$. Then there is a family $\omega_t, t \geqslant 0$, of symplectic forms that all tame J and have cohomology class*

$$[\omega_t] = [\omega_0] + t\,\mathsf{PD}([C]),$$

where $\mathsf{PD}([C])$ is Poincaré dual to $[C]$.

For negative curves a somewhat reverse procedure exists, called negative inflation or deflation.

Theorem 1.8 (Deflation)**.** *Let J be an ω_0-tamed almost complex structure on a symplectic 4-manifold (X,ω_0) that admits an embedded J-holomorphic curve C with $[C]\cdot[C]=-m$. Then there is a family ω_t of symplectic forms that all tame J and have cohomology class*

$$[\omega_t]=[\omega_0]+t\,\mathsf{PD}([C])$$

for all $0\leqslant t<\dfrac{\omega_0([C])}{m}$.

2. Elliptic geometrically ruled surfaces

2.1. **General remarks.** A complex surface X is ruled means that there exists a holomorphic map $\pi:X\to Y$ to a Riemann surface Y such that each fiber $\pi^{-1}(y)$ is a rational curve; if, in addition, each fiber is irreducible, then X is called geometrically ruled. A ruled surface is obtained by blowing up a geometrically ruled surface. Note however that a geometrically ruled surface needs not be minimal (the blow up of $\mathbb{CP}^2$, denoted by $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, is de facto a unique example of a geometrically ruled surface that is not a minimal one). Unless otherwise noted, all ruled surfaces are assumed to be geometrically ruled. One can speak of the genus of the ruled surface X, meaning thereby the genus of Y. We thus have rational ruled surfaces, elliptic ruled surfaces and so on.

Topologically, there are two types of orientable S^2-bundles over a Riemann surface: the product $S^2\times Y$ and the nontrivial bundle $S^2\tilde{\times}Y$. The product bundle admits sections Y_{2k} of even self-intersection number $[Y_{2k}]^2=2k$, and the skew-product admits sections Y_{2k+1} of odd self-intersection number $[Y_{2k+1}]^2=2k+1$. We will choose the basis $\boldsymbol{Y}=[Y_0],\boldsymbol{S}=[\mathrm{pt}\times S^2]$ for $\mathsf{H}_2(S^2\times Y;\mathbb{Z})$, and use the basis $\boldsymbol{Y}_-=[Y_{-1}],\boldsymbol{Y}_+=[Y_1]$ for $\mathsf{H}_2(S^2\tilde{\times}Y;\mathbb{Z})$. To simplify notations, we denote both the classes $\boldsymbol{S}$ and $\boldsymbol{Y}_+-\boldsymbol{Y}_-$, which are the fiber classes of the ruling, by $\boldsymbol{F}$. Further, the class $\boldsymbol{Y}_++\boldsymbol{Y}_-$, which is a class for a bisection of X, will be of particular interest for us, and will be widely used in forthcoming computations; we denote this class by $\boldsymbol{B}$. Throughout this paper we will freely identify homology and cohomology by Poincaré duality.

Clearly, we have $[Y_{2k}]=\boldsymbol{Y}+k\boldsymbol{F}$ and $[Y_{2k+1}]=\boldsymbol{Y}_++k(\boldsymbol{Y}_++\boldsymbol{Y}_-)$. This can be seen by evaluating the intersection forms for these 4-manifolds on the given basis:

$$\mathcal{Q}_{S^2\times Y}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \mathcal{Q}_{S^2\tilde{\times}Y}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Observe that these forms are non-isomorphic. That is why the manifolds $S^2 \times Y$ and $S^2 \tilde{\times} Y$ are non-diffeomorphic. One more way to express the difference between them is to note that the product $S^2 \times Y$ is a spin 4-manifold, but the skew-product $S^2 \tilde{\times} Y$ is not of that kind. This divergence, however, is sort of fragile: only one blow-up is needed to make them diffeomorphic: $S^2 \times Y \# \overline{\mathbb{CP}}^2 \simeq S^2 \tilde{\times} Y \# \overline{\mathbb{CP}}^2$.

This section is mainly about the non-spin elliptic ruled surface $S^2 \tilde{\times} T^2$. When study this manifold we sometimes use the notations $\boldsymbol{T}_+$ and $\boldsymbol{T}_-$ instead of $\boldsymbol{Y}_+$ and $\boldsymbol{Y}_-$ for the standard homology basis in $\mathsf{H}^2(S^2 \tilde{\times} T^2; \mathbb{Z})$.

From the algebro-geometric viewpoint every such X is a holomorphic $\mathbb{CP}^1$-bundle over a Riemann surface Y whose structure group is $\boldsymbol{PGl}(2, \mathbb{C})$. Biholomorphic classification of ruled surfaces is well understood, at least for low values of the genus. Below we recall part of the classification of elliptic ruled surfaces given by Atiyah in [At-2]; this being the first step towards understanding almost complex geometry of these surfaces. We also provided a short summary of Suwa's results: *i)* an explicit construction of a complex analytic family of ruled surfaces, where one can see the jump phenomenon of complex structures, see §2.3, *ii)* an examination of those complex surfaces which are both ruled and admit an elliptic pencil, see *Theorem 2.3*.

It maybe should be mentioned that no matter what complex structure we are dealing with, the formula describing the anticanonical class of a geometrically ruled surface is as follows:

$$K_X^*(S^2 \times Y) = 2\boldsymbol{Y} + \chi(Y)\boldsymbol{S}, \quad K_X^*(S^2 \tilde{\times} Y) = (1 + \chi(Y))\boldsymbol{Y}_+ + (1 - \chi(Y))\boldsymbol{Y}_-. \tag{2.1}$$

The symplectic geometry of ruled surfaces has been extensively studied by many authors, see [AGK, Li-Li, Li-Liu-1, Li-Liu-2] and references therein. Ruled surfaces are of great interest from the symplectic point of view mainly because of the following significant result due to Lalonde-McDuff, see [La-McD].

Theorem 2.1 (The classification of ruled 4-manifolds)**.** *Let X be oriented diffeomorphic to a minimal rational or ruled surface, and let $\xi \in \mathsf{H}^2(X)$. Then there is a symplectic form (even a Kähler one) on X in the class ξ iff $\xi^2 > 0$. Moreover, any two symplectic forms in class ξ are diffeomorphic.*

Thus all symplectic properties of ruled surfaces depend only on the cohomology class of a symplectic form. In particular, the symplectic blow-up of a symplectic ruled surface depends only on the size of the exceptional curve; this does not hold for a general symplectic 4-manifold, where one can possibly find two symplectic balls of the same volume that are not translated to each other by means of a symplectomorphism.

Let (X,ω_μ) be a symplectic ruled 4-manifold $(S^2\tilde{\times}T^2,\omega_\mu)$, where ω_μ is a symplectic structure of the cohomology class $[\omega_\mu] = \boldsymbol{T}_+ - \mu\boldsymbol{T}_-$, $\mu \in (-1,1)$. Here the coefficient of $\boldsymbol{T}_+$ is 1 because one can always rescale a symplectic form ω on X to get $\int_{\boldsymbol{T}_+}\omega = 1$. By *Theorem 2.1* (X,ω_μ) is well-defined up to symplectomorphism. As promised in the introduction, we will prove that $\pi_0(\mathscr{S}ymp^*(X,\omega_\mu))$ is trivial.

Given $\mu > 0$, then the space (X,ω_μ) contains a symplectic (-1)-torus. This simplifies life drastically. Following McDuff [McD-B], we will show that the group $\pi_0(\mathscr{S}ymp^*(X,\xi))$ coincides with the mapping class group of certain diffeomorphisms, see *Lemma 2.16*; the latter group can be computed to be trivial by standard topological technique, see *Proposition 2.12*.

When $\mu \leqslant 0$, we first embed $\mathcal{J}(X,\omega_\mu)$ in the bigger space $\mathcal{J}$. We then show that *i)* $\pi_1(\mathscr{D}iff(X)) \to \pi_1(\mathcal{J})$ is epimorpic, *ii)* $\mathcal{J}(X,\omega_\mu)$ is complenet of $\mathcal{J}$ to a certain codimension 2 divisor in $\mathcal{J}$, and finally prove that $\pi_1(\mathscr{D}iff(X)) \to \pi_1(\mathcal{J}(X,\omega_\mu))$ is epimorpic as well.

Let (X,ω_λ) be a symplectic ruled 4-manifold $(S^2\times T^2,\omega_\lambda)$, where $[\omega_\lambda] = \boldsymbol{T} + \lambda\boldsymbol{S}$, $\lambda \in (0,+\infty)$. We will prove $\pi_0(\mathscr{S}ymp^*(X,\omega_\lambda)) = 0$ for every λ positive. The reader is invited to look at the proof for $S^2\tilde{\times}T^2$ first. Then it will be explained how this proof can be adapted for $S^2\times T^2$.

As it was mentioned, the 4-manifold $S^2\times T^2$ is spin. We say "spin case" when indicate that we are considering $S^2\times T^2$. The manifold $S^2\tilde{\times}T^2$ is no spin and unless otherwise noted, it is the one we refer to when say "non-spin case".

To study homotopy groups one needs to choose some basepoint. We first take the connected component of $\Omega(X,\omega_\mu)$ that contains ω_μ and use the same notation $\Omega(X,\omega_\mu)$ for this component. We do the same for $\mathcal{J}(X,\omega_\mu)$. Now $\Omega(X,\omega_\mu)$ is connected, and the group $\pi_1(\Omega(X,\omega_\mu))$ does not depend on a basepoint. We have an obvious choice of a basepoint for $\Omega(X,\omega_\mu)$, since X is equipped with the form ω_μ. What remains is to choose a basepoint for $\mathcal{J}(X,\omega_\mu)$. By *Theorem 2.1* every symplectic form on X is Kähler, so there exists at least one integrable complex structure $J \in \mathcal{J}(X,\omega_\mu)$ such that (X,ω_μ,J) is a Kähler manifold. It seems aesthetically correct to choose J to be a basepoint of $\mathcal{J}(X,\omega_\mu)$ for our forthcoming computations.

2.2. **Classification of complex surfaces ruled over elliptic curves.** Here we very briefly describe possible complex structures on elliptic ruled surfaces and study some of their properties.

Let X be diffeomorphic to either $S^2 \times Y^2$ or $S^2 \tilde{\times} Y^2$. The Enriques-Kodaira classification of complex surfaces (see e.g.[BHPV]) ensures the following:

(1) Every complex surface X of this diffeomorphism type is algebraic and hence Kähler.
(2) Every such complex surface X is ruled, i.e. there exists a holomorphic map $\pi\colon X \to Y$ such that Y is a complex curve, and each fiber $\pi^{-1}(y)$ is an irreducible rational curve. Note that, with the single exception of $\mathbb{CP}^1 \times \mathbb{CP}^1$, a ruled surface admits at most one ruling.

It was shown by Atiyah [At-2] that every holomorphic $\mathbb{CP}^1$-bundle over a curve Y with structure group the projective group $\boldsymbol{PGl}(2,\mathbb{C})$ admits a holomorphic section, and hence the structure group of such bundle can be reduced to the affine group $\boldsymbol{Aff}(1,\mathbb{C}) \subset \boldsymbol{PGl}(2,\mathbb{C})$.

All of what is said works perfectly for any ruled surface, no matter the genus. Keep in mind, however, that everything below is for genus one surfaces. It was Atiyah who gave a classification of ruled surfaces with the base elliptic curve. The description presented here is taken from [Sw].

Theorem 2.2 (Atiyah)**.** *Every $\boldsymbol{PGl}(2,\mathbb{C})$-bundle over an elliptic curve can expressed uniquely as one of the following:*

i) *a $\mathbb{C}^*$-bundle of nonpositive degree,*
ii) $\mathcal{A}$,
iii) $\mathcal{A}^{Spin}$,

where $\mathcal{A}^{Spin}$ and $\mathcal{A}$ are affine bundles.

We shall proceed with a little discussion of these bundles:

i) With a bit luck the structure group of a $\mathbb{CP}^1$-bundle can be reduced further to $\mathbb{C}^* \subset \boldsymbol{Aff}(1,\mathbb{C}) \subset \boldsymbol{PGl}(2,\mathbb{C})$. We now give an explicit description of such bundles.

Let $y \in Y$ be a point on the curve Y, and let $\{V_0, V_1\}$ be an open cover of Y such that $V_0 = Y \setminus \{y\}$ and V_1 is a small neighbourhood of y, so the domain $V_0 \cap V_1 =: \hat{V}$ is a punctured disk. We choose a multivalued coordinate u on Y centered at y.

A surface X_k associated to the line bundle $\mathscr{O}(ky)$ (or if desired, a $\mathbb{C}^*$-bundle) can be described as follows:

$$X := \big(V_0 \times \mathbb{CP}^1\big) \cup \big(V_1 \times \mathbb{CP}^1\big) \,/ \sim,$$

where $(u, z_0) \in V_1 \times \mathbb{CP}^1$ and $(u, z_1) \in V_2 \times \mathbb{CP}^1$ are identified iff $u \in \hat{V},\ z_1 = z_0 u^k$. Here z_0, z_1 are inhomogeneous coordinates on $\mathbb{CP}^1$'s.

Clearly, the biholomorphism $(z_0, u) \to (z_0^{-1}, u)$, $(z_1, u) \to (z_1^{-1}, u)$ maps X_k to X_{-k}. Thus it is sufficient to consider only values of k that are nonpositive.

There is a natural $\mathbb{C}^*$-action on X_k via $g \cdot (z_0, u) := (gz_0, u)$, $g \cdot (z_1, u) := (gz_1, u)$ for each $g \in \mathbb{C}^*$. The fixed point set of this action consists of two mutually disjoint sections Y_k and Y_{-k} defined respectively by $z_0 = z_1 = 0$ and $z_0 = z_1 = \infty$. We have $[Y_k]^2 = k$ and $[Y_{-k}]^2 = -k$.

It is very well known that any line bundle L of degree $\mathsf{deg}\,(L) = k \neq 0$ is isomorphic to $\mathscr{O}(ky)$ for some $y \in Y$. Thus all the ruled surfaces associated with line bundles of non-zero degree k are biholomorphic to one and the same surface X_k.

On the other hand, the parity of the degree of the underlying line bundle is a topological invariant of a ruled surface. More precisely, a ruled surface X associated with a line bundle L is diffeomorphic to $Y \times S^2$ for $\mathsf{deg}\,(L)$ even, and to $Y \tilde{\times} S^2$ for $\mathsf{deg}\,(L)$ odd.

ii) Again, we start with an explicit description of the ruled surface $X_{\mathcal{A}}$ associated with the affine bundle $\mathcal{A}$. Let $\{V_0, V_1, \hat{V}\}$ be the open cover of Y as before, u be a coordinate on Y centered at y, and z_0, z_1 be fiber coordinates. Define

$$X_{\mathcal{A}} := \big(V_0 \times \mathbb{CP}^1\big) \cup \big(V_1 \times \mathbb{CP}^1\big) \,/ \sim,$$

where $(z_0, u) \sim (z_0, u)$ for $u \in \hat{V}$ and $z_0 = z_1 u + u^{-1}$.

There is an obvious section Y_1 defined by the equation $z_0 = z_1 = \infty$, but in contrast to $\mathbb{C}^*$-bundles, the surface $X_{\mathcal{A}}$ contains no section disjoint from that one. This can be shown by means of direct computation in local coordinates, but one easily deduce this from *Theorem* 2.3 below.

We will make repeated use of the following geometric characterization of $X_{\mathcal{A}}$, whose proof is given in [Sw], see *Theorem 5*.

Theorem 2.3. *The surface $X_{\mathcal{A}}$ associated with the affine bundle $\mathcal{A}$ has fibering of smooth elliptic curves. The elliptic structure is uniquely specified by $X_{\mathcal{A}}$: the base curve is the* *rational one, a general fiber is a torus in class* $2\boldsymbol{Y}_+ + 2\boldsymbol{Y}_-$*, there are 3 double fibers, and there are no other multiple fibers.*

The following corollary will be used later. The reader is invited to look at [McD-D] for the definition of the Gromov invariants and some examples of their computation.

