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Mathematical Methods for $4d$ $\mathcal{N} = 2$ QFTs

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Part I

General Overview

1 Introduction and Motivation

In this work we study different aspects of $4d$ $\mathcal{N} = 2$ superconformal field theories. Not only we accurately define what we mean by a $4d$ $\mathcal{N} = 2$ superconformal field theory, but we also invent and apply new mathematical methods to classify these theories and to study their physical content. Therefore, although the origin of the subject is physical, our methods and approach are rigorous mathematical theorems: the physical picture is useful to guide the intuition, but the full mathematical rigor is needed to get deep and precise results. No familiarity with the physical concept of Supersymmetry (SUSY) is need to understand the content of this thesis: everything will be explained in due time. The reader shall keep in mind that the driving force of this whole work are the consequences of SUSY at a mathematical level. Indeed, as it will be detailed in part II, a mathematician can understand a $4d$ $\mathcal{N} = 2$ superconformal field theory as a complexified algebraic integrable system. The geometric properties are very constrained: we deal with special Kähler geometries with a few other additional structures (see part II for details). Thanks to the rigidity of these structures, we can compute explicitly many interesing quantities: in the end, we are able to give a coarse classification of the space of "action" variables of the integrable system, as well as a fine classification – only in the case of rank $k = 1$ – of the spaces of "angle" variables.

We were able to classify conical special Kähler geometries via a number of deep facts of algebraic number theory, diophantine geometry and class field theory: the perfect overlap between mathematical theorems and physical intuition was astonishing. And we believe we have only scratched the surface of a much deeper theory: we can probably hope to get much more information than what we already discovered; of course, a deeper study of the subject – as well as its generalizations – is required.

A $4d$ $\mathcal{N} = 2$ superconformal field theory can thus be defined by its geometric structure: its scaling dimensions, its singular fibers, the monodromy around them and so on. But giving a proper and detailed definition is only the beginning: one may be interested in exploring its physical content. In particular, we are interested in supersymmetric quantities such as BPS states, framed BPS states and UV line operators. These quantities, thanks to SUSY, can be computed independently of many parameters of the theory: this peculiarity makes it possible to use the language of category theory to analyze the aforementioned aspects. As it will be proven in part V, to each $4d$ $\mathcal{N} = 2$ superconformal field theory we can associate a web of categories, all connected by functors, that describe the BPS states, the framed BPS states (IR) and the UV line operators. Hence, following the old ideas of 't Hooft, it is possible to describe the phase space of gauge theories via categories, since the vacuum expectation values of such line operators are the order parameters of the confinement/deconfinement phase transitions. Mathematically, the (quantum) cluster algebra of Fomin and Zelevinski is the structure needed. Moreover, the analysis of BPS objects led us to a deep understanding of generalized S-dualities. Not only were we able to precisely define – abstractly and generally – what the S-duality group of a $4d$ $\mathcal{N} = 2$ superconformal field theory should be, but we were also able to write a computer algorithm to obtain these groups in many examples (with very high accuracy).

The structure of the thesis is organized as follows: we start with a more detailed introduction of the various topics analyzed during my PhD years in SISSA: this is the remaining content of part I. Part II contains all the mathematical background needed to understand the geometry of generic $4d$ $\mathcal{N} = 2$ superconformal field theories. Part II has its roots in the seminal papers of Seiberg and

Witten [223, 224] and it includes many progresses of the last decade. We develop our framework with great details: after the main definitions are settled, we pass onto part III. In this part we give a fine classification of all rank $k = 1$ theories: our main tool is the Kodaira-Néron model and the Mordell-Weil theorem. Indeed, it turns out that the classification of physical objects can be found inside the tables classifying all MW lattices with certain properties (see section 8.1 for more details). The totally explicit description of the rank one case is very useful to have simple examples always at hand.
Part IV contains the coarse classification of the scaling dimensions of the Coulomb Branch. The study of the monodromy around normal rays as well as many deep theorems of class field theory are needed to prove the rules of the classification. In section 12 we provide the full list of scaling dimensions up to rank 4.
Part V employs a different language: the language of category theory. This language is very general and totally well defined. Unfortunately, the categories we have found – although they describe the BPS physics very well – are heavily based on the existence of a BPS quiver (the quiver of a 2d theory associated to the $4d$ $\mathcal{N} = 2$ SCFT via the the 2d/4d correspondence, as described in appendix A). We are currently trying to remove this strong hypothesis. Our starting point would be to find a categorical classification of the discretely gauged rank one theories of III: this work is still in progress.
Finally, in part VI we provide the proof and the description of the algorithm to compute the generalized S-duality groups as well as the vacuum expectation values of the UV line operators. Many explicit examples are provided: it is manifest how these methods can be used to tackle problems which are too hard for other techniques.
Eventually we provide some additional details is the appendices as well as a full bibliography.

2 Geometric classification

Following the seminal papers by Seiberg and Witten [223, 224], in the last years a rich landscape of four-dimensional $\mathcal{N} = 2$ superconformal field theories (SCFT) had emerged, mostly without a weakly-coupled Lagrangian formulation [58, 59, 66, 79, 80, 90, 110, 123, 125, 126, 141, 157, 190, 243, 244, 249]. It is natural to ask for a map of this vast territory, that is, for a classification of unitary 4d $\mathcal{N} = 2$ SCFTs. The work in this direction follows roughly two approaches: the first one aims to partial classifications of $\mathcal{N} = 2$ SCFTs having some specific construction [90, 123, 141, 157, 244] or particular property [67, 110]. The second approach, advocated in particular by the authors of refs. [19–22, 26, 27], relates the classification of $\mathcal{N} = 2$ SCFTs to the geometric problem of classifying the *conic special geometries* (CSG) which describe their IR physics along the Coulomb branch M *à la* Seiberg and Witten [223, 224]. The first part of this thesis belongs to this second line of thought: it is meant to be a contribution to the geometric classification of CSG with applications to $\mathcal{N} = 2$ SCFT.

Comparison with other classification problems in complex geometry suggests that, while describing all CSG up to isomorphism may be doable when M has very small dimension,[1] it becomes rapidly intractable as we increase the rank k. A more plausible program would be a *coarse-grained*

[1] The complex dimension k of the Coulomb branch M is also known as the *rank* of the SCFT. We shall use the two terms interchangeably.

classification of the CSG not up to isomorphism but rather up to some kind of "birational" equivalence, that is, neglecting the details of the geometry along "exceptional" loci of positive codimension. This is the point of view we adopt in our analysis.

The classification problem is in a much better shape than one may expect. Indeed, while *a priori* the Coulomb branch M is just a complex analytic space, it follows from the local properties of special Kähler geometry that a CSG must be a complex cone over a base K which is a normal projective variety. K is (birational to) an algebraic variety of a very special kind: a simply-connected *log-Fano* [150] with Picard number one and trivial Hodge diamond (a special instance of a *Mori dream space* [148]).

In practice, in the coarse-grained classification we limit our ambition to the description of the allowed rings $\mathscr{R}$ of holomorphic functions on M (the Coulomb branch chiral rings). Several distinct (deformation-types of) SCFTs have the same $\mathscr{R}$ but differ in other respects as their flavor symmetry. The simplest example of a pair of distinct SCFT with the same $\mathscr{R}$ is given by $SU(N)$ SQCD with $2N$ fundamentals and $\mathcal{N} = 2^*$ with the same gauge group; see refs. [19–22] and part III of the present thesis for additional examples in rank 1.
The chiral ring $\mathscr{R}$ of a SCFT is graded by the value of the $U(1)_R$ charge (equal to the dimension Δ for a chiral operator). The general expectation[2] is that the Coulomb branch chiral ring is a graded free polynomial ring,

$$\mathscr{R} = \mathbb{C}[u_1, u_2, \cdots, u_k]. \tag{2.1}$$

This is equivalent[3] to saying that the log-Fano K is a *weighted projective space* (WPS). The last statement is only slightly stronger than the one in the previous paragraph: all WPS are simply-connected log-Fano with Picard number one and trivial Hodge diamond [100]. Conversely, a *toric* log-Fano with these properties is necessarily a (fake[4]) weighted projective space [48]. Then it appears that the log-Fano varieties which carry all the structures implied by special geometry form to a class of manifolds only slightly more general than the WPS. This explains why many $\mathcal{N} = 2$ models have free chiral rings.

Assuming (2.1), the information encoded in the ring $\mathscr{R}$ is just the k-tuple $\{\Delta_1, \Delta_2, \cdots, \Delta_k\}$ of the $U(1)_R$ charges of its free generators u_i. Even if the ring is non-free the spectrum of dimensions of $\mathscr{R}$ is a basic invariant of the SCFT. The coarse-grained classification of CSG then aims to list the allowed Coulomb branch dimension k-tuples $\{\Delta_1, \cdots, \Delta_k\}$ for each rank $k \in \mathbb{N}$. An even less ambitious program is to list the finitely-many real numbers Δ which may be the dimension of a Coulomb branch generator in a $\mathcal{N} = 2$ SCFT of rank at most k. In a unitary theory, all Δ's are rational numbers ≥ 1. If the chiral ring is non-free, but with a finite free covering, we easily reduce to the above case.

Non-free chiral rings. After the submission of our paper [52], the article [42] appeared in the arXiv where examples of $\mathcal{N} = 2$ theories with non-free chiral rings are constructed. Those examples are in line with our geometric discussion being related to the free ring case by a finite quotient (gauging). Most of the discussion of the present thesis applies to these more general situation as

[2] See however [15] for a discussion of the phenomena which would appear if this is not the case. We shall briefly elaborate on this topic in the third *caveat* of §. 5.1.1.
[3] Since the dimensions Δ are positive rationals.
[4] All *fake* WPS are quotients of WPS by finite Abelian groups.

well (many arguments are formulated modulo finite quotients). The only point where we assume that the ring $\mathscr{R}$ is free[5], in order to simplify the analysis, is in showing that the Coulomb branch contains "many" normal rays. Our argument in the present form requires just *one* normal ray, so the request of "many" of them is rather an overkill. It is relatively straightforward to extend the details of our analysis to finite quotients.

In the rest of this **Introduction** we present a non-technical survey of our main results for both the list of allowed dimensions and dimension k-tuples. In particular, §. 2.3 contains a heuristic derivation of the Universal Dimension Formula.

2.1 Coulomb branch dimensions Δ

In section 4 we present a very simple recursive algorithm to produce the list of (putative) dimensions Δ allowed in a rank-k CSG for all $k \in \mathbb{N}$. The dimension lists for ranks up to 13 are presented in the tables of section 6. After the completion of our paper [52], ref. [24] appeared on the arXiv where the list of dimensions for $k = 2$ is also computed. Our results are in perfect agreement with theirs.

Let us describe some general property of the set of dimensions in given rank k. The number of allowed Δ's is not greater than a certain Number-Theoretical function $\boldsymbol{N}(k)$ of the rank k

$$\#\left\{\Delta \in \mathbb{Q}_{\geq 1}\;\middle|\;\begin{array}{l}\Delta \equiv \text{dimension of a Coulomb branch}\\ \text{free generator in a CSG of rank} \leq k\end{array}\right\} \leq \boldsymbol{N}(k). \tag{2.2}$$

There is evidence that $\leq$ may be actually replaced by an equality sign.

$\boldsymbol{N}(k)$ is a rather peculiar function: it is stationary for "most" $k \in \mathbb{N}$, $\boldsymbol{N}(k) = \boldsymbol{N}(k-1)$ and, while the ratio

$$\nu(2k) = \big(\boldsymbol{N}(k) - \boldsymbol{N}(k-1)\big)/2k \tag{2.3}$$

is "typically" a small integer, it takes all integral value ≥ 2 infinitely many times. The first few values of $\boldsymbol{N}(k)$ are listed in table 1. $\boldsymbol{N}(k)$ may be written as a Stieltjes integral of the function[6] $\boldsymbol{N}(k)_{\text{int}}$ which counts the number of *integral* dimensions Δ at rank k

$$\boldsymbol{N}(k) = 2\int_0^{k+\epsilon} x\, d\boldsymbol{N}\big(x\big)_{\text{int}}. \tag{2.4}$$

From this expression we may easily read the large rank asymptotics[7] of $\boldsymbol{N}(k)$

$$\boldsymbol{N}(k) = \frac{2\,\zeta(2)\,\zeta(3)}{\zeta(6)}\,k^2 + o(k^2) \quad \text{as } k \to \infty. \tag{2.5}$$

As mentioned above, we expect this bound to be optimal, that is, the actual number of Coulomb dimensions to have the above behavior for large k.

[5] A part for the explicit examples of course.
[6] For the precise definition of $\boldsymbol{N}(x)_{\text{int}}$ for *real* positive x, see §. 10.4.2.
[7] $\zeta(s)$ is the Riemann zeta-function.

k	1	2	3	4	5	6	7	8	9	10	11	12
$\boldsymbol{N}(k)$	8	24	48	88	108	180	180	276	348	448	492	732
k	13	14	15	16	17	18	19	20	21	22	23	24
$\boldsymbol{N}(k)$	732	788	848	1072	1072	1360	1360	1720	1888	2020	2112	2640

Table 1: Values of the function $\boldsymbol{N}(k)$ for ranks $k \leq 25$ ($\boldsymbol{N}(25) = \boldsymbol{N}(24)$).

The number $\boldsymbol{N}(k)$ is vastly smaller than the number of isoclasses of CSG of rank $\leq k$, showing that the coarse-grained classification is dramatically simpler than the fine one. The counting (2.2) should be compared with the corresponding one for Lagrangian $\mathcal{N}=2$ SCFTs

$$\#\left\{\Delta \in \mathbb{Q}_{\geq 1} \;\middle|\; \begin{array}{l}\Delta \equiv \text{dimension of a Coulomb branch free}\\ \text{generator in a } \textit{Lagrangian model} \text{ of rank} \leq k\end{array}\right\} = \left\lfloor \frac{3\,k}{2} \right\rfloor, \quad k \geq 15, \tag{2.6}$$

which confirms the idea that the Lagrangian dimensions have "density zero" in the set of all $\mathcal{N}=2$ Coulomb dimensions. The Lagrangian dimensions are necessarily integers; the number of allowed *integral* dimension at rank k is (not greater than)

$$\boldsymbol{N}(k)_{\text{int}} = \frac{2\,\zeta(2)\,\zeta(3)}{\zeta(6)}\,k + o(k) \quad \text{as } k \to \infty \tag{2.7}$$

so, for large k, roughly 38.5% of all allowed *integral* dimensions may be realized by a Lagrangian SCFT. Remarkably, for $k \geq 15$ the ratio

$$\boldsymbol{\varrho}(k) = \frac{\#\big\{\text{Lagrangian dimensions in rank } k\big\}}{\#\big\{\text{integral dimensions in rank } k\big\}} \tag{2.8}$$

is roughly independent of k up to a few percent modulation, see e.g. table 2.

2.2 Dimension k-tuples and Dirac quantization of charge

The classification of the dimension k-tuples $\{\Delta_1, \cdots, \Delta_k\}$ allowed in a rank-k CSG contains much more information than the list of the individual dimensions Δ. Indeed, the values of the dimensions of the various operators in a given SCFT are strongly correlated. The list of dimension k-tuples may also be explicitly determined recursively in k using our Universal Dimension Formula.

The problem may be addressed at two levels: there is a simple algorithm which produces, for a given k, a finite list of would-be dimension k-tuples. However there are subtle Number Theoretical aspects, and some of these k-tuples are consistent only under special circumstances. The tricky point is as follows: a special geometry is, in particular, an analytic family of *polarized* Abelian varieties. The polarization corresponds physically to the Dirac electro-magnetic pairing Ω, which is an integral, non-degenerate, skew-symmetric form on the charge lattice. Usually one assumes this polarization to be *principal*, that is, that all charges which are consistent with Dirac quantization are permitted. But physics allows Ω to be non-principal [103] at the cost of introducing additional selection rules on the values of the electro-magnetic charges and fluxes (see §. 5.1.1 for details).

k	**1**	15	16	17	18	19	20	21	22	23
$\boldsymbol{\varrho}(k)$	**0.4**	0.3859	0.375	0.3906	0.375	0.3888	0.3703	0.3647	0.375	0.3777
k	24	25	26	27	28	29	30	31	32	∞
$\boldsymbol{\varrho}(k)$	0.3564	0.3663	0.3786	0.3809	0.3888	0.3909	0.3781	0.3865	0.3779	**0.3858**

Table 2: Values of the ratio $\boldsymbol{\varrho}(k)$ (up to four digits) for various values of the rank $k \in \mathbb{N}$, including $k = 1$ and the asymptotic value for $k = \infty$.

The deep arguments of ref. [30] suggest that Ω should be principal for a $\mathcal{N} = 2$ QFT which emerges from a consistent quantum theory of gravity in some decoupling limit. It turns out that only a subset of the dimension k-tuples produced by the simple algorithm are consistent with a principal polarization; the others may be realized only in generalized special geometries endowed with suitable *non*-principal polarization i.e. to be consistent they require additional selection rules on the electro-magnetic charges. Therefore one expects that such Coulomb dimensions would not appear in $\mathcal{N} = 2$ SCFT having a stringy construction. On the other hand, the Jacobian of a genus g curve carries a canonical principal polarization; thus the special geometry of a SCFT with such "non-principal" Coulomb dimensions cannot be described by a Seiberg-Witten *curve.*

To determine the dimension k-tuples which are compatible with a principal polarization is a subtle problem in Number Theory. For instance, the putative dimension list in rank 2 contains the two pairs[8] $\{12, 6\}$ and $\{12, 8\}$ (resp. the two pairs $\{10/7, 8/7\}$ and $\{12/7, 8/7\}$) but only the first one is consistent with a principal polarization. The pair $\{12, 6\}$ corresponds to rank 2 Minahan-Nemeshansky (MN) of type E_8 [35,129] (resp. $\{10/7, 8/7\}$ to Argyres-Douglas (AD) model of type A_4); since this model has a stringy construction, the Number Theoretic subtlety is consistent with the physical arguments of [30].

The reason why four of the putative rank-2 pairs $\{\Delta_1, \Delta_2\}$ are not consistent with a principal Ω looks rather exoteric at first sight: while the ideal class group of the number field $\mathbb{Q}[\zeta]$ (ζ a primitive 12-th root of unity) is trivial, the *narrow* ideal class group of its totally real subfield $\mathbb{Q}[\sqrt{3}]$ is $\mathbb{Z}_2$, and the narrow class group is an obstruction to the consistency of such dimension pairs in presence of a principal polarization (a hint of why this group enters in the game will be given momentarily in §. 2.3). To see which one of the two pairs $\{12, 6\}$ or $\{12, 8\}$ survives, we need to understand the action of the narrow ideal class group; it turns out that Class Field Theory properly selects the physically expected dimensions $\{12, 6\}$. We regard this fact as a non-trivial check of our methods.

Remark 2.2.1. Let us give a rough physical motivation for the role of Class Field Theory in our problem. It follows from the subtle interplay between the dynamical breaking of the SCFT $U(1)_R$ symmetry[9] in the supersymmetric vacua and the Dirac quantization of charge. Along the Coulomb branch M, the $U(1)_R$ symmetry should be spontaneously broken. But there are special holomorphic subspaces $M_n \subset M$ which parametrize SUSY vacua where a discrete subgroup $\mathbb{Z}_n \subset U(1)_R$ remains unbroken. Assuming eqn.(2.1), the locus

$$\big\{u_i = 0 \text{ for } i \neq i_0\big\} \subset M \qquad (2.9)$$

[8] Note that all three numbers 12, 8, and 6 are allowed as *single* dimensions in rank 2 even if Ω is principal.
[9] Properly speaking, what we call "$U(1)_R$" is the quotient group $U(1)_R/\langle(-1)^F\rangle$ acting effectively on the bosons.

is such a subspace M_n with n the order of $1/\Delta_{i_0}$ in $\mathbb{Q}\backslash\mathbb{Z}$. To the locus M_n one associates the rational group-algebra $\mathbb{Q}[e^{2\pi i R/\Delta_{i_0}}]$ of the unbroken R-symmetry $\mathbb{Z}_n$. The chiral ring $\mathscr{R}$ is then a module of this group-algebra (of non-countable dimension). Replace $\mathscr{R}$ by the much simpler subring $\mathscr{I}\subset\mathscr{R}$ of chiral operators of integral $U(1)_R$ charge; $\mathscr{I}=\mathscr{R}^{\mathbb{M}}$ where $\mathbb{M}$ is the quantum monodromy of the SCFT [60, 65]. Since $\mathsf{Proj}\,\mathscr{I}\cong\mathsf{Proj}\,\mathscr{R}$ [140] there is no essential loss of information in the process. $\mathscr{I}$ is a $\mathbb{C}$-algebra; Dirac quantization is the statement that $\mathscr{I}$ is obtained from a $\mathbb{Q}$-algebra $\mathscr{I}_\mathbb{Q}$ by extension of scalars, $\mathscr{I}\cong\mathscr{I}_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{C}$. An element of $\mathscr{I}_\mathbb{Q}$ is simply a holomorphic function which locally restricts to an element of $\mathbb{Q}[a^i,b_j]$, where (a^i,b_j) are the periods of special geometry well-defined modulo $Sp(2k,\mathbb{Z})$. E.g. if our model is a Lagrangian SCFT with gauge group G of rank k, $\mathscr{I}_\mathbb{Q}=\mathbb{Q}[a^i,b_j]^{\mathrm{Weyl}(G)}$. $\mathscr{I}_\mathbb{Q}$ is a module of $\mathbb{Q}[e^{2\pi i R/\Delta_{i_0}}]$ of just countable dimension. By Maschke theorem [106] $\mathscr{I}_\mathbb{Q}$ is a countable sum of Abelian number fields

$$\mathscr{I}_\mathbb{Q}=\bigoplus_\alpha \mathbb{F}_\alpha. \tag{2.10}$$

Being Abelian, the fields $\mathbb{F}_\alpha$ are best studied by the methods of Class Field Theory. On the other hand, the Coulomb dimensions $\Delta(\phi)$ are just the characters of $U(1)_R$ appearing in $\mathscr{R}$

$$\chi_\phi\colon e^{2\pi i t R}\mapsto e^{2\pi i t\Delta(\phi)}\in\mathbb{C}^\times,\qquad \text{for } \phi\in\mathscr{R} \text{ of definite dimension.} \tag{2.11}$$

Focusing on the subspace $M_n\subset M$, the characters $\{\chi_\phi\}$ induce characters of the unbroken subgroup $\mathbb{Z}_n$. Hence the Coulomb dimensions $\Delta(\phi)$ may be read from the decomposition of $\mathscr{R}$ into characters of $\mathbb{Z}_n$. If all Coulomb dimensions are integral, $\mathscr{R}=\mathscr{I}_\mathbb{Q}\otimes_\mathbb{Q}\mathbb{C}$, and the last decomposition is obtained from the one in (2.10) by tensoring with $\mathbb{C}$, so that we may read $\Delta(\phi)$ directly from the Number Theoretic properties of the $\mathbb{F}_\alpha$. The same holds in the general case, *mutatis mutandis.* In part IV we shall deduce the list of allowed Coulomb dimensions by a detailed geometric analysis, but the final answer is already given by eqn.(2.10) when supplemented with the obvious relation between the rank k of the SCFT and the degrees of the number fields $\mathbb{F}_\alpha$.

2.3 Rank 1 and natural guesses for $k\geq 2$

The case of rank one is well known [26]. The allowed Coulomb dimensions are

$$\Delta=1,\ 2,\ 3,\ 3/2,\ 4,\ 4/3,\ 6,\ 6/5, \tag{2.12}$$

$\Delta=1$ corresponds to the free (Lagrangian) theory, $\Delta=2$ to interacting Lagrangian models (i.e. $SU(2)$ gauge theories), and all other dimensions to strongly interacting SCFTs. A crucial observation is that the list of dimensions (2.12) is organized into orbits of an Abelian group $H_\mathbb{R}$. For $k=1$ the group is simply $H_\mathbb{R}\cong\mathbb{Z}_2$ generated by the involution ι

$$\iota\colon\Delta\mapsto\Delta'=\frac{1}{\langle 1-\Delta^{-1}\rangle},\qquad \left|\ \begin{array}{l}\text{where, for } x\in\mathbb{R},\ \langle x\rangle \text{ denotes the real}\\ \text{number equal } x \bmod 1 \text{ with } 0<x\leq 1.\end{array}\right. \tag{2.13}$$

The Lagrangian models correspond to the fixed points of $H_\mathbb{R}$, $\Delta=1,2$. (2.14)

There are dozens of ways to prove that eqn.(2.12) is the correct set of dimensions for $k=1$

$\mathcal{N}=2$ SCFT; each argument leads to its own interpretation[10] of this remarkable list of rational numbers and of the group $H_{\mathbb{R}}$. Each interpretation suggests a possible strategy to generalize the list (2.12) to higher k. We resist the temptation to focus on the most elegant viewpoints, and stick ourselves to the most obvious interpretation of the set (2.12):

Fact. *The allowed values of the Coulomb dimension* Δ *for rank* 1 $\mathcal{N}=2$ *SCFTs, eqn.*(2.12)*, are in one-to-one correspondence with the elliptic conjugacy classes in the rank-one duality-frame group,* $Sp(2,\mathbb{Z})\equiv SL(2,\mathbb{Z})$*. Lagrangian models correspond to central elements (which coincide with their class). The group* $H_{\mathbb{R}}\cong GL(2,\mathbb{Z})/SL(2,\mathbb{Z})$ *permutes the distinct* $SL(2,\mathbb{Z})$*-conjugacy classes which are conjugate in the bigger group* $GL(2,\mathbb{Z})$*.*

By an *elliptic* conjugacy class we mean a conjugacy class whose elements have finite order. There are several ways to check that the above **Fact** is true. The standard method is comparison with the Kodaira classification of exceptional fibers in elliptic surfaces [166]. Through the homological invariant [166], Kodaira sets the (multiplicity 1) exceptional fibers in one-to-one correspondence with the quasi-unipotent conjugacy classes of $SL(2,\mathbb{Z})$. In dimension 1 the homological invariant of a CSG must be semi-simple. Since quasi-unipotency and semi-simplicity together imply finite order, **Fact** follows. The trivial conjugacy class of 1 corresponds to the free SCFT, the class of the central element -1 to $SU(2)$ gauge theories, and the regular elliptic classes to strongly-coupled models with no Lagrangian formulation. The map between (conjugacy classes of) elliptic elements of $SL(2,\mathbb{Z})$ and Coulomb branch dimensions Δ is through their modular factor $(c\tau+d)$ evaluated at their fixed point[11] τ in the upper half-plane $\mathfrak{h}$. Explicitly:

$$\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in SL(2,\mathbb{Z})\ \text{elliptic}\longrightarrow \Delta=\frac{2\pi i}{\mathbf{log}(c\tau+d)},\qquad \left|\ \begin{matrix}\text{where } \tau \text{ is a solution to}\\ a\tau+b=\tau(c\tau+d),\end{matrix}\right. \tag{2.15}$$

and $\mathbf{log}\, z$ is the branch of the logarithm such that $\mathbf{log}(e^{2\pi i x})=2\pi i\langle x\rangle$ for $x\in\mathbb{R}$ (cfr. eqn.(2.13) for the notation). The action of $\iota\in H_{\mathbb{R}}$ (eqn.(2.13)) is equivalent to[12] $\tau\leftrightarrow\bar\tau$, i.e. $\mathbf{log}(c\tau+d)/2\pi i\leftrightarrow \langle 1-\mathbf{log}(c\tau+d)/2\pi i\rangle$.

The basic goal of the coarse-grained classification of $\mathcal{N}=2$ SCFT is to provide the correct generalization of the above **Fact** to arbitrary rank k. The natural guess is to replace the rank-one duality group $SL(2,\mathbb{Z})$ by its rank-k counterpart, i.e. the Siegel modular group $Sp(2k,\mathbb{Z})$, and consider its finite-order conjugacy classes. Now the fixed-point modular factor, $C\boldsymbol{\tau}+D$, is a $k\times k$ unitary matrix with eigenvalues λ_i, ($i=1,\dots,k$ and $|\lambda_i|=1$), to which we may tentatively associate the k-tuple $\{\Delta_i\}_{i=1}^k$ of would-be Coulomb dimensions

$$\Delta_i=1+\frac{2\pi i-\mathbf{log}\,\lambda_i}{\mathbf{log}\,\lambda_1}\in\mathbb{Q}_{\geq 1}\quad\text{for } i=1,2,\dots,k, \tag{2.16}$$

[10] To mention just a few: the set of $\mathbb{Z}_2$-orbits in eqn.(2.12) is one-to-one correspondence with: *(i)* Coxeter labels of the unique node of valency >2 in an affine Dynkin graph which is also a star; *(ii)* Coxeter numbers of semi-simple rank-2 Lie algebras; *(iii)* degrees of elliptic curves written as complete intersections in WPS, *(iv)* and so on.

[11] The locus of fixed points in $\mathfrak{H}$ of an elliptic element of the modular group $SL(2,\mathbb{Z})$ is not empty and connected, see **Lemma 10.2.1**. Note that $(c\tau+b)$, being a root of unity, is independent of the chosen τ in the fixed locus.

[12] $\bar\tau$ is in the lower half-plane; to write everything in the canonical form, one should conjugate it to a point in the upper half-plane by acting with the proper orientation-reversing element of $GL(2,\mathbb{Z})$.

giving a putative 1-to-k correspondence[13] between $Sp(2k,\mathbb{R})$-conjugacy classes of elliptic elements in the Siegel group $Sp(2k,\mathbb{Z})$ and would-be dimension k-tuples. The candidate correspondence (2.16) reduces to the Kodaira one for $k=1$, and is consistent with the physical intuition of **Remark 2.2.1**.

It turns out that for $k \geq 2$ the guess (2.16) is morally correct, but there are many new phenomena and subtleties with no counterpart in rank 1, so the statement of the correspondence should be taken with *a grain of salt,* and supplemented with the appropriate limitations and specifications, as we shall do in part IV. In particular, the same k-tuple $\{\Delta_i\}_{i=1}^k$ is produced by a number $\leq k$ of distinct conjugacy classes; in facts, the geometrically consistent k-tuples are those which appear precisely k times (properly counted).

In rank $k \geq 2$ the notion of "duality-frame group" is subtle. The Siegel modular group $Sp(2k,\mathbb{Z})$ is the arithmetic group preserving the principal polarization. If Ω is not principal, $Sp(2k,\mathbb{Z})$ should be replaced by the arithmetic group $S(\Omega)_{\mathbb{Z}}$ which preserves it

$$S(\Omega)_{\mathbb{Z}} = \Big\{ m \in GL(2k,\mathbb{Z}) \colon m^t \Omega m = \Omega \Big\}. \tag{2.17}$$

$Sp(2k,\mathbb{Z})$ and $S(\Omega)_{\mathbb{Z}}$ are commensurable arithmetic subgroups of $Sp(2k,\mathbb{Q})$ [197]. If a SCFT has a non-principal polarization Ω, its Coulomb dimensions are related to the elliptic conjugacy classes in $S(\Omega)_{\mathbb{Z}}$ which (in general) lead to different eigenvalues λ_i and Coulomb dimensions Δ_i.

As in the $k=1$ case, the dimension k-tuples $\{\Delta_i\}_{i=1}^k$ form orbits under a group. The most naive guess is that this is the "automorphism group" of eqn.(2.16), $\mathbb{Z}_k \rtimes \mathbb{Z}_2^k$, where the first factor cyclically permutes the λ_i while $\mathbb{Z}_2^k$ is the straightforward generalization of ι for $k=1$:

$$\iota_j \colon \mathbf{log}\,\lambda_i/2\pi i \longmapsto \begin{cases} \langle 1 - \mathbf{log}\,\lambda_j/2\pi i \rangle & \text{for } i=j \\ \mathbf{log}\,\lambda_i/2\pi i & \text{otherwise,} \end{cases} \qquad j=1,2,\dots,k. \tag{2.18}$$

However, in general, this action would not map classes in $Sp(2k,\mathbb{Z})$ to classes in the same group but rather in some other arithmetic group $S(\Omega)_{\mathbb{Z}}$. The proper generalization of the $k=1$ case requires to replace[14] $\mathbb{Z}_2^k$ by the Abelian group $H_{\mathbb{R}}$ which permutes the $Sp(2k,\mathbb{R})$-conjugacy classes of elliptic elements of the Siegel modular group $Sp(2k,\mathbb{Z})$ which are conjugate in $GL(2k,\mathbb{C})$. $H_{\mathbb{R}}$ is a subgroup of the group (2.18), i.e. we have an exact sequence

$$1 \to H_{\mathbb{R}} \to \mathbb{Z}_2^k \to C \to 1 \tag{2.19}$$

for some 2-group C. In the simple case when the splitting field $\mathbb{K}$ of the elliptic element has class number 1, C is just the narrow class group $C_{\Bbbk}^{\text{nar}}$ of its maximal totally real subfield $\Bbbk \subset \mathbb{K}$. For instance, the dimension pair $\{12,6\}$ (the $k=2$ E_8 MN model mentioned before) is reproduced by eqn.(2.16) for $\mathbf{log}\,\lambda_1 = 2\pi i/12$ and $\mathbf{log}\,\lambda_2 = 14\pi i/12$; applying naively eqn.(2.18) we would get the dimension pair $\iota_2\{12,6\} = \{12,8\}$. However in this case $\iota_2 \notin H_{\mathbb{R}}$, and the $H_{\mathbb{R}}$-orbit of $\{12,6\}$ does not contain $\{12,8\}$ which then is not a valid dimension pair for $k=2$ for the duality-frame group

[13] Since we have a k-fold choice of which eigenvalue we wish to call λ_1.

[14] $\mathbb{Z}_2^k$ is the group which permutes the $Sp(2k,\mathbb{R})$-conjugacy classes of elliptic elements of the *real* group $Sp(2k,\mathbb{R})$ which are conjugate in $GL(2k,\mathbb{C})$. However some real conjugacy class may have no integral element, and only a subgroup survives over $\mathbb{Z}$. This implies that $H_{\mathbb{R}}$ is indeed a subgroup of $\mathbb{Z}_2^k$.

$Sp(2k,\mathbb{Z})$ but it is admissible if the duality-frame group is $S(\Omega)_\mathbb{Z}$ with $\det\Omega = 2$ or larger.

Remark 2.3.1. The generalization to higher k of the $k = 1$ criterion (2.14) for the Coulomb dimensions to be consistent with a weakly-coupled Lagrangian description is as follows. Let $\iota \in H_\mathbb{R}$ be given by

$$\iota = \iota_1\,\iota_2\,\cdots\,\iota_k. \tag{2.20}$$

If a dimension k-tuple $\{\Delta_1, \cdots, \Delta_k\}$ may be realized by a Lagrangian SCFT then it is left invariant by ι up to a permutation of the Δ_i. The inverse implication is probably false.

2.4 Springer Theory of reflection groups

The proposed dimension formula (2.16) may look puzzling at first. Being purportedly universal, it should, in particular, reproduce the correct dimensions for a weakly-coupled Lagrangian SCFT with gauge group an arbitrary semi-simple Lie group G. By the non-renormalization and Harrish-Chandra theorems [122], in the Lagrangian case the dimension k-tuple $\{\Delta_i\}$ is just the set $\{d_i\}$ of the degrees of the Casimirs of G (its exponents $+1$). Thus eqn.(2.16), if correct, implies a strange universal formula for the degrees of a Lie algebra which looks rather counter-intuitive from the Lie theory viewpoint.

The statement that (2.16) is the correct degree formula (not just for Weyl groups of Lie algebras, but for all finite reflection groups) is the main theorem in the Springer Theory of reflection groups [38, 83, 237], We shall see in §. 10.3.5 that the correspondence between our geometric analysis of the CSG and Springer Theory is more detailed than just giving the right dimensions. In particular, Springer Theory together with weak-coupling QFT force us to use in eqn.(2.16) the universal determination of the logarithm we call **log** (see after eqn.(14.2)), which is therefore implied by conventional Lagrangian QFT.

In other words, the proposed Universal Dimension Formula (2.16) may also be obtained using the following

Strategy. *Write the usual dimension formula valid for* <u>*all*</u> *weakly-coupled Lagrangian SCFTs in a clever way, so that it continues to make sense even for non-Lagrangian SCFT, i.e. using only intrinsic physical data such as the breaking pattern of* $U(1)_R$. *This leads you to eqn.*(2.16). *Then claim the formula to have general validity.*

This is the third heuristic derivation of (2.16) after the ones in §.1.2 and 1.3. The sheaf-theoretic arguments of §. 10.3.4 will make happy the pedantic reader (at least we hope). It will also supplement (2.16) all the required specifications and limitations.

Remark 2.4.1. Inverting the argument, we may say that our analysis of the CSG yields a (simpler) transcendental proof of the classical Springer results.

3 Introduction to rank $k = 1$ theories

The classification of all 4d $\mathcal{N} = 2$ SCFTs of rank k may be (essentially) reduced to the geometric problem of classifying all dimension k special geometries [19, 20, 26, 27, 103, 104, 223, 224]. This classification is naturally organized in two distinct steps. At the *coarse-grained* level one lists the allowed k-tuples $\{\Delta_1, \Delta_2, \cdots, \Delta_k\}$ of dimensions of operators generating the Coulomb branch (see refs. [18, 24, 52] for recent progress on this problem). Then we have the *fine* classification of the physically inequivalent models belonging to each coarse-grained class, that is, the list of the distinct QFT which share the same dimension k-tuple. Theories in the same coarse-grained class differ by invariants like the flavor symmetry group, the conformal charges k_F, a, c, and possibly by subtler aspects. For $k = 1$ the fine classification has been worked out by Argyres *et al.* in a series of remarkable papers [19–23]. To restrict the possibilities, these authors invoke some physically motivated conjectures like "planarity", "absence of dangerous irrelevant operators", and "charge quantization".

The purpose of part III is to revisit the fine classification for $k = 1$, introducing new ideas and techniques which we hope may be of help for a future extension of the fine classification beyond $k = 1$. In the process we shall greatly simplify and clarify several points of the $k = 1$ case and provide proofs of (versions of) the above conjectures.

We borrow the main ideas from Diophantine Geometry[15]. We hope the reader will share our opinion that Special Arithmetic [53] is a very beautiful and deep way of thinking about Special Geometry.

Traditionally, Special Geometry is studied through its Weierstrass model. In this note we advocate instead the use of the Kodaira-Néron model, which we find both easier and more powerful. The Kodaira-Néron model $\mathcal{E}$ of the (total) space X of a non-trivial[16] rank-1 special geometry is a (smooth compact) relatively minimal, elliptic surface[17], with a zero section S_0, which happens to be rational (so isomorphic to $\mathbb{P}^2$ blown-up at 9 points). $\mathcal{E}$ is equipped with a marked fiber F_∞ which must be *un*stable[18], that is, as a curve F_∞ is *not* semi-stable in the Mumford sense. Comparing with Kodaira classification, we get 11 possible F_∞: seven of them correspond to the (non-free) Coulomb branch dimensions Δ allowed in a rank-1 SCFT, and the last four to the possible *non*-zero values of the β-function in a rank-1 asymptotically-free $\mathcal{N} = 2$ theory. The fact that $\mathcal{E}$ is rational implies *inter alia* the "planarity conjecture", that is, the chiral ring $\mathscr{R}$ is guaranteed to be a polynomial ring (of transcedence degree 1), $\mathscr{R} = \mathbb{C}[u]$.

Basic arithmetic gagdets associated to $\mathcal{E}$ are its Mordell-Weil group $\mathsf{MW}(\mathcal{E})$ of "rational" sections, its finite-index sub-group $\mathsf{MW}(\mathcal{E})^0$ of "narrow" sections, and its finite sub-set of "integral" sections, all sections being exceptional (-1)-curves on $\mathcal{E}$. $\mathsf{MW}(\mathcal{E})^0$ is a finitely-generated free Abelian group i.e. a lattice. This lattice is naturally endowed with a positive-definite, symmetric, integral pairing

$$\langle -, - \rangle_{NT} \colon \mathsf{MW}(\mathcal{E})^0 \times \mathsf{MW}(\mathcal{E})^0 \to \mathbb{Z} \tag{3.1}$$

induced by the Néron-Tate (canonical) height. The root system Ξ_∞ of the flavor group F may be identified with a certain finite sub-set of $\mathsf{MW}(\mathcal{E})^0$, and is completely determined by the above arithmetic data (we shall give a sketchy picture of Ξ_∞ momentarily). Given this identification, the

[15] For a survey see [178].

[16] *Non-trivial* means that X is not the product of an open curve C with a fixed elliptic curve E (equivalently: X has at least one singular fiber); physically, *non-trivial* means the 4d $\mathcal{N} = 2$ theory is not free.

[17] A complex surface is said to be *elliptic* if it has a holomorphic fibration over a curve, $\mathcal{E} \to C$, whose *generic* fiber is an elliptic curve.

[18] *Un*stable fibers are also known as *additive* fibers. In this part of thesis we shall use mostly the latter name.

list of possible flavor symmetries in rank-1 $\mathcal{N}=2$ QFT is read directly from the well-known tables of Mordell-Weil groups for rational elliptic surfaces, see ref. [205] or the nice book [221].

The result are (of course) consistent with Argyres *et al.* [19–23].

The Diophantine language is useful for other questions besides classifying the flavor groups. First it clarifies the subtler distinctions between inequivalent geometries which have the same flavor group and invariants k_F, a, c. Second, from the arithmetic point of view the gauging of a discrete symmetry is a *base change* (an extension of the ground field K over which the elliptic curve is defined). All consistent base changes are listed in table 6 of [156]; from that table one recovers the discrete gauging classification.

3.1 A sketch of flavor symmetry in rank-1 4d $\mathcal{N}=2$ QFT

A rational (-2)-curve $\mathcal{C}$ on a rational elliptic surface $\mathcal{E}$ is called an E_8*-root curve* if it is disjoint from the zero section S_0, i.e. iff it satisfies the three conditions

$$\mathcal{C}\cdot S_0=0,\qquad \mathcal{C}^2=-2,\qquad K_{\mathcal{E}}\cdot\mathcal{C}=0. \tag{3.2}$$

The name "E_8-root curve" stems from the following fact. Consider the "most generic" situation[19] where all fibers of the elliptic surface are *irreducible* curves. Physically, such a geometry describes a general mass-deformation of the Minahan-Nemeshanski (MN) SCFT [190] with flavor symmetry $\mathsf{F}=E_8$. On such a surface, $\mathcal{E}_{MN8}$, there are precisely 240 E_8-root curves $\mathcal{C}_a$ ($a=1,\dots,240$) with the property that their intersection pairing

$$\mathcal{C}_a\cdot\mathcal{C}_b=-\big\langle\alpha_a,\,\alpha_b\big\rangle_{\text{Cartan}},\qquad a,b=1,\cdots,240, \tag{3.3}$$

where the α_a are the roots of E_8 and $\langle -,-\rangle_{\text{Cartan}}$ is the bilinear form induced by the E_8 Cartan matrix. In other words, the classes of these 240 rational curves form an E_8 root system in[20] $H_2(\mathcal{E},\mathbb{R})^\perp$. This is the usual way we understand geometrically the presence of a flavor E_8 symmetry in this particular Minahan-Nemeshanski theory.

The crucial observation is that the E_8-root (-2)-curves are in one-to-one correspondence with a special class of exceptional (-1)-curves on $\mathcal{E}$, namely the finite set of *integral* elements of the Mordell-Weil group $\mathsf{MW}(\mathcal{E})$. Indeed, $\mathcal{C}$ is an E_8-root curve if and only if the (-1)-curve ($\equiv$ section of $\mathcal{E}$)

$$S=\mathcal{C}+S_0+F \tag{3.4}$$

is *integral* in $\mathsf{MW}(\mathcal{E})$. In (3.4) S_0 and $F=-K_{\mathcal{E}}$ are the divisors of the zero section and a fiber, respectively, $=$ being equality in the Néron-Severi (or Picard) group.

Away from this "generic" situation, three competing mechanisms become operative:

Symmetry lift. Part of the original E_8 root system gets lost. An elliptic surface $\mathcal{E}$ with a reducible fiber has less than 240 integral sections and hence less than 240 E_8-root curves satisfying (3.2). Some of the E_8 roots simply are no longer there. For instance, the elliptic surface $\mathcal{E}_{MN7}$ describing a generic mass deformation of the E_7 Minahan-Nemeshanski SCFT has only 126+56=182 integral sections and hence only 182 E_8-root curves;

[19] More precisely, the "most generic" *unstable* elliptic surface $\mathcal{E}$.

[20] $H_2(\mathcal{E},\mathbb{R})^\perp$ denotes the subspace of classes orthogonal to the fiber and the zero section S_0.

Symmetry obstruction. Some of the E_8-root curves present in $\mathcal{E}$ do not correspond to symmetries because they are obstructed by the symplectic structure Ω of special geometry. An E_8-root curve $\mathcal{C}$ leads to a root of the flavor symmetry F if and only if, for all irreducible components of the fibers $F_{u,\alpha}$,

$$F_{u,\alpha} \cdot \mathcal{C} = 0. \tag{3.5}$$

If our rank-1 model is not E_8 MN, some fiber of $\mathcal{E}$ should be reducible, and hence we get extra conditions (3.5) from the additional irreducible components. These conditions project some E_8-curves out of the root system of F. For instance, 56 of the 182 E_8-root curves of E_7 MN do not satisfy (3.5), and we remain with only 126 "good" curves making the E_7 root system. The other 56 E_8-root curves on $\mathcal{E}_{MN7}$ yield instead the weights of the fundamental representation **56** of E_7.

If a (-2)-curve $\mathcal{C}$ with the properties (3.2) satisfies (3.5) we say that it lays *in good position* in the Néron-Severi lattice. Only E_8-curves in good position contribute to the flavor symmetry. The corresponding sections (3.4) are precisely the ones which are both integral and narrow;

Symmetry enhancement. Some integral sections which are not of the form (3.4) – and hence not related to the "generic" symmetry – but lay in good position in the Néron-Severi group, get promoted to roots of the flavor Lie algebra $\mathfrak{f} = \mathfrak{Lie}(\mathsf{F})$. When both kinds of roots are present – the ones inherited from the "original" E_8 as well as the ones arising from enhancement – the last (first) set makes the long (short) roots of a non-simply-laced Lie algebra.

In some special models the symmetry enhancement has a simple physical meaning. These $\mathcal{N} = 2$ QFTs may be obtained by gauging a discrete (cyclic) symmetry of a parent theory. This situation is described geometrically by a branched cover between the corresponding elliptic surfaces

$$f\colon \mathcal{E}_{\text{parent}} \to \mathcal{E}_{\text{gauged}}. \tag{3.6}$$

The (-1)-curves associated with the enhanced symmetries of the gauged QFT, when pulled back to the parent ungauged geometry $\mathcal{E}_{\text{parent}}$, take the form (3.4) for some honest E_8-root curve $\mathcal{C}_a \subset \mathcal{E}_{\text{parent}}$ laying in good position. Thus, at the level of the parent theory the "enhanced" symmetries are just the "obvious" flavor symmetries inherited from E_8-roots. The deck group of (3.6), $\mathsf{Gal}(f)$ (the symmetry being gauged), is a subgroup of $\text{Aut}(\mathcal{E}_{\text{parent}})$ and acts on the parent root system by isometries of its lattice. The root system of $\mathcal{E}_{\text{gauged}}$ is obtained by "folding" the Dynkin graph of $\mathcal{E}_{\text{parent}}$ by its symmetry $\mathsf{Gal}(f)$. In these particular examples the general arithmetic construction of the flavor symmetry F out of the Mordell-Weil group $\mathsf{MW}(\mathcal{E}_{\text{gauged}})$ is equivalent to the physical relation between the flavor symmetries of the gauged and ungauged QFTs.

For examples of flavor root lattices, see §.8.4.3. For examples of diagram foldings see §.9.2.

4 The categorical approach

The BPS objects of a supersymmetric theory are naturally described in terms of ($\mathbb{C}$-linear) triangle categories [8] and their stability conditions [44]. The BPS sector of a given physical theory $\mathcal{T}$ is described by a plurality of different triangle categories $\mathfrak{T}_{(a)}$ depending on:[21]

[21] The index a take values in some index set I.

a) the class of BPS objects (particles, branes, local or non-local operators,...) we are interested in;

b) the physical picture (fundamental UV theory, IR effective theory,...);

c) the particular engineering of $\mathcal{T}$ in QFT/string/M-/F-theory.

The diverse BPS categories $\mathfrak{T}_{(a)}$ are related by a web of exact functors, $\mathfrak{T}_{(a)} \xrightarrow{\mathsf{c}_{(a,b)}} \mathfrak{T}_{(b)}$, which express physical consistency conditions between the different physical pictures and BPS objects. The simplest instance is given by two different engineerings of the same theory: the duality $\mathcal{T} \leftrightarrow \mathcal{T}'$ induces equivalences of triangle categories $\mathfrak{T}_{(a)} \xrightarrow{\mathsf{d}_{(a)}} \mathfrak{T}'_{(a)}$ for all objects and all physical descriptions $a \in I$. An example is mirror symmetry between IIA and IIB string theories compactified on a pair of mirror Calabi-Yau 3-folds, $\mathcal{M}$, $\mathcal{M}^\vee$ which induces on the BPS branes *homological mirror symmetry*, that is, the equivalences of triangle categories [155]

$$D^b(\mathsf{Coh}\,\mathcal{M}) \cong D^b(\mathsf{Fuk}\,\mathcal{M}^\vee), \qquad D^b(\mathsf{Coh}\,\mathcal{M}^\vee) \cong D^b(\mathsf{Fuk}\,\mathcal{M}).$$

In the same way, the functor relating the IR and UV descriptions of the BPS sector may be seen as *homological Renormalization Group,* while the functor relating particles and branes may be seen as describing properties of the combined system.

A duality induces a family of equivalences $\mathsf{d}_{(a)}$, one for each category $\mathfrak{T}_{(a)}$, and these equivalences should be compatible with the functors $\mathsf{c}_{(a,b)}$, that is, they should give an equivalence of the full web of categories and functors. Our philosophy is that the study of equivalences of the full functorial web is a very efficient tool to detect dualities. We shall focus on the case of $4d$ $\mathcal{N} = 2$ QFTs, but the strategy has general validity. We are particularly concerned with S-dualities, i.e. auto-dualities of the theory $\mathcal{T}$ which act non trivially on the UV degrees of freedom.

Building on previous work by several people[22], we present our proposal for the triangle categories describing different BPS objects, both from the UV and IR points of view, and study the functors relating them. This leads, in particular, to a categorical understanding of the S-duality groups and of the VEV of UV line operators. The categorical language unifies in a systematic way all aspects of the BPS physics, and leads to new powerful techniques to compute SUSY protected quantities in $\mathcal{N} = 2$ $4d$ theories. We check in many explicit examples that the results obtained from this more abstract viewpoint reproduce the ones obtained by more traditional techniques. However the categorical approach may also be used to tackle problems which look too hard for other techniques.

Main triangle categories and functors. The basic example of a web of functors relating distinct BPS categories for $4d$ $\mathcal{N} = 4$ QFT is the following exact sequence of triangle categories (Theorem 5.6 of [162]):

$$0 \to D^b\Gamma \xrightarrow{\mathsf{s}} \mathfrak{Per}\,\Gamma \xrightarrow{\mathsf{r}} \mathcal{C}(\Gamma) \to 0, \tag{4.1}$$

where (see §. 13 for precise definitions and details):

- Γ is the *Ginzburg algebra* [132] of a quiver with superpotential [9] associated to the $\mathcal{N} = 2$ theory at hand;

[22] References to previous work are provided in the appropriate sections of the thesis.

- $D^b\Gamma$ is the *bounded derived* category of Γ. $D^b\Gamma$ may be seen as a "universal envelope" of the categories describing, in the deep IR, the BPS particle spectrum in the several BPS chambers. To discuss states in the IR we need to fix a Coulomb vacuum u; this datum defines a stability condition Z_u on $D^b\Gamma$. The category which describes the BPS particles in the u vacuum is the subcategory of $D^b\Gamma$ consisting of objects which are semi-stable for Z_u. The BPS particles arise from (the quantization of the moduli of) the simple objects in this subcategory. Its Grothendieck group $K_0(D^b\Gamma)$ is identified with the Abelian group of the IR additive conserved quantum numbers (electric, magnetic, and flavor charges) which take value in the lattice $\Lambda \cong K_0(D^b\Gamma)$. $D^b\Gamma$ is a *3-Calabi-Yau* (3-CY)[23] triangle category, which implies that its Euler form

$$\chi(X,Y) \equiv \sum_{k\in\mathbb{Z}} (-1)^k \ \dim \mathrm{Hom}_{D^b\Gamma}(X, Y[k]), \qquad X,\ Y \in D^b\Gamma$$

is a *skew-symmetric* form $\Lambda \times \Lambda \to \mathbb{Z}$ whose physical meaning is the Dirac electro-magnetic pairing between the charges $[X]$, $[Y] \in \Lambda$ carried by the states associated to the stable objects X, $Y \in D^b\Gamma$;

- $\mathcal{C}(\Gamma)$ is the *cluster category* of Γ which describes[24] the BPS UV line operators. This identification is deeply related to the Kontsevitch-Soibelmann wall-crossing formula [171], see [69,162]. The Grothendieck group $K_0(\mathcal{C}(\Gamma))$ then corresponds to the, additive as well as multiplicative, UV quantum numbers of the line operators. These quantum numbers, in particular the *multiplicative* ones follow from the analysis by 't Hooft of the quantum phases of a $4d$ non-Abelian gauge theory being determined by the *topology of the gauge group* [144–147]. 't Hooft arguments are briefly reviewed in §. 14.2: the UV line quantum numbers take value in a finitely generated Abelian group whose torsion part consists of two copies of the fundamental group of the gauge group while its free part describes flavor. The fact that $K_0(\mathcal{C}(\Gamma))$ is automatically equal to the correct UV group, as predicted by 't Hooft (detecting the precise topology of the gauge group!), yields convincing evidence for the proposed identification, see §. 15.3.2. $\mathcal{C}(\Gamma)$ is a 2-CY category, and hence its Euler form induces a *symmetric* form on the additive UV charges, which roughly speaking has the form

$$K_0(\mathcal{C}(\Gamma))/K_0(\mathcal{C}(\Gamma))_{\text{torsion}} \bigotimes K_0(\mathcal{C}(\Gamma))/K_0(\mathcal{C}(\Gamma))_{\text{torsion}} \to \mathbb{Z},$$

but whose precise definition is slightly more involved[25] (see §. 13.9). We call this pairing the *Tits form* of $\mathcal{C}(\Gamma)$. Its physical meaning is simple: while in the IR the masses break (generically) the flavor group to its maximal torus $U(1)^f$, in the deep UV the masses become irrelevant and the flavor group gets enhanced to its maximal non-Abelian form F. Then the UV category should see the full F and not just its Cartan torus. The datum of the group F may be given as its weight lattice together with its Tits form; the cluster Tits form is equal to

[23] See §. 13 for precise definitions. *Informally,* a triangle category is k-CY iff it behaves as the derived category of coherent sheaves, $D^b\,\mathsf{coh}\,\mathcal{M}_k$, on a Calabi-Yau k-fold $\mathcal{M}_k$.

[24] This is slightly imprecise. Properly speaking, the line operators correspond to the generic objects on the irreducible components of the moduli spaces of isoclasses of objects of $\mathcal{C}(\Gamma)$.

[25] The subtleties in the definition are immaterial when the QFT is UV superconformal (as contrasted to asymptotically-free) and all chiral operators have integral dimensions.

the Tits form of the non-Abelian flavor group F, and we may read F directly from the cluster category. In fact the cluster category also detects the global topology of the flavor group, distinguishing (say) $SO(N)$ and $\mathrm{Spin}(N)$ flavor groups.[26] For objects of $\mathcal{C}(\Gamma)$ there is also a weaker notion of 'charge', taking value in the lattice Λ of electric/magnetic/flavor charges, namely the *index*, which is the quantity referred to as 'charge' in many treatments. Since $\mathcal{C}(\Gamma)$ yields an UV description of the theory, there must exist relations between its mathematical properties and the physical conditions assuring UV completeness of the associated QFT. We shall point of some of them in §. 15.4;

- $\mathfrak{Per}\,\Gamma$ is the *perfect derived category* of Γ. From eqn.(4.1) we see that, morally speaking, the triangle category $\mathfrak{Per}\,\Gamma$ describes all possible BPS IR object generated by the insertion of UV line operators, dressed (screened) by particles, in all possible vacua. This rough idea is basically correct. Perhaps the most convincing argument comes from consideration of class $\mathcal{S}$ theories, where we have a geometric construction of the perfect category $\mathfrak{Per}\,\Gamma$ [213] as well as a detailed understanding of the BPS physics [126,127]. In agreement with this identification, the Grothendieck group $K_0(\mathfrak{Per}\,\Gamma)$ is isomorphic to the IR group Λ. $\mathfrak{Per}\,\Gamma$ is not CY, instead the Euler form defines a *perfect* pairing

$$K_0(D^b\Gamma)\bigotimes K_0(\mathfrak{Per}\,\Gamma)\to\mathbb{Z};$$

- the exact functor r in eqn.(4.1) may be seen as the homological (inverse) RG flow.

Dualities. The (self)-dualities of an $\mathcal{N}=2$ theory should relate BPS objects to BPS objects of the same kind, and hence should be (triangle) auto-equivalences of the above categories which are consistent with the functors relating them (e.g. s, r in eqn.(4.1)). We may describe the physical situation from different viewpoints. In the IR picture one would have the putative 'duality' group $\mathrm{Aut}\,D^b\Gamma$; however a subgroup acts trivially on all observables [51], and the physical IR 'duality' group is[27]

$$\mathcal{S}_{\mathrm{IR}}\equiv \mathrm{Aut}\,D^b\Gamma\Big/\big\{\text{physically trivial autoequivalences}\big\}=\mathrm{Auteq}\,D^b\Gamma\rtimes\mathrm{Aut}(Q). \tag{4.2}$$

In the UV (that is, at the operator level) the natural candidate 'duality' group is

$$\mathcal{S}_{\mathrm{UV}}\equiv \mathrm{Aut}\,\mathcal{C}(\Gamma)\Big/\big\{\text{physically trivial}\big\}$$

From the explicit description of $\mathrm{Aut}\,D^b\Gamma$ (see §.5) we learn that $\mathcal{S}_{\mathrm{IR}}$ extends to a group of autoequivalences of $\mathfrak{Per}\,\Gamma$ which preserve $D^b\Gamma$ (by definition). Hence the exact functor $\mathsf{r}\colon\mathfrak{Per}\,\Gamma\to$

[26] In facts, the cluster Grothendieck group $K_0(\mathcal{C}(\Gamma))$ should contain even more detailed informations on the flavor. For instance, in $SU(2)$ gauge theory with N_f flavors the states of even magnetic charge are in tensor representations of the flavor $SO(2N_f)$ while states of odd magnetic charge are in spinorial representation of Spin(8); $K_0(\mathcal{C}(\Gamma))$ should know the correlation between the parity of the magnetic charge and $SO(2N_f)$ vs. $\mathrm{Spin}(2N_f)$ flavor symmetries (and it does).

[27] For the precise definition of $\mathrm{Auteq}\,D^b\Gamma$, see §. 5. $\mathrm{Aut}(Q)$ is the group of automorphisms of the quiver Q modulo the subgroup which fixes all nodes.

$\mathcal{C}(\Gamma)$ in eqn.(4.1) induces a group homomorphism

$$\mathcal{S}_{\rm IR} \xrightarrow{r} \mathcal{S}_{\rm UV},$$

whose image is

$$\mathbb{S} = r\Big(\operatorname{Auteq} D^b\Gamma\Big) \rtimes \operatorname{Aut} Q. \tag{4.3}$$

$\mathbb{S}$ is a group of auto-equivalences whose action is defined at the operator level, that is, independently of a choice of vacuum. They are equivalences of the full web of BPS categories in eqn.(4.1). Thus $\mathbb{S}$ is the natural candidate for the role of the (extended) *S-duality group* of our $\mathcal{N}=2$ model. Indeed, in the examples where we know the S-duality group from more conventional considerations, it coincides with our categorical group $\mathbb{S}$. In this survey we take equation (4.3) as the definition of the S-duality group.

Clearly, the essential part of $\mathbb{S}$ is the group $r(\operatorname{Auteq} D^b\Gamma)$. It turns out that precisely this group is an object of central interest in the mathematical literature which provides an explicit combinatorial description of it [115]. This combinatorial description is the basis of an algorithm for computer search of S-dualities, see §. 17. If our $\mathcal{N}=2$ theory is not too complicated (that is, the ranks of the gauge and flavor group are not too big) the algorithm may be effectively implemented on a laptop, see §. 17 for explicit examples.

For class $\mathcal{S}$ theories, the above combinatorial description of S-duality has a nice geometric intepretation as the (tagged) mapping class group of the Gaiotto surface, in agreement with the predictions of [123] (see also [13]), see §. 5.2. More generally, for class $\mathcal{S}$ theories all categorical constructions have a simple geometric realization which makes manifest their physical meaning.

The IR group $\mathcal{S}_{\rm IR}$ may be understood in terms of duality walls, see §.16.3.

Cluster characters and vevs of line operators. The datum of a Coulomb vacuum u defines a map

$$\langle - \rangle_u \colon \operatorname{GenOb}(\mathcal{C}\big(\Gamma)\big) \to \mathbb{C},$$

given by taking the VEV in the vacuum u of the UV line operator associated to a given *generic* object of the cluster category $\mathcal{C}(\Gamma)$. Physically, the renormalization group implies that the map $\langle - \rangle_u$ factors through the (Laurent) ring $\mathbb{Z}[\mathcal{L}]$ of line operators in the effective (Abelian) IR theory. The associated map

$$\operatorname{GenOb}(\mathcal{C}\big(\Gamma)\big) \to \mathbb{Z}[\mathcal{L}]$$

is called a *cluster character* and is well understood in the mathematical literature. Thus the theory of cluster character solves (in principle) the problem of computing the VEV of arbitrary BPS line operators (see §.19).

Part II

Geometric structure of a 4d $\mathcal{N}=2$ SCFT

5 Special cones and *log*-Fano varieties

In this section we review special geometry and related topics to set up the scene. The first three subsections contain fairly standard material; our suggestion to the experts is to skip them (except for the *disclaimer* in §.5.1.1). Later subsections describe basic properties of conic special geometries (CSG) which were not previously discussed in the literature: we aim to establish that a CSG is an affine (complex) cone over a special kind of normal projective manifold: a simply-connected log-Fano with minimal Hodge numbers.

5.1 Special geometric structures

In this thesis by a "special geometry" we mean a holomorphic integrable system with a Seiberg-Witten (SW) meromorphic differential [103, 104, 120]:

Definition 1. By a *special geometry* we mean the following data:

D1: A holomorphic map $\pi\colon X \to M$ between two normal complex analytic manifolds, X and M, of complex dimension $2k$ and k, respectively, whose generic fiber is (analytically isomorphic to) a principally polarized Abelian variety. π is required to have a zero-section. The closed analytic set $\mathcal{D} \subset M$ at which the fiber degenerates is called the *discriminant.* The dense open set $M^\sharp \equiv M \setminus \mathcal{D}$ is called the *regular locus*;

D2: A meromorphic 1-form (1-current) λ on X (the Seiberg-Witten (SW) differential) such that $d\lambda$ is a holomorphic symplectic form on X, with respect to which the fibers of π are Lagrangian.