Corollary 2.4. $\mathsf{Gr}(\boldsymbol{Y}_+ + \boldsymbol{Y}_-) = 3$.

Proof. There are no smooth curves in class $\boldsymbol{Y}_+ + \boldsymbol{Y}_-$ apart from those three curves which are the double fibers of the elliptic fibration described above. Since each of these curves is a double fiber of an elliptic fibering, its normal bundle is isomorphic to the square root of the trivial bundle and hence the transversality property holds for it, see *§1.7* and [McD-E]. □

Based on this theorem, Suwa then gives another construction of $X_{\mathcal{A}}$. We mention this construction here because it appears to have interest for the sequel.

Let us identify Y with the quotient $\mathbb{C}/\Lambda$, where Λ is a lattice in $\mathbb{C}$, a discrete additive subgroup $\mathbb{Z}\oplus\mathbb{Z}\subset\mathbb{C}$. Consider the representation

$$\mathbb{Z}\oplus\mathbb{Z}\to \boldsymbol{PGl}(2,\mathbb{C})\colon (n,m)\to f^n g^m, \tag{2.2}$$

where $f, g \in \boldsymbol{PGl}(2,\mathbb{C})$ form a pair of nonidentity distinct commuting involutions, say

$$f\colon z\to -z, \quad g\colon z\to \frac{1}{z}. \tag{2.3}$$

The product $\mathbb{C}\times\mathbb{CP}^1$ can be equipped with a free $\mathbb{Z}^2$ action in such a way that $\mathbb{Z}^2$ acts as a translation on the the first factor, and by automorphisms f, g on the second one.

It is clear that the quotient $X_{\mathcal{A}} := \mathbb{C}\times_{\mathbb{Z}^2}\mathbb{CP}^1$ is an elliptic ruled surface equipped with the ruling $X_{\mathcal{A}} \to \mathbb{C}/\Lambda$. This surface is non-spin, for an explanation, see [Sw], where this is proved by constructing a section for $X_{\mathcal{A}}$ of odd self-intersection number, see also [McD-Sa-1] for one different explanation, see *Exercises 6.13 and 6.14.*

Further, because $\mathbb{Z}^2$ acts by elements of order 2, this gives rise to an effective action of $\mathcal{T} = \mathbb{C}/2\Lambda$, which is a complex torus, on $X_{\mathcal{A}}$. The desired fibering is constructed by means of this action. It consists of regular fibers, where the action is free, and of three multiple fibers, whose isotropy groups correspond to the three pairwise different order two subgroups of $\mathcal{T}$.

iii) The ruled surface associated to $\mathcal{A}^{Spin}$ is diffeomorphic to $S^2\times T^2$, thus it is not discussed here, but see [Sw].

Summarizing our above observations we see that $X \cong S^2 \tilde{\times} T^2$ admits countably many complex structures. These structures are as follows:

- the structures $J \in \mathscr{J}_{1-2k}, k > 0$, such that the ruled surface (X, J), which is biholomorphic to X_{1-2k}, contains a section of self-intersection number $1-2k$, and
- the affine structures $J \in \mathscr{J}_{\mathcal{A}}$ such that the ruled surface $J \in (X, J)$, which is biholomorphic to $X_{\mathcal{A}}$, contains no sections of negative self-intersection number but does contain a triple of smooth bisections.

2.3. **One family of ruled surfaces over elliptic base.** Here is a construction of a one-parametric complex-analytic family $p\colon \mathfrak{X} \to \mathbb{C}$ of non-spin elliptic ruled surfaces, such that the surfaces $p^{-1}(t), t \neq 0$, are biholomorphic to $X_{\mathcal{A}}$ and $p^{-1}(0) \cong X_{-1}$.

As before, we take a point y on Y, let u be a coordinate of the center y, and put $\{V_0, V_1, \hat{V}\}$ to be an open cover for Y such that $V_0 := Y \setminus \{y\}$, V_1 is a small neighbourhood of y, and $\hat{V} := V_0 \cap V_1$. Further, let Δ be a complex plane, and let t be a coordinate on it.

We construct the complex 3-manifold $\mathfrak{X}$ by patching $\Delta \times V_0 \times \mathbb{CP}^1$ and $\Delta \times V_1 \times \mathbb{CP}^1$ in such a way that $(t, z_0, u) \sim (t, z_0, u)$ for $u \in \hat{V}$ and $z_0 = z_1 u + t u^{-1}$.

The preimage of 0 and 1 under the natural projection $p\colon \mathfrak{X} \to \Delta$ are biholomorphic respectively to X_{-1} and $X_{\mathcal{A}}$. In fact, it is not hard to see that for each $t \neq 0$, the surface $p^{-1}(t)$ is biholomorphic to $X_{\mathcal{A}}$ as well. One way to prove this is to use the $\mathbb{C}^*$-action on $\mathfrak{X}$

$$g \cdot (t, z_0, u) := (tg^{-1}, gz_0, u),\, g \cdot (t, uz_1, u) := (tg^{-1}, gz_1, u) \quad \text{for each } g \in \mathbb{C}.$$

This proves even more than we desired, namely, that there exists a $\mathbb{C}^*$-action on $\mathfrak{X}$ such that for each $g \in \mathbb{C}^*$ we get a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\cdot g"] \arrow[d, "p"'] & \mathfrak{X} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "\cdot g"'] & \mathbb{C}, \end{tikzcd} \tag{2.4}$$

where $\mathfrak{X} \xrightarrow{\cdot g} \mathfrak{X}$ denotes the biholomorphism induced by $g \in \mathbb{C}^*$.

The construction of the complex-analytic family $\mathfrak{X}$ is due to Suwa, see [Sw]. However, the existence of the $\mathbb{C}^*$-action on $\mathfrak{X}$ was not mentioned explicitly in his paper.

2.4. **Embedded curves and almost complex structures.** In §2.2 the classification for non-spin elliptic ruled surfaces was given. It turns out that this classification can be extended to the almost complex geometry of $S^2\tilde{\times}T^2$.

Let X be diffeomorphic to $S^2\tilde{\times}T^2$, and let $\mathcal{J}(X)$ be the space of almost complex structures on X that are tamed by some symplectic form; the symplectic forms need not be the same. Here we use the short notation $\mathcal{J}$ for $\mathcal{J}(X)$.

Given $k>0$, let $\mathcal{J}_{1-2k}(X)$ (for shorten, we will refer to it by $\mathcal{J}_{1-2k}$) be the subset of $J\in\mathcal{J}$ consisting of elements that admit a smooth irreducible J-holomorphic elliptic curve in the class $\boldsymbol{T}_+ - k\boldsymbol{F}$. By virtue of *Lemma 1.6* $\mathcal{J}_{1-2k}$ forms a subvariety of $\mathcal{J}$ of real codimension $2{\cdot}(2k-1)$.

Further, define $\mathcal{J}_{\mathcal{A}}(X)$ (or $\mathcal{J}_{\mathcal{A}}$, for short) be the subset $J\in\mathcal{J}$ of those element for which there exists a smooth irreducible J-holomorphic elliptic curve in the class $\boldsymbol{B}$.

By pretty straightforward computation one can show that the sets $\mathcal{J}_{1-2k}$ are mutually disjoint, and each $\mathcal{J}_{1-2k}$ is disjoint from $\mathcal{J}_{\mathcal{A}}$. Further, it is not hard to see that $\mathcal{J}_{-1}\subset\overline{\mathcal{J}}_{\mathcal{A}}$ and $\mathcal{J}_{1-2(k+1)}\subset\overline{\mathcal{J}}_{1-2k}$, where $\overline{\mathcal{J}}_{1-2k}$ is for the closure of $\mathcal{J}_{1-2k}$. A less trivial fact is that

$$\mathcal{J}=\mathcal{J}_{\mathcal{A}}+\bigsqcup_{k=1}^{\infty}\mathcal{J}_{1-2k}, \tag{2.5}$$

it can be also stated as follows. Here both "+" and "$\bigsqcup$" are for the disjoint union.

Proposition 2.5 (cf. *Lemma 4.2* in [McD-B]). *Let (X,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}T^2$. Then every ω-tamed almost complex structure J admits a smooth irreducible J-holomorphic representative in either $\boldsymbol{B}$ or $\boldsymbol{T}_+ - k\boldsymbol{F}$ for some $k>0$.*

Proof. The proof is analogous to the one of *Lemma 4.2* in [McD-B]. Observe that the expected codimension for the class $\boldsymbol{B}$ is zero. By *Lemma 2.4* we have $\mathsf{Gr}(\boldsymbol{T}_+ + \boldsymbol{T}_-)>0$. Hence, $\mathcal{J}_{\mathcal{A}}$ is an open dense subset of $\mathcal{J}$, and, thanks to the Gromov compactness theorem, for each $J\in\mathcal{J}$ the class $\boldsymbol{B}$ has at least one J-holomorphic representative, possibly singular, reducible or having multiple components.

By virtue of *Theorem 2.8*, no matter what J was chosen, our manifold X admits the smooth J-holomorphic ruling π by rational curves in class $\boldsymbol{F}$.

Since $\boldsymbol{B}\cdot\boldsymbol{F}>0$, it follows from positivity of intersections that any J-holomorphic representative B of the class $\boldsymbol{B}$ must either intersect a J-holomorphic fiber of π or must contain this fiber completely.

a) First assume that B is irreducible. Then it is of genus not greater than 1 because of the adjunction formula. Assume that B is of genus zero, then it gives some nontrivial element in $\pi_2(X)$. However, one can use this homotopy exact sequence for π

$$0 = \pi_3(Y) \longrightarrow \pi_2(S^2) \longrightarrow \pi_2(X) \longrightarrow \pi_2(Y) = 0 \tag{2.6}$$

to deduce that every spherical homology class of X is proportional to $\boldsymbol{F}$. This contradicts our assumption that B has no fiber components. Therefore B is of genus one. It remains to apply the adjunction formula one more time to conclude that B is smooth, i.e. $J \in \mathcal{J}_{\mathcal{A}}$.

b) The curve B is reducible but contains no irreducible components which are the fibers of π. Then it contains precisely two components B_1 and B_2, since $\boldsymbol{B} \cdot \boldsymbol{F} = 2$. Both the curves B_1 and B_2 are smooth sections of π, and hence $[B_i] = \boldsymbol{T}_+ + k_i \boldsymbol{F}$, $i = 1, 2$. Since $[B_1] + [B_2] = \boldsymbol{B}$, it follows that $k_1 + k_2 = -1$, and hence either k_1 or k_2 is negative. Thus we have that either B_1 or B_2 is a smooth J-holomorphic section of negative self-intersection index.

c) If some of the irreducible components of B are in the fibers class $\boldsymbol{F}$, then one can apply arguments similar to that used in *a)* and *b)* to prove that the part B' of B which contains no fiber components has a section of negative self-intersection index as a component. □

When X is diffeomorphic to $S^2 \times T^2$, a somewhat similar statement holds. Namely:

Proposition 2.6 (cf. *Lemma 4.2* in [McD-B]). *Let (X, ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times T^2$. Then every ω-tamed almost complex structure J admits a smooth irreducible J-holomorphic representative in either $\boldsymbol{T}$ or $\boldsymbol{T} - k\,\boldsymbol{F}$ for some $k > 0$.*

Proof. The key ingredient here is the fact that $\mathsf{Gr}(\boldsymbol{T}) = 2$. This number was calculated by McDuff in [McD-E]. From here, one proceeds analogously to the proof of *Proposition 2.5*. □

In other words, we show that $\mathcal{J}$ decomposes as follows

$$\mathcal{J} = \mathcal{J}_{\mathcal{A}} + \bigsqcup_{k=1}^{\infty} \mathcal{J}_{-2k}, \tag{2.7}$$

thus completing the analogy with the non-spin case. Here $\mathcal{J}_{-2k} \subset \mathcal{J}$ is for the subset of almost-complex structures on X that admit a smooth pseudoholomorphic representative in class $\boldsymbol{T} - k\,\boldsymbol{F}$, and $\mathcal{J}_{\mathcal{A}}$ consists of structures that have a pseudoholomorphic curve of class $\boldsymbol{T}$. Importantly, $\mathcal{J}_{-2k}$ forms a submanifold of $\mathcal{J}$ of real codimension $2 \cdot 2k$. In particular, $\pi_1(\mathcal{J}_{\mathcal{A}}) \to \pi_1(\mathcal{J})$ is isomorphic in the spin case.

2.5. **Rulings and almost complex structures.** Let X be a ruled surface equipped with a ruling $\pi\colon X \to Y$, and let J be an almost complex structure on X. We shall say that J *is compatible with the ruling* $\pi\colon X \to Y$ if each fiber $\pi^{-1}(y)$ is J-holomorphic.

I take the opportunity to thank D. Alekseeva for sharing her proof of the following statement.

Proposition 2.7. *Let $\mathcal{J}(X,\pi)$ be the space of almost complex structures on X compatible with π.*

i) $\mathcal{J}(X,\pi)$ is contractible.

ii) Any structure $J \in \mathcal{J}(X,\pi)$, as well as any compact family $J_t \in \mathcal{J}(X,\pi)$, is tamed by some symplectic form.

Proof. *i)* Let be $\mathbb{J}(\mathbb{R}^4,\mathbb{R}^2)$ be the space of linear maps $J\colon \mathbb{R}^4 \to \mathbb{R}^2$ such that $J^2 = -\mathit{id}$ and $J(\mathbb{R}^2) = \mathbb{R}^2$, i.e. it is the space of linear complex structures preserving $\mathbb{R}^2$. In addition, we assume $\mathbb{R}^4$ and $\mathbb{R}^2$ are both oriented and each $J \in \mathbb{J}(\mathbb{R}^4,\mathbb{R}^2)$ induces the given orientations for both $\mathbb{R}^4$ and $\mathbb{R}^2$. We now prove the space $\mathbb{J}(\mathbb{R}^4,\mathbb{R}^2)$ is contractible.

Indeed, let us take $J \in \mathbb{J}(\mathbb{R}^4,\mathbb{R}^2)$. Fix two vectors $e_1 \in \mathbb{R}^2$ and $e_2 \in \mathbb{R}^4 \setminus \mathbb{R}^2$. The vectors e_1 and Je_1 form a positively oriented basis for $\mathbb{R}^2$. Therefore Je_1 is in the upper half-plane for e_1. Further, the vectors e_1, Je_1, e_2, Je_2 form a positively oriented basis for $\mathbb{R}^4$. Therefore Je_2 is in the upper half-space for the hyperplane spanned on e_1, Je_1, e_2.

We see that the space $\mathbb{J}(\mathbb{R}^4,\mathbb{R}^2)$ is homeomorphic to the direct product of two half-spaces, and hence it is for sure contractible.

To finish the proof of *i)* we consider the subbundle $V_x := \mathsf{Ker}\, d\pi(x) \subset T_xX$, $x \in X$, of the tangent bundle TX of X. Every $J \in \mathcal{J}(X,\pi)$ is a section of the bundle $\mathbb{J}(TX,V) \to X$ whose fiber over $x \in X$ is the space $\mathbb{J}(T_xX,V_x)$. Since the fibers of $\mathbb{J}(TX,V)$ are contractible; it follows that the space of section for $\mathbb{J}(TX,V)$ is contractible as well.

ii) Again, we start with some linear algebra. Let V be a 2-subspace of $W \cong \mathbb{R}^4$, and let $J \in \mathbb{J}(W,V)$. Choose a 2-form $\tau \in \Lambda^2(W)$ such that the restriction $\tau|_V \in \Lambda^2(V)$ of τ to V is positive with respect to the J-orientation of V, i.e. $\tau(\xi, J\xi) > 0$. Clearly, the subspace $H := \mathsf{Ker}\,\tau \subset W$ is a complement to V. Further, let $\sigma \in \Lambda^2(V)$ be any 2-form such that $\sigma|_V$ vanishes, but $\sigma|_H$ does not. If H is given the orientation induced by σ, then the J-orientation of W agrees with that defined by the direct sum decomposition $W \cong V \oplus H$. We now prove that J is tamed by $\tau + K\sigma$ for $K > 0$ sufficiently large.

It is easy to show that there exists a basis $e_1, e_2 \in V$, $e_3, e_4 \in H$ for W such that J takes the form

$$J = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrix Ω of $\tau + K\sigma$ with respect to this basis is block-diagonal, say

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & K\sigma + \ldots \\ 0 & 0 & -K\sigma + \ldots & 0 \end{pmatrix} \text{ for } \sigma > 0.$$

It remains to check that the matrix ΩJ is positive definite, i.e. $(\xi, \Omega J \xi) > 0$. A matrix is positive definite iff its symmetrization is positive definite. It is straightforward to check that $\Omega J + (\Omega J)^t$ is of that kind for K large enough.

Let us go back to the ruled surface X. The theorem of Thurston [Th] (see also *Theorem 6.3* in [McD-Sa-1]) ensures the existence of a closed 2-form τ on X such that the restrictions of τ to each fiber $\pi^{-1}(y)$ is non-degenerate. Choose an area form σ on Y. By the same reasoning as before, any $J \in \mathcal{J}(X, \pi)$ is tamed by $\tau + K\pi^*\sigma$ for K large enough. $\square$

The following theorem by McDuff motivates the study of compatible almost complex structures, see *Lemma 4.1* in [McD-B].

Theorem 2.8. *Let X be an irrational ruled surface, and let $J \in \mathcal{J}(X)$. Then there exists a unique ruling $\pi\colon X \to Y$ such that $J \in \mathcal{J}(X, \pi)$.*

2.6. **Diffeomorphisms.** Let X be diffeomorphic to either $Y \times S^2$ or $Y \tilde{\times} S^2$, and let $\pi\colon X \to Y$ be a smooth ruling; here Y is any Riemann surface. Further, let $\mathsf{Fol}(X)$ be the space of all smooth foliations of X by spheres in the fiber class $\boldsymbol{F}$.

The group $\mathscr{D}\!\mathit{iff}(X)$ acts transitively on $\mathsf{Fol}(X)$ as well as the group $\mathscr{D}\!\mathit{iff}_0(X)$ acts transitively on a connected component $\mathsf{Fol}_0(X)$ of $\mathsf{Fol}(X)$. This gives rise to a fibration sequence

$$\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X) \to \mathscr{D}\!\mathit{iff}_0(X) \to \mathsf{Fol}_0(X),$$

where $\mathcal{D}$ is the group of fiberwise diffeomorphisms of X. By the definition of $\mathcal{D}$ there exists a projection homomorphism $\tau\colon \mathcal{D} \to \mathscr{D}\!\mathit{iff}(Y)$ such that for every $F \in \mathcal{D}$ we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi} \\ Y & \xrightarrow[\tau(F)]{} & Y, \end{array} \tag{2.8}$$

which induces the corresponding commutative diagram for homology

$$\begin{array}{ccc} \mathsf{H}_1(X;\mathbb{Z}) & \xrightarrow{F_*} & \mathsf{H}_1(X;\mathbb{Z}) \\ {\scriptstyle\pi_*}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_*} \\ \mathsf{H}_1(Y;\mathbb{Z}) & \xrightarrow[\tau(F)_*]{} & \mathsf{H}_1(Y;\mathbb{Z}). \end{array} \tag{2.9}$$

Notice that $\tau(F)$ is isotopic to the identity only if $\tau(F)_* = \mathit{id}$. Since π_* is an isomorphism, it follows that the subgroup $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ of $\mathcal{D}$ is mapped by τ to $\mathscr{D}\!\mathit{iff}_0(Y)$, so we end up with the restricted projection homomorphism

$$\tau\colon \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X) \to \mathscr{D}\!\mathit{iff}_0(Y). \tag{2.10}$$

Since we shall exclusively be considering this restricted homomorphism, we use the same notation τ for this.

Given an isotopy $f_t \subset \mathscr{D}\!\mathit{iff}_0(Y), f_0 = \mathit{id}$, one can lift it to an isotopy $F_t \in \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X), F_0 = \mathit{id}$ such that $\tau(F_t) = f_t$. This immediately implies that the inclusion $\mathsf{Ker}\,\tau \in \mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ induces an epimorphism

$$\pi_0(\mathsf{Ker}\,\tau) \to \pi_0(\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)). \tag{2.11}$$

Because of this property we would like to look at the group $\mathsf{Ker}\,\tau$ in more detail, but first introduce some useful notion.

Let X be a smooth manifold, and let f be a self-diffeomorphism X. Define the *mapping torus* $T(X, f)$ as the quotient of $X \times [0,1]$ by the identification $(x,1) \sim (f(x),0)$. For the diffeomorphism f to be isotopic to identity it is necessary to have the mapping torus diffeomorphic to $T(X, \mathit{id}) \cong X \times S^1$.

Let us go back to the group $\mathsf{Ker}\,\tau$ that consists of bundle automorphisms of $\pi\colon X \to Y$. Let $F \in \mathsf{Ker}\,\tau$ be a bundle automorphism of π, and let γ be a simple closed curve on Y. By F_γ denote the restriction of F to $\pi^{-1}(\gamma) \cong S^1 \times S^2$. The mapping torus $T(\pi^{-1}(\gamma), F_\gamma)$ is either diffeomorphic to $S^2 \times T$ or $S^2 \tilde{\times} T$. In the later case we shall say that the automorphism F is *twisted* along γ.

Lemma 2.9. *Let X be diffeomorphic to either $Y \times S^2$ or $Y \tilde{\times} S^2$, and let $F \in \mathsf{Ker}\,\tau$. Then F is isotopic to the identity through $\mathsf{Ker}\,\tau$ iff Y contains no curve for F to be twisted along.*

Proof. A closed orientable genus g surface Y has a cell structure with one cell, $2g$ 1-cells, and one 2-cell. Clearly, F can be isotopically deformed to *id* over the 0-skeleton of Y. The obstruction for extending this isotopy to the 1-skeleton of Y is a well-defined cohomology class $c(F) \in \mathsf{H}^1(X;\mathbb{Z}_2)$; the obstruction cochain $c(F)$ is the cochain whose value on a 1-cell e equals 1 if F is twisted along e and 0 otherwise. It is evident that $c(F)$ is a cocycle.

By assumption $c(F) = 0$. Consequently there is an extension of our isotopy to an isotopy over a neighbourhood of the 1-skeleton of Y, but such an isotopy always can be extended to the rest of Y. □

A short way of represent the issue algebraically is by means of the *obstruction homomorphism*

$$c\colon \mathsf{Ker}\,\tau \to \mathsf{H}^1(X;\mathbb{Z}_2) \tag{2.12}$$

defined in the lemma; any two elements $F, G \in \mathsf{Ker}\,\tau$ are isotopic to each other through $\mathsf{Ker}\,\tau$ iff $c(F) = c(G)$.

Lemma 2.10. *Let X be diffeomorphic to $S^2 \times Y$, and let $F \in \mathsf{Ker}\,\tau$, then Y contains no curve for F to be twisted along. This means that the obstruction homomorphism is the null homomorphism.*

Proof. The converse would imply that the mapping torus $T(X,F)$ is not spin, but $T(X, id) \cong S^2 \times Y \times S^1$ is a spin 5-manifold. □

The following result is due to McDuff [McD-B], but a different proof follows by combining *Lemma* 2.10 with *Lemma* 2.9.

Proposition 2.11. *Let X be diffeomorphic to $S^2 \times Y^2$, then the group $\mathcal{D} \cap \mathscr{D}\!\mathit{iff}_0(X)$ is connected.*

In what follows we need a non-spin analogue of this Proposition for the case of elliptic ruled surfaces.

Proposition 2.12. *Let X be diffeomorphic to $S^2\tilde{\times}T^2$, then the group $\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X)$ is connected.*

Proof. Fix any cocycle $c\in \mathsf{H}^1(X;\mathbb{Z}_2)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$, then we claim there exists $F\in \mathsf{Ker}\,\tau$ such that $c(F)=c$ and, moreover, F is isotopic to *id* through diffeomorphisms in $\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X)$. It follows from Suwa's model, see §*2.2*, that the automorphism group for the complex ruled surface $X_{\mathcal{A}}$ contains the complex torus $\mathcal{T}$ as a subgroup. By construction, it is clear that $\mathcal{T}$ is a subgroup of $\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X)$. Besides that, the 2-torsion subgroup $\mathcal{T}_2\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$ of $\mathcal{T}$ is a subgroup of $\mathsf{Ker}\,\tau$. We trust the reader to check $\mathcal{T}_2$ is mapped isomorphically by the obstruction homomorphism to $\mathsf{H}^1(X;\mathbb{Z}_2)$.

The algebra behind this argument is expressed by a commutative diagram

$$\begin{array}{ccccc} \mathcal{T}_2 & \xrightarrow{i} & \mathsf{Ker}\,\tau & \xrightarrow{j} & \mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0(\mathcal{T}_2) & \xrightarrow{i_*} & \pi_0(\mathsf{Ker}\,\tau) & \xrightarrow{j_*} & \pi_0(\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X)), \end{array} \tag{2.13}$$

where i_* is an isomorphism, $j_*\circ i_*$ is the null homomorphism, and therefore j_* is the null homomorphism as well. But we already know that j_* is an isomorphism, and hence $\pi_0(\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X))$ is trivial. □

2.7. **Finishing the proof.** Here is the part where a proof of *Theorem 0.2* comes. We split it into a few pieces. Let X be the symplectic ruled 4-manifold $(S^2\tilde{\times}T^2,\omega_\mu)$, $[\omega_\mu]=\boldsymbol{T}_+-\mu\boldsymbol{T}_-$. Recall that the space $\mathcal{J}(X)$ was defined, see §*2.4*, to be the space of almost complex structures on X that are tamed by some symplectic form. Here this definition is somewhat modified; here we take the connected component of $\mathcal{J}(X)$ which contains the space $\mathcal{J}(X,\omega_\mu)$; the same applies to $\mathcal{J}_{\mathcal{A}}$ and $\mathcal{J}_{1-2k}$.

Lemma 2.13. $\mathcal{J}_{\mathcal{A}}\subset\mathcal{J}(X,\omega_\mu)$ *for every* $\mu\in(-1,1)$.

Proof. Given $\mu\in(-1,1)$, we first take some symplectic form ω taming J, and then rescale it in order to get $\int_{\boldsymbol{T}_+}\omega=1$. Set $\eta:=\int_{\boldsymbol{T}_-}\omega$.

i) If $\eta\geqslant\mu$, we inflate ω along a smooth J-holomorphic curve of class $\boldsymbol{B}$ (such a curve indeed exists because $J\in\mathcal{J}_{\mathcal{A}}$), so we obtain a form ω_t of the class $[\omega_t]:=[\omega]+t\boldsymbol{B}$. It is easy to check that $\int_{\boldsymbol{T}_-}\omega_t=\eta-t$ and $\int_{\boldsymbol{T}_+}\omega_t=1+t$. We rescale ω_t one more time to get $\hat{\omega}_t:=\frac{\omega_t}{1+t}$. We obtain $\int_{\boldsymbol{T}_-}\hat{\omega}_t=\frac{\eta-t}{1+t}$ and $\int_{\boldsymbol{T}_+}\hat{\omega}_t=1$, which means one should inflate till $t=\frac{\eta-\mu}{\mu+1}$. □

ii) If $\eta<\mu$, we inflate ω along a J-sphere in class $\boldsymbol{F}$ to get $\int_{\boldsymbol{T}_+}\frac{\omega_t}{1+t}=1$ and $\int_{\boldsymbol{T}_-}\frac{\omega_t}{1+t}=\frac{\eta+t}{1+t}$, and so we have to inflate till $t=\frac{\mu-\eta}{1-\mu}$.

Lemma 2.14. *$\mathcal{J}_{\mathcal{A}}=\mathcal{J}(X,\omega_\mu)$ for every $\mu\in(-1,0]$.*

Proof. It is clear that $\mathcal{J}(X,\omega_\mu)$ does no contain the structures $\mathcal{J}_{1-2k}$ for $\mu\in(-1,0]$, and hence by (2.5) and *Lemma* 2.13 the proof follows. $\square$

This means that there is no topology change for the space $\mathcal{J}(X,\omega_\mu)$ when μ is being varied in $(-1,0]$. In particular,

$$\pi_1(\mathcal{J}(X,\omega_\mu))=\pi_1(\mathcal{J}_{\mathcal{A}}(X))\quad\text{for } \mu\in(-1,0].$$

Lemma 2.15. *$\mathcal{J}_{1-2k}\subset\mathcal{J}(X,\omega_\mu)$ iff $\mu\in\left(1-\dfrac{1}{k},1\right)$.*

Proof. If $J\in\mathcal{J}_{1-2k}$, then it admits a smooth J-holomorphic representative in class $\boldsymbol{T}_+-k\boldsymbol{F}$. Now the "only if" part is obvious. Indeed, if J is ω-tamed, then $\int_{\boldsymbol{T}_+-k\boldsymbol{F}}\omega=1-k+k\mu>0$. The "if" can be proved by deflating along $\boldsymbol{T}_+-k\boldsymbol{F}$ and inflating along $\boldsymbol{F}$. $\square$

Combining *Lemma* 2.13 with *Lemma* 2.15, as well as the fact that the higher codimension submanifolds $\mathcal{J}_{2k-1}$, $k\geqslant 2$ do not affect the fundamental group of $\mathcal{J}(X,\omega_\mu)$, we see that there is no topology change in $\pi_1(\mathcal{J}(X,\omega_\mu))$ when μ is being varied in $(0,1)$, i.e. we have

$$\pi_1(\mathcal{J}(X,\omega_\mu))=\pi_1(\mathcal{J}(X))\quad\text{for } \mu\in(0,1). \tag{2.14}$$

To derive the symplectic mapping class group the expected way is to use the fundamental diagram (0.9) which requires to know the image of the homomorphism

$$\nu_*\colon \pi_1(\mathscr{D}\!\mathit{iff}_0(X))\to\pi_1(\mathcal{J}(X)). \tag{2.15}$$

Lemma 2.16. *ν_* is an epimorphism.*

Proof. Though the map $\nu\colon\mathscr{D}\!\mathit{iff}_0(X)\to\mathcal{J}(X)$ is not fibration, it can be extended to a homotopy fibration

$$\mathscr{D}\!\mathit{iff}_0(X)\to\mathcal{J}(X)\to\mathsf{Fol}_0(X), \tag{2.16}$$

where the last arrow is a homotopy equivalence, see *Theorem* 2.8 and *Proposition* 2.7. Thus we end up with the homotopy exact sequence

$$\ldots\to\pi_1(\mathscr{D}\!\mathit{iff}_0(X))\to\pi_1(\mathcal{J}(X))\to\pi_0(\mathcal{D}\cap\mathscr{D}\!\mathit{iff}_0(X)). \tag{2.17}$$

If X is of genus 1, the group $\pi_0(\mathcal{D}\cap\mathscr{Diff}_0(X))$ is trivial by *Propositions* 2.11 and 2.12. This finishes the proof. $\square$

The following corollary will not be used in the remainder of this thesis, but it is a very natural application of *Lemma 2.16*.

Corollary 2.17. *The space $\mathcal{J}(X)$ is homotopy simple. In other words, $\pi_1(\mathcal{J}(X))$ is abelian and acts trivially on $\pi_n(\mathcal{J}(X))$.*

Proof. The proof is straightforward from *Lemma 2.16*. $\square$

By virtue of the fundamental diagram (0.9) and (2.14), *Lemma 2.16* immediately implies

Proposition 2.18. $\pi_0(\mathscr{Symp}^*(X,\omega_\mu)) = 0$ *for every* $\mu \in (0,1)$.