5.1.1 Three crucial *caveats* on the definition

The one given above is the definition which is natural from a geometric perspective. However in the physical applications one also considers slightly more general situations which may easily be reduced to the previous one. This aspect should be kept in mind when making comparison of our findings with existing results in the physics literature. We stress three aspects:

Multivalued symplectic forms. In **Definition 1** X is *globally* a holomorphic integrable system with a well-defined holomorphic symplectic form $d\lambda$. Since the overall phase of the SW differential λ is not observable, in the physical applications sometimes one also admits geometries in which λ is well-defined only up to (a locally constant) phase, see [26] for discussion and examples. Let C be the Coulomb branch of such a generalized special geometry; there is an unbranched cover of the regular locus, $M^\sharp \to C^\sharp$, on which λ is univalued. $M^\sharp$ is the regular locus of a special geometry in the sense of **Definition 1**. The cover branches only over the discriminant $\mathcal{D}$. Dually, we have an embedding of chiral rings $\mathscr{R}_C \hookrightarrow \mathscr{R}_M$. Away from the discriminant, there is little difference between the two descriptions: working in C we identify vacua having the same physics, while in M we declare them to be distinct states (with the same physical properties). In the first picture we consider non-observable the chiral operators which distinguish the physically equivalent Coulomb vacua, that is, $\mathscr{R}_C \cong \mathscr{R}_M^G$, where G is the (finite) deck group of the covering. $\mathscr{R}_C$ is still free iff G is a reflection group [82, 232] acting homogeneously; in this case, the dimensions of its generators are multiples of the ones for $\mathscr{R}_M$.

The two special geometries C and M may lead to different ways of resolving the singularities along $\mathcal{D}$, and hence they may correspond to physical inequivalent theories in the "same" coarse class. It may happen that we may attach physical sense only to the chiral sub-ring $\mathscr{R}_C$. To compare our results with those of papers which allow multivalued λ, one should first pull-back their geometries to a cover on which the holomorphic symplectic form is univalued.

Non-principal polarizations. In **Definition 1** the generic fiber of $X \to M$ is taken to be a *principally* polarized Abelian variety. As already stressed in the Introduction, we may consider non principal-polarization. This means that not all electric/magnetic charges and fluxes consistent with Dirac quantization are present in the system [103]. This is believed not to be possible in theories arising as limits of consistent quantum theories containing gravity [30]. Every non-principally polarized Abelian variety has an isogenous principally polarized one [139, 180].

We see the polarization of the regular fiber X_u as a primitive,[28] integral, non-degenerate, pairing [149, 180]

$$\langle -,-\rangle\colon H_1(X_u,\mathbb{Z})\times H_1(X_u,\mathbb{Z})\to\mathbb{Z} \tag{5.1}$$

which has the physical interpretation of the Dirac electro-magnetic. We may find generators γ_i of the electro-magnetic charge lattice $H_1(X_u,\mathbb{Z})$ so that the matrix $\Omega_{ij}\equiv\langle\gamma_i,\gamma_j\rangle$ takes the (unique) canonical form [203]

$$\Omega=\begin{bmatrix}0 & e_1\\ -e_1 & 0\end{bmatrix}\bigoplus\begin{bmatrix}0 & e_2\\ -e_2 & 0\end{bmatrix}\bigoplus\cdots\bigoplus\begin{bmatrix}0 & e_k\\ -e_k & 0\end{bmatrix},\qquad e_i\in\mathbb{N},\ e_i\mid e_{i+1},\quad e_1\equiv 1. \tag{5.2}$$

The polarization is principal iff $e_i=1$, i.e. $\det\Omega=1$. Physically, the integers e_i are *charge multipliers*: (in a suitable duality frame) the allowed values of the i-th electric charge are integral multiples of e_i.

If Ω is principal, the duality-frame group is the Siegel modular group $Sp(2k,\mathbb{Z})$, while in general it is the commensurable arithmetic group $S(\Omega)_{\mathbb{Z}}$, eqn.(2.17). Since, as mentioned in the Introduction, the Coulomb dimensions $\{\Delta_i\}$ are related to the possible elliptic subgroups of the duality-frame group, there is a correlation between the set of charge multipliers $\{e_i\}$ and the set of dimensions $\{\Delta_i\}$. The simplest instance of this state of affairs appears in rank 2: the set of dimensions $\{12, 8\}$ is not allowed for Ω principal, but it is permitted when the charge multiplier e_2 is (e.g.) 2 or 3.

Non-normal Coulomb branches and "non-free" chiral rings. In **Definition 1** the Coulomb branch M is taken to be *normal* as an analytic space, that is, we see the Coulomb branch as a ringed space $(M,\mathcal{O}_M)$ where M is a Hausdorf topological space and $\mathcal{O}_M$ is the structure sheaf whose local sections are the local holomorphic functions. Being *normal* means that the stalks $\mathcal{O}_{M,x}$ at all points $x\in M$ are domains which are integrally closed in the stalk $\mathcal{M}_x$ of the sheaf of germs of meromorphic functions [134, 136]. Geometrically this is the convenient and natural definition; indeed, there is no essential loss of generality since we may always replace a non-normal analytic space M_0 by its normalization M: just replace the structure sheaf $\mathcal{O}_{M_0}$ with its integral closure $\mathcal{O}_M$ and the topological space M_0 by the analytic spectrum M of $\mathcal{O}_M$ [136]. Roughly speaking, passing

[28] That is, the matrix of the form $\Omega_{ij}\in\mathbb{Z}(2k)$ satisfies $\gcd_{i,j}\{\Omega_{ij}\}=1$.

to the normalization just enlarges the ring of the holomorphic functions from global sections of $\mathcal{O}_{M_0}$ to global sections of $\mathcal{O}_M$. In facts, the normalization corresponds to the maximal extension of the ring of local holomorphic functions compatible with $\mathcal{O}_{M_0}$-coherence. Thus, geometrically, a non-normal Coulomb branch just amounts to "forget" some (local) holomorphic function. The simplest example of a non-normal analytic space is the plane cuspidal cubic whose ring of regular functions is $\mathbb{C}[u_1, u_2]/(u_1^2 - u_2^3)$. Its normalization is the affine line with ring $\mathbb{C}[t]$, corresponding to the parametrization $u_1 = t^3$, $u_2 = t^2$. In this example the normalization ring $\Gamma(M, \mathcal{O}_M)$ has the (topological) basis $1, t, t^2, t^3, \cdots$ while the basis of the non-normal version, $\Gamma(M_0, \mathcal{O}_{M_0})$, is $1, t^2, t^3, \cdots$ where one "forgets" the function t.

From the physical side the situation is subtler. We define the *(geometric) chiral ring* $\mathscr{R}$ to be the Frechét ring of the global holomorphic functions $\mathscr{R} \equiv \Gamma(M, \mathcal{O}_M)$. This geometric ring may or may not coincide with the *physical* chiral ring $\mathscr{R}_{\text{ph}}$, defined as the ring of holomorphic functions on $M^\sharp \subset M$ which may be realized as vacuum expectation values of a chiral operator. Clearly $\mathscr{R}_{\text{ph}} \subset \mathscr{R}$, and we get the physical ring by "forgetting" some holomorphic function. Then $\mathscr{R}_{\text{ph}} = \Gamma(M_{\text{ph}}, \mathcal{O}_{\text{ph}})$ where the stalks of $\mathcal{O}_{\text{ph}}$ are domains[29] which may or may not be integrally closed. In the second case the physics endows the Coulomb branch with the structure of a *non-normal analytic space* $(M_{\text{ph}}, \mathcal{O}_{\text{ph}})$. Geometrically it is natural to replace it with its normalization $(M, \mathcal{O}_M)$ while proclaiming that only a subring $\mathscr{R}_{\text{ph}}$ of the chiral ring $\mathscr{R}$ is a ring of physical operators. Notice that the full geometric ring $\mathscr{R}$ may be a free polynomial ring, $\mathbb{C}[u_1, \cdots, u_k]$, while the physical ring $\mathscr{R}_{\text{ph}}$ is a non-free finitely-generated ring, as the example of the cuspidal cubic shows.

The putative "non-free" Coulomb branch geometries of ref. [15] arise this way: they are non-normal analytic spaces whose normalization has a free polynomial ring of regular functions, $\mathbb{C}[u_1, \cdots, u_k]$. That is, the "non-free" chiral rings are obtained from free geometric rings by forgetting some holomorphic functions of $\mathscr{R}$. The physical rationale for "forgetting" functions is the unitarity bound. In a CSG $\mathscr{R}$ is graded by the conformal dimension Δ, and unitarity requires that a non-constant *physical* holomorphic function has $\Delta \geq 1$. Hence one is naturally led to the proposal

$$\mathscr{R}_{\text{ph}} = \mathscr{R}_{\Delta \geq 1} \equiv \mathbb{C} \cdot \mathbf{1} \oplus \Big\{ \phi \in \mathscr{R} \ : \ \Delta(\phi) \geq 1 \Big\} \subset \mathscr{R}. \tag{5.3}$$

If $0 < \Delta(\phi) < 1$ for some $\phi \in \mathscr{R}$, $\mathscr{R}_{\text{ph}}$ defines a non-normal structure sheaf $\mathcal{O}_{\text{ph}}$ and the physical ring is non-free. Is this fancy possibility actually realized?

The equations determining the dimensions Δ_i for the normalization M of a CSG (satisfying our regularity conditions), deduced in §. 10.3.4 below, always have a (unique) solution such that $\Delta(\phi) \geq 1$ for all $\phi \in \mathscr{R}$, $\phi \neq \mathbf{1}$, with equality precisely when ϕ is a free field. Indeed, with the **log** determination of the logarithm, the formula (2.16) expresses Δ_i as 1 plus a manifestly non-negative quantity. To produce $\mathscr{R}_{\text{ph}} \neq \mathscr{R}$, we may try to replace **log** by some bizzarre branch of the logarithm, with the effect that $\Delta(\phi) \to \Delta(\phi)_{\text{new}} = 2 - \Delta(\phi)$, so that an element with $1 < \Delta(\phi) < 2$ would be reinterpreted as having the dimension $0 < \Delta(\phi)_{\text{new}} < 1$. However this is extremely unnatural and gruesome since it will spoil the universality of the prescription to compute the dimension Δ that better be the same one for all SCFT and all chiral operators (the correct prescription should be the unique one which reproduces the correct results for Lagrangian QFT, see §.2.4.

[29] Because the Coulomb branch is reduced.

Assuming universality, the fancy possibilities of ref.[15] cannot be realized, and we shall neglect them for the rest of this thesis. If the reader is aware of physical motivations for their existence and wants to study them, he needs only to perform the non-universal analytic continuation of the relevant formulae.

For most of the thesis we focus on special geometries in the sense of **Definition 1**, with $\mathscr{R}_{\rm ph} = \mathscr{R}$ and principally polarized fibers. Occasionally we comment on the modifications required for non-principal Ω.

5.1.2 Review of implied structures

The data **D1**, **D2** imply the existence of several canonical geometric structures. We recall just the very basic ones (many others may be obtained by the construction in **S5**):

S1: *(polarized local system)* A local constant sheaf Γ on $M^\sharp$ with stalk $\cong \mathbb{Z}^{2k}$ equipped with a skew-symmetric form $\langle -,-\rangle\colon \Gamma\times\Gamma\to\mathbb{Z}$ under which $\Gamma\simeq\Gamma^\vee$. Γ is given by the holomogy of the fiber $\Gamma_u = H_1(\pi^{-1}(u),\mathbb{Z})$ with the intersection form given by the principal polarization;

S2: *(flat Gauss-Mannin connection)* On the holomorphic bundle $\mathcal{E}=\Gamma^\vee\otimes\mathcal{O}_M$ over $M^\sharp$, we have the flat holomorphic connection $\nabla^{\rm GM}$ defined by the condition that the local sections of $\Gamma^\vee$ are holomorphic;

S3: *(monodromy representation)* $m\colon \pi_1(M^\sharp)\to Sp(2k,\mathbb{Z})$;

S4: *(Hodge bundle)* $\mathcal{V}\to M^\sharp$: it is the holomorphic sub-bundle of $\mathcal{E}$ whose fibers are (1,0) cohomology classes, i.e. $\mathcal{V}_u = H^0(\pi^{-1}(u),\Omega^1)$. The flat connection $\nabla^{\rm GM}$ of $\mathcal{E}$ induces the sub-bundle (holomorphic) connection ∇^H on $\mathcal{V}$ [55, 137]. Note that $\Gamma^\vee$ acts by translation on $\mathcal{V}$ and that $\mathcal{V}/\Gamma^\vee\cong X^\sharp\equiv\pi^{-1}(M^\sharp)$;

S5: *(period map and the family of homogeneous bundles)* the period matrix $\boldsymbol{\tau}_{ij}$ of the Abelian variety $\pi^{-1}(u)$ is a complex symmetric matrix with positive imaginary part well defined up to $Sp(2k,\mathbb{Z})$ equivalence; hence $\boldsymbol{\tau}$ defines the holomorphic map:

$$\boldsymbol{\tau}\colon M^\sharp\to Sp(2k,\mathbb{Z})\big\backslash Sp(2k,\mathbb{R})/U(k). \tag{5.4}$$

The period map $\boldsymbol{\tau}$ yields a universal construction of many other canonical geometrical objects on $M^\sharp$. We limit ourselves to a special class of holomorphic ones. The Griffiths period domain $Sp(2k,\mathbb{R})/U(k)$ is an open domain in its complex Griffiths compact dual [55, 137, 138]

$$Sp(2k,\mathbb{R})/U(k)\subset Sp(2k,\mathbb{C})/P(k), \qquad \text{where } P(k)\subset Sp(2k,\mathbb{C}) \text{ is the Siegel parabolic subgroup.} \tag{5.5}$$

By general theory, to every $P(k)$-module (in particular to all $U(k)$-modules) we associate a holomorphic vector bundle over the compact dual equipped with a unique metric, complex structure, and connection having an explicit Lie theoretic construction [55]. These bundles, metrics, and connections may be restricted to the period domain and then pulled back to $M^\sharp$ *via* $\boldsymbol{\tau}$ to get God-given bundles, metrics, and connections on $M^\sharp$. All the quantities of "special geometry" (including the Kähler metric) arise in this way from Lie theoretic gadgets.

For instance, the Hodge bundle $\mathcal{V}$ (resp. the flat bundle $\mathcal{E}$) is the pull back of a homogenous bundle, and the connections ∇^H and $\nabla^{\rm GM}$ are the pull-back of the corresponding canonical connections on the symmetric space (5.5);

S6: *(periods and local special coordinates)* Let $U \subset M^\sharp$ be simply connected. We trivialize Γ in U choosing local sections making a canonical symplectic basis (A^i, B_j), $i, j = 1, \dots, k$,

$$\langle A^i, A^j\rangle = \langle B_i, B_j\rangle = 0, \quad \langle A^i, B_j\rangle = \delta^i{}_j. \tag{5.6}$$

The local special (holomorphic) coordinates a^i and their duals b^i are (in U)

$$a^i = \langle A^i, \lambda\rangle, \qquad b_i = \langle B_i, \lambda\rangle. \tag{5.7}$$

Writing

$$(\mathcal{V}|_U)_{\rm smooth} \equiv (\Gamma^\vee \otimes \mathbb{R})|_U = U \times \Big(A^i x_i + B_j y^j\Big), \qquad (y^j, x_i) \in \mathbb{R}^{2k}, \tag{5.8}$$

the holomorphic symplectic form becomes

$$\sigma = d\lambda = da^i \wedge dx_i + db_i \wedge dy^i = da^i \wedge \left(dx_i + \frac{\partial b_j}{\partial a^i}\, dy^j\right). \tag{5.9}$$

Since the holomorphic coordinates along the fiber are $z_i = x_i + \tau_{ij} y^j$ we get

$$\boldsymbol{\tau}_{ij} = \frac{\partial b_i}{\partial a^j}. \tag{5.10}$$

Since $\boldsymbol{\tau}_{ij}$ is symmetric, locally there exists a prepotential (holomorphic) function $\mathcal{F}(a^j)$ such that

$$b_i = \frac{\partial \mathcal{F}(a^j)}{\partial a^i}, \qquad \boldsymbol{\tau}_{ij}(a) = \frac{\partial^2 \mathcal{F}}{\partial a^i \partial a^j}. \tag{5.11}$$

S7: *(the dual bundle $\mathcal{V}^\vee \simeq \mathcal{E}/\mathcal{V}$)* This is yet another bundle whose metric and connection is given by the general construction in **S5**. It coincides with $TM^\sharp$, so it yields the geometry of the base. On the intersection of two special coordinate charts U, U' we have:

$$\begin{bmatrix} b' \\ a' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix}, \quad \begin{bmatrix} y' \\ x' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-t} \begin{bmatrix} y \\ x \end{bmatrix}, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2k, \mathbb{Z}), \tag{5.12}$$

so that the periods (a^i, b_i) are flat sections of $\mathcal{E}^\vee \simeq \mathcal{E}$. The holomorphic tangent bundle $TM^\sharp$ is then identified with the quotient bundle $\mathcal{E}/\mathcal{V}$ (again the pull-back of a homogeneous bundle). In particular, the flat connection $\nabla^{\rm GM}$ induces canonically a quotient bundle connection ∇^Q on $\mathcal{E}/\mathcal{V}$, that is, on $TM^\sharp$. Taking the differential and using $db_i = \boldsymbol{\tau}_{ij}\, da^i$ we get the modular transformation of the $k \times k$ period matrix $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = (A\boldsymbol{\tau} + B)(C\boldsymbol{\tau} + D)^{-1}; \tag{5.13}$$

S8: *(the hyperKähker structures on $X^\sharp$ and $\mathcal{V}$)* On the total space of $\mathcal{V}$, equivalently of the flat real bundle $\Gamma \otimes \mathbb{R}$, there is a hyperKähler structure (I^a, g) invariant under translation by local sections of $\Gamma \otimes \mathbb{R}$; then (I^a, g) descends to a hyperKähler structure on the total space of $\mathcal{H}^\sharp$ [64]. The complex structure of $\mathcal{V}$ is the $\zeta = 0$ one in hyperKähler $\mathbb{P}^1$-family of complex structures. We give the hyperKähler structure by presenting the explicit $\mathbb{P}^1$-family of local holomorphic Darboux coordinates $X^a(\zeta) = (q^i(\zeta), p_i(\zeta))$ satisfying the reality condition [124]

$$X^a(\zeta) = -\overline{X^a(-1/\bar{\zeta}\,)}, \tag{5.14}$$

such that the holomorphic symplectic form in complex structure $\zeta \in \mathbb{P}^1$ is

$$\Omega(\zeta) = -\frac{i}{2\,\zeta}\,\omega_+ + \omega_3 - \frac{i}{2}\,\zeta\,\omega_- = dp_i(\zeta) \wedge dq^i(\zeta), \quad \text{where } \omega_\pm = \omega_1 \pm \omega_3, \tag{5.15}$$

and ω_α $(\alpha = 1, 2, 3)$ are the three Kähler forms. We have

$$q^i(\zeta) = \frac{1}{\zeta}\,a^i + iy^i + \zeta\,\overline{a}^i, \quad p_i(\zeta) = \frac{1}{\zeta}\,b_i + ix_i + \zeta\,\bar{b}_i, \tag{5.16}$$

hence

$$\omega_+ = 2\Big(da^i \wedge dx_i - db_i \wedge dy^i\Big), \qquad \omega_3 = db_i \wedge d\bar{a}_i + d\bar{b}_i \wedge da^i. \tag{5.17}$$

S9: *(fiber metric and Chern connection on $\mathcal{V}$)* Restricting the hyperKähler metric along the fibers, we get a Hermitian metric and associated Chern connection on the holomorphic bundle $\mathcal{V}$. By uniqueness of the homogeneous connection, is coincides with the sub-bundle connection ∇^H. The Hermitian metric is simply $\|z\|^2 = y^{ij} z_i \bar{z}_j$ where y^{ij} is the inverse matrix of $y_{ij} = 2\,\mathrm{Im}\,\boldsymbol{\tau}_{ij}$;

S10: *(Special Kähler metric on $M^\sharp$ and its global Kähler potential)* In the same way, restricting the hyperKähler metric on $\mathcal{V}$ to the zero section (which is a holomorphic subspace in $\zeta = 0$ complex structure) we get a Kähler metric on $M^\sharp$ whose Kähler form is the restriction of ω_3. The restriction to the zero-section yields a Kähler metric on $M^\sharp$ with Kähler form

$$\omega_3|_{M^\sharp} = \Big(\boldsymbol{\tau}_{ij}(a) - \overline{\boldsymbol{\tau}}_{ij}(\bar{a})\Big) da^i \wedge d\bar{a}^j, \qquad \boldsymbol{\tau}_{ij}(a) \equiv \partial_{a_i}\partial_{a_j}\mathcal{F}, \qquad \mathrm{Im}\,\boldsymbol{\tau}_{ij}(a) > 0. \tag{5.18}$$

We note that the assumption of the existence of a SW differential implies the existence of a globally defined Kähler potential on $M^\sharp$:

$$\Phi = -i\big(b_i\,\bar{a}^i - a^i\,\bar{b}_i\big). \tag{5.19}$$

Again, by uniqueness of the homogeneous connection, the Levi-Civita connection ∇^{LC} of this Kähler metric is the quotient-bundle connection ∇^Q. In other words, all the relevant connections are just projections of the flat one.

S11: *(The cubic symmetric form of the infinitesimal Hodge deformation)* This is a symmetric holomorphic cubic form of type (3,0)

$$\odot^3\, TM^\sharp \to \mathcal{O}_M, \tag{5.20}$$

describing the infinitesimal deformation of Hodge structure (of the Abelian fiber) in the sense of Griffiths [137]. Locally in special coordinates it is given just by

$$T_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a^i \partial a^j \partial a^k}. \tag{5.21}$$

Remark 5.1.1. For $k = 1$ a special structure is, in particular, a surface fibered over a curve whose general section is an elliptic curve. Hence the possible local behaviors (i.e. degenerations of fibers) are described by the classical Kodaira papers [166]. In his terminology, **S3** is called the *homological invariant* and **S5** the *analytic invariant*.

Example 1 ($k = 1$ locally flat special structures[30])**.** In this thesis we are interested in conic special structures. Since in real dimension 2 all metric cones are locally flat, for $k = 1$ we are reduced to study flat special geometries whose discriminant is a single point. The hyperKähler manifold $X^\sharp$ then is locally isometric to $\mathbb{R}^4 \cong \mathbb{H}$, and the singular hyperKähler geometry should be of the form $\mathbb{C}^2/G$ with G a finite subgroup of $SU(2)$. One checks that conformal invariance requires the group G to correspond *via* the McKay correspondence to an affine Dynkin graph which is a star, that is, D_4, E_6, E_7, or E_8. Before resolving the singularity, the spaces X_{sing} are the well-known Du Val singular hypersurfaces[31] in $\mathbb{C}^3$ [28]

$$\begin{aligned} &D_4\colon y^2 - h_3(x,u) = 0, && E_6\colon y^2 - 4\,x^3 + u^4 = 0,\\ &E_7\colon y^2 - 4\,x^3 + u^3 x = 0, && E_8\colon y^2 - 4\,x^3 + u^5 = 0. \end{aligned} \tag{5.22}$$

That $u\colon X_{\text{sing}} \to \mathbb{C}$ is an elliptic fibration (with section) is obvious by reinterpreting Du Val singularities as the Weierstrass model of a family of elliptic curve parametrized by u. The crepant resolution X of X_{sing} is given by the corresponding ALE space. For each of the four special geometries we have *a priori* two distinct special structures. Indeed, we have two dual choices for the SW differential λ: I) a holomorphic section of $\mathcal{V}$ with no zero in $M^\sharp \equiv \{u \neq 0\}$ which vanish in the limit $u \to 0$ to order at most 1, or II) a holomorphic section of $\mathcal{V}^\vee$ with the same properties. Note that these properties fix λ uniquely up to an irrelevant overall constant. In terms of the Weierstrass model the two dual choices read:

$$\text{I)}\ \lambda = u\,\frac{dx}{y}, \qquad\qquad \text{II)}\ \lambda = \frac{x\,dx}{y}. \tag{5.23}$$

The corresponding Coulomb branch dimensions are

(5.24)

	D_4	E_6	E_7	E_8
I)	2	3	4	6
II)	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{6}{5}$

which is the correct list of (non-free) Δ's for $k = 1$. The periods can be easily computed using

[30] We shall return several times to this **Example** in the section. The present discussion is meant as a mere appetizer.

[31] In eqn.(5.22) $h_3(x,u)$ stands for a homogeneous cubic polynomial in x, u.

Weierstrass elliptic functions[32]. As an example, we write them for E_8:

$$\text{I)}\quad \begin{bmatrix} a \\ b \end{bmatrix} \equiv u \begin{bmatrix} e^{i\pi/3}\omega_1 \\ \omega_1 \end{bmatrix} = \frac{\Gamma(\frac{1}{3})^3}{4\pi} \begin{bmatrix} e^{i\pi/3} \\ 1 \end{bmatrix} u^{1/6}, \qquad \text{II)}\quad \begin{bmatrix} b \\ a \end{bmatrix} \equiv \begin{bmatrix} e^{i\pi/3}\eta_3 \\ \eta_3 \end{bmatrix} = \frac{2\pi^2 e^{-\pi i/3}}{\sqrt{3}\,\Gamma(\frac{1}{3})^3} \begin{bmatrix} e^{i\pi/3} \\ 1 \end{bmatrix} u^{5/6}$$

from which it is obvious that the dimension of u is 6 and respectively 6/5. In the dual choice the role of a and b get interchanged, since the non-trivial element of $H_{\mathbb{R}}$ inverts the sign of the polarization. Of course, the resolutions of the singularity at $u = 0$ are different in the two cases, the exceptional locus being Kodaira exceptional fiber of type II^* and II, respectively. The periods of dx/y and $x\,dx/y$ scale with opposite power of u by the Legendre relation.

5.1.3 Rigidity principle and reducibility

A basic trick of the trade is that global properties in special geometry fix everything. This principle is known as "the Power of Holomorphy" [226]; mathematicians call it *rigidity.*

Proposition 5.1.1 (Rigidity principle [55])**.** *Two special geometries with the same* compact *base* M*, isomorphic monodromy representations, and isomorphic fibers over* one *point, are equivalent.*

Thus the monodromy representation **S3** essentially determines the special structure. In particular, if the monodromy representation splits $m = m_1 \oplus m_2$ (over[33] $\mathbb{Z}$) then the special geometry is a product.

5.1.4 Curvature properties of special geometry

Let $\mathcal{W}$ be a holomorphic Hermitian vector bundle with Chern connection ∇. We consider a holomorphic sub-bundle $\mathcal{S} \subset \mathcal{W}$ and the quotient bundle $\mathcal{Q} = \mathcal{W}/\mathcal{S}$ equipped with the sub-bundle and quotient connections $\nabla^{\mathcal{S}}$ and $\nabla^{\mathcal{Q}}$, respectively. The curvature of $\nabla^{\mathcal{S}}$ (resp. $\nabla^{\mathcal{Q}}$) is bounded above (resp. below) by the one of ∇, see [139] page 79 or [55]. Applying this principle to $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{E}/\mathcal{V}$ we get:

Proposition 5.1.2. *The curvature of the bundle* $\mathcal{V}$ *is* non-positive, *while the curvature of the Kähler metric on* $M^\sharp$ *is* non-negative *(in facts,* positive*). In particular, the Ricci curvature of* $M^\sharp$ *is non-negative,* $R_{i\bar{\jmath}} \geq 0$ *and it vanishes iff* $M^\sharp$ *is locally flat.*

Let us give an alternative proof of the last statement.

Proof. In a Kähler manifold the Ricci form is[34] $\rho = -i\partial\bar{\partial}\log\det g$. Thus from (5.18)

$$\rho = -i\partial\bar{\partial}\log\det\operatorname{Im}\boldsymbol{\tau}_{kl}(a) \equiv \boldsymbol{\tau}^*\Omega \tag{5.25}$$

where Ω is the (positive) Kähler form on the locally Hermitian space in eqn.(5.4). Note that $R_{i\bar{\jmath}} = 0$ only at critical points of the period map $\boldsymbol{\tau}$. By Sard theorem, the set of periods $\boldsymbol{\tau}_{ij}(a)$ at which $R_{i\bar{\jmath}} = 0$ has zero measure. In particular $R_{i\bar{\jmath}} \equiv 0$ means $\boldsymbol{\tau}$ = (a constant map), so $M^\sharp$ is locally flat. □

[32] Notations as in DLMF §.23 [206].
[33] If the splitting is over $\mathbb{Q}$, the geometry is a product up to an isogeny in the fiber.
[34] Cfr. [37] eqn.(2.98).

Remark 5.1.2. This result may also be understood as follows. The total space of the holomorphic integrable system, X, is hyperKähler, so carries a Ricci-flat metric. M is a complex subspace, and the Ricci curvature of its induced metric is minus the curvature of the determinant of the normal bundle whose Hermitian metric is $(\det \operatorname{Im} \boldsymbol{\tau})^{-1}$.

Sectional and isotropic curvatures. From the above **Proposition** it is pretty obvious that all sectional curvatures of a special Kähler metric are non-negative. A stronger property is that all its *isotropic curvatures* are non-negative. Indeed, we claim an even stronger statement, that is, that the *curvature operators* are non-negative at all points p.

Definition 2. Let X be a Riemannian n-fold with tangent space T_pX at $p \in X$. The *curvature operator* at p is the self-adjoint linear operator

$$\boldsymbol{R}\colon \wedge^2 T_pX \to \wedge^2 T_pX, \tag{5.26}$$

given by the Riemann tensor. We say that X has *positive* (resp. *weakly positive*) *curvature operators* iff the eigenvalues of $\boldsymbol{R}$ are positive (resp. non-negative) at all $p \in X$.

The claim follows from the explicit form of the Riemann tensor

$$R_{i\bar{j}k\bar{l}} = T_{ikm}\, \bar{T}_{\bar{j}\bar{l}\bar{n}}\, g^{m\bar{n}}, \tag{5.27}$$

where T is the cubic symmetric form of the infinitesimal Hodge deformation (structure **S11**).

Sphere theorems. The positivity of the curvature operators has dramatic implications for the topology of X. We collect here some results which we shall use later in the thesis:

Theorem 1 (Meyers [199])**.** *Let X be a complete Riemannian manifold of metric g whose Ricci curvature satisfies $R \geq \lambda^2 g$ with $\lambda > 0$ a constant. Then X is compact with diameter $d(X) \leq \pi/\lambda$. Applying the result to the Riemannian universal cover $\widetilde{X}$ of X, we conclude that $\pi_1(X)$ is finite.*

Remark 5.1.3. There is a version of Meyers theorem which applies to orbifolds, see **Corollary 21** in [40] or **Corollary 2.3.4** in [252]. In case of Riemannian orbifolds *complete* should be understood as *complete as a metric space.* The version in [252] states that a metrically complete Riemannian orbifold X, whose Ricci curvature satisfies $R \geq \lambda^2 g$, is compact with a diameter $d(X) \leq \pi/\lambda$.

Theorem 2 (Synge [239])**.** *An even dimensional compact orientable manifold with positive sectional curvature is simply-connected.*

Remark 5.1.4. Again the result extends to Riemannian orbifolds, see **Corollary 2.3.6** of [252], so that an even dimensional orientable complete Riemannian orbifold with positive sectional curvature is simply connected.

Theorem 3 (Micallef-Moore [185], Böhm-Wilking [39])**.** *Let X be a compact n-dimensional Riemannian* orbifold*. If X has positive curvature operators it is diffeomorphic to a space form S^n/G, S^n being the sphere and G a finite subgroup of $SO(n+1)$.*

The special Kähler manifolds M have just *weakly* positive curvature operators (and are typically non-compact). However, taking **Theorems 1**, **3** together, one gets the rough feeling that the non-flat special Kähler manifolds are "close" to being locally spheres. The statement will become precise under the assumption that M is also a cone.

5.1.5 Behavior along the discriminant

We need to understand the behavior near the discriminant locus $\mathcal{D} \subset M$ where the fiber degenerates, that is, some periods (a_i, b^j) vanish. Physically this means that along the discriminant locus some additional light degrees of freedom appear, so that the IR description in terms of the massless fields parametrizing M becomes incomplete and breaks down.

The singular behavior is best understood in terms of properties of the period map $\boldsymbol{\tau}$. We see the discriminant $\mathcal{D}$ as an effective divisor $\mathcal{D} = \sum_i n_i S_i$, where S_i are the irreducible components and $M^\sharp = M \setminus \operatorname{Supp} \mathcal{D}$. The behavior of the period map as we approach a generic point s of an irreducible component S_i is described by three fundamental results: the **strong monodromy Theorem** [55,137,138], the SL_2**-orbit Theorem** [220], and the **invariant cycle Theorem** [220]. In a neighborhood U of $s \in S_i$, we may find complex coordinates $z_1, \cdots, z_k$ so that, locally in U, S_i is given by $z_1 = 0$. Then we have $U \cap M^\sharp \cong \Delta^* \times \Delta^{k-1}$ where Δ (resp. Δ^*) stands for the unit disk (resp. the punctured unit disk). We write p for the period map $\boldsymbol{\tau}$ restricted to $\Delta^* \times (z_2, \cdots, z_k) \subset \Delta^* \times \Delta^{k-1}$, and $\mathfrak{h}$ for the upper half-plane seen as the universal cover of Δ^* via the map $\tau \mapsto q(\tau) \equiv e^{2\pi i \tau}$. We have the commutative diagram

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\ \tilde{p}\ } & Sp(2k,\mathbb{R})/U(k) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \text{can}} \\ \Delta^* & \xrightarrow{\ p\ } & Sp(2k,\mathbb{Z})\backslash Sp(2k,\mathbb{R})/U(k) \end{array} \tag{5.28}$$

where $\tilde{p}$ is the lift of the (restricted) period map. Let γ be the generator of $\pi_1(\Delta^* \times \Delta^{k-1}) \cong \mathbb{Z}$ and $m \equiv m(\gamma)$ the corresponding monodromy element (cfr. **S3**). Then

$$\tilde{p}(\tau + 1) = m \cdot \tilde{p}(\tau). \tag{5.29}$$

Let $d(\cdot,\cdot)$ be the distance function defined by the standard invariant metric on the symmetric space $Sp(2k,\mathbb{R})/U(k)$ and $d_P(\cdot,\cdot)$ the distance with respect to the usual Poincaré metric in $\mathfrak{h}$; the inequalities on the curvatures together with the Schwarz lemma imply

Proposition 5.1.3 (Strong monodromy theorem [55,137,138])**.** *The lifted period map $\tilde{p}$ is distance-decreasing*

$$d(\tilde{p}(x), \tilde{p}(y)) \leq d_P(x, y). \tag{5.30}$$

Then the monodromy m is quasi-unipotent, *i.e. there are minimal integers $r \geq 1$, $0 \leq s \leq k$ such that*

$$(m^r - 1)^{s+1} = 0. \tag{5.31}$$

Equivalently (by Kronecker theorem) m has spectral radius 1 *(Mahler measure 1).*

All eigenvalues of m are r-th roots of unit. Since $m \in Sp(2k,\mathbb{Z})$, its minimal polynomial $M(z)$

is a product of cyclotomic polynomials $\Phi_d(z)$

$$M(z)=\prod_{d|r}\Phi_d(z)^{s_d},\qquad s_d\in\mathbb{N}. \tag{5.32}$$

The monodromy m is *semi-simple* iff $s=0$, that is, if $s_d\in\{0,1\}$ for all d. We say that m is *regular* iff all its eigenvalues are distinct, i.e. iff $s_d\in\{0,1\}$ and $\sum_{d|r}s_d=2k$.

The case of m semi-simple. Semi-simplicity of m has the following consequence:

Proposition 5.1.4 ([55, 137, 138])**.** *The period map $p\colon \Delta^*\to Sp(2k,\mathbb{Z})\backslash Sp(2k,\mathbb{R})/U(k)$ may be extended holomorphically to the origin if and only if $s=0$.*

In other words, along an irreducible component S_i of $\mathcal{D}$ whose monodromy element m is semi-simple the period matrix $\boldsymbol{\tau}_{ij}$ is defined and regular even if the Abelian fiber itself degenerates. The Kähler metric $d^2s=2\,\mathrm{Im}\,\boldsymbol{\tau}_{ij}\,da^i\otimes d\bar a^j$ is singular along S_i since the periods a^i are not valid local coordinates at this locus. The singularity is of the mildest possible kind: just a cyclic orbifold singularity. We illustrate the situation along a semi-simple component S_i of the discriminant $\mathcal{D}$ in a typical example.

Example 2 (Non-Lagrangian $k=1$ SCFT[35])**.** In these $k=1$ models, the period of the elliptic fiber τ is frozen in an orbifold (elliptic) point of the modular fundamental domain $\mathfrak{h}/SL(2,\mathbb{Z})$, i.e. either $\tau=e^{2\pi i/3}$ or $\tau=i$ depending on the model. Thus the Kähler metric is flat and M is locally *isometric* to $\mathbb{R}^2$

$$ds^2=2\,\mathrm{Im}\,\tau\,da\,d\bar a\xrightarrow{\textit{local}\text{ isometry}}dr^2+r^2\,d\theta^2,\qquad r^2=2\,\mathrm{Im}\,\tau\,|a|^2. \tag{5.33}$$

The coordinate r is globally defined, since r^2 is the momentum map of the $U(1)$ action given by R-symmetry. On the other hand, the period of the canonically conjugate angle θ needs not to be 2π (which corresponds to the free SCFT). The period of the angle θ is related to the Coulomb dimension Δ by the identification $\theta\sim\theta+2\pi/\Delta$. Hence, if the theory is not free, $\Delta\neq 1$, at the tip of the cone we have a cyclic orbifold singularity. We note that the unitary bound $\Delta\geq 1$ (with equality iff the SCFT is free) becomes (period of θ) $\leq 2\pi$. Thus unitarity requires the curvature at the tip to be *non-negative* and we may smooth out the geometry by cutting away the region $r\leq\epsilon$ and gluing back a *positively* curved disk. This is consistent with our discussion of the curvature in special geometry in §.5.1.4. This example shows that the curvature inequalities apply also to the δ-function curvature concentrated at orbifold points and their relation to physical unitarity.

We state this as a

Physical principle. *The unitarity bounds guarantee that the δ-function curvatures associated to the angular deficits at orbifold points are consistent with the positivity of curvatures required by special Kähler geometry.*

We quote another useful result:

[35] This is a special case of **Example 1**.

Proposition 5.1.5 ([55]). *Suppose that the period map $\boldsymbol{\tau}$ factors trough a quasi-projective variety K*

$$\begin{array}{ccc} M & \xrightarrow{\quad\boldsymbol{\tau}\quad} & Sp(2k,\mathbb{Z})\backslash Sp(2k,\mathbb{R})/U(k) \\ & \searrow & \nearrow_{p} \\ & K & \end{array} \tag{5.34}$$

and that the discriminant of p is a SNC[36] *divisor with semi-simple monodromies. If p is* not *the constant map, p is* proper. *Its image is a closed analytic subvariety containing $p(K^\sharp)$ as the complement of an analytic set.*

m **non semi-simple.** We now turn to the case in which the monodromy m is *not* semi-simple. Again we consider the neighborhood $U \cong \Delta^* \times \Delta^{k-1}$ considered around eqn.(10.51) and pull-back all structures to its universal cover $U_{\text{uni}} \cong \mathfrak{h} \times \Delta^{k-1}$. By the strong monodromy theorem there exist minimal integers $r \geq 1$, $s \geq 0$ such that $(m^r - 1)^{s+1} = 0$. In the non semi-simple case $s \geq 1$. Then,

$$\tilde{p}(\tau + r) = (1 + T) \cdot \tilde{p}(\tau) \quad \text{with } T = m^r - 1 \text{ and } T^{s+1} = 0, \tag{5.35}$$

$$\text{so that } \; N \equiv \log(1 + T) = \sum_{n=1}^{s} \frac{(-1)^{n-1}}{n}\, T^n \; \text{ is well defined.} \tag{5.36}$$

$N^s \neq 0$ and $N^{s+1} = 0$. The nilpotent operator N defines the weight filtration of a mixed Hodge structure in the sense of Deligne [87] to which we shall return momentarily; more elementarily, by the Jacobson-Morozov theorem [153, 195, 197] the rational matrix N defines a polynomial homomorphism $\phi\colon SL(2,\mathbb{Q}) \to Sp(2k,\mathbb{Q})$ such that N is the image of the raising operator of the $\mathfrak{sl}(2,\mathbb{Q})$ Lie algebra. ϕ induces a period map $\mathring{p}\colon \mathfrak{h} \to Sp(2k,\mathbb{R})/U(k)$ which is the simplest solution to the functional equation (5.35):

$$\mathring{p}(\tau) = e^{\tau N/r} \cdot p_0 \qquad p_0 \text{ globally defined in } U_r, \tag{5.37}$$

where $\sigma\colon U_r \to U$ is the local r-fold cover

$$\sigma\colon U_r \equiv \mathfrak{h}/(\tau \sim \tau + r) \times \Delta^{k-1} \longrightarrow \mathfrak{h}/(\tau \sim \tau + 1) \times \Delta^{k-1} \cong U. \tag{5.38}$$

The SL_2-orbit theorem [220] states that the actual period map p differs from the Lie-theoretic map $\mathring{p}(\tau)$ by exponentially small terms $O(q^{1/r})$ as $\tau \to i\infty$ ($q \equiv e^{2\pi i \tau}$). A physicists studying the corresponding (2,2) supersymmetric σ-model states the theorem saying that $\mathring{p}(\tau)$ is the perturbative solution, valid asymptotically as the coupling $4\pi/\operatorname{Im}\tau \to 0$, and this perturbative solution receives corrections only by instantons which are suppressed by the exponentially small (fractional) instanton counting parameter $q^{1/r}$. To physicists working in 4d $\mathcal{N} = 2$ QFT , the SL_2-orbit theorem is familiar as a fundamental result by Seiberg [225].

Let $0 \neq x \in U_r$; ϕ decomposes $\Gamma_x \otimes \mathbb{Q}$ into irreducible representations of $SL_2(\mathbb{Q})$; the highest weight $\mathbb{Q}$-cycles ψ are defined by the condition $N\psi = 0$; all other $\mathbb{Q}$-cycles are obtained from these ones by acting on them with the SL_2 lowering operator. Since $\boldsymbol{\tau}$ is the period map of a degenerating weight 1 Hodge structure, it follows from the Deligne weight filtration (or by the Clemens-Schmid

[36] SNC = simple normal crossing.

sequence, see **Corollary 2** in [196]), that only spin 0 and spin 1/2 representations are presents, that is, $N^2 = 0$. More precisely, we have a weight filtration of $\mathbb{Q}$-spaces

$$W_0 \subset W_1 \subset W_2 \equiv \Gamma_x \otimes \mathbb{Q}, \qquad W_0 = \operatorname{im} N, \quad W_1 = \ker N, \tag{5.39}$$

such that $N\colon W_2/W_1 \to W_0$ is an isomorphism, and the polarization $\langle -, - \rangle$ induces a perfect pairing between W_2/W_1 and W_0 as well as of W_1/W_0 with itself [220] (of course, these statements are just the usual selection rules for angular momentum).

We pull-back the local family of Abelian varieties $X|_U$ to the r-fold cover U_r; we get the family $\pi\colon \sigma^* X|_U \to U_r$. By construction, the monodromy of the pulled back family is $m_\sigma \equiv m^r = e^N$, so the monodromy invariant $\mathbb{Q}$-cycles are precisely the ones in W_1. The invariant cycle theorem guarantees that all 1-cycle $\gamma_x \in \Gamma_x$ invariant under the monodromy there is a homologous 1-cycle $\hat{\gamma}$ in $\sigma^* X|_U$ in the total space of the (local) family and all 1-cycles in the total space are of this form. Then W_2/W_1 consists of vanishing cycles, so that the corresponding periods vanish as $q^{1/r} \to 0$ i.e. $a_{\text{van}} \propto q^{1/r}$ and then eqn.(15.3.3) says that the periods along the "spin-0" cycles W_1/W_0 are regular as $q \to 0$ while the ones in W_0 (which are dual to the vanishing ones under the Dirac pairing) go as

$$a^D_{\text{van}} \propto a_{\text{van}} \log a_{\text{van}}. \tag{5.40}$$

The conclusion we got is totally trivial from the physical side. The special geometry along the Coulomb branch is the IR description obtained integrating out the massive degrees of freedom; the singularities arise because at certain loci in M some additional degree of freedom becomes massless. One gets the leading singularity by computing the correction to the low energy coupling by loops of light fields, see the discussion in §. 5.4 of the original paper by Seiberg and Witten [224]. The mixed Hodge variation formula (5.40) is just their eqn.(5.10).

Thus, a part for the need to go to the local r-fold cover U_r, at a generic point of an irreducible component S_i of the discriminant $\mathcal{D}$ we do not get singularities worse than physically expected. The singularity in eqn.(5.40) is mild:

R1 the squared-norm of the SW differential

$$\begin{aligned} \Phi(q) \stackrel{\text{def}}{=} i\Big(\int_{A^i} \lambda \int_{B_i} \bar{\lambda} - \int_{A^i} \bar{\lambda} \int_{B_i} \lambda\Big) = \\ = \text{const.}\, |q|^{2/r}(-\log|q|) + \text{regular as } q \to 0, \end{aligned} \tag{5.41}$$

while *not* smooth along the discriminant, extends *continuously* to $\mathcal{D}$;

R2 Its differential $d\Phi$, while singular in U, becomes continuous (non-smooth) when pulled back to the local r-fold cover U_r;

R3 the points on the discriminant, $q = 0$, are at a finite distance from smooth points. Indeed, on the local cover U_r the metric is modeled on $ds^2 = (-\log|q|)|dq^{1/r}|^2$ which is length decreasing with respect the flat metric $|dz|^2$, $z = (-\log|q|)^{1/2} q^{1r}$. On U the metric is asymptotically conical. In particular, M remain complete as a metric space;

R4 the integral of the Ricci curvature on the r-fold *covering* disk $|q^{1/r}| < \epsilon$ vanishes as $\epsilon \to 0$, i.e. there is no δ-like curvature concentrated on the discriminant, *except* for the obvious $\mathbb{Z}_r$-

orbifold singularity implied by the covering quotient $U_r \to U_r/\mathbb{Z}_r \equiv U$. Thus all arguments based on curvature bounds work as in the semi-simple case. We have already remarked that orbifold singularities do not spoil the curvature bounds (cfr. **Physical principle**).

All the above statements hold at all points of the discriminant (and not just at generic points along a smooth component) when $\mathcal{D}$ is a SNC divisor [220]. While this is generically the situation, the special geometry describing a particular SCFT with the mass deformations switched off may well be non generic. If the SCFT admits "enough" mass/relevant deformations, we can make $\mathcal{D}$ to be SNC by an arbitrarily small perturbation which cannot change the qualitative aspects of the physics. Even in SCFTs without (enough) deformations, it is very likely that — while the singularities may be more severe than the SNC ones — the four regularity conditions **R1-R4** still hold. Indeed, **R3** has been advocated by Gukov, Vafa and Witten as a necessary condition for a sound SCFT [141]. In the rest of the thesis we shall make the

Mild assumption. *Our special geometry satisfies* **R1**-**R4***.*

Remark 5.1.5. We may look at the singularities also from the point of view of the hyperKähler geometry of the total space X. Since hyperKähler manifolds are in particular Calabi-Yau, the discussions of refs. [141,241] directly apply with similar conclusions. Note that the statements hold also for hyperKähler *orbifolds.*

5.2 Some facts about complex orbifolds

In the last subsection we found that the analytic space M typically has cyclic orbifold singularities. Here we collect some well known facts about complex orbifolds that we shall need below.

Proposition 5.2.1 (See e.g. [43])**.** *The locally ringed space $(Z, \mathcal{O}_Z)$ associated to a complex orbifold has the following properties:*

i) $(Z, \mathcal{O}_Z)$ is a reduced normal analytic space;

ii) the singular locus $\Sigma(Z)$ is a closed reduced complex subspace of Z and has complex codimension at least 2 in Z;

iii) the smooth locus Z_{reg} is a complex manifold and a dense open subset of Z;

iv) Z is $\mathbb{Q}$-factorial.[37]

In particular, under our mild assumption, the Coulomb branch M is a $\mathbb{Q}$-factorial reduced normal analytic space.

We stress that the singular set in the orbifold sense of Z, $S(Z)$, may be actually larger than the singular locus of the underlying analytic space, $\Sigma(Z)$, see the discussion in ref. [43]. The case of maximal discrepancy between the two sets is given by the following:

Proposition 5.2.2. *Let G be a Shephard-Todd group ($\equiv$ a finite complex reflexion group* [82, 83, 232]*). Then the analytic space underlying the orbifold $\mathbb{C}^n/G$ is smooth, in fact isomorphic to $\mathbb{A}^n$.*

We shall also need the orbifold version of the Kodaira embedding theorem:

Theorem 4 (Kodaira-Baily [29])**.** *Let Z be a compact complex orbifold and suppose Z has a positive orbi-bundle $\mathcal{L}$. Then Z is a projective algebraic variety.*

[37] An analytic space is $\mathbb{Q}$-factorial if all Weil divisor has a multiple which is a Cartier divisor.

5.3 Structures on cones

We review the geometry of metric cones in a language suited for our purposes.

5.3.1 Riemannian cones

A *metric (Riemannian) cone* over the (connected, Riemannian) base B is the *warped product*

$$\mathbb{R}_{>0} \times_{r^2} B \equiv C(B), \tag{5.42}$$

that is, $C(B)$ is the product space $\mathbb{R}_{>0} \times B$ equipped with the metric

$$ds^2 = dr^2 + r^2\, \gamma_{ab}(y)\, dy^a dy^b \tag{5.43}$$

where $ds_y^2 = \gamma_{ab}(y)\, dy^a dy^b$ is a metric on B (y^a being local coordinates in B). We shall write $\overline{C(B)}$ for the singular space obtained by adding the tip of the cone $r = 0$ to $\gamma(B)$, endowed with the obvious topology. Note that the radial coordinate r is a globally defined continuous real function on $\overline{C(B)}$ taking all non-negative values. A cone $\mathbb{C}(B)$ possesses the following canonical (global) structures: the plurisubharmonic function r^2 and the concurrent vector field $E = r\partial_r$ (*Euler field*) which satisfy the following properties

$$\pounds_E r^2 = 2r^2, \qquad \pounds_E g = 2g, \qquad E(dr^2) = 2r^2, \qquad 2E_i = \nabla_i r^2. \tag{5.44}$$

that is

$$\nabla_i \nabla_j r^2 = \nabla_i E_j + \nabla_j E_i = 2g_{ij}. \tag{5.45}$$

Proposition 5.3.1 ([250]). *Conversely, if the Riemannian manifold (C, g) has a vector field E whose dual form is closed and $\pounds_E g = 2g$, there exist coordinates such that the metric takes the conical form* (5.43)*.*

Corollary 5.3.1. *Let C_1, C_2 be metric cones. Then $C_1 \times C_2$ is a metric cone with Euler vector $E_1 + E_2$.*

Definition 3. By a *good* cone we mean a cone $C(B) = \mathbb{R}_{>0} \times_{r^2} B$ with B smooth and complete. For a good cone, the only possibly singular point is the tip of the cone $r = 0$. Note that a non-trivial product of metric cones is never good unless one of the factors is $\mathbb{R}^k$ with the flat metric.

Remark 5.3.1. On a smooth Riemannian manifold the two notions of geodesic completeness and metric-space completeness coincide (Hopf-Rinow theorem [37]). This is not longer true in presence of singularities. **Example 2** illustrates the point: the Minahan-Nemeshanski geometry is complete in the metric space sense, but certainly not in the geodesic one. The singular Riemannian spaces which are "physically acceptable" better be complete as metric space. This is part of regularity assumption **R3**.

For later use, we give the well-known formulae relating the curvatures of $C(B)$ and its base B. We write R_{ijkl} (resp. R_{ij}) for the Riemann (Ricci) tensor of $C(B)$ and B_{abcd} (resp. B_{ab}) for the Riemann (Ricci) tensor of B.

Lemma 5.3.1. *One has*

$$R_{abcr} = R_{arbr} = 0 \tag{5.46}$$

$$R_{abc}{}^{d} = B_{abc}{}^{d} - \gamma_{ac}\,\delta_b^d + \gamma_{bc}\,\delta_a^d \tag{5.47}$$

$$R_{ab} = B_{ab} - (\dim B - 1)\gamma_{ab}. \tag{5.48}$$

5.3.2 Singular Kähler cones: the Stein property

Now suppose the Riemannian manifold M is both Kähler (with complex structure I) and a (metric) cone, $M \cong \overline{C(B)}$. Eqn.(5.45) implies that $\Phi = r^2$ is a *globally defined* Kähler potential assuming all values $0 \leq \Phi < \infty$.

In the applications to $\mathcal{N} = 2$ SCFT we have in mind, the Kähler metric on the cone M is singular even away from the tip $r = 0$. We specify the class of geometries we are interested in.

Definition 4. By a *singular Kähler cone M* we mean the following: **1)** M is an analytic space (which we may assume to be normal[38]) with an open everywhere dense smooth complex submanifold $M^\sharp = M \setminus \mathcal{D}$. **2)** On $M^\sharp$ there is a smooth conical Kähler metric (in particular, $M^\sharp$ is preserved by the $\mathbb{C}^\times$ action generated by the holomorphic Euler vector $\mathcal{E}$, see eqn.(5.52)). **3)** The global Kähler potential $\Phi \equiv r^2$ on $M^\sharp$ extends as a continuous function to all M (cfr. regularity condition **R1**). Then the Kähler form $i\partial\overline{\partial} r^2$ extends to M as a *positive (1,1) current.* The continuous function r^2 is then plurisubharmonic in the sense of ref. [182].

Proposition 5.3.2. *A singular Kähler cone with a compact base B is a* Stein analytic space.

Indeed, r^2 is a continuous plurisubharmonic function which is an exhaustion for M. The statement is then the Narasimhan singular version of Oka theorem (see e.g. page 48 of [182]).

Then the singular cone M is Stein and Cartan's **Theorem A** and **Theorem B** apply [119, 135, 139]. Below we shall exploit this fact in several ways.

In the physical applications we have in mind, the Fréchet ring [201] of global holomorphic functions, $\mathscr{R} = \Gamma(M, \mathcal{O}_M)$, is the Coulomb branch chiral ring, our main object of interest. M being Stein implies that $\mathscr{R}$ contains "many" functions: around all points of M we may find local coordinate systems given by global holomorphic functions, and $\mathscr{R}$ separates points, i.e. given two distinct points we may find a global holomorphic function which takes on these two points any two pre-assigned complex values. Affine varieties over $\mathbb{C}$ are in particular Stein [119]. The converse is not true in general, but it holds under some mild additional conditions [254]. We shall see that that the M's which are Coulomb branches of $\mathcal{N} = 2$ SCFTs are always affine.

5.3.3 Kähler cones: local geometry at smooth points, Sasaki manifolds

Specializing the results of §.5.3.1, in the (open everywhere dense) smooth locus $M^\sharp$ we have[39]

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi, \qquad \nabla_i \partial_j \Phi = 0, \qquad \Phi = r^2. \tag{5.49}$$

[38] An analytic space $(M, \mathcal{O}_M)$ is *normal* iff the stalks of its structure sheaf $\mathcal{O}_M$ are integrally closed, i.e. valuation rings. If M is not normal, replace it with its normalization.

[39] Factor 2 mismatches arise from different conventions in the real vs. complex case.

In particular, the real vector $R = IE$, or in components

$$R^i = i g^{i\bar{j}} \partial_{\bar{j}} \Phi, \qquad R^{\bar{i}} = -i g^{\bar{i}j} \partial_j \Phi, \tag{5.50}$$

is a *Killing vector*. The physical interpretation of this geometric result is as follows: a Kählerian cone may be used as a target space of a (classical) 3d supersymmetric σ-model. The fact that it is Kähler means the model is $\mathcal{N} = 2$ supersymmetric, while the fact that it is a cone means that is *classically* conformally invariant [36]; the two statements together imply that the model has classically $\mathcal{N} = 2$ superconformal symmetry hence a $U(1)_R$ R-symmetry which is part of the algebra. The action of $U(1)_R$ on the scalars is given by a (holomorphic) Killing vector which is R. We note that

$$[E, R] = 0, \qquad R(dr^2) = 0 \tag{5.51}$$

so that $R = R^a(y)\partial_{y_a}$ is in facts a Killing vector for the metric ds_y^2 on the base B whose norm is 1, i.e. $R^a R_a = 1$. For a holomorphic function h on a conic Kähler manifold the actions of E and R (in physical language: their dimension and R-charge) are related by

$$\pounds_R h = i \pounds_E h \quad \Longleftrightarrow \quad \pounds_{\bar{\mathcal{E}}} h = 0, \quad \text{where } \mathcal{E} = (E - iR)/2, \tag{5.52}$$

which physically says that these two quantum numbers should be equal for a chiral superconformal operator. We refer to $\mathcal{E}$ as the holomorphic Euler vector.

Remark 5.3.2. By definition, a cone $\mathbb{R}_{>0} \times_{r^2} B$ is Kählerian if and only if its base B is *Sasaki* [43]. The base B is in particular a K-contact manifold whose *Reeb vector* is R.

5.3.4 Geometric "F-maximization"

We pause a second to digress on a different topic, namely F-maximization in 3d [154]. A problem one encounters in studying SCFT is the exact determination of the R-charge which enters in the superconformal algebra. For (classical) σ-models with conic Kähler target spaces [36], this is the problem of identifying the Reeb Killing vector R between the family of Killing vectors with the appropriate action on the supercharges $\pounds_V Q = \pm\frac{1}{2}Q$. The general such Killing vector has the form $V = R + F$ with F a 'flavor' Killing symmetry. One has the following:

Claim. *Let M be a Kähler cone and V a Killing vector on M which acts on supercharges as $\pounds_V Q = \pm\frac{1}{2}Q$. Then (point-wise)*

$$\|V\|^2 \geq \Phi \equiv r^2, \tag{5.53}$$

with equality iff V is the Reeb vector R. That is, the true superconformal R-charge extremizes the square-norm.

Remark 5.3.3. The reader may easily check that this purely geometric fact *is really* F-maximization for the partition function on S^3 of the corresponding 3d σ-model in the classical limit $\hbar \to 0$. By considering the low-energy effective theory on the moduli space of SUSY vacua of a (quantum) 3d $\mathcal{N} = 2$ SCFT, one reduces the general case to this statement.

5.3.5 Quasi-regular Sasaki manifolds

In the physical applications we are mainly interested in Kähler cones with *quasi-regular* Sasaki bases B, that is, with compact Reeb vector orbits, so that R generates a (locally free) *compact* group of isometries $U(1)_R$ which we identify with the R-symmetry group which must be compact on physical grounds. B is *regular* if in addition, the $U(1)_R$ isometries act freely. We collect here some useful results.

Proposition 5.3.3 (see e.g. [43])**.** *B is a* quasi-regular *Sasakian manifold. Then*

i) The Reeb leaves are geodesic;

ii) B is a principal $U(1)$ orbi-bundle $B \to K$;

iv) the base K is Kähler orbifold;

v) If the flow is regular *K is a Kähler manifold and B a principal S^1-bundle.*

Indeed, K is just the symplectic quotient of M with respect to the Hamiltonian $U(1)_R$ flow, r^2 being its momentum map, as eqn.(5.50) shows.

5.4 Conical special geometries

We may introduce distinct notions of "conical special geometry". The weakest one corresponds to a holomorphic integrable system $X \to M$ whose Kählerian base M (with Kähler form (5.18)) happens to be a metric cone. A slightly stronger notion requires M to be a cone and the full set of geometric structures **S1-S11** to be equivariant with respect to the Euler action $\pounds_E$ (or $\pounds_{\mathcal{E}}$). An even stronger notion requires in addition that the base of B of M is a *quasi-regular* Sasaki manifold.[40] By a *conical special geometry* (CSG) we shall mean the strongest notion together with the regularity conditions **R1-R4**.

5.4.1 Weak special cones

Locally in the good locus $M^\sharp \subset M$ we may write the special structure in terms of special complex coordinates and holomorphic prepotential $\mathcal{F}(a)$. By the assumption of the existence of a SW differential, we know that there is a globally defined Kähler potential

$$\Phi = i\big(a^i \bar{b}_i - \bar{a}^i b_i\big). \tag{5.54}$$

If M is also a cone, so r^2 is also a globally defined Kähler potential, and

$$r^2 = \Phi + ih - i\bar{h} \tag{5.55}$$

for some local holomorphic function h with global real part. The metric is conic iff the vector dual to the $(0,1)$ form $\bar{\partial}(i\Phi + \bar{h})$ is holomorphic. Locally this happens if there are constants (c^i, d_j) such

[40] We do not know if the special cones in the slightly stronger sense are automatically special cones in the strongest sense.

that

$$b_i + d_i = \tau_{ij}(a)\big(a^j + c^j\big) \quad \text{and} \quad r^2 = i(a^i + c^i)(\bar{b}_i + \bar{d}_i) - i(\bar{a}^i + \bar{c}^i)(b_i + d_i). \tag{5.56}$$

The slighter stronger notion of special cone corresponds to the case $(c^i, d_j) = 0$ (so $r^2 \equiv \Phi$); in other words, to get a slightly stronger special cone out of a weak one we simply absorb the constants in the definition of the periods a^i, b_i by a shift $\delta\lambda = c^i\,dx_i + d_i\,dy^i$ of the SW differential. Then

$$\pounds_E \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b \\ a \end{bmatrix}, \tag{5.57}$$

which means that locally we can find a prepotential $\mathcal{F}(a)$ which is homogeneous of degree 2 in the special coordinates a^i. Indeed, in the slightly strong conic case, from eqn.(5.57) the Euler (Reeb) vector have the local expression

$$E = a^i\partial_{a_i} + \bar{a}^i\partial_{\bar{a}^i}, \qquad\qquad \pounds_E\mathcal{F} = 2\mathcal{F}, \tag{5.58}$$

$$R = ia^i\partial_{a_i} - i\bar{a}^i\partial_{\bar{a}^i}, \qquad\qquad \pounds_R\mathcal{F} = 2i\mathcal{F}. \tag{5.59}$$

The vector E agrees in the overlaps between two special coordinate patches if the condition $\pounds_E\mathcal{F} = 2\mathcal{F}$ holds in one of the two patches (and then also in the other, up to a constant). Indeed, the second eqn.(5.58) implies

$$b_i = \tau_{ij}(a)\,a^j. \tag{5.60}$$

Now, on the overlap

$$\begin{bmatrix} b' \\ a' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{5.61}$$

so that

$$a'^i = \big(C^{ik}\tau_{kj} + D^i{}_j\big)a^j, \qquad \frac{\partial a'^i}{\partial a^j} = C^{ik}\tau_{kj} + D^i{}_j, \tag{5.62}$$

then

$$\mathcal{E}' = a'^i\,\partial_{a'^i} = a^i\,\partial_{a^i} = \mathcal{E}, \tag{5.63}$$

and the holomorphic Euler vector $\mathcal{E}$ is globally defined. The fact that the periods a are (locally defined) holomorphic functions, fixes their transformation under the Killing-Reeb vector R

$$Ra = ia, \qquad Rb = ib. \tag{5.64}$$

Let Φ be the global Kähler potential (5.54).