In order to compute the group $\pi_0(\mathscr{Symp}^*(X,\omega_\mu))$ for $\mu \in (-1,0)$ it is necessary to know better the fundamental group of $\mathcal{J}_{\mathcal{A}}$. The space $\mathcal{J}_{\mathcal{A}}$ is the complement to (the closure of) the codimension 2 divisor $\mathcal{J}_{-1}$ in the ambient space $\mathcal{J}(X)$. We denote by i the inclusion

$$i\colon \mathcal{J}_{\mathcal{A}}(X) \to \mathcal{J}(X). \tag{2.18}$$

By *Lemma 2.16* every loop $J(t) \in \pi_1(\mathcal{J}_{\mathcal{A}})$ can be decomposed into a product $J(t) = J_0(t)\cdot J_1(t)$, where $J_0(t) \in \mathsf{Im}\,\nu_*$, and $J_1(t) \in \mathsf{Ker}\,i_*$.

Those loops which lie in $\mathsf{Ker}\,i_*$ could contribute drastically to the symplectic mapping class group via the corresponding elliptic twists. But this is what will not happen, because the following holds.

Lemma 2.19. $\mathsf{Ker}\,i_* \subset \mathsf{Im}\,\nu_*$. *Here ν_* is for $\nu_*: \pi_1(\mathscr{Diff}_0(X)) \to \pi_1(J_{\mathcal{A}}(X))$. This homomorphism is well defined, because the space $J_{\mathcal{A}}$ is invariant w.r.t. to the natural action of $\mathscr{Diff}_0(X)$.*

Proof. Choose some $J_* \in \mathcal{J}_{-1}$, and let Δ be a 2-disk which intersects $\mathcal{J}_{-1}$ transversally at the single point J_*. Denote by $J(t)$ the boundary of Δ. By *Lemma 2.20* one simply needs to show that the homotopy class of $J(t)$ comes from the natural action of $\mathscr{Diff}_0(X)$ on $\mathcal{J}_A$, and the lemma will follow.

If J_* is integrable, then one can choose Δ such that $J(t)$ is indeed an oribit of the action of a certain loop in $\mathscr{Diff}_0(X)$, see the description of the complex-analytic family constructed in *§2.3*. Thus it remains to check that every structure $\mathcal{J}_* \in \mathcal{J}_{-1}$ can be deformed to be integrable through structures on $\mathcal{J}_{-1}$. This will be proved by *Lemma 2.21* below. $\square$

Lemma 2.20. *Let $x, y \in \mathcal{J}_{\mathcal{A}}$, and let $H(t) \in \mathcal{J}_{\mathcal{A}}, t \in [0,1]$ be a path joining them such that $H(0) = x, H(1) = y$. If a loop $J(t) \in \pi_1(\mathcal{J}_{\mathcal{A}}, y), t \in [0,1]$ lies in the image of $\pi_1(\mathscr{Diff}_0(X), \mathit{id}) \to \pi_1(\mathcal{J}_{\mathcal{A}}, y)$, then $H^{-1} \cdot J \cdot H \in \pi_1(\mathcal{J}_{\mathcal{A}}, x)$ lies in the image of $\pi_1(\mathscr{Diff}_0(X), \mathit{id}) \to \pi_1(\mathcal{J}_{\mathcal{A}}, x)$.*

Proof. Without loss of generality we assume that there exists a loop $f(t) \in \pi_1(\mathscr{Diff}_0, \mathit{id})$ such that $J(t) = f_*(t)J(0)$. Let H_s be the piece of the path H that joins the points $H(0) = x$ and $H(s)$. To prove the lemma it remains to consider the homotopy

$$J(s,t) := H_s^{-1} \cdot f_*(t)H(s) \cdot H_s, \tag{2.19}$$

where $J(1,t) = H^{-1} \cdot J \cdot H$ and $J(0,t) = f_*(t)H(0)$. $\square$

Lemma 2.21. *Every connected component of $\mathcal{J}_{-1}$ contains at least one integrable structure.*

Proof. Take a structure $J \in \mathcal{J}_{-1}$, and denote by C the corresponding smooth elliptic curve in class $[C] = \boldsymbol{T}_-$. Let $\pi\colon X \to C$ be the ruling such that $J \in \mathcal{J}(X, \pi)$, see *Theorem 2.8*. Apart from the section given by C, we now choose one more smooth section C_1 of π such that C_1 is disjoint from C; the section C_1 need not be holomorphic, but be smooth. We claim that there exists a unique $\mathbb{C}^*$-action on X such that

(a) it is fiberwise, i.e. this diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\cdot g"] \arrow[d, "p"'] & \mathfrak{X} \arrow[d, "p"] \\ \mathbb{C} \arrow[r, "\cdot g"'] & \mathbb{C}, \end{tikzcd} \tag{2.20}$$

commute for each $g \in \mathbb{C}^*$,

(b) it acts on the fibers of π by means of biholomorphisms, and

(c) it fixes both C and C_1.

The complement $X - C_1$ is a $\mathbb{C}$-bundle with C being the zero-section; we keep the notation π for the projection $X - C_1 \to C$. Evidently, this bundle inherits the $\mathbb{C}^*$-action described above. Let us consider the unitary subgroup $\boldsymbol{U}(1)$-subgroup of $\mathbb{C}^*$, then the disk bundle $\pi\colon X - C_1 \to C$ posesses a unique $\boldsymbol{U}(1)$-invariant connection, which gives rise to a holomorphic structure (integrable complex structure) J_1 on $X - C_1$, see [Gr-Ha]. Note that this structure agrees with J on the fibers of π.

Further, there is a unique compactification of $X - C_1$ by C_1 such that J_1 extends to this compactification X with C_1 being a holomorphic section, so (X, J_1) is a complex ruled surface with $J_1 \in \mathcal{J}(X,\pi)$, and, moreover, J_1 coincides with J when is restricted to the bundle $TX|_C$ over C.

By *Proposition 2.7* there is a symplectic form ω taming both structures J and J_1. Given a symplectic curve, say C, in X, and an almost complex structure, say J, defined along C (i.e. on $TX|_C$) and tamed by ω. There exists an ω-tamed almost complex structure on X which extends the given one. Moreover, such an extension is homotopically unique. In particular, one can always construct a family J_t joining J and J_1 such that C is keep being J_t-holomorphic, and the lemma is proved. $\square$

Summarizing the results of *Lemma 2.16* and *Lemma 2.19* we obtain

Lemma 2.22. *$\pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(\mathcal{J}(X))$ is epimorphic.*

Again, it is implied by the fundamental diagram (0.9) that the following holds.

Proposition 2.23. *$\pi_0(\mathscr{S}\!\mathit{ymp}^*(X,\omega_\mu)) = 0$ for every $\mu \in (-1,0]$.*

Together with *Proposition 2.18*, this statement covers what is claimed in *Theorem 0.2*.

Now let X be a symplectic ruled 4-manifold $(S^2 \times T^2, \omega_\lambda)$, $[\omega_\lambda] = \boldsymbol{T} + \lambda \boldsymbol{S}$, $\lambda \in (0,+\infty)$.

Lemma 2.24 (cf. *Lemma 2.13*)**.** *$\mathcal{J}_{\mathcal{A}} \subset \mathcal{J}(X,\omega_\lambda)$ for every $\lambda \in (0,+\infty)$.*

Proof. The proof is straightforward once we observe that we can rescale a symplectic form and inflate it along a curve in class $\boldsymbol{T}$. I left the details to the reader with an easy heart. $\square$

For completeness of exposition, we now give without a proof an analogue of *Lemma 2.15* for the spin case, though we do not need it for the sequel.

Lemma 2.25 (cf. *Lemma 2.15*)**.** *$\mathcal{J}_{-2k} \subset \mathcal{J}(X,\omega_\lambda)$ iff $\lambda \in (k,+\infty)$.*

Going back to the proof of the theorem, we note that the isomorphisms $\pi_1(\mathcal{J}_{\mathcal{A}}) \to \pi_1(\mathcal{J}(X,\omega_\lambda))$ and $\pi_1(\mathcal{J}_{\mathcal{A}}) \to \pi_1(\mathcal{J})$ are induced respectively by the inclusions $\mathcal{J}_{\mathcal{A}} \subset \mathcal{J}(X,\omega_\lambda)$ and $\mathcal{J}_{\mathcal{A}} \subset \mathcal{J}$. Both these isomorphisms are equivariant with respect to the natural action of $\mathscr{D}\!\mathit{iff}(X)$. Therefore, it follows from *Lemma 2.16* that $\pi_1(\mathscr{D}\!\mathit{iff}(X)) \to \pi_1(\mathcal{J}_{\mathcal{A}})$ is epimorphic, and so is $\pi_1(\mathscr{D}\!\mathit{iff}(X)) \to \pi_1(\mathcal{J}(X,\omega_\lambda))$. This finishes the proof for the spin case.

2.8. **Appendix: a note on $\mathscr{D}\!\mathit{iff}_0(S^2\tilde{\times}T^2)$.** As a byproduct of §2.2, one can prove the following interesting (and simple) statement on the topology of the diffeomorphism group of $S^2\tilde{\times}T^2$.

Proposition 2.26. *Let X be diffeomorphic to $S^2\tilde{\times}T^2$. Take some point $p \in X$ and consider an evaluation map $\mathsf{ev}^p\colon \mathscr{D}\!\mathit{iff}_0(X) \to X$ given by*

$$\mathsf{ev}^p\colon f \to f(p).$$

Then $\mathsf{ev}^p_*\colon \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(X)$ is a monomorphism with the cokernel isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. It follows from Suwa's model, see §2.2, that the group $\mathscr{D}\!\mathit{iff}_0(X)$ contains a 2-torus $\mathcal{T}$ such that the restriction of ev^p to $\mathcal{T}$ induces the monomorphism $\mathsf{ev}^p_*\colon \pi_1(\mathcal{T}) \to \pi_1(X)$ with $\pi_1(X)/\mathsf{ev}^p_*(\pi_1(\mathcal{T})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In other words, $\mathsf{ev}^p_*(\pi_1(\mathcal{T}))$ contains every element that is divisible by 2. Thus, it remains to show that every element of the image of $\mathsf{ev}^p_*\colon \pi_1(\mathscr{D}\!\mathit{iff}_0(X)) \to \pi_1(X)$ is even.

Suppose, contrary to our claim, that there is a family $h_t\colon I \to C(X,X)$, $h(0) = h(1) = \mathsf{id}$, of continuous maps from X to X such that the loop $\alpha\colon I \to X$, $\alpha(t) := h_t(p)$ gives rise to an odd element of $\pi_1(X)$. Without loss of generality, we assume that $[\alpha] \in \pi_1(X)$ is prime.

Let $\beta\colon I \to X$ be a path with $\beta(0) = \beta(1) = p$ such that $[\alpha],[\beta]$ generate $\pi_1(X)$.

Define a map $H\colon S^1 \times X \to X$ to be $H(t,x) := h_t(x)$, $t \in S^1$, $x \in X$.

Let $\pi\colon X \to T^2$ be a ruling, and let $\gamma := \pi(\beta)$, $q := \pi(p)$.

We now consider the restriction of H to $S^1 \times \pi^{-1}(\gamma)$, and denote it by the same letter H. Note that $S^1 \times \pi^{-1}(\gamma)$ is diffeomorphic to $S^2 \times T^2$. We claim that the induced map $H_*\colon \pi_k(S^1 \times \pi^{-1}(\gamma)) \to \pi_k(X)$ is an isomorphism for every $k \geqslant 0$.

Indeed, it is evident that H_* is isomorphic for $k = 0$.

To see H_* is isomorphic for $k = 1$, we observe that the inclusion

$$i\colon S^1 \times \beta \to S^1 \times \pi^{-1}(\gamma)$$

induces an isomorphism of π_1, and so does the composition $H \circ i$; this is because $H(S^1 \times p) = \alpha$, $H(0 \times \beta) = \beta$.

To see H_* is isomorphic for $k > 1$, we observe that the inclusion

$$j\colon 0 \times \pi^{-1}(q) \to S^1 \times \pi^{-1}(\gamma)$$

induces an isomorphism of π_k, $k > 1$, and so does the composition $H \circ j$; this is because H maps the 2-sphere $0 \times \pi^{-1}(q)$ homeomorphically onto $\pi^{-1}(q)$. By using the homotopy exact sequence for the bundle $\pi\colon X \to T^2$, the interested reader will be able to verify that the inclusion $\pi^{-1}(q) \to X$ leads to the isomorphisms $\pi_k(\pi^{-1}(q)) \cong \pi_k(X)$, $k > 1$.

Then $H\colon S^2 \times T^2 \to S^2 \tilde{\times} T^2$ induces a weak homotopy equivalence, thus, by virtue of Whitehead's theorem, inducing a homotopy equivalence between $S^2 \times T^2$ and $S^2 \tilde{\times} T^2$. These manifolds, however, are not homotopy equivalent, since they have non-isomorphic cohomology rings. This contradiction finishes the proof. $\square$

3. Exotic symplectomorphisms

Recall that it was conjectured by McDuff and Salamon that every symplectomorphism of a geometrically ruled surface that is smoothly isotopic to the identity is isotopic to the identity within the symplectomorphism group. Inspite of this conjecture, which is very likely to be true, there are many 4-manifold for which forgetful homomorphism (0.1) is not injective. For example, if Σ is a smooth Lagrangian sphere in X, then there exists a symplectomorphism $T_\Sigma : X \to X$, called *symplectic Dehn twist along* Σ, such that T_Σ^2 is smoothly isotopic to the identity. However, the following theorem due to Seidel asserts that T_Σ^2 is not symplectically isotopic to the identity provided that X is sufficiently generic, see [Sei2].

Theorem 3.1 (Seidel)**.** *Let (X,ω) be a closed simply-connected minimal symplectic 4-manifold that is neither rational nor ruled. Suppose that $\dim \mathsf{H}^2(X,\mathbb{Q}) \geqslant 3$; then for every Lagrangian sphere $\Sigma \subset X$, the squared Dehn twist T_Σ^2 is not symplectically isotopic to the identity.*

Moreover, it was shown by Evans [Ev] that it is possible for iterated and composed Dehn twists to form a sophisticated and interesting symplectic mapping class group.

Theorem 3.2 (Evans, [Ev]). *Let X be a rational surface diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. Then there exists a symplectic form ω on X such that*

$$\pi_0 \mathscr{S}ymp^*(X,\omega) \cong \pi_0 \mathscr{D}iff^+(S^2,5),$$

where $\mathscr{D}iff^+(S^2,5)$ is the group of orientation-preserving diffeomorphisms of S^2 fixing 5 points.

This being said, a little is known about the existence of non-trivial symplectomorphisms (smoothly but not symplectically isotopic to the identity) for 4-manifolds. There are plenty symplectic 4-manifolds which do not admit Lagrangian spheres, yet they are believed to have non-trivial reduced symplectic mapping class groups.

In my joint work with Shevchishin [S-S] we introduced and studied a new class of symplectomorphisms which we call ***elliptic twists***. We showed that not only Lagrangian (-2)-spheres but also symplectic (-1)-tori give rise to certain elements in symplectic mapping class groups and we considered examples for which these new elements were proved to be of infinite order.

Let (X,ω_0) be a symplectic 4-manifold which contains a symplectically embedded torus $C \subset X$ of self-intersection number (-1). Assume C is oriented in such a way that $\int_C \omega_0 > 0$. Then, by using Gompf's symplectic sum, one can construct a new symplectic form ω on X such that $\int_C \omega \leqslant 0$. In our joint paper with Shevchishin [S-S] we introduced an ω-symplectomorphism $E_C : (X,\omega) \to (X,\omega)$ which we call the ***elliptic twist along*** C.

A careful reader might noticed that a symplectic $S^2 \tilde{\times} T^2$ may contain a symplectic (-1)-torus, and so may admit an elliptic twist for a suitable symplectic form. What was shown in the previous subsection was the triviality of this elliptic twist. This was an "unfortunate" example where an elliptic twist turns out to be symplectically trivial. Surprisingly, this phenomemon seems to be exceptional and it disapperes after one blow-up stabilization.