Lemma 5.4.1. *For a (slightly strong) special cone M we have:*

1) *The function Φ in eqn.*(5.54) *is the squared-norm of the Euler (and Reeb) vector*

$$r^2 = \Phi = \|E\|^2 = \|R\|^2; \tag{5.65}$$

2) the conic relations (5.44) *take the form*

$$\partial_{\bar{\imath}}\Phi = g_{\bar{\imath}j}\, E^j. \tag{5.66}$$

Remark 5.4.1. Eqn.(5.66) together with **R2** imply that the real (complex) analytic vector fields E, R (resp. $\mathcal{E}$, $\overline{\mathcal{E}}$) on $M^\sharp$ extend to *continuous* fields in M and their $\mathbb{C}^\times$-action makes sense in M.

Remark 5.4.2. Let M be a special cone. Then the symplectic quotient K (cfr. **Proposition 5.3.3**) is a *wrong sign* projective special Kähler manifold [120]. By definition, projective special Kähler manifolds are the geometries appearing in $\mathcal{N}=2$ supergravity (as contrasted to $\mathcal{N}=2$ gauge theory); "wrong sign" means that K corresponds to supergravity with an unphysical sign for the Newton constant.

5.4.2 Properties of the Reeb flow/foliation: the Reeb period

As already anticipated, the special cones M which arise as Coulomb branches of $\mathcal{N}=2$ SCFTs have the property that the flow of the Reeb vector field R yields a $U(1)_R$ action on M, i.e. the Reeb leaves [43] are compact in M. This must be so because the exponential map[41] $t \mapsto \exp(2\pi t R)$ should implement the superconformal $U(1)_R$ symmetry which is compact in a regular SCFT. The action is automatically locally free, since the Reeb vector has constant norm 1 and hence does not vanish anywhere.[42] The statement that R generates a locally-free $U(1)_R$ action is equivalent to the statement that its basis B is a *quasi-regular* Sasaki orbifold.

We now give our final definition:

Definition 5. By a *conical special geometry* (CSG) we mean a complex analytic integrable system $X \to M$ with SW differential λ such that the base M (with the Kähler metric **S10**) is a singular Kähler cone (satisfying **R1**-**R4**) such that the restriction to $M^\sharp$ of its Euler vector E satisfies

$$\pounds_E\lambda - \lambda = d\varrho, \qquad \varrho \text{ meromorphic}, \tag{5.67}$$

while its base B is a quasi-regular Sasaki orbifold.

Note that this means (on $M^\sharp$)

$$\lambda = a^i\, dx_i + b_i\, dy^i + d\varrho'. \tag{5.68}$$

Exponential action of the Reeb field R. Quasi-regularity requires the existence of a minimal positive real number $\alpha > 0$ such that the Reeb exponential map

$$\exp(2\pi\alpha R)\colon M \to M \tag{5.69}$$

is the identity diffeomorphism. If the base B is compact, the map $\exp(2\pi\alpha t R)$ may have fixed points in $M \setminus \{r=0\}$ only for a finite set of rational values $0 < t < 1$.

[41] By $\exp(2\pi t R)$ we always mean the finite isometry of M or B generated by the Reeb Killing field of parameter $2\pi t$.

[42] Of course, this geometric statement also follows from unitarity of the SCFT.

Definition 6. The real number $\alpha > 0$ is called the *Reeb period* of the CSG.

The Reeb period is a basic invariant for a 4d $\mathcal{N} = 2$ SCFT.

Reeb period and Coulomb dimensions. The chiral ring $\mathscr{R} = \Gamma(M, \mathcal{O}_M)$ of a CSG is (the Fréchet closure of) a graded ring, the grading of a (homogeneous) global holomorphic function h being given by its dimension $\Delta(h) \in \mathbb{R}$

$$\pounds_{\mathcal{E}} h = \Delta(h)\, h. \tag{5.70}$$

In the chiral ring of a unitary SCFT, except for the constant function 1 which has dimension zero, all other dimensions $\Delta(h)$ should be strictly positive for h to be regular at the tip. The Reeb exponential map then yields

$$\exp(2\pi t R) \cdot h = e^{2\pi i t \Delta(h)}\, h \qquad t \in \mathbb{R} \tag{5.71}$$

and the definition (5.69) implies

$$\Delta(h) \in \frac{1}{\alpha}\, \mathbb{N}, \tag{5.72}$$

so $\mathscr{R}$ is graded by the semigroup $\mathbb{N}/\alpha$. We claim that α is a positive rational number ≤ 1 (so that the dimensions of all chiral operators, but the identity, are rational numbers ≥ 1). Indeed, the map $e^{2\pi\alpha R}$ acts on the periods as $(a^i, b_j) \mapsto e^{2\pi i\alpha}(a^i, b_j)$. From (5.69) we deduce that the initial and final periods are equivalent up to the action of an element m of the monodromy group. Then $e^{2\pi i\alpha} \equiv \lambda$ is an eigenvalue of a monodromy, hence a root of unity. Therefore $\alpha \in \mathbb{Q}_{>0}$ and $\alpha = \log(\lambda)/2\pi i$. The requirement that the curvature at the tip of the cone is non-negative forces us to use the **log** determination of the logarithm, see below eqn.(14.2).

The order r of $1/\alpha$ in $\mathbb{Q}/\mathbb{Z}$ coincides with the order of the quantum monodromy $\mathbb{M}$ which is well-defined in virtue of the Kontsevitch-Soibelman wall-crossing formula, see discussion in [60, 65].

5.4.3 Local geometry on $K^\sharp$

In a special cone, the discriminant locus is preserved by the $\mathbb{C}^\times$ action generated by the vector fields E and R. Hence the singular locus on its base B is preserved by the R-flow, and so is its smooth locus $B^\sharp$. By our definition, $B^\sharp$ must be Sasaki quasi-regular. Then the Hamiltonian quotient yields a Kähler orbifold (manifold if $B^\sharp$ is regular) $K^\sharp \equiv B^\sharp/U(1)$ of dimension $(k-1)$ (cfr. **Proposition 5.3.3**). The Kähler potential of $K^\sharp$ is

$$\log \Phi = \log \|R\|^2 = \log \|\lambda\|^2, \tag{5.73}$$

where the SW differential λ is seen as a section of a holomorphic line sub-bundle $\mathcal{L} \subset \mathcal{V}$. The period matrix $\boldsymbol{\tau}_{ij}(a)$ is homogeneous of degree zero, $\pounds_R \boldsymbol{\tau}_{ij}(a) = 0$ so the (restriction of the) period map $\boldsymbol{\tau}|_{M^\sharp}$ factors trough $K^\sharp$. We write $p\colon K^\sharp \to Sp(2k, \mathbb{Z})\backslash\mathfrak{H}_k$ for the period map so defined.

Lemma 5.4.2. *The Ricci form on $K^\sharp$ is*

$$\rho = k\,\omega + p^*\Omega \geq k\,\omega, \tag{5.74}$$

where $\omega = i\partial\overline{\partial}\log\Phi$ *is the Kähler form on* $K^\sharp$ *and* Ω *the Kähler form of the Siegel upper half-space* $\mathfrak{H}_k$ (compare **Proposition 5.1.2**).

5.4.4 Special cones with smooth bases

The bases B of the CSG interesting for the physical applications are seldom smooth (as Riemannian spaces). However, we first consider the case of B a regular smooth Sasaki manifold and then discuss what may change in our conclusion in presence of singularities.

Combing **Lemma 5.3.1** and **Proposition 5.1.2** we get the inequality

$$B_{ab} \geq 2(k-1)\,\gamma_{ab}. \tag{5.75}$$

In view of Meyer theorem (**Theorem 1**) we conclude that for $k \equiv \dim_{\mathbb{C}} M > 1$ the base B of the cone is compact, $\pi_1(B)$ is finite, and the diameter of B is bounded

$$d(B) \leq \pi/\sqrt{2(k-1)}. \tag{5.76}$$

In other words: if M is a special cone of dimension $k > 1$ over a *smooth* base B,

$$M = \widetilde{M}/G \tag{5.77}$$

with $\widetilde{M}$ a *simply-connected* Kähler cone with compact base of diameter $\leq \pi/\sqrt{2(k-1)}$ and G is a *finite* group acting freely.

Corollary 5.4.1. *M a special Kähler cone over a smooth base $B \Rightarrow M$ is Stein.*

Indeed B is compact, and then **Proposition 5.3.2** applies.

Proposition 5.4.1. *Let X be a special cone whose Sasaki base B is smooth and regular. Then its Hamiltonian reduction K (see* **Proposition 5.3.3***) is a smooth, compact, and* simply-connected *Kähler manifold.*

Indeed, K is compact, smooth, and, being complex, oriented of even real dimension. Its sectional curvatures are positive. Then the last statement follows from Synge theorem (**Theorem 2**). In facts, since B is compact by Meyer theorem, **Theorem 3** yields an even stronger statement:

Proposition 5.4.2. *A smooth base B of a CSG is* diffeomorphic *to S^{2k-1}/G for some freely acting finite subgroup $G \subset U(k)$.*

5.4.5 Relation to Fano manifolds

Recall that a Fano manifold X is a smooth projective variety whose anticanonical line bundle $-K_X$ is ample. We have:

Proposition 5.4.3. *Under the assumptions of* **Proposition 5.4.1***, the Kähler manifold K is a* Fano projective manifold.

Proof. K is smooth by assumption and compact by Meyers theorem. By **Lemma 8.30** the Ricci form is $\geq (\dim K + 1)\omega$, ω being the (positive) Kähler form. Thus the anti-canonical is ample, hence K is projective (by Kodaira embedding theorem [139]) and Fano. $\square$

Proposition 5.4.4. *Under the assumptions of* **Proposition 5.4.1** *and* $k > 1$*:*

i) the universal cover $\widetilde{B}$ *of the base manifold* B *is* homeomorphic *to* S^{2k-1}*;*

*ii) the rank of the Picard group (*Picard number*) of the Fano manifold* K *is* 1*:*

$$\varrho(K) \equiv \operatorname{rank}\operatorname{Pic}(K) = 1; \tag{5.78}$$

iii) the Hodge diamond of K *is* $h^{p,q}(K) = \delta^{p,q}$*.*

Proof. $\widetilde{B}$ is compact and simply connected by Meyers' theorem. From **Lemma 5.3.1** and eqn.(5.27) we see that the eigenvalues of the curvature operators are bounded below by 1. Using **Theorem 3** we get *i)*. Then $B \equiv \widetilde{B}/\pi_1(B)$ has real cohomology

$$H^q(B,\mathbb{R}) = \begin{cases} \mathbb{R} & \text{for } q = 0, 2k-1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.79}$$

Since B is Sasaki-regular, it is a principal $U(1)$-bundle over K

$$\begin{array}{ccc} S^1 & \longrightarrow & B \\ & & \downarrow \\ & & K \end{array} \qquad \text{and } K \text{ is simply-connected (}\textbf{Proposition 5.4.1}\text{).} \tag{5.80}$$

To this fibration we apply the Leray spectral sequence of de Rham cohomology.[43] We get

$$H^q(K) = \begin{cases} \mathbb{R} & q = 0, 2, \cdots, 2(k-1) \\ 0 & \text{otherwise.} \end{cases} \tag{5.81}$$

This shows *iii)*. To get *ii)* note that the exponential exact sequence yields the implication

$$H^1(K) = 0 \quad \Longrightarrow \quad \operatorname{Pic}(K) \cong H^2(K,\mathbb{Z}), \tag{5.82}$$

and then *ii)* follows from *iii)*. □

We have a much stronger statement:

Proposition 5.4.5. *Under the assumptions of* **Proposition 5.4.1***, the Fano manifold* $K \cong \mathbb{P}^{k-1}$ *and the period map* p *is constant.*

Before proving this **Proposition** we define the *index* $\iota(F)$ of a Fano manifold F whose canonical divisor we write K_F. The index $\iota(F)$ is the largest positive integer such that $-K_F/\iota(F)$ is a (ample) divisor [150]. Under the assumptions of **Proposition 5.4.1** we have a line bundle $\mathcal{L} \to K$ such that ω is its Chern class (up to normalization). Thus from **Lemma 8.30** we get:

Lemma 5.4.3. *Under the assumptions of* **Proposition 5.4.1** *we have* $\iota(K) \geq \dim K + 1$ *with equality iff the period map* $\boldsymbol{\tau}$ *is constant.*

[43] The computation is world-for-world identical to Example 14.22 in [41].

Proof. The Picard number of K is 1, and we have

$$[p^*\Omega] = \delta[\omega] \quad \text{for some } \delta \in \mathbb{Q},\ \text{with } \delta \geq 0,\ \text{and } = \text{ only if } p \text{ is constant,} \tag{5.83}$$

so

$$\iota(K) = \dim K + 1 + \delta \geq \dim K + 1 \tag{5.84}$$

with equality iff p =constant. □

Then **Proposition 5.4.5** follows from this **Lemma** and a basic fact from the theory of Fano varieties: the index of a Fano manifold F cannot exceed $\dim F + 1$, and if $\iota(F) = \dim F + 1$ then $F \cong \mathbb{P}^{\dim F}$ [150].

Corollary 5.4.2. *Suppose we have a special cone of dimension $k > 1$. If its base B is a smooth, complete, regular Sasaki manifold, the period map $\boldsymbol{\tau}$ is constant and $M = \mathbb{C}^k$ with the a Kähler metric.* That is: if the geometry is regular except for the singularity at the vertex of the cone, then the SCFT is free, as expected.

Remark 5.4.3. The result does not hold for $k = 1$. All $k = 1$ Minahan-Nemeshanski geometries satisfy the other assumptions, yet the SCFT is not free. The essential point is that for $k > 1$ the assumptions imply regularity in codimension 1, whereas the tip of the cone is automatically a codimension 1 singularity for $k = 1$.

Remark 5.4.4. The last result also follows from rigidity. If K is smooth and compact, the monodromy group is trivial, hence (by uniqueness) the period map should be the constant one.

5.4.6 Properties of non-smooth CSG

In the previous sub-section we considered the case in which the geometry of the special cone is totally regular away from its vertex. We got only the (known) $k = 1$ geometries and free theories for $k > 1$. In all other cases there are singularities in $M \setminus \{0\}$ of the kind consistent with **R1-R4**.

Since $B^\sharp$ is only quasi-regular, we have a locus $N \subset B^\sharp$ on which $U(1)_R$ does not act freely. We write $\mathring{B} = B^\sharp \setminus N$. For $k > 1$, $\mathring{K} = \mathring{B}/U(1)$ is a open dense Kähler sub-manifold of the singular space K, in facts $\mathring{K} = K \setminus D$, for some divisor D. Since the period matrix $\boldsymbol{\tau}_{ij}$ is homogeneous of degree zero, the period map $\boldsymbol{\tau}$ factors through K. We consider its restriction to the regular subspace, $\boldsymbol{\tau}\colon \mathring{K} \to Sp(2k, \mathbb{Z})\backslash\mathfrak{H}_k$. All the considerations in §. 5.1.5 apply to this map; in particular, we have a monodromy representation $\mathring{m}\colon \pi_1(\mathring{K}) \to Sp(2k, \mathbb{Z})$. Along an irreducible component D_i of D such that the monodromy is semi-simple, we may extend $\boldsymbol{\tau}$ holomorphically and K has along D_i only a $\mathbb{Z}_r$ cyclic orbifold singularity: indeed, the geometry becomes smooth after the local base change $U \to U_r$, cfr. §. 5.1.5. Otherwise, the monodromy along D_i satisfies $(m^r - 1)^2 = 0$; the base change $U \to U_r$ sets the local geometry in the form discussed around eqn.(5.40). Locally U_r has the form $\Delta^* \times \Delta^{k-2}$ and we may choose local coordinates so that the Kähler form take the form (here $|q| < 1$)

$$i\partial\overline{\partial}\log\Phi \approx i\partial\overline{\partial}\log\left(|q|^2\big(1 - \log|q|\big) + \sum_{i=2}^{k-1}|x_i|^2\right). \tag{5.85}$$

We may modify this metric to

$$i\partial\overline{\partial}\log\left(|q|^2\Big(1-\log|q|\cdot f_C(-\log|q/\epsilon|^2)\Big)+\sum_{i=2}^{k-1}|x_i|^2\right) \tag{5.86}$$

where $f_C(x)\colon \mathbb{R}\to\mathbb{R}$ is a smooth function which equals 1 for $x\leq 0$ and has all derivatives vanishing as $x\to+\infty$. The new Kähler form (5.86) is smooth in U_r and agrees with the original one for $|q|\geq\epsilon$; one checks that one may choose the local deformation so that the metric and the curvatures remain positive. Of course, it is no longer a special Kähler metric. The point we wish to argue is that K admits a non-special *orbifold* Kähler metric, with only $\mathbb{Z}_r$ orbifold singularities, whose Ricci form satisfies a bound of the form $R_{i\bar{\jmath}}\geq k g_{i\bar{\jmath}}$. The same conclusion applies to M (the Ricci tensor being non-negative in this case); then the basis B is also regular except for cyclic orbifolds singularities. We may apply to B the orbifold version of Meyers [40, 252]): a metrically complete Riemannian orbifold X, whose Ricci curvature satisfies $R\geq\lambda^2 g$, is compact with a diameter $d(X)\leq\pi/\lambda$. Once we are assured that B, albeit singular, is compact we conclude that M is Stein. Moreover, from the Synge theorem for orbifolds [252], we see that K, albeit no smooth, is still compact and simply-connected.

$\mathcal{L}$ is now a line orbi-bundle which is still positive by eqn.(5.74), Kodaira-Baily embedding theorem (**Theorem 4**) guarantees that K is a normal projective algebraic variety with at most cyclic orbifold singularities. The anticanonical divisor $-K_K$ is now a Weyl divisor which is *ample* as a $\mathbb{Q}$-Cartier divisor. A normal projective variety with ample anti-canonical $\mathbb{Q}$-divisor having only cyclic singularities is a *log*-Fano variety, to be defined momentarily. The Picard number ϱ is 1 as in the smooth case. Indeed, B is still diffeomorphic to a generalized Lens space S^{2k-1}/G (see **Proposition 5.4.2**) and the finite group G centralizes the Reeb $U(1)$ action, so that the orbifold K is homeomorphic to a finite quotient of $\mathbb{P}^{k-1}$, and hence rank $H^2(K,\mathbb{Z})=1$. The index $\iota(F)$ of a *log*-Fano variety F is the greatest positive *rational* such that $-K_F=\iota(F)\,H$ for some Cartier divisor H (called the *fundamental divisor*). By a theorem of Shokurov [150], the index of a *log*-Fano F is at most $\dim F+1$. This is not necessarily in contradiction with eqn.(5.74) since now $\mathcal{L}$ is just a $\mathbb{Q}$-divisor. Let σ be the smaller positive rational number so that $\mathcal{L}^\sigma$ is a *bona fide* line bundle. Then

$$\iota(K)=\frac{\dim K+1+\delta}{\sigma}\geq\frac{\dim K+1}{\sigma}. \tag{5.87}$$

The theorem of Shokurov then yields $1\leq\sigma$ with equality if and only if our special cone M is $\mathbb{C}^k$ with the flat metric, i.e. we are talking about the free SCFT. Shokurov inequality is one of the many geometrical properties underlying the physical fact that saturating the unitarity bound means to be free. We see that the main difference between the smooth case of the previous subsection and the general is that in the second case G acts properly discontinuously on S^{2k-1} but not freely. The previous argument giving $\varrho=\operatorname{rank}\operatorname{Pic}(K)=1$ extends to this (slightly) more general case (indeed, K is simply-connected and topologically a finite quotient of $\mathbb{P}^{k-1}$).

5.4.7 *log*-Fano varieties

A singular Fano variety whose only singularities are cyclic orbifold ones is a *log*-Fano variety. We recall the relevant definitions [150]:

Definition 7. **1)** A Weil divisor D is called $\mathbb{Q}$-*Cartier* if there exists $m \in \mathbb{N}$ such that mD is a Cartier divisor. **2)** A normal variety X is $\mathbb{Q}$-*factorial* if all Weil divisors are $\mathbb{Q}$-Cartier. (Note that if X is normal, the canonical divisor K_X is well-defined in the Weil sense[44]). **3)** A normal variety X is said to have *terminal, canonical, log terminal,* or *log canonical singularities* iff its canonical divisor K_X is $\mathbb{Q}$-Cartier and there exists a projective birational morphism (whose exceptional locus has normal crossing[45]) $f\colon V \to X$ from a smooth variety V such that:

$$K_V = f^* K_X + \sum_i a_i\, E_i, \qquad \text{where } E_i \text{ are the } \textit{prime exceptional divisors,} \text{ i.e. the irriducibile components of the exceptional locus of } f \text{ of codimension 1} \tag{5.88}$$

and, respectively:

$$\begin{aligned} &a_i > 0 \quad \text{terminal,} && a_i \geq 0 \quad \text{canonical,}\\ &a_i > -1 \quad \text{log-terminal,} && a_i \geq -1 \quad \text{log-canonical.}\end{aligned} \tag{5.89}$$

4) A normal projective variety X with only log-terminal singularities whose anti-canonical divisor $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor is called a *log-Fano variety.* **5)** The greatest rational $\iota(X) > 0$ such that $-K_X = \iota(X)\, H$ for some (ample) Cartier divisor H is called the *index* of X and H is called a *fundamental divisor.* **6)** The *degree* of a Fano variety X is the self-intersection index $d(X) = H^{\dim X}$.

Comparing our discussion in §.5.4.6 with the above definitions, we conclude (cfr. **Proposition 7.5.33** of [43]):

Corollary 5.4.3. *The symplectic quotient $K = M/\!\!/U(1)$ of a CSG is a* log-*Fano variety with Picard number one and Hodge diamond $h^{p,q} = \delta^{p,q}$. Moreover, in the smooth sense, $K \cong \mathbb{P}^{k-1}/G$ for some finite group G.*

Thus a CSG M is a complex cone over a normal projective variety of a very restricted kind; in particular, M is affine and a *quasi-cone* in the sense of §.3.1.4 of [100].

Example 3. A large class of examples of log-Fano varieties with the properties in the **Corollary** is given by the weighted projective spaces (WPS) [100]. Il $\boldsymbol{w} = (w_1, w_2, \cdots, w_k) \in \mathbb{N}^k$ is a system of weights, $\mathbb{P}(\boldsymbol{w})$ is

$$\big(\mathbb{C}^k \setminus \{0\}\big)/\sim \quad \text{where} \quad (z_1, \cdots, z_k) \sim (\lambda^{w_1} z_1, \cdots, \lambda^{w_k} z_k)\ \forall\, \lambda \in \mathbb{C}^\times. \tag{5.90}$$

All such spaces are log-Fano with just cyclic orbifold singularities, simply-connected, have Hodge numbers $h^{p,q} = \delta^{p,q}$, and are isomorphic to $\mathbb{P}^{k-1}/G$ for some finite Abelian group G. More generally, all quasi-smooth complete intersections in weigthed projective space of degree $d < \sum_i w_i$ and dimension at least 3 are log-Fano, simply-connected, and have $\varrho = 1$ [100] but typically their Hodge numbers $h^{p,q} \not\equiv \delta^{p,q}$. A slightly more general class of examples of projective varieties satisfying all conditions above is given by the *fake* weighted projective spaces [48]. A fake WPS is canonically the quotient of a WPS by a finite Abelian group. One shows that a log-Fano with Picard number one which is also *toric* is automatically a fake WPS.

[44] Indeed, X is non-singular in codimension 1, so a canonical divisor over the smooth open set $X_{\text{smooth}} \subset X$ can be extended as a Weil divisor to X.

[45] Such resolutions exists by Hironaka theorem [143].

Example 4. Let $G \subset PGL(k, \mathbb{C})$ be a finite subgroup; then $\mathbb{P}^{k-1}/G$ is a ($\mathbb{Q}$-factorial) *log*-Fano variety with $\varrho = 1$ [150]. By §. 5.4.6, K is always *diffeomorphic* to such a variety.

6 The chiral ring $\mathscr{R}$ of a $\mathcal{N} = 2$ SCFT

6.1 General considerations

In the previous section we reviewed the general properties of the CSG describing the Coulomb branch M of a 4d $\mathcal{N} = 2$ SCFT in which we have switched off all mass and relevant deformations so that the conformal symmetry is only spontaneously broken by the non-zero expectation values of some chiral operators $\langle \phi_i \rangle_x \neq 0$ in the susy vacuum $x \in M$. Although the Kähler metric on M has singularities in codimension 1, due to additional degrees of freedom becoming massless or symmetries getting restored, these singularities are assumed to be mild and the underlying complex space M is regular[46]. The analysis of local special geometry on the open, everywhere dense, regular set $M^\sharp = M \setminus \mathcal{D}$ only determines the special geometry up to "birational equivalence" since the singular fibers over the discriminant locus $\mathcal{D}$ may be resolved in different ways; different consistent resolutions give non-isomorphic CSG which correspond to distinct physical models. For instance, $SU(2)$ SQCD with $N_f = 4$ and $SU(2)$ $\mathcal{N} = 2^*$ have "birational equivalent" CGS whose total fiber spaces X are certainly different in codimension 1; other examples of such "birational" pairs in rank one are given by Minahan-Nemeshansky E_r SCFTs and Argyres-Wittig models [25] with the same Coulomb branch dimension but smaller flavor groups. The basic invariant of an equivalence class of SCFTs is the chiral ring $\mathscr{R} = \Gamma(M, \mathcal{O}_M)$.

The common lore is that $\mathscr{R}$ is (the Fréchet completion of) a free graded polynomial ring $\mathscr{R} = \mathbb{C}[u_1, \cdots, u_k]$ whose grading is given by the action of the holomorphic Euler field, $\pounds_{\mathcal{E}} u_i = \Delta_i u_i$. The set of rational number $\{\Delta_1, \cdots, \Delta_k\}$ are the Coulomb dimensions. This lore may be equivalently stated by saying that the Hamiltonian reduction $K = M /\!\!/ U(1)_R$ is (birational to) the weighted projective space (WPS) $\mathbb{P}(\Delta_1, \cdots, \Delta_k)$. In the previous section we deduced from special geometry several detailed properties of M and K which are consistent with this idea: M is affine while K is a normal projective log-Fano variety, which is simply-connected and has Hodge numbers $h^{p,q} = \delta^{p,q}$. These are quite restrictive requirements on an algebraic variety, and they hold automatically for all WPS; indeed they almost characterize such spaces. For instance, if one could argue that K (or M) is a toric variety then these properties would imply that K is a fake WPS [48], and in particular a finite Abelian quotient of a WPS.

We sketch some further arguments providing additional evidence that K is (birational to) a finite quotient of a WPS (possibly non-Abelian). The monodromy and SL_2-orbit theorems describe the asymptotical behavior of the special Kähler metric on M as we approach the discriminant $\mathcal{D}$ (at least for $\mathcal{D}$ SNC). As argued in §. 5.4.6, we can modify the metric to a Kähler metric which is smooth in $M \setminus \{0\}$, agrees with the original one outside a tubular neighborhood of $\mathcal{D}$ of size ϵ, is conical and Kähler with non-negative curvatures. Of course the new metric is no longer special Kähler and cannot be written in terms of a holomorphic period map; nevertheless it is a nice regular metric on the complex manifold $M \setminus \{0\}$ with all the good properties. Working in the smooth category, and using the sphere theorems (cfr. §.5.1.4), we conclude that the Riemannian base B of the cone

[46] Regularity of the underlying analytic space M in codimension 1 already follows from the fact that we are free to assume it to be normal.

is diffeomorphic to a space form S^{2k-1}/G for some finite G acting freely, hence to S^{2k-1} up to a finite cover. The finite cover M' of M is a Riemannian cone over S^{2k-1} is diffeomorphic to $\mathbb{R}^{2k} \cong \mathbb{C}^k$. Moreover we know that M' is Stein, in fact a normal affine variety. If we could say that M', being an affine variety diffeomorphic to $\mathbb{C}^k$ is biholomorphically equivalent to $\mathbb{C}^k$, we would be done. Unfortunately, for $k \geq 3$ there are several examples of *exotic* $\mathbb{C}^k$, that is, affine algebraic spaces which are diffeomorphic to $\mathbb{C}^k$ but not biholomorphic to $\mathbb{C}^k$ [253]. However, no example is known of an affine variety over $\mathbb{C}$ which is diffeomorphic to $\mathbb{C}^k$ but not birational to $\mathbb{C}^k$; the conjecture which states that there are none being still open [253]. Hence, assuming the conjecture to be true, we conclude that M' is *birational* to $\mathbb{C}^k$. On the other hand, M' is not just a complex space diffeomorphic to $\mathbb{C}^k$, it has additional properties as (e.g.) a holomorphic Euler vector $\mathcal{E}$ whose spectrum is an additive semigroup $\mathsf{spec}(\mathcal{E}) \subset \{0\} \cup \mathbb{Q}_{\geq 1}$, and the condition of non-negative bisectional curvatures. Were not for the singularity at the tip of the cone, this last property by itself would guarantee[47] that M' is analytically isomorphic to $\mathbb{C}^k$.

There is still the question of the relation between the Coulomb branch M and its finite cover M'. Since $M \cong M'/G$, the statement $M = M'$ is equivalent to $G = 1$, that is (for $k > 1$) that after smoothing the discriminant locus by local surgery M becomes simply-connected. This holds in the sense that the monodromy along a cycle not associated to the discriminant is trivial (since the period map factors through K).

All the above results and considerations, while short of a full mathematical proof, provide convincing evidence for the general expectation that M is birational to $\mathbb{C}^k$, so for the purposes of our "birational classification" we may take M to be just $\mathbb{C}^k$. $\mathbb{C}^\times$ acts on M through the exponential action of the Euler field $\mathcal{E}$. Again, an action of $\mathbb{C}^\times$ on $\mathbb{C}^k$ is guaranteed to be linear only for $k \leq 3$ [253]; in larger dimensions exotic actions do exist. However, in the present case the action is linear in the smooth sense; under the assumption that the complex structure is the standard one for $\mathbb{C}^k$, we have a linear action, that is, the chiral ring has the form

$$\mathscr{R} = \mathbb{C}[u_1, \cdots, u_k]. \tag{6.1}$$

where the u_i can be chosen to be eigenfunctions of $\mathcal{L}_{\mathcal{E}}$, i.e. homogeneous of a certain degree Δ_i. Then K is the WPS $\mathbb{P}(\Delta_i)$. Thus, under some mild regularity conditions and modulo the plausible conjecture that a smooth affine variety M whose underlying C^∞-manifold is diffeomorphic to $\mathbb{C}^n$ is actually bimeromorphic to $\mathbb{C}^n$, we conclude that the common lore is correct up to birational equivalence (and, possibly, finite covers). For the rest of this thesis we shall assume this to be the case without further discussion.

Remark 6.1.1. It is interesting to compare the above discussion with the one in ref. [15]. The previous argument is based on the idea that the unitary bound $\Delta(u) > 1$ implies that we may smooth out the metric while preserving the curvature inequalities. In ref. [15] they find that if the chiral ring is not free there must be a local parameter u with $\Delta(u) < 1$, so if there exists such an

[47] The following theorem (a special case of Yau's conjecture [78,251]) holds:

Theorem (Chau-Tam [77]). *N a complete non-compact Kähler manifold with non-negative and bounded bi-sectional curvature and maximal volume growth. N is biholomorphic to $\mathbb{C}^n$.*

M with the modified metric satisfy non-negativity of bisectional curvatures and maximal volume growth, but the curvatures (if non zero) blow up as $r \to 0$, and geodesic completeness fails at the vertex.

u the argument leading to (6.1) cannot be invoked. Clearly u should not be part of the physical chiral ring $\mathscr{R}_{\rm ph}$. However, in a Stein space the local parameter may always be chosen to be a global holomorphic function, so $u \in \Gamma(M, \mathcal{O}_M)$, which is our definition of the "chiral ring". Going through the examples of ref. [15] one sees that their Coulomb branches are *non-normal analytic spaces,* that is, they have the same underlying topological space $M_{\rm top}$ by a different structure sheaf $\mathcal{O}_{\rm ph} \neq \mathcal{O}_M$ whose stalks are not integrally closed. In this case the physical chiral ring $\mathscr{R}_{\rm ph} = \Gamma(M, \mathcal{O}_{\rm ph})$ may be identified with a subring of the geometrical one $\mathscr{R} = \Gamma(M, \mathcal{O}_M)$. (See also the third *caveat* in §.5.1.1 and §.10.3.1 below).

To fully determine the graded rings $\mathscr{R}$ which may arise as Coulomb chiral rings of a CSG, it remains to determine the allowed dimension k-tuples $\{\Delta_1, \dots, \Delta_k\}$. We shall address this question in section 10. Before going to that, we consider the simplest possible CSG just to increase the list of explicit examples on which we may test the general ideas.

6.2 The simplest CSG: constant period maps

For $k = 1$ all CSG have constant period map $\boldsymbol{\tau} = \text{const}$. For $k > 1$ constant period maps are still a possibility, although very special. For instance, this happens in a Lagrangian SCFT in the limit of extreme weak coupling (classical limit). This is good enough to compute the Coulomb dimensions Δ_i, since they are protected by a non-renormalization theorem (or just by the fact that they are $U(1)_R$ characters and $U(1)_R$ is not anomalous).

The Kähler metric is locally flat, i.e. has the local form

$$ds^2 = \text{Im}\, \boldsymbol{\tau}_{ij}\, da^i d\bar{a}^j. \tag{6.2}$$

Then $M = \mathbb{C}^k/\mathcal{G}$ where $\mathcal{G} \subset \text{Fix}(\boldsymbol{\tau}) \subset Sp(2k, \mathbb{Z})$ is a subgroup of the isotropy group $\text{Fix}(\boldsymbol{\tau})$ of $\boldsymbol{\tau}$. $\text{Fix}(\boldsymbol{\tau})$, being discrete and compact, is finite. Then

$$\mathbb{C}[a_1, \cdots, a_k]^{\mathcal{G}} \subset \mathscr{R}. \tag{6.3}$$

We stress that (in general) we have just an *inclusion* not an equality: in **Example 2** the three rank-1 Argyres-Douglas SCFT (respectively of types A_2, A_3 and D_4) have

$$\mathbb{C}[a]^{\mathbb{Z}_{m+1}} = \mathbb{C}[u^m] \subsetneq \mathbb{C}[u] \equiv \mathscr{R}, \tag{6.4}$$

with $m = 5, 4, 2$ respectively. Note, however, that we have still

$$K = \mathsf{Proj}\, \mathscr{R} \cong \mathsf{Proj}\, \mathbb{C}[a]^{\mathcal{G}} \tag{6.5}$$

in these cases. The point is that K does not fix uniquely the chiral ring, unless we specify its orbifold structure, see the discussion in [216]. We shall study the orbifold behavior in more generality in the next section. Here we limit ourselves to the very simplest case in which

$$\mathscr{R} \cong \mathbb{C}[a_1, \cdots, a_k]^{\mathcal{G}} \tag{6.6}$$

as graded $\mathbb{C}$-algebras. The Shephard-Todd-Chevalley theorem [82, 232] states that $\mathbb{C}[a_1, \cdots, a_k]^{\mathcal{G}}$

is a graded polynomial ring if and only if $\mathcal{G}$ is a finite (complex) reflection group[48]. In such a case the Coulomb branch dimensions Δ_i coincide with the degrees of the fundamental invariants u_i of the group $\mathcal{G}$ such that $\mathbb{C}[a_1, \cdots, a_k]^{\mathcal{G}} = \mathbb{C}[u_1, \cdots, u_k]$. However, not all finite reflection groups can appear in special geometry, since only subgroups of the Siegel modular group, $\mathcal{G} \subset Sp(2k, \mathbb{Z})$, are consistent with Dirac quantization.[49] In §.10.2 we review the well known:

Fact. *Let $\mathcal{G} \subset Sp(2k, \mathbb{Z})$ be a* finite *subgroup of the Siegel modular group. The subset* $\mathrm{Fix}(\mathcal{G}) \subset \mathfrak{H}_k$ *of points in the Siegel upper half-space fixed by $\mathcal{G}$ is a* non empty, connected *complex submanifold.*

Thus, if $\mathcal{G}$ is a finite reflection subgroup of $Sp(2k, \mathbb{Z})$ there is at least one period $\boldsymbol{\tau}$ with $\boldsymbol{\tau} \subset \mathrm{Fix}(\mathcal{G})$ and the quotient by $\mathcal{G}$ of $\mathbb{C}^k$ with the flat metric (6.2) makes sense (as a orbifold). All such metrics are obtained by continuous deformations of a reference one, and so they belong to a unique deformation-type. The dimension of the fixed locus, $\mathsf{d} = \dim_{\mathbb{C}} \mathrm{Fix}(\mathcal{G})$, is given by eqn.(10.40).

The finite complex reflection groups (and their invariants u_i) have been fully classified by Shephard-Todd [130, 181, 232]. They are direct products of irreducible finite reflection groups. The list of irreducible finite complex reflection groups is[50] [130, 181, 232]:

1) The cyclic groups $\mathbb{Z}_m$ in degree 1;

1) the symmetric group S_{k+1} in the degree k representation (i.e. the Weyl group of A_k);

2) the groups $G(m, d, k)$ where $k > 1$ is the degree, $m > 1$ and $d \mid m$;

3) 34 sporadic groups denoted as $G_4, G_5, \cdots, G_{37}$ of degrees ≤ 8.

For our purposes, we need to classify the embeddings of the Sphephard-Todd (ST) groups of rank-k into the Siegel modular group $Sp(2k, \mathbb{Z})$ modulo conjugacy. We shall say that a ST group is *modular* iff it has at least one such embedding. A degree-k ST group $\mathcal{G}$ which preserves a lattice $L \subset \mathbb{C}^k$ is called *crystallographic* [101, 130]; clearly a modular ST group is in particular crystallographic.

Let $\mathbb{K}$ be the character field of an irreducible reflection group[51] and $\mathfrak{O}$ its ring of integers. One shows that $\mathcal{G} \subset GL(k, \mathfrak{O})$ [181], so $\mathcal{G}$ is crystallographic iff $\mathbb{K}$ is either $\mathbb{Q}$ or an *imaginary* quadratic field $\mathbb{Q}[\sqrt{-d}]$. The crystallographic Shephard-Todd groups are listed in table 3 together with their character field $\mathbb{K}$. The groups with $\mathbb{K} = \mathbb{Q}$ are just the irreducible real crystallographic groups, namely the Weyl groups of simple Lie algebras. Rank-k Weyl groups are obviously subgroups of $Sp(2k, \mathbb{Z})$. Let us consider the case $\mathbb{K} = \mathbb{Q}[\sqrt{-d}]$; we choose an embedding $\mathcal{G} \hookrightarrow GL(k, \mathfrak{O})$; since $\mathcal{G}$ is finite and absolutely irreducible, it preserves an Hermitian form with coefficients in $\mathfrak{O}$, of the form $H_{ij}\psi^i\bar{\psi}^j$ where[52] $\psi^i \equiv x^i + \zeta y^i \in \mathfrak{O}^k$, $(x^i, y^j) \in \mathbb{Z}^{2k}$, and $\bar{H}_{ij} = H_{ji}$. Hence it preserves the skew-symmetric form with rational coefficients

$$\frac{1}{\zeta - \bar{\zeta}} H_{ij}\, \psi^i \wedge \bar{\psi}^j. \tag{6.7}$$

[48] A degree-k *reflection group* $\mathcal{G}$ is a concrete group of $k \times k$ matrices generated by reflections, i.e. by matrices $g \in \mathcal{G}$ such that $\dim \mathrm{coker}(g - 1) = 1$. In particular, a reflection group comes with a defining representation V, whose dimension k is called the *degree* of the reflection group.

[49] More generally, subgroups of duality-frame groups $S(\Omega)_{\mathbb{Z}}$.

[50] In the classification one does not distinguish a group and its complex conjugate since the two are conjugate in $GL(V)$.

[51] Note that if $\mathbb{K}$ is the character field of an irreducible finite complex reflection group $\mathcal{G}$, then $\mathsf{Gal}(\mathbb{K}/\mathbb{Q})$ is Abelian. Moreover, if $\mathcal{G}$ is crystallographic, its class number is $h(\mathbb{K}) = 1$.

[52] $\zeta \in \mathfrak{O}$ is an integer of $\mathbb{Q}[\sqrt{-d}]$ such that $\{1, \zeta\}$ is a basis of the ring $\mathfrak{O}$ as a $\mathbb{Z}$-module. Clearly $\bar{\zeta} \neq \zeta$.

Clearing denominators, we get a non-degenerate integral skew-symmetric form Ω on $\mathbb{Z}^{2k}$ which is preserved by $\mathcal{G}$ whose entries have no non-trivial common factor. The given embedding $\mathcal{G} \hookrightarrow GL(k, \mathfrak{O})$ induces an embedding $\mathcal{G} \hookrightarrow Sp(2k, \mathbb{Z})$ iff $\det \Omega = 1$; in particular, we have the necessary condition

$$\det H \text{ is a } k\text{-th power in } \mathbb{Q}. \tag{6.8}$$

Example 5. In the **Introduction** we claimed that there is no rank-2 CSG with principal polarization and Coulomb dimensions $\{\Delta_1, \Delta_2\} = \{12, 8\}$. However, the crystallographic ST group G_8 has degrees[53] $\{12, 8\}$ so $\mathbb{C}^2/G_8$ is potentially a candidate counter-example to our claim. The only way out is to show that $G_8 \not\subset Sp(4, \mathbb{Z})$, so that this particular CSG is consistent only for suitable non-principal polarizations. Up to conjugacy, there is a unique embedding $G_8 \hookrightarrow GL(2, \mathbb{Z}[i])$ [114], generated by two reflections r_1, r_2, with invariant Hermitian form H:

$$r_1 = \begin{bmatrix} i & i \\ 0 & 1 \end{bmatrix}, \qquad r_2 = \begin{bmatrix} 1 & 0 \\ -1 & i \end{bmatrix}, \qquad H = \begin{bmatrix} 2 & 1-i \\ 1+i & 2 \end{bmatrix} \tag{6.9}$$

$\det H = 2$ is not a square in $\mathbb{Q}$, and hence the polarization is non-principal with *charge multipliers* $(e_1, e_2) = (1, 2)$ (cfr. eqn.(5.2)). Thus G_8 is no contradiction to our claim. We shall return to this in section 5.

For the groups $G(m, d, k)$ with $m = 3, 4, 6$ the usual monomial basis is both defined over $\mathfrak{O}$ and orthonormal, so $\det H = 1$ and they are all subgroups of $Sp(2k, \mathbb{Z})$. However, for special values of (m, d, k) we may have more than one inequivalent embeddings in $Sp(2k, \mathbb{Z})$:

Lemma 6.2.1 ([114]). *Consider $G(m, d, k)$ with $m = 3, 4, 6$ and $(m, d, k) \neq (m, m, 2)$; there is a single conjugacy class of embeddings $G(m, d, k) \hookrightarrow Sp(2k, \mathbb{Z})$ except for $m = 3, 4 \equiv p^s$ ($p = 3, 2$ is a prime), $1 < d \mid p^s$, and $p \mid k$ in which case we have $p + 1$ inequivalent embeddings.*

Remark 6.2.1. This statement is the complex analogue of the usual GSO projection of string theory [212]. Indeed, consider the *real* reflection group $G(2, 2, k) \equiv \text{Weyl}(D_k)$ and count the number of inequivalent embeddings[54] $G(2, 2, k) \hookrightarrow SO(k, \mathbb{Z}) \subset Sp(2k, \mathbb{Z})$, or equivalently the number of maximal local subalgebras of the $Spin(2k)_k$ chiral current algebra. We have just one, generated by the free fermion fields $\psi^a(z)$ unless $p \equiv 2 \mid k$ in which case we have $3 \equiv 2 + 1$ of them, the additional ones being the two GSO projections of opposite chirality. In the complex case, chirality is replaced by a $\mathbb{Z}_p$ symmetry. The proof is essentially the same as in the GSO case [114].

Example 6. A Lagrangian SCFT with gauge group $\prod_i G_i$ at extreme weak coupling has a CSG which asymptotically takes this constant period form with $\mathcal{G} = \prod_i \text{Weyl}(G_i)$ in the standard reflection representation.

Example 7. The model associated to the group $G(m, 1, k)$ ($m = 3, 4, 6$) has the simple physical interpretation of representing (the birational class of) the rank-k MN E_r SCFT for $r = 6, 7, 8$, respectively. Indeed, geometrically the quotient of $\mathbb{C}^k$ by $G(m, 1, k)$ is the same as taking the k-fold

[53] In fact $\{12, 8\}$ are even regular degrees in the sense of Springer theory.

[54] In addition there is an embedding $G(2, 2, k) \hookrightarrow Sp(2k, \mathbb{Z})$ which does not factor through $SO(k, \mathbb{Z})$.

$\mathbb{Q}$	ST names	$\mathbb{Z}_2$	$G(2,1,k)$	$G(2,2,k)$	$G(6,6,2)$	S_{k+1}	G_{28}	G_{35}	G_{36}	G_{37}	
	Coxeter	A_1	B_k	D_k	G_2	A_k	F_4	E_6	E_7	E_8	
$\mathbb{Q}(\zeta_3)$	ST names	$\mathbb{Z}_3$	$\mathbb{Z}_6$	$G(m,d,k)$	G_4	G_5	G_{25}	G_{26}	G_{32}	G_{33}	G_{34}
	notes			$m=3,6;\ d \mid m$ $(m,d,k) \neq (m,m,2)$							
$\mathbb{Q}(i)$	ST names	$\mathbb{Z}_4$	$G(4,d,k)$	G_8	G_{29}	G_{31}					
	notes		$d \mid 4$ $(4,d,k) \neq (4,4,2)$								
$\mathbb{Q}(\sqrt{-2})$	ST names	G_{12}									
$\mathbb{Q}(\sqrt{-7})$	ST names	G_{24}									

Table 3: The Shephard-Todd irriducible complex cristallographic groups; ζ_3 stands for a primitive third root of 1. The excluded cases are $G(3,3,2) \equiv \text{Weyl}(A_2)$, $G(6,6,2) \equiv \text{Weyl}(G_2)$, and $G(4,4,2)$, which is conjugate in $U(2)$ to $\text{Weyl}(B_2)$.

symmetric product of quotient of $\mathbb{C}$ by $\mathbb{Z}_m$. Correspondingly, the isotropy group of the diagonal period matrix, $\text{Fix}(e^{2\pi i/m}\,\mathbf{1}_k) \subset Sp(2k,\mathbb{Z})$, is $G(m,1,k)$ (see [111] for a discussion in the $k=3$ case). The dimensions are $\{\Delta, 2\Delta, 3\Delta, \cdots, k\Delta\}$ with $\Delta = m$.

Example 8. The CSG geometries associated to the groups $G(m,d,k)$ are just $\mathbb{Z}_d$ covers of the previous one. We have a new operator $\mathcal{O}$ of dimension km/d such that $\mathcal{O}^d$ is the MN operator of maximal dimension.

Example 7 may be generalized. One has a CSG M and takes the n-th symmetric power. This works well if M has dimension 1, but in general the resulting geometry may be more singular than permitted.

In general, when we have a CSG of the form $\mathbb{C}^k/\mathcal{G}$, where $\mathcal{G} = \prod_a \mathcal{G}_a$ with $\mathcal{G}_a$ irreducible reflection groups, and, in addition, $\mathscr{R} = \mathbb{C}[a_1, \cdots, a_k]^{\mathcal{G}}$, the Coulomb dimensions are equal to the degrees of $\mathcal{G}$. The period $\boldsymbol{\tau}$, being symmetric, transforms in the symmetric square of the defining representation V of the reflection group $\mathcal{G} \subset Sp(2k,\mathbb{Z})$ (see §.10.2 below). Hence the dimension of the space of allowed deformations of $\boldsymbol{\tau}$, that is, the dimension d of the conformal manifold of a constant-period CSG is given by the multiplicity of the trivial representation in $\odot^2 V$. By Schur-Frobenius,

$$\mathsf{d} = \frac{1}{2\,|\mathcal{G}|} \sum_{g \in \mathcal{G}} \Big(\chi_V(g)^2 + \chi_V(g^2) \Big) = \left[\begin{array}{l} \text{multiplicity of 2} \\ \text{as a weight of } \mathcal{G} \end{array} \right. = \left[\begin{array}{l} \#\ \text{irreducible factors } \mathcal{G}_a \\ \text{which are Weyl groups,} \end{array} \right. \tag{6.10}$$

since a reflection group has a degree 2 invariant iff it is defined over the reals (and hence, if crystallographic, should be a Weyl group). The physical interpretation of this result is that such a CSG represents a gauge theory with gauge group the product of all simple Lie groups whose Weyl groups are factors of $\mathcal{G}$ coupled to some other intrinsically strongly interacting SCFT associated to the complex factor groups $\mathcal{G}_a$ (as well as hypermultiplets in suitable representations of the gauge group). The d marginal deformations of the geometry are precisely the d Yang-Mills couplings which are associated to d chiral operators with $\Delta = 2$. Note that the Yang-Mills couplings may be taken as weak as we please.

Thus the Lagrangian models and higher MN SCFTs already account for all CSG with constant period map up (at most) to finitely many exceptional ones.

Part III

Fine classification of the SW geometry in rank $k = 1$

7 Special geometry and rational elliptic surfaces

7.1 Summary of the previous definitions

In the literature there are several "morally equivalent" definitions of "Special Geometry". In order not to confuse the reader, we recall here explicitly the definitions we are using in this thesis.

We start from the most basic, physically defined, object: the *chiral ring* $\mathscr{R}$, i.e. the ring of all (quantum) chiral operators in the given 4d $\mathcal{N} = 2$ theory. $\mathscr{R}$ is a commutative associative $\mathbb{C}$-algebra with unit and also a finitely-generated domain.

Remark 7.1.1. *A priori* we do not require $\mathscr{R}$ to be a free polynomial ring; this fact will be proven below (in the case of interest). Neither we assume $\mathscr{R}$ to be normal, i.e. in principle we allow for the "exotic" possibilities discussed in ref. [15], but rule them out (in rank 1) as a result of the analysis.

Definition 8. The *Coulomb branch* M of $\mathscr{R}$ is the complex-analytic variety M underlying the affine scheme $\mathsf{Spec}\,\mathscr{R}$. Its complex dimension is called the *rank* of $\mathscr{R}$. We write $\mathbb{C}(M)$ for the function field of M i.e. the field of fractions of the domain $\mathscr{R}$.

Remark 7.1.2. In rank-1, the normalization $\mathscr{R}^{\text{nor}}$ of the chiral ring $\mathscr{R}$ is a Dedekind domain, so morally "$\mathscr{R}^{\text{nor}}$ behaves like the ring of integers $\mathbb{Z}$". This is the underlying reason why classification in rank-1 is so simple.

Definition 9. Let $\mathscr{R}$ be a finitely-generated domain over $\mathbb{C}$ of dimension k. A *special geometry* (SG) over $\mathsf{Spec}\,\mathscr{R}$ is a quadruple $(\mathscr{R}, X, \Omega, \pi)$ where:

a) X is a complex space of dimension $2k$ and Ω a holomorphic symplectic form on X;

b) $\pi\colon X \to M$ is a holomorphic fibration, with base the Coulomb branch M of $\mathscr{R}$, such that the fibers $F_u \equiv \pi^{-1}(u)$ are Lagrangian, i.e. $\Omega|_{F_u} = 0$ for $u \in M$;

c) π has a (preferred) section $s_0\colon M \to X$. We write $S_0 := s_0(M)$ for its image;

d) the fiber F_η over the generic point η of M is (isomorphic to) a polarized Abelian variety. The restriction $S_0|_{F_\eta}$ is the zero in the corresponding group.

In other words, a special geometry is a (polarized) Abelian variety over the function field $\mathbb{C}(M)$ which, as a variety over $\mathbb{C}$, happens to be symplectic with Lagrangian fibers.

Remark 7.1.3. The Coulomb branch M is an open space, so the definition of special geometry should be supplemented by appropriate "boundary conditions" at infinity. Physically, the requirement is that the geometry should be asymptotic to the UV behavior of either a unitary SCFT or an asymptotically free QFT. In the context of rank-1 special geometries, this condition (dubbed *UV completeness*) will be made mathematically precise in §.7.3.

Definition 10. A *Seiberg-Witten (SW) differential* λ on a special geometry is a meromorphic one-form λ on X such that $d\lambda = \Omega$. We are only interested in special geometries admitting SW differentials. We shall say that a special geometry is *SW complete* iff it admits "enough" SW differentials, that is, all infinitesimal deformations of the symplectic structure Ω may be induced by infinitesimal deformations of λ and viceversa.

7.2 Rank-1 special geometries as rational elliptic surfaces

Kodaira-Néron models. Let $\mathscr{R}$ be a rank-1 chiral ring and $\eta \in \mathsf{Spec}\,\mathscr{R}$ the generic point of its Coulomb branch. The fiber over η, F_η, is open and dense in X, and may be identified with its "good" locus of smooth fibers. In rank-1, F_η is (in particular) an elliptic curve $E(\mathbb{C}(M))$ defined over the function field $\mathbb{C}(M)$ of transcendence degree 1. By a *model* of the elliptic curve $E(\mathbb{C}(M))$ we mean a morphism $\pi\colon \mathcal{E} \to C$ between an algebraic surface $\mathcal{E}$ and a curve C whose generic fiber is isomorphic to the elliptic curve $E(\mathbb{C}(M))$ (i.e. to F_η). All models are birationally equivalent, and contain the same amount of information. Most of the literature on Special Geometry uses the minimal Weierstrass model, $y^2 = x^3+ax+b$, $(a, b \in \mathbb{C}(M))$, which is easy to understand but has the drawback that it is not smooth (in general) as a complex surface. A better tool is the Kodaira-Néron model given by a (relatively minimal[55]) smooth compact surface $\mathcal{E}$ fibered over a smooth compact curve C such that $\mathbb{C}(M) \cong \mathbb{C}(C)$. The Kodaira-Néron model always exists for one-dimensional function fields [167, 168, 187, 221, 234], and is unique up to isomorphism. In particular, the smooth model exists for all rank-1 special geometries. By definition, the generic fiber of $\pi\colon \mathcal{E} \to C$ is a smooth elliptic curve, and $\mathcal{E}$ is a smooth, relatively minimal, (compact) elliptic surface having a section. The geometry of such surfaces is pretty well understood, see e.g. [33, 167, 168, 191, 221, 234]. Note that, having a section, the surface $\mathcal{E}$ cannot have multiple fibers.

We say that an elliptic surface is *trivial* iff $\mathcal{E} \cong E \times C$, that is, iff its fibers are all smooth elliptic curves. This trivial geometry corresponds to a free $\mathcal{N} = 2$ QFT. We shall esclude the trivial case from now on, that is, for the rest of the thesis we assume that at least one[56] fiber of $\mathcal{E}$ is singular. Special geometries with this property will be called *non-free*. In the non-free case [33, 168, 191, 221],

$$q(\mathcal{E}) \equiv h^{0,1}(\mathcal{E}) = 0. \tag{7.1}$$

The non-smooth fibers which may appear in $\mathcal{E}$ are the ones in the Kodaira list, see table 4.

Remark 7.2.1. (*Weierstrass vs. Kodaira-Néron*) The (minimal) Weierstrass model is obtained from the smooth Kodaira-Néron surface, $\mathcal{E}$, by blowing-down all components of the *reducible* fibers which do not cross the reference section S_0. If all exceptional fibers are irreducible (i.e. of Kodaira types I_1 and II) the two models coincide, and the flavor group is the "generic" E_8. Otherwise the blowing-down introduces singularities in the Weierstrass geometry. From the Weierstrass viewpoint, the information on the flavor group is contained in these singularities, which are most easily analyzed by blowing-up them. By construction, this means working with the Kodaira-Néron model.

The chiral ring $\mathscr{R}$ is free. Since $\mathbb{C}(C) \cong \mathbb{C}(M)$, we have

$$M = C \setminus \mathsf{supp}\, D_\infty \tag{7.2}$$

for some effective divisor D_∞. Then

$$X \cong \mathcal{E} \setminus \mathsf{supp}\, \pi^*(D_\infty). \tag{7.3}$$

[55] An elliptic surface is *relatively minimal* if its fibers do not contain exceptional -1 rational curves.

[56] If $\mathcal{E}$ has singular fibers, it has at least 2 of them. If it has precisely 2 singular fibers, its functional invariant is constant, and the special geometry describes an interacting SCFT with no mass deformation.

In order to be a special geometry, X must be symplectic (the fibers of π are then automatically Lagrangian). From eqn.(10.40) we have

$$\Omega \in \Gamma(\mathcal{E}, K_{\mathcal{E}}(\pi^* D_\infty)), \tag{7.4}$$

so that[57] $K_{\mathcal{E}}(\pi^* D_\infty) \sim \mathcal{O}_{\mathcal{E}}$, or

$$K_{\mathcal{E}} = -\pi^*[D_\infty]. \tag{7.5}$$

We recall Kodaira's formula for the canonical divisor $K_{\mathcal{E}}$ of an elliptic surface with no multiple fibers (see e.g. §.V.12 of [33])

$$K_{\mathcal{E}} = \pi^* \mathcal{L}, \quad \text{where } \mathcal{L} \text{ is a line bundle on } C \text{ of degree } p_g(\mathcal{E}) + 2g(C) - 1, \tag{7.6}$$

where $p_g(\mathcal{E}) \equiv h^{2,0}(\mathcal{E}) \geq 0$ (resp. $g(C) \equiv h^{1,0}(C) \geq 0$) is the geometric genus of $\mathcal{E}$ (resp. C). Comparing eqns.(7.5)(7.6), yields $\mathcal{L} \sim -D_\infty$; since D_∞ is effective, $\deg \mathcal{L} < 0$, which is consistent with eqn.(7.6) only if

$$p_g(\mathcal{E}) = g(C) = 0 \quad \Longrightarrow \quad C \cong \mathbb{P}^1 \text{ and } \deg D_\infty = 1, \tag{7.7}$$

so that D_∞ consists of a single point on $\mathbb{P}^1$ which we denote as ∞. The Coulomb branch is

$$M = \mathbb{P}^1 \setminus \infty = \mathbb{C}, \tag{7.8}$$

and its ring of regular functions is $\mathscr{R} \cong \mathbb{C}[u]$.

The functional invariant $\mathscr{J}$. The elliptic fibration $\pi\colon \mathcal{E} \to \mathbb{P}^1$ yields a rational function (called the *functional invariant* of $\mathcal{E}$ [167, 168])

$$\mathscr{J} : \mathbb{P}^1 \to \mathbb{P}^1, \qquad u \mapsto J(\tau_u), \tag{7.9}$$

where (for $u \in \mathbb{P}^1 \equiv \pi(\mathcal{E})$) $\tau_u \in \mathfrak{H}$ is the modulus of the elliptic curve $\pi^{-1}(u)$ and $J(z) \equiv j(z)/1728$, $j(z)$ being the usual modular invariant [230]. The function $\mathscr{J}$ determines $\mathcal{E}$ up to quadratic transformations [191]. A quadratic transformation consists in flipping the type of an *even* number of fibers according to the rule

$$I_b \leftrightarrow I_b^*, \quad II \leftrightarrow IV^*, \quad III \leftrightarrow III^*, \quad IV \leftrightarrow II^*. \tag{7.10}$$

Scale-invariant vs. mass-deformed special geometries. As we shall see momentarily, the special geometries associated to scale-invariant $\mathcal{N} = 2$ SCFT are precisely the ones described by a *constant* function $\mathscr{J}$. Mass-deformed geometries instead have functional invariants of positive degree, $\deg \mathscr{J} > 0$. Our approach applies uniformly to both situations.

The surface $\mathcal{E}$ is rational. The divisor $-K_{\mathcal{E}}$ is effective, so all plurigenera vanish (i.e. $\mathcal{E}$ has Kodaira dimension $\kappa(\mathcal{E}) = -\infty$). Since $q(\mathcal{E}) = 0$, $\mathcal{E}$ is rational by the Castelnuovo criterion [33].

[57] Here $\sim$ denotes linear equivalence.

The other numerical invariants of $\mathcal{E}$ are [33, 191]:

$$\text{topological Euler number } e(\mathcal{E}) = 12, \quad b_1(\mathcal{E}) = 0, \quad b_2(\mathcal{E}) = h^{1,1}(\mathcal{E}) = \varrho(\mathcal{E}) = 10. \tag{7.11}$$

The Néron-Severi group $\mathsf{NS}(\mathcal{E}) \cong \mathsf{Pic}(\mathcal{E})/\mathsf{Pic}(\mathcal{E})^0$ is then a unimodular (odd) lattice of signature $(1, 9)$. In facts $\mathcal{E}$, being a relatively minimal rational elliptic surface with section, is just $\mathbb{P}^2$ blown-up at 9 points (see **Theorem 5.6.1** of [85] or §. VIII.1 of [191]). Note that the Kodaira-Néron surfaces of all rank-1 special geometries have the same topological type allowing for a uniform discussion of them. This does not hold in the Weierstrass approach, since the blowing down kills cohomology classes in a model dependent fashion.

The class F of any fiber is $-K_{\mathcal{E}}$. By the moving lemma $F^2 = 0$, so $K_{\mathcal{E}}^2 = 0$. Let S_0 be the zero section. One has $K_{\mathcal{E}} \cdot S_0 = -F \cdot S_0 = -1$. Then, by adjunction,

$$-2 = 2\, g(\mathbb{P}^1) - 2 = S_0^2 + K_{\mathcal{S}} \cdot S_0 \quad \Longrightarrow \quad S_0^2 = -1, \tag{7.12}$$

so that the zero section S_0 is an exceptional (-1)-line. Contracting it we get a weak degree 1 del Pezzo surface, see §. 8.8.3 of [100].

The Euler number of $\mathcal{E}$ is the sum of the Euler numbers of its singular fibers. So

$$12 = e(\mathcal{E}) = \sum_{u \in U} e(F_u), \tag{7.13}$$

where $U \subset \mathbb{P}^1$ is the finite set of points with a non-smooth fiber. The Euler numbers $e(F)$ for the various types of singular fibers are listed in table 4. Note that for all additive* fibers $e(F^*) \geq 6$, so eqn.(7.13) implies that we can have at most one additive* fiber with the single exception of $\{I_0^*, I_0^*\}$ which is (the Kodaira-Néron model of) the special geometry of $\mathcal{N} = 4$ SYM with gauge group $SU(2)$. Since a quadratic transformation preserves the parity of the number of $*$, the function $\mathscr{J}$ specifies completely $\mathcal{E}$ if there are no additive* fibers, while if there is one such fiber we are free to flip the type of the additive* fiber and of precisely one other fiber (possibly regular) by the rule 7.10. This process is called *transfer of* $*$ [192].

$\mathcal{E}$ and the symplectic structure of (X, Ω). We write $F_\infty = \pi^{-1}(\infty)$ for the fiber at infinity. Then

$$X = \mathcal{E} \setminus F_\infty. \tag{7.14}$$

From eqn.(7.4) we see that the pair $(\mathcal{E}, F_\infty)$ uniquely fixes the symplectic structure Ω up to overall normalization. Physically, the overall constant may be seen as a choice of mass unit.

Moduli of rational elliptic surfaces with given singular fibers. The rational elliptic surfaces with a given set of singular fiber types, $\{F_u\}_{u \in U}$, are in one-to-one correspondence with the rational functions $\mathscr{J}$ consistent with the given fiber types $\{F_u\}_{u \in U}$ modulo the action of $\mathrm{Aut}(\mathbb{P}^1) \equiv PSL(2, \mathbb{C})$. We adopt the convention that the number of fibers of a given Kodaira type is denoted by the corresponding lower-case roman numeral, so (say) iii stands for the number of fibers of type III while iv^* for the number of fibers of type IV^*. We also write s, a°, and a^* for, respectively, the total number of semi-stable, additive$^\circ$, and additive* singular fibers (cfr. table 4).

category	type	$e(F)$	$m(F)$	$m(F)^{(1)}$	$o(F)$	$u(F)$	$d(F)$	semi-simple	$R(F)$
stable (regular)	I_0	0	1	1	0	0	0	✓	
semi-stable	$I_{b\geq 1}$	b	b	b	-	-	b	no	A_{b-1}
additive°	II	2	1	1	1	0	0	✓	
	III	3	2	2	0	1	0	✓	A_1
	IV	4	3	3	2	0	0	✓	A_2
additive*	I_0^*	6	5	4	0	0	0	✓	D_4
	$I_{b\geq 1}^*$	$6+b$	$5+b$	4	-	-	b	no	D_{4+b}
	II^*	10	9	1	2	0	0	✓	E_8
	III^*	9	8	2	0	1	0	✓	E_7
	IV^*	8	7	3	1	0	0	✓	E_6

Table 4: Kodaira fibers and their numerical invariants. I_0 is the *regular* (generic) fiber, all other types are *singular*. Additive fibers are also called *unstable*. Additive fibers come in two categories: *un-starred* and *starred* ones. A fiber is simply-connected iff it is additive; then $e(F) = m(F)+1$. A fiber type is *reducible* if it has more than one component, i.e. $m(F) > 1$. A fiber F is *semi-simple* iff the local monodromy at F is semi-simple. The last column yields the intersection matrix of the non-identity component of the reducible fibers. $m(F)^{(1)}$ is the number of simple components in the divisor F_u equal to the order of the center of the simply-connected Lie group in the last column.

As already anticipated, we distinguish two kinds of geometries:

scale-invariant: $\mathscr{J}$ is constant, that is, the coupling τ_u does not depend on the point u in the Coulomb branch M, and the special geometry is scale-invariant. The fiber configurations of the elliptic surfaces with $\mathscr{J}$ constant which satisfy the physical requirement of UV completeness (see §.7.3) are listed in table 5. Each of the first three elliptic surfaces describe *two* distinct $\mathcal{N} = 2$ SCFT, having Coulomb branch dimension Δ and $\Delta/(\Delta - 1)$, depending on which of the two singular fibers is placed at ∞: see §.7.3, in particular eqn.(7.33);

mass-deformed: $\mathscr{J}$ has positive degree $d > 0$ and satisfies the following properties [168,191] (cfr. table 4):

- $\mathscr{J}$ has a pole of order b at fibers of types I_b, I_b^*;
- the order of zero $\nu_0(F)$ of $\mathscr{J}$ at a fiber of type F is $\nu_0(F) = o(F) \bmod 3$;
- the order of zero $\nu_1(F)$ of $\mathscr{J} - 1$ at a fiber of type F is $\nu_1(F) = u(F) \bmod 2$.

Since the degree of $\mathscr{J}$ is $d = \sum_u d(F_u)$, $\mathscr{J}$ is constant iff all fibers are semi-simple.

From table 4 we see that

$$e(F) = d(F) + 2o(F) + 3u(F) + \begin{cases} 0 & F \in \{\text{semi-stable}\} \cup \{\text{additive}^\circ\} \\ 6 & F \in \{\text{additive}^*\}. \end{cases} \tag{7.15}$$

Suppose $d > 0$, and let a_i be the positions of the poles, b_j the positions of 0's and c_k the positions

$\{II^*, II\}$	$\{III^*, III\}$	$\{IV^*, IV\}$	$\{I_0^*, I_0^*\}$

Table 5: List of singular fiber configurations for $\mathscr{J}$ *constant* containing at most one additive$^\circ$ fiber [192]. They describe SCFTs with masses and relevant perturbations switched off.

of 1's of $\mathscr{J}$. We have

$$\mathscr{J}(z) = A\frac{\prod_{j=1}^{n_0}(z-b_j)^{\nu_0(j)}}{\prod_{i=1}^{p}(z-a_i)^{b(i)}} = 1 + B\frac{\prod_{k=1}^{n_1}(z-c_k)^{\nu_1(k)}}{\prod_{i=1}^{p}(z-a_i)^{b(i)}}, \tag{7.16}$$

where n_0 (resp. n_1) is the (maximal) number of *distinct* 0's (resp. 1's)

$$n_0 = ii + iv + ii^* + iv^* + \frac{d - \sum_i o(F_i)}{3}, \qquad n_0 = iii + iii^* + \frac{d - \sum_i u(F_i)}{2}, \tag{7.17}$$

and p is the number of non-semi-simple fibers, $p \equiv s + \sum_{b\geq 1} i_b^*$. $PSL(2,\mathbb{C})$ allows to fix three points; the number of effective parameters is then $n_0 + n_1 + p - 1$, while the equality of the two expressions in (7.16) yields $d+1$ relations. Thus the space of rational functions has dimension $\mu \equiv n_0 + n_1 + p - d - 2$, or

$$\mu + i_0^* = s + a^\circ + 2a^* - \frac{1}{6}\Big[d + 2\sum_i o(F_i) + 3\sum_i u(F_i) + 6a^*\Big] - 2 \tag{7.18}$$

The number of fibers of a given type is restricted by eqn.(7.13). Using (7.15)

$$12 = d + \sum_i (2o(F_i) + 3u(F_i)) + 6a^*, \tag{7.19}$$

so that

$$\mu + i_0^* = s + a^\circ + 2a^* - 4. \tag{7.20}$$

Hurwitz formula applied to the covering $\mathscr{J} : \mathbb{P}^1 \to \mathbb{P}^1$ implies [191]

$$\mu + i_0^* \geq 0. \tag{7.21}$$

A fiber configuration $\{F_u\}_{u\in U}$ which violates the bound (7.21) cannot be realized geometrically. The bound is saturated if and only if: *i)* $\mathscr{J}$ is a Belyi function[58] [174,247] and *ii)* the order of the zeros of $\mathscr{J}$ (resp. of $\mathscr{J} - 1$) is ≤ 3 (resp. ≤ 2) [191].

The number of parameters from which a $d > 0$ special geometry (X, Ω) depends is

$$\boldsymbol{n} \equiv \mu + i_0^* + 1 \equiv s + a^\circ + 2a^* - 3, \tag{7.22}$$

where the term i_0^* arises from the choice of the locations where we insert the I_0^* fibers (by quadratic transformation of some regular fiber I_0) and the $+1$ is the overall scale of Ω.

[58] Recall that a function $C \to \mathbb{P}^1$ (C a compact Riemann surface) is a (normalized) *Belyi function* iff it ramifies only over the three points $\{0, 1, \infty\}$.

F	$I_{b<2}$	$I_{b\geq 2}$	I_b^* b even	I_b^* b odd	II, II^*	III, III^*	IV, IV^*
$Z(F)$	$\{0\}$	$\mathbb{Z}/b\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	$\{0\}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$

Table 6: The Abelian group $Z(F)$ of a Kodaira fiber of type F.

ADE **and all that.** The exceptional fibers F_u are in general reducible with $m(F_u)$ irreducible components $F_{u,\alpha}$, see table 4. The divisor of $\pi^{-1}(u)$ has the form

$$\big(\pi^{-1}(u)\big) = \sum_{\alpha=0}^{m(F_u)-1} n_\alpha\, F_{u,\alpha}, \tag{7.23}$$

where the n_α are positive integers. A component $F_{u,\alpha}$ is said to be *simple* iff $n_\alpha = 1$. The numbers of simple components for each fiber type, $m(F)^{(1)}$, are listed in table 4. By the moving lemma we have

$$F \sim \sum_{\alpha=0}^{m(F_u)-1} n_\alpha\, F_{u,\alpha} \quad \text{and} \quad F_{u,\alpha}\cdot F = 0 \;\; \text{for all } u,\alpha. \tag{7.24}$$

Let S_0 be the zero section. Since $F_u\cdot S_0 = 1$ for all u, the section S_0 intersects a single component of the fiber F_u which must be simple. This component is said to be the *identity component,* and will be denoted as $F_{u,0}$. Forgetting the identity component $F_{u,0}$, we remain with the set $F_{u,\alpha}$, $\alpha = 1,\cdots,m(F_u)-1$ of irreducible divisors whose intersection matrix

$$F_{u,\alpha}\cdot F_{u,\beta} = -C(F_u)_{\alpha\beta} \qquad \alpha,\beta = 1,\cdots,m(F_u)-1 \tag{7.25}$$

is minus the Cartan matrix $C(F_u)$ of the ADE root system $R(F_u)$. The root systems $R(F)$ for the various fiber types are listed in the last column of table 4. One has

$$\operatorname{rank} R(F) = e(F) - \begin{cases} 1 & F\in \text{semi-stable}\\ 2 & F\in \text{additive}^\circ \cup \text{additive}^*. \end{cases} \tag{7.26}$$

To each $R(F)$ we associate a finite Abelian group

$$Z(F) = \Gamma_{\substack{\text{weigth}\\ \text{lattice}}} \Big/ \Gamma_{\substack{\text{root}\\ \text{lattice}}} \tag{7.27}$$

isomorphic to the center of the simply-connected Lie group associated to $R(F)$. From the table we see that $|Z(F)| = m(F)^{(1)}$, and indeed, $Z(F)$ acts freely and transitively on the simple components of a reducible fiber. See table 6.

Allowed fiber configurations and Dynkin theorem. A fundamental problem is to list the configurations of singular fibers, $\{F_u\}_{u\in U}$ which are realized by some rational elliptic surface. There are 379 fiber configurations which satisfy eqn.(7.13). Of these 100 cannot be geometrically realized, most of them because they violate the Hurwitz bound (7.21). For the list of those which *can* be realized see refs. [192, 210].

The realizable fiber configurations may be understood in Lie-theoretic terms. From its numerical invariants, eqns.(7.7)(7.11), we infer that $\mathcal{E}$, seen as a compact topological 4-fold, has intersection form $H_2(\mathcal{E},\mathbb{Z})\times H_2(\mathcal{E},\mathbb{Z})\to\mathbb{Z}$ isomorphic to $\boldsymbol{U}\oplus E_8^-$, where E_8^- stands for the E_8 root lattice with the opposite quadratic form (see §.8.2 for details). The classes of the non-identity components of the reducible fibers belong to the E_8^- part, so that homology yields an embedding of roots lattices [192, 205, 210, 221]

$$\bigoplus_{\substack{\text{reducible}\\ \text{fibers } F_u}} R(F_u) \rightarrowtail E_8. \tag{7.28}$$

Two such embeddings are equivalent if they are conjugate by the Weyl group Weyl(E_8). The classification of all inequivalent embeddings was given by Dynkin [109]. There are 70 root systems which may be embedded in E_8, all but 5 of them in an unique way. The special 5 have two inequivalent embeddings each. They are

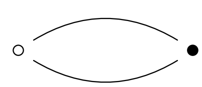
$$A_7, \quad A_3\oplus A_3, \quad A_5\oplus A_1, \quad A_3\oplus A_1\oplus A_1, \quad A_1\oplus A_1\oplus A_1\oplus A_1. \tag{7.29}$$

Three out of the 70 sub-root systems cannot be realized geometrically because they violate Euler's bound (7.13). The full list of allowed singular fiber configurations, $\{F_u\}_{u\in U}$, is then obtained by consider the various ways of producing a given allowed embedding of a root system in E_8.

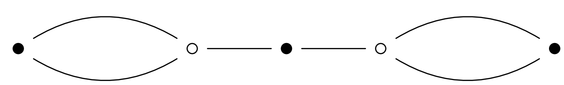
Aside: *Dessin d'enfants.* When the bound (7.21) is saturated, the functional invariant $\mathscr{J}$ is (in particular) a Belyi function. Belyi functions are encoded in their Grothendieck *dessin d'enfants* [174, 247]. Since it is often easier to work with *dessins* than with functions, we recall that story even if we don't need it.[59] A Belyi function f is a holomorphic map from some Riemann surface Σ to $\mathbb{P}^1$ which is branched only over the three points 0, 1 and ∞. If a Belyi functions exists, Σ and f are defined over the a number field. The *dessin* of f is a graph $G\subset\Sigma$ which is the inverse image of the segment $[0,1]\subset\mathbb{P}^1$. The inverse images of 0 (resp. 1) are represented by white[60] nodes $\circ$ (black nodes $\bullet$). The coloring makes G into a bi-partite graph. G is a connected graph whose complement, $\Sigma\setminus G$ is a disjoint union of disks in one-to-one correspondence with the inverse images of ∞.

If the bound (7.21) is saturated, all white (black) nodes have valency at most 3 (2).

Example 9. The *dessins* of Argyres-Douglas of type A_2 and of pure $SU(2)$ SYM are (the first one is drawn in a chart of $\mathbb{P}^1$ around ∞)

Argyres-Douglas A_2: 

(7.30)

pure SYM: 

These are special instances of *double flower dessins* [174] so that the special geometry for these QFTs is *rational* (i.e. defined over $\mathbb{Q}$).

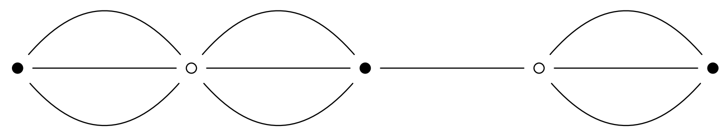
If the bound (7.21) is not saturated, so that the space $\mathcal{S}(\{F_u\})$ of rational functions $\mathscr{J}$ has

[59] For a survey see [247].

[60] We use the coloring convention of [247]. Ref. [174] uses the opposite convention.

positive dimension, and $\deg \mathscr{J} \geq 2$, we may still found some *exceptional* points $\mathcal{P}_\sigma \subset \mathcal{S}(\{F_u\})$ where $\mathscr{J}$ becomes a Belyi function (however the nodes will have larger valency).

Example 10. Consider the fiber configuration $\{II; I_4, I_1^6\}$ which corresponds to the Argyres-Wittig SCFT [25] with $\Delta = 6$ and flavor symmetry $Sp(10)$. It has $\mu = 4$, that is, $\boldsymbol{n} = 5 \equiv \operatorname{rank} \mathfrak{sp}(10)$. The bound (7.21) is far from being saturated, but nevertheless there is a dimension 1 locus in the space of mass parameters where the model is described by the *dessin*



(7.31)

7.3 UV completeness and the fiber F_∞ at infinity

As already mentioned, the possible fibers F_∞ at ∞ are rescricted by the condition of "UV completeness". Heuristically this means that we can make sense out of the QFT without introducing extra degrees of freedom at infinite energy (they would play the role of Pauli-Vilards regulators that we cannot get rid off). This translates in the condition that F_∞ is simply-connected, hence *additive* ($\equiv$ unstable). There are only 11 additive fibers which can appear in a rational elliptic surface

$$\overbrace{II,\ III,\ IV,\ II^*,\ III^*,\ IV^*,\ I_0^*}^{\text{semisimple}},\ \overbrace{I_1^*, I_2^*, I_3^*, I_4^*}^{\text{non-semisimple}}. \tag{7.32}$$

$I_{b\geq 7}^*$ are ruled out because their Euler number > 12 and the fiber configurations $\{I_6^*\}$, $\{I_5^*, I_1\}$ because they have $\mu = -2$ and -1 respectively. The seven semi-simple fibers in (7.32) correspond to the seven UV asymptotic special geometries[61] for a non-free SCFT, which are labelled by the dimension Δ of the chiral operator parametrizing the Coulomb branch:

additive semi-simple F_∞	II	III	IV	II^*	III^*	IV^*	I_0^*
Δ	6	4	3	6/5	4/3	3/2	2

(7.33)

while the 4 non semi-simple ones describe the possible UV behavior of asymptotically-free theories. Note that the correspondence between fiber type at infinity, F_∞, and the Coulomb branch dimension, Δ, is the *opposite* of the usual one since the monodromy at infinity M_∞ in the Coulomb branch is related to the local monodromy around the fiber at infinity, $M(F_\infty)$, by an inversion of orientation

$$M_\infty = M(F_\infty)^{-1}. \tag{7.34}$$

This is consistent with the usual statements in the SCFT context, since in the zero-mass limit $\mathcal{E}$ becomes a constant geometry with fiber configuration $\{F_\infty; F_0\}$ with

$$M_\infty = M(F_0) \equiv M(F_\infty)^{-1}, \tag{7.35}$$

and in the literature it is usually stated the zero-mass limiting correspondence $F_0 \leftrightarrow \Delta$.

[61] By "UV asymptotic special geometry" we mean the behavior of the geometry for large $u \in \mathbb{C}$.

Asymptotically free QFTs. $F_\infty = I_b^*$ yields the UV asymptotic special geometry of $SU(2)$ SYM coupled to $N_f = 4 - b$ fundamentals. This relation implies both the UV geometrical bound $b \leq 4$ and the physical UV bound $N_f \leq 4$, and illustrates as the additive reduction of the fiber at infinity captures the physical idea of UV completeness (i.e. $\beta \leq 0$).

$SU(2)$ with N_f fundamentals and generic masses corresponds to the fiber configuration $\{I^*_{4-N_f}; I_1^{N_f+2}\}$. Using eqn.(7.22) we see that the number of parameters on which this geometry depends is

$$\boldsymbol{n}(N_f) = N_f + 1 \tag{7.36}$$

which is the physically correct number: the masses and the Yang-Mills scale Λ for $N_f \leq 3$, the masses and the coupling constant g_{YM} for $N_f = 4$ (which correspond to $+i_0^*$ in eqn.(7.22)).

$\{I_4^*; I_1^2\}$ is the only fiber configuration with $F_\infty = I_4^*$ [210]; it corresponds to an *extremal* rational elliptic surface [193] (defined over $\mathbb{Q}$). Thus pure $SU(2)$ SYM is unique in its UV class. There are two configurations with $F_\infty = I_3^*$, $\{I_3^*; I_1^3\}$ and $\{I_3^*, II, I_1\}$; the second one will be ruled out in §.8.3.1 on the base that is has no "enough" SW differentials. Hence $SU(2)$ SQCD with $N_f = 1$ is also unique in its UV class. There are six configurations with $F_\infty = I_2^*$, three of which are ruled out by the same argument. The remaining 3 are either the standard SQCD or special cases of it. Finally, there are 13 configurations with $F_\infty = I_1^*$; 8 of them are ruled out as before, while 5 look like special instances of SQCD with $N_f = 3$.

The UV asymptotics of the special geometry. The behavior of the periods $(b(u), a(u))$ as we approach $u = \infty$ for each of the 11 allowed fibers at infinity, eqn.(7.32), may be read (including the sub-leading corrections!) in table (VI.4.2) of [191]. If u is a standard coordinate on the Coulomb branch, as $u \to \infty$ the special geometry periods behave as

$$\big(b(u), a(u)\big) = \big(u\, r_2(1/u), u\, r_1(1/u)\big) \qquad u \text{ large}, \tag{7.37}$$

where the functions $r_1(t)$, $r_2(t)$ are listed in the table of ref. [191]. In the particular case of a geometry which is UV asymptotic to a SCFT, F_∞ is semi-simple, and $a(u) \simeq u^{1/\Delta}$ with Δ as in eqn.(7.33), confirming the correspondence $F_\infty \leftrightarrow \Delta$.

The "generic" massive deformation. As an example, let us consider the generic configuration with a marked fiber F_∞ of one type in eqn.(7.32), i.e. $\{F_\infty; I_1^{12-e(F_\infty)}\}$, which is always geometrically realized. The number of parameters $\boldsymbol{n}(F_\infty)$ in the geometry is

$$\boldsymbol{n}(F_\infty) = 12 - e(F_\infty) - \begin{cases} 2 & F_\infty \in \{\text{additive}^\circ\} \\ 1 & F_\infty \in \{\text{additive}^*\}, \end{cases} \tag{7.38}$$

which precisely matches the number of physical relevant+marginal deformations for the theory with Coulomb dimension Δ having the largest possible flavor symmetry of rank

$$\text{rank}\, \mathfrak{f} = 8 - \text{rank}\, R(F_\infty) \equiv 10 - e(F_\infty). \tag{7.39}$$

8 SW differentials vs. Mordell-Weil lattices

We have not yet enforced one crucial property of the special geometries relevant for $\mathcal{N}=2$ QFT, namely the existence of Seiberg-Witten (SW) differentials with the appropriate properties. In this section we consider the restrictions on the pair $(\mathcal{E}, F_\infty)$ coming from this requirement.

8.1 SW differentials and horizontal divisors

A SW differential λ is a meromorphic one-form on the total space $X = \mathcal{E} \setminus F_\infty$ or, with non-trivial residue along a simple normal-crossing effective divisor D_{SW}, such that $d\lambda = \Omega$ in X. Let $D_{SW} = \sum_i D_i$, be the decomposition of D_{SW} into prime divisors. Standard residue formulae [88, 139, 242] yield the following equality in cohomology [224] (see [104] for a nice discussion in the present context)

$$[\Omega] = \sum_i \mu_i [D_i], \tag{8.1}$$

where the complex coefficients μ_i are linearly related to the masses m_a living in the Cartan subalgebra $\mathfrak{h}$ of the flavor Lie algebra $\mathfrak{f} = \mathfrak{Lie}(\mathsf{F})$ [223, 224]. For the relation of this statement to the Duistermaat-Heckman theorem in symplectic geometry, see [104]. We may rewrite (11.2.1) in terms of the independent mass parameters m_a as

$$[\Omega] = \sum_{a=1}^{\mathrm{rank}(\mathfrak{f})} m_a [L_a], \tag{8.2}$$

for certain *non* effective divisors L_a on X. The surface $\mathcal{E}$ (with a choice of zero section S_0) has an involution corresponding to taking the negative in the associated Abelian group. Since λ is odd under this involution, the divisors L_a belong to the odd cohomology [224].

The closure in the smooth elliptic surface $\mathcal{E}$ of the divisors D_i, L_a (originally defined in the open quasi-projective variety $X \subset \mathcal{E}$) yields divisors on $\mathcal{E}$ which we denote by the same symbols.

A divisor on an elliptic surface $\pi\colon \mathcal{E} \to M$ contained (resp. not contained) in a fiber is called *vertical* (resp. *horizontal*) [191, 221]. The divisors D_i, L_a cannot be contained in a fiber F of $\mathcal{E}$, since the masses are well-defined at all generic points $u \in M$ and u independent.[62] We conclude that the divisors D_i, L_a are horizontal. Since the fibers are Lagrangian and the m_a independent, eqn.(8.2) implies[63]

$$\Omega\big|_{F_{u,\alpha}} = 0 \quad \Longrightarrow \quad F_{u,\alpha} \cdot L_a = 0 \ \text{ for all } a,\ u,\ \alpha. \tag{8.3}$$

Thus, to determine the flavor symmetry F associated to a given special geometry $(\mathcal{E}, F_\infty)$, preliminarly we have to understand the geometry of its horizontal divisors. In the next subsection we review this elegant topic. We shall resume the discussion of Special Geometry in §. 8.3.

8.2 Review: Néron-Severi and Mordell-Weil groups

The Néron-Severi group. We see the divisors D_i, L_a on $\mathcal{E}$ as elements of the Nerón-Severi group $\mathsf{NS}(\mathcal{E})$, the group of divisors on $\mathcal{E}$ modulo algebraic equivalence. For all projective

[62] A more formal argument is as follows. The primitive divisors contained in the fibers are compact analytic submanifolds of X, hence as cohomology classes have type $(1,1)$ while Ω has type $(2,0)$.

[63] Again, this also follows from type considerations.

variety Y, the Néron-Severi group $\mathsf{NS}(Y)$ is a finitely-generated Abelian group [139, 221]. Its rank, $\varrho(Y) := \operatorname{rank} \mathsf{NS}(Y)$, is called the *Picard number* of Y.

In the case of a projective surface S, the intersection pairing $\langle -, - \rangle$ endows[64]

$$\mathsf{Num}(S) := \mathsf{NS}(S)/\mathsf{NS}(S)_{\text{tors}} \tag{8.4}$$

with a non-degenerate, symmetric, integral, bilinear pairing of signature $(1, \varrho(S) - 1)$ having the same parity as the first Chern class. In other words, $(\mathsf{Num}(S), \langle -, - \rangle)$ is a non-degenerate *lattice*.

For an elliptic surface $\mathcal{E}$, the Néron-Severi group is torsion-free, so $\mathsf{Num}(\mathcal{E}) = \mathsf{NS}(\mathcal{E})$, and the Néron-Severi group is itself a lattice.

If, in addition, the elliptic surface $\mathcal{E}$ is *rational*, we have the further identification with the Picard group: $\mathsf{Num}(\mathcal{E}) = \mathsf{NS}(\mathcal{E}) = \mathsf{Pic}(\mathcal{E})$, that is, linear and numerical equivalence coincide. In this case $p_g(\mathcal{E}) = 0$, $\varrho(\mathcal{E}) = 10$, and $\mathsf{NS}(\mathcal{E})$ is an (odd) unimodular lattice of signature $(1, 9)$; by general theory it is isomorphic to

$$\boldsymbol{U} \oplus E_8^-, \tag{8.5}$$

where $\boldsymbol{U}$ is the rank 2 lattice with Gram matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \tag{8.6}$$

and E_8^- is the opposite[65] of the E_8 root lattice (its Gram matrix is *minus* the Cartan matrix of E_8). E_8^- is the *unique* negative-definite, even, self-dual lattice of rank 8 [230]. The sublattice $\boldsymbol{U}$ in (8.5) is spanned by the zero section S_0 and the fiber F.

The Néron-Severi group $\mathsf{NS}(\mathcal{E})$ of a rational elliptic surface contains an obvious subgroup, called the *trivial group,* $\mathsf{Triv}(\mathcal{E})$, generated by the zero section S_0 and all the vertical divisors, that is, the irreducible divisors $F_{u,\alpha}$ contained in some fiber F_u. The rank of the trivial group is

$$\operatorname{rank} \mathsf{Triv}(\mathcal{E}) = 2 + \sum_{u \in U} \big(m(F_u) - 1\big) \geq 2 \tag{8.7}$$

where $U \subset \mathbb{P}^1$ is the finite set of points at which the fiber is not smooth and $m(F_u)$ is the number of irreducible components $F_{u,\alpha}$ of the fiber at u (see table 4). The only relations between the vertical divisors $F_{u,\alpha}$ are $\sum_\alpha n_\alpha \, F_{u,\alpha} = F$, from which we easily get eqn.(8.7). In facts, $\mathsf{Triv}(\mathcal{E})$ is the lattice

$$\boldsymbol{U} \oplus R^-, \tag{8.8}$$

where R^- is the lattice generated by all irreducible components of the fibers which do not meet the zero section S_0. As reviewed in the previous section, the opposite lattice R of R^- is the direct (i.e. orthogonal) sum of the roots lattices of ADE type associated to each reducible fiber (see last column of table 4)

$$R = \bigoplus_{u \colon m(F_u) > 1} R(F_u). \tag{8.9}$$

[64] The free Abelian group $\mathsf{Num}(S)$ is the group of divisors modulo *numerical* equivalence.
[65] Given a lattice L, by its *opposite* lattice L^- we mean the same Abelian group endowed with a bilinear pairing which is *minus* the original one.

Definition 11. The orthogonal complement $R^\perp$ of R in E_8 is called the *essential* lattice Λ.

The group $Z(\mathcal{E})$. The intersection form $\langle -,-\rangle$ yields a map

$$\mathsf{NS}(\mathcal{E}) \to R^\vee := \mathrm{Hom}(R,\mathbb{Z}), \tag{8.10}$$

and, passing to the quotient, (cfr. eqn.(7.27))

$$\gamma\colon \mathsf{NS}(\mathcal{E}) \to R^\vee/R \equiv \bigoplus_{u\colon m(F_u)>1} Z(F_u) =: Z(\mathcal{E}). \tag{8.11}$$

We note that

$$\mathsf{Triv}(\mathcal{E}) \subset \ker\gamma. \tag{8.12}$$

E_8-root curves. A rational curve $\mathcal{C}\subset\mathcal{E}$ is said to be a *E_8-root curve* iff its class $\mathcal{C}\in\mathsf{NS}(\mathcal{E})$ is a root of the E_8^- lattice (cfr. eqn.(8.5)). In other words, $\mathcal{C}$ is a E_8-root curve iff the following three conditions are satisfied

$$F\cdot\mathcal{C} \equiv -K_{\mathcal{E}}\cdot\mathcal{C} = 0, \qquad S_0\cdot\mathcal{C}=0, \qquad \mathcal{C}^2=-2. \tag{8.13}$$

An *E_8-root curve* is a particular case of a (-2)-curve [100]. It is clear that a rational elliptic surface $\mathcal{E}$ may have at most 240 E_8-root curves (240 being the number of roots of E_8).

The Mordell-Weil group of sections. As discussed in section 2, a rank-1 special geometry is, in particular, an elliptic curve E/K defined over the field of rational functions $K\equiv\mathbb{C}(u)$. The Mordell-Weil group $\mathsf{MW}(E/K)$ of an elliptic curve E defined over some field K is the group $E(K)$ of its points which are "rational" over K, that is, whose coordinates lay in K and not in some proper field extension [56, 178, 187, 235]. When K is a number field, the Mordell-Weil theorem of Diophantine Geometry states[66] that the Abelian group $E(K)$ is finitely-generated [56,178,187,235]. When K (as in our case) is a function field defined over $\mathbb{C}$, the Mordell-Weil theorem must be replaced by the Néron-Lang one [175, 178]: there is an Abelian variety B over $\mathbb{C}$ of dimension ≤ 1 (an Abelian variety of dimension zero being just the trivial group 0), and an injective map defined over K [179]

$$\mathrm{tr}_{K/\mathbb{C}}\colon B\to E, \qquad \text{(the \textit{trace} map)} \tag{8.14}$$

such that the quotient group $E(K)/\mathrm{tr}_{K/\mathbb{C}}(B)$ is finitely generated.

We may rephrase the above Diophantine statements in geometric language in terms of our Kodaira-Néron model, which is a rational elliptic surface $\pi\colon\mathcal{E}\to\mathbb{P}^1$ with a reference section $s_0\colon\mathbb{P}^1\to\mathcal{E}$. The (scheme-theoretic) closure in $\mathcal{E}$ of a point of E defined over $\mathbb{C}(u)$ is the same as a section of π. Thus the set of all sections of π is an Abelian group (with respect to fiberwise addition) isomorphic to the "abstract" Mordell-Weil group $\mathsf{MW}(\mathcal{E})\equiv\mathsf{MW}(E/K)$. The preferred section S_0 (the image of s_0) plays the role of zero in this group.

[66] The Mordell-Weil and the Néron-Lang theorems are stated in general for arbitrary Abelian varieties, not just for elliptic curves.

The Abelian variety $B/\mathbb{C}$ is non-trivial iff the fibers F_u of $\mathcal{E}$ are all isomorphic elliptic curves; in this case $\mathcal{E} \cong B \times \mathbb{P}^1$ and the special geometry is *trivial*. As before, we focus on non-trivial geometries where $B = 0$. Then the group $\mathsf{MW}(\mathcal{E})$ is finitely generated by the Néron-Lang theorem.

A section S defines a horizontal divisor on $\mathcal{E}$. By Abel theorem, addition in $\mathsf{MW}(\mathcal{E})$ corresponds to addition in $\mathsf{NS}(\mathcal{E})/\mathsf{Triv}(\mathcal{E}) \equiv \mathsf{Pic}(\mathcal{E})/(\text{vertical classes})$

$$S_1 + S_2 = S_3 \text{ in } \mathsf{MW}(\mathcal{E}) \quad \Longleftrightarrow \quad (S_1) + (S_2) = (S_3) \text{ in } \mathsf{NS}(\mathcal{E})/\mathsf{Triv}(\mathcal{E}), \tag{8.15}$$

so that, in our special case, the Néron-Lang theorem follows from the finite-generation of the Néron-Severi group.

The basic result is

Theorem 5 (Thm. (VII.2.1) of [191], Thm. 6.5 of [221])**.** *Let $\mathcal{E}$ be a (relatively minimal) rational elliptic surface. The following sequence (of finitely-generated Abelian groups) is exact*

$$0 \to \mathsf{Triv}(\mathcal{E}) \to \mathsf{NS}(\mathcal{E}) \xrightarrow{\beta} \mathsf{MW}(\mathcal{E}) \to 0. \tag{8.16}$$

In particular, the Shioda-Tate formula holds

$$\operatorname{rank} \mathsf{MW}(\mathcal{E}) = 8 - \sum_{u \in U} \big(m(F_u) - 1 \big). \tag{8.17}$$

In addition, using (8.12)*, the map γ factors through $\mathsf{MW}(\mathcal{E})$ so we get a map*

$$\gamma \colon \mathsf{MW}(\mathcal{E}) \to Z(\mathcal{E}) \tag{8.18}$$

which is injective on the torsion subgroup.

Remark 8.2.1. The involution of $\mathcal{E}$ acts on the Abelian group $\mathsf{MW}(\mathcal{E})$ as $S \mapsto -S$. Hence the even cohomology is in the kernel of β.

From eqns.(8.5)(8.8)(8.9) we see that (after flipping the overall sign!!)

$$\mathsf{MW}(\mathcal{E}) \cong E_8 \Big/ R \equiv E_8 \Bigg/ \bigoplus_{m(F_u)>1} R(F_u)\ . \tag{8.19}$$

The exact sequence (8.16) does not split (in general). However it does split once tensored with $\mathbb{Q}$. Then we define $\mathsf{NS}(\mathcal{E})_\mathbb{Q} := \mathsf{NS}(\mathcal{E}) \otimes \mathbb{Q}$. The orthogonal projection

$$\Phi_\mathbb{Q} \colon \mathsf{MW}(\mathcal{E}) \to \mathsf{NS}(\mathcal{E})_\mathbb{Q}, \tag{8.20}$$

splits β. Explicitly [221],

$$\Phi_\mathbb{Q} \colon S \longmapsto S - S_0 + (1 + S \cdot S_0)F + \sum_{u \in U} \sum_{\alpha,\beta=1}^{m(F_u)-1} F_{u,\alpha}\, C(u)^{-1}_{\alpha\beta} (F_{u,\alpha} \cdot S) \in \mathsf{NS}(\mathcal{E})_\mathbb{Q}, \tag{8.21}$$

whose image (by construction) is contained in the essential subspace (cfr. **Definition 11**)

$$\Lambda_{\mathbb{Q}} \subset E_8 \otimes \mathbb{Q} \subset \mathsf{NS}(\mathcal{E})_{\mathbb{Q}}. \tag{8.22}$$

In eqn.(8.21) $C(u)_{\alpha\beta}$ is the Cartan matrix of the ADE root system $R(F_u)$, cfr. table 4.

The map $\Phi_{\mathbb{Q}}$ induces on $\mathsf{MW}(\mathcal{E})/\mathsf{MW}(\mathcal{E})_{\rm tors}$ a $\mathbb{Q}$-valued positive-definite symmetric pairing, called the *Néron-Tate pairing*

$$\langle S_1, S_2\rangle_{\rm NT} = \langle \Phi_{\mathbb{Q}}(S_1), \Phi_{\mathbb{Q}}(S_2)\rangle_{E_8\otimes\mathbb{Q}} \in \mathbb{Q}. \tag{8.23}$$

The corresponding quadratic form $S \mapsto \mathsf{h}(S) \equiv \langle S, S\rangle_{\rm NT}$ is known as the Néron-Tate (or canonical) *height.* In terms of the intersection pairing $\cdot$ we have [221]

$$\langle S_1, S_2\rangle_{\rm NT} = 1 + S_1\cdot S_0 + S_2\cdot S_0 - S_1\cdot S_2 - \sum_{m(F_u)>1} C(u)^{-1}_{\alpha\beta}\,(F_{u\alpha}\cdot S_1)(F_{u\beta}\cdot S_2) \in \frac{1}{m}\mathbb{Z} \tag{8.24}$$

where $m = \mathrm{lcm}(m(F_u)^{(1)}))$. $\mathsf{MW}(\mathcal{E})/\mathsf{MW}(\mathcal{E})_{\rm tors}$ equipped with the Néron-Tate pairing is called the *Mordell-Weil lattice* [221].

Remark 8.2.2. From the facts that $K_{\mathcal{E}} = -F$ and $S^2 = -\chi(\mathcal{E}) = -1$, we see that all sections S are, in particular, (rational) (-1)-curves.

The narrow Mordell-Weil group. There is an important finite-index torsion-free subgroup of $\mathsf{MW}(\mathcal{E})$, the *narrow* Mordell-Weil group, $\mathsf{MW}(\mathcal{E})^0$, consisting of the sections which at all reducible fibers intersect the same component $F_{u,0}$ as S_0, so that the sum in the RHS of eqn.(8.21) vanishes. The sum in eqn.(8.24) also vanishes if either S_1 or S_2 is narrow. Thus the Néron-Tate pairing is $\mathbb{Z}$-valued when restricted to $\mathsf{MW}(\mathcal{E})^0$. More generally, the pairing of a section in $\mathsf{MW}(\mathcal{E})^0$ with *any* section in $\mathsf{MW}(\mathcal{E})$ is an integer. Indeed one has the isomorphisms of lattices [221]

$$\mathsf{MW}(\mathcal{E})^0 \cong \Lambda, \qquad \mathsf{MW}(\mathcal{E})/\mathsf{MW}(\mathcal{E})_{\rm tors} \cong \Lambda^\vee. \tag{8.25}$$

One shows that $\mathsf{MW}(\mathcal{E})^0 = \ker\gamma$ [191].

Integral sections. Given a (fixed) particular model of an elliptic curve $E/\Bbbk$ over a number field $\Bbbk$, say an explicit curve in $\mathbb{A}^2_{\Bbbk}$, we may consider, besides the points which are "rational" over $\Bbbk$, also the points which are "integral" over $\Bbbk$, that is, whose coordinates belong to the Dedekind domain $\mathfrak{O}_{\Bbbk}$ of algebraic integers in $\Bbbk$. While the "rational" points of $E/\Bbbk$ form a (typically infinite) finitely-generated group, its "integral" points form a *finite set* (Siegel theorem [235]).

The integer ring $\mathfrak{O}_{\mathbb{C}(u)}$ of the rational function field $\mathbb{C}(u)$ is, of course, the Dedekind domain of polynomials in u, $\mathbb{C}[u]$. The analogy with Siegel theorem in Number Theory suggests to look for sections given by polynomials. Of course, "integrality" is a model-dependent statement. If we focus on the elliptic curves over the rational field $\mathbb{C}(u)$ which are relevant for Special Geometry, and describe them through their *minimal* Weierstrass model, $y^2 = x^3 + a(u)x + b(u)$, the correct statement is that the integral sections are the ones of the form $(x, y) = (p(t), q(t))$ where $p(t)$ (resp. $q(t)$) is a polynomial of degree at most 2 (resp. 3) [221].

From the vantage point of the Kodaira-Néron model the notion of integral section becomes simpler:

Definition 12. A section $S \in \mathsf{MW}(\mathcal{E})$ is said to be *integral* if it does not intersect the zero section, i.e. $S \cdot S_0 = 0$.

Siegel theorem still holds [221]:

Proposition 8.2.1. *$\mathcal{E}$ a (relatively minimal) rational elliptic surface. There are only finitely many integral sections (at most 240) and they generate the full Mordell-Weil group.*

Indeed, from eqn.(8.24) we see that if S is integral

$$\mathsf{h}(S) = \langle S, S \rangle_{\mathrm{NT}} = 2 - \sum_{u,\alpha} C(F_u)^{-1}_{\alpha,\alpha} \, (F_{u,\alpha} \cdot S), \leq 2 \tag{8.26}$$

so that all integral sections have square-norms ≤ 2. Since there are only finitely many such elements in the lattice $\Lambda^\vee$ and the torsion subgroup $\subseteq \Lambda^\vee / \Lambda$ is finite, the statement follows.

Lemma 8.2.1. *If $S \in \mathsf{MW}(\mathcal{E})$ satisfies any two of the following three conditions, it also satisfies the third one:*

1) S is narrow: $S \in \mathsf{MW}(\mathcal{E})^0$;

2) S is integral: $S \cdot S_0 = 0$;

3) S has Néron-Tate height 2: $\mathsf{h}(S) = 2$.

Proof. From eqn.(8.24), the narrow condition implies $\mathsf{h}(S) = 2 + 2\, S \cdot S_0 \geq 2$ with equality if and only if $S \cdot S_0 = 0$. From eqn.(8.26) the integral condition implies $\mathsf{h}(S) \leq 2$ with equality if and only if S is narrow. □

The following observation is crucial:

Proposition 8.2.2. *$\pi \colon \mathcal{E} \to \mathbb{P}^1$ a (relatively minimal) rational elliptic surface. Let S be an* integral *section of π. Then the divisor*

$$\mathcal{C} = S - S_0 - F \tag{8.27}$$

is an E_8-root curve.

Proof. We have to check the three conditions in eqn.(8.13)

$$\begin{aligned} &F \cdot (S - S_0 - F) = 0, \qquad S_0(S - S_0 - F) = S_0 \cdot S = 0, \\ &(S - S_0 - F)^2 = -2 - 2\, S \cdot S_0 = -2. \end{aligned} \tag{8.28}$$

So $\mathcal{C}$ is an actual rational curve on the surface $\mathcal{E}$ which represents in $\mathsf{NS}(\mathcal{E})$ a root of the lattice E_8^- (cfr. eqn.(8.5)). □

Note that to an integral section there are associated both a (-1)-curve S and an E_8-root (-2)-curve $\mathcal{C}$. If, in addition, S is narrow,

$$F_{u,\alpha} \cdot \mathcal{C} = 0 \quad \text{for all } u,\ \alpha. \tag{8.29}$$

We say that an E_8-root curve is *in good position* in the Néron-Severi lattice if it satisfies eqn.(8.29). E_8-root curves in good position are in one-to-one correspondence with the integral-narrow sections of π.

8.3 Arithmetics of SW differentials

We return to the study of rank-1 special geometries and their SW differentials.

8.3.1 The "no dangerous irrelevant operator" property

Let us consider a special geometry $X_0 = \mathcal{E}_0 \setminus F_\infty$ described by a certain rational function $\mathscr{J}_0$ consistent with a given fiber configuration $\{F_\infty; F_i\}$. From eqn.(8.2) and the discussion following it, we see that X_0 carries a symplectic form Ω_0 such that (in cohomology)

$$[\Omega_0] \in \Lambda_{\mathbb{C}} \equiv \Lambda \otimes \mathbb{C}. \tag{8.30}$$

Now let us slightly deform the rational function $\mathscr{J} = \mathscr{J}_0 + \delta\mathcal{J}$, in a way consistent with the given fiber configuration $\{F_\infty; F_i\}$, while keeping fixed the fiber at infinity (i.e. the asymptotic geometry as $u \to \infty$, see discussion around eqn.(7.37)). Since we keep fixed the UV geometry, the deformation $X_0 \to X$ should correspond to a small change of masses and relevant couplings.

The deformed manifold X is smoothly equivalent to X_0; so we may identify the cohomology groups $H^2(X, \mathbb{C}) \cong H^2(X_0, \mathbb{C})$ and compare the symplectic forms in cohomology [104]. The variation $\delta[\Omega] = [\Omega] - [\Omega_0]$ computes the modification of the masses induced by the variation $\delta\mathcal{J}$ of Kodaira's functional invariant. Eqn.(8.30) identifies the space of mass parameters with a subspace of the essential vector space $\Lambda_{\mathbb{C}}$.

It is natural to require our geometry to have "enough" mass deformations (or equivalently "enough" SW differentials) to span all $\Lambda_{\mathbb{C}}$, that is, to require that no mass deformation is forbidden or obstructed. This requirement formalizes the physical idea that we are probing all genuine IR deformations of our QFT, and not arbitrarily restricting the parameters to some special locus in coupling space. We call this condition *SW completeness.* The main goal of this subsection is to show the following

Claim. *In rank-1, SW completeness implies the property "no dangerous irrelevant operators" conjectured in refs.* [19–23].

Proof. The statement of SW completeness says that the total number $\boldsymbol{n}$ of deformation of an UV complete geometry should be equal to the dimension of the space $\Lambda_{\mathbb{C}}$ plus the number of relevant/marginal operators. In formulae

$$\boldsymbol{n} - \dim \Lambda_{\mathbb{C}} = \begin{cases} 1 & \text{if } \Delta \leq 2 \\ 0 & \text{otherwise.} \end{cases} \tag{8.31}$$

From eqn.(7.26)

$$\dim \Lambda_{\mathbb{C}} = 8 - \sum_u \operatorname{rank} R(F_u) = 8 - \sum_u e(F_u) + s + 2a^\circ + 2a^* = s + 2a^\circ + 2a^* - 4. \tag{8.32}$$

while, from eqn.(7.22),

$$\boldsymbol{n} = s + a^\circ + 2a^* - 3, \tag{8.33}$$

so that the LHS of eqn.(8.31) is simply

$$1 - a^\circ \tag{8.34}$$

from which we see that $a^\circ = 1$ if $\Delta > 2$ and $a^\circ = 0$ otherwise. □

Comparing with §.7.3 we get

Fact. *In a non-constant, UV and SW complete, rank-1 special geometry, an* additive$^\circ$ *fiber (i.e. types II, III, and IV) may be present in $\mathcal{E}$ only as the fiber at infinity F_∞. In this case the $\mathcal{N} = 2$ QFT is a mass-deformation of a SCFT with $\Delta = 6, 4$ and 3, respectively.*

This statement has identical implications for the classification program (in rank-1) as the "no dangerous irrelevant operator" conjecture of ref. [19–23].

8.3.2 The flavor lattice (elementary considerations)

In the previous subsection we have identified $\Lambda_\mathbb{C}$ with the complexification $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \otimes \mathbb{C}$ of the flavor Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{f}$. The dimensions of the two spaces agree for SW complete geometries.

Inside the Cartan algebra $\mathfrak{h}$ we have natural lattices, such as the weight and roots lattices of $\mathfrak{f}$. These lattices are endowed with a positive-definite symmetric pairing with respect to which the Weyl group $\mathrm{Weyl}(\mathfrak{f})$ acts by isometries. Moreover, in $\mathfrak{h}$ we may distinguish finitely many vectors playing special roles, such as the co-roots, the roots, and the fundamental weights.

In order for the identification $\Lambda_\mathbb{C} \leftrightarrow \mathfrak{h}_\mathbb{C}$ to be fully natural, the above discrete structures should be identifiable in $\Lambda_\mathbb{C}$ too. In $\Lambda_\mathbb{C}$ there exist canonical lattices, like Λ, $\Lambda^\vee$ and their sub- and over-lattices, as well as a natural positive-definite symmetric pairing, i.e. the Néron-Tate height $\langle -, - \rangle_{\mathrm{NT}}$. These lattices also contains a special finite sub-set, namely the integral sections.

In particular, to a given fiber configuration $\{F_u\}_{u \in U}$ we may associate the group $O(\mathsf{MW}(\mathcal{E})^0)$ of isometries of the narrow Mordell-Weil lattice $\mathsf{MW}(\mathcal{E})^0$. Then, consistency yields

Necessary condition. *Let $\mathfrak{f}$ be the flavor Lie algebra associated to a rank-1 (UV and SW complete) special geometry, and let* $\mathrm{Weyl}(\mathfrak{f})$ *be its Weyl group. Then*

$$\mathrm{Weyl}(\mathfrak{f}) \subseteq O(\mathsf{MW}(\mathcal{E})^0). \tag{8.35}$$

This condition does not fix $\mathfrak{f}$ uniquely. For instance, let $\mathsf{MW}(\mathcal{E})^0 \cong D_4$, so that

$$O(\mathsf{MW}(\mathcal{E})^0) \cong \mathrm{Weyl}(D_4) \rtimes \mathfrak{S}_3, \tag{8.36}$$

where the symmetric group $\mathfrak{S}_3$ acts by Spin(8) triality. Then $O(\mathsf{MW}(\mathcal{E})^0) \cong \mathrm{Weyl}(F_4)$, while the subgroup $\mathrm{Weyl}(D_4) \rtimes \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathrm{Weyl}(C_4) \cong \mathrm{Weyl}(B_4)$, so in this case the above condition leaves us with 4 possible irreducible $\mathfrak{f}$, namely B_4, C_4, D_4 and F_4, and a few more reducible candidates.

In order to unfold the ambiguity, we need to understand the flavor root *system* and not just its root *lattice*. This issue will be discussed in the next subsection. The obvious guess is that the finite set of integral sections will play the major role.

F_∞	II	III	IV	I_0^*	II^*	III^*	IV^*	I_1^*	I_2^*
$R(F_\infty)$	-	A_1	A_2	D_4	E_8	E_7	E_6	D_5	D_6
$\mathsf{MW}(\mathcal{E})^0 \equiv \Lambda$	E_8	E_7	E_6	D_4	-	A_1	A_2	A_3	$A_1 \oplus A_1$
$\delta(F_\infty)$	0	$\frac{1}{2}$	$\frac{2}{3}$	1	2	$\frac{3}{2}$	$\frac{4}{3}$	$1,\ \frac{5}{4}$	$1,\ \frac{3}{2}$
integral repr.	-	$\mathbf{56}$	$\mathbf{27} \oplus \overline{\mathbf{27}}$	$\mathbf{8}_c \oplus \mathbf{8}_s \oplus \mathbf{8}_c$	-	$\mathbf{2}$	$\mathbf{3} \oplus \overline{\mathbf{3}}$	$\mathbf{6} \oplus \mathbf{4} \oplus \overline{\mathbf{4}}$	$\mathbf{4} \oplus \mathbf{2}_L \oplus \mathbf{2}_R$

Table 7: Flavor symmetries and integral representations for "general" deformations. Note that in the non-semisimple cases $\delta(F_\infty)$ takes two distinct values (cfr. [221] page 124). For $I_{b\leq 2}^*$ the integral representation is given by $\mathsf{vector} \oplus \mathsf{spinor}_L \oplus \mathsf{spinor}_R$ of $SO(8-2b)$.

In simple situations the correct physical flavor symmetry may be easily guessed from the narrow Mordell-Weil lattice $\mathsf{MW}(\mathcal{E})^0$. However, in general, one needs the precise treatment in terms of roots systems described in the next subsection. Here we present the simplest possibile situation (i.e. maximal symmetry for the given Δ) where naive ideas suffice.

Example 11 (Maximal flavor symmetry)**.** Let us consider fiber configurations of the form $\{F_\infty; I_1^{12-e(F_\infty)}\}$ where F_∞ is one of the 7 semi-simple additive fibers in eqn.(7.33) or $I_{b\leq 4}^*$ in the asymptotic-free case. These configurations are the "general deformations" of the SCFT associated to the given fiber at infinity, in the sense that they yield the family of elliptic surfaces depending on the largest number of parameters. Thus $\{F_\infty; I_1^{12-e(F_\infty)}\}$ is the configuration which, for a given Coulomb dimension Δ (encoded in F_∞), maximizes the rank of the flavor group, see eqn.(8.32). In this case all fibers are irreducible except (possibly) the fiber at infinity. The Mordell-Weil group is torsionless [221] and thus

$$\mathsf{MS}(\mathcal{E}) \cong (\mathsf{MS}(\mathcal{E})^0)^\vee \equiv \Lambda^\vee. \tag{8.37}$$

Standard facts about lattices [221] yield

Lemma 8.3.1. *Let F_∞ be one of the 7 semi-simple additive fiber types in eqn.(7.32) or $I_{b\leq 2}^*$ and $R(F_\infty)$ the corresponding ADE root system (table 4). Let $\Lambda = R(F_\infty)^\perp$ be its orthogonal complement in the E_8 lattice (i.e. the* essential *lattice). Then Λ is an irreducible root lattice of type ADE, except for $F_\infty = I_2^*$ where Λ is the root lattice of $\mathfrak{so}(4) = A_1 \oplus A_1$. ($\Lambda^\vee$ is then the corresponding ADE weight lattice). See table 7. Moreover,*

$$\mathsf{MS}(\mathcal{E})/\mathsf{MS}(\mathcal{E})^0 \equiv \Lambda^\vee/\Lambda = R(F_\infty)^\vee/R(F_\infty) \equiv Z(F_\infty) \equiv Z(\mathcal{E}), \tag{8.38}$$

is the center of the corresponding (simply-connected) ADE Lie group.

Remark 8.3.1. Note that for $\{F_\infty; I_1^{12-e(F_\infty)}\}$ adding/deleting $*$ on the fiber at ∞ simply interchanges the two orthogonal sub-lattices $R(F_\infty) \leftrightarrow \mathsf{MW}(\mathcal{E})^0$.

In the $\{F_\infty; I_1^{12-e(F_\infty)}\}$ case, for all sections $S \in \mathsf{MW}(\mathcal{E})$

$$\mathsf{h}(S) \equiv \langle S, S\rangle_{\mathrm{NT}} = 2 + 2\, S\cdot S_0 - \begin{cases} 0 & \text{if } S \in \mathsf{MW}(\mathcal{E})^0 \\ \delta(F_\infty) & \text{if } S \notin \mathsf{MW}(\mathcal{E})^0 \end{cases} \tag{8.39}$$

see table 7. For $F_\infty = II$, $\mathsf{MW}(\mathcal{E})^0 \equiv \mathsf{MW}(\mathcal{E})$, so the second case in (8.39) does not appear.

The roots of the ADE lattice $\mathsf{MW}(\mathcal{E})^0$ are narrow of height 2 hence integral sections by **Lemma 8.2.1** which are related to *E_8-root curves* by **Proposition 8.2.2**. Being narrow, they are automatically in *good position.* It is known that the flavor Lie algebra $\mathfrak{f} \equiv \mathfrak{Lie}(\mathsf{F})$ of the "maximally symmetric" models is the simply-laced Lie algebra $\mathsf{MW}(\mathcal{E})^0$. Thus the roots of the flavor algebra simply correspond to the E_8-root curves in good position for the fiber configuration $\{F_\infty; I_1^{12-e(F_\infty)}\}$.

The Mordell-Weil group $\mathsf{MW}(\mathcal{E})$ is the weight lattice of the Lie algebra in the third row of table 7, and the integral sections which are not ADE roots form the weights of the representation in the last row of the table. These sections correspond to (-2)-curves which are not in good position. They form a Weyl invariant set of weights. Note that the 'integral representation' of F in the last row of the table is precisely the one carried by the BPS hypermultiplets which are stable in the regime $u \to \infty$. For instance, for $\{I_b^*, I_1^{6-b}\}$, which corresponds to $SU(2)$ SQCD with $N_f = 4 - b$, we get $\mathsf{F} = SO(2N_f)$ and the hypers (quarks, monopoles, and dyons [223,224]) belong to the vector and left/right spinor representations.

Example 12. In **Example 11** we excluded two possible fibers at ∞, I_4^* and I_3^*. The first one, which corresponds to pure SYM, has a flavor group of rank 0. The second one, i.e. $SU(2)$ SQCD with $N_f = 1$ (cfr. §. 7.3), has a flavor group of rank 1. However, in this case the flavor group is *not* semi-simple, but rather the Abelian group $SO(2)$ (baryon number) which does *not* correspond to a root system. Correspondingly, in this instance the essential lattice is *not* a root lattice but rather [221]

$$\Lambda = \langle 4 \rangle, \qquad \mathsf{MW}(\mathcal{E}) \equiv \Lambda^\vee = \langle 1/4 \rangle, \tag{8.40}$$

where $\langle \ell \rangle$ stands for the group $\mathbb{Z}$ endowed with the quadratic form $\mathsf{h}(n) = \ell\, n^2$. One has $\delta(I_3^*) = 1$, or $\frac{7}{4}$, so that the integral sections correspond to the elements of $\langle 1/4 \rangle$ having height 1 or $\frac{1}{4}$. They correspond to $U(1) \cong SO(2)$ baryon charges ± 1 and $\pm\frac{1}{2}$, which are the correct values for quarks and, respectively, dyons in $N_f = 1$ SQCD.

8.4 The flavor root system

8.4.1 The root system associated to the Mordell-Weil lattice

The Mordell-Weil lattices contain a canonical root system that we now define.

As reviewed above, for a rational elliptic surface $\mathcal{E}$ we have

$$\begin{array}{ccccc} \mathsf{MW}(\mathcal{E})^0 & \subset & \mathsf{MW}(\mathcal{E})/\mathsf{MW}(\mathcal{E})_{\text{tors.}} & \subset & \mathsf{NS}(\mathcal{E})^-_{\mathbb{Q}} \\ \| & & \| & & \| \\ \Lambda & \subset & \Lambda^\vee & \subset & \boldsymbol{U}_{\mathbb{Q}} \oplus (E_8 \otimes \mathbb{Q}) \end{array} \tag{8.41}$$

Λ, $\Lambda^\vee$ being equipped with the Néron-Tate pairing and $\mathsf{NS}(\mathcal{E})^-_{\mathbb{Q}}$ with *minus* the intersection pairing. The embeddings in (8.41) are isometries. We consider the sublattice of "integral points" in $\Lambda^\vee$

$$\Lambda_{\mathbb{Z}} := \Lambda^\vee \cap \mathsf{NS}(\mathcal{E})^- \subset \Lambda^\vee. \tag{8.42}$$

A vector $s \in \Lambda_{\mathbb{Z}}$, being an element of $\mathsf{NS}(\mathcal{E})^-$, defines a divisor $D(s)$ unique up to linear equivalence. An element $\lambda \in \Lambda^\vee$ defines a section $S(\lambda)$ unique up to torsion.

The *level* of $s \in \Lambda_{\mathbb{Z}}$ is the largest positive integer $k(s)$ such that

$$\hat{s} \equiv \frac{1}{k(s)}\, s \in \Lambda^{\vee}. \tag{8.43}$$

We have,

$$\langle s, s\rangle_{\rm NT} = k(s)^2\, \mathsf{h}(\hat{s}) \qquad \forall\, s \in \Lambda_{\mathbb{Z}} \tag{8.44}$$

$$\langle \lambda, s\rangle_{\rm NT} = k(s)\, \langle \lambda, \hat{s}\rangle_{\rm NS} = -S(\lambda)\cdot D(s) \in \mathbb{Z} \qquad \forall\, s \in \Lambda_{\mathbb{Z}},\ \lambda \in \Lambda^{\vee}, \tag{8.45}$$

In particular, $\Lambda_{\mathbb{Z}} \subset \Lambda$.

Definition 13. The *MW root system* $\Xi \subset \Lambda_{\mathbb{Z}}$ is the set of elements $s \in \Lambda_{\mathbb{Z}}$ such that

$$\mathsf{h}(s)/k(s) \equiv k(s)\, \mathsf{h}(\hat{s}) = 2 \quad \Rightarrow \quad \langle s, s\rangle_{\rm NT} = 2\, k(s). \tag{8.46}$$

For each $s \in \Xi$ we consider the reflection

$$r_s \colon \lambda \mapsto \lambda - \frac{2\langle \lambda, s\rangle_{\rm NT}}{\langle s, s\rangle_{\rm NT}}\, s. \tag{8.47}$$

Lemma 8.4.1. *Let $s \in \Xi$. The reflection r_s:*

1) *is an isometry of $\Lambda^{\vee}$;*

2) *preserves the lattice $\Lambda_{\mathbb{Z}}$;*

3) *preserves the level $k(s')$ of $s' \in \Lambda_{\mathbb{Z}}$.*

Proof. **1)** It suffice to show that $r_s(\lambda)$ is a linear combination of elements of $\Lambda^{\vee}$ with integral coefficients. For all $s \in \Xi$ and $\lambda \in \Lambda^{\vee}$,

$$\frac{2\langle \lambda, s\rangle_{\rm NT}}{\langle s, s\rangle_{\rm NT}}\, s = \frac{2\langle \lambda, s\rangle_{\rm NT}}{2\, k(s)}\, k(s)\, \hat{s} = -S(\lambda)\cdot D(s)\, \hat{s}. \tag{8.48}$$

2) We have to show that

$$\frac{2\langle s', s\rangle_{\rm NT}}{\langle s, s\rangle_{\rm NT}} \in \mathbb{Z} \quad \text{for all } s \in \Xi,\ s' \in \Lambda_{\mathbb{Z}}. \tag{8.49}$$

Now

$$\frac{2\langle s', s\rangle_{\rm NT}}{\langle s, s\rangle_{\rm NT}} = \frac{2k(s)\langle s', \hat{s}\rangle_{\rm NT}}{2k(s)} = -D(s')\cdot S(\hat{s}) \in \mathbb{Z} \tag{8.50}$$

3) Indeed, $r_s(s') = k(s')\, r_s(\hat{s}')$ where $r_s(\hat{s}') \in \Lambda^{\vee}$ by **1)**. □

From this **Lemma** it follows that the finite set Ξ is a reduced root system canonically associated to the Mordell-Weil group.

The restricted root system of $(\mathcal{E}, F_\infty)$. In our set-up, we have a marked additive fiber $F_\infty \in \mathcal{E}$. We consider the subset of $\Xi_\infty \subset \Xi$ such that

$$s \in \Xi_\infty \quad \Longleftrightarrow \quad s \in \Xi \text{ and } S(\hat{s}) \text{ crosses } F_\infty \text{ in the identity component.} \tag{8.51}$$

From (8.47) we see that Ξ_∞ is also a root system. Indeed, for all $s \in \Xi_\infty$, $s' \in \Xi$ and $\alpha \geq 1$,

$$F_{\infty,\alpha} \cdot S(r_s(\hat{s}')) = F_{\infty,\alpha} \cdot S(\hat{s}') - \langle \hat{s}', s \rangle_{\rm NT} \, F_{\infty,\alpha} \cdot S(\hat{s}) \equiv F_{\infty,\alpha} \cdot S(\hat{s}'). \tag{8.52}$$

Explicit formulae for divisors. Let $s \in \Lambda_{\mathbb{Z}}$ be an element of level $k(s)$, and write $\hat{S}$ for $S(\hat{s})$. Then the $D(s)$, $S(s)$ are the divisors

$$D(s) = k(s)\, \Phi_{\mathbb{Q}}(\hat{S}) \in \Lambda_{\mathbb{Z}} \subset \mathsf{NS}(\mathcal{E}) \tag{8.53}$$

$$S(s) = k(s)\big(\Phi_{\mathbb{Q}}(\hat{S}) + F\big) + S_0 \in \mathsf{NS}(\mathcal{E}). \tag{8.54}$$

$S(s)$ is an exceptional (-1)-curve, i.e. $K_{\mathcal{E}} \cdot S(s) = S(s)^2 = -1$, namely a section.

Remark 8.4.1. All $s \in \Lambda^\vee \cong \mathsf{MW}(\mathcal{E})/\mathsf{MW}(\mathcal{E})_{\rm tor}$ corresponding to *narrow-integral* sections are elements of Ξ_∞ corresponding to "short" roots (height = 2). Conversely, all roots of height 2 arise from narrow-integral sections. Let $\hat{S}$ be a non-narrow integral section which is narrow at ∞, and $k(\hat{S})$ the smallest integer such that $k(\hat{S})\,\hat{S} \in \Lambda$. If $\hat{S}$ satisfies the criterion

$$k(\hat{S})\, \mathsf{h}(\hat{S}) = 2, \tag{8.55}$$

then $k(\hat{S})\,\hat{S} \in \Xi_\infty$.