Theorem 3.3. *Let Z be $S^2 \tilde{\times} T^2 \# \overline{\mathbb{CP}}^2$, then there exist a symplectic form ω on Z and three (-1)-tori C_1, C_2, and C_3 in Z such that the elliptic twists E_{C_i} are well-defined and none of them is symplectically isotopic to the identity. Moreover, each symplectomorphism E_{C_i} has infinite order in the reduced symplectic mapping class group.*

Note that $Z \cong S^2 \tilde{\times} T^2 \# \overline{\mathbb{CP}}^2$ contains no Lagrangian spheres. This is due to homological obstructions. Thus it is impossible in principle for E_{C_i} to be decomposed into Dehn twists. The rest of the section is devoted to prove this theorem.

The section uses the concepts and technical results developed in the previous section, so it cannot be read independently.

3.1. **Elliptic twists.** We start with a symplectic 4-manifold (X,ω) which contains an embedded symplectic torus C of self-intersection number (-1), $[C]^2 = -1$.

We choose an appropriate ω-tamed almost-complex structure J_* for which the torus C is pseudoholomorphic. One can think of J_* as a point of the subspace $\mathcal{D}_{[C]} \subset \mathcal{J}(X,\omega)$ of those almost-complex structures which admit a pseudoholomorphic curve in class $[C]$. In what follows, we refer to $\mathcal{D}_{[C]}$ as the *elliptic divisorial locus* for the class $[C]$. The term *divisorial locus* is taken from the fact that the subspace $\mathcal{D}_{[C]}$ in some neighbourhood of J_* locally behaves as a submanifold of real codimension 2 of $\mathcal{J}(X,\omega)$ provided the J_*-holomorphic curve C is smooth, see e.g. [Iv-Sh-1].

Further, we shall say that an almost-complex structure J_* satisfies the *wall-crossing property* if

i) there exists a small 2-disc $\Delta \subset \mathcal{J}(X,\omega)$ which intersects $\mathcal{D}_{[C]}$ transversally at J_*,

ii) there exists a cohomology class $\xi \in \mathsf{H}^2(X;\mathbb{R})$, such that $\int_{[C]} \xi \leqslant 0$ and such that every $J \in \Delta \backslash \{J_*\}$ is tamed by some symplectic form θ_J with the cohomology class $[\theta_J] = \xi$.

One observes that all the forms θ_J, $J \in \Delta\backslash\{J_*\}$, are deformation equivalent to each other through symplectic forms of cohomology class ξ. With this understood, one applies Moser's theorem to show that the forms θ_J, $J \in \Delta\backslash\{J_*\}$, are isotopic to each other.

Let us use the parameter t for the points on the boundary $\partial\Delta$ of the disc Δ and $J(t)$ for the structure parameterized by $t \in \partial\Delta$. We thus have $\psi(\theta_{J(t)}) = J(t)$, where ψ is defined in (0.6). Set $\theta := \theta_{J(0)}$. Obviously, the space $\mathcal{J}(X,\theta)$ does not contain J_*, as well as no other points of $\mathcal{D}_{[C]}$, but it does contain $\Delta\backslash\{J_*\}$. Therefore the loop $J(t)$, which is a boundary of the disc Δ, is locally non-contractible in $\mathcal{J}(X,\theta)$ and potentially gives certain element in $\pi_1(J(X,\theta))$ which does not lie in the image of the homomorphism ν_* from the fundamental diagram (0.9).

Therefore, the symplectomorphism $E_C := \partial(\theta_{J(t)})$, which is well-defined up to symplectic isotopy, may give a nontrivial element in $\pi_0(\mathscr{S}ymp^*(X,\omega))$. We call E_C the *elliptic twist along* C.

3.2. **When do elliptic twists occur?** In order to study elliptic twists, we better have examples of 4-manifolds where the wall-crossing property can be easily verified. Here is a series of such 4-manifolds.

Let X be a symplectic 4-manifold containing an embedded symplectic torus C of self-intersection number 0, i.e. $[C]^2 = 0$. Choose a tamed almost-complex structure for which C is pseudoholomorphic, so C becomes a smooth elliptic curve. We then deform this structure slightly to make it integrable in some tubular neighbourhood of C, and assume that a sufficiently small neighbourhood of C has an elliptic fibering with C being a multiple fiber of multiplicity $m > 1$.

Let $g : \Delta \to X$ be a holomorphic embedding of a small complex 2-disc Δ into X such that $g(\Delta)$ intersects C transversally at a single point P. Choose a complex coordinate t on Δ such that $g(0) = P$. Consider the product $\Delta \times X$ and the embedding $h : \Delta \to \Delta \times X$ for which $h(\Delta)$ is the diagonal of $\Delta \times g(\Delta)$.

We now define a 3-fold $\mathbf{Z}$ to be the blow-up of $\Delta \times X$ along $h(\Delta)$. There is a natural mapping $\mathbf{Z} \to \Delta \times X \to \Delta$, where the first arrow is the contraction map, while the second one is the projection map. Because of this, our 3-fold $\mathbf{Z}$ forms a complex-analytic family Z_t, $t \in \Delta$, of small deformations of Z_0, where Z_0 is X blown-up at the point P.

Let E_t be the exceptional (-1)-curve in Z_t, and let $\sigma_t : Z_t \to X$ be a map that contracts E_t to a point $Q_t \in X$, $g(t) = Q_t$. We denote by $\boldsymbol{E}$ the homology class of the exceptional lines E_t, which is the same for each $t \in \Delta$. Since C is a multiple fiber, there are no curves in homology class $[C]$ passing through Q_t, provided t is nonzero. However, since C has the multiplicity m, for each $t \neq 0$ there exists a unique elliptic curve in homology class $m[C]$ such that it does pass through Q_t.

We thus claim that:

i) Z_0 contains a unique smooth elliptic curve in homology class $[C] - \boldsymbol{E}$, which is the strict transform of the curve C in X, and *ii)* Z_t, $t \neq 0$, contains no elliptic curves in homology class $[C] - \boldsymbol{E}$. However, what Z_t contains is a smooth elliptic curve C_m in homology class $m[C] - \boldsymbol{E}$, which is the strict transform of a certain elliptic curve in X.

Now let $\mathcal{D}_{[C]-\boldsymbol{E}}$ be the elliptic divisorial locus for $[C]-\boldsymbol{E}$. The manifold Z_0 corresponds to some point $J_0 \in \mathcal{D}_{[C]-\boldsymbol{E}}$, as well as the family Z_t corresponds to some 2-disc J_t, $t \in \Delta$, which intersects $\mathcal{D}_{[C]-\boldsymbol{E}}$ transversally at the single point J_0.

To see J_0 satisfies the wall-crossing property it is needed to construct a family θ_t, $t \in \Delta - J_0$, of cohomologous symplectic forms on Z such that J_t is θ_t-tamed and $\int_{[C]-\boldsymbol{E}} \theta_t \leqslant 0$.

We construct them as follows: let ω be some symplectic form on Z_0 which is taming the almost-complex structure J_0. Clearly, the almost-complex structures J_t are ω-tamed for $|t|$ small enough. We have $\int_{[C]} \omega > 0$, but $\int_{[C]-\boldsymbol{E}} \omega > 0$, and so the form ω should be changed.

Every Z_t, $t \neq 0$, contains the smooth elliptic curve C_m, which is in class $m[C] - \boldsymbol{E}$. Thus the negative inflation technique can be used to deform ω to be a form θ_t such that

$$\int_{m[C]-\boldsymbol{E}} \theta_t = \varepsilon \quad \text{for } \varepsilon > 0 \text{ arbitrary small.}$$

Recall that the negative inflation deformation does not violate the taming condition for Z_t, and it is being performed in a small neighbourhood of the curve C_m, so it does not affect the symplectic area of the curve C, see [Bu]. We thus have

$$\int_{[C]-\boldsymbol{E}} \theta_t = \varepsilon - (m-1) \int_{[C]} \omega \tag{3.1}$$

and hence if we take m sufficiently large, we can make the area of the class $[C] - \boldsymbol{E}$ as negative as desired.

3.3. **Rational (-1)-curves.** Let (Z, ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2 \tilde{\times} Y^2 \# \overline{\mathbb{CP}}^2$. Here we study homology classes in $\mathsf{H}_2(Z;\mathbb{Z})$ that can be represented by a symplectically embedded (-1)-sphere. Given a symplectically embedded (-1)-sphere A, it satisfies

$$[A]^2 = -1, \quad c_1([A]) = 1. \tag{3.2}$$

A simple computation shows that there are two homology classes satisfying (3.2), namely, $[A] = \boldsymbol{E}$ and $[A] = \boldsymbol{F} - \boldsymbol{E}$.

The following lemma will be used in the sequel often without any specific reference.

Lemma 3.4. *Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}Y^2\#\overline{\mathbb{CP}}^2$. Then for every choice of ω-tamed almost-complex structure J, both the classes $\boldsymbol{E}$ and $\boldsymbol{F}-\boldsymbol{E}$ are represented by smooth rational J-holomorphic curves.*

Proof. Given an arbitrary ω-tamed almost-complex structure J, the exceptional class $\boldsymbol{F}-\boldsymbol{E}$ is represented by either a smooth J-holomorphic curve or by a J-holomorphic cusp-curve A of the form $A=\sum m_iA_i$ where each A_i stands for a rational curve occuring with the multiplicity $m_i \geqslant 1$. Clearly, we have

$$0<\int_{A_i}\omega<\int_A\omega. \tag{3.3}$$

Because $c_1(\boldsymbol{F}-\boldsymbol{E})=1$, there exists at least one irreducible component of the curve A, say A_1, with $c_1([A_1])\geqslant 1$.

Note that spherical homology classes in $\mathsf{H}_2(Z;\mathbb{Z})$ are generated by $\boldsymbol{F}$ and $\boldsymbol{E}$. Hence, we have $[A_1]=p\boldsymbol{F}-q\boldsymbol{E}$, which particularly implies $[A_1]^2=-q^2\leqslant 0$, with an equality iff $[A_1]=p\boldsymbol{F}$. But the latter is prohibited by (3.3) because

$$\int_{\boldsymbol{F}-\boldsymbol{E}}\omega>0. \tag{3.4}$$

Therefore, we have $[A_1]^2\leqslant -1$. Further, one may use the adjunction formula to obtain that A_1 is a smooth rational curve with $[A_1]^2=-1$ and $c_1(A_1)=1$. Note that it is not possible for A_1 to be in the class $\boldsymbol{F}-\boldsymbol{E}$ because of (3.3). Hence, we have $[A_1]=\boldsymbol{E}$.

Take another irreducible component, say A_2. If A_2 does not intersect A_1, then $[A_2]=p\boldsymbol{F}$, which contradicts (3.3). Thus A_2 intersects A_1, positively. Hence, $[A_2]=p\boldsymbol{F}-q\boldsymbol{E}$ for q positive. The same argument works for the other irreducible components $A_2,A_3,\ldots$ of the curve A. But note that $[A_2]\cdot[A_3]<0$, and hence there are no other components of A, except A_1 and A_2. We thus have $m_2[A_2]=\boldsymbol{F}-(m_1+1)\boldsymbol{E}$ for $m_1,m_2\geqslant 1$. The class $\boldsymbol{F}-(m_1+1)\boldsymbol{E}$ is prime, and hence $m_2=1$. Further, this class cannot be represented by a rational curve, which can be easily checked using the adjunction formula. We thus proved the lemma for the class $\boldsymbol{F}-\boldsymbol{E}$; the case of $\boldsymbol{E}$ is analogous. □

This lemma leads to the following generalization of *Theorem 2.8* for ruled but not geometrically ruled symplectic 4-manifolds.

Lemma 3.5. *Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2\tilde{\times}Y^2\#\overline{\mathbb{CP}}^2$, and let J be an ω-tamed almost-complex structure. Then Z admits a singular ruling given by a proper projection $\pi:Z\to Y$ onto Y such that* *i) there is a singular value $y^* \in Y$ such that π is a spherical fiber bundle over $Y - y^*$, and each fiber $\pi^{-1}(y)$, $y \in Y - y^*$, is a J-holomorphic smooth rational curve in class $\boldsymbol{F}$;*

ii) the fiber $\pi^{-1}(y^*)$ consists of the two exceptional J-holomorphic smooth rational curves in classes $\boldsymbol{F} - \boldsymbol{E}$ and $\boldsymbol{E}$.

Proof. By *Lemma* 3.4 our manifold Z contains a unique smooth J-holomorphic rational curve E representing the class $\boldsymbol{E}$, and a unique smooth J-holomorphic rational curve E' representing the class $\boldsymbol{F} - \boldsymbol{E}$.

Denote by S the union of the curves E and E'. To prove the lemma it suffices to check that the complement $Z - S$ to the singular curve S in Z is fibered by smooth J-holomorphic rational curves of class $\boldsymbol{F}$.

Choose any point $P \in Z - S$. Take a C^0-small perturbation $\tilde{J}$ of J which is integrable a small neighbourhood $\mathcal{U}(S)$ of the curve S and such that S remains $\tilde{J}$-holomorphic; the structures J and $\tilde{J}$ coincide away from $\mathcal{U}(S)$, and the neighbourhood $\mathcal{U}(S)$ can be chosen small enough to not contain the point P.

Let X be the blow-down of E from Z. To finish the proof we check that X contains a unique smooth pseudoholomorphic curve in class $\boldsymbol{F}$ that pass through P. But that is what *Theorem* 2.8 states. □

3.4. **Straight structures.** Let $Z \cong S^2 \tilde{\times} T^2 \# \overline{\mathbb{CP}}^2$ be a complex ruled surface, and let E be a smooth rational (-1)-curve in $\boldsymbol{E} \in \mathsf{H}_2(Z;\mathbb{Z})$. The blow-down of E from Z, which is a non-spin geometrically ruled genus one surface, will be denoted by X. The surface Z is said to be an *affine surface* if X is biholomorphic to the surface $X_{\mathcal{A}}$, see subsection 2.2.

Let $p \in X$ be the image of E under the contraction map. Recall that X_A contains the triple of bisections, which are smooth elliptic curves in class $\boldsymbol{B} \in \mathsf{H}_2(X;\mathbb{Z})$. The surface Z is called *straight affine surface* if there is no bisection passing through p in X. In other words, a straight affine surface contains a triple of smooth curves in homology class $\boldsymbol{B}$, while a non-straight affine surface contains a smooth elliptic (-1)-curve in class $\boldsymbol{B} - \boldsymbol{E} \in \mathsf{H}_2(Z;\mathbb{Z})$. We remark that it follows from *Theorem* 2.3 that straight affine surfaces can be characterized as those for which there exists a smooth elliptic (-1)-curve in homology class $2\boldsymbol{B} - \boldsymbol{E} \in \mathsf{H}_2(Z;\mathbb{Z})$.

Let π be the ruling of X, and let S be the fiber of π that pass through p. When Z is affine, there are three bisection $B_i \subset X$, each of which intersects S at precisely two distinct points. The following result was established in subsection 2.2, when Suwa's model for $X_{\mathcal{A}}$ was described.

Lemma 3.6. *There exists a complex coordinate s on S such that the intersection points $B_i \cap S$ are as follows:*

$$B_1 \cap S = \{0, \infty\}, \quad B_2 \cap S = \{-1, 1\}, \quad B_3 \cap S = \{-i, i\}. \tag{3.5}$$

We then claim

Lemma 3.7. *There exists a complex-analytic family Z_s of affine surfaces depending on a parameter s ranging over $\mathbb{CP}^1$ such that for s equals one of these exceptional values*

$$\{0, \infty\}, \quad \{-1, 1\}, \quad \{-i, i\},$$

the surface Z_s is not a straight affine surface, while for other parameter values, Z_s is straight affine.

Proof. Let X be a ruled surface of the $X_{\mathcal{A}}$ complex type. Choose any fiber F of the ruling of X. Now consider the complex submanifold $F \times \mathbb{CP}^1 \subset X \times \mathbb{CP}^1$, and denote by S the diagonal in $F \times \mathbb{CP}^1$. We construct $\mathbf{Z}$ as the blow-up of $X \times \mathbb{CP}^1$ along S. The 3-fold $\mathbf{Z}$ forms the complex-analytic family $\mathbf{Z} \to S$ that was claimed to exist in the lemma. $\square$

The notion of the straight affine complex structure can be generalized to almost-complex geometry as follows. Choose a tamed almost-complex structure $J \in \mathcal{J}(Z)$. We will call J straight affine, or simply ***straight***, if each J-holomorphic representative in class $\boldsymbol{B} \in \mathsf{H}_2(Z;\mathbb{Z})$ is smooth. Clearly, the space of straight structures $\mathcal{J}_{\mathrm{st}}(Z)$ is an open dense submanifold in $\mathcal{J}(Z)$. Instead of $\mathcal{J}(Z)$ or $\mathcal{J}_{\mathrm{st}}(Z)$ we write $\mathcal{J}$ and $\mathcal{J}_{\mathrm{st}}$ for short. This definition of the straightness is motivated by the following lemma the proof of which is left to the reader because it is similar to the proof of *Proposition* 2.5, (but the modified version of *Theorem* 2.8 given by *Lemma* 3.5 should be used).

Lemma 3.8. *Let (Z,ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2 \tilde{\times} T^2 \# \overline{\mathbb{CP}}^2$, and let J be an ω-tamed almost-complex structure. Then every J-holomorphic representative in class $\boldsymbol{B}$ is either irreducible smooth or contains a smooth component in one of the classes $\boldsymbol{T}_+ - k\boldsymbol{F}$, $\boldsymbol{B} - \boldsymbol{E}$.*

Similarly to *Proposition* 2.5 this lemma leads to a natural stratification of the space $\mathcal{J}$ of tamed almost-complex structures. Namely, this space can be presented as the disjoint union

$$\mathcal{J} = \mathcal{J}_{\mathrm{st}} + \mathcal{D}_{\boldsymbol{T}_-} + \mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}} + \dots,$$

where $\mathcal{D}_{\boldsymbol{T}_-}$ and $\mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}}$, which are submanifolds of real codimension 2 in $\mathcal{J}$, are the elliptic divisorial locuss for respectively the classes $\boldsymbol{T}_-$ and $\boldsymbol{B}-\boldsymbol{E}$; we denoted by "+" the disjoint union. Here we omitted the terms of real codimension greater than 2, because they do not affect the fundamental group of $\mathcal{J}$.

Coming to the symplectic side of straightness, we claim that if a symplectic form ω on Z satisfies both these two *period conditions*

$$\int_{\boldsymbol{T}_-} \omega < 0, \quad \int_{\boldsymbol{B}-\boldsymbol{E}} \omega < 0, \tag{3.6}$$

then $\mathcal{J}(Z,\omega) \subset \mathcal{J}_{\mathrm{st}}$. Moreover, a somewhat inverse statement holds, at least for integrable structures.

Lemma 3.9. *Every complex straight affine surface Z has a symplectic form which tames the given complex structure on Z and satisfies the period conditions. Moreover, given a compact connected family Z_t (for example, a path) of straight affine structures, the cohomology class of a symplectic form can be chosen to be the same for each complex structure from the family.*

Proof. We first check that a complex straight affine surface Z has a taming symplectic form θ such that θ satisfies the period conditions.

If Z is affine then it is the surface $X_{\mathcal{A}} \cong S^2 \tilde{\times} T^2$ blown-up once. Since $X_{\mathcal{A}}$ admits a symplectic structure which satisfies the first period condition, then so does Z. Further, the second period condition can be achieved by means of deflation along a smooth elliptic curve in class $2\boldsymbol{B}-\boldsymbol{E}$; such a curve indeed exists thanks to the straightness of Z.

Let K be the parameter space for our family Z_t, $t \in K$, and let $I \subset K$, $I = \{t_1, t_2, \dots\}$ be a countable dense subset of K.

We proved that there exists a set of symplectic forms θ_I parametrized by I, so that θ_{t_i}, $t_i \in I$ tames the corresponding Z_{t_i} and satisfies the period conditions.

Let $U_{t_i} \in K$ be a sufficiently small neighbourhood of t_i such that for each $t \in U_{t_i}$

$$\theta_{t_i} \text{ tames the complex structure in } Z_t.$$

Clearly, the set $U_{t_i}, t_i \in I$ forms an open cover of K. Since K is compact, one take a finite subcover $U_{t_i}, t_i \in I'$.

The forms $\theta_{I'}$ are not necessarily cohomologous because they may have different integrals on the homology class $\boldsymbol{E}$. Set $\varepsilon_{t_i} := \int_{\boldsymbol{E}} \theta_{t_i}$, $t_i \in I'$, and $\varepsilon := \min \varepsilon_{t_i}$. We now deflate (Z_{t_i}, θ_{t_i}) along the homology class $\boldsymbol{E}$ to get $\int_{\boldsymbol{E}} \theta_{t_i} = \varepsilon$. Thank to this deflation the forms $\theta_{I'}$ become cohomologous and still do satisfy the period conditions.

Finally, set $\hat{\theta}(t) := \sum_{I'} \rho_{t_i}(t)\theta_{t_i}$, where the functions $\rho_{t_i} = \rho_{t_i}(t)$ is a partition of unity for the finite open cover $U_{t_i}, t_i \in I'$ of K. What remains is to verify that Z_t is tamed by $\hat{\theta}(t)$ for every $t \in K$. Pick some $t^* \in K$, then there are but finitely many charts $U_{t_1}, \ldots, U_{t_p}$ that contains the point $t^* \in K$. That is why $\hat{\theta}(t^*) = \rho_{t_1}(t^*)\theta_{t_1} + \ldots + \rho_{t_p}(t^*)\theta_{t_p}$. Since each of $\theta_{t_1}, \ldots, \theta_{t_p}$ tames Z_{t^*}, then so does $\hat{\theta}(t^*)$. □

3.5. **Refined Gromov invariants.** In this subsection, we work with an almost-complex manifold (Z, J) equipped with a straight structure $J \in \mathcal{J}_{\mathrm{st}}$, i.e. every J-holomorphic curve of class $\boldsymbol{B} \in \mathsf{H}_2(Z;\mathbb{Z})$ in Z is smooth. We also note that such a curve is not multiply-covered, because the homology class $\boldsymbol{B}$ is prime. The universal moduli space $\mathcal{M}(\boldsymbol{B};\mathcal{J}_{\mathrm{st}})$ of embedded non-parametrized pseudoholomorphic curves of class $\boldsymbol{B}$ is a smooth manifold, and the natural projection $\mathsf{pr} : \mathcal{M}(\boldsymbol{B};\mathcal{J}_{\mathrm{st}}) \to \mathcal{J}_{\mathrm{st}}$ is a Fredholm map, see [Iv-Sh-1, McD-Sa-3]. Given a generic $J \in \mathcal{J}_{\mathrm{st}}$, the preimage $\mathsf{pr}^{-1}(J)$ is canonically oriented zero-dimensional manifold, see [Tb] where it is explained how this orientation is chosen. Further, the oriented bordism class of $\mathsf{pr}^{-1}(J)$ does not depends on any particular choice of $J \in \mathcal{J}_{\mathrm{st}}$. Therefore the degree of pr can be defined to be equal to $\mathsf{Gr}(\boldsymbol{B})$.

It is stated by *Corollary 2.4* that $\mathsf{Gr}(\boldsymbol{B}) = 3$, and hence Z contains not one but several curves in class $\boldsymbol{B}$. Once we restricted almost-complex structures to those with the straightness property, the following modification of Gromov invariants can be proposed: given the image G of a certain homomorphism $\mathbb{Z}^2 \to \mathsf{H}_1(X;\mathbb{Z})$, instead of counting pseudoholomorphic curves C such that $[C] = \boldsymbol{B}$, we will count curves C such that $[C] = \boldsymbol{B}$ and the embedding $i : C \hookrightarrow X$ satisfies $\mathsf{Im}\, i = G$. The definitions of Gromov invariants $\mathsf{Gr}(\boldsymbol{B}, G)$, moduli space $\mathcal{M}(\boldsymbol{B}, G;\mathcal{J}_{\mathrm{st}})$, and so forth are completely analogous to those in "usual" Gromov's theory.

Suppose J is an integrable straight affine structure, then the complex surface (Z, J) contains precisely 3 smooth elliptic curves C_1, C_2, and C_3 in homology class $\boldsymbol{B}$. We denote by G_k the subgroup of $\mathsf{H}_1(X;\mathbb{Z})$ generated by cycles on C_k; these subgroups G_k

are pairwise distinct, as it can be deduced, for example, from Suwa's model of $X_{\mathcal{A}}$, see subsection 2.2.

It is clear now that the space $\mathcal{M}(\boldsymbol{B};\mathcal{J}_{\mathrm{st}})$ is disconnected and can be presented as the union

$$\mathcal{M}(\boldsymbol{B};\mathcal{J}_{\mathrm{st}}) = \bigsqcup_{k=1}^{3} \mathcal{M}(\boldsymbol{B}, G_k;\mathcal{J}_{\mathrm{st}}). \tag{3.7}$$

We define the *moduli space of bisections* to be the fiber product

$$\mathcal{M}_{3B} = \{(m_1, m_2, m_3) \mid m_k \in \mathcal{M}(\boldsymbol{B}, G_k;\mathcal{J}_{\mathrm{st}}),\ \mathsf{pr}(m_1) = \mathsf{pr}(m_2) = \mathsf{pr}(m_3))\}.$$

Similarly to $\mathcal{M}(\boldsymbol{B};\mathcal{J}_{\mathrm{st}})$, the moduli space $\mathcal{M}_{3B}$ is a smooth manifold equipped with the projection $\mathsf{pr} : \mathcal{M}_{3B} \to \mathcal{J}_{\mathrm{st}}$, which is a smooth map of degree one. We close this section by stating an obvious property of the projection map that we shall use in the sequel.

Lemma 3.10. *The projection map $\mathsf{pr} : \mathcal{M}_{3B} \to \mathcal{J}_{st}$ is a diffeomorphism, when is restricted to integrable straight affine complex structures.*

3.6. **A cocycle on $\mathcal{M}_{3B}$.** The map $\nu : \mathscr{D}\!\mathit{iff}_0(Z) \to \mathcal{J}_{\mathrm{st}}$ defined by

$$\mathscr{D}\!\mathit{iff}_0(Z) \xrightarrow{\nu} \mathcal{J}(Z,\omega) : f \to f_*J, \tag{3.8}$$

can be naturally lifted to a map $\mathscr{D}\!\mathit{iff}_0(Z) \to \mathcal{M}_{3B}$. Indeed, take a point $m \in \mathcal{M}_{3B}$, which is a quadruple $[J, B_1, B_2, B_3](m)$ consisting of an almost-complex structure $J(m) \in \mathcal{J}_{\mathrm{st}}$ on Z and of the triple of smooth $J(m)$-holomorphic elliptic curves $B_1(m)$, $B_2(m)$, and $B_3(m)$ in Z. Then one can define

$$\mathscr{D}\!\mathit{iff}_0(Z) \xrightarrow{\nu} \mathcal{M}_{3B} : f \to [f_*J, f(B_1), f(B_2), f(B_3)]. \tag{3.9}$$

Here we construct a cocycle $\Lambda \in \mathsf{H}^1(\mathcal{M}_{3B};\mathbb{Q}^2)$ such that this homomorphism

$$\pi_1(\mathscr{D}\!\mathit{iff}_0(Z)) \xrightarrow{\nu_*} \pi_1(\mathcal{M}_{3B}) \xrightarrow{\Lambda} \mathbb{Q}^2 \tag{3.10}$$

is the null-homomorphism.

To start we consider the tautological bundle $\mathcal{Z} \cong \mathcal{M}_{3B} \times Z$ over $\mathcal{M}_{3B}$ whose fiber over a point $m \in \mathcal{M}_{3B}$ is the almost-complex manifold $(Z, J(m))$.

It was claimed in *Lemma* 3.4 that every almost-complex manifold $(Z, J(m))$ contains a unique smooth rational (-1)-curve $S(m)$ in class $\boldsymbol{F} - \boldsymbol{E}$. Thus one can associate to $\mathcal{Z}$ an auxiliary bundle $\mathcal{S}$ whose fiber over $m \in \mathcal{M}_{3B}$ is the rational curve $S(m)$.

Note that each $B_i(m)$ intersects $S(m)$ at precisely 2 distinct points denoted by $P_{i,1}$ and $P_{i,2}$. Hence we can mark out 3 distinct pairs of points $(P_{i,1}, P_{i,2})$, $i = 1,2,3$ on each fiber $S(m)$ of $\mathcal{S}$.

Besides that, every $(Z, J(m))$ contains a unique smooth rational curve $E(m)$ in class $\boldsymbol{E}$. The curve $E(m)$ intersects $S(m)$ at precisely one point, say $Q(m)$. This point Q does not coincide with any of the point $P_{i,1}, P_{i,2}$, because $J(m)$ is assumed to be a straight one. Therefore $\mathcal{S}$ can be considered as a fiber bundle over $\mathcal{M}_{3B}$ whose fiber is the rational curve $S(m)$ with 7 distinct marked points, partially ordered in such a way that the first six points form the three ordered pairs, points inside every pair are not ordered, and the last point is of number seven.

One more fiber bundle, or better to say, a covering, we work with is the bundle $\mathcal{N}$ whose fiber over $m \in \mathcal{M}_{3B}$ consists of the six points $P_{i,1}(m), P_{i,2}(m)$, $i = 1,2,3$, so $\mathcal{N}$ is a covering space of the covering group $G := \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This covering is not necessary trivial, i.e. it is possible for $\mathcal{N}$ to have a monodromy along certain loop in $\mathcal{M}_{3B}$.

Let $p : \widetilde{\mathcal{M}}_{3B} \to \mathcal{M}_{3B}$ be the Galois covering, so the pullback $p^*\mathcal{N}$ has no monodromy. Then the bundle $\widetilde{\mathcal{S}} := p^*\mathcal{S}$ can be considered as a fiber bundle over $\widetilde{\mathcal{M}}_{3B}$ whose fiber over $m \in \widetilde{\mathcal{M}}_{3B}$ is the rational curve $S(m)$ with 7 marked points, distinct and ordered.

Denote by $\mathring{S}$ the punctured projective line $\mathbb{CP}^1 - \{0, 1, i\}$. We now construct a map $\lambda : \widetilde{\mathcal{M}}_{3B} \to \mathring{S}$ as follows: choose $m \in \widetilde{\mathcal{M}}_{3B}$, and consider the corresponding fiber $S(m)$ of the bundle $\widetilde{\mathcal{S}}$. Let $P_{i,1}, P_{i,2}$, and Q be the corresponding marked points on $S(m)$; then there is a unique complex coordinate s on $S(m)$ such that $s(P_{1,1}) = 0, s(P_{2,1}) = 1, s(P_{3,1}) = i$. We set $\lambda(m) := s(Q)$.

The following obvious property of λ will be used soon.

Lemma 3.11. *The map λ is invariant w.r.t. to the natural action of $\mathscr{D}\!i\!f\!f_0$ on $\widetilde{\mathcal{M}}_{3B}$.*

Now choose a basis for $\mathsf{H}_1(\mathring{S}; \mathbb{Q}) \cong \mathbb{Q}^2$ consisting of two small loops going respectively around the points $s = 0$ and $s = 1$. One can think of the induced map $\lambda_* : \mathsf{H}_1(\widetilde{\mathcal{M}}_{3B}; \mathbb{Q}) \to \mathsf{H}_1(\mathring{S}; \mathbb{Q})$ as a $\mathbb{Q}^2$-valued 1-cocycle on $\widetilde{\mathcal{M}}_{3B}$.