Remark 8.4.2. We have $\operatorname{rank} \Xi_\infty \leq \operatorname{rank} \Lambda$. When the inequality is strict, F has an Abelian factor $U(1)^a$ with $a = \operatorname{rank} \Lambda - \operatorname{rank} \Xi_\infty$, cfr. **Example 12**.

8.4.2 SW differentials and flavor

In §. 8.3.2 we considered the polar divisor of λ up to algebraic (or linear) equivalence. In doing this we lost some information about the actual curves $\mathcal{S}_i \subset \mathcal{E}$ along which the SW differential λ has poles. We know that these curves must be sections of $\pi \colon \mathcal{E} \to \mathbb{P}^1$, i.e. $F \cdot \mathcal{S}_i = 1$. We may take one of the $\mathcal{S}_i$, say $\mathcal{S}_0$ as the zero section $S_0 \equiv \mathcal{S}_0$. The divisors dual to the free mass parameters (cfr. eqn.(8.2)) then take the form $L_a \sim \mathcal{S}_a - \mathcal{S}_0$ for $a > 0$. The L_a should be trivial at infinity (since the masses are UV irrelevant), that is, the sections $\mathcal{S}_i$ should cross F_∞ in the identity component[67], $F_{\infty,\alpha} \cdot \mathcal{S}_i = \delta_{\alpha,0}$. Eqn.(8.3) yields

$$i) \quad L_a \equiv \mathcal{S}_a - \mathcal{S}_0 \in \Lambda_{\mathbb{Z}} \oplus \mathbb{Z}F, \qquad ii) \quad F_{\infty,\alpha} \cdot (L_a + \mathcal{S}_0) = 0 \qquad \alpha \geq 1. \tag{8.56}$$

We have to determine the sections $\mathcal{S}_i$ (equivalently, the divisors L_a satisfying *i)* and *ii)*), which may actually appear in the polar divisor of λ. From comparison with E_8 Minahan-Nemeshanski we know that L_a is allowed to be an E_8-root (-2)-curve. Note that eqn.(8.56) enforces the condition that L_a is in good position. If L_a is a E_8-root satisfying (8.56), the associated (-1)-curve $\mathcal{S}_a$ is an integral-narrow section hence an element of Ξ_∞ of height 2.

However the integral-narrow sections cannot be the full story, since the set of integral-narrow sections does not behave properly under covering maps (discrete gaugings in the QFT language). In the next section we shall discuss the functorial properties of the Mordell-Weil lattices under such coverings. There it will be shown that a natural finite set of sections which contains the

[67] We call such sections *narrow at* ∞.

Δ	6	4	3	2
F_∞	II	III	IV	I_0^*
R	A_3	$A_3 \oplus A_1$	$A_3 \oplus A_2$	$A_3 \oplus D_4$
$\mathsf{MW}(\mathcal{E})^0$	D_5	$A_3 \oplus A_1$	$(D_5 : A_2)$	$\langle 4 \rangle$
$\mathsf{MW}(\mathcal{E})$	$D_5^\vee$	$A_3^\vee \oplus A_1^\vee$	$(D_5 : A_2)^\vee$	$\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$

Table 8: Lattices for fibers $\{F_\infty; I_4, I_1^{8-e(F_\infty)}\}$. $(D_5 : A_2)$ stands for the orthogonal complement of the lattice A_2 in D_5 (it cannot be written as a direct sum of root and rank 1 lattices). The Mordell-Weil groups are read from the table attached to **Theorem 8.7** of [221].

integral-narrow ones and behaves well under covering maps is the set Ξ defined in §. 8.4.1. As we have seen, Ξ is automatically a root system in $\Lambda_{\mathbb{R}}$. The condition (8.56) restricts further to the subsystem Ξ_∞. Therefore consistency leaves us with just one possible conclusion:

The root system of the flavor Lie group F *is* Ξ_∞.

This statement is checked in §.8.4.3 in (essentially all) examples.

Remark 8.4.3 (Abelian flavor symmetries)**.** The general situation is similar to $SU(2)$ SQCD with $N_f = 1$. In that model the rank of the flavor Lie algebra is 1, but the set of roots is empty since: *i)* by definition, in $\{I_3^*, I_1^3\}$ there are no non-narrow sections which are narrow at ∞, and *ii)* $\Lambda = \langle 4 \rangle$ so no narrow section is integral. This is the correct result for a $U(1)$ flavor symmetry. On the other hand, the integral section which are not narrow at ∞ give baryon numbers of BPS states as we commented in **Example 12**.

Remark 8.4.4 (Maximal symmetry again)**.** In the configuration $\{F_\infty, I_1^{12-e(F_\infty)}\}$, all sections narrow at ∞ are narrow. So the roots are just the elements of Λ which have height 2, and the root system is the unique simply-laced one with root lattice Λ, see third row of table 7.

8.4.3 More examples of flavor root systems

Example 13 (Fiber configurations $\{F_\infty; I_4, I_1^{8-e(F_\infty)}\}$)**.** We assume the presence of a single semi-stable fiber of type I_4. This restricts the additive fiber at ∞ to 4 possible types as in table 8. For $F_\infty = II$ and III the narrow Mordell-Weil groups are the root lattices $\mathsf{MW}(\mathcal{E})^0 = D_5$ and $A_3 \oplus A_1$, respectively, and the full Mordell-Weil group is the corresponding weight lattice. The 40 roots of D_5 (resp. 14 roots of $A_3 \oplus A_1$) correspond to narrow-integral sections and are roots of $\mathfrak{f}$. An integral[68] non-narrow section S has Néron-Tate height

$$\mathsf{h}(S) = 2 - \delta(I_4) = \begin{cases} 1 & k(S) = 2 \\ 5/4 & k(S) = 4, \end{cases} \tag{8.57}$$

and the criterion (8.55) is satisfied only by the sections of height 1 which have square-length 4. For $F_\infty = II$ there are 10 such roots of square-length 4, one for each vector weight in $D_5^\vee$. For $F_\infty = III$ there are 6 of them in correspondence with the vector weights of $A_3 \cong \mathfrak{so}(6)$. We conclude:

[68] A non-integral section has height ≥ 3.

- The flavor Lie algebra of $\{II; I_4, I_1^6\}$ has a root system consisting of 40 roots of square-length 2 and 10 roots of square-length 4, and a Weyl group $\mathrm{Weyl}(D_5) \rtimes \mathbb{Z}/2\mathbb{Z}$. The special geometry describes a $\Delta = 6$ SCFT with flavor group (isogeneous to) $Sp(10)$;

- The flavor Lie algebra of $\{III; I_4, I_1^6\}$ has a root system consisting of 14 roots of square-length 2 and 6 roots of square-length 4, and a Weyl group $(\mathrm{Weyl}(A_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mathrm{Weyl}(A_2)$. The geometry describes a $\Delta = 4$ SCFT with flavor group (isogeneous to) $Sp(6) \times Sp(2)$.

If $F_\infty = I_0^*$, $R = A_3 \oplus D_4$, and $\Lambda = \langle 4 \rangle$. There are no integral narrow sections, and the roots are in one-to-one correspondence with the elements of the Mordell-Weil group of Néron-Tate height 1. We conclude:

- $\{I_0^*; I_4, I_1^2\}$ describes a $\Delta = 2$ SCFT with $\mathsf{F} = Sp(2)$, namely $SU(2)$ $\mathcal{N} = 2^*$.

If $F_\infty = IV$, the narrow Mordell-Weil and the full Mordell-Weil groups are rank 3 dual lattices Λ, $\Lambda^\vee$ with respective Gram matrices [221]

$$\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 4 \end{bmatrix} \qquad \frac{1}{12}\begin{bmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ 2 & 2 & 4 \end{bmatrix} \tag{8.58}$$

Short roots are in one-to-one correspondence with elements of the first lattice of Néron-Tate height 2 while long roots are in correspondence with elements of the second lattice with height 1. There are 4 short roots $[\pm 1, 0, 0], [0, \pm 1, 0] \in \Lambda$ and 4 long roots $[\epsilon_1, \epsilon_2, -(\epsilon_1 + \epsilon_2)/2] \in \Lambda^\vee$, $(\epsilon_1, \epsilon_2 = \pm 1)$ making the root system of $Sp(4)$. Since the flavor group has rank 3, and there is no root associated to the last component of the Cartan subalgebra, we conclude that F has an Abelian factor (compare **Remark 8.4.3**). Then

- $\{IV; I_4, I_1^4\}$ describes a $\Delta = 3$ SCFT with $\mathsf{F} = Sp(4) \times U(1)$.

Example 14 (The configuration $\{II; IV^*, I_1^2\}$). In this case $R = E_6$ (table 4), $\Lambda = A_2$ and $\mathsf{MW}(\mathcal{E}) = A_2^\vee$. The 6 roots of A_2 yield short roots of the flavor algebra $\mathfrak{f}$. Let us consider the roots arising from the integral non-narrow sections. An integral non-narrow section S has Néron-Tate height (cfr. table 7)

$$\mathsf{h}(S) = 2 - \delta(IV^*) = \frac{2}{3} \qquad \text{and} \quad k(S) = 3. \tag{8.59}$$

The criterion (8.55) is satisfied, and the integral non-narrow sections correspond to long roots of square-length $2 \cdot 3 = 6$. The long roots are then in 1-to-1 correspondence with the elements of height $\frac{2}{3}$ in $A_2^\vee$, whose number is 6. The 6 roots of square-length 2 together with the 6 roots of square-length 6 form the root system of G_2. Therefore

- $\{II; IV^*, I_1^2\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = G_2$.

Example 15 (The configuration $\{II; I_0^*, I_1^4\}$). In this case $R = D_4$ (table 4), $\Lambda = D_4$ and $\mathsf{MW}(\mathcal{E}) = D_4^\vee$. The 24 roots of D_4 yield short roots of the flavor algebra $\mathfrak{f}$ of square-length 2. The integral non-narrow sections S have Néron-Tate height (cfr. table 7)

$$\mathsf{h}(S) = 2 - \delta(I_0^*) = 1, \qquad \text{and} \quad k(S) = 2, \tag{8.60}$$

and correspond to long roots in correspondence with the 24 elements of the lattice $D_4^\vee$ of height 1. We have 24 short roots of height 2 and 24 long roots of height 4; thus

- $\{II; I_0^*, I_1^4\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = F_4$.

Example 16 (The configuration $\{II; I_1^*, I_1^3\}$)**.** In this case $R = D_5$ (table 4), $\Lambda = A_3$ and $\mathsf{MW}(\mathcal{E}) = A_3^\vee$. The 12 roots of A_3 yield short roots of the flavor algebra $\mathfrak{f}$ of square-length 2. An integral non-narrow section S has Néron-Tate height

$$\mathsf{h}(S) = 2 - \delta(I_1^*) = \begin{cases} 1 & k(S) = 2 \\ 3/4 & k(S) = 4, \end{cases} \tag{8.61}$$

corresponding, respectively, to the vector and spinor representations of $\mathfrak{so}(6) \cong A_3$. The first line satisfies (8.55) and lead to 6 long roots of square-norm 4. Then

- $\{II; I_1^*, I_1^3\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = Sp(6)$.

Example 17 (The configuration $\{II; I_4^2, I_1^2\}$)**.** In this case $R = A_3 \oplus A_3$. This is a subtle case since two distinct Mordell-Weil lattices may be realized [221] (cfr. eqn.(7.29))

$$1) \qquad \Lambda = A_1 \oplus A_1 \qquad \mathsf{MW}(\mathcal{E}) = \Lambda^\vee \oplus \mathbb{Z}/2\mathbb{Z} \tag{8.62}$$

$$2) \qquad \Lambda = \langle 4 \rangle \oplus \langle 4 \rangle \qquad \mathsf{MW}(\mathcal{E}) = \Lambda^\vee. \tag{8.63}$$

Let us consider the two possibilities in turn.

1) We have 4 square-length 2 roots from the integral-narrow sections. The non-narrow sections have $k(S) = 2$. We have the 4 roots of square-length 4 associated to the elements $(\pm\frac{1}{2}, \pm\frac{1}{2}) \in A_1^\vee \oplus A_1^\vee$. In total we get the root system of $Sp(4)$.

2) There are no roots from the narrow sections. An integral section S which is narrow at one of the two I_4 fibers has Néron-Tate height

$$\mathsf{h}(S) = 2 - \delta(I_4) = \begin{cases} 1 & k(S) = 2 \\ 5/4 & k(S) = 4, \end{cases} \tag{8.64}$$

and those which are narrow at both

$$\mathsf{h}(S) = 1 - \delta(I_4) - \delta(I_4)' = \begin{cases} 1/4 & k(S) = 4 \\ 1/2 & k(S) = 4. \end{cases} \tag{8.65}$$

The criterion (8.55) is satisfied by the sections of Néron-Tate height 1 which have square-length 4 (there are 4 of them), and by those of height 1/2 which have square-length 8 (other 4). Rescaling the length by a factor $1/\sqrt{2}$, we recognize again the root system of $Sp(4)$. Thus

- $\{II; I_4^2, I_1^2\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = Sp(4)$. However, it looks like we have *two* distinct theories with these properties.

Example 18 (The configuration $\{II; I_2^*, I_1^2\}$)**.** In this case we have $R = D_6$, $\Lambda = A_1 \oplus A_1$ and $\Lambda^\vee = A_1^\vee \oplus A_1^\vee \cong \mathfrak{so}(4)$. The 4 roots of Λ are roots of square-length 2 while the integral sections in the **4** of $\mathfrak{so}(4)$ give roots of square-lenght 4.

- $\{II; I_2^*, I_1^2\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = Sp(4)$.

Example 19 ($\{II; I_1^*, I_3\}$)**.** $R = D_5 \oplus A_2$ and $\Lambda = \langle 12\rangle$, $\mathsf{MW}(\mathcal{E}) = \langle 1/12\rangle$. The 2 sections with $\mathsf{h} = 1/3$ have $k(S) = 6$ so are roots of square-length $12 = 4(\sqrt{3})^2$.

- $\{II; I_1^*, I_3\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = Sp(2)$.

Example 20 ($\{II; I_2, IV^*\}$)**.** $R = E_6 \oplus A_1$ and $\Lambda = \langle 6\rangle$, $\mathsf{MW}(\mathcal{E}) = \langle 1/6\rangle$. The 2 integral section with $\mathsf{h}(S) = 2/3$ have $k(S) = 3$ and hence are roots of square-length $3 \cdot 2 = 6$. Rescaling the length, we get the root system of $\mathsf{F} = Sp(2)$, but it looks like a specialization of the G_2 model.

Example 21 ($\{II; I_1, I_3^*\}$)**.** $R = D_7$ and $\Lambda = \langle 4\rangle$, $\mathsf{MW} = \langle 1/4\rangle$. The 2 sections with $\mathsf{h}(S) = 1$ have $k(S) = 2$ and we get a $Sp(2)$ flavor group.

Example 22 ($\{II; I_4, I_2^2, I_1^2\}$)**.** $R = A_3 \oplus A_1 \oplus A_1$ and $\Lambda = A_3$, $\mathsf{MW} = A_3^\vee \oplus \mathbb{Z}/2\mathbb{Z}$. We have 12 roots of length 2 from the narrow sections, and 6 roots of length 4.

- $\{II; I_4, I_2^2, I_1^2\}$ describes a $\Delta = 6$ SCFT with $\mathsf{F} = Sp(6)$.

Example 23 ($\{III; I_0^*, I_1^3\}$)**.** Here $R = A_1 \oplus D_4$, $\Lambda = A_1 \oplus A_1 \oplus A_1$. We have the 6 roots of the Λ and the 3×4 vectors of the three $A_1 \otimes A_1$ subalgebras. The flavor algebra has 6 short and 12 long roots hence

- $\{III; I_0^*, I_1^3\}$ describes a $\Delta = 4$ SCFT with $\mathsf{F} = Spin(7)$.

Example 24 ($\{III; I_1^*, I_1^2\}$)**.** Here $R = A_1 \oplus D_5$, $\Lambda = A_1 \oplus \langle 4\rangle$. We have the 2 roots of A_1 and the 2 sections $\mathsf{h}(S) = 1$ with $k(S)$. The two sets of roots are orthogonal[69]

- $\{III; I_1^*, I_1^2\}$ describes a $\Delta = 4$ SCFT with $\mathsf{F} = SU(2) \times Sp(2)$.

Example 25 ($\{IV; I_0^*, I_1^2\}$)**.** $R = D_4 \oplus A_2$, $\Lambda = A_2[2]$ and $\mathsf{MS}(\mathcal{E}) = A_2^\vee[1/2]$. There are no narrow-integral sections. The integral sections which are narrow at ∞ correspond to the image of the A_2 roots in $A_2^\vee[1/2]$ which have $\mathsf{h}(S) = 1$ and $k(S) = 2$, so they are flavor roots and form a A_2 system.

- $\{IV; I_0^*, I_1^2\}$ describes a $\Delta = 3$ SCFT with $\mathsf{F} = SU(3)$.

Example 26 ($\{II; III^*, I_1\}$)**.** In this case $R = E_7$, $\Lambda = A_1$ and $\mathsf{MS}(\mathcal{E}) = A_1^\vee$. We have two roots from the two narrow-integral sections. Non narrow integral sections have height 1/2 and level 2, so they do not produce any new root and $\mathsf{F} = SU(2)$.

8.5 Classification

The moduli space of the rational elliptic surfaces is connected; thus all geometries with a given fiber at infinity F_∞ may be obtained as degenerate limits of the "maximally symmetric" geometry $\{F_\infty, I_1^{12-e(F_\infty)}\}$. It is thus important to have a criterion to establish when a geometry should be considered just a special case or limit of a previous one, in which we have simply frozen some mass deformation, and when it corresponds to a "new" geometry describing a different $\mathcal{N} = 2$ QFT. A reasonable criterion is that we have a distinct geometry along a sub-locus $\mathcal{M}' \subset \mathcal{M}$ in moduli

[69] Of course $SU(2) \cong Sp(2)$; however we use write the two factor groups in different ways to emphasize the different role of the two symmetries in the Mordell-Weil group.

space whenever along $\mathcal{M}'$ there are exceptional (-1)-curves associated to flavor roots which are not present away from $\mathcal{M}'$. In other words, "new theories" with the same Δ correspond to loci of enhanced symmetry.

Example 27. Let us consider the family of fiber configurations $\{II; I_b, I_1^{10-b}\}$, of special geometries with $\Delta = 6$. We have

b	1	2	3	4	5	6	7	8	9
R	$-$	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
Λ	E_8	E_7	E_6	D_5	D_4	A_3	$*$	$\langle 8\rangle$	0

A new (-1)-curve with the needed properties arises at $b = 4$ where the "unbroken" subgroup $SO(10) \subset E_6$ get enhanced to $Sp(10) \not\subset E_6$. Then $b = 5, 6$ return to the subgroup symmetry and 7 and 8 to groups like $SU(2) \times U(1)$ and $U(1)$.

The evidence suggests that the above geometric criterion in terms of (-1) curves produces roughly the same restrictions as the physically motivated "Dirac quantization constraint" used by the authors of ref. [19–23]. In fact, the geometric criterion is slightly weaker than the physical one, and this aspect deserves further investigation.

The pattern emerging from the "arithmetic" perspective of the present thesis then essentially agrees with the more direct methods of [19–23].

9 Base change and discrete gaugings

In ref. [23] the non-simply-laced flavor symmetries are understood as a result of the gauging of a discrete symmetry in a parent $\mathcal{N} = 2$ theory. In the arithmetic language this translates into functorial properties under base change [167,191,221]. In Diophantine terms, ungauging the discrete symmetry means passing from the original special geometry (seen as an elliptic curve E over the field $K = \mathbb{C}(u)$) to the special geometry described by the elliptic curve E', defined over a finite-degree extension K' of K. E' is given by the fibered product

$$E' := E \otimes_K K'. \tag{9.1}$$

K' is the function field of some curve C, and the extension from $\mathbb{C}(u)$ to K' arises from a morphisms $f\colon C \to \mathbb{P}^1$. The Kodaira-Néron model of E' is an elliptic surface $\pi\colon \mathcal{E}' \to C$. For our purposes we are interested in the case $C = \mathbb{P}^1$.

Given a rational map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ and a rational elliptic surface $\pi\colon \mathcal{E} \to \mathbb{P}^1$ with section, we may pull-back the elliptic fibration through f producing a new elliptic surface with section, $f^*\mathcal{E}$, not necessarily rational, on which the deck group of f acts by automorphisms.

Suppose our relatively minimal rational elliptic surface $\mathcal{E}$ has an automorphism $\alpha\colon \mathcal{E} \to \mathcal{E}$ which induces the automorphism $\tau\colon \mathbb{P}^1 \to \mathbb{P}^1$ on its base. If $\mathrm{ord}(\alpha) = \mathrm{ord}(\tau) = n$, $\mathcal{E}$ is the pull-back of another relatively minimal rational elliptic surface $\mathcal{E}'$ *via* the map

$$f_n\colon z \to z^n \equiv u \tag{9.2}$$

$\deg f$	5	4	3	2	2	2
$F_\infty^{(1)}$	II	II	II	II	III	IV
$F_\infty^{(2)}$	II^*	IV^*	I_0^*	IV	I_0^*	IV^*

Table 9: Possible fibers at infinity in UV complete base changes.

(we locate the fixed points of τ at 0 and ∞), see **Theorem 5.1.1** of [156].

In the physical terminology, $\mathcal{E}'$ is the rational elliptic surface which describes the special geometry of the QFT obtained by gauging a discrete symmetry $\mathbb{Z}_n$ of the parent QFT associated to $\mathcal{E}$. Table (VI.4.1) of [191] yields the change in fiber type under arbitrary local base changes. Table 6 of [156] lists all possible rational elliptic surfaces which can be obtained as the pull-back of another rational elliptic surface. However not all such coverings are meaningful QFT gaugings, since, in addition, we need to impose UV and SW completeness on the geometries[70].

UV and SW completeness. Let $f\colon z \mapsto z^n$ be a cover inducing a discrete gauging of the special geometry $\mathcal{E}^{(1)}$. The functional invariants of the two geometries $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)} = f^*\mathcal{E}^{(1)}$ are simply related: $\mathscr{J}^{(2)} = f^*\mathscr{J}^{(1)}$. From this relation we read the change in fiber types which affects only the fibers F_0 and F_∞ over the branching points of f in agreement with the local rules of [191]. Semi-simplicity is preserved by base change. Since u is the Coulomb branch coordinate, UV completeness requires

$$\Delta(F_\infty^{(1)}) = \deg f \cdot \Delta(F_\infty^{(2)}). \tag{9.3}$$

For $\deg f > 1$ we have only three possibilities $F_\infty^{(1)} = II, III, IV$. This yields the restrictions in table 9 which should be supplemented by the conditions arising from SW completeness. Comparing with table 5 of [156] we see that the configurations satisfying the criterion are[71]:

- in degree 5 none;
- in degree 4 the single cover $\{IV^*; I_1^4\} \to \{II; III^*, I_1\}$;
- in degree 3 the single cover $\{I_0^*; I_1^6\} \to \{II; IV^*, I_1^2\}$;
- in degree 2 with $F_\infty^{(1)} = II$ there are seven pairs which include as covered surface the types $\{II; I_0^*, I_1^4\}$, $\{II; I_1^*, I_1^3\}$, $\{II; I_1^*, I_3\}$, $\{II; I_2^*, I_1^2\}$, $\{II; I_3^*, I_1\}$;
- in degree 2 with $F_\infty^{(1)} = III$ five pairs which include as covered surface the types $\{III; I_0^*, I_1^3\}$, $\{III; I_1^*, I_1^2\}$, $\{III; I_1^*, I_2\}$, $\{II; I_2^*, I_1\}$;
- in degree 2 with $F_\infty^{(1)} = IV$ a single cover[72] with $\{IV^*, I_1^4\} \to \{IV; I_0^*, I_1^2\}$.

[70] And possibly "Dirac quantization".
[71] For brevity we list only the covered types which satisfy the "Dirac quantization" condition.
[72] The type $\{IV; I_1^*, I_1\}$ admits a double cover of type $\{IV^2, I_2^2\}$ which does not satisfy the SW completeness criterion.

For simplicity in the rest of this section we focus on the first cover in each of the above items (they are the more interesting anyhow). They have the property that the fiber $F_0^{(2)} \equiv I_0$ is smooth and hence $F_0^{(1)} \in \text{additive}^* \cap \text{semi-simple}$. For the five coverings we have respectively,

$$F_0^{(1)} = III^*,\ IV^*,\ I_0^*,\ I_0^*,\ I_0^*. \tag{9.4}$$

In each case $F_0^{(1)}$ is the only reducible fiber over $u \neq \infty$.

Lemma 9.0.1. *Let S be an integral non-narrow section, narrow at ∞, of an elliptic surface which is the base of one of the above 5 coverings $\mathcal{E}^{(2)} \to \mathcal{E}^{(1)}$. One has*

$$\mathsf{h}(S) = \frac{2}{\deg f}. \tag{9.5}$$

Moreover $k(S) = \deg f$, except for the first degree-4 cover where $k(S) = \deg f/2$.

Remark 9.0.1. The first case corresponds to **Example 26** which does not present peculiarities.

9.1 Functoriality under base change

Base change yields a commuting diagram

$$\begin{array}{ccc} \mathcal{E}_2 & \overset{\mathcal{F}}{\dashrightarrow} & \mathcal{E}_1 \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{P}^1 & \underset{f}{\longrightarrow} & \mathbb{P}^1 \end{array} \tag{9.6}$$

where $\mathcal{F}$ is a rational map. Base change (9.1) induces a map of Mordell-Weil groups

$$f^\sharp \colon \mathsf{MW}(\mathcal{E}_1) \to \mathsf{MW}(\mathcal{E}_2). \tag{9.7}$$

At the level of divisors $f^\sharp S$ is the closure of $\mathcal{F}^* S$. The Kodaira formula yields

$$\mathcal{F}^* K_{\mathcal{E}^{(1)}} = \deg f \cdot K_{\mathcal{E}^{(2)}} \tag{9.8}$$

Since $S_0^{(2)} = f^\sharp S_0^{(1)}$, $f^\sharp$ maps integral sections into integral sections (as expected from the Number Theoretic analogy). One has [221]

$$\langle f^\sharp S, f^\sharp S' \rangle_{\text{NT}} = \deg f \cdot \langle S, S' \rangle_{\text{NT}}, \tag{9.9}$$

so the pull-back of a narrow-integral section has height $2 \deg f$.

Conversely, let $S \in \mathsf{MS}(\mathcal{E}_1)$ be an integral section with $\deg f \cdot \mathsf{h}(S) = 2$. Its pull-back $f^\sharp S$ would be an integral section on $\mathcal{E}^{(2)}$ of Néron-Tate height 2, that is, an integral-narrow section associated to an E_8-root curve in good position.

Comparing with **Lemma 9.0.1** we see that in these examples the root system $\Xi_\infty(\mathcal{E}^{(1)})$ is composed by elements which either are associated to E_8-root curves in good position on $\mathcal{E}^{(1)}$ or such that there is a cover under which they become associated to E_8-root curves in good position. There are rare situations in which the full set of elements of Λ whose pull-back is associated to an

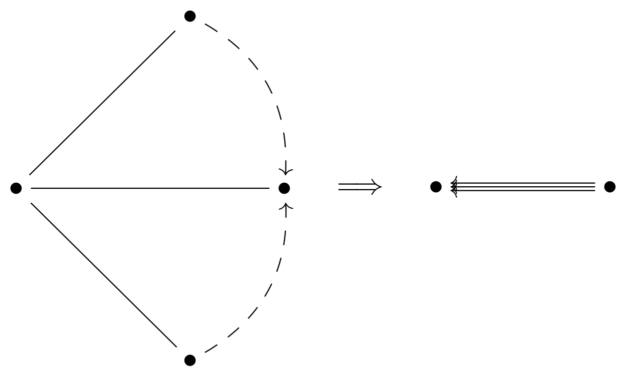


Figure 1: The G_2 Dynkin graph as a folding of the D_4 one.

E_8-root curve is a *non-reduced* root system (see **Example 26**). Our prescription of considering the minimal level instead of the degree of the cover reduces the root system to the correct one.

9.2 Explicit examples

We conclude with a couple of explicit examples.

Example 28. We consider the $\Delta = 6$ QFT with the non-simply-laced flavor group G_2, already discussed in **Exercise 14** from the point of view of the Mordell-Weil root system. The Dynkin graph of G_2 is obtained from the one of D_4 by folding it, that is, by taking the quotient by the cyclic subgroup $\mathbb{Z}/3\mathbb{Z}$ of its automorphism group $\mathfrak{S}_3$, see figure 1. One expects that the G_2 model is a $\mathbb{Z}/3\mathbb{Z}$ gauging of a model with $D_4 \rtimes \mathbb{Z}/3\mathbb{Z}$ flavor symmetry. The special geometry of the parent QFT should be the pull-back by the cyclic cover $z \mapsto z^3$ of the G_2 one. Let us check this idea by explicitly constructing the two geometries.

For $a, b \in \mathbb{C}$, let A be a root of the quadratic equation

$$A^2 + 2(a+b)A + (a-b)^2 = 0, \tag{9.10}$$

and set $c = (A + a + b)/2 = \pm\sqrt{ab}$. Consider the two rational functions

$$\mathscr{J}_1(z) = A\,\frac{z}{(z-a)(z-b)} = 1 - \frac{(z-c)^2}{(z-a)(z-b)} \tag{9.11}$$

$$\mathscr{J}_2(w) = A\,\frac{w^3}{(w^3-a)(w^3-b)} = \mathscr{J}_1(w^3). \tag{9.12}$$

Clearly, they are related by the base change $w \to z = w^3$ branched over $w = 0, \infty$.

The function $\mathscr{J}_2(w)$ describes a rational elliptic surface of type $\{I_0^*; I_1^6\}$ with the fiber at infinity of type I_0^* such that $\mathscr{J}(\infty) = \mathscr{J}(0) = 0$ while the pole form two orbits under the $\mathbb{Z}/3\mathbb{Z}$ group $w \mapsto e^{2\pi i/3}w$. Therefore, $\mathscr{J}_2(w)$ describes a very special point in the moduli space of of $SU(2)$ SQCD with $N_f = 4$ where $\tau = e^{2\pi i/3}$ and the hyper masses are invariant under a $\mathbb{Z}/3\mathbb{Z}$ symmetry. w is a global coordinate on the $SU(2)$ Coulomb branch and has dimension $\Delta = 2$. The monodromy at infinity corresponds to $w \mapsto e^{2\pi i}w$, and is $m(I_0^*) \equiv -1 \in SL(2, \mathbb{Z})$.

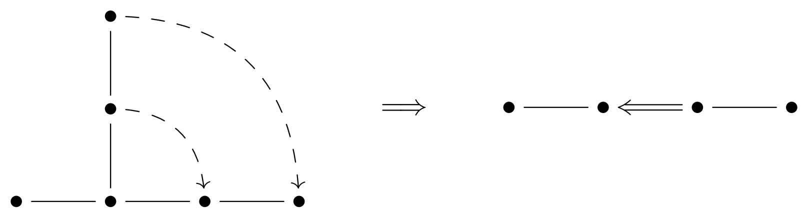


Figure 2: Diagram folding $E_6 \to F_4$.

The function $\mathscr{J}_1(z)$ has two zeros of order 1 and two simple poles. It describes a rational elliptic surface of type $\{II; IV^*, I_1^2\}$; the additive° fiber II should be at infinity (cfr. (8.3.1)), so this function describes a special geometry with $\Delta = 6$. This is obvious, since $z \equiv w^3$ has dimension $3 \cdot 2 = 6$ while $w \to e^{2\pi i} w$ is equivalent to $z \to e^{6\pi i} z$, that is, the two monodromies at infinity are related as $M_2 = M_1^3$, which corresponds to the identity $M(II)^3 = M(I_0^*)$. The fiber at the second branch point of the cover, zero, is IV^* and again $M(IV^*)^3 = M(I_0) = 1$.

Since the covering theory has $SO(8) \rtimes \mathbb{Z}/3\mathbb{Z}$ symmetry and the covered one a G_2 flavor symmetry and the deck group is $\mathbb{Z}/3\mathbb{Z}$, this geometry precisely corresponds to the diagram folding of figure 1.

We note that

$$Z(\mathcal{E}_1)/Z(\mathcal{E}_1)_\infty = \mathbb{Z}/3\mathbb{Z}, \qquad Z(\mathcal{E}_2)/Z(\mathcal{E}_2)_\infty = 0. \tag{9.13}$$

In the G_2 geometry the group $\mathbb{Z}/3\mathbb{Z}$ acts on the sub-group of sections narrow at infinity, the trivial representation corresponding to the subgroup of narrow sections.

$$f^* \colon \mathsf{MW}(\mathcal{E}_1) \to \mathsf{MW}(\mathcal{E}_2)$$

Example 29. We consider the rational elliptic surface of type $\{II; I_0^*, I_1^4\}$ which describes a (mass deformed) $\Delta = 6$ SCFT with $\mathsf{F} = F_4$. Its functional invariant has the form

$$\mathscr{J}_1(z) = A \, \frac{(z-b)^3}{\prod_i (z - a_i)}. \tag{9.14}$$

Writing $z = w^2$ we get on the double cover a function $\mathscr{J}_2(w)$ corresponding to a surface of fiber type $\{IV; I_1^8\}$, that is, the $\Delta = 3$ model with $\mathsf{F} = E_6$ at a certain $\mathbb{Z}/2\mathbb{Z}$ symmetric point. The corresponding diagram folding is represented in figure 2.

Part IV

Coarse classification of the SW geometries in all ranks

10 The Coulomb dimensions Δ_i

Now we come to the main focus of this part of the thesis, namely to get geometric restrictions on the spectra of Coulomb branch dimensions Δ_i for any rank.

We may address two different problems:

Problem 1. *For $k \in \mathbb{N}$ specify the set*

$$\Xi(k) = \left\{ \Delta \in \mathbb{Q}_{\geq 1} \;\; \begin{array}{l} \textit{such that: there is a CSG } M \\ \textit{with } \dim M \leq k \textit{ and a generator} \\ u \textit{ of } \mathscr{R} \equiv \mathbb{C}[M] \textit{ with } \pounds_{\mathcal{E}} u = \Delta\, u \end{array} \right\} \tag{10.1}$$

Problem 2. *For $k \in \mathbb{N}$ determine the set*

$$\Lambda(k) = \left\{ \Big(\Delta_1, \Delta_2, \cdots, \Delta_k\Big) \in \big(\mathbb{Q}_{\geq 1}\big)^k \;\; \begin{array}{l} \textit{such that: there is a CSG } M \textit{ with} \\ \mathscr{R} \cong \mathbb{C}[u_1, u_2, \cdots, u_k],\;\; \pounds_{\mathcal{E}} u_i = \Delta_i\, u_i \end{array} \right\} \tag{10.2}$$

The solution to the **Problem 2** contains vastly more information than the answer to **Problem 1**, since there are strong correlations between the dimensions Δ_i of a given CSG and $\Lambda(k)$ is a rather small subset of $\Xi(k)^k$. However **Problem 1** is much simpler, and its analysis is a first step in answering **Problem 2**. We give a solution in the form of a necessary condition: $\Xi(k) \subset \widehat{\Xi}(k)$, where $\widehat{\Xi}(k)$ is a simple explicit set of rational numbers. There are reasons to believe that the discrepancy between the two sets $\Xi(k)$ and $\widehat{\Xi}(k)$ is small and vanishes as $k \to \infty$. In facts $\Xi(k) = \widehat{\Xi}(k)$ for $k = 1, 2$, and the equality may hold for all k.

To orient our ideas, we discuss a few special instances as a warm-up for the general case.

10.1 Warm-up: revisiting some well-understood cases

10.1.1 Rank-one again

We know that

$$\Xi(1) \equiv \Lambda(1) = \Big\{1, 2, 3, 3/2, 4, 4/3, 6, 6/5\Big\}. \tag{10.3}$$

where 1 corresponds to the free theory and 2 to a $SU(2)$ (Lagrangian) SCFT. We have already obtained this result from several points of view. In rank-1 the period map is automatically constant and we have $M \cong \mathbb{C}/\mathcal{G}$, with $\mathcal{G}$ a rank-1 modular ST group. However, for a complex $\mathcal{G}$, there are two possible values of Δ, as we stressed already several times. This correspond to the fact that, in this case, we have two distinct embeddings $m\colon \mathcal{G} \hookrightarrow SL(2, \mathbb{Z})$ modulo conjugacy.

A rank-1 ST group is cyclic. Let σ be a generator and consider

$$m(\sigma) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}) \text{ elliptic}, \quad \tau \in \mathfrak{h} \text{ a fixed point } a\tau + b = \tau(c\tau + d). \tag{10.4}$$

The explicit matrices $m(\sigma)$ corresponding to the 8 elliptic conjugacy classes in $SL(2, \mathbb{Z})$ are written in the first column of Table I of Kodaira [166] (omitting the non semi-simple ones). Under the

fiber type	Δ	modular factor	fiber type	Δ	modular factor
regular	1	1	I_0^*	2	$e^{\pi i}$
II	6/5	$e^{5\pi i/3}$	II^*	6	$e^{\pi i/3}$
III	4/3	$e^{3\pi i/2}$	III^*	4	$e^{\pi i/2}$
IV	3/2	$e^{4\pi i/3}$	IV^*	3	$e^{2\pi i/3}$

Table 10: Modular factors for the Kodaira fibers with semi-simple holonomy invariant.

action of σ, the period a transforms with the modular factor

$$\sigma\colon a \to a' = (c\tau + d)a \quad \text{while} \quad \sigma\colon u \to e^{2\pi i}u, \tag{10.5}$$

since u is an univalued function. $(c\tau + d)$ is a character of the cyclic group $\mathcal{G}$, hence a root of unity. Let α be the unique real number $0 < \alpha \leq 1$ such that $e^{2\pi i\alpha} = c\tau + d$. Clearly the only functional dependence $a = a(u)$ consistent with (10.5) such that $a \to 0$ as $u \to 0$ (the tip of the cone) is

$$a = u^\alpha \quad \Longrightarrow \quad \Delta(u) = \frac{1}{\alpha} \geq 1. \tag{10.6}$$

Using the matrices in Table I in [166] one recovers the well-known result, see table 10.

Our basic strategy is to mimic this analysis of the $k = 1$ for general k. Before doing that, we discuss another special case in two different ways: first we review the conventional approach and then recover the same results by reducing the analyis to the $k = 1$ case. This will give the first concrete example of the basic strategy of the present thesis.

10.1.2 Hypersurface singularities in F-theory and 4d/2d correspondence

There is a special class of simple (typically non-Lagrangian) 4d $\mathcal{N} = 2$ SCFTs which are engineered in F-theory out of an isolated quasi-homogeneous dimension-3 hypersurface singularity with $\hat{c} < 2$ (see eqn.(10.8)) [60, 141]. Their SW geometry (in absence of mass deformations) is given by the hypersurface $\mathcal{F} \subset \mathbb{C}^4$ of equation

$$F(x_1, x_2, x_3, x_4) \equiv W(x_1, x_2, x_3, x_4) + \sum_{a=0}^{k-1} u_a\, \phi_a(x_i) = 0, \tag{10.7}$$

where $W(x_i)$ is a quasi-homogeneous polynomial

$$\begin{aligned} &W(\lambda^{q_i} x_i) = \lambda\, W(x_i), \quad \forall\, \lambda \in \mathbb{C}, \\ &\text{with } q_i \equiv \deg x_i \in \mathbb{Q} \text{ and } \hat{c} \equiv \sum_{i=1}^{4} (1 - 2q_i) < 2, \end{aligned} \tag{10.8}$$

while $\phi_a(x_i)$ are the elements of the (2,2) chiral ring $\mathcal{R} \equiv \mathbb{C}[x_i]/(\partial_i W)$ having degree $< \hat{c}/2$. In particular $\phi_0 \equiv 1$ is the identity operator. The number of ϕ_a's with $\deg \phi_a < \hat{c}/2$ is equal to the rank k of the corresponding 4d $\mathcal{N}=2$ SCFT and the complex parameters u_a in eqn.(10.7) are the Coulomb branch (global) homogeneous coordinates. One has

$$\deg u_a = 1 - \deg \phi_a \qquad \text{where } 0 \leq \deg \phi_a < \hat{c}/2 < 1. \tag{10.9}$$

The regularity condition of ref.[141], $\hat{c} < 2$, ensures that the u_a's have positive degree. The SW 3-form is the obvious one

$$\Omega = PR\left(\frac{dx_1 \wedge \cdots \wedge dx_4}{F}\right), \tag{10.10}$$

where PR stands for "Poincaré residue" [139]. At the conformal point, $u_a = 0$, Ω has degree

$$\sum_{i=1}^{4} q_i - 1 = 1 - \frac{\hat{c}}{2} > 0. \tag{10.11}$$

Since Ω (by definition) has $U(1)_R$ charge q equal 1, for all chiral object ϕ with a definite degree, we have

$$\Delta(\phi) \equiv q(\phi) = \frac{\deg \phi}{1 - \hat{c}/2}, \tag{10.12}$$

in particular,

$$\Delta(u_a) = \frac{1 - \deg \phi_a}{1 - \hat{c}/2} \equiv 1 + \frac{\hat{c}/2 - \deg \phi_a}{1 - \hat{c}/2}. \tag{10.13}$$

To get the Coulomb branch dimensions $\Delta(u_a)$, it remains to determine the k rational numbers

$$t_a \equiv \frac{\hat{c}}{2} - \deg \phi_a, \quad a = 0, 1, \dots, k-1, \quad \text{in particular } t_0 = \frac{\hat{c}}{2}. \tag{10.14}$$

There are many ways of doing this, including pretty trivial ones. Here we shall compute the t_a's in a way that seems unnaturally complicate: but recall that we are doing this computation as a warm-up, meaning that we wish to perform this elementary computation in a way which extends straightforwardly to the general case where simple minded methods fail.

Picard-Lefschetz analysis [61]. One convenient viewpoint is the 4d/2d correspondence of ref.[60]. One considers the 2d (2,2) Landau-Ginzburg model with superpotential $F(x_i)$ and uses the techniques of tt^* geometry [62,63] to compute 2d quantities which are then reinterpreted in the 4d language. In 2d $\hat{c}$ is one-third the Virasoro central charges and $\deg \phi$ is the R-charge (in the 2d sense) of the chiral object ϕ.

In the 2d approach, the t_a's are computed using the Picard-Lefschetz theory [28] (see [63] for a survey in the present language). Consider the family of hypersurfaces $\mathcal{F}_z = \{F(x_i) = z\}$ parametrized by $z \in \mathbb{C}$; we have $H_3(\mathcal{F}_z, \mathbb{Z}) \cong \mathbb{Z}^\mu$, where μ is the Milnor number of the singularity $W = 0$ (i.e. the dimension of the (2,2) chiral ring $\mathcal{R}$). The *classical monodromy* H of the quasi-homogeneous singularity W is given by the lift on the homology of the fiber, $H_3(\mathcal{F}_z, \mathbb{Z})$, along the closed loop in the base $z = \rho\, e^{2\pi i t}$ ($t \in [0,1]$ and $\rho \gg 1$). Concretely, H is a $\mu \times \mu$ integral matrix acting on the lattice $\mathbb{Z}^\mu$ whose action on $\mathbb{C}^\mu \equiv \mathbb{Z}^\mu \otimes \mathbb{C}$ is semi-simple of spectral radius 1 [63]. Let $\Phi \subset \mathbb{Z}^\mu$ be the sublattice fixed (element-wise) by H, and consider the quotient lattice $\Gamma = \mathbb{Z}^m/\Phi$ which has rank $2k$. H induces an automorphism $\overline{H}$ of Γ. The intersection form in the homology of the hypersurface $\mathcal{F}_z$, induces a non-degenerate, integral, skew-symmetric pairing $\langle -,-\rangle$ on Γ, preserved by $\overline{H}$. Then

$$\overline{H} \in Sp(2k,\mathbb{Q}) \tag{10.15}$$

In simple examples the induced polarization $\langle -,-\rangle$ is principal, and one has $\overline{H} \in Sp(2k,\mathbb{Z})$; in the general case we reduce to this situation by a suitable isogeny in the intermediate Jacobian of $\mathcal{F}_z$. Moreover, $\overline{H}$ is semi-simple of spectral radius 1 so (by Kronecker's theorem) it has a finite order ℓ. Its eigenvalues are of the form $\{\exp(2\pi i\alpha_a), \exp(2\pi i(1-\alpha_a))\}$ for some $0<\alpha_a\leq 1$, $a=0,1,\dots,k-1$ with $\ell\alpha_a \in \mathbb{N}$. These eigenvalues are related to the t_a by the 2d spectral-flow relation [63]

$$\mathsf{Spectrum}\,\overline{H} = \Big\{e^{2\pi i\alpha_a}, e^{2\pi i(1-\alpha_a)}\Big\} \equiv \Big\{e^{\pm 2\pi i t_a}\Big\}. \tag{10.16}$$

Since $0<t_a<\hat{c}/2<1$, for each index $a=0,1,\dots,k-1$ we have two possibilities

$$t_a = \alpha_a \;\; \text{or} \;\; 1-\alpha_a, \tag{10.17}$$

where we relabel the indices of t_a so that $t_0=\max t_a$. Thus knowing the spectrum of $\overline{H}$ fixes the t_a's up to a 2^k-fold ambiguity corresponding to choosing for each a one of the two possible values (10.17). Let us explain the origin of this ambiguity. For simplicity of illustration we assume the characteristic polynomial $P(z)$ of $\overline{H}$ to be irreducible over $\mathbb{Q}$; in this case the spectrum uniquely fixes $\overline{H}$ up to conjugacy in $GL(2k,\mathbb{Q})$. However two physical systems described by monodromy matrices $\overline{H}_1, \overline{H}_2 \in Sp(2k,\mathbb{Z})$ are physically equivalent iff they are related by a change of duality frame i.e. iff the corresponding reduced monodromies $\overline{H}_1, \overline{H}_2$ are conjugate in the smaller group $Sp(2k,\mathbb{Z})$: the unique $GL(2k,\mathbb{Q})$-conjugacy class of $\overline{H}$ decomposes[73] into 2^k distinct $Sp(2k,\mathbb{Q})$-conjugacy classes in one-to-one correspondence with the inequivalent choices of the t_a's. (The conjugacy classes over the integral group, $Sp(2k,\mathbb{Z})$, are trickier, and will be discussed in section 5). The Picard-Lefschetz theory has a canonical symplectic structure (i.e. the intersection form in homology) and hence a canonical choice of the $\{t_a\}$. The spectrum of Coulomb branch dimension for the SCFT engineered by the singularity is given by plugging these canonical $\{t_a\}$ in the expressions

$$\Big\{\Delta(u_a)\Big\} = \left\{1+\frac{t_a}{1-t_0}\right\}, \quad \text{where } t_0 = \max\{t_a\}. \tag{10.18}$$

The ray analysis. Let us rephrase the above Picard-Lefschetz analysis in a different language. We return to eqn.(10.7) and consider the Coulomb branch M of the associated $\mathcal{N}=2$ SCFT; we see M as the affine cone over the WPS with homogeneous coordinates $(u_0,u_1,\cdots,u_{k-1})$ the coordinate u_a having grade $\Delta(u_a)$. We focus on the closed one-dimensional sub-cone M_0 parametrized by the VEV u_0 of the chiral operator of largest dimension $\Delta(u_0)$,

$$M_0 = \Big\{u_1 = u_2 = \cdots = u_{k-1}\Big\} \subset M, \tag{10.19}$$

[73] Since $P(z)$ is irreducible, $\mathbb{Q}[\overline{H}]$ is an Abelian number field $\mathbb{K}$ of degree $2k$. Let $\Bbbk$ be its maximal real subfield (of degree k); the elements $\xi\in\Bbbk$ may be written as the rational matrices $\xi(\overline{H}) \equiv \sum_s a_s(\overline{H}+\overline{H}^{-1})^s$ with $a_s\in\mathbb{Q}$. Then the map

$$\Omega \to \Omega' = \Omega\,\xi(\overline{H})$$

identifies the set of inequivalent $\mathbb{Q}$-symplectic structures compatible with H with the multiplicative group $\Bbbk^\times/\Bbbk^\times_{\text{tot.pos.}} \cong \mathbb{Z}_2^k$. See section 5 for more details.

which is preserved by the $\mathbb{C}^{\times}$-action generated by the Euler field $\mathcal{E}$. Metrically, M_0 is a flat cone and the restricted period map $\boldsymbol{\tau}|_{M_0}$ is constant. The only difference with respect to the rank-one case of §. 10.1.1 is that the monodromy $\overline{H}$ around the tip of the cone M_0 is now valued in $Sp(2k,\mathbb{Z})$ instead of $Sp(2,\mathbb{Z})$.

A Coulomb vacuum $x \in M_0 \subset M$ preserves a discrete subgroup of $U(1)_R$ of the form $\mathbb{Z}_n$ where n is the order of $1/\Delta(u_0)$ in $\mathbb{Q}/\mathbb{Z}$. Since in an interacting unitary theory $\Delta(u_0) > 1$, this subgroup is never trivial, and $n > 2$ (i.e. the unbroken subgroup is complex) if $\Delta(u_0) \neq 2$ (i.e. unless u_0 is superficially marginal). The $\mathbb{Z}_n$ symmetry unbroken along the locus $M_0 \subset M$ is generated by

$$\exp\big(2\pi i R/\Delta(u_0)\big), \tag{10.20}$$

and the action of this operator on the cohomology of a (regular) fiber of the special geometry is given by $\overline{H}\, e^{2\pi i(1-\hat{c}/2)}$, the extra factor $e^{2\pi i(1-\hat{c}/2)}$ corresponding to the spectral flow in the (2,2) language [63]. Thus

$$\begin{aligned}&\mathsf{Spectrum}\,\overline{H} = \Big\{\exp\Big[\pm 2\pi i\Big(\hat{c}/2 - 1 + \Delta(u_a)/\Delta(u_0)\Big)\Big]\Big\}\\ &\Rightarrow\quad t_a = \hat{c}/2 - 1 + \Delta(u_a)/\Delta(u_0), \qquad\qquad t_0 = \hat{c}/2,\\ &\text{and } t_a = (1-t_0)(\Delta(u_a)-1),\end{aligned} \tag{10.21}$$

which gives back (10.18). Thus, in the context of 4d $\mathcal{N} = 2$ SCFTs engineered by hypersurface singularities, the classical monodromy computation of the dimensions $\Delta(u_a)$ may be rephrased as a local analysis along the sub-locus M_0 in the Coulomb branch M of vacua which leave unbroken the largest possible discrete subgroup of $U(1)_R$. The local analysis on M_0 is essentially identical to the $k = 1$ case.

The classical monodromy approach is unsatisfactory in two ways:

a) it works for a particular class of SCFTs;

b) is not *democratic* in the following sense: we have many sub-loci in the Coulomb branch M over which some discrete R-symmetry is restored, but the classical monodromy (when applicable) applies to just one of them (the one with the largest unbroken symmetry).

In order to solve **Problems 1, 2** we need a generalization of the classical monodromy construction which may be applied uniformly to all loci in the Coulomb branch with some unbroken discrete R-symmetry, and to all SCFT, while reducing to the classical Picard-Lefschetz theory when we consider the locus of largest unbroken $U(1)_R$ symmetry of a SCFT engineered by a F-theory singularity.

Suppose such a generalization exists. Along the Coulomb branch of a typical SCFT we have several loci with enhanced (discrete) R-symmetry; each such locus produces a list of Δ_a. Then we get the highly non-trivial constraint that the dimension set $\{\Delta_a\}$ should be the same independently of which special locus we use to compute it. On the other hand, the agreement of the dimensions computed along different loci in M is convincing evidence of the correctness of the method.

In the remaining part of this note we describe the generalized method, and check its consistency is a variety of examples. Although we could present the algorithm already at this stage in the form of an educated guess inspired by the classical Picard-Lefschetz formulae, we prefer to deduce it mathematically from scratch. Before doing that, we need some elementary preparation.

10.2 Cyclic subgroups of Siegel modular groups I

We saw already in the $k=1$ case that an important ingredient in the classification of all possible dimension sets $\{\Delta_a\}$ is the list of all embeddings of the cyclic group $\mathbb{Z}_n$ (or more generally of a finite group G) into the Siegel modular group $Sp(2k,\mathbb{Z})$ modulo symplectic conjugacy in $Sp(2k,\mathbb{Z})$ (cfr. discussion around eqn.(10.17)).

We start by establishing a fact, already mentioned in §.6.2, which applies to all subgroups of the Siegel modular group, cyclic or otherwise.

Lemma 10.2.1. *Let $\mathcal{G}\subset Sp(2k,\mathbb{Z})$ be a finite subgroup. The fixed locus in the Siegel upper half-space $\mathfrak{H}_k\equiv\{\boldsymbol{\tau}\in\mathbb{C}(k)\,|\,\boldsymbol{\tau}=\boldsymbol{\tau}^t,\ \text{Im}\,\boldsymbol{\tau}>0\}$*

$$\text{Fix}(\mathcal{G})=\Big\{\boldsymbol{\tau}\in\mathfrak{H}_k\cong Sp(2k,\mathbb{R})/U(k)\;\Big|\;g\cdot\boldsymbol{\tau}=\boldsymbol{\tau},\ \ \forall\, g\in\mathcal{G}\Big\} \tag{10.22}$$

is non-empty and connected.

Remark 10.2.1. The proof shows that the **Lemma** holds for all duality-frame groups $S(\Omega)_\mathbb{Z}$.

Proof. Being finite, $\mathcal{G}$ is compact. Hence $\mathcal{G}\subset K$ for some maximal compact subgroup $K\subset Sp(2k,\mathbb{R})$. All maximal compact subgroups in $Sp(2k,\mathbb{R})$ are conjugate to the standard one, the isotropy group of $i\mathbf{1}_k\in\mathfrak{H}_k$, that is, there is $R\in Sp(2k,\mathbb{R})$

$$K=\left\{R^{-1}\begin{bmatrix}A & B\\ C & D\end{bmatrix}R\;\middle|\;Ai+B=i(Ci+D)\in U(k)\right\}. \tag{10.23}$$

The Cayley transformation C maps biholomorphically the Siegel upper half-space $\mathfrak{H}_k$ into the Siegel disk [49, 57]

$$\mathfrak{D}_k=\big\{\boldsymbol{w}\in\mathbb{C}(k)\;\big|\;\boldsymbol{w}^t=\boldsymbol{w},\ 1-\boldsymbol{w}\boldsymbol{w}^*>0\big\}, \tag{10.24}$$

taking $\boldsymbol{\tau}=i\mathbf{1}_k$ to the origin $\boldsymbol{w}=0$ and conjugating the standard maximal compact subgroup into the diagonal subgroup. Then

$$CR\mathcal{G}R^{-1}C^{-1}\subset\left\{\begin{bmatrix}U & 0\\ 0 & U^*\end{bmatrix},\quad U\in U(k)\right\}. \tag{10.25}$$

Consider the embedding $\mathcal{U}\colon\mathcal{G}\hookrightarrow U(k)$ sending g into the upper-left block of $CRgR^{-1}C^{-1}$; we write V for the corresponding degree-k unitary representation. The action of $\mathcal{G}$ on the Siegel disk $\mathfrak{D}_k$ is linear

$$\boldsymbol{w}\mapsto\mathcal{U}(g)\,\boldsymbol{w}\,\mathcal{U}(g)^t,\qquad g\in\mathcal{G}, \tag{10.26}$$

i.e. the Cayley-rotated period $\boldsymbol{w}$ transforms in the symmetric square representation $\odot^2V$. The fixed locus of $\mathcal{G}$ is the intersection of $\mathfrak{D}_k\subset\odot^2V$ with the linear subspace

$$(\odot^2V)^{\mathcal{U}(\mathcal{G})}\subset\odot^2V \tag{10.27}$$

of trivial representations whose dimension d is as in eqn.(10.40); in particular Fix($\mathcal{G}$) is non-empty and connected. In the special case that $\odot^2 V$ does not contain the trivial representation, the fixed locus reduces to the origin in $\mathfrak{D}_k$, and hence is an isolated point. Mapping back to the Siegel upper half-space $\mathfrak{H}_k$, the fixed point is

$$\boldsymbol{\tau} = R^{-1}(i\,\mathbf{1}_k) \equiv (iA_R + B_R)(iC_R + D_R)^{-1}, \qquad R^{-1} \equiv \begin{bmatrix} A_R & B_R \\ C_R & D_R \end{bmatrix} \in Sp(2k, \mathbb{R}). \tag{10.28}$$

In the general case

$$\text{Fix}(\mathcal{G}) = R^{-1}C^{-1}\big((\odot^2 V)^{\mathcal{U}(\mathcal{G})}\big) \cap \mathfrak{H}_k. \tag{10.29}$$

□

If $\boldsymbol{\tau} \in \text{Fix}(\mathcal{G})$ we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \boldsymbol{\tau} & \bar{\boldsymbol{\tau}} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau} & \bar{\boldsymbol{\tau}} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C\boldsymbol{\tau}+D & 0 \\ 0 & C\bar{\boldsymbol{\tau}}+D \end{bmatrix}, \quad \forall \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{G} \subset Sp(2k, \mathbb{Z}), \tag{10.30}$$

and the embedding $\mathcal{U}\colon \mathcal{G} \hookrightarrow U(k)$ is given by the modular factor $C\boldsymbol{\tau}+D \in U(k)$, cfr. the discussion of the $k=1$ case around eqn.(10.4). This is the same factor appearing in the transformation of the a-periods, $a \to (C\boldsymbol{\tau}+D)a$, and is the one which controls the Coulomb dimensions, as we saw in the $k=1$ case. Implicitly we have already used these facts in the discussion of the CSG with constant period map, §. 6.2.

Let us specialize to the case in which $\mathcal{G}$ is a cyclic group $\mathbb{Z}_n$ generated by a matrix $m \in Sp(2k, \mathbb{Z})$. m is called *regular* iff Fix(m) is an isolated point, i.e. iff $\odot^2 V$ does not contain the trivial representation. Note that this is a weaker notion than m being regular as an element of the Lie group $Sp(2k, \mathbb{R})$ which requires the characteristic polynomial of m to be square-free; we shall refer to the last situation as *strongly regular*. The spectrum of the unitary matrix $\mathcal{U}(m) = C\boldsymbol{\tau}+D$ is a set of k n-th roots of unity, $\{\zeta_1, \dots, \zeta_k\}$, and m is regular iff

$$\zeta_i\zeta_j \neq 1 \quad \text{for all} \quad 1 \leq i, j \leq k. \tag{10.31}$$

The eigenvalues of the $2k \times 2k$ matrix m are $\{\zeta_1, \dots, \zeta_k\} \cup \{\zeta_1^{-1}, \dots, \zeta_k^{-1}\}$ the union being disjoint iff m is regular. Let ψ_ζ be the normalized eigenvector of m associated to the eigenvalue ζ. The symplectic matrix Ω of $Sp(2k, \mathbb{Z})$ corresponds to the 2-form

$$\Omega = i\sum_i \psi_{\zeta_i} \wedge \psi_{\zeta_i^{-1}} \tag{10.32}$$

Thus, for m regular, the splitting of the spectrum of m in the two disjoint sets $\mathsf{Spectrum}\,\mathcal{U}(m)$ and $\mathsf{Spectrum}\,\overline{\mathcal{U}(m)}$ may be read directly from Ω: the eigenvalue ζ belongs to the spectrum of $\mathcal{U}(m)$ iff the term $\psi_\zeta \wedge \psi_{\zeta^{-1}}$ appears in $-i\Omega$ with the $+$ sign, otherwise $\zeta^{-1} \in \mathsf{Spectrum}\,\mathcal{U}(m)$.

We shall present a much more detailed discussion of the cyclic subgroups of the Siegel modular group in section 5 below.

10.3 The Universal Dimension Formula

10.3.1 Normal complex rays in CSG

As anticipated at the end of §.10.1.1, our goal is to reduce the determination of the Δ_a's to the analysis of *one-dimensional* conic complex geometries. In this subsection we introduce the basic construction. We start with a definition:

Definition 14. Let M be the Kähler cone of a CSG with holomorphic Euler vector $\mathcal{E} = (E - iR)/2$. A *complex ray* $M_* \subset M$ is a closed one-dimensional complex subspace preserved by the action of $\mathcal{E}$. A complex ray $M_* \subset M$ is called *normal* iff it is normal as an analytic subspace of M.

Thus, a complex ray M_* is the orbit under the Lie group generated by E,R of a point $x \neq 0$ in M. Equipped with the induced metric, M_* is a Kähler cone, hence locally flat. Restricting our considerations to a ray makes physical sense; indeed

Proposition 10.3.1. *Let M be a Kähler cone and $M_* \subset M$ a complex ray. Then M_* is* totally geodesic in M.

Proof. Since M is a Kähler cone, we have the holomorphic field $\mathcal{E} \equiv E + iR$ such that $\nabla_{\bar{i}}\mathcal{E}^j = 0$ while $\nabla_i \mathcal{E}^j = \delta_i^j$. With respect to the induced Kähler metric, M_* is also a Kählerian cone with a holomorphic Euler vector $\mathcal{E}_* = \mathcal{E}|_{M_*}$. $\mathcal{E}_*$, $\bar{\mathcal{E}}_*$ span the real tangent bundle TM_*. Let ∇^* be the Levi-Civita connection of the induced metric on M_*. The second fundamental form of M_* in M is

$$\begin{aligned} &I\!I(\mathcal{E}_*, \mathcal{E}_*) = \nabla_{\mathcal{E}}\mathcal{E}\big|_{M_*} - \nabla^*_{\mathcal{E}_*}\mathcal{E}_* = \mathcal{E}\big|_{M_*} - \mathcal{E}_* = 0 \\ &\text{then also} \quad I\!I(\mathcal{E}_*, \bar{\mathcal{E}}_*) = I\!I(\bar{\mathcal{E}}_*, \bar{\mathcal{E}}_*) = 0. \end{aligned} \tag{10.33}$$

□

The chiral ring of M is the ring of global homomorphic functions, $\mathscr{R} = \Gamma(M, \mathcal{O}_M)$. The linear differential operator $\mathcal{E}$ acts on $\mathscr{R}$ and we write S for its spectrum, i.e. the set of dimensions of chiral operators. $\mathscr{R}$ is the Frechét completion of a finitely generated graded ring. If $\phi \in \mathscr{R}$ is a homogeneous element of degree $\Delta(\phi)$, we have $\mathcal{E}\phi = \Delta(\phi)\phi$. In the same way we write $\mathscr{R}_*$ for the ring of holomorphic functions on the ray M_* and S_* for the spectrum of $\mathcal{E}_*$ in $\mathscr{R}_*$.