In order to get an element in $\mathsf{H}^1(\mathcal{M}_{3B}; \mathbb{Q}^2)$ we average the cocycle $\Lambda \in \mathsf{H}^1(\widetilde{\mathcal{M}}_{3B}; \mathbb{Q}^2)$ over the action of G on $\widetilde{\mathcal{M}}_{3B}$. We keep the notation Λ for this new element in $\mathsf{H}^1(\mathcal{M}_{3B}; \mathbb{Q}^2)$; the statement below follows from *Lemma 3.11*

Corollary 3.12. $\mathsf{Im}\,\nu_* \subset \mathsf{Ker}\,\Lambda$.

3.7. **Loops in $\mathcal{J}_{\mathrm{st}}$.** Here the group $\pi_1(\mathcal{J}_{\mathrm{st}})$ will be proved to contain infinite order elements; the elements to be presented will not lie in the image of the homomorphism $\nu_* : \pi_1(\mathscr{D}\!\mathit{iff}_0(Z)) \to \pi_1(\mathcal{J}_{\mathrm{st}})$.

The first ingredient we use is the complex-analytic family Z_s, $s \in S$, where $S \cong \mathbb{CP}^1$, given by *Lemma* 3.7. The surface Z_s satisfies the straightness property for all but finitely many $s \in S$, see subsection 3.4; there are these six *exceptional values*

$$\{0, \infty\}, \quad \{-1, 1\}, \quad \{-i, i\},$$

for which the corresponding surface Z_s violates the mentioned property. These exceptional surfaces, which are affine but not straight affine, are of interest for us because they contain a smooth elliptic curve of class $\boldsymbol{B} - \boldsymbol{E}$, and hence they will correspond to the points of the elliptic divisorial locus $\mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}}$.

Choose a closed path $s(t) \in S$ avoiding the exceptional values. Since $S \cong \mathbb{CP}^1$ is simply-connected, there exists a disc Δ which bounds $s(t)$; note that, in general, such a disc cannot be mapped into S in such way to avoid the exceptional points.

Since Δ is the disc, the family Z_s is smoothly trivial when is restricted to Δ. Once a trivialization for, say $Z_{s(0)}$, is chosen, one can map Δ into the space $\mathcal{J}_{\mathrm{st}}$ by extending the trivialization for $Z_{s(0)}$ to the trivialization for the family Z_s over Δ; note that such an extension is not unique, though all possible extensions are homotopic to each other. We also note that if the mapping of Δ into S was chosen to be transversal to the exceptional values of s, then the constructed mapping of Δ into $\mathcal{J}_{\mathrm{st}}$ would be transversal to the elliptic divisorial locus $\mathcal{D}_{\boldsymbol{B}-\boldsymbol{E}}$. Denote by $J(t)$ the loop in $\mathcal{J}_{\mathrm{st}}$ which bounds Δ.

Recall that there is a smooth map $\mathsf{pr} : \mathcal{M}_{3B} \to \mathcal{J}_{\mathrm{st}}$ of degree 1, see subsection 3.5. Since $J(t)$ consists of integrable structures, the preimage $m(t) := \mathsf{pr}^{-1}(J(t))$ is a loop in $\mathcal{M}_{3B}$, see *Lemma* 3.10. The key property we need is:

Lemma 3.13. *If a loop $J(t)$ is homotopic to zero in $\mathcal{J}_{st}$, then $m(t)$ is homologous to zero in $\mathcal{M}_{3B}$. In particular, we have $\Lambda(m(t)) = 0$.*

Proof. Let Δ be a disc which bounds $J(t)$. By *Theorem* 1.5 we can arrange that Δ is transverse to pr, and the preimage $\mathsf{pr}^{-1}(J(t))$ is a smooth orientable surface that bounds $m(t)$. □

This lemma together with *Corollary* 3.12 imply

Lemma 3.14. *If $J(t) \in \operatorname{Im} \nu_*$, then $\Lambda(m(t)) = 0$.*

Now set $s(t) = \varepsilon e^{it}$, where $\varepsilon > 0$ small enough, and consider the corresponding loops $J(t) \in \mathcal{J}_{\mathrm{st}}$ and $m(t) \in \mathcal{M}_{3B}$. To compute $\Lambda(m(t))$ we note that $\widetilde{m}(t) := p^{-1}(m(t)) \in \widetilde{\mathcal{M}}_{3B}$ consists of eight distinct closed curves; one of them is

$$(P_{1,1}(t), P_{1,2}(t), P_{2,1}(t), P_{2,2}(t), P_{3,1}(t), P_{3,2}(t), Q(t)) = (0, \infty, 1, -1, i, -i, \varepsilon e^{it}).$$

with respect to some trivialization of the bundle $\mathcal{S}$.

We divide these curves into two groups:

$$\widetilde{m}_{1,k}(t) = (0, \infty, \dots, \varepsilon e^{it}),\ \widetilde{m}_{2,k}(t) = (\infty, 0, \dots, \varepsilon e^{it}), \quad k = 1, \dots, 4.$$

For each k we have that $\lambda \circ \widetilde{m}_{1,k}(t) \in \mathring{S}$ is a small simple closed path going around the point $z = 0$, while $\lambda \circ \widetilde{m}_{2,k}(t) \in \mathring{S}$ is a small path around $z = \infty$. For the homology class $[\lambda \circ m_{1,k}] \in \mathsf{H}_1(\mathring{S}; \mathbb{Q})$ we have $[\lambda \circ m_{1,k}] = (1, 0)$ with respect to the chosen basis for $\mathsf{H}_1(\mathring{S}; \mathbb{Q})$, while $[\lambda \circ m_{2,k}] \in \mathsf{H}_1(\mathring{S}; \mathbb{Q})$ clearly vanishes. It follows that

$$\Lambda(m(t)) \neq 0.$$

If $s(t)$ was given by

$$s(t) = 1 + \varepsilon e^{it} \quad \text{or} \quad s(t) = i + \varepsilon e^{it},$$

then a similar argument would work to prove that $\Lambda(m(t)) \neq 0$.

3.8. **Let's twist again.** Here we outline the proof of *Theorem* 3.3, referring the reader to the previous subsections for details.

We start with a symplectic 4-manifold Z diffeomorphic to $S^2 \tilde{\times} Y^2 \# \overline{\mathbb{CP}}^2$ for which the symplectic mapping class group mapping class group will be proved to contain elliptic twists.

Let X be a complex surface biholomorphic to $X_{\mathcal{A}}$, see subsection 2.2. By *Theorem* 2.3 we know that X contains a triple of smooth elliptic curves C_1, C_2, and C_3 in homology class $\boldsymbol{B} \in \mathsf{H}_2(X; \mathbb{Z})$, which are bisections of the corresponding ruling. Therefore, the procedure given in subsection 3.2 can be applied to prove the existence of elliptic twist for $X \# \overline{\mathbb{CP}}^2$.

In subsection 3.7 the corresponding three loops, say J_{C_1}, J_{C_2}, and J_{C_3}, are constructed as loops contained in the space $\mathcal{J}_{\mathrm{st}}$ of the straight almost-complex structures. We then prove these loops do not lie in the image of $\nu_* : \pi_1(\mathscr{Diff}_0(Z)) \to \pi_1(\mathcal{J}_{\mathrm{st}})$, see subsection 3.7. Because these loops consist of integrable structures, it follows from *Lemma* 3.9 there exists a symplectic form, say θ, such that $J_{C_i} \in \mathcal{J}(Z,\theta)$ and the inclusion $\mathcal{J}(Z,\theta) \subset \mathcal{J}_{\mathrm{st}}$ holds.

Since the inclusion $\mathcal{J}(Z,\theta) \subset \mathcal{J}_{\mathrm{st}}$ is equivariant w.r.t. to the natural action of $\mathscr{D}\!\mathit{iff}_0(Z)$ on these spaces, it follows that J_{C_i} do not lie in the image of $\nu_* : \pi_1(\mathscr{D}\!\mathit{iff}_0(Z)) \to \mathcal{J}(Z,\theta)$. Therefore the elements $\psi(J_{C_i})$ are not in the kernel of $\partial : \pi_1(\mathcal{J}(Z,\theta)) \to \pi_0(\mathscr{S}\!\mathit{ymp}^*(Z,\theta))$ and the theorem follows.

4. Spin Lorentzian cobordisms

This section studies necessary and sufficient conditions for the existence of cobordisms between closed smooth manifolds of arbitrary dimensions such that the structure group of the cobordism is $Spin(1,n)_0$, where the subscript indicates the connected component of the identity. The reader is invited to look at Milnor's book on characteristic classes [Mil-St] for basics on cobordisms. The results of this section have appeared in my recent joint work with Torres [S-T].

4.1. **Preliminaries.**

A *cobordism* is a triple $(M;N_1,N_2)$ that consists of a smooth compact $(n+1)$-manifold M with non-empty boundary $\partial M = N_1 \sqcup N_2$, where N_1 and N_2 are smooth closed n-manifolds. The cobordism relation splits manifolds into equivalence classes, which are called *cobordism classes*. The cobordism class of a manifold N is usually denoted by $[N]$. The set of cobordism classes of n-manifolds $\Omega_n^{\boldsymbol{O}}$ is an abelian group with respect to the disjoint union operation $[N_1]+[N_2] = [N_1] \sqcup [N_2]$. The zero of the group is simply the class of an empty manifold. We also have $\partial(N \times I) = N \sqcup N$, which implies that every element of the group is of order 2. The group $\Omega_n^{\boldsymbol{O}}$ is called the *nth unoriented (co)bordism group*.

Cobordism relation can be extended to manifolds equipped with some additional structures, which leads to new important groups for the manifolds theory; the most known of them are:

i) Oriented cobordism group $\Omega_n^{\boldsymbol{SO}}$. Here we say that two oriented closed n-manifolds N_1 and N_2 are cobordant if there is an oriented $(n+1)$-manifold M with non-empty boundary $\partial M = N_1 \sqcup \overline{N}_2$, where $\overline{N}_2$ is for N_2 with the orientation reversed. Hence, $[\overline{N}] = -[N]$ in $\Omega_n^{\boldsymbol{SO}}$. The group $\Omega_n^{\boldsymbol{SO}}$ is much more interesting than $\Omega_n^{\boldsymbol{O}}$ simply because elements of $\Omega_n^{\boldsymbol{SO}}$ generally do not have order 2, i.e. $[\overline{N}] \neq [N]$.

ii) Spin cobordism group Ω_n^{Spin}. Let ξ be an oriented real rank n vector bundle over a manifold M. The bundle ξ has a spin structure if it admits a trivialization over the 1-skeleton of M that extends over the 2-skeleton. A spin structure is a homotopy class of such trivialization. Denote by $\mathfrak{spin}\,\xi$ the set of all spin structures of ξ. It is known that the group $\mathsf{H}^1(M;\mathbb{Z}_2)$ acts transitively and, if $\mathsf{rank}\,\xi > 1$, freely on the $\mathfrak{spin}\,\xi$, see e.g. [Law-Mich, Sc].

If ϵ is a trivial line bundle over M, then there is a natural mapping

$$\mathfrak{spin}\,\xi \to \mathfrak{spin}\,\xi \oplus \epsilon \tag{4.1}$$

given by sending a trivialization $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ to $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n, \boldsymbol{v}$, where $\boldsymbol{v}$ is a non-vanishing section of ϵ that agrees with the given orientation of ϵ; note that $\boldsymbol{v}$ is unique up to homotopy. It is known that (4.1) is equivariant with respect to the action of $\mathsf{H}^1(M;\mathbb{Z}_2)$ and hence it is one-to-one.

A spin manifold will mean a manifold M, together with a spin structure on the tangent bundle TM. We denote by $\mathfrak{spin}\,M$ the set of such spin structures. If M is a manifold with boundary, then every spin structure on M can be restricted to $N\partial M$. Indeed, if TM is spin, then so is $TM|_{\partial M}$. By using an outward-pointing trivialization we decompose $TM|_{\partial M} = T_{\partial M} \oplus N_{\partial M/M}$, where $T_{\partial M}$ is the tangent bundle of ∂M, and $N_{\partial M/M}$ is the normal bundle of ∂M inside M; the latter bundle is a trivial line bundle. Therefore one can map

$$\mathfrak{spin}\,M \to \mathfrak{spin}\,\partial M.$$

by applying (4.1).

However, it is not true that a spin structure on ∂M can be extended to be a spin structure on the whole M. This gives rise to a new cobordism relation. We say that two oriented spin closed n-manifolds N_1 and N_2 are cobordant if there exists a spin manifold M such that $\partial M = N_1 \sqcup \overline{N}_2$ and the spin structure M agrees with the given structures on N_1 and N_2. This new requirement is somewhat restrictive; for instance, $\Omega_0^{\boldsymbol{SO}} = \Omega_0^{Spin} = 1$, while $\Omega_1^{\boldsymbol{SO}} = 1$ and $\Omega_1^{Spin} = \mathbb{Z}_2$.

iii) Complex cobordism group $\Omega_n^{\boldsymbol{U}}$. A naive approach to define the cobordism relation between complex manifolds fails because the manifold-membrane M is odd-dimensional and hence is not complex. That is why the notion of a complex structure will be weaken, a stably almost-complex structure on a manifold M will be defined to be an almost-complex structure on $TM \oplus \epsilon^k$. The notion of complex cobordisms can now be easily defined for stably complex manifolds. We will not go into the details here.

The cobordism theory was one of the most significant and central part of topology in the twentieth century; many mathematical works should be mentioned here, e.g. [And, Buch, N1, N2, Q, W1] and the references therein.

4.2. **Problem statement.** Following [Cham] and [Rein], we define Lorentzian and spin Lorentzian cobordisms as follows. Define a *Lorentzian cobordism* between closed smooth n-manifolds N_1 and N_2 to be a quadruple $((M; N_1, N_2), g)$ that consists of *i)* a cobordism $(M; N_1, N_2)$, *ii)* a nonsingular Lorentzian metric (M, g) with a timelike non-vanishing vector field $\boldsymbol{v}$; this can be expressed by saying that M is *time-orientable*, and *iii)* we want the boundary $\partial M = N_1 \sqcup \overline{N}_2$ to be spacelike, i.e. $(N_1, g|_{N_1})$ and $(N_2, g|_{N_2})$ are Riemannian manifolds, where $g|_{N_i}$ is for the restriction of g to N_i. Clearly, having a non-vanishing vector field $\boldsymbol{v}$ such that $\boldsymbol{v}$ is interior normal on N_1 and exterior normal on N_2 is necessary for being Lorentzian cobordant. It turns out that this condition is also sufficient, see [Yod-1]. In the presence of such non-vanishing vector $\boldsymbol{v}$, one defines a Lorentzian metric by

$$g(\boldsymbol{\xi}, \boldsymbol{\eta}) := g_R(\boldsymbol{\xi}, \boldsymbol{\eta}) - \frac{2g_R(\boldsymbol{\xi}, \boldsymbol{v})g_R(\boldsymbol{\eta}, \boldsymbol{v})}{g_R(\boldsymbol{v}, \boldsymbol{v})}, \tag{4.2}$$

where (M, g_R) is some Riemannian metric. It is easy to see that $\boldsymbol{v}$ is timelike with respect to the given Lorentzian metric. Further, one can use the hypothesis of transversality of $\boldsymbol{v}$ at the boundary to show this boundary is Riemannian, see [Yod-1].

The tangent bundle TM of an orientable and time-orientable spacetime M splits as follows

$$TM \cong \xi \oplus \epsilon,$$

where $\epsilon \subset TM$ is a trivial line subbundle spanned by $\boldsymbol{v}$ and $\xi \cong TM/\epsilon$. It is evident that

$$\xi|_{\partial M} \cong T_{\partial M},$$

where $T_{\partial M}$ is the tangent bundle to ∂M. Therefore if ξ has a spin structure, then so does $T_{\partial M}$. If ξ has a spin structure we shall say that M admits *a $Spin(1;n)_0$-structure*. Given this definition of $Spin(1;n)_0$-structure, it is easy to see that a $Spin(1;n)_0$-structure on an orientable and time-orientable n-manifold M induces a canonical spin structure on its boundary. This gives rise to a somewhat new cobordism relation.

A $Spin(1;n)_0$-Lorentzian cobordism, or, simply, a *spin Lorentzian cobordism* between two spin manifolds N_1 and N_2 is a Lorentzian cobordism $((M;N_1;N_2);g)$ such that TM/ϵ admits a spin structure this structure agrees with the given spin structures on N_1 and N_2.

It is easy to conclude from what we discussed here that the following holds

Proposition 4.1. *Closed spin manifolds N_1 and N_2 are spin Lorentzian cobordant iff they are spin cobordant and Lorentzian cobordant.*

The problems to be considered in this section are the classification problem for n-manifolds up to $Spin(1,n)_0$-Lorentizan cobordism and the computation problem for the corresponding cobordism groups. The classification problem was studied by several physicists and mathematicians, see e.g. [G-H-1, G-H-2, Rein, Sor, Yod-1, Yod-2]. However, it seems there is no complete solution to this problem achieved so far. The purpose of this note is to give a more or less satisfatory criterion for two spin manifolds to be spin Lorentzian cobordant.

Theorem 4.2. *Let N_1 and N_2 be closed spin n-manifolds. They are $Spin(1,n)_0$-Lorentizan cobordant iff they are spin cobordant and*

i) $n \bmod 2 = 0$, $\chi(N_1) = \chi(N_2)$;

ii) $n \bmod 8 = 7$, nothing else is required;

iii) $n \bmod 8 = 1,3,5$, $u(N_1) = u(N_2)$, where by $u(N)$ we denote the Kervaire semicharacterisitc of a manifold N

$$u(N) = \sum_{i=0}^{(n-1)/2} \beta_i(N) \bmod 2,$$

where $\beta_i(N)$ is the ith Betti number of N.

Corollary 4.3. *There exists a spin manifold M with boundary $\partial M = S^n$ equipped with a non-vanishing vector field transversal to the boundary iff $n \bmod 8 = 7$.*

It is important to emphasize that the question whether or not two spin manifolds are spin bordant is solved, see [And] for a complete answer.

The case n even is due to Sorkin [Sor] and Reinhart [Rein], while the case n odd was partially treated by Gibbons and Hawking [G-H-1] for 3-manifolds, but, as we shall see, their approach perfectly works for other dimensions. The proof of *Theorem* 4.2 is given in the following section.

4.3. **Classification.** Here the coefficient group for homology cycles is assumed to be $\mathbb{Z}_2$. We refer the reader to [St, Pr] for Steenrod's squares discussion. See [Mil-St] on Wu's classes.

Let $(M; N_1, N_2)$ be a spin cobordism between N_1 and N_2, and we set $n := \mathsf{dim}\, N_i$. The following cases will be considered separately.

i) $n = 2k$. It was observed by Sorkin [Sor], see also [Rein], that a cobordism $(M; N_1, N_2)$ is Lorentzian iff $\chi(N_1) = \chi(N_2)$.

ii) $n = 8k + 7$. Again, it was shown in [Rein, Sor] that $(M; N_1, N_2)$ is Lorentzian iff $\chi(M) = 0$. We now want to modify $(M; N_1, N_2)$ to make $\chi(M) = 0$ but keep M spin.

To this end, we simply go from M to $M \# \mathbb{HP}^{2k+2} \# k\mathbb{T}^{n+1}$ to increase $\chi(M)$ by 1 and to $M \# \mathbb{HP}^{2k+2} \# (k+1)\mathbb{T}^{n+1}$ to decrease $\chi(M)$ by 1. Note that $\mathbb{HP}^{2k+2}$ is spin and $\chi(\mathbb{HP}^{2k+2}) = 2k + 3$. To justify these changes of the Euler characteristic we recall that the Euler characteristic of a connected sum of $(n+1)$-manifolds M_1 and M_2 is given by $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^{n+1})$. Note also that the connected sum of spin manifolds inherits a spin structure from the summands.

iii) $n = 8k + 1, 8k + 3, 8k + 5$. We first prove a few lemmas.

Lemma 4.4. *Let M be a closed spin $2m$-manifold, $m \bmod 4 \neq 0$. Then the cup product pairing $\cup \colon \mathsf{H}^m(M) \otimes \mathsf{H}^m(M) \to \mathsf{H}^{2m}(M)$ is skew-symmetric.*

Proof. Let us consider the map $\mathsf{H}^m(M) \to \mathsf{H}^{2m}(M) \colon x \to x^2$. If we work with $\mathbb{Z}_2$ as coefficient group, then this map is linear. Since the cup product $\mathsf{H}^m(M) \otimes \mathsf{H}^m(M) \to \mathsf{H}^{2m}(M)$ is non-degenerate for M closed, it follows that there exists a unique "characteristic" class $v^m \in H^m(M)$ such that $x^2 = x \cup v$. This class is called the *mth Wu class*. It was shown by Hopkins and Singer [Hop-Sin] that $v^m = 0$ if $m \bmod 4 \neq 0$ for M spin. Hence $x^2 = 0$. $\square$

Lemma 4.5. *Let M be a closed spin $2m$-manifold with boundary, $m \bmod 4 \neq 0$. Then the cup product pairing $\cup \colon \mathsf{H}^m(M, \partial M) \otimes \mathsf{H}^m(M, \partial M) \to \mathsf{H}^{2m}(M, \partial M)$ is skew-symmetric.*

Proof. Let us denote ∂M by A. Again, we need to prove that $x^2 = 0$ for every $x \in \mathsf{H}^m(M, A)$. It is known, see e.g. [Ker], that the pairing $\mathsf{H}^m(X, A) \otimes \mathsf{H}^m(X) \to \mathsf{H}^{2m}(X, A)$ is *completely orthogonal*. A pairing is completely orthogonal means that either of the first two groups involved is isomorphic to the group of all homomorphisms of the other into the third. Thus one can conclude that there exists a unique class $s^m \in \mathsf{H}^m(M)$ such that $x^2 = x \cup s$ for every $x \in \mathsf{H}^m(M, A)$.

Let P be the manifold obtained by matching together two copies of M along the copies of A, and let $i\colon M \to P$ be the natural inclusion. It was proved by Kervaire [Ker] that

$$s^m = i^* v^m, \tag{4.3}$$

where v^m is the mth Wu class of P. The manifold P is spin, provided M is spin. By *Lemma* 4.4 $v^m = 0$ and hence $s^m = 0$. □

The following lemma was proved by Gibbons and Hawking [G-H-1] for $m = 2$, and by Geiges [Gei] for $m = 3$. Note that the proof of Geiges works not only form spin 6-manifold M but also for an arbitrary 6-manifold with boundary. The proof for the general case is essentially the same as for $m = 2, 3$, but it is given below for completeness of the exposition.

Lemma 4.6. *Let M be a closed spin $2m$-manifold with boundary, $m \bmod 4 \neq 0$. Then*

$$\chi(M) + u(\partial M) = 0 \bmod 2.$$

Proof. Let us denote ∂M by A. Now consider the pair exact sequence

$$0 \to \mathsf{H}^0(M, A) \to \mathsf{H}^0(M) \to \mathsf{H}^0(A) \to \ldots \to \mathsf{H}^m(M, A) \to \mathsf{H}^m(M) \to \ldots, \tag{4.4}$$

and define W to be the image of $\mathsf{H}^2(M, A) \to \mathsf{H}^2(M)$ under the last homomorphism, so we have

$$0 \to \mathsf{H}^0(M, A) \to \mathsf{H}^0(M) \to \mathsf{H}^0(A) \to \ldots \to \mathsf{H}^m(M, A) \to W \to 0. \tag{4.5}$$

Because of exactness, the alternating sum of the dimensions of these vector spaces over $\mathbb{Z}_2$ must vanish.

$$\sum_{i=0}^{m} \mathsf{dim}\,\mathsf{H}^i(M, A) + \sum_{i=0}^{m-1} \mathsf{dim}\,\mathsf{H}^i(M) + \sum_{i=0}^{m-1} \mathsf{dim}\,\mathsf{H}^i(A) + \dim W = 0. \tag{4.6}$$

A more careful computation, taking in account the Poincaré duality isomorphism $\mathsf{H}_i(M) \cong \mathsf{H}^{2m-i}(M, A)$, will tell

$$\chi(M) + u(A) + \mathsf{dim}\, W = 0. \tag{4.7}$$

Since the pairing $\mathsf{H}^m(X, A) \otimes \mathsf{H}^m(X) \to \mathsf{H}^{2m}(X, A)$ is completely orthogonal, it follows that the restriction of $\mathsf{H}^m(X, A) \otimes \mathsf{H}^m(X, A) \to \mathsf{H}^{2m}(X, A)$ to W is non-degenerate. By *Lemma* 4.5 the latter form is skew-symmetric. Hence the dimension of W is even. This finishes the proof. □

We now go back to prove part *iii)* of the theorem. The "only if" part follows from *Lemma* 4.6. Assume that N is a spin Lorentzian boundary, $\partial M = N$. Since N is a Lorentzian boundary, it follows that $\chi(M) = 0$. Then apply *Lemma* 4.6 to conclude $u(N) = 0$.

To prove the "if" we observe that $\chi(M)$ is even, provided $u(N_1) + u(N_2) = 0$. In order to make $\chi(M)$ be equal to zero, we modify M to

i) $M\#p\mathbb{T}^{n+1}\#q(S^2 \times \mathbb{HP}^{2k})$ for $n = 8k+1$;

ii) $M\#p\mathbb{T}^{n+1}\#q(S^2 \times S^2 \times \mathbb{HP}^{2k})$ for $n = 8k+3$;

iii) $M\#p\mathbb{T}^{n+1}\#q(S^2 \times \mathbb{HP}^{2k+1})$ for $n = 8k+5$.

The exceptional case $n = 1$ is left to the reader.

4.4. **The group structure.** The set L_n of $Spin(1,n)_0$-Lorentizan cobordism classes of manifolds is an abelian group w.r.t. to the disjoint union operation. This was mentioned but not proved in the introduction.

Proposition 4.7. *L_n is an abelian group.*

Proof. It is clear that L_n is an abelian semigroup, so the only nontrivial thing to check is the invertibility property. Let N be a spin closed n-maniolfd. Then there exists a spin structure on $\overline{N}$ such that $[N]+[\overline{N}] = 0$ in Ω_n^{Spin}. We claim that there exists an integer p such that $[N]+[\overline{N}]+p[S^n] = 0$ in L_n, see *Theorem* 4.2.

i) If n is even, then set $p := 2\chi(N)$.

ii) If n is odd, then $[N]+[\overline{N}] = 0$ in L_n because $u(N \sqcup \overline{N}) = 0$. Here $p = 0$.

Therefore the desired inverse is given by $[\overline{N}]+p[S^n]$. □

Let K_n be the group defined by the exact sequence

$$0 \to K_n \to L_n \to \Omega_n^{Spin} \to 0. \tag{4.8}$$

Here is a simple corollary of the main theorem.

Corollary 4.8. *$K_{2k} \cong \mathbb{Z}$, $K_{8k+7} = 0$, and for the other dimensions this is $\mathbb{Z}_2$. In each case, the group is generated by the sphere.*

4.5. **The ring structure.** The multiplication given by the Cartesian product gives rise to the (commutative) ring structure on cobordism classes. We denote the spin cobordism ring by Ω_*^{Spin} and the spin Lorentzian cobordism ring by L_*. Let K_* be a ring defined by the sequence

$$0 \to K_* \to L_* \to \Omega_*^{Spin} \to 0. \tag{4.9}$$

Here is one more corollary form *Theorem 4.2*.

Corollary 4.9. *The ring K_* is a product ring*

$$K_* = K_*^{\text{even}} \bigtimes K_*^{\text{odd}},$$

where K_*^{even} is a commutative ring generated by $x_0, x_2, \dots$ with the multiplication given by $x_{2p}x_{2q} := 2x_{2p+2q}$, and K_*^{odd} is commutative ring generated by $x_1, x_3, \dots$, where $2x_{2p+1} = 0$, $x_{8k+7} = 0$, and the multiplication is trivial. Here x_i is the cobordism class of S^i.

Proof. Let M be a closed manifold of even dimension, and let N be a closed manfiold of odd dimension. Suppose that both M and N are spin boundaries, then the product $M \times N$ is a spin Lorentzian boundary. This follows from *Theorem 4.2* and *Lemma 4.10* below. Further, the equalities $x_{2p}x_{2q} - 2x_{2p+2q} = 0$ and $x_{2p+1}x_{2q+1} = 0$ follows from the formula $\chi(N_1 \times N_2) = \chi(N_1)\chi(N_2)$ and *Theorem 4.2*. $\square$

The following lemma gives a tool to prove the statement above.

Lemma 4.10. *Let M be a closed manifold of even dimension, and let N be closed manifold of odd dimension. Then*

$$u(M \times N) = u(N)\chi(M) \bmod 2.$$

Proof. Set $2m := \operatorname{dim} M$, $2l + 1 := \operatorname{dim} N$, and $\beta_{p,q} := \beta_p(M)\beta_q(N)$, where β_i is as usual for ith Betti number. Recall that $u(M \times N)$ is

$$u(M \times N) = \sum_{i \leqslant m+l} \beta_i(M \times N).$$

Combining this with the Künneth formula we get

$$u(M \times N) = \sum_{i \leqslant m+l} \sum_{p+q=i} \beta_{p,q}.$$

Write this sum as follows:

$$\sum_{i \leqslant m+l} \sum_{p+q=i} \beta_{p,q} = \sum_{\substack{p>m \\ p+q \leqslant m+l}} + \sum_{\substack{p=m \\ q \leqslant l}} + \sum_{\substack{p<m \\ p-q>m-l \\ p+q \leqslant m+l}} + \sum_{\substack{q \leqslant l \\ p-q \leqslant m-l-2 \\ p+q \leqslant m+l}} + \sum_{\substack{q>l \\ p-q \leqslant m-l-2 \\ p+q \leqslant m+l}} .$$

Using Poincaré duality $\beta_{p,q} = \beta_{m-p,q}$, $\beta_{p,q} = \beta_{p,m-q}$ one can show that

$$\sum_{\substack{p>m\\ p+q\leqslant m+l}} = \sum_{\substack{p<m\\ p-q\geqslant m-l\\ p+q\leqslant m+l}} \quad \text{and} \quad \sum_{\substack{q>l\\ p-q\leqslant m-l-2\\ p+q\leqslant m+l}} = \sum_{\substack{q\leqslant l\\ p-q\leqslant m-l-2\\ p+q\leqslant m+l}} .$$

Here we illustrate the above equality for $n = 9$, $m = 4$, see (4.10).

$$\begin{pmatrix} \beta_{00} & \beta_{01} & \beta_{02} & \beta_{03} & \beta_{04} & \beta_{05} & \beta_{06} & * & * & * \\ \beta_{10} & \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} & \beta_{15} & * & * & * & * \\ \beta_{20} & \beta_{22} & \beta_{22} & \beta_{23} & \beta_{24} & * & * & * & * & * \\ \beta_{30} & \beta_{31} & \beta_{32} & \beta_{33} & * & * & * & * & * & * \\ \beta_{40} & \beta_{41} & \beta_{42} & * & * & * & * & * & * & * \end{pmatrix} \tag{4.10}$$

It follows that

$$u(M \times N) = \sum_{\substack{p=m\\ q\leqslant l}} = \beta_m(M) \sum_{q=0}^{l} \beta_q(N) \bmod 2.$$

On the other hand, we have

$$u(N)\chi(M) = \chi(M) \sum_{q=0}^{l} \beta_q(N).$$

By Poincaré duality $\chi(M) = \beta_m(M) \bmod 2$. This finishes the proof. $\square$

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