Proposition 10.3.2. *Let $M_* \subset M$ be a* <u>*normal*</u> *complex ray of the special cone M. Then $S_* \subset S$. Moreover, $\mathscr{R}_*$ is the Frechét completion of the graded ring $\mathbb{C}[u_*]$ where the single generator u_* has dimension $1 \leq \Delta_* \in S_*$, and there is a generator u of $\mathscr{R}$ having the same dimension Δ_*.*

Proof. M is Stein, and $M_* \subset M$ is a closed analytic subspace. Hence, by Cartan's extension theorem [119, 135] we have an epimorphism of chiral rings

$$\mathscr{R} \xrightarrow{i^*} \mathscr{R}_* \to 0, \tag{10.34}$$

given by restriction, which preserves the $\mathcal{E}$ action. Hence $S_* \subset S$. In this argument we do not use the fact that $M_* \subset M$ is normal. However, if M_* is not normal, there is a subtlety in the above statement. Let $i \colon M_* \to M$ be the closed inclusion. Cartan's theorem states

$$\Gamma(M, \mathcal{O}_M) \xrightarrow{i^*} \Gamma(M_*, i^*\mathcal{O}_M) \to H^1(M, \ker i^*) \equiv 0, \tag{10.35}$$

where $i^*\mathcal{O}_M$ is the structure sheaf of M_* seen as an analytic subspace of M. Iff M_* is normal, $i^*\mathcal{O}_M$ coincides with the structure sheaf of its normalization $\mathcal{O}_{M_*}$ (i.e. with the integral closure of $i^*\mathcal{O}_M$ stalk-wise); its global sections $\Gamma(M_*, \mathcal{O}_{M_*})$ correspond to the "intrinsic" notion of holomorphic functions on M_*, while $\Gamma(M_*, i^*\mathcal{O}_M)$ is the subset of holomorphic functions on M_* seen as a concrete sub-space of M. We illustrate this subtlety in **Example 30** below. Then, although the result is valid in general, to apply it for $M_* \subset M$ not normal we need to be able to distinguish holomorphic functions in the two different senses. If M_* is normal, it coincides with its own normalization, and the two notions coincide.

Since M_* is one-dimensional and normal, it is smooth.[74] On M_* we have a $\mathbb{C}^\times$ action $\zeta \mapsto \zeta^{\mathcal{E}}$, acting transitively on $M_* \setminus \{0\}$. Then M_* is analytically (hence algebraically) a copy of $\mathbb{C}$ on which $\zeta^{\mathcal{E}}$ acts by automorphisms fixing the origin. Let u_* be a standard coordinate on this $\mathbb{C}$; the polynomial ring $\mathbb{C}[u_*]$ is dense in $\Gamma(M_*, \mathcal{O}_{M_*})$ and graded by $\deg u_* = \mathcal{E} u_* > 0$.

If $w \in \mathscr{R}$ is a homogeneous holomorphic function with $\Delta(w) \notin S_*$ we must have $w|_{M_*} \equiv 0$. Let $\{v_i\}_{i=1}^k$ be a set of homogeneous generators of (a dense subring of) the global chiral ring $\mathscr{R}$. Not all restrictions $\{v_i|_{M_*}\}_{i=1}^k$ may vanish identically since in a Stein manifold the ring of homolorphic functions separates points [135]. Let $u \in \{v_i\}$ be a generator of $\mathscr{R}$ with $u|_{M_*} \not\equiv 0$ of smallest degree d_*. All (non constant) homogenous elements $f \in \mathscr{R}$ either restrict to zero $f|_{M_*} \equiv 0$ or have a degree $\geq d_*$. Since $\mathscr{R}_* \cong \mathbb{C}[u_*]$, for some u_* having minimal positive degree in $\mathscr{R}_*$, and u_* is the restriction of a function in $\mathscr{R}$, we conclude that we may choose $u_* = u|_{M_*}$. Therefore the dimension $d_* \equiv \Delta_*$ of the generator u_* of $\mathscr{R}_*$, is equal to the dimension $\Delta(u)$ of the generator u of the full chiral ring $\mathscr{R}$. □

Example 30. We illustrate the subtle point in the proof of the above **Proposition**. We consider $SU(3)$ SQCD with $N_f = 6$ at weak coupling and zero quark masses. In this case the Coulomb branch ring is $\mathbb{C}[u, v]$ where the generators u and v have dimensions 2 and 3. The (reduced) complex rays are

$$M_u = \{v = 0\}, \quad M_v = \{u = 0\}, \quad M_{(\alpha)} = \{v^2 = \alpha\, u^3\},\ \alpha \in \mathbb{C}^\times. \tag{10.36}$$

M_u and M_v are normal with $\mathscr{R}_u = \mathbb{C}[u]$, $\mathscr{R}_v = \mathbb{C}[v]$; we see that the generators of the two normal ray rings, u and v, are generators of the full chiral ring $\mathscr{R}$. Instead, $M_{(\alpha)}$ is a plane cubic with a cusp, which is the simplest example of a non-normal variety (see e.g. §.4.3 of [112]). The ring of holomorphic functions in the subspace sense is not integrally closed. The integral closure of the ray ring contains the function v/u (indeed, $(v/u)^2 = \alpha u$) which is holomorphic in the normalization and has dimension 1, a dimension $\notin S$.

Remark 10.3.1. Note that the non-normal rays $M_{(\alpha)}$ in the above example correspond to the "non-free" geometry discussed in ref. [15].

10.3.2 Unbroken R-symmetry along a ray

The statements in §.10.3.1 reduce the computation of the Coulomb dimensions of the generators of $\mathscr{R}$ to local computations at normal complex rays in the conical Coulomb branch M. This leads to the following two questions: 1) are there *enough* normal complex rays to compute all Coulomb dimensions? 2) how we characterize the *normal* complex rays?

[74] As a complex space; the Kähler metric has a conical singularity at the tip.

In **Example 30** we see that the two normal complex cones in the Coulomb branch M of $SU(3)$ SQCD are precisely the loci of Coulomb vacua with an unbroken discrete subgroup of $U(1)_R$, namely $\mathbb{Z}_3$ for M_v and $\mathbb{Z}_2$ for M_u. All non-normal rays consist of vacua which completely break $U(1)_R$ (except at the tip where the full $U(1)_R$ is restored). This characterization of normal rays by unbroken R-symmetry holds in general.

To a complex ray M_* there is associated a rational positive number α_*, namely the smallest positive number such that

$$\exp(2\pi\alpha_* R) = \mathrm{Id}_{M_*}. \tag{10.37}$$

$\alpha_* \in \mathbb{Q}_{>0}$ since the R-symmetry group $U(1)_R$ is compact. In the physical language this means that a subgroup of $U(1)_R$ is unbroken:

Fact. *A Coulomb vacuum $x \in M_* \subset M$ preserves a discrete $\mathbb{Z}_n \subset U(1)_R$ R-symmetry where n is the order of α_* in $\mathbb{Q}/\mathbb{Z}$. We say that M_* is an* elliptic ray *iff $n > 2$.*

In section 3 we presented some evidence that the chiral ring is a free polynomial ring (or simply related to such a ring). In this case M is parametrized by weighted homogeneous coordinates u_i having weights $\Delta_i \in \mathbb{Q}_{\geq 1}$. Then the complex ray along the i-th axis

$$M_i \equiv \Big\{(u_1, u_2, \cdots, u_k) = (0, 0, \cdots, \overset{i\text{-th}}{u}, 0, \cdots, 0),\ u \in \mathbb{C}\Big\} \subset M \tag{10.38}$$

is *normal* and has $\alpha_i = 1/\Delta_i$. If $\Delta_i = r_i/s_i > 1$ with $(r_i, s_i) = 1$, the order of the residual R-symmetry is $n_i = r_i \geq 2$ with equality iff $\Delta_i = 2$, i.e. iff u_i is the vev of a (superficially) marginal operator. In particular, if $\mathscr{R}$ is a free polynomial ring we do have enough normal rays.

Let M_* be a normal ray and $x \in M_* \setminus \{0\}$. We consider the closed R-orbit

$$x(t) = \exp(2\pi\alpha_* t R) \cdot x \in M_*, \qquad t \in [0, 1]. \tag{10.39}$$

The monodromy of this path is an element m_* of the modular group $Sp(2k, \mathbb{Z})$ which is independent of x modulo conjugacy. We have

$$\det[z - m_*] = \prod_{\ell | n} \Phi_\ell(z)^{s(\ell)}, \qquad s(\ell) \in \{0, 1, 2, \cdots\}, \quad \sum_{\ell | n} s(\ell)\,\phi(n) = 2k, \tag{10.40}$$

since $\exp(2\pi n\alpha_* R)$ acts trivially on the periods. We say that M_* is regular if its monodromy m_* is *strongly* regular. Regularity is equivalent to $s(\ell) \in \{0, 1\}$ (so $s(1) = s(2) = 0$ in the regular case). Regularity implies that the fixed period $\boldsymbol{\tau}$ is unique. More crucially, it means that the ray is not part of the "bad" discriminant locus, $M_* \not\subset \mathcal{D}_{\text{bad}}$. Here $\mathcal{D}_{\text{bad}}$ is the union of the irreducible components of $\mathcal{D}$ with a non-semi-simple monodromy; along $\mathcal{D}_{\text{bad}}$ the period matrix $\boldsymbol{\tau}$ degenerates in agreement with the SL_2-orbit theorem, see §. 5.1.5. Indeed, strong regularity implies that all the eigenvalues of m_* are distinct, and no non-trivial Jordan block may be present. More generally, we split the product in (10.40) in the square-free factor $\prod_{s(\ell)=1} \Phi_\ell(z)$ and the complementary factor $\prod_{s(\ell)>1} \Phi_\ell(z)^{s(\ell)}$. m is conjugate in $Sp(2k, \mathbb{Q})$ to a block-diagonal matrix of the form $m_{\text{reg}} \oplus m_{\text{comp}}$ with $m_{\text{reg}} \in Sp(2k_{\text{reg}}, \mathbb{Z})$ strongly regular. Then, up to isogeny,[75] $\boldsymbol{\tau} = \boldsymbol{\tau}_{\text{reg}} \oplus \boldsymbol{\tau}_{\text{comp}}$ for a unique

[75] These statements follow from the Poincaré total reducibility theorem.

$\boldsymbol{\tau}_{\text{reg}}$. The regular rank k_{reg} of M_* is

$$k_{\text{reg}} = \frac{1}{2} \sum_{\ell\colon s(\ell)=1} \phi(\ell). \tag{10.41}$$

Thus, locally at the ray, the family of Abelian varieties $X \to M_*$ splits (modulo isogeny) in a product $X_1 \times X_2 \to M_*$, and M_* is not in the "bad" discriminant of the first factor (but it may be for the second one).

Let M_* be a normal ray which is also regular. The real function $r^2 = \text{Im}\, \boldsymbol{\tau}_{ij}\, a^i \bar{a}^j$ is smooth and non-zero on M_*; since $\text{Im}\, \boldsymbol{\tau}_{ij}$ is the unique fixed period, which is non singular on M_* (in particular, bounded), it means that there exists a $\mathbb{C}$-linear combination a_* of the periods a^i which does not vanish on M_* (more precisely, a_* is well defined on a finite cyclic cover of M_* branched at the tip). Applying m_* to a_*, we see that $e^{2\pi i \alpha_*} \equiv e^{2\pi i/\Delta(u_*)}$ belongs to the spectrum of m_*. Since m_* is the lift of the generator of the unbroken subgroup $\mathbb{Z}_n \subset U(1)_R$, we conclude that $e^{2\pi i \alpha_*}$ is a n-root of unity. In facts, it should be a primitive root, otherwise the unbroken symmetry would be smaller. Comparing with the $k = 1$ case, and using **Proposition 10.3.2** we learn that $1/\alpha_*$ is the dimension of a generator of the full chiral ring $\mathscr{R}$. Now suppose that M_* is not regular. Taking into account only the "good" block, that is, focusing on the first local family, $X_1 \to M_*$, we reduce to the regular situation. The argument applies to the irregular block too, as long as $\text{Im}\, \boldsymbol{\tau}_{ij}$ is not singular on M_*. In general we may decompose $\text{Im}\, \boldsymbol{\tau}_{ij}$ in a regular block and one which is in the modular orbit of $i\infty$. The argument works as long as the regular block is non-trivial, that is as long as along M_* not all photons decouple.

Then

Fact. *If in M there is a* normal *ray with residual R-symmetry $\mathbb{Z}_n$, there should be a generator of $\mathscr{R}$ with $\Delta = n/s$, $s \in (\mathbb{Z}/n\mathbb{Z})^\times$ i.e. $e^{2\pi i/\Delta}$ is a primitive n-th root of unity. $e^{2\pi i/\Delta}$ is an eigenvalue of a quasi-unipotent element $m_* \in Sp(2k, \mathbb{Z})$, and hence $\phi(n) \leq 2k$.*

Remark 10.3.2. Above we assumed the polarization Ω to be principal. In the general case, we replace the Siegel modular group with the relevant duality-frame group $S(\Omega)_{\mathbb{Z}}$. The conclusion $\phi(n) \leq 2k$ being still valid.

10.3.3 The Stein tubular neighborhood of a complex ray

A ray $M_* \subset M$ is a closed analytic subset, hence a Stein submanifold of the Stein manifold M, and the restriction map $\mathscr{R} \to \mathscr{R}_*$ is essentially surjective (the image is dense in the Frechét sense). The Docquier-Grauert theorem [119] guarantees that we can find a Stein tubular (open) neighborhood $M_\circ$ of M_* in M. There is a holomorphic retraction of $M_\circ$ onto M_*, and $M_\circ$ is biholomorphic to a neighborhood of the zero section in the normal bundle of M_* in M [119]. The fact that $M_\circ$ retracts holomorphically to M_*, means that the monodromy group of the special geometry restricted to $M_\circ$ is just the cyclic group generated by m_*. In particular, the a-periods are well-defined on an unbranched cover $\widetilde{M}_\circ \setminus \{0\} \to M_\circ \setminus \{0\}$ with Galois (deck) group $\mathbb{Z}_n$ generated by m_*.

Since $(M, \mathcal{O}_M)$ is a normal analytic space, so is $(M_\circ, \mathcal{O}_M|_{M_\circ})$. We have the restriction morphisms

$$\mathscr{R} \xrightarrow{\text{mono}} \mathscr{R}_\circ \xrightarrow{\text{epi}} \mathscr{R}_*. \tag{10.42}$$

Lemma 10.3.1. *The map $\mathscr{R} \to \mathscr{R}_\circ$ preserves the spectrum of $\mathcal{E}$.*

Proof. Suppose on the contrary that there is an eigenvalue λ of $\mathcal{E}_\circ$ which is not in the spectrum S of $\mathcal{E}$, and let $f \in \mathscr{R}_\circ$ be a non-zero eigenfunction. Let $x \in M_\circ$ be such that $f(x) \neq 0$, and extend f by homogeneity on the closed complex ray generated by x, $M_x \equiv \overline{\zeta^{\mathcal{E}} \cdot x}$. Since $i_x \colon M_x \to M$ is a closed embedding and $f \in \Gamma(M_x, i_x^* \mathcal{O}_M)$, by Cartan's theorem f extends to a function in $\Gamma(M, \mathcal{O}_M)$ and then $\lambda \in S$. □

10.3.4 Tubular neighborhoods and the Universal Dimension Formula

Identifing the tubular neighborhood $M_\circ$ of the normal ray M_* with a neighborhood of the zero-section in the normal bundle, we may introduce homogeneous complex coordinates

$$(u, v_1, v_2, \cdots, v_{k-1}) \tag{10.43}$$

such that $M_* \subset M_\circ$ is given by the analytic set $v_1 = v_2 = \cdots = v_{k-1} = 0$, while $u = u_*$ is the coordinate along the normal ray M_*. Indeed, the additional coordinates v_i are just linear coordinates along the fibers of the holomorphic normal bundle. The v_i are globally defined in $M_\circ$ since the holomorphic normal bundle of M_* is holomorphically trivial. This follows from a result of Grauert (cfr. **Theorem 5.3.1**(iii) of [119]) since $\dim_{\mathbb{C}} M_* = 1$.

We saw in the previous subsection that u is homogeneous of degree $1/\alpha_*$. Along the ray M_* only a complex-linear combination of the a-periods, a_*, does not vanish. In the tubular neighborhood $M_\circ$ all k linear combinations of the a-periods are not (identically) zero. In a *conical* special geometry the a periods transform through the modular factors

$$a' = (C\boldsymbol{\tau} + D)a, \tag{10.44}$$

where $\boldsymbol{\tau}$ is the (constant) period matrix on M_*. Hence, if m_* is a regular elliptic element of $Sp(2k, \mathbb{Z})$ and $(e^{2\pi i\alpha_*}, e^{2\pi i\beta_1}, \cdots, e^{2\pi i\beta_{k-1}})$ is the spectrum of $C\boldsymbol{\tau} + D$, with $e^{2\pi i\alpha_*}$ a primitive n-th root of unity, we may find complex-linear combinations of a-periods a_*, a_i in $M_\circ$ which diagonalize the action of m_*

$$\begin{aligned} a'_* &= e^{2\pi i\alpha_*}\, a_*, \quad 0 < \alpha_* \leq 1 \\ a'_s &= e^{-2\pi i\beta_s} a_s \quad s = 1, \ldots, k-1, \quad 0 \leq \beta_s < 1, \end{aligned} \tag{10.45}$$

where a_* is the linear combination non-zero along the ray M_*.

Fact. *The generalization of the $k = 1$ equation* (10.6) *to the tubular neighborhood $M_\circ$ of a* regular normal *ray M_* is*

$$a_* \propto u^{\alpha_*}, \tag{10.46}$$

$$a_s \propto v_s\, u^{-\beta_s} \quad \textit{for } s = 1, \ldots, k-1. \tag{10.47}$$

Proof. Eqn.(10.46) is just the previous result along the normal ray M_*. Let us consider the $\mathbb{C}$-periods a_s vanishing along M_*. The a-periods should transform with the correct monodromy m_* along the path

$$u \to e^{2\pi i t} u, \quad t \in [0, 1], \quad v_s = \text{const} \tag{10.48}$$

cfr. eqn.(10.45). Thus we must have

$$a_s = f_s(v)\, u^{\alpha_s} \qquad \text{with } f_s(0) = 0 \text{ and } 2\pi i \alpha_s = \log\big(e^{-2\pi i \beta_s}\big). \tag{10.49}$$

It remains to fix the functions $f_s(v)$ and the branch of the logarithm giving the correct value of α_s. The holomorphic symplectic form $\Omega = da^i \wedge (dx_i - \boldsymbol{\tau}_{ij}\, dy^i)$ should have maximal rank when restricted to M_*, so that $df_1 \wedge \cdots \wedge df_{k-1}|_{M_*} \neq 0$ and hence

$$f_s = A_{st}\, v_t + \text{higher order} \qquad \det A \neq 0. \tag{10.50}$$

We may set A_{st} to the identity matrix by a linear redefinition of the v_s. Since the f_s are homogeneous functions, the higher order corrections vanish. Finally the branch of the logarithm is fixed to $\alpha_s \equiv -\beta_s$ by the requirement that the dimensions satisfy the unitary bound $\Delta[v_s] \geq 1$. □

Then the k dimensions of the generators of $\mathscr{R}$ are

$$\Delta_i \equiv \Delta(v_i) = \begin{cases} 1 + \beta_i/\alpha_* & i = 1, \dots, k-1 \\ 1/\alpha_* & i = k, \end{cases} = 1 + \frac{\beta_i}{1 - \beta_k} \tag{10.51}$$

where we set $v_k \equiv u_*$ and $\beta_k = 1 - \alpha_*$.

If M_* is normal but non regular, we cannot determine all dimensions from an analysis in the neighborhood $M_\circ$ of M_* but only as many as its regular rank k_{reg}.

Remark 10.3.3. These formulae have the following natural property. Let m_* be weakly regular, that is, some eigenvalue ζ of $\mathcal{U}(m_*)$ have multiplicity $s > 1$. Assume a_* is an eigenperiod associated to ζ; then $\alpha_* = \alpha$ while $(s-1)$ β's are equal $1-\alpha$. The dimensions of the s operators associated to the eigenvalue ζ are all equal to $1/\alpha$, without distinction between the operator parametrizing the ray and the operators parametrizing its normal bundle. This property guarantees that we get the correct dimension spectrum for SCFT whose Coulomb branch is birational to a product of identical cones, so that the largest dimension $\Delta_{\max}$ is degenerate. This happen e.g. in class $\mathcal{S}[A_1]$ SCFTs where $\{\Delta_i\} = \{2, 2, 2, \cdots, 2\}$.

The (universal) dimension formula (10.51), if correct, should pass three crucial consistency checks:

a) it should reproduce the well-known formulae for constant period maps, in particular for all weakly-coupled Lagrangian SCFTs.

b) it should reproduce the Picard-Lefschetz results for models engineered by hypersurface singularities in F-theory;

c) it should produce the same spectrum of dimensions independently of which normal regular ray $M_* \subset M$ we consider;

The third requirement is quite strong, and it seems *a priori* quite unlike that such a strong property may be acutally true. We perform the three checks in turn.

10.3.5 Relation to Springer Theory

We have to check that the "abstract" dimension formula (10.51) reproduces the obvious dimensions for a weakly-coupled Lagrangian SCFT and more generally for all CSG with constant period maps of the form $M = \mathbb{C}^k/\mathcal{G}$ for a degree-k ST group $\mathcal{G}$ whose chiral ring $\mathscr{R}$ coincides with the ring of polynomial invariants (see §.6.2).

That eqn.(10.51) correctly reproduces the ST degrees d_i as dimensions Δ_i of the generators of $\mathscr{R}$ is a deep result in the Springer Theory of regular elements in finite reflection groups [38,83,237].

We recall the definitions: let the finite group $\mathcal{G}$ act as a reflection group on the $\mathbb{C}$-space V. A vector $v \in V$ is said to be *regular* iff it does not lay in a reflection hyperplane. An element $g \in \mathcal{G}$ is said to be *regular* if it has a regular eigenvector v. The *regular degrees* of $\mathcal{G}$ are a (minimal) set of integer numbers such that the order of all regular elements of $\mathcal{G}$ is a divisor of an element of the set and conversely all divisors of these numbers are the order of a regular element. Then

Theorem 6 (see [83, 237])**.** *Let ζ be a primitive d-root of unity. Let $g \in \mathcal{G}$ be regular with regular eigenvector $v \in V$ and related eigenvalue ζ. Denote by W the ζ-eigenspace*

$$W = \{x \in V \mid gx = \zeta x\}. \tag{10.52}$$

Then:

(i) d is the order of g, and g has eigenvalues $\zeta^{1-d_1}, \zeta^{1-d_2}, \cdots, \zeta^{1-d_k}$, where d_i are the degrees of $\mathcal{G}$;

(ii) $\dim W = \#\{i \mid d$ *is a divisor of* $d_i\}$*;*

(iii) the restriction to W of the centralizer of g in $\mathcal{G}$ defines an isomorphism onto a reflection group in W whose degrees are the d_i divisible by d and whose order is $\prod_{d|d_i} d_i$;

(iv) the conjugacy class of g consists of all elements of $\mathcal{G}$ having $\dim W$ *eigenvalues ζ.*

One can show that an integer is regular iff it divides as many degrees as co-degrees [130].

Remark 10.3.4. All irreducible ST groups have at least one regular degree. In facts, they have either 1 or 2 regular degrees, except for the Weyl groups of E_6 and E_8 which have *three* regular degrees each, respectively $\{8, 9, 12\}$ and $\{20, 24, 30\}$. If $\mathcal{G}$ is a Weyl group, the Coxeter number h is one of the regular degrees. See refs. [83, 130] for tables of regular degrees.

Corollary 10.3.1. *Going through the the tables* [83, 130]*, one sees that part (ii) of the* **Theorem** *implies that for an irreducible* crystallographic *Shephard-Todd group* $\dim W = 1$ *for all regular degrees d. In other words, in the crystallographic case, the only degree which is an integral multiple of the regular degree d is d itself.*

Let us re-interpret **Theorem 6** in the context of the constant-period class of CSG's discussed in §. 6.2, with Coulomb branch $M = \mathbb{C}^k/\mathcal{G}$, $\mathcal{G}$ a degree-k crystallographic ST group, and chiral ring $\mathscr{R} = \mathbb{C}[a^1, \cdots, a^k]^{\mathcal{G}}$. For simplicity, we take $\mathcal{G}$ irreducible.

By definition, $v \in \mathbb{C}^k$ is a regular vector iff it does not lay on a reflection hyperplane, i.e. if its projection $\tilde{v}$ in the Coulomb branch $M = \mathbb{C}^k/\mathcal{G}$ does not belong to the discriminant locus $\mathcal{D} \subset M$, i.e. iff $\tilde{v} \in M^\sharp$, the smooth locus.

Let d be a regular weight of $\mathcal{G}$ (so $d \equiv d_{i_0}$ for some i_0). By definition, there is an element $m_{i_0} \in \mathcal{G}$ of order precisely d_{i_0}. Let v be a regular eigenvector of m_{i_0} corresponding to the primitive d_{i_0}-th root[76] $\zeta = e^{2\pi i/d_{i_0}}$. The (closure of the) $\mathbb{C}^\times$-orbit of $\tilde{v} \in M^\sharp$, $M_v \subset M$, is a complex ray not lying in the discriminant $\mathcal{D}$ (more precisely, intersecting $\mathcal{D}$ only at the tip).

We claim that M_v is also normal. Recall that $\mathscr{R} = \mathbb{C}[a^1, \cdots, a^k]^{\mathcal{G}} \equiv \mathbb{C}[u_1, \cdots, u_k]$ by the Shephard-Todd-Chevalley theorem. Homogeneity implies

$$u_i\big|_{M_v} = c_i\,(\lambda v)^{d_i} \quad \lambda \in \mathbb{C}, \tag{10.53}$$

for some constants c_i. Applying m_{i_0} on both sides, and using **Corollary 10.3.1** i.e.

$$\frac{d_i}{d_{i_0}} \in \mathbb{N} \quad \Longrightarrow \quad i \equiv i_0, \tag{10.54}$$

we conclude that

$$M_v \equiv \big\{u_i = 0 \text{ for } i \neq i_0\big\} \equiv M_{i_0}, \tag{10.55}$$

is automatically a normal right of the form in eqn.(10.38). This also shows that m_{i_0} is the monodromy along the normal ray M_v, which is then regular. This is exactly the set up in which we deduced the universal dimension formula (10.51). From item *(i)* in the **Theorem 6** we see that

$$\beta_i = (d_i - 1)/d \quad \text{and} \quad \alpha_* = 1 - \beta_{i_0} = 1/d. \tag{10.56}$$

The universal formula (10.51) then yields

$$\Delta_i = 1 + d\,\beta_i = d_i, \tag{10.57}$$

which is the correct result.

Thus the formula (10.51) is nothing else than the plain extension to the non-Lagrangian SCFT of the usual formula, valid for all Lagrangian SCFT, following from the standard supersymmetric non-renormalization theorems.

10.3.6 Recovering Picard-Lefschetz for hypersurface singularities

We consider the hypersurface

$$F \equiv \left\{ W(x_1, x_2, x_3, x_4) + \sum_{a=0}^{k-1} u_a\,\phi_a(x_i) = 0 \right\} \subset \mathbb{C}^4, \tag{10.58}$$

where u_a has dimension $\Delta_0(1 - \deg\phi_a)$, Δ_0 being the relative normalization of the R-charges in the 4d and 2d sense under the 4d/2d correspondence [60]. We consider the ray parametrized by the coupling u_0 of the 2d identity operator

$$M_0 = \{u_1 = \cdots = u_{k-1} = 0\} \subset M. \tag{10.59}$$

[76] This is a special choice for complex $\mathcal{G}$; in the case of a Lagrangian SCFT, i.e. if $\mathcal{G}$ is a Weyl group, this choice does not imply any loss of generality.

The corresponding monodromy m_0, along the path $u_0 \to e^{2\pi i t} u_0$, $t \in [0,1]$ is, by construction, the one induced on $H_1(F,\mathbb{Z})/\mathrm{rad}\,\langle -,-\rangle$ by the classical monodromy H of the hypersurface singularity, that is, ($f \equiv \mathrm{rank}\,\mathrm{rad}\,\langle -,-\rangle$)

$$\det[z-H] = (z-1)^f \;\det[z-m_0], \qquad \det[z-m_0] = \prod_{\ell\geq 2} \Phi_\ell(z)^{s(\ell)}. \tag{10.60}$$

Thus the spectrum of m_0 is

$$\mathrm{spec}(m_0) = \Big\{ e^{2\pi i(q_a - \hat{c}/2)} \colon q_a \; U(1)_R \text{ charge of 2d chiral primaries with } q_a \neq \hat{c}/2 \Big\}. \tag{10.61}$$

This way of writing implicitly selects a special embedding of m_0 into $Sp(2k,\mathbb{Z})$ as well as the eigenvalue which corresponds to the non trivial period a_0 on M_0; these are the canonical choices dictated by Picard-Lefschetz theory. This choice yields

$$\alpha_0 = 1 - \frac{\hat{c}}{2}, \qquad \beta_a = \frac{\hat{c}}{2} - q_a, \quad 0 < q_a < \hat{c}/2, \tag{10.62}$$

so that,

$$\Delta_0 = \frac{1}{1-\hat{c}/2}, \qquad \Delta_a = 1 + \frac{\hat{c}/2 - q_a}{1-\hat{c}/2} \equiv \frac{1-q_a}{1-\hat{c}/2}, \quad 0 < q_a < \hat{c}/2, \tag{10.63}$$

which precisely yields back the Picard-Lefschetz formula (10.13) ($q_a \equiv \deg\phi_a$ by definition).

Example 31 (Complete intersections of singularities)**.** The previous argument applies to all SCFT which have a 4d/2d correspondent in the sense of [60]; in the general case the classical monodromy of the singularity H should be replaced by the (2,2) quantum monodromy as defined in [63]. The eigenvalues of the (2,2) quantum mondromy have the form $e^{2\pi i(q_a - \hat{c}/2)}$ where q_a are the $U(1)_R$ charges of the 2d chiral operators [63]. For instance, this result applies to the models engineered by the complete intersection of two hypersurface singularities in $\mathbb{C}^5$ [244]

$$W_1(x_i) + \sum_\alpha u_{1,\alpha}\, \phi_{1,\alpha}(x_i) = W_2(x_i) + \sum_\alpha u_{2,\alpha}\, \phi_{2,\alpha}(x_i) = 0, \tag{10.64}$$

where $W_a(x_i)$ are quasi-homogeneous of degree $d_1 \equiv 1$ and d_2 (we assume $d_1 \leq d_2$ with no loss) $W_1(\lambda^{w_i} x_i) = \lambda W_1(x_i)$, $W_2(\lambda^{w_i} x_i) = \lambda^d W_2(x_i)$ and $\phi_{a,\alpha}(x_i)$ is a basis of the admissible deformations of the equations.[77] This correspond to a 2d model with superpotential [244]

$$\mathcal{W}(x_i) = W_1(x_i) + \sum_\alpha u_{1,\alpha}\, \phi_{1,\alpha}(x_i) + \Lambda\Big(W_2(x_i) + \sum_\alpha u_{2,\alpha}\, \phi_{2,\alpha}(x_i) \Big) \tag{10.65}$$

where Λ is an extra chiral field of $U(1)_R$ charge $q_\Lambda = 1 - d_2$. The scaling dimension of the parameters $u_{a,\alpha}$ is $1 - q_{a,\alpha}$ where $q_{\alpha,\alpha}$ is the charge of the corresponding operator perturbing $\mathcal{W}$, i.e. $\phi_{1,\alpha}$ and $\Lambda\phi_{2,\alpha}$, respectively. One has $1 - \hat{c}/2 = \sum_i w_i - d_1 - d_2$ and so [244]

$$\Delta(u_{a,\alpha}) = \frac{d_a - \deg\phi_{a,\alpha}(x_i)}{\sum_i w_i - \sum_a d_a}. \tag{10.66}$$

[77] That is, a basis of the Jacobian ring.

Example 32 (The DZVX models)**.** A third class of 4d SCFT with a nice 2d correspondent is the one constructed in [90] parametrized by a pair (affine star, simply-laced Lie algebra). Our formulae yield the correct dimension spectrum by construction.

10.3.7 Consistency between different normal rays

Let us consider a simple (but instructive) class of examples, the Argyres-Douglas (AD) models of type A_{N-1} with N odd. The special geometry of the A_{N-1} AD model corresponds to the intermediate Jacobian of the following family of hypersurfaces in $\mathbb{C}^4$

$$y^2 + w^2 + z^2 = x^N + \sum_{a=0}^{(N-3)/2} u_a\, x^a, \tag{10.67}$$

the period matrix $\boldsymbol{\tau}$ being degenerate on the locus where the discriminant of the polynomial in the RHS vanishes. The coefficient u_a ($a = 0, 1, \dots, [(N-3)/2]$) have degree $2(N-a)$, so we have a map

$$M \setminus \{0\} \to \mathbb{P}\big(2N, 2N-2, \cdots, N+3\big), \tag{10.68}$$

through which the period map $\boldsymbol{\tau}$ factorizes. We consider the normal rays

$$M_a = \{u_b = 0,\ b \neq a\} \subset M \qquad a = 0, 1, \cdots, (N-3)/2. \tag{10.69}$$

Along the ray M_0 we have an unbroken R-symmetry $\mathbb{Z}_{2N}$ with a (strongly) regular monodromy

$$\det[z - m_0] = \frac{z^N + 1}{z + 1}. \tag{10.70}$$

Instead along the ray M_1 the unbroken R-symmetry is $\mathbb{Z}_{2(N-1)}$ again with regular monodromy

$$\det[z - m_1] = z^{N-1} + 1. \tag{10.71}$$

Along a ray M_a with $a \geq 2$ the discriminant of the polynomial in the RHS of (10.67) vanishes, and our considerations apply only to the regular factor. The unbroken R-symmetry is $\mathbb{Z}_{2(N-a)}$ and we have

$$\Big(\det[z - m_a]\Big)_{\text{regular factor}} = \begin{cases} (z^{N-a} + 1)/(z+1) & a \text{ even} \\ z^{N-a} + 1 & a \text{ odd.} \end{cases} \tag{10.72}$$

For each a we have an embedding

$$m_a \in Sp(2k_{a,\text{reg}}, \mathbb{Z}) \equiv Sp(2[(N-a)/2], \mathbb{Z}) \tag{10.73}$$

specified by the spectrum of $(C\boldsymbol{\tau} + D)_{\text{reg.block}}$ which consists of $k_{a,\text{reg}}$ out of the $2k_{a,\text{reg}}$ roots of the characteristic polynomial of m_a: it is a set of $2(N-a)$-th roots of unity satisfying the condition in equation (10.31). We also need to specify which one of the $k_{a,\text{reg}}$ roots corresponds to the non-trivial period a_a on the ray M_a. There is a "canonical" Picard-Lefschetz choice. We have (here

$0 \leq a \leq (N-1)/2)$

a	ζ	embeding	α	β_ℓ	Δ_ℓ
0	$e^{\pi i/N}$	$\zeta^{2N-2\ell+1}$	$(N+2)/2N$	$(2\ell-1)/2N$	$(N+2\ell+1)/(N+2)$
				$\ell \neq (N-3)/2$	$1 \leq \ell \leq (N-1)/2$
1	$e^{\pi i/(N-1)}$		$(N+2)/(2N-2)$	$(2\ell-1)/(2N-2)$	$(N+2\ell+1)/(N+2)$
				$\ell \neq (N-3)/2$	$1 \leq \ell \leq (N-1)/2$
a	$e^{\pi i/(N-a)}$	$\zeta^{2(N-a-\ell)+1}$	$(N+2)/(2N-2a)$	$(2\ell-1)/(2N-2a)$	$(N+2\ell+1)/(N+2)$
				$\ell \neq (N-2a-1)/2$	$1 \leq \ell \leq [(N-a)/2]$

we see that the several rays M_a yield mutually consistent results for the spectra of dimension. M_0 and M_1 yield the full set of dimensions (which agrees between the two rays), while $M_{2\ell}$ and $M_{2\ell+1}$ yield a partial list of $k-\ell$ out of the k dimensions. (But the formal "analytic continuation" gives the full correct set of dimensions at all normal rays).

Remark 10.3.5. Note that the list of the k_{reg} dimensions computed from a M_a is a k_{reg}-tuple of dimensions which is allowed for a rank k_{reg} SCFT. In particular, the dimension of the operator parametrizing a ray with $k_{\text{reg}} = 1$ should be in the one-dimensional list $\{1, 2, 3, 4, 6, 3/2, 4/3, 6/5\}$.

Example 33. Consider the Argyres-Douglas models of type D_5 and D_6 which have $k = 2$. The ray parametrized by the operator of the largest dimension corresponds to Picard-Lefschetz theory and is regular, while the one associated to the operator of lesser dimension is non-regular. We deduce by the previous **Remark** that for these models the smaller of the two Coulomb dimensions should belong to the $k = 1$ list. Indeed it is 6/5 for type D_5, and 4/3 for type D_6. This statement may be generalized in the form of inter-SCFT consistency conditions relating the spectrum of dimensions in different SCFT. For $\mathfrak{g} \in ADE$ we write $\{\Delta\}_{\mathfrak{g}}$ for the set of Coulomb branch dimensions of the Argyres-Douglas model of type $\mathfrak{g}$ and we write $\{\Delta\}_{\mathfrak{g}}^{(s)}$ for the subset obtained from $\{\Delta\}_{\mathfrak{g}}$ by omitting the s largest dimensions. Then we have the relations

$$\{\Delta\}_{A_n} = \{\Delta\}_{D_{n+3}}^{(1)}. \tag{10.74}$$

We present a more complicated example of such consistency condition between Coulomb dimensions in different ranks.

Example 34 (Argyres-Douglas of type E_8)**.** This SCFT has $k = 4$ with Coulomb dimensions $\Delta_1 = 15/8$, $\Delta_2 = 3/2$ $\Delta_3 = 5/4$ and $\Delta_4 = 9/8$. Since $\frac{1}{2}\phi(15) = 4$ from the ray M_1 we should be able to compute all 4 dimensions (this is classical Picard-Lefschetz for the E_8 minimal singularity). For the other normal rays we have consistency requirements saying that a subset of the dimension set $\{15/8, 3/2, 5/4, 9/8\}$ should be the dimension k' tuple for the appropriate regular rank $k' < k$. Since $\frac{1}{2}\phi(3) = 1$, from M_2 we compute the single dimension 3/2 which is in the $k = 1$ list. $\frac{1}{2}\phi(5) = 2$ so from M_3 we get a pair of dimensions in the $k = 2$ list, namely $\{5/4, 3/2\}$. Finally $\frac{1}{2}\phi(9) = 3$ and 3 dimensions out of 4 should form a dimension triple for rank-3. This is consistent since both $\{9/8, 15/8, 3/2\}$ and $\{9/8, 3/2, 5/4\}$ are in the $k = 3$ list.

10.3.8 A comment on the conformal manifold

As further evidence of the correctness of the universal dimension formula (10.51), let us consider the conformal manifold, that is, the space of moduli (deformations) of the CSG. We have already discussed the special case in which the period map is constant while the chiral ring is the invariant subring $\mathbb{C}[a_1, \dots, a_k]^{\mathcal{G}}$ (see §. 6.2). In that case the dimension of the conformal manifold was given by the number of generators of the chiral ring of dimension 2. In the general case, the allowed continuous deformations of the CSG are described by the rigidity principle (**Proposition 5.1.1**). Since the monodromy representation is a discrete datum, a continuous deformation of the CSG is uniquely determined by the deformation it induces on a single Abelian fiber X_u. Suppose that the CSG under considerations has a normal ray with semi-simple monodromy m_*. We focus on a fiber over a point in the ray. If the monodromy is regular, by definition there is no deformation of the fiber, that is, the conformal manifold reduces to an isolated point. On the other hand, m_* regular implies that no chiral operator has dimension 2. Indeed, in an interacting theory a chiral operator of dimension 2 is necessarily a generator of $\mathscr{R}$. Suppose the dimension of the i-th generator is 2. Then from eqn.(10.51)

$$1 + \frac{\beta_i}{1 - \beta_0} = 2 \quad \Rightarrow \quad e^{-2\pi i \beta_i}\, e^{-2\pi i \beta_0} = \zeta_i \zeta_0 = 1, \tag{10.75}$$

contradicting the assumption that m_* is regular. If m_* is semi-simple, but not regular, the same argument shows that the dimension d of the conformal manifold is $\leq$ the number of generators of $\mathscr{R}$ having dimension 2. This is the physically expected result: the number d of exactly marginal operators is not greater than the number of chiral operators of dimension 2. Note that this is just an inequality since being m_*-invariant is only a necessary condition for a deformation to be allowed. Some of the m_*-invariant deformations may be obstructed by other elements of the monodromy group; comparison with the constant period map case shows that the obstruction comes from the non-semisimple part of the monodromy group. The non-semisimple part of the monodromy measures the effective one-loop beta-function of the QFT (cfr. discussion around eqn.(5.40)). Thus the dimension formula (10.51) is consistent with the physical expectations on the conformal manifold.

10.4 The set $\Xi(k)$ of allowed dimensions

We write $\Xi(k) \subset (\mathbb{Q}_{\geq 1})$ for the set of all rational numbers which appear as dimension Δ of a generator of the chiral ring $\mathscr{R}$ in a CSG of rank $\leq k$. Clearly $\Xi(k)$ is monotonic in k: $\Xi(k-1) \subset \Xi(k)$. If M is a rank k_2 CSG, its symmetric power $M^{[k_1]}$ – if not too singular – should also be a CSG. This rather sloppy argument would suggest that the set $\Xi(k)$ also satisfies the following requirement: for all $k_1, k_2 \in \mathbb{N}$

$$\bigcup_{s=1}^{k_2} s \cdot \Xi(k_1) \subset \Xi(k_1 k_2), \tag{10.76}$$

where $s \cdot \Xi(k)$ stands for the set of rationals obtained by multiplying all rationals in $\Xi(k)$ by the integer s.

To determine $\Xi(k)$ it suffices to give the difference of the sets for two successive ranks k; we already know that $\Xi(1) = \{1, 2, 3, 4, 6, 3/2, 4/3, 6/5\}$. The **Fact** stated at then end of §.10.3.2

implies

$$\Xi(k) \subseteq \widehat{\Xi}(k) \equiv \left\{ \frac{\ell}{s} \in \mathbb{Q}_{\geq 1} \;\middle|\; \phi(\ell) \leq 2k,\ (\ell, s) = 1 \right\}. \tag{10.77}$$

A priori this is just an inclusion, that is, the conditions we got insofar are just necessary conditions. However, experience with the first few k's suggests that the two sets $\Xi(k)$ and $\widehat{\Xi}(k)$ are pretty close, and likely equal. The discrepancy (if there is any) is expected to vanish as k increases. Note that

$$\mathbb{Q}_{\geq 1} = \bigcup_{k=1}^{\infty} \widehat{\Xi}(k), \tag{10.78}$$

that is, all rational numbers ≥ 1 appear in the list for some (large enough) rank k. For instance if Δ is a *large integer,* the Coulomb dimension Δ will first appear in rank $k_{\min}$ [218]

$$k_{\min} = \frac{1}{2}\,\phi(\Delta) > \frac{1}{2}\,\frac{\Delta}{e^{\gamma}\,\log\log\Delta + \frac{A}{\log\log\Delta}}, \quad \text{where } A = 2.50637, \tag{10.79}$$

while (for comparison) the minimal rank for a Lagrangian SCFT is

$$k_{\text{min,Lag.}} = \Delta/2 > \phi(\Delta)/2. \tag{10.80}$$

Eqn.(10.77) is shown by recursion in k. For $k = 1$ we know that it is true with $\subseteq$ replaced by $=$. We consider the rays $M_* \subset M$ generated by the vev of a single chiral field of dimension Δ; along M_* a R-symmetry $\mathbb{Z}_\ell$ is preserved, ℓ being the order of $1/\Delta$ in $\mathbb{Q}/\mathbb{Z}$, so that $1/\Delta = s/\ell$ with $1 \leq s \leq \ell$ and $(s, \ell) = 1$. We have $\ell = 1, 2$ iff $\Delta = 1, 2$. Now let m_* be the corresponding element of the monodromy. $m_*^\ell = 1$. m_* acts on the $\mathbb{C}$-period a_* non-zero on M_* by a primitive ℓ-th root of unity, hence the cyclotomic polynomial $\Phi_\ell(z)$ divides $\det[z - m_*]$, so that $\phi(\ell) = \deg \Phi_\ell(z) \leq \deg\det[z - \mu_*] = 2k$. The monotonicity of $\widehat{\Xi}(k)$ is obvious. Eqn.(10.76) for $\widehat{\Xi}(k)$ is equivalent to the inequality $\phi(\ell k_2) \leq \phi(\ell)\, k_2$ which holds by Euler's formula

$$\frac{\phi(\ell k_2)}{\phi(\ell)\, k_2} = \prod_{\substack{p \mid k_2 \\ p \nmid \ell}} \left(1 - \frac{1}{p}\right) \leq 1. \tag{10.81}$$

For bookkeeping, it is convenient to list the (candidate) "new-dimensions" at rank k

$$\mathfrak{N}(k) \equiv \widehat{\Xi}(k) \setminus \widehat{\Xi}(k-1). \tag{10.82}$$

Example 35. For instance, for $k = 2$

$$\mathfrak{N}(2) = \left\{5,\ \frac{5}{2},\ \frac{5}{3},\ \frac{5}{4},\ 8,\ \frac{8}{3},\ \frac{8}{5},\ \frac{8}{7},\ 10,\ \frac{10}{3},\ \frac{10}{7},\ \frac{10}{9},\ 12,\ \frac{12}{5},\ \frac{12}{7},\ \frac{12}{11}\right\}. \tag{10.83}$$

Remark 10.4.1. After the completion of our paper [52], the paper [24] appeared on the arXiv in which $\Xi(2)$ is also determined. The agreement with (10.83) is perfect.

The new-dimension sets $\mathfrak{N}(k)$ up to $k = 13$ are listed in table 14.

10.4.1 The number of allowed dimensions at given rank k

The number of elements of $\mathfrak{N}(k)$ (resp. the number of *integers* in $\mathfrak{N}(k)$) is

$$|\mathfrak{N}(k)| = 2k \cdot \nu(2k) \qquad \text{resp.} \qquad |\mathfrak{N}(k)|_{\text{int.}} = \nu(2k), \tag{10.84}$$

where the Number-Theoretic function $\nu(d)$ is the *totient multiplicity* of $d \in \mathbb{N}$ [219], that is, the number of solutions to $\phi(x) = d$. A positive integer d is called a *totient* iff it belongs to the range of ϕ, i.e. if $\nu(d) > 0$; an integer d is called a *nontotient* if it not a totient, i.e. if $\nu(d) = 0$. Thus, if $2k$ is a *nontotient* there are no new-dimensions in rank k, $\mathfrak{N}(k) = \emptyset$, and $\widehat{\Xi}(k) = \widehat{\Xi}(k-1)$. The first few even nontotients are (see sequence A005277 in OEIS [204])

$$14,\ 26,\ 34,\ 38,\ 50,\ 62,\ 68,\ 74,\ 76,\ 86,\ 90,\ 94,\ 98,\ 114,\ \cdots \tag{10.85}$$

hence

$$\begin{aligned} &\mathfrak{N}(7) = \mathfrak{N}(13) = \mathfrak{N}(17) = \mathfrak{N}(25) = \mathfrak{N}(31) = \mathfrak{N}(34) = \mathfrak{N}(37) = \\ &\quad = \mathfrak{N}(38) = \mathfrak{N}(43) = \mathfrak{N}(45) = \mathfrak{N}(47) = \mathfrak{N}(49) = \mathfrak{N}(57) = \cdots = \emptyset. \end{aligned} \tag{10.86}$$

The first few valued of the totient multiplicity $\nu(2k)$ (for $k \in \mathbb{N}$) are (see sequences A014197 or A032446 in OEIS [204])

$$\nu(2k) = 3,\ 4,\ 4,\ 5,\ 2,\ 6,\ 0,\ 6,\ 4,\ 5,\ 2,\ 10,\ 0,\ 2,\ 2,\ 7\,,0,\ 8,\ 0,\ 9,\ 4,\ 3,\ 2,\ 11,\ 0,\ \cdots \tag{10.87}$$

Properties of $\nu(d)$. We list some useful properties of the function $\nu(d)$ [219]:

1) Nontotients have density 1 in $\mathbb{N}$, i.e. totients are "sparse" (density zero) so for "most" ranks there are *no* new-dimensions. More precisely, let $N(x)$ be the number of totients less or equal x,

$$N(x) = \#\big\{m \;\big|\; \exists\, n \in \mathbb{N} \colon \phi(n) = m,\ \text{and } m \leq x\big\}, \tag{10.88}$$

then for all $\varepsilon > 0$ there exists $x(\varepsilon) \in \mathbb{N}$ such that

$$N(x) < \frac{x}{(\log x)^{1-\epsilon}} \quad \text{for } x > x(\varepsilon). \tag{10.89}$$

2) Every integer has a multiple which is a nontotient.
3) The function $\nu(d)$ takes all integral values ≥ 2 infinitely times.
4) The Carmichael conjecture (still open) states that $\nu(d) \neq 1$ for all d; the conjecture is known to be true for $d < 10^{10^{10}}$.
5) One has

$$k \text{ odd} \quad \Rightarrow \quad |\mathfrak{N}(k)| \leq 8k. \tag{10.90}$$

The cardinality $\boldsymbol{N}(k)$ of the set $\widehat{\Xi}(k)$. $\boldsymbol{N}(k)$ may be written in the form of a Stieltjes integral

$$\boldsymbol{N}(k) \equiv |\widehat{\Xi}(k)| = 2 + \sum_{\ell=1}^{k} 2\ell\, \nu(2\ell) = 2 + \int_{2-\epsilon}^{2k+\epsilon} x\, dV(x). \tag{10.91}$$

where $V(x)$ is the Erdös-Bateman Number-Theoretic function [34, 219]

$$V(x) = \sum_{n \leq x} \nu(n), \tag{10.92}$$

whose values for $x \in \mathbb{N}$ are given by the sequence A070243 in OEIS [204]. Note that $V(2k)$ is also the number $\boldsymbol{N}(k)_{\text{int}}$ of *integral* elements of $\widehat{\Xi}(k)$

$$\boldsymbol{N}(k)_{\text{int}} \equiv \big|\, \widehat{\Xi}(k) \cap \mathbb{N} \,\big| = V(2k). \tag{10.93}$$

The first few values are

$$\begin{aligned} \boldsymbol{N}(k)_{\text{int}} = 5,\ 9,\ 13,\ 18,\ 20,\ 26,\ 26,\ 32,\ 36,\ 41,\ 43,\ 53,\ 53,\ 54,\ 57,\ 64,\ 64, \\ 72,\ 72,\ 81,\ 85,\ 88,\ 90,\ 101,\ 101,\ 103,\ 105,\ 108,\ 110,\ 119,\ 119,\ 127,\ \cdots \end{aligned} \tag{10.94}$$

For large x [219][78]

$$V(x) = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\, x + o\!\left(\frac{x}{(\log x)^C}\right) \qquad \forall\, C > 0, \tag{10.95}$$

$$\frac{\zeta(2)\,\zeta(3)}{\zeta(6)} = 1.9435964\ldots \tag{10.96}$$

so that as $k \to \infty$

$$\boldsymbol{N}(k) = \frac{2\,\zeta(2)\,\zeta(3)}{\zeta(6)}\, k^2 + o(k^2). \tag{10.97}$$

Various expressions for the error term may be found in ref. [34, 219].

To construct the putative new-dimension sets $\mathfrak{N}(k)$ we only need to solve the equation $\phi(x) = 2k$ for all $k \geq 2$. There is an explicit algorithm to solve recursively this equation for a given k once we know the solutions for all $k' < k$ [184].

10.4.2 Analytic expressions of $\boldsymbol{N}(k)$ and $\boldsymbol{N}(k)_{\text{int}}$

We define $\boldsymbol{N}(x)_{\text{int}}$ for *real* x as

$$\boldsymbol{N}(x)_{\text{int}} = V(2x) \equiv \sum_{n \leq 2x} \nu(n). \tag{10.98}$$

[78] $\zeta(s)$ is the Riemann zeta-function.

Note that $\boldsymbol{N}(1/2)_{\rm int}=2$ corresponding to the rank-1 Lagrangian SCFT. With this convention, eqn.(2.4) holds[79]

$$\boldsymbol{N}(k)=2\int\limits_0^{k+\epsilon} x\, d\boldsymbol{N}(x)_{\rm int}. \tag{10.99}$$

To obtain an analytic formula for $\boldsymbol{N}(x)_{\rm int}$ one starts from the Dirichlet generating function $\mathscr{N}(s)$ (chap. 1 of [151]) for the totient multiplicities $\nu(m)$. $\mathscr{N}(s)$ has an Euler product

$$\mathscr{N}(s):=\sum_{m=1}^\infty\frac{\nu(m)}{m^s}=\sum_{n=1}^\infty\frac{1}{\phi(n)^s}=\prod_{p:\ \text{prime}}\left(1+\frac{p^s}{(p-1)^s\,(p^s-1)}\right). \tag{10.100}$$

The function $\mathscr{N}(s)/\zeta(s)$ is analytical in the half-plane $\mathrm{Re}\, s>0$, and takes the value $\zeta(2)\zeta(3)/\zeta(6)$ at $s=1$ [34]. Then the subtracted generating function

$$\widetilde{\mathscr{N}}(s):=\mathscr{N}(s)-\frac{\zeta(2)\,\zeta(3)}{\zeta(6)}\,\frac{s}{s-1}, \tag{10.101}$$

is analytic in the full half-plane $\mathrm{Re}\, s>0$. The totient multiplicities $\nu(m)$ are obtained from $\widetilde{\mathscr{N}}(s)$ by taking the inverse Mellin transform of eqn.(10.100). Thus (cfr. [34] §. 6)

$$\boldsymbol{N}(x)_{\rm int}=\frac{2\,\zeta(2)\,\zeta(3)}{\zeta(6)}\,x+\lim_{t\to\infty}\int\limits_{2-it}^{2+it}\frac{(2x+\epsilon)^s}{s}\,\widetilde{\mathscr{N}}(s)\,\frac{ds}{2\pi i}. \tag{10.102}$$

(note the prescription for this convergent but not absolutely convergent integral). From eqn.(10.102) the asymptotic formula eqn.(2.7) follows by deforming the integration contour to a suitable path along which $\mathrm{Re}\, s<1$ [34].

10.5 Dimension k-tuples $\{\Delta_1,\dots,\Delta_k\}$

As discussed in §. 10.3.4, from the local analysis at a regular (normal) ray we get the full k-tuple of dimensions $\{\Delta_1,\dots,\Delta_k\}$, a prototypical case being the Picard-Lefschetz theory of a SCFT engineered by F-theory on a singularity. More generally, from local considerations on a ray of regular rank $k_{\rm reg}$ we get $k_{\rm reg}$ out of the k dimensions. The number k-tuples of allowed dimensions in presence of rays of regularity $k_{\rm reg}\geq 2$ is much less than $|\Xi(k)|^k$ due to correlations between the dimensions of the various chiral operators of a given SCFT.

At a regular ray we have

$$\det[z-m_*]=\Phi_{d_1}(z)\Phi_{d_2}(z)\cdots\Phi_{d_t}(z) \tag{10.103}$$

where the factors are all distinct. Two embeddings $m_*\hookrightarrow Sp(2k,\mathbb{Z})$ are equivalent if they are conjugate in $Sp(2k,\mathbb{Z})$. A weaker notion of equivalence is conjugacy in $Sp(2k,\mathbb{R})$. It follows from the considerations in §. 10.2 that a $Sp(2k,\mathbb{R})$ conjugacy class is characterized by a subset of k out of

[79] The $+\epsilon$ prescription is needed since $\boldsymbol{N}(x)$ is not a continuous function but rather a function of bounded total variation on compact subsets of $\mathbb{R}$.

$\Phi_3\Phi_6$	$\boldsymbol{e^{2\pi i/3}}, e^{10\pi i/6}$	$\{3,3/2\}$	$\boldsymbol{e^{2\pi i/6}}, e^{4\pi i/3}$	$\{6,3\}$
	$\boldsymbol{e^{4\pi i/3}}, e^{10\pi i/6}$	$\{3/2,5/4\}$	$\boldsymbol{e^{2\pi i/6}}, e^{2\pi i/3}$	$\{6,5\}$
$\Phi_4\Phi_6$	$\boldsymbol{e^{2\pi i/4}}, e^{10\pi i/6}$	$\{4,5/3\}$	$\boldsymbol{e^{2\pi i/6}}, e^{6\pi i/4}$	$\{6,5/2\}$
Φ_5	$\boldsymbol{\zeta}, \zeta^3$	$\{5,3\}$	$\boldsymbol{\zeta^2}, \zeta^4$	$\{5/2,3/2\}$
	$\boldsymbol{\zeta}, \zeta^2$	$\{5,4\}$	$\boldsymbol{\zeta^3}, \zeta^4$	$\{5/3,4/3\}$
	$\boldsymbol{\zeta^2}, \zeta$	$\{5/2,3\}$	$\boldsymbol{\zeta^4}, \zeta^3$	$\{5/4,3/2\}$
Φ_8	$\boldsymbol{\zeta}, \zeta^5$	$\{8,4\}$	$\boldsymbol{\zeta^3}, \zeta^7$	$\{8/3,4/3\}$
	$\boldsymbol{\zeta}, \zeta^3$	$\{8,6\}$	$\boldsymbol{\zeta^5}, \zeta^7$	$\{8/5,6/5\}$
	$\boldsymbol{\zeta^3}, \zeta$	$\{8/3,10/3\}$	$\boldsymbol{\zeta^7}, \zeta^5$	$\{8/7,10/7\}$
	$\boldsymbol{\zeta^5}, \zeta$	$\{8/5,12/5\}$	$\boldsymbol{\zeta^7}, \zeta^3$	$\{8/7,12/7\}$
Φ_{10}	$\boldsymbol{\zeta}, \zeta^7$	$\{10,4\}$	$\boldsymbol{\zeta^3}, \zeta^9$	$\{10/3,4/3\}$
	$\boldsymbol{\zeta}, \zeta^3$	$\{10,8\}$	$\boldsymbol{\zeta^7}, \zeta^9$	$\{10/7,8/7\}$
	$\boldsymbol{\zeta^3}, \zeta$	$\{10/3,4\}$	$\boldsymbol{\zeta^9}, \zeta^7$	$\{10/9,4/3\}$
Φ_{12}	$\boldsymbol{\zeta}, \zeta^7$	$\{12,6\}$	$\boldsymbol{\zeta^5}, \zeta^{11}$	$\{12/5,6/5\}$

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Φ_{12}	$\boldsymbol{\zeta}, \zeta^5$	$\{12,8\}$	$\boldsymbol{\zeta^7}, \zeta^{11}$	$\{12/7,8/7\}$

Table 11: List of inequivalent embeddings of regular finite cyclic subgroups of $Sp(4,\mathbb{Z})$ that lead to dimensions $\{\Delta_1,\Delta_2\} \in \widehat{\Xi}(2)^2$. For the characteristic polynomial Φ_d, ζ stands for the standard primitive d-root, $\zeta = e^{2\pi i/d}$. The boldface root is the one associated with the non-zero period a_* along the regular ray. The dimension of the operator spanning the ray M_* is always the *first* one in the ordered pair $\{\Delta_1,\Delta_2\}$.

the $2k$ roots of the characteristic polynomial (10.103) (namely the spectrum of $\mathcal{U}(m_*)$), $\{\zeta_1,\cdots,\zeta_k\}$, having the property that $\zeta_i\zeta_j \neq 1$ for all $1 \leq i,j \leq k$ (cfr. eqn.(10.31)). Once given the spectrum $\{\zeta_i,\cdots,\zeta_k\}$ of $\mathcal{U}(m_*)$, we have to select which one of the k roots is the eigenvalue associated to the non-zero (multivalued) period a_* on M_*. Renumbering the roots so that this is the first one, we have

$$e^{2\pi i\alpha_*} = \zeta_1, \quad e^{-2\pi i\beta_j} = \zeta_j, \quad j \geq 2, \tag{10.104}$$

the dimension k-tuple $\{\Delta_i\}$ then being given by eqn.(10.51). It may happen that some of the dimensions in $\{\Delta_i\}$ so found do not belong to $\widehat{\Xi}(k)$; such a k-tuple should be discarded.

A priori, there are 2^k ways of splitting the spectrum of m_* into two non-overlapping sets satisfying eqn.(10.31). Taking into account the k choices of the root we call ζ_1, this yield $k\cdot 2^k$ possibilities for $\{\alpha_*,\beta_i\}$ for a given characteristic polynomial of the form (10.103). However, some of these possibilities come with a restriction: it is not true in general that we may find arithmetical embeddings $m_* \hookrightarrow Sp(2k,\mathbb{Z})$ which realize as spectrum of $\mathcal{U}(m_*)$ all the subsets of roots consistent with eqn.(10.31). The simplest example is provided by the regular embeddings

$$\mathbb{Z}_{12} \hookrightarrow Sp(4,\mathbb{Z}), \qquad \det[z - m_*] = \Phi_{12}(z). \tag{10.105}$$

In this case the sets $\{e^{2\pi i/12}, e^{14\pi i/12}\}$ and $\{e^{10\pi i/12}, e^{22\pi i/12}\}$ are realized as $\mathsf{Spectrum}\ \mathcal{U}(m_*)$, while $\{e^{2\pi i/12}, e^{10\pi i/12}\}$ and $\{e^{14\pi i/12}, e^{22\pi i/12}\}$ are *not* realized. Thus the allowed dimension sets depends on subtle Number-Theoretical aspects of the classification of all inequivalent embedding $\mathbb{Z}_n \hookrightarrow Sp(2k,\mathbb{Z})$; this topic will be addressed in section 5. There we shall justify the above claim on the embeddings $\mathbb{Z}_{12} \hookrightarrow Sp(4,\mathbb{Z})$. In section 5 we shall also see that the two spectra $\{e^{2\pi i/12}, e^{10\pi i/12}\}$ and $\{e^{14\pi i/12}, e^{22\pi i/12}\}$ may be realized by an embedding in $\mathbb{Z}_{12} \hookrightarrow S(\Omega)_\mathbb{Z}$ where Ω is a polarization with charge multiplies $(1, e_2)$ with $e_2 \geq 2$; this result agrees with **Example 5**.

Summarizing: a conjugacy class of regular embeddings $\mathbb{Z}_n \hookrightarrow Sp(2k,\mathbb{Z})$ is a candidate for the monodromy m_* at a normal (regular) ray $M_* \subset M$ along which the discrete subgroup $\mathbb{Z}_n \subset U(1)_R$ is unbroken. The datum of the conjugacy class, together with a choice of ζ_1, produces a candidate dimension k-tuple $\{\Delta_1, \cdots, \Delta_k\}$ by eqn.(10.51). It is not guaranteed that $\{\Delta_1, \cdots, \Delta_k\} \in \widehat{\Xi}(k)^k$, and k-tuples which do not belong to $\widehat{\Xi}(k)^k$ should be discarded. At this point we must also impose consistency between the various normal rays M_*. This reduce the list of allowed k-tuple even further. While we have no proof that all the surviving k-tuples are actually realized by some CSG, the experience suggests that this algorithm produces few "spurious" k-tuples, *if any.* As an illustration, we now run the algorithm in detail for $k=2$. Ranks $k=3$ and $k=4$ may be found in the tables of section 6.

10.5.1 $\{\Delta_1, \Delta_2\}$ for rank-2 SCFTs

The possible regular characteristic polynomial are $\Phi_3\Phi_4$, $\Phi_3\Phi_6$, $\Phi_4\Phi_6$, Φ_5, Φ_8, Φ_{10}, Φ_{12}. In table 11 we write only the embeddings of the corresponding regular cyclic groups in $Sp(4,\mathbb{Z})$ which lead to dimensions $\{\Delta_1, \Delta_2\} \in \widehat{\Xi}(2)^2$. For instance, $\Phi_3\Phi_4$ does not have any embedding with this property. We have written separately the dimensions pairs associated to embeddings in groups $S(\Omega)_\mathbb{Z}$ with Ω a non-principal polarization. The fine points on the conjugacy classes of elliptic elements in the Siegel modular group $Sp(2k,\mathbb{Z})$ will be discussed in the next section; table 11 summarizes the results of that analysis in the special case $k=2$.

Consistency between the two rays $u_2 = 0$ and $u_1 = 0$ leads us to consider three situations:

RR pairs of dimensions which appear twice in table 11, once in the form $\{\Delta_1, \Delta_2\}$ and once in the form $\{\Delta_2, \Delta_1\}$ corresponding to the case of both rays being strongly regular;

RN a pair of dimensions $\{\Delta_1, \Delta_2\}$ where the second one $\Delta_2 \in \Xi(1)$ corresponding to one strongly regular and one weakly regular of irregular ray;

NN $\{\Delta_1, \Delta_2\} \in \Xi(1)^2$ two weakly regular/irregular rays.

All rank-2 CSG with a conformal manifold of positive dimension must be of type **NN**. This holds, in particular, for the weakly coupled Lagrangian models which have

$$\{\Delta_1, \Delta_2\} = \{2,2\},\ \{2,3\},\ \{2,4\},\ \text{or}\ \{2,6\}. \tag{10.106}$$

The list of dimension pairs satisfying **RR** is rather short: $\{3/2, 5/4\}$ and $\{10/7, 8/7\}$ which correspond, respectively, to AD of types A_5 and A_4. We have 17 dimension pairs of type **RN** which may be read from table 11; all of them but $\{5,4\}$, $\{8,6\}$ and $\{10/3,4\}$ have already appeared

$\Delta[v]$	$\Delta[u]$	$y^2 = \cdots$	M_v	M_u
10/7	8/7	$x^5 + ux + v$	$RI\ \mathbb{Z}_{10}$	$RI\ \mathbb{Z}_8$
8/5	6/5	$x^5 + ux^2 + vx$	$RI\ \mathbb{Z}_8$	$Ir\ \mathbb{Z}_6$
5/2	3/2	$x^5 + ux^2 + 2uvx + v^2$	$RI\ \mathbb{Z}_5$	$Ir\ \mathbb{Z}_3$
4	2	Lagrangian SCFT		
10	4	$x^5 + (ux + v)^3$	$RI\ \mathbb{Z}_{10}$	$Ir\ \mathbb{Z}_4$
3/2	5/4	$x^6 + ux + v$	$RR\ \mathbb{Z}_6$	$RI\ \mathbb{Z}_5$
5/3	4/3	$x^6 + ux^2 + vx$	$RI\ \mathbb{Z}_5$	$Ir\ \mathbb{Z}_4$
3	2	Lagrangian SCFT		
5	3	$x^6 + x(ux + v)^2$	$RI\ \mathbb{Z}_5$	$Ir\ \mathbb{Z}_3$
10/3	4/3	$v^{-1}[x^6 + v^2x(ux + 2v)]$	$RI\ \mathbb{Z}_{10}$	
6	2	Lagrangian SCFT		

Table 12: Geometries in refs. [26, 27] with univalued symplectic structure. First two columns give the Coulomb dimensions, third the family of hyperelliptic curves, third and fourth the regularity/irregularity of the rays along the axes together with the corresponding unbroken R-symmetry.

in the literature as Coulomb dimensions of some CSG. There are three **RN** dimension pairs with $\Delta_i < 2$: $\{8/5, 6/5\}$ (AD of type D_5), $\{5/3, 4/3\}$ (AD of type D_6) and $\{10/9, 4/3\}$ on which we shall comment in the next subsection.

10.5.2 Comparing with Argyres *et al.* refs. [26, 27]

The authors of refs. [26, 27] have given a (possibly partial) classification of the dimension pairs $\{\Delta_1, \Delta_2\}$ which may appear in a rank 2 SCFT using quite different ideas. It is interesting to compare their list with the present arguments. To perform the comparison, we need to keep in mind the two *caveat* in §.5.1.1.

Let us recall the framework of [26, 27]. They start from the fact that all families of rank 2 *principally polarized* Abelian varieties are families of Jacobians of genus 2 hyperelliptic curves which they write in two ways

$$y^2 = v^{-r/s}\Big(x^5 + \cdots\Big) \quad \text{and} \quad y^2 = v^{-r/s}\Big(x^6 + \cdots\Big), \tag{10.107}$$

where $\cdots$ stand for certain polynomials in x, u, v depending on the particular CSG which are listed in [26, 27]. u, v are the "global" (in their sense) coordinates in the conical Coulomb branch, with v the operator of larger dimension. $0 \leq r/s \leq 1$ is a rational number written in minimal terms, i.e. $(r, s) = 1$. Their SW differential has the form

$$\lambda \equiv v\,\frac{dx}{y} + u\,\frac{x\,dx}{y} + u,\ v \text{ independent.} \tag{10.108}$$

The two rays $M_v = \{u = 0\}$ and $M_u = \{v = 0\}$ preserve a discrete R-symmetry which may be read for each CSG from the explicit polynomials in the large parenthesis. From the same expressions

$\Delta[v^{\lvert 1-r/s\rvert}]$	$\Delta[u]$	$y^2 = \cdots$
4/3	10/9	$v^{-2/5}[x^5 + v(5ux^2 - 15vx - 6uv)]$
8/5	6/5	$v^{-1/3}[x^5 + vx(2ux + 3v)]\ \mathbb{Z}_8$
8/5	6/5	$v^{-1/3}[x^2 - 4v][x^3 - 2v(3x + 2u)]$
5/2	3/2	$v^{-1/3}[x^5 + v(2ux + 3v)^2]\ \mathbb{Z}_5$
8/5	6/5	$v^{-2/3}[x^5 + v^2x(ux + 3v)]$
5/2	3/2	$v^{-2/3}[x^5 + v^2(ux + v)^2]$
4	2	$v^{-1/2}[x^5 + vx(ux + 2x)^2]$
10	4	$v^{-1/2}[x^5 + v(ux + 2v)^3]$
10/9	4/3	$v^{-1/2}[x^6 + vx(3ux + 4v)]$
2	2	$v^{-1/2}[x^6 + v(3ux + 4v)^2]$
10/3	4/3	$v^{-3/2}[x^6 + v^3x(ux + 4v)]$
5/2	3	$v^{-2/3}[x^6 + vx(2ux + 3v)^2]$
6	2	$v^{-3/2}[x^6 + v^3(ux + 4v)^2]$
5	3	$v^{-4/3}[x^6 + v^2x(ux + 3v)^2]$

Table 13: The geometries in refs. [26, 27] with multivalued symplectic structure. First two columns contain the cover Coulomb dimensions; third one the hyperelliptic curves.

we may read if these rays are regular irreducible (RI), regular reducible (RR), or irregular (Ir).

We may distinguish their geometries in two classes. The first one is when $r/s \in \mathbb{Z}$, that is, the global pre-factor in the RHS of eqn.(10.108) is a univalued function of v. The geometries with this property listed in refs. [26, 27] are recalled in table 12 (we do not bother to discuss the Lagrangian models since the agreement with our results is obvious in this case). The first two columns are the dimensions of v and u as listed in refs. [26, 27]. We see that in all cases these dimensions belong to the intersection of the sets of dimensions associated with the cones M_v, M_u, yielding perfect agreement with our approach. Note that only dimensions consistent with a principal polarization appear, since this is an assumption in [26, 27].

The second class of geometries is when $r/s \notin \mathbb{Z}$, that is, a multi-valued prefactor. As discussed in [26] these geometries lead to a susy central charge Z which is well-defined up to a (locally constant) unobservable phase. Here the first remark of §.5.1.1 applies: to get a univalued SW differential λ we need to go to a finite cover where a suitable fractional power of v becomes univalued. Then we consider as global coordinates on the cover the functions $(v^{(1-r/s)}, u)$ and compare their dimensions with the ones in our table. This leads to the dimension list in table 13; we see that all dimension pairs agree with our table on the nose.

Of course, if the physically correct Coulomb branch is the geometrically natural covering which has a well-defined holomorphic symplectic structure Ω or its quotient considered by Argyres et al. it is a question of physics not of geometry. There is one aspect that suggests that quotient of [26, 27] is the physical Coulomb branch: the dimension pair $\{10/9, 4/3\}$ enters in their list (twice) only through quotient CSG. The dimensions of the two quotient geometries are, respectively,

$\{20/9, 10/9\}$ and $\{20/9, 4/3\}$. Now, the both covering dimensions are < 2, and there is evidence that a consistent SCFT with all $\Delta_i < 2$ should be an Argyres-Douglas model of type ADE; since $\{10/9, 4/3\}$ does not correspond to such a model, we are inclined to think that physics requires to take a discrete quotient of the geometrically natural geometry, as the authors of [26, 27] do.

11 Elliptic conjugacy classes in Siegel modular groups

Listing the dimension k-tuples $\{\Delta_1, \cdots, \Delta_k\}$ has been reduced to understand the conjugacy classes of finite order elements inside the Siegel modular group $Sp(2k, \mathbb{Z})$ or, in case of more general polarizations (non-trivial charge multipliers e_i, see eqn.(5.2)) in the commensurable arithmetic group $S(\Omega)_{\mathbb{Z}}$. In this section we give an explicit description of such classes. Readers not interested in Number Theoretic subtleties may skip the section.

11.1 Preliminaries

We write Ω for the $2k \times 2k$ symplectic matrix in normal form and $\langle -, - \rangle$ for the corresponding skew-symmetric bilinear pairing.

11.1.1 Elements of $Sp(2k, \mathbb{Z})$ with spectral radius 1

$m \in Sp(2k, \mathbb{Z})$ has (spectral) radius 1 iff its characteristic polynomial is a product of cyclotomic ones

$$\det[z - m] = \prod_{d \in I} \Phi_d(z)^{s_d} \qquad \begin{array}{l} I = \{d_1, \cdots, d_{|I|}\} \subset \mathbb{N}, \\ s_d \in \mathbb{Z}_{\geq 1}, \quad \displaystyle\sum_{d \in I} s_d\, \phi(d) = 2k, \end{array} \tag{11.1}$$

that is, if all its eigenvalues are roots of unit. An element m of spectral radius 1 is *semi-simple* (over $\mathbb{C}$) iff its minimal polynomial is *square-free*, i.e.

$$\prod_{d \in I} \Phi_d(m) = 0. \tag{11.2}$$

A semisimple element m of radius 1 has finite order, $m^N = 1$ with $N = \mathrm{lcm}\{d \in I\}$. Conversely, all elements of finite order are semi-simple of radius 1.

Lemma 11.1.1. *Let $m \in Sp(2k, \mathbb{Z})$ be of finite order.*
1) *There exists $R \in Sp(2k, \mathbb{Q})$ which sets m in a block-diagonal form over $\mathbb{Q}$*

$$R\, m\, R^{-1} = \mathrm{diag}(m_1, \cdots, m_{|I|}), \quad \Phi_d(m_d) = 0, \quad m_d \in Sp(s_d \phi(d), \mathbb{Z}). \tag{11.3}$$

2) *Suppose that no ratio d_i/d_j $(i \neq j)$ is a prime power. Then in* (11.3) *we may choose $R \in Sp(2k, \mathbb{Z})$. More generally, if ℓ is a prime such that $\ell^r \neq d_i/d_j$ for all i, j and $r \neq 0$, we may choose $R \in Sp(2k, \mathbb{Z}_\ell)$.*

Proof. For each $d \in I$ we define the integral $2k \times 2k$ matrices

$$\Pi_d = \prod_{a \in (\mathbb{Z}/d\mathbb{Z})^\times} \prod_{\substack{e \in I\\ e \neq d}} \Big[m^{-a\phi(e)t(e)/2}\, \Phi_e(m^a)^{t(e)} \Big], \qquad \text{where } t(e) = \begin{cases} 2 & e = 1,2 \\ 1 & \text{otherwise.} \end{cases}. \tag{11.4}$$

Since $m\Omega = \Omega m^{-t}$, and

$$(1/z)^{\phi(e)t(e)/2}\, \Phi_e(z)^{t(e)} = z^{\phi(e)t(e)/2}\, \Phi_e(1/z)^{t(e)}, \tag{11.5}$$

we have $\Pi_d \Omega = \Omega \Pi_d^t$. Up to a rational multiple, the Π_d form a complete set of orthogonal idempotents over $\mathbb{Q}$

$$\mathbf{1} = \sum_{d \in I} \frac{\Pi_d}{\varrho_d}, \qquad \frac{\Pi_d}{\varrho_d} \cdot \frac{\Pi_e}{\varrho_e} = \delta_{d,e}\, \frac{\Pi_d}{\varrho_d} \tag{11.6}$$

compatible with the skew-symmetric pairing Ω. Then they split over $\mathbb{Q}$ the representation in the block diagonal form of item **1)**. The splitting is over $\mathbb{Z}$ iff the ϱ_d are ± 1. We have

$$\varrho_d = \pm \prod_{\substack{e \in I\\ e \neq d}} \mathrm{R}(\Phi_d, \Phi_e)^{t(e)} \tag{11.7}$$

where $\mathrm{R}(P, Q)$ stands for the resultant of the two polynomials P and Q. Under the assumption that d/e, e/d are not prime powers, $\varrho_d = \pm 1$ [7]. In facts, ϱ_d is divisible only by the primes p such that there is $e \in I$ with $d/e = p^r$, $0 \neq r \in \mathbb{Z}$ [7]. □

In other words, if no ratio d_i/d_j is a non-trivial prime power, all embeddings $m \hookrightarrow Sp(2k, \mathbb{Z})$ are block-diagonal up to equivalence. If some d_i/d_j is a prime power, in addition to the block-diagonal ones, we may have other inequivalent embeddings. We shall return to this aspect after studying the case that the minimal polynomial is irreducible over $\mathbb{Q}$.

11.1.2 Regular elliptic elements of the Siegel modular group

We recall that a finite-order element $m \in Sp(2k, \mathbb{Z})$ is regular iff the eigenvalues $\{\zeta_1, \cdots, \zeta_k\}$ of $\mathcal{U}(m) \equiv C\boldsymbol{\tau} + D$ satisfy $\zeta_i \zeta_j \neq 1$. The spectrum $\mathfrak{S} \equiv \{\zeta_i\}$ of $\mathcal{U}(m)$ will be called the *spectral invariant* of the regular elliptic element $m \in Sp(2k, \mathbb{Z})$. It is a subset of k roots, $\{\zeta_i\}$, out of the $2k$ ones of $\det[z - m]$ with the property

$$\zeta_i \zeta_j \neq 1 \qquad i, j = 1, \ldots, k. \tag{11.8}$$

Remark 11.1.1. Two regular elliptic elements $m, m' \in Sp(2k, \mathbb{Z})$ which are conjugate in $Sp(2k, \mathbb{R})$ but not in $Sp(2k, \mathbb{Z})$ have the same spectral invariant, $\mathfrak{S} = \mathfrak{S}'$ but their fixed points $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ are inequivalent periods. Two elements are fully equivalent (and should be identified) iff they are conjugate in $Sp(2k, \mathbb{Z})$. However the dimension spectrum $\{\Delta_i\}$ depends only on the spectral invariant $\mathfrak{S}$ of the monodromy and hence only on its $Sp(2k, \mathbb{R})$-conjugacy class.

11.1.3 The spectral invariant as a sign function

We focus on a $m \in Sp(2k, \mathbb{Z})$ whose minimal polynomial is $\mathbb{Q}$-irreducible,

$$\det[z - m] = \Phi_d(z)^s, \quad s\,\phi(d) = 2k, \quad s \in \mathbb{N}. \tag{11.9}$$

We fix a primitive d-root, ζ, and write $\mathbb{K} \equiv \mathbb{Q}[\zeta]$ for the corresponding cyclotomic field and $\Bbbk = \mathbb{Q}[\zeta + \zeta^{-1}]$ for its maximal totally real subfield, $\mathsf{Gal}(\mathbb{K}/\Bbbk) = \{\pm 1\}$.

Let $\psi_1^\alpha \in \mathbb{K}$ ($\alpha = 1, \ldots, s$) be a basis of the ζ-eigenspace of the matrix m. Let $\sigma \in \mathsf{Gal}(\mathbb{K}/\mathbb{Q}) \cong (\mathbb{Z}/d\mathbb{Z})^\times$; then $\psi_\sigma^\alpha \equiv \sigma(\psi_1^\alpha)$ form a basis of the $\sigma(\zeta)$-eigenspace. We write $\langle \psi_1^\alpha, \psi_{-1}^\beta \rangle = t^{\alpha\beta} \in \mathbb{K}(s)$. Without loss of generality, we may assume $t^{\alpha\beta}$ to be diagonal $t^{\alpha\beta} = t_\alpha \delta^{\alpha\beta}$ with $t_\alpha \in \mathbb{K}$. t_α is odd (i.e. purely imaginary) $\bar{t}_\alpha = -t_\alpha$. The symplectic structure is given by a 2-form

$$2\,\Omega = i \sum_\alpha \sum_{\sigma \in \mathsf{Gal}(\mathbb{K}/\mathbb{Q})} \sigma(t_\alpha)\, \psi_{-\sigma}^\alpha \wedge \psi_\sigma^\alpha. \tag{11.10}$$

Thus m is the direct sum of s (possibly inequivalent) embeddings $\mathbb{Z}_d \to Sp(\phi(d), \mathbb{R})$. For each summand we define the <u>odd</u> *sign (function)*

$$\mathsf{sign}_\alpha \colon \mathsf{Gal}(\mathbb{K}/\mathbb{Q}) \to \{\pm 1\}, \qquad \sigma \mapsto \frac{\sigma(t_\alpha)}{i|\sigma(t_\alpha)|}. \tag{11.11}$$

The spectral invariant of the α-th summand is

$$\mathfrak{S}_\alpha \equiv \Big\{ \zeta^\sigma \;\Big|\; \sigma \in \mathsf{Gal}(\mathbb{K}/\mathbb{Q}) \text{ such that } \mathsf{sign}_\alpha(\sigma) = +1 \Big\}. \tag{11.12}$$

It is obvious that $\mathfrak{S}_\alpha$ satisfies condition (11.8). We shall denote by the same symbol, $\mathfrak{S}_\alpha$, both the spectral invariant and the corresponding sing function. A semi-simple element m satisfying (11.9) is regular iff the spectral invariant is the same for all its direct summands, i.e. $\mathfrak{S}_\alpha = \mathfrak{S}_\beta$.

Remark 11.1.2. For a single regular block with characteristic polynomial $\Phi_d(z)$ the number of sub-sets $\mathfrak{S} \subset \{\zeta^a \colon a \in (\mathbb{Z}/d\mathbb{Z})^\times\}$ satisfying condition (11.8) is $2^{\phi(d)/2}$. However it is not true (in general) that all such sub-sets (i.e. sign functions) are realized as spectral invariant of some embedding $\mathbb{Z}_d \to Sp(\phi(d), \mathbb{Z})$. For instance, for $d = 12$ we have $2^{\phi(12)/2} = 4$, but only 2 sign functions are produced by actual embeddings. The set $\{\mathfrak{S}\}$ of sign functions which do are realized satisfies the obvious condition

$$\sigma \mapsto \mathsf{sign}(\sigma) \in \{\mathfrak{S}\} \;\Rightarrow\; \sigma \mapsto \mathsf{sign}(\tau\sigma) \in \{\mathfrak{S}\} \qquad \forall\, \tau \in \mathsf{Gal}(\mathbb{K}/\mathbb{Q}). \tag{11.13}$$

In particular if sign is realized $-\mathsf{sign}$ is also realized.

In §.10.5 that the list of possible Coulomb branch dimensions $\{\Delta_1, \cdots, \Delta_k\}$ is determined from the $Sp(2k, \mathbb{R})$-conjugacy classes of regular elements of $Sp(2k, \mathbb{Z})$ (or the corresponding arithmetic group for non principal polarizations) through their sign function invariant $\mathfrak{S}$. Then our main problem at this point is to understand the set $\{\mathfrak{S}\}$ of signs which do are realized for a given polarization. This is the next task.

11.2 Cyclic subgroups of integral matrix groups

In this section we review the theory of the embedding of cyclic groups into groups of matrices having integral coefficients in a language convenient for our purposes (also providing explicit expressions for the matrices). See also [186, 203]. Our basic goal is to describe the set $\{\mathfrak{S}\}$ of signs which do appear and more generally the regular elliptic elements of the Siegel modular group.

11.2.1 Embeddings $\mathbb{Z}_n \hookrightarrow GL(2k, \mathbb{Z})$ vs. fractional ideals

We focus on a single block, that is, we consider a matrix $m \in GL(2k, \mathbb{Z})$ with minimal polynomial $\Phi_n(z)$. Then $2k \equiv s\,\phi(n)$ for some $s \in \mathbb{N}$.

Notations. We fix once and for all a primitive n-root of unity $\zeta \in \mathbb{C}$, and write $\mathbb{K} \equiv \mathbb{Q}[\zeta]$ for the n-th cyclotomic field, $\mathfrak{O} \equiv \mathbb{Z}[\zeta]$ for its rings of integers, $\Bbbk \equiv \mathbb{Q}[\zeta + \zeta^{-1}]$ for its maximal real subfield, and $\mathfrak{o} \equiv \mathbb{Z}[\zeta + \zeta^{-1}]$ for the ring of algebraic integers in $\Bbbk$. $\mathsf{Gal}(\mathbb{K}/\Bbbk) \cong \mathbb{Z}_2$, the non-trivial element ι being complex conjugation, $\iota(x) = \bar{x}$. $\mathsf{Gal}(\Bbbk/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times/\{\pm 1\}$. We write $\breve{n}$ for the *conductor* of the field $\mathbb{K}$:

$$\breve{n} = \begin{cases} n & \text{if } n \neq 2 \mod 4 \\ n/2 & \text{otherwise.} \end{cases} \tag{11.14}$$

We write $C_\mathbb{K}$ ($C_\Bbbk$) for the group of ideal classes in $\mathbb{K}$ (resp. in $\Bbbk$). N will denote the relative norm $\mathbb{K} \to \Bbbk$ extended to the groups of fractional ideals $\mathfrak{I}_\mathbb{K} \xrightarrow{N} \mathfrak{I}_{\,\Bbbk}$ in the usual way [121, 188].

The embedding $\mathbb{Z}_n \hookrightarrow GL(2k, \mathbb{Z})$ makes $\mathbb{Z}^{2k}$ into a finitely-generated torsion-less $\mathfrak{O}$-module $\mathcal{M}$, multiplication by ζ being given by m. Conversely, any finitely-generated torsion-less $\mathfrak{O}$-module $\mathcal{M}$ defines an embedding $\mathbb{Z}_n \hookrightarrow GL(2k, \mathbb{Z})$ where $2k$ is the rank of $\mathcal{M}$ seen as a (free) $\mathbb{Z}$-module. The ring of cyclotomic integers $\mathfrak{O}$ is a Dedekind domain. The following statement holds for all such domains:

Proposition 11.2.1 (see [121])**.** *A finitely-generated torsion-less module $\mathcal{M}$ over the Dedekind domain $\mathfrak{O}$ has the form $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_s$, where $\mathfrak{a}_i$ are fractional ideals in $\mathbb{K}$. Two modules $\mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_s$ and $\mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots \oplus \mathfrak{b}_t$ are isomorphic if and only if $s = t$ and the ideal class $\prod_i \mathfrak{a}_i \mathfrak{b}_i^{-1}$ is trivial. In particular we may always set $\mathcal{M} = \mathfrak{O} \oplus \cdots \oplus \mathfrak{O} \oplus \mathfrak{a} \equiv (1)^{\oplus(s-1)} \oplus \mathfrak{a}$. Then the class of $\mathfrak{a}$ yields a one-to-one correspondence*

$$\big\{GL(2k, \mathbb{Z})\text{-conjugacy classes of embeddings } \mathbb{Z}_n \hookrightarrow GL(2k, \mathbb{Z})\big\} \xleftrightarrow{\text{1-1}} C_\mathbb{K}. \tag{11.15}$$

In order to describe the explicit embeddings we sketch the proof in the special case $s = 1$.

Proof. Let $\mathfrak{a} \subset \mathbb{K}$ be a fractional ideal. In particular $\mathfrak{a}$ is a torsion-free finitely generated $\mathbb{Z}$-module, hence a lattice isogeneous to $\mathfrak{O}$, and thus of rank $2k$. Choosing generators, we may write $\mathfrak{a} = \bigoplus_{a=1}^{2k} \mathbb{Z}\,\omega_a$, with $\omega_a \in \mathbb{K}$. Now $\zeta\,\omega_a \in \mathfrak{a}$, and hence there is an *integral* $2k \times 2k$ matrix m such that

$$\zeta\omega_a = m_{ab}\,\omega_b. \tag{11.16}$$

The minimal polynomial of m is the n-th cyclotomic polynomial, $\Phi_n(m) = 0$. Thus the matrix m yields an explicit embedding of $\mathbb{Z}_n$ into $GL(2k, \mathbb{Z})$. Had we chosen a different set of generators for

$\mathfrak{a}$, ω'_a, we would had gotten an integral matrix m' which differs from m by conjugacy in $GL(2k,\mathbb{Z})$. Indeed, $\omega'_a = A_{ab}\,\omega_b$, for some $A \in GL(2k,\mathbb{Z})$. Thus the map $\mathfrak{a} \mapsto$ (conjugacy class of m) is independent of all choices. By construction, the vector $\boldsymbol{\omega} \equiv (\omega_1,\cdots,\omega_{2k}) \in \mathbb{K}^{2k}$ is the eigenvector of m associated to the eigenvalue ζ. The eigenvector associated to the eigenvalue $\sigma(\zeta)$, $\sigma \in \mathsf{Gal}(\mathbb{K}/\mathbb{Q})$, is the $\sigma(\boldsymbol{\omega})$.

Conversely, if $m \in GL(2k,\mathbb{Z})$ with minimal polynomial Φ_n, consider an eigenvector $\boldsymbol{\omega} \equiv (\omega_1,\cdots,\omega_{2k}) \in \mathbb{K}^{2k}$ associated to the eigenvalue ζ, and set $\mathfrak{a} = \bigoplus_{a=1}^{2k} \mathbb{Z}\omega_a$. Clearly, if $\boldsymbol{\omega}$ is such an eigenvector so is $\mu\,\boldsymbol{\omega}$ for all $\mu \in \mathbb{K}^\times$. Hence $\mathfrak{a}$ and $(\mu)\mathfrak{a}$ ($\mu \in \mathbb{K}^\times$) describe the same conjugacy class of integral matrices m, that is, the conjugacy class of m depends only on the class of the fractional ideal $\mathfrak{a}$ in $\mathbb{C}_{\mathbb{K}} = \mathfrak{I}_{\mathbb{K}}/(\mathbb{K}^\times)$. □

The action of ζ on the module $\bigoplus_{i=1}^{s} \mathfrak{a}_i$ is unitary for the natural Hermitian form

$$\langle a_i, b_i \rangle = \sum_{i=1}^{s} \mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\bar{a}_i b_i). \tag{11.17}$$

Remark 11.2.1. A $\mathfrak{O}$-module $\mathcal{M}$ gives a unitary representation of $\mathbb{Z}_n$ on the associated $\mathbb{C}$-space $V_{\mathcal{M}} = \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{C}$ which corresponds to the natural embedding $GL(2k,\mathbb{Z}) \subset GL(2k,\mathbb{C})$.

11.2.2 The dual embedding

Given an embedding of $\mathbb{Z}_n \hookrightarrow GL(2k,\mathbb{Z})$, generated by the integral matrix m, we have a second embedding, the *dual one,* where the generator is represented by the integral matrix $(m^t)^{-1}$. If we write the matrices in an unitary basis of $V_{\mathcal{M}}$ (instead of an integral one), the two representations of $\mathbb{Z}_n$ are related by complex conjugation.

Let $\mathcal{M} = \bigoplus_i \mathfrak{a}_i$ be an $\mathfrak{O}$-module associated to the embedding m as in **Proposition 11.2.1**; then the dual $\mathfrak{O}$-module $\mathcal{M}^\vee$ is associated to the dual embedding $(m^t)^{-1}$. $\mathcal{M}^\vee$ is uniquely determined up to isomorphism.

Lemma 11.2.1. $\mathcal{M}^\vee = \bigoplus_i \mathfrak{a}_i^*$ *where* $\mathfrak{a}_i^*$ *is the dual (*a.k.a. *complementary) fractional ideal of* $\mathfrak{a}_i$ *(with respect to* (11.17)*). One has* [142, 176]

$$\mathfrak{a}_i^* = \frac{1}{(\,\overline{\Phi'_n(\zeta)}\,)\,\bar{\mathfrak{a}}_i} \tag{11.18}$$

where $\Phi'_n(z)$ *is the derivative of* $\Phi_n(z)$ *and* $\bar{\mathfrak{a}}_i$ *is the complex conjugate ideal of* $\mathfrak{a}_i$.

Note that

$$\mathfrak{a}^{**} = \left(\frac{\Phi'_n(\zeta)}{\overline{\Phi'_n(\zeta)}}\right)\mathfrak{a} = \mathfrak{a}, \qquad \text{since } \frac{\Phi'_n(\zeta)}{\overline{\Phi'_n(\zeta)}} \text{ is a unit in } \mathfrak{O}. \tag{11.19}$$

In APPENDIX B.1 we show a some properties of the map $\mathfrak{a} \to \mathfrak{a}^*$ which greatly simplify the computations. In particular:

Lemma 11.2.2. *For all fractional ideal* $\mathfrak{a}$ *of* $\mathbb{K}$ *we have*

$$\mathfrak{a}^* = \varrho/\bar{\mathfrak{a}}, \text{ for a certain } \varrho \in \mathbb{K}^\times \text{ with } \iota(\varrho) = -\varrho. \tag{11.20}$$

If $\breve{n}$ is not the power of an odd prime, we may alternatively choose ϱ to be real by multiplying it by a purely imaginary unit, e.g. $(\zeta - \zeta^{-1})$ for $\breve{n} \neq 2^r$ or i for $\breve{n} = 2^r$.

11.2.3 Complex, real, quaternionic

At the level of underlying $\mathbb{C}$-linear representations, $V_{\mathcal{M}} \cong V_{\mathcal{M}}^\vee \equiv V_{\mathcal{M}^\vee}$. An *anti*-linear morphism $R\colon V_{\mathcal{M}} \to V_{\mathcal{M}^\vee}$ is said to be a *real* (resp. *quaternionic*) structure iff it squares to $+1$ (resp. -1) [46]. A real (resp. quaternionic) structure embeds the matrix m in the orthogonal (resp. symplectic) group. To get an embedding $\mathbb{Z}_n \hookrightarrow Sp(2k,\mathbb{Z})$ we need a quaternionic structure defined over $\mathbb{Z}$. First of all, this requires $\mathcal{M}$ and $\mathcal{M}^\vee$ to be isomorphic as $\mathfrak{O}$-modules. Writing $\mathcal{M} = (1)^{\oplus(s-1)} \oplus \mathfrak{a}$, we must have

$$(1)^{\oplus(s-1)} \oplus \mathfrak{a} \cong (\varrho)^{\oplus(s-1)} \oplus \varrho\, \bar{\mathfrak{a}}^{-1}. \tag{11.21}$$

which implies that

$$N\mathfrak{a}\cdot\mathfrak{O} = \mathfrak{a}\,\bar{\mathfrak{a}} = \big(\eta\big). \tag{11.22}$$

Since the natural map $C_{\Bbbk} \to C_{\mathbb{K}}$, $[\mathfrak{b}] \mapsto [\mathfrak{b}\cdot\mathfrak{O}]$ is injective [177], the fractional ideal $N\mathfrak{a}$ is principal in $\Bbbk$, that is, $N\mathfrak{a} = (\eta)$ for some $\eta \in \Bbbk^\times$.

From eqn.(11.21) we see that the construction of the embedding is essentially reduced to the case $s = 1$. From now on we specialize to this case, so that $\mathcal{M} \equiv \mathfrak{a}$, $\mathcal{M}^\vee = \mathfrak{a}^*$. Fix a $\mathbb{Z}$-basis[80] $\{\omega_a\}$ of $\mathfrak{a}$ and let ϕ^a be the dual basis of $\mathfrak{a}^*$, i.e. $\langle\omega_a, \phi^b\rangle = \delta_a{}^b$. If $\mathfrak{a}$ satisfies condition (11.22), $\mathfrak{a}^* = \varrho/\bar{\mathfrak{a}} = \varrho\eta^{-1}\mathfrak{a}$, and we write

$$\mathfrak{a}^* = \lambda_v\, \mathfrak{a} \quad \text{where } \lambda_v = v\varrho/\eta^{-1} \quad \text{with } v \text{ a unit of } \mathfrak{O}. \tag{11.23}$$

Then $\{\lambda_v\, \omega_a\}$ is also a $\mathbb{Z}$-basis of the dual fractional ideal $\mathfrak{a}^*$ and there is a a matrix $J_{ab} \in GL(2k,\mathbb{Z})$ (depending on the unit v) such that

$$\lambda_v\, \omega_a = J_{ab}\, \phi^b. \tag{11.24}$$

One has

$$\bar{\lambda}_v^{-1} J_{ab} = \bar{\lambda}_v^{-1}\langle J_{ac}\phi^c, \omega_b\rangle = \langle\omega_a, \omega_b\rangle = \langle\omega_a, J_{bc}\phi^c\rangle\lambda^{-1} = J_{ba}\,\lambda_v^{-1} \tag{11.25}$$

i.e. the integral unimodular matrix J_{ab} is skew-symmetric (resp. symmetric) if the unit v is such that λ_v is purely imaginary (resp. real). In the first case J_{ab} is a principal integral symplectic structure, hence similar over $\mathbb{Z}$ to the standard one Ω, i.e. $J = h^t\Omega h$ for some $h \in GL(2k,\mathbb{Z})$. In the second case J is a unimodular symmetric quadratic form. Thus for a fixed fractional ideal $\mathfrak{a}$, we find a symplectic structure (i.e. an embedding in $Sp(2k,\mathbb{Z})$) per each choice of the unit v such that λ_v is purely imaginary. We shall count the inequivalent ones in the next subsection.

Since $\eta \in \Bbbk^\times$ is always real, and ϱ was chosen to be purely imaginary (cfr. **Lemma 11.2.2**) J_{ab} is skew-symmetric iff v is real, and symmetric iff it is purely imaginary. In particular, the two obvious choices $v = \pm 1$, always produce embeddings $\mathbb{Z}_n \hookrightarrow Sp(2k,\mathbb{Z})$ $(2k \equiv \phi(n))$.

[80] That is, a set of generators of $\mathfrak{a}$ seen as a free $\mathbb{Z}$-module.

11.2.4 Conjugacy classes of embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$

Regular embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ exist for all $n \geq 3$. Indeed, the condition $N\mathfrak{a}$ principal is trivially satisfied if $\mathfrak{a}$ itself is principal. Thus the trivial ideal class (1) yields regular embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ for all $n \geq 3$. We proceed as follows: we fix an embedding (m, J) associated to the ideal (1) and call it the *reference* embedding. All inequivalent embeddings are obtained by acting on the reference one (m, J) with a certain Abelian group H defined in the next **Proposition**. A subgroup of H is easy to describe: in eqn.(11.23) we may choose a different *real* unit $v \in \mathfrak{O}$ and still get an invariant quaternionic structure; this is the same as multiplying $\eta \in \Bbbk^\times$ by a unit of $\mathfrak{o}$. In this way we get new (inequivalent) embeddings (m, J') for a given fractional ideal $\mathfrak{a}$: they correspond to embeddings which are conjugate over $GL(\phi(n), \mathbb{Z})$ but not over the subgroup $Sp(\phi(n), \mathbb{Z})$. On the other hand, under $\mathfrak{a} \to \mu\mathfrak{a}$ with $\mu \in \mathbb{K}^\times$ we have $\eta \to \eta\, N\mu$, hence the image of η in the group $\Bbbk^\times/N\mathbb{K}^\times$ is independent of the choice of the representative ideal $\mathfrak{a}$ in the ideal class. To describe also the embeddings belonging to different $GL(\phi(n), \mathbb{Z})$ conjugacy classes, it is convenient to consider the group

$$L = \ker\Big(\mathfrak{I}_{\mathbb{K}} \xrightarrow{N} C_{\Bbbk},\ \mathfrak{a} \mapsto [N\mathfrak{a}]\Big) \tag{11.26}$$

of fractional ideal classes in $\mathbb{K}$ whose relative norm is principal in $\Bbbk$. Then we have group

$$K = \big\{(\mathfrak{a}, \eta) \in L \times \Bbbk^\times \ :\ N\mathfrak{a} = (\eta)\big\} \tag{11.27}$$

and the group homomorphism

$$\pi\colon K \to C_{\mathbb{K}} \times \Bbbk^\times/N\mathbb{K}^\times, \qquad (\mathfrak{a}, \eta) \mapsto ([\mathfrak{a}], [\eta]). \tag{11.28}$$

The above discussion shows the

Proposition 11.2.2 (see [186]). *Let $n \geq 3$. The (Abelian) group*

$$H \equiv \operatorname{Im}\pi \subset C_{\mathbb{K}} \times \Bbbk^\times/N\mathbb{K}^\times \tag{11.29}$$

acts freely and transitively on the set of the $Sp(\phi(n), \mathbb{Z})$-conjugacy classes of embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$. In particular, the number of $Sp(\phi(n), \mathbb{Z})$-conjugacy classes is

$$\big|H\big| \equiv \big|\ker\big(C_{\mathbb{K}} \xrightarrow{N} C_{\Bbbk}\big)\big| \times \big|\boldsymbol{u}/N\boldsymbol{U}\big|, \tag{11.30}$$

where $\boldsymbol{U}$ (resp. $\boldsymbol{u}$) is the group of unities of $\mathfrak{O}$ (resp. $\mathfrak{o}$).

Let $h = |C_{\mathbb{K}}|$ and $h^+ = |C_{\Bbbk}|$ be the class numbers of the fields $\mathbb{K}$ and $\Bbbk$, respectively. The map $N\colon C_{\mathbb{K}} \to C_{\Bbbk}$ is surjective [cite], and the ratio

$$h^- = h/h^+ = \big|\ker(C_{\mathbb{K}} \xrightarrow{N} C_{\Bbbk})\big| \tag{11.31}$$

is called the *relative class number* of $\mathbb{K}$. h^- is much easier to compute that either h or h^+ (it has an explicit expression in terms of generalized Bernoulli numbers [177, 245]). It turns out that $h^- = 1 \Leftrightarrow h = h^+ = 1$ [245]. For $n \leq 22$ the relative class number is $h^- = 1$, while for large n we have the asymptotic behavior [245]

$$\log h^- \sim \frac{1}{4}\,\phi(n)\,\log \breve{n} \qquad n \to \infty. \tag{11.32}$$

$h^- = 1$ iff the conductor $\breve{n}$ is one of the numbers [245]

$$\begin{aligned}\breve{n} =& 1,3,4,5,7,8,9,11,12,13,15,16,17,19,20,21,\\ & 24,25,27,28,32,33,35,36,40,44,45,48,60,84.\end{aligned} \tag{11.33}$$

When $h^- = 1$, all inequivalent embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ arise from the same fractional ideal, and hence are all conjugate in the larger group $GL(\phi(n), \mathbb{Z})$.

To compute the second factor in (11.30) we consider the group of units in the relevant fields.

Group of units. We write $\boldsymbol{\mu}$ for the group generated by the roots of unity in $\mathbb{K}$

$$\boldsymbol{\mu} = \big\{ \pm \zeta^k \big\}. \tag{11.34}$$

The *Hasse unit index* Q of the cyclotomic field $\mathbb{K}$ is

$$Q \equiv [\boldsymbol{U} : \boldsymbol{\mu}\,\boldsymbol{u}]. \tag{11.35}$$

Proposition 11.2.3 (see [142])**.** *Let $\mathbb{K}$ be a cyclotomic field of conductor $\breve{n}$. One has*

$$Q = \begin{cases} 1 & \breve{n} \text{ is a prime power} \\ 2 & \text{otherwise.} \end{cases} \tag{11.36}$$

Moreover, $\boldsymbol{U}/(\boldsymbol{\mu}\,\boldsymbol{u}) \cong N\boldsymbol{U}/\boldsymbol{u}^2$ and then

$$[N\boldsymbol{U} : \boldsymbol{u}^2] = Q. \tag{11.37}$$

In other words, if $Q = 1$ all $\{N\varepsilon \,:\, \varepsilon \in \boldsymbol{U}\}$ are squares in $\boldsymbol{u}$, while for $Q = 2$ only half of them are squares.

Remark 11.2.2. Let the conductor $\bar{n}$ be divisible by two distinct primes. Let us describe explicitly a generator of the group $N\boldsymbol{U}/\boldsymbol{u}^2 \cong \mathbb{Z}_2$. $(1-\zeta)$ is a unit in $\mathbb{K}$, and

$$N(1-\zeta) = 2 - \zeta - \zeta^{-1} = 2(1 - \cos(2\pi/n)) = 4\sin^2(\pi/n) \in \Bbbk, \tag{11.38}$$

while its square root $2\sin(\pi/n)$ is not in $\Bbbk$. Hence, for n divisible by two distinct primes,

$$N\boldsymbol{U} = N(1-\zeta)^a\,\boldsymbol{u}^2, \quad a = 0, 1. \tag{11.39}$$

In particular $\varepsilon_1 \equiv N(1-\zeta)$ is a fundamental unit of $\Bbbk$.

For $n \geq 3$, let $k = \phi(n)/2$. From Dirichlet unit theorem [121] we know that

$$\boldsymbol{u} = \Big\{ \pm\, \varepsilon_1^{s_1} \varepsilon_2^{s_2} \cdots \varepsilon_{k-1}^{s_{k-1}},\ s_a \in \mathbb{Z} \Big\} \tag{11.40}$$

where ε_a, $a = 1, \dots, k-1$ are the real positive fundamental units. For $Q = 1$ we have

$$\boldsymbol{u}/N\boldsymbol{U} \cong \boldsymbol{u}/\boldsymbol{u}^2 = \Big\{ \pm\, \varepsilon_1^{s_1} \varepsilon_2^{s_2} \cdots \varepsilon_{k-1}^{s_{k-1}},\ s_a \in \mathbb{Z}/2\mathbb{Z} \Big\} \cong \mathbb{Z}_2^k. \tag{11.41}$$

while for $Q = 2$

$$\boldsymbol{u}/N\boldsymbol{U} \cong \boldsymbol{u}/\big(\mathbb{Z}_2 \times \boldsymbol{u}^2\big) = \Big\{ \pm\, \varepsilon_2^{s_2} \cdots \varepsilon_{k-1}^{s_{k-1}},\ s_a \in \mathbb{Z}/2\mathbb{Z} \Big\} \cong \mathbb{Z}_2^{k-1}. \tag{11.42}$$

In conclusion,

Corollary 11.2.1. *Let $n \geq 3$. The number of $Sp(\phi(n), \mathbb{Z})$-conjugacy classes of embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ is*

$$\frac{2^{\phi(n)/2}}{Q}\, h^- \equiv |\boldsymbol{\mu}| \prod_{\chi\ odd} \big(- B_{1,\chi} \big), \tag{11.43}$$

where $Q = 1, 2$ is the Hasse unit index, h^- the relative class number of the cyclotomic field $\mathbb{K}$, and $B_{1,\chi}$ the first Bernoulli number of the odd Dirichlet character χ.

However, to fully solve our problem we need to know also when two distinct conjugacy classes are conjugate in the larger group $Sp(\phi(n), \mathbb{R})$ (or, equivalently, in $Sp(\phi(n), \mathbb{Q})$).

11.2.5 Embeddings $\mathbb{Z}_n \hookrightarrow S(\tilde{\Omega})_{\mathbb{Z}}$ for $\tilde{\Omega}$ non-principal

The symplectic matrix J defined in eqn.(11.24), for λ_v as in (11.23) (with v a unit of $\mathfrak{o}$), corresponds to a principal polarization, i.e. J is an integral skew-symmetric matrix with $\det J = 1$. Let $0 \neq \kappa \in \mathfrak{o}$ and consider the matrix J^κ defined by

$$J^\kappa_{ab}\, \phi^b = \kappa \lambda\, \omega_a. \tag{11.44}$$

If κ is a unit, J^κ is a principal-polarization. For κ just integer in $\Bbbk$, J^κ is an integral skew-symmetric matrix with determinat

$$\det J^\kappa = \big(N_{\Bbbk/\mathbb{Q}}\, \kappa \big)^2. \tag{11.45}$$

11.2.6 $Sp(\phi(n), \mathbb{Q})$-conjugacy classes

As we saw in §. 11.1.3, the $Sp(\phi(n), \mathbb{R})$-conjugacy classes are distinguished by the sign of the corresponding (integral) symplectic structure sign_σ. Then we need to understand the action of the group of $Sp(\phi(n), \mathbb{Z})$-conjugacy classes of embeddings, H, on the sign function which we see as a map $\mathsf{Gal}(\Bbbk/\mathbb{Q}) \to \mathbb{Z}_2$. The set of such maps form a group isomorphic to $\mathbb{Z}_2^{\phi(n)/2}$. Then we have a well-defined homomorphism of Abelian groups

$$\mathfrak{s} \colon H \longrightarrow \mathbb{Z}_2^{\phi(n)/2}, \qquad ([\mathfrak{a}], [\eta]) \longmapsto \left\{ \mathsf{sign}_{([\mathfrak{a}],[\eta])} \colon \sigma \mapsto \frac{\sigma(\eta)}{|\sigma(\eta)|} \right\} \tag{11.46}$$

An element $([\mathfrak{a}], [\eta]) \in H$ changes the $Sp(\phi(n), \mathbb{Z})$ conjugacy class of m without changing its $Sp(\phi(n), \mathbb{R})$-conjugacy class iff it belongs to the kernel of $\mathfrak{s}$, that is,

Corollary 11.2.2 (Midorikawa [186]). *The group $H_{\mathbb{R}} \equiv H/\ker \mathfrak{s}$ acts freely and transitively on the $Sp(\phi(n), \mathbb{R})$-conjugacy classes of embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$.*

An element $\eta \in \Bbbk^\times$ is said to be *totally positive* iff $\sigma(\eta) > 0$ for all $\sigma \in \mathsf{Gal}(\Bbbk/\mathbb{Q})$; the set of all totally positive elements $\Bbbk_+^\times \subset \Bbbk^\times$ form a subgroup while (for $n \geq 3$) [121]

$$\Bbbk_+^\times/\Bbbk^\times \cong \mathbb{Z}_2^{\phi(n)/2}. \tag{11.47}$$

Comparing with eqn.(11.29), we see that

$$\ker \mathfrak{s} = H \bigcap \Big(C_{\mathbb{K}} \times \Bbbk_+^\times / N\mathbb{K}^\times\Big). \tag{11.48}$$

The group of principal fractional ideals (η) with $\eta \in \Bbbk_+^\times$ is a subgroup of the group of all principal fractional ideals. The quotient $\mathfrak{I}_\Bbbk/(\Bbbk_+^\times)$ is called the *narrow-ideal class*, $C_\Bbbk^{\text{nar}}$. Likewise we have the subgroup of totally positive units $\boldsymbol{u}_+ \subset \boldsymbol{u}$; from the ray class exact sequence [121, 189]

$$1 \to \boldsymbol{u}/\boldsymbol{u}_+ \to \mathbb{Z}_2^{\phi(n)/2} \to C_\Bbbk^{\text{nar}}/C_\Bbbk \to 1, \tag{11.49}$$

we get $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^{\phi(n)/2-a}$, $C_\Bbbk^{\text{nar}}/C_\Bbbk \cong \mathbb{Z}_2^a$ for some $0 \leq a \leq \phi(n)/2 - 1$ ($a \geq 1$ when $Q = 2$). Then

$$\big|\ker \mathfrak{s}\big| = \big|\ker\big(C_{\mathbb{K}} \xrightarrow{N} C_\Bbbk^{\text{nar}}\big)\big| \times \big|\boldsymbol{u}_+/N\boldsymbol{U}\big|, \tag{11.50}$$

and the number of $Sp(\phi(n), \mathbb{R})$-conjugacy classes of embeddings $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ (i.e. the number of possible sign assignments in the integral symplectic structure is

$$\big|H_{\mathbb{R}}\big| = \frac{|\ker(C_{\mathbb{K}} \xrightarrow{N} C_\Bbbk)|}{|\ker(C_{\mathbb{K}} \xrightarrow{N} C_\Bbbk^{\text{nar}})|} \cdot \big|\boldsymbol{u}/\boldsymbol{u}_+\big|. \tag{11.51}$$

Hence $\ker(C_{\mathbb{K}} \xrightarrow{N} C_\Bbbk)/\ker(C_{\mathbb{K}} \xrightarrow{N} C_\Bbbk^{\text{nar}}) \cong \mathbb{Z}_2^b$, with $b \leq a$ while $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^{\phi(n)/2-a}$ and

$$H_{\mathbb{R}} \cong \mathbb{Z}_2^{\phi(n)/2+(b-a)} \tag{11.52}$$

so that the number of $Sp(\phi(n), \mathbb{R})$ inequivalent embeddings is $2^{\phi(n)/2+(b-a)} \leq 2^{\phi(n)/2}$.

Corollary 11.2.3. *Let the class number of* $\mathbb{K}$, $h_{\mathbb{K}}$, *be* odd. *Then*

$$H_{\mathbb{R}} \cong \boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^{\phi(n)/2-a}, \qquad b = 0. \tag{11.53}$$

From eqn.(11.52) we see that if $b < a$ not all signatures of the symplectic structure may be realized. Indeed, from eqn.(11.49) we see that all signatures are realized iff the kernel of the natural map $C_\Bbbk^{\text{nar}} \to C_\Bbbk$ is contained in the image of N. We mention a few known facts on a:

a) (Weber) if $n = 2^r$, $\boldsymbol{u}_+ = \boldsymbol{u}^2$, that is, $a = 0$;

b) (Kummer, Shimura [200, 233]) if $\breve{n}$ is a prime, $a = 0$ if and only if the class number of $\mathbb{K}$ is odd;

c) of course $a > 0$ if $\breve{n}$ is divisible by two distinct primes.

Thus, for instance, if $h_{\mathbb{K}}$ is odd and $\breve{n}$ composite $\neq 2^r$, not all signs of the symplectic form are realizable.

Remark 11.2.3. If $h_{\mathbb{K}} = 2$ we have $H \cong \mathbb{Z}_2 \times \boldsymbol{u}/N\boldsymbol{U}$, since we must have[81] $h_{\Bbbk} = 1$; if (in addition) $\breve{n}$ is not a prime power, $H \cong \mathbb{Z}_2^{1+\phi(n)/2-1} \cong \mathbb{Z}_2^{\phi(n)/2}$. This happens e.g. for $n = 39, 56, 78$. For these three instances the conductor is divisible by just two distinct primes, and hence (by a result of Sinnott [236]) $\boldsymbol{u}$ coincides with the group of cyclotomic units.

11.2.7 The sign function

From (11.25) we see that the sign function is

$$\mathsf{sign}\colon \sigma \mapsto \frac{\sigma(\lambda_v)}{i|\sigma(\lambda_v)|} = \frac{\sigma(\varrho)}{i|\sigma(\varrho)|}\cdot\frac{\sigma(\eta)}{|\sigma(\eta)|}. \tag{11.54}$$

Comparing eqn.(11.44) with eqn.(11.47), we conclude

Corollary 11.2.4. *All sign functions (i.e. all spectral invariants) are realized for some arithmetic embedding $\mathbb{Z}_n \hookrightarrow Sp(\tilde{\Omega}, \mathbb{Z})$ with $\tilde{\Omega}$ a non-necessarily principal polarization.*

A few examples are in order. We have seen above that if

$$(**) \qquad h_{\mathbb{K}} \text{ is odd and the conductor } \breve{n} \text{ is either a power of 2 or an odd prime} \tag{11.55}$$

then *all* $2^{\phi(\breve{n})/2}$ signs are realized with Ω principal. Let us consider the first few n's which do not satisfy these condition $(**)$. The first one is 9.

Example 36. $\boxed{n = 9}$ In this case $h_{\mathbb{K}} = Q = 1$ so there are $2^{\phi(n)/2} = 8$ distinct $Sp(6, \mathbb{Z})$-conjugacy classes of order 9 elements. Since 9 is a prime power, $\boldsymbol{u}$ is the group of the the real cyclotomic units, that is,

$$\boldsymbol{u} = \pm(\zeta + \zeta^8)^{\mathbb{Z}}\,(\zeta^4 + \zeta^5) \equiv \pm\, u^{\mathbb{Z}}\, v^{\mathbb{Z}}. \tag{11.56}$$

The sign table for the three elements $\sigma_a \in \mathsf{Gal}(\Bbbk/\mathbb{Q})$ are

	σ_1	σ_2	σ_4
u	$+$	$+$	$-$
v	$-$	$+$	$+$

(11.57)

Thus $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^3 \cong \boldsymbol{u}/N\boldsymbol{U}$; hence all $Sp(6, \mathbb{Z})$-conjugacy classes are distinct as $Sp(6, \mathbb{R})$-conjugacy classes and all 8 signs are realized (cfr. also [111]).

Example 37. $\boxed{n = 12}$ In this case $h_{\mathbb{K}} = 1$ and $Q = 2$, so we have only $2^{\phi(12)/2}/Q = 2$ inequivalent embeddings over $Sp(4, \mathbb{Z})$. They correspond to $\eta = \pm 1$. One has

$$\overline{\Phi_{12}'(e^{2\pi i/12})} = 2\sqrt{3}\, e^{-2\pi i/3}, \tag{11.58}$$

so as ϱ we may choose

$$-\, i\, e^{-2\pi i/3}/(2\sqrt{3}\, e^{-2\pi i/3}) \equiv -i/(2\sqrt{3}) = \frac{1}{2(\zeta^4 - \zeta^{-4})}. \tag{11.59}$$

[81] Indeed, $2 \mid h_{\Bbbk} \Rightarrow 2 \mid h^-$ [cite] so that $2 \mid h_{\Bbbk} \Rightarrow 4 \mid h_{\mathbb{K}}$.

Thus for $n = 12$ the sign function is

$$\begin{aligned}&(\mathbb{Z}/12\mathbb{Z})^\times \to \{\pm 1\}, \\ &a \mapsto i^{a-1},\end{aligned} \qquad \text{i.e.} \qquad \begin{aligned}&1 \mapsto -1, \quad 5 \mapsto +1,\\ &7 \mapsto -1, \quad 11 \mapsto +1.\end{aligned} \tag{11.60}$$

The group of units of $\Bbbk \equiv \mathbb{Q}[\sqrt{3}]$ is $\boldsymbol{u} = \pm(2-\sqrt{3})^{\mathbb{Z}}$, and $\boldsymbol{u}/\boldsymbol{u}_+ \cong \{\pm 1\}$. Thus only two spectral invariants out of four are realized by embeddings $\mathbb{Z}_{12} \to Sp(4,\mathbb{Z})$ (as expected) namely $\{\zeta, \zeta^7\}$ and $\{\zeta^5, \zeta^{11}\}$ (where $\zeta = e^{2\pi i/12}$).

Remark 11.2.4. Since the ST group G_8 has degrees $\{12, 8\}$, according to the dimension formulae of section 4, its elements of order 12 should have an embedding with spectral invariant $\{\zeta, \zeta^5\}$. We saw in §. 6.2 (**Example 5**) that the G_8-invariant polarization has $\det \tilde{\Omega} = 2^2$. This polarization has the form in §. 11.2.5 with $\kappa = 1 + \sqrt{3} \in \mathfrak{o}$ which is not totally positive (its norm is negative) $N_{\Bbbk/\mathbb{Q}}(1+\sqrt{3}) = -2$. This illustrates **Corollary 11.2.4**.

Example 38. $\boxed{n = 15}$ Again $h_{\mathbb{K}} = 1$ and we have $2^{\phi(15)/2}/Q = 8$ different $Sp(8,\mathbb{Z})$-conjugacy classes. We have ($\zeta \equiv e^{2\pi i/15}$)

$$\Phi'_{15}(\zeta) = 15\,\frac{\zeta^{-1}(\zeta-1)}{(\zeta^5-1)(\zeta^3-1)} \quad \text{we choose} \quad \varrho = -\frac{1}{15}\,\zeta^3(\zeta^2-1)(\zeta^{10}-1)(\zeta^{12}-1), \tag{11.61}$$

and then the signs of the reference embedding are

$$\frac{\sigma_a(\varrho)}{i\,|\sigma_a(\varrho)|} = \begin{cases} +1 & a = 1,2 \\ -1 & a = 4,7 \end{cases} \xrightarrow{\text{spectral inv.}} \quad \{\zeta, \zeta^2, \zeta^8, \zeta^{11}\}. \tag{11.62}$$

Writing $\xi = \zeta + \zeta^{-1}$, we have (according to Mathematica)

$$\boldsymbol{u} = \pm(-1+3\xi-\xi^3)^{\mathbb{Z}}\,(2+3\xi-\xi^2-\xi^3)^{\mathbb{Z}}\,(-1+4\xi-\xi^3)^{\mathbb{Z}} = \pm u_1^{\mathbb{Z}}\,u_2^{\mathbb{Z}}\,u_3^{\mathbb{Z}}, \tag{11.63}$$

whose signs are

	σ_1	σ_2	σ_4	σ_7
u_1	$-$	$+$	$-$	$+$
u_2	$-$	$+$	$+$	$-$
u_3	$+$	$+$	$-$	$-$

(11.64)

so that $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^3$, i.e. all $Sp(8,\mathbb{Z})$ classes correspond to $Sp(8,\mathbb{R})$ classes, with signs:

$$\{++--\},\ \{-++-\},\ \{-+-+\},\ \{++++\},\ \{--++\},\ \{+--+\},\ \{+-+-\},\ \{----\},$$

that is, the spectral invariants

$$\begin{gathered}\{\zeta,\zeta^2,\zeta^{11},\zeta^8\}, \quad \{\zeta^{14},\zeta^2,\zeta^4,\zeta^8\}, \quad \{\zeta^{14},\zeta^2,\zeta^{11},\zeta^7\}, \quad \{\zeta,\zeta^2,\zeta^4,\zeta^7\},\\ \{\zeta^{14},\zeta^{13},\zeta^4,\zeta^7\}, \quad \{\zeta,\zeta^{13},\zeta^{11},\zeta^7\}, \quad \{\zeta,\zeta^{13},\zeta^4,\zeta^8\}, \quad \{\zeta^{14},\zeta^{13},\zeta^{11},\zeta^8\}.\end{gathered} \tag{11.65}$$

Example 39. $\boxed{n = 20}$ Again $h_{\mathbb{K}} = 1$ and we have $2^{\phi(20)/2}/Q = 8$ different $Sp(8,\mathbb{Z})$-conjugacy classes. We have ($\zeta \equiv e^{2\pi i/20}$)

$$\Phi'_{20} = \frac{10\,\zeta^8}{\zeta+\zeta^{-1}} \quad \text{we choose} \quad \varrho = \frac{1}{10}(\zeta+\zeta^{-1})\zeta^5. \tag{11.66}$$

Then

$$\frac{\sigma_a(\varrho)}{i\,|\sigma_a(\varrho)|} = \begin{cases} +1 & a=1,7 \\ -1 & a=3,9 \end{cases} \xrightarrow{\text{spectral inv.}} \{\zeta,\zeta^{17},\zeta^7,\zeta^{11}\}. \tag{11.67}$$

Again, with $\xi = \zeta+\zeta^{-1}$

$$\boldsymbol{u} = \pm(1+\xi)^{\mathbb{Z}}\,(2-\xi^2)^{\mathbb{Z}}\,(1-3\xi+\xi^3)^{\mathbb{Z}} = \pm u_1^{\mathbb{Z}}\,u_2^{\mathbb{Z}}\,u_3^{\mathbb{Z}}, \tag{11.68}$$

with signs

	σ_1	σ_3	σ_7	σ_9
u_1	+	+	−	−
u_2	−	+	+	−
u_3	+	−	+	−

(11.69)

so, again $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^3$ and $Sp(8,\mathbb{Z})$ and $Sp(8,\mathbb{R})$ conjugacy classes coincide.

Example 40. $\boxed{n=21}$ Again $h_{\mathbb{K}}=1$ and we have $2^{\phi(21)/2}/Q = 32$ different $Sp(12,\mathbb{Z})$-conjugacy classes. We have ($\zeta \equiv e^{2\pi i/21}$)

$$\Phi'_{21}(\zeta) = 21\,\frac{\zeta^{-1}(\zeta-1)}{(\zeta^7-1)(\zeta^3-1)} \quad \text{we choose} \quad \varrho = -\frac{1}{21}\,\zeta^9(\zeta^2-1)(\zeta^{14}-1)(\zeta^{18}-1), \tag{11.70}$$

and then the signs of the reference embedding are

$$\frac{\sigma_a(\varrho)}{i\,|\sigma_a(\varrho)|} = \begin{cases} +1 & a=1 \\ -1 & a=2,4,5,8,10 \end{cases} \xrightarrow{\text{spectral inv.}} \{\zeta,\zeta^{19},\zeta^{17},\zeta^{16},\zeta^{13},\zeta^{11}\}. \tag{11.71}$$

Writing $\xi = \zeta+\zeta^{-1}$, we have (according to Mathematica)

$$\begin{aligned} \boldsymbol{u} &= \pm\xi^{\mathbb{Z}}\,(2-\xi^2)^{\mathbb{Z}}\,(2-4\xi^2+\xi^4)^{\mathbb{Z}}\,(3-8\xi-\xi^2+6\xi^3-\xi^5)^{\mathbb{Z}}\,(2-5\xi-\xi^2+5\xi^3-\xi^5)^{\mathbb{Z}} \\ &= \pm u_1^{\mathbb{Z}}\,u_2^{\mathbb{Z}}\,u_3^{\mathbb{Z}}\,u_4^{\mathbb{Z}}\,u_5^{\mathbb{Z}}, \end{aligned} \tag{11.72}$$

whose signs are

	σ_1	σ_2	σ_4	σ_5	σ_8	σ_{10}
u_1	+	+	+	+	−	−
u_2	−	−	+	+	−	−
u_3	+	−	+	+	−	+
u_4	+	+	−	+	+	−
u_5	−	+	−	+	−	−

(11.73)

so that $u_1u_2u_3u_4u_5 \in \boldsymbol{u}_+$ and $\boldsymbol{u}/\boldsymbol{u}_+ \cong \mathbb{Z}_2^{\phi(21)/2-1} \equiv \mathbb{Z}_2^5$, and only 32 out of the possible 64 signs are actually realized. The allowed spectral invariants may be read from the above tables.

11.3 Explicit matrices

We now write explicitly the integral matrices yielding a reference embedding $\mathbb{Z}_n \hookrightarrow Sp(\phi(n), \mathbb{Z})$ on which we act with the groups H or $H_\mathbb{R}$ to get the inequivalent embeddings over $Sp(\phi(n), \mathbb{Z})$ and $Sp(\phi(n), \mathbb{R})$, respectively.

Let $\mathfrak{a}$ be a fractional ideal of $\mathbb{K}$ such that $N\mathfrak{a} = (\eta)$, $\eta \in \Bbbk^\times$. We write $k = \phi(n)/2$ and choose generators of the free $\mathbb{Z}$-module $\mathfrak{a}$, $\mathfrak{a} = \bigoplus_{a=1}^{2k} \mathbb{Z}\omega_a$. Define the dual vector $(\phi^a) \in \mathbb{K}^{2k}$ by the condition

$$\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}\big(\omega_a \,\bar{\phi}^b\big) = \delta_a{}^b. \tag{11.74}$$

By definition, the dual ideal is $\mathfrak{a}^* = \bigoplus_{a=1}^{2k} \mathbb{Z}\phi^a$. Since $\mathfrak{a}^* = \lambda\mathfrak{a}$ with λ purely imaginary (cfr. eqn.(11.23)), there exists $\Lambda \in GL(2k, \mathbb{Z})$ such that

$$\Lambda^{ab}\omega_b = \lambda^{-1}\phi^b, \qquad \Lambda_{ba}\phi^b = \bar{\lambda}\omega_a, \tag{11.75}$$

where Λ_{ab} is the inverse of Λ^{ab}; the second equation being a consequence of the first in view of (11.74)). Then

$$((\Lambda^t)^{-1}\Lambda)_a{}^b\, \omega_b = \lambda^{-1}\bar{\lambda}\,\omega_a = -\omega_a, \tag{11.76}$$

and the integral matrix Λ is antisymmetric with determinant 1, hence similar over the integers to the standard symplectic matrix Ω.

Each ideal class $[\mathfrak{a}] \in C_\mathbb{K}^- = \ker(C_\mathbb{K} \xrightarrow{N} \mathbb{C}_\Bbbk)$ yields an embedding $\mathbb{Z}_n \to GL(2k, \mathbb{Z})$ which is quaternionic with respect to $2^k/Q$ inequivalent (over $\mathbb{Z}$) symplectic structures. To get the reference embedding, let us consider the trivial class in $C_\mathbb{K}^-$; as a representative ideal we take $\mathfrak{O} \equiv (1)$ itself.

As a $\mathbb{Z}$-basis of $\mathfrak{O}$ we take $\omega_x = \zeta^{x-1}$ with $x = 1, \dots, 2k$. It is convenient to re-label the elements of this basis. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ be the prime decomposition of n. Choose primitive $p_i^{r_i}$-th roots of 1, ζ_i. Then $\zeta = \prod_i \zeta_i$ is a primitive n-th root. By the Chinese remainder theorem, there exist integers e_i, $i = 1, \dots, s$, such that

$$e_i = \delta_{ij} \mod p_j^{r_j} \quad \forall\, i, j = 1, \dots, s. \tag{11.77}$$

We write the index $x = 1, 2, \dots, 2k$ uniquely as

$$\begin{gathered} x - 1 = \sum_i e_i\big((a_i - 1)p^{r_i - 1} + (\alpha_i - 1)\big) \\ \text{with } a_i = 1, \dots, p_i - 1 \text{ and } \alpha_i \in \mathbb{Z}/p_i^{r_i - 1}\mathbb{Z}. \end{gathered} \tag{11.78}$$

Then the basis ζ^{x-1} is re-written as a tensor product over the primes $p_i \mid n$

$$\omega_{a_i\alpha_i} = \prod_i \zeta_i^{p_i^{r_i - 1}(a_i - 1) + (\alpha_i - 1)}, \qquad \omega_{\boldsymbol{a}\,\boldsymbol{\alpha}} = \bigotimes_{i=1}^s (\omega_i)_{a_i\alpha_i} \tag{11.79}$$

It is convenient to realize the action of m as multiplication by a different primitive n-root

$$\zeta' = \prod_i \zeta_i^{p_i^{r_i-1}+1}. \tag{11.80}$$

With these conventions, the action of $\mathbb{Z}_n \cong \prod_i \mathbb{Z}_{p_i^{r_i}}$ explicitly factorizes in the product of the action of the factor groups $\mathbb{Z}_{p_i^{r_i}}$

$$\zeta' \, \omega_{\boldsymbol{a}\,\boldsymbol{\alpha}} = \bigotimes_{i=1}^{s} (m_i)_{a_i\alpha_i}{}^{b_j\beta_j}\, \omega_{b_j\beta_j} \quad \text{that is} \quad m = \bigotimes_{i=1}^{s} m_i. \tag{11.81}$$

We have to discuss separately the matrices m_i associated to an odd prime and the one associated to 2 (if present). For p_i odd, each m_i factorizes in the matrix $m_{(p)}$ yielding the reference embedding $\mathbb{Z}_p \to GL(p-1,\mathbb{Z})$ times the $p_i^{r_i-1}$-circulant

$$m_i = m_{(p_i)} \otimes C_{p_i,r_i}, \tag{11.82}$$

where $m_{(p)}$ (resp. $C_{p,r}$) is the $(p-1)\times(p-1)$ matrix (resp. $p^{r-1}\times p^{r-1}$)

$$m_{(p)} = \left[\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline -1 & -1 & \cdots & -1 \end{array}\right], \qquad C_{p,r} = \left[\begin{array}{c|ccc} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \hline 1 & 0 & \cdots & 0 \end{array}\right] \tag{11.83}$$

For $p_1 = 2$, m_1 is just the scalar -1 for $r_1 = 1$. For $r_1 \geq 2$, m_1 is the tensor product of the 2×2 matrix $m_{(2)}$ yielding the embedding $\mathbb{Z}_4 \to GL(2,\mathbb{Z})$ (the 2×2 matrix of the form on the left of (11.83)) with the 2^{r_1-2}-circulant.

Likewise, the Hermitian structure factorizes

$$\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}\big(\omega_{a_i\alpha_i}\,\bar{\omega}_{b_i,\beta_i}\big) = \prod_i \Big(p_i^{r_i-1}\, T_{i,a_ib_i}\, \delta^{(p_i^{r_i-1})}_{\alpha_i\beta_i}\Big). \tag{11.84}$$

where

$$T_{i,ab} = p_i\,\delta_{ab} - v_a v_b, \quad v_a = (1,1,\dots,1), \qquad \delta^{(\ell)}_{\alpha\beta} = \begin{cases} 1 & \alpha \equiv \beta \mod \ell \\ 0 & \text{otherwise,} \end{cases} \tag{11.85}$$

and we used [54]

$$\mathrm{Tr}_{\mathbb{K}/\mathbb{Q}}(\zeta^t) \equiv \sum_{\ell\in(\mathbb{Z}/n\mathbb{Z})^\times} \zeta^{\ell t} = \frac{\phi(n)}{\phi(n/(n,t))}\,\mu(n/(n,t)), \tag{11.86}$$

where $\mu(x)$ is the Möbius function. Let ζ_i be the primitive p^{r_i} we have chosen and set $\xi_i = \zeta_i^{p_i^{r_i-1}}$ (a primitive p-root). We write

$$\omega_i^{a_i\,\alpha_i} = (\xi_i^{a_i-1} - \xi_i^{-1})\zeta_i^{\alpha_i-1} \tag{11.87}$$

The dual basis of $\omega_{\boldsymbol{a}\,\boldsymbol{\alpha}}$ (cfr. (11.79)) is

$$\phi^{\boldsymbol{a}\,\boldsymbol{\alpha}} = \frac{1}{n}\bigotimes_{i=1}^{s}(\omega_i)^{a_i\,\alpha_i}. \tag{11.88}$$

Using the reference ϱ described in APPENDIX B.1, for s *odd* the symplectic matrix Λ of our reference embedding, is simply the tensor product of the reference symplectic matrices Λ_i for each prime $p_i|n$, $\Lambda = \bigotimes_i \Lambda_i$; for p_i odd

$$\Lambda_i = U_i \otimes \mathbf{1}_{p_i^{r_i-1}}, \quad U_i = \left[\begin{array}{c|c} 0 & -J_i \\ \hline J_i^t & 0 \end{array}\right] \quad \text{where } J_i \text{ is the } (p_i-1)/2 \times (p_i-1)/2 \text{ Jordan block of eigenvalue } -1, \tag{11.89}$$

while for $p_1 = 2$ we have $\Lambda_1 = 1$ if $r_1 = 1$ and otherwise

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \mathbf{1}^{r_1-2}. \tag{11.90}$$

To set this matrix in the standard form Ω, it suffices to replace the above $\mathbb{Z}$-basis with the $\mathbb{Z}$-basis

$$\bigotimes_{i=1}^{s}(\tilde{\omega}_i)_{a_i,\alpha_i} \quad \text{where } \tilde{\omega}_{a_i,\alpha_i} = \begin{cases} \omega_{a_i,\alpha_i} & 1 \leq a_i < p_i/2 \\ \phi^{a_i,\alpha_i} & p_i/2 < a_i < p_i. \end{cases} \tag{11.91}$$

Therefore, for s odd the reference embedding $\mathbb{Z}_n \cong \prod_{i=1}^s \mathbb{Z}_{p_i^{r_i}}$ in $Sp(\phi(n),\mathbb{Z})$ is simply the tensor product of the embeddings of the factor groups $\mathbb{Z}_{p_i^{r_i}} \to Sp(\phi(p_i^{r_i},\mathbb{Z})$.

For s even the above tensor product produces an orthogonal rather than a symplectic embedding since $\bigotimes_i \Lambda_i$ is symmetric; to get an antisymmetric pairing, we multiply it by the reference imaginary unit of **Lemma B.1.2**. We get

$$\Lambda = \begin{cases} m^{n/4}\bigotimes_{i=1}^s \Lambda_i & 4 \mid n \\ (m-m^{-1})\bigotimes_{i=1}^s \Lambda_i & \text{otherwise.} \end{cases} \tag{11.92}$$

One checks that

$$m\Lambda m^t = \Lambda. \tag{11.93}$$

This completes the explicit description of the reference embedding. Now we act with the group H on it to get all other inequivalent embeddings.

Multiplying by an element of $\boldsymbol{u}/N\boldsymbol{U}$. This subgroup of H does not change the matrix m but only the symplectic matrix Λ. Since $\mathfrak{o} = \mathbb{Z}[\zeta+\zeta^{-1}]$, each element of $v \in \boldsymbol{u}/N\boldsymbol{U}$ may be represented by a polynomial

$$p_v(\zeta+\zeta^{-1}) = \sum_{i=0}^{\phi(n)/2-1} c(v)_i(\zeta+\zeta^{-1})^i \quad \text{with } c(v)_i \in \mathbb{Z}. \tag{11.94}$$

Then the change in the matrix Λ produced by multiplication by v is simply

$$\Lambda \to \Lambda_v \equiv p_v(m+m^{-1})\,\Lambda = \Lambda\, p_v(m+m^{-1})^t \quad\Longrightarrow\quad m\Lambda_v m^t = \Lambda_v. \tag{11.95}$$

Replacing (1) **with a non-principal fractional ideal.** In a Dedekind domain $\mathfrak{O}$, an ideal $\mathfrak{a}$ which is not principal may be generated by two elements, that is, has the form

$$\mathfrak{a} = x\,\mathfrak{O} + y\,\mathfrak{O}, \qquad x, y \in \mathfrak{O}. \tag{11.96}$$

Let ω_a a $\mathbb{Z}$-basis of $\mathfrak{O}$ and ϖ_a a $\mathbb{Z}$=basis of $\mathfrak{a}$. There are integral matrices X, Y such that

$$x\,\omega_a = X_a^{\ b}\,\varpi_b, \qquad y\,\omega_a = Y_a^{\ b}\,\varpi_b. \tag{11.97}$$

The matrix M yielding the action of ζ' in the basis ϖ_a is $M = X^{-1}mX = Y^{-1}mY$. The condition that $N\mathfrak{a}$ is principal then implies that the induced Λ is defined over $\mathbb{Z}$.

11.4 Reducible minimal polynomial

For completeness, we give some additional details on the case that the minimal polynomial of the elliptic element $m \in Sp(2k,\mathbb{Z})$ is reducible over $\mathbb{Q}$

$$M(z) = \Phi_d(z)\,\Phi_n(z), \qquad n > d. \tag{11.98}$$

If n/d is not a prime power, all embeddings are conjugate to a block-diagonal one $m_d \oplus m_n$, see **Lemma 11.1.1**. Suppose that $n/d = p^r$ with p prime while $(d,p) = 1$. We still have the block-diagonal embeddings, and all embeddings are conjugate to block-diagonal ones over $\mathbb{Q}$. Thus there is an element $R \in Sp(2k,\mathbb{Q})$ which splits the $\mathbb{Z}[m]$-module V and the symplectic structure

$$\begin{aligned} &R^{-1}V = \mathfrak{a} \oplus \mathfrak{b} \quad \text{with} \quad \mathfrak{a} \in \mathfrak{I}_{\mathbb{Q}[\zeta_n]}, \ \ \mathfrak{b} \in \mathfrak{I}_{\mathbb{Q}[\zeta_d]}, \\ &R^{-1}mR \text{ acts as as multiplication by } \zeta_n \times \zeta_d, \\ &R^t\,\Omega\,R = J_n \oplus J_d. \end{aligned} \tag{11.99}$$

R must have the form

$$R = \sum_{s=0}^{(\phi(n)+\phi(d))/2-1} a_s \left(m + m^{-1}\right)^s, \qquad a_s \in \mathbb{Z} \tag{11.100}$$

for certain coefficients a_s.

12 Tables of dimensions for small k

In this section we present some sample tables of both dimensions and dimension k-tuples for small values of the rank k.

12.1 New-dimension lists for $k \leq 13$

In table 14 we list the new-dimension sets $\mathfrak{N}(k)$, for ranks $1 \leq k \leq 13$. They have been computed using the defining formula:

$$\mathfrak{N}(k) := \left\{ \frac{l}{s} \in \mathbb{Q}_{\geq 1} \ : \ \phi(l) = 2k, \ (l,s) = 1 \right\}. \tag{12.1}$$

The set of dimensions allowed in rank k is contained in the set

$$\widehat{\Xi}(k) = \bigcup_{\ell=1}^{k} \mathfrak{N}(k). \tag{12.2}$$

From eqns.(10.84)(10.87), the cardinalities of the new-dimension sets $\mathfrak{N}(k)$ are:

k	1	2	3	4	5	6	7	8	9	10	11	12	13
$\lvert\mathfrak{N}(k)\rvert$	2+6	16	24	40	20	72	0	96	72	100	44	240	0
$\lvert\mathfrak{N}(k)\rvert_{\text{int}}$	2+3	4	4	5	2	6	0	6	4	5	2	10	0

(12.3)

12.2 Dimension k-tuples: USE OF THE TABLES

Tables of all ALLOWED dimension k-tuples become quite long pretty soon as we increase k. For conciseness we list only the STRONGLY REGULAR dimension k-tuples from which one can infer all allowed k-tuples. The tables of STRONGLY REGULAR k-tuples contain the basic informations needed to check whether a proposed dimension k-tuple $\{\Delta_1, \cdots, \Delta_k\}$ is consistent or not with the arguments of the present section. By definition, a STRONGLY REGULAR dimension k-tuples is a set of dimensions as computed using eqn.(10.51) along a normal ray $M_* \subset M$ with strongly regular monodromy m_* (i.e. such that the characteristic polynomial of m_* is square-free). In turn, the STRONGLY REGULAR monodromies may be distinguished in two kinds: the ones consistent with a principal polarization, $m_* \in Sp(2k, \mathbb{Z})$, and those associated to a suitable non principal polarization, $m_* \in S(\Omega)_{\mathbb{Z}}$ ($\det \Omega \neq 1$). The complete list of all ALLOWED dimension k-tuples is then recovered from the tables of the STRONGLY REGULAR ones by the algorithm described in section 4.5.1 which we review in the next subsection.

In tables 15, 16, and 17 we present the list of the strongly regular dimension k-tuples for $k = 3$ and $k = 4$. For $k = 3$ we list both the principal (table 15) and non-principal 3-tuples (table 16), while for $k = 4$ we limit ourselves to the principal ones (table 17).

12.2.1 The algorithm to check admissibility of a given dimension k-tuple

Suppose we are given a would-be dimension k-tuple, $\{\Delta_1, \cdots, \Delta_k\}$, written in non-increasing order $\Delta_i \geq \Delta_{i+1}$, and we wish to determine whether it is consistent with the geometric conditions discussed in the present thesis. In order to answer the question, we focus on the k normal rays in the Coulomb branch

$$M_i = \big\{ u_j = 0 \text{ for } i \neq j \big\} \subset M, \qquad i = 1, 2, \ldots, k. \tag{12.4}$$

The monodromy along M_i, m_i, may be either regular or irregular. From a regular monodromy m_i we may read all k dimensions using the Universal Formula (10.51). Experience with explicit examples (e.g. the ones having constant period map or those engineered in F-theory) suggests that the ray M_1 associated with the chiral operator u_1 of largest dimension $\Delta_1 = \max_i \Delta_i$ always has a regular monodromy.[82] However m_1 may be just *weakly* regular; in this case eqn.(10.51) still applies but the corresponding k-tuple is not listed in the tables, and we need to follow the procedure described below. Recall that we have defined the *regular rank* $k_{\text{reg},1} \leq k$ of the monodromy matrix m_1 to be one-half the degree of the square-free part of its characteristic polynomial.

To run the algorithm, one begins writing the rational number Δ_1 in minimal form, $\Delta_1 \equiv n_1/r_1$ with $(n_1, r_1) = 1$; we may assume $n_1 > 2$ by the argument in §.10.3.2. Let ℓ_1 be the largest integer such that $\Delta_{\ell_1} = \Delta_1$, that is, the multiplicity of the largest dimension. Δ_1 is a new-dimension in some rank $k_1 = \frac{1}{2}\phi(n_1)$ and $k_1\ell_1 \leq k$. If $k_1 = k$, the monodromy m_1 is automatically strongly regular, and hence the full k-tuple should appear in the tables of strongly regular k-tuples under the characteristic polynomial (C.P.) Φ_{n_1}. More generally, if $k_1\ell_1 = k$, the dimension k-tuple is the union of ℓ_1 strongly regular dimension k_1-tuples for Φ_{n_1}. If $\ell_1 k_1 < k$, the k-tuple is the union of ℓ_1 strongly regular k_1-tuples and a residual $(k - \ell_1 k_1)$-tuple $\{\Delta_i\}_{i\in A_1}$ ($A_1 \subset \{1, \dots, k\}$). Under the assumption that m_1 is (weakly) regular we have

$$\beta_i = \frac{\Delta_i - 1}{\Delta_1}, \qquad i \in A_1, \tag{12.5}$$

Let $s_i \in \mathbb{N}$ be the order of β_i in $\mathbb{Q}/\mathbb{Z}$. The multiplicity $\ell(s)$ of each integer $s \in \mathbb{N}$ in $\{s_i\}_{i\in A_1}$ should be an integral multiple of $\phi(s)/2$, and $\{\beta_i\}_{i\in A_1} \cup \{1 - \beta_i\}_{i\in A_1}$ should consist in the union of $2\ell(s_i)/\phi(s_i)$ copies of each set $B(s_i) = \{s_i/r,\ (s_i, r) = 1,\ 1 < r < s_i\}$. This corresponds to a characteristic polynomial

$$\det[z - m_1] = \Phi_{n_1}(z)^{\ell_1} \prod_{s\in\mathbb{N}} \Phi_s(z)^{2\ell(s)/\phi(s)}. \tag{12.6}$$

If our candidate k-tuple satisfies all these requirements at the ray M_1, we next consider consistency conditions at the rays M_2, M_3, etc. along the lines of §.10.3.7. The arguments are parallel to the ones for M_1 except that now we do not expect the monodromy to be fully regular (not even in the weak sense) so that only a sub $k_{\text{reg},i}$-tuple of dimensions is fixed at each ray. This still yields non trivial consistency conditions as in the examples of §.10.3.7.

The algorithm is longer to explain than to run. We illustrate the method in a typical example.

Example 41. In rank 4 the following (non strongly regular) 4-tuple exists[83]

$$\big\{14, 10, 8, 4\big\}. \tag{12.7}$$

Let us apply the procedure to it. The largest dimension, $\Delta_1 = 14$ has multiplicity 1 and is a rank 3 new-dimension (see table 14); then 3 out of the 4 dimensions (12.7) should form a strongly regular 3-tuple to be found in table 15 under Φ_{14}. Indeed, there we find $\{14, 10, 4\}$. The set of residual

[82] If there are $\ell > 1$ chiral generators of largest dimension, we have a $\mathbb{P}^{\ell-1}$ family of normal rays associated to this dimension. In this case the expectation is that the *generic* ray in the family has regular monodromy.

[83] We thank Jacques Distler for suggesting this example.

dimensions is $\{8\}$ (i.e. $A_1 = \{3\}$) and

$$\beta_3 = \frac{7}{14} \equiv \frac{1}{2}, \qquad s_3 = 2, \qquad 2\ell(s_3)/\phi(2) \equiv 2, \tag{12.8}$$

so that the 4-tuple (12.7) is consistent with m_1 being weakly regular with

$$\det[z - m_1] = \Phi_{14}(z)\,\Phi_2(z)^2. \tag{12.9}$$

Next we consider the second dimension $\Delta_2 = 10$. From table 14 we see that it is a rank 2 new-dimension, hence we expect that two out of the four dimensions (12.7) form a regular 2-tuple of the form $\{10, *\}$. Indeed in table 11 we find both $\{10, 8\}$ and $\{10, 4\}$. $\Delta_3 = 8$ is a rank 2 new-dimension, and $\{8, 4\}$ is a regular 2-tuple. Finally $\Delta_4 = 4$ is a rank 1 new-dimension (and hence a regular 1-tuple). Thus the 4-tuple (12.7) satisfies all the requirements.

12.3 Constructions of the lists

The procedure to determine the lists is the one explained in sections 4, 5 which we sum up here. We start by computing ρ in the case of a cyclic group with an indecomposable characteristic polynomial $\Phi(z)$. We start the algorithm with

$$\rho_{temp} := \frac{1}{\bar{\Phi}(\xi)'}. \tag{12.10}$$

If ρ_{temp} is purely imaginary, then $\rho = \rho_{temp}$, otherwise we find the unit u such that $u\rho_{temp}$ is purely imaginary.[84] Once ρ is computed, we get the initial signs for each element σ_i of the Galois group associated to $\Phi(z)$. From the positive signs we compute the dimension tuple. All the other signs can be computed exploiting the fundamental units of the cyclotomic field (using MATHEMATICA or PARI [209] when the former software fails): to each generator of the unit group we associate a sign tuple (the signs of the Galois elements). From the signs of ρ, it is easy to compute all possible signs by repeatedly multiplying the sings amongst one another. This is the algorithm to get all dimensions tuples.

The embedding may be obtained by a direct sum of lower order cyclic elements. In this case, we get the product of cyclotomic polynomials

$$\Phi_{d_1, d_2, \cdots, d_s}(z) := \Phi_{d_1}(z)\, \cdots\, \Phi_{d_s}(z). \tag{12.11}$$

The procedure is similar to the above: we first compute all the signs separately for each factor – using the above algorithm – and then we put them all together to compute the full list of dimensions.

Particular attention must be payed to those products in which the ratio of the conductors of the cyclotomic factors is a (power of a) prime number, e.g. $\Phi_{12}\Phi_4$ in the rank 3 case. In this situation, the cyclic group representation is no longer irreducible, and thus the theory of Dedekind domains of rank 1 cannot be applied: ρ is no longer a number but rather a matrix. Since this branch of number theory is not well developed, we preferred the explicit construction of the symplectic matrices Λ's. We only consider the action of the group $H = \boldsymbol{u}/N\boldsymbol{U}$ on the initial embedding, whose signs are still defined by those of ρ (the corresponding matrix shall be called m and is given in subsection 11.3).

[84] This algorithm exploits the fact that the group of units has a finite number of generators (called *fundamental units*).

The action of H only modifies the symplectic matrix: the signs of the characteristic polynomial of the new symplectic matrix, evaluated at the cyclotomic roots, give the sign changes to be applied to the original signs of ρ. Hence, the problem is to find the group H. It is a very had task to find this group: fortunately we know how it acts on the symplectic matrices:

$$\Lambda_v := p_v(m + m^{-1})\Lambda, \qquad \forall\, v \in H, \tag{12.12}$$

where p_v is a polynomial with integer coefficients of maximal rank $\phi(n) - 1$. Thus, we can write an algorithm that looks for as many p_v's as possible: every time we find one we check whether Λ_v is principal (i.e. of unit determinant) and symplectic; once these two conditions are matched, we add the sign tuple to our results. In the end, we compute all the dimensions starting from the signs of ρ and we multiply the signs with those explicitly found by our algorithm. If we find all possible signs, then the final result is definitive. In general, if we do not find all signs, we can only state our results with high confidence.

In table 15 we list the fully regular 3-tuples for $k = 3$. The first column yields the characteristic polynomial of the embedding and the second column the corresponding dimension 3-tuples.

Table 17 contains the fully regular 4-tuples for $k = 4$.

Table 14: New-dimension sets for ranks $1 \leq k \leq 13$.

Rank k	$\mathfrak{N}(k)$
1	$1, 2, 3, \frac{3}{2}, 4, \frac{4}{3}, 6, \frac{6}{5}$
2	$\frac{12}{11}, \frac{10}{9}, \frac{8}{7}, \frac{5}{4}, \frac{10}{7}, \frac{8}{5}, \frac{5}{3}, \frac{12}{7}, \frac{12}{5}, \frac{5}{2}, \frac{8}{3}, \frac{10}{3}, 5, 8, 10, 12$
3	$\frac{18}{17}, \frac{14}{13}, \frac{9}{8}, \frac{7}{6}, \frac{14}{11}, \frac{9}{7}, \frac{18}{13}, \frac{7}{5}, \frac{14}{9}, \frac{18}{11}, \frac{7}{4}, \frac{9}{5}, \frac{9}{4}, \frac{7}{3}, \frac{18}{7}, \frac{14}{5}, \frac{7}{2}, \frac{18}{5}, \frac{9}{2}, \frac{14}{3}, 7, 9, 14, 18$
4	$\frac{30}{29}, \frac{24}{23}, \frac{20}{19}, \frac{16}{15}, \frac{15}{14}, \frac{15}{13}, \frac{20}{17}, \frac{16}{13}, \frac{24}{19}, \frac{30}{23}, \frac{15}{11}, \frac{24}{17}, \frac{16}{11}, \frac{20}{13}, \frac{30}{19}, \frac{30}{17}, \frac{16}{9}, \frac{20}{11}, \frac{24}{13}, \frac{15}{8}, \frac{15}{7}, \frac{24}{11}, \frac{20}{9}, \frac{16}{7}, \frac{30}{13}, \frac{30}{11}, \frac{20}{7}, \frac{16}{5}, \frac{24}{7}, \frac{15}{4}, \frac{30}{7}, \frac{24}{5}, \frac{16}{3}, \frac{20}{3}, \frac{15}{2}, 15, 16, 20, 24, 30$
5	$\frac{22}{21}, \frac{11}{10}, \frac{22}{19}, \frac{11}{9}, \frac{22}{17}, \frac{11}{8}, \frac{22}{15}, \frac{11}{7}, \frac{22}{13}, \frac{11}{6}, \frac{11}{5}, \frac{22}{9}, \frac{11}{4}, \frac{22}{7}, \frac{11}{3}, \frac{22}{5}, \frac{11}{2}, \frac{22}{3}, 11, 22$
6	$\frac{42}{41}, \frac{36}{35}, \frac{28}{27}, \frac{26}{25}, \frac{21}{20}, \frac{13}{12}, \frac{21}{19}, \frac{28}{25}, \frac{26}{23}, \frac{42}{37}, \frac{36}{31}, \frac{13}{11}, \frac{28}{23}, \frac{21}{17}, \frac{26}{21}, \frac{36}{29}, \frac{13}{10}, \frac{21}{16}, \frac{42}{31}, \frac{26}{19}, \frac{36}{25}, \frac{13}{9}, \frac{42}{29}, \frac{28}{19}, \frac{26}{17}, \frac{36}{23}, \frac{21}{13}, \frac{13}{8}, \frac{28}{17}, \frac{42}{25}, \frac{26}{15}, \frac{42}{23}, \frac{13}{7}, \frac{28}{15}, \frac{36}{19}, \frac{21}{11}, \frac{21}{10}, \frac{36}{17}, \frac{28}{13}, \frac{13}{6}, \frac{42}{19}, \frac{26}{11}, \frac{42}{17}, \frac{28}{11}, \frac{13}{5}, \frac{21}{8}, \frac{36}{13}, \frac{26}{9}, \frac{28}{9}, \frac{42}{13}, \frac{13}{4}, \frac{36}{11}, \frac{26}{7}, \frac{42}{11}, \frac{21}{5}, \frac{13}{3}, \frac{36}{7}, \frac{26}{5}, \frac{21}{4}, \frac{28}{5}, \frac{13}{2}, \frac{36}{5}, \frac{42}{5}, \frac{26}{3}, \frac{28}{3}, \frac{21}{2}, 13, 21, 26, 28, 36, 42$
7	None
8	$\frac{60}{59}, \frac{48}{47}, \frac{40}{39}, \frac{34}{33}, \frac{32}{31}, \frac{17}{16}, \frac{40}{37}, \frac{34}{31}, \frac{32}{29}, \frac{48}{43}, \frac{60}{53}, \frac{17}{15}, \frac{48}{41}, \frac{34}{29}, \frac{32}{27}, \frac{40}{33}, \frac{17}{14}, \frac{60}{49}, \frac{34}{27}, \frac{60}{47}, \frac{32}{25}, \frac{40}{31}, \frac{48}{37}, \frac{17}{13}, \frac{34}{25}, \frac{48}{35}, \frac{40}{29}, \frac{32}{23}, \frac{60}{43}, \frac{17}{12}, \frac{60}{41}, \frac{34}{23}, \frac{40}{27}, \frac{32}{21}, \frac{17}{11}, \frac{48}{31}, \frac{34}{21}, \frac{60}{37}, \frac{48}{29}, \frac{32}{19}, \frac{17}{10}, \frac{40}{23}, \frac{34}{19}, \frac{32}{17}, \frac{17}{9}, \frac{40}{21}, \frac{48}{25}, \frac{60}{31}, \frac{60}{29}, \frac{48}{23}, \frac{40}{19}, \frac{17}{8}, \frac{32}{15}, \frac{34}{15}, \frac{40}{17}, \frac{17}{7}, \frac{32}{13}, \frac{48}{19}, \frac{60}{23}, \frac{34}{13}, \frac{48}{17}, \frac{17}{6}, \frac{32}{11}, \frac{40}{13}, \frac{34}{11}, \frac{60}{19}, \frac{17}{5}, \frac{60}{17}, \frac{32}{9}, \frac{40}{11}, \frac{48}{13}, \frac{34}{9}, \frac{17}{4}, \frac{48}{11}, \frac{40}{9}, \frac{32}{7}, \frac{60}{13}, \frac{34}{7}, \frac{60}{11}, \frac{17}{3}, \frac{40}{7}, \frac{32}{5}, \frac{34}{5}, \frac{48}{7}, \frac{17}{2}, \frac{60}{7}, \frac{48}{5}, \frac{32}{3}, \frac{34}{3}, \frac{40}{3}, 17, 32, 34, 40, 48, 60$
9	$\frac{54}{53}, \frac{38}{37}, \frac{27}{26}, \frac{19}{18}, \frac{27}{25}, \frac{38}{35}, \frac{54}{49}, \frac{19}{17}, \frac{54}{47}, \frac{38}{33}, \frac{27}{23}, \frac{19}{16}, \frac{38}{31}, \frac{27}{22}, \frac{54}{43}, \frac{19}{15}, \frac{38}{29}, \frac{54}{41}, \frac{27}{20}, \frac{19}{14}, \frac{38}{27}, \frac{27}{19}, \frac{54}{37}, \frac{19}{13}, \frac{38}{25}, \frac{54}{35}, \frac{19}{12}, \frac{27}{17}, \frac{38}{23}, \frac{27}{16}, \frac{19}{11}, \frac{54}{31}, \frac{38}{21}, \frac{54}{29}, \frac{19}{10}, \frac{27}{14}, \frac{27}{13}, \frac{19}{9}, \frac{54}{25}, \frac{38}{17}, \frac{54}{23}, \frac{19}{8}, \frac{27}{11}, \frac{38}{15}, \frac{27}{10}, \frac{19}{7}, \frac{54}{19}, \frac{38}{13}, \frac{19}{6}, \frac{54}{17}, \frac{27}{8}, \frac{38}{11}, \frac{19}{5}, \frac{27}{7}, \frac{54}{13}, \frac{38}{9}, \frac{19}{4}, \frac{54}{11}, \frac{27}{5}, \frac{38}{7}, \frac{19}{3}, \frac{27}{4}, \frac{38}{5}, \frac{54}{7}, \frac{19}{2}, \frac{54}{5}, \frac{38}{3}, \frac{27}{2}, 19, 27, 38, 54$
10	$\frac{66}{65}, \frac{50}{49}, \frac{44}{43}, \frac{33}{32}, \frac{25}{24}, \frac{50}{47}, \frac{33}{31}, \frac{44}{41}, \frac{66}{61}, \frac{25}{23}, \frac{66}{59}, \frac{44}{39}, \frac{25}{22}, \frac{33}{29}, \frac{50}{43}, \frac{33}{28}, \frac{44}{37}, \frac{25}{21}, \frac{50}{41}, \frac{66}{53}, \frac{44}{35}, \frac{33}{26}, \frac{50}{39}, \frac{25}{19}, \frac{33}{25}, \frac{66}{49}, \frac{50}{37}, \frac{25}{18}, \frac{66}{47}, \frac{44}{31}, \frac{33}{23}, \frac{25}{17}, \frac{50}{33}, \frac{44}{29}, \frac{66}{43}, \frac{25}{16}, \frac{66}{41}, \frac{50}{31}, \frac{44}{27}, \frac{33}{20}, \frac{50}{29}, \frac{33}{19}, \frac{44}{25}, \frac{66}{37}, \frac{25}{14}, \frac{50}{27}, \frac{66}{35}, \frac{44}{23}, \frac{25}{13}, \frac{33}{17}, \frac{33}{16}, \frac{25}{12}, \frac{44}{21}, \frac{66}{31}, \frac{50}{23}, \frac{25}{11}, \frac{66}{29}, \frac{44}{19}, \frac{33}{14}, \frac{50}{21}, \frac{33}{13}, \frac{44}{17}, \frac{50}{19}, \frac{66}{25}, \frac{25}{9}, \frac{66}{23}, \frac{44}{15}, \frac{50}{17}, \frac{25}{8}, \frac{33}{10}, \frac{44}{13}, \frac{66}{19}, \frac{25}{7}, \frac{50}{13}, \frac{66}{17}, \frac{33}{8}, \frac{25}{6}, \frac{50}{11}, \frac{33}{7}, \frac{44}{9}, \frac{66}{13}, \frac{50}{9}, \frac{25}{4}, \frac{44}{7}, \frac{33}{5}, \frac{50}{7}, \frac{33}{4}, \frac{25}{3}, \frac{44}{5}, \frac{66}{7}, \frac{25}{2}, \frac{66}{5}, \frac{44}{3}, \frac{33}{2}, \frac{50}{3}, 25, 33, 44, 50, 66$

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Rank k	$\mathfrak{N}(k)$
11	$\frac{46}{45}, \frac{23}{22}, \frac{46}{43}, \frac{23}{21}, \frac{46}{41}, \frac{23}{20}, \frac{46}{39}, \frac{23}{19}, \frac{46}{37}, \frac{23}{18}, \frac{46}{35}, \frac{23}{17}, \frac{46}{33}, \frac{23}{16}, \frac{46}{31}, \frac{23}{15}, \frac{46}{29}, \frac{23}{14}, \frac{46}{27}, \frac{23}{13}, \frac{46}{25}, \frac{23}{12},$ $\frac{23}{11}, \frac{46}{21}, \frac{23}{10}, \frac{46}{19}, \frac{23}{9}, \frac{46}{17}, \frac{23}{8}, \frac{46}{15}, \frac{23}{7}, \frac{46}{13}, \frac{23}{6}, \frac{46}{11}, \frac{23}{5}, \frac{46}{9}, \frac{23}{4}, \frac{46}{7}, \frac{23}{3}, \frac{46}{5}, \frac{23}{2}, \frac{46}{3}, 23, 46$
12	$\frac{90}{89}, \frac{84}{83}, \frac{78}{77}, \frac{72}{71}, \frac{70}{69}, \frac{56}{55}, \frac{52}{51}, \frac{45}{44}, \frac{39}{38}, \frac{35}{34}, \frac{70}{67}, \frac{45}{43}, \frac{39}{37}, \frac{56}{53}, \frac{35}{33}, \frac{52}{49}, \frac{84}{79}, \frac{78}{73}, \frac{72}{67}, \frac{90}{83}, \frac{35}{32}, \frac{45}{41},$ $\frac{56}{51}, \frac{78}{71}, \frac{52}{47}, \frac{72}{65}, \frac{39}{35}, \frac{35}{31}, \frac{90}{79}, \frac{39}{34}, \frac{70}{61}, \frac{84}{73}, \frac{52}{45}, \frac{78}{67}, \frac{90}{77}, \frac{72}{61}, \frac{84}{71}, \frac{45}{38}, \frac{70}{59}, \frac{56}{47}, \frac{35}{29}, \frac{52}{43}, \frac{45}{37}, \frac{39}{32},$ $\frac{72}{59}, \frac{70}{57}, \frac{90}{73}, \frac{56}{45}, \frac{84}{67}, \frac{39}{31}, \frac{90}{71}, \frac{52}{41}, \frac{78}{61}, \frac{84}{65}, \frac{35}{27}, \frac{56}{43}, \frac{72}{55}, \frac{70}{53}, \frac{78}{59}, \frac{45}{34}, \frac{90}{67}, \frac{39}{29}, \frac{35}{26}, \frac{72}{53}, \frac{56}{41}, \frac{70}{51},$ $\frac{84}{61}, \frac{39}{28}, \frac{52}{37}, \frac{45}{32}, \frac{78}{55}, \frac{84}{59}, \frac{56}{39}, \frac{45}{31}, \frac{35}{24}, \frac{72}{49}, \frac{78}{53}, \frac{90}{61}, \frac{52}{35}, \frac{70}{47}, \frac{56}{37}, \frac{35}{23}, \frac{90}{59}, \frac{84}{55}, \frac{72}{47}, \frac{45}{29}, \frac{39}{25}, \frac{52}{33},$ $\frac{84}{53}, \frac{35}{22}, \frac{78}{49}, \frac{45}{28}, \frac{70}{43}, \frac{78}{47}, \frac{72}{43}, \frac{52}{31}, \frac{39}{23}, \frac{56}{33}, \frac{90}{53}, \frac{70}{41}, \frac{45}{26}, \frac{72}{41}, \frac{39}{22}, \frac{84}{47}, \frac{52}{29}, \frac{70}{39}, \frac{56}{31}, \frac{78}{43}, \frac{90}{49}, \frac{35}{19},$ $\frac{70}{37}, \frac{78}{41}, \frac{90}{47}, \frac{52}{27}, \frac{56}{29}, \frac{35}{18}, \frac{72}{37}, \frac{39}{20}, \frac{84}{43}, \frac{45}{23}, \frac{45}{22}, \frac{84}{41}, \frac{39}{19}, \frac{72}{35}, \frac{35}{17}, \frac{56}{27}, \frac{52}{25}, \frac{90}{43}, \frac{78}{37}, \frac{70}{33}, \frac{35}{16}, \frac{90}{41},$ $\frac{78}{35}, \frac{56}{25}, \frac{70}{31}, \frac{52}{23}, \frac{84}{37}, \frac{39}{17}, \frac{72}{31}, \frac{45}{19}, \frac{70}{29}, \frac{90}{37}, \frac{56}{23}, \frac{39}{16}, \frac{52}{21}, \frac{72}{29}, \frac{78}{31}, \frac{70}{27}, \frac{45}{17}, \frac{78}{29}, \frac{35}{13}, \frac{84}{31}, \frac{52}{19}, \frac{39}{14},$ $\frac{45}{16}, \frac{72}{25}, \frac{84}{29}, \frac{90}{31}, \frac{35}{12}, \frac{56}{19}, \frac{70}{23}, \frac{52}{17}, \frac{90}{29}, \frac{78}{25}, \frac{72}{23}, \frac{35}{11}, \frac{45}{14}, \frac{56}{17}, \frac{84}{25}, \frac{78}{23}, \frac{45}{13}, \frac{52}{15}, \frac{39}{11}, \frac{84}{23}, \frac{70}{19}, \frac{56}{15},$ $\frac{72}{19}, \frac{35}{9}, \frac{39}{10}, \frac{90}{23}, \frac{45}{11}, \frac{78}{19}, \frac{70}{17}, \frac{72}{17}, \frac{56}{13}, \frac{35}{8}, \frac{84}{19}, \frac{78}{17}, \frac{52}{11}, \frac{90}{19}, \frac{39}{8}, \frac{84}{17}, \frac{56}{11}, \frac{90}{17}, \frac{70}{13}, \frac{72}{13}, \frac{39}{7}, \frac{45}{8},$ $\frac{52}{9}, \frac{35}{6}, \frac{56}{9}, \frac{70}{11}, \frac{45}{7}, \frac{84}{13}, \frac{72}{11}, \frac{90}{13}, \frac{78}{11}, \frac{52}{7}, \frac{84}{11}, \frac{70}{9}, \frac{39}{5}, \frac{90}{11}, \frac{35}{4}, \frac{39}{4}, \frac{72}{7}, \frac{52}{5}, \frac{78}{7}, \frac{56}{5}, \frac{45}{4}, \frac{35}{3},$ $\frac{90}{7}, \frac{72}{5}, \frac{78}{5}, \frac{84}{5}, \frac{52}{3}, \frac{35}{2}, \frac{56}{3}, \frac{39}{2}, \frac{45}{2}, \frac{70}{3}, 35, 39, 45, 52, 56, 70, 72, 78, 84, 90$
13	None

Table 15: Strongly regular principal 3-tuples for rank 3

C.P.	strongly regular principal 3-tuples
Φ_7	$\left\{\frac{7}{6}, \frac{4}{3}, \frac{5}{3}\right\}, \left\{\frac{7}{6}, \frac{4}{3}, \frac{3}{2}\right\}, \{7,3,5\}, \{7,3,4\}, \{7,6,5\}, \{7,6,4\}, \left\{\frac{6}{5}, \frac{7}{5}, \frac{9}{5}\right\},$ $\left\{\frac{6}{5}, \frac{7}{5}, \frac{8}{5}\right\}, \left\{\frac{3}{2}, \frac{7}{2}, 3\right\}, \left\{\frac{3}{2}, \frac{7}{2}, \frac{5}{2}\right\}, \left\{4, \frac{7}{2}, 3\right\}, \left\{4, \frac{7}{2}, \frac{5}{2}\right\}, \left\{\frac{4}{3}, \frac{5}{3}, \frac{7}{3}\right\}, \left\{\frac{5}{4}, \frac{3}{2}, \frac{7}{4}\right\},$ $\left\{\frac{4}{3}, \frac{8}{3}, \frac{7}{3}\right\}, \left\{\frac{5}{4}, \frac{9}{4}, \frac{7}{4}\right\}, \left\{3, \frac{5}{3}, \frac{7}{3}\right\}, \left\{\frac{5}{2}, \frac{3}{2}, \frac{7}{4}\right\}, \left\{3, \frac{8}{3}, \frac{7}{3}\right\}, \left\{\frac{5}{2}, \frac{9}{4}, \frac{7}{4}\right\}$
Φ_9	$\left\{\frac{9}{8}, \frac{3}{2}, \frac{5}{4}\right\}, \{9,5,8\}, \{9,5,3\}, \{9,6,8\}, \{9,6,3\}, \left\{\frac{6}{5}, \frac{9}{5}, \frac{12}{5}\right\}, \left\{\frac{6}{5}, \frac{9}{5}, \frac{7}{5}\right\},$ $\left\{\frac{5}{4}, \frac{9}{4}, \frac{3}{2}\right\}, \left\{3, \frac{9}{4}, \frac{3}{2}\right\}, \left\{\frac{3}{2}, 3, \frac{9}{2}\right\}, \left\{\frac{3}{2}, \frac{7}{2}, \frac{9}{2}\right\}, \left\{\frac{8}{7}, \frac{12}{7}, \frac{9}{7}\right\}, \left\{5, 3, \frac{9}{2}\right\}, \left\{5, \frac{7}{2}, \frac{9}{2}\right\}$
Φ_{14}	$\{14,6,12\}, \{14,6,4\}, \{14,10,12\}, \{14,10,4\}, \left\{\frac{10}{9}, \frac{14}{9}, \frac{4}{3}\right\}, \left\{\frac{6}{5}, \frac{14}{5}, \frac{8}{5}\right\},$ $\left\{\frac{18}{5}, \frac{14}{5}, \frac{8}{5}\right\}, \left\{\frac{4}{3}, \frac{8}{3}, \frac{14}{3}\right\}, \left\{\frac{4}{3}, 4, \frac{14}{3}\right\}$
Φ_{18}	$\left\{\frac{18}{7}, \frac{12}{7}, \frac{8}{7}\right\}, \left\{\frac{12}{5}, \frac{18}{5}, \frac{6}{5}\right\}, \{12,14,18\}, \{12,6,18\}, \{8,14,18\}, \{8,6,18\}$

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C.P.	strongly regular principal 3-tuples
$\Phi_{3,5}$	$\left\{3, \frac{8}{5}, \frac{14}{5}\right\}, \left\{\frac{8}{3}, 5, 4\right\}, \left\{\frac{8}{3}, 5, 3\right\}, \left\{\frac{14}{9}, \frac{4}{3}, \frac{5}{3}\right\}, \left\{\frac{14}{9}, \frac{7}{3}, \frac{5}{3}\right\}, \left\{\frac{8}{3}, \frac{3}{2}, \frac{5}{2}\right\}, \left\{\frac{8}{3}, 3, \frac{5}{2}\right\}$
$\Phi_{4,5}$	$\left\{\frac{9}{4}, 5, 4\right\}, \left\{\frac{9}{4}, 5, 3\right\}, \left\{\frac{9}{4}, \frac{4}{3}, \frac{5}{3}\right\}, \left\{\frac{9}{4}, \frac{7}{3}, \frac{5}{3}\right\}$
$\Phi_{5,6}$	None
$\Phi_{3,8}$	None
$\Phi_{4,8}$	$\left\{\frac{4}{3}, \frac{7}{6}, \frac{3}{2}\right\}, \left\{4, \frac{3}{2}, \frac{5}{2}\right\}, \left\{4, \frac{3}{2}, \frac{7}{2}\right\}, \left\{4, \frac{9}{2}, \frac{5}{2}\right\}, \left\{4, \frac{9}{2}, \frac{7}{2}\right\}, \left\{\frac{9}{7}, \frac{8}{7}, \frac{10}{7}\right\}, \left\{\frac{9}{7}, \frac{8}{7}, \frac{12}{7}\right\},$ $\{3, 8, 4\}, \{3, 8, 6\}, \{7, 8, 4\}, \{7, 8, 6\}, \left\{\frac{7}{5}, \frac{6}{5}, \frac{8}{5}\right\}, \left\{\frac{5}{3}, \frac{4}{3}, \frac{8}{3}\right\}, \left\{\frac{7}{5}, \frac{12}{5}, \frac{8}{5}\right\},$ $\left\{\frac{5}{3}, \frac{10}{3}, \frac{8}{3}\right\}, \left\{3, \frac{4}{3}, \frac{8}{3}\right\}, \left\{3, \frac{10}{3}, \frac{8}{3}\right\}$
$\Phi_{6,8}$	$\left\{\frac{7}{3}, 8, 4\right\}, \left\{\frac{7}{3}, 8, 6\right\}, \left\{\frac{7}{3}, \frac{6}{5}, \frac{8}{5}\right\}, \left\{\frac{7}{3}, \frac{12}{5}, \frac{8}{5}\right\}$
$\Phi_{3,10}$	None
$\Phi_{4,10}$	$\left\{\frac{7}{2}, \frac{10}{3}, 4\right\}, \left\{\frac{7}{2}, \frac{10}{3}, \frac{4}{3}\right\}, \left\{\frac{7}{2}, 4, 10\right\}, \left\{\frac{7}{2}, 8, 10\right\}$
$\Phi_{6,10}$	$\left\{6, \frac{14}{5}, \frac{8}{5}\right\}, \left\{\frac{14}{9}, \frac{10}{3}, 4\right\}, \left\{\frac{14}{9}, \frac{10}{3}, \frac{4}{3}\right\}, \left\{\frac{8}{3}, 4, 10\right\}, \left\{\frac{8}{3}, 8, 10\right\}$
$\Phi_{3,12}$	$\left\{\frac{9}{5}, \frac{12}{5}, \frac{6}{5}\right\}, \{5, 6, 12\}, \{9, 6, 12\}$
$\Phi_{4,12}$	$\left\{4, \frac{8}{3}, \frac{14}{3}\right\}, \left\{4, \frac{8}{3}, \frac{4}{3}\right\}, \left\{4, \frac{14}{3}, \frac{10}{3}\right\}, \left\{4, \frac{10}{3}, \frac{4}{3}\right\}, \left\{\frac{10}{7}, \frac{12}{7}, \frac{18}{7}\right\},$ $\left\{\frac{10}{7}, \frac{12}{7}, \frac{8}{7}\right\}, \left\{\frac{8}{5}, \frac{12}{5}, \frac{6}{5}\right\}, \left\{\frac{14}{5}, \frac{12}{5}, \frac{6}{5}\right\}, \{4, 6, 12\}, \{4, 12, 8\},$ $\{10, 12, 8\}, \left\{\frac{14}{11}, \frac{18}{11}, \frac{12}{11}\right\}, \{10, 6, 12\}, \left\{\frac{4}{3}, \frac{14}{9}, \frac{10}{9}\right\}$
$\Phi_{6,12}$	$\left\{6, \frac{9}{2}, \frac{3}{2}\right\}, \left\{\frac{9}{7}, \frac{12}{7}, \frac{18}{7}\right\}, \left\{\frac{7}{5}, \frac{12}{5}, \frac{6}{5}\right\}, \left\{3, \frac{12}{5}, \frac{6}{5}\right\}, \{3, 6, 12\}$
$\Phi_{3,4,6}$	$\left\{3, \frac{7}{4}, \frac{3}{2}\right\}, \left\{3, \frac{7}{4}, \frac{7}{2}\right\}, \left\{\frac{7}{3}, 4, \frac{5}{3}\right\}, \left\{3, \frac{5}{2}, 6\right\}, \left\{5, \frac{5}{2}, 6\right\}$

Table 16: Strongly regular non-principal 3-tuples

C.P.	strongly regular non-principal 3-tuples
$\Phi_{3,12}$	$\left\{3, \frac{9}{4}, \frac{5}{4}\right\}, \{5, 12, 8\}, \{9, 12, 8\}$
$\Phi_{6,12}$	$\left\{6, \frac{7}{2}, \frac{3}{2}\right\}, \{3, 12, 8\}, \left\{\frac{9}{7}, \frac{12}{7}, \frac{8}{7}\right\}$

Table 17: Strongly regular principal 4-tuples for rank 4

C.P.	strongly regular principal 4-tuples
Φ_{15}	$\left\{\frac{15}{2}, \frac{9}{2}, 3, 8\right\}, \left\{\frac{15}{2}, 5, 3, \frac{3}{2}\right\}, \left\{\frac{5}{4}, \frac{15}{8}, \frac{3}{2}, \frac{9}{8}\right\}, \left\{\frac{9}{7}, \frac{15}{7}, \frac{18}{7}, \frac{8}{7}\right\}, \left\{\frac{20}{7}, \frac{15}{7}, \frac{18}{7}, 3\right\},$ $\left\{\frac{3}{2}, 3, \frac{15}{4}, \frac{5}{4}\right\}, \{3, 8, 12, 15\}, \left\{\frac{8}{7}, \frac{3}{2}, \frac{9}{7}, \frac{15}{14}\right\}, \{3, 9, 5, 15\},$ $\{14, 8, 5, 15\}, \{14, 9, 12, 15\}$
Φ_{16}	$\left\{\frac{16}{15}, \frac{6}{5}, \frac{4}{3}, \frac{8}{5}\right\}, \{16, 4, 6, 8\}, \{16, 4, 6, 10\}, \{16, 4, 12, 8\}, \{16, 4, 12, 10\},$ $\{16, 14, 6, 8\}, \{16, 14, 6, 10\}, \{16, 14, 12, 8\}, \{16, 14, 12, 10\}, \left\{\frac{14}{13}, \frac{16}{13}, \frac{18}{13}, \frac{20}{13}\right\},$ $\left\{\frac{14}{13}, \frac{16}{13}, \frac{24}{13}, \frac{20}{13}\right\}, \left\{\frac{4}{3}, \frac{16}{3}, \frac{8}{3}, \frac{10}{3}\right\}, \left\{\frac{4}{3}, \frac{16}{3}, \frac{8}{3}, 4\right\}, \left\{\frac{4}{3}, \frac{16}{3}, \frac{14}{3}, \frac{10}{3}\right\}, \left\{\frac{4}{3}, \frac{16}{3}, \frac{14}{3}, 4\right\},$ $\left\{6, \frac{16}{3}, \frac{8}{3}, \frac{10}{3}\right\}, \left\{6, \frac{16}{3}, \frac{8}{3}, 4\right\}, \left\{6, \frac{16}{3}, \frac{14}{3}, \frac{10}{3}\right\}, \left\{6, \frac{16}{3}, \frac{14}{3}, 4\right\}, \left\{\frac{12}{11}, \frac{14}{11}, \frac{16}{11}, \frac{18}{11}\right\},$ $\left\{\frac{12}{11}, \frac{14}{11}, \frac{16}{11}, \frac{20}{11}\right\}, \left\{\frac{6}{5}, \frac{8}{5}, \frac{16}{5}, \frac{12}{5}\right\}, \left\{\frac{6}{5}, \frac{8}{5}, \frac{16}{5}, \frac{14}{5}\right\}, \left\{\frac{12}{11}, \frac{24}{11}, \frac{16}{11}, \frac{18}{11}\right\}, \left\{\frac{12}{11}, \frac{24}{11}, \frac{16}{11}, \frac{20}{11}\right\},$ $\left\{\frac{6}{5}, \frac{18}{5}, \frac{16}{5}, \frac{12}{5}\right\}, \left\{\frac{6}{5}, \frac{18}{5}, \frac{16}{5}, \frac{14}{5}\right\}, \left\{4, \frac{8}{5}, \frac{16}{5}, \frac{12}{5}\right\}, \left\{4, \frac{8}{5}, \frac{16}{5}, \frac{14}{5}\right\}, \left\{4, \frac{18}{5}, \frac{16}{5}, \frac{12}{5}\right\},$ $\left\{4, \frac{18}{5}, \frac{16}{5}, \frac{14}{5}\right\}, \left\{\frac{10}{9}, \frac{4}{3}, \frac{14}{9}, \frac{16}{9}\right\}, \left\{\frac{8}{7}, \frac{10}{7}, \frac{12}{7}, \frac{16}{7}\right\}, \left\{\frac{10}{9}, \frac{4}{3}, \frac{20}{9}, \frac{16}{9}\right\}, \left\{\frac{8}{7}, \frac{10}{7}, \frac{18}{7}, \frac{16}{7}\right\},$ $\left\{\frac{8}{7}, \frac{20}{7}, \frac{12}{7}, \frac{16}{7}\right\}, \left\{\frac{8}{7}, \frac{20}{7}, \frac{18}{7}, \frac{16}{7}\right\}, \left\{\frac{8}{3}, \frac{4}{3}, \frac{14}{9}, \frac{16}{9}\right\}, \left\{\frac{8}{3}, \frac{4}{3}, \frac{20}{9}, \frac{16}{9}\right\}$
Φ_{20}	$\{20, 4, 12, 8\}, \{20, 4, 10, 14\}, \{20, 18, 12, 14\}, \{20, 18, 10, 8\}, \left\{\frac{4}{3}, \frac{20}{3}, \frac{14}{3}, \frac{10}{3}\right\},$ $\left\{\frac{4}{3}, \frac{20}{3}, 4, \frac{16}{3}\right\}, \left\{\frac{12}{11}, \frac{14}{11}, \frac{20}{11}, \frac{18}{11}\right\}, \left\{\frac{30}{11}, \frac{14}{11}, \frac{20}{11}, \frac{24}{11}\right\}, \left\{\frac{8}{7}, \frac{10}{7}, \frac{18}{7}, \frac{20}{7}\right\},$ $\left\{\frac{14}{13}, \frac{30}{13}, \frac{24}{13}, \frac{20}{13}\right\}, \left\{\frac{8}{7}, \frac{24}{7}, \frac{16}{7}, \frac{20}{7}\right\}$
Φ_{24}	$\{24, 8, 14, 6\}, \{24, 8, 12, 20\}, \{24, 18, 14, 20\}, \{24, 18, 12, 6\}, \left\{\frac{8}{7}, \frac{24}{7}, \frac{20}{7}, \frac{12}{7}\right\},$ $\left\{\frac{30}{7}, \frac{24}{7}, \frac{18}{7}, \frac{12}{7}\right\}, \left\{\frac{12}{11}, \frac{18}{11}, \frac{24}{11}, \frac{30}{11}\right\}, \left\{\frac{14}{13}, \frac{20}{13}, \frac{24}{13}, \frac{18}{13}\right\}, \left\{\frac{6}{5}, \frac{12}{5}, \frac{18}{5}, \frac{24}{5}\right\}$
Φ_{30}	$\left\{\frac{30}{7}, \frac{24}{7}, \frac{18}{7}, \frac{8}{7}\right\}, \{8, 14, 20, 30\}, \{8, 18, 12, 30\}, \{24, 14, 12, 30\}, \{24, 18, 20, 30\}$
$\Phi_{3,7}$	$\left\{\frac{10}{3}, \frac{7}{2}, \frac{5}{2}, 4\right\}, \left\{\frac{10}{3}, \frac{7}{2}, \frac{5}{2}, \frac{3}{2}\right\}, \left\{\frac{10}{3}, \frac{7}{2}, 3, 4\right\}, \left\{\frac{10}{3}, \frac{7}{2}, 3, \frac{3}{2}\right\}, \left\{\frac{16}{9}, \frac{5}{3}, \frac{7}{3}, 3\right\},$ $\left\{\frac{16}{9}, \frac{5}{3}, \frac{7}{3}, \frac{4}{3}\right\}, \left\{\frac{16}{9}, \frac{8}{3}, \frac{7}{3}, 3\right\}, \left\{\frac{16}{9}, \frac{8}{3}, \frac{7}{3}, \frac{4}{3}\right\}, \left\{\frac{10}{3}, 3, 4, 7\right\}, \left\{\frac{10}{3}, 3, 5, 7\right\},$ $\left\{\frac{10}{3}, 6, 4, 7\right\}, \left\{\frac{10}{3}, 6, 5, 7\right\}, \left\{\frac{16}{9}, \frac{4}{3}, \frac{3}{2}, \frac{7}{6}\right\}, \left\{\frac{16}{9}, \frac{4}{3}, \frac{5}{3}, \frac{7}{6}\right\}$

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Table 17 – continued from previous page

C.P.	strongly regular principal 4-tuples
$\Phi_{4,7}$	$\{\frac{15}{8},\frac{7}{6},\frac{3}{2},\frac{4}{3}\}, \{\frac{15}{8},\frac{7}{6},\frac{5}{3},\frac{4}{3}\}, \{\frac{15}{8},\frac{3}{2},\frac{5}{2},\frac{7}{2}\}, \{\frac{15}{8},\frac{3}{2},3,\frac{7}{2}\}, \{\frac{15}{8},4,\frac{5}{2},\frac{7}{2}\}, \{\frac{15}{8},4,3,\frac{7}{2}\}$
$\Phi_{6,7}$	None
$\Phi_{3,9}$	$\{\frac{3}{2},\frac{7}{6},\frac{5}{3},\frac{4}{3}\}, \{\frac{3}{2},\frac{7}{3},\frac{5}{3},\frac{4}{3}\}, \{3,\frac{4}{3},\frac{7}{3},\frac{10}{3}\}, \{3,\frac{4}{3},\frac{7}{3},\frac{5}{3}\}, \{3,\frac{4}{3},\frac{8}{3},\frac{10}{3}\}, \{3,\frac{4}{3},\frac{8}{3},\frac{5}{3}\},$ $\{4,9,5,8\}, \{4,9,5,3\}, \{4,9,6,8\}, \{4,9,6,3\}, \{\frac{7}{4},\frac{9}{8},\frac{3}{2},\frac{15}{8}\}, \{\frac{7}{4},\frac{9}{8},\frac{3}{2},\frac{5}{4}\}, \{7,9,5,8\},$ $\{7,9,5,3\}, \{7,9,6,8\}, \{7,9,6,3\}, \{\frac{8}{5},\frac{6}{5},\frac{9}{5},\frac{12}{5}\}, \{\frac{8}{5},\frac{6}{5},\frac{9}{5},\frac{7}{5}\}, \{\frac{7}{4},\frac{5}{4},\frac{9}{4},\frac{3}{2}\},$ $\{\frac{7}{4},3,\frac{9}{4},\frac{3}{2}\}, \{\frac{5}{2},\frac{5}{4},\frac{9}{4},\frac{3}{2}\}, \{\frac{5}{2},3,\frac{9}{4},\frac{3}{2}\}, \{\frac{5}{2},\frac{3}{2},3,\frac{9}{2}\}, \{\frac{5}{2},\frac{3}{2},\frac{7}{2},\frac{9}{2}\}, \{\frac{10}{7},\frac{8}{7},\frac{12}{7},\frac{9}{7}\},$ $\{\frac{5}{2},5,3,\frac{9}{2}\}, \{\frac{5}{2},5,\frac{7}{2},\frac{9}{2}\}, \{\frac{10}{7},\frac{15}{7},\frac{12}{7},\frac{9}{7}\}, \{4,\frac{3}{2},3,\frac{9}{2}\}, \{4,\frac{3}{2},\frac{7}{2},\frac{9}{2}\}, \{4,5,3,\frac{9}{2}\},$ $\{4,5,\frac{7}{2},\frac{9}{2}\}$
$\Phi_{4,9}$	None
$\Phi_{6,9}$	$\{\frac{5}{2},\frac{9}{5},\frac{12}{5},\frac{6}{5}\}, \{\frac{5}{2},\frac{9}{5},\frac{7}{5},\frac{6}{5}\}, \{\frac{7}{4},\frac{7}{2},\frac{9}{2},5\}, \{\frac{7}{4},\frac{7}{2},\frac{9}{2},\frac{3}{2}\}, \{\frac{7}{4},3,\frac{9}{2},5\}, \{\frac{7}{4},3,\frac{9}{2},\frac{3}{2}\},$ $\{\frac{5}{2},6,8,9\}, \{\frac{5}{2},6,3,9\}, \{\frac{5}{2},5,8,9\}, \{\frac{5}{2},5,3,9\}$
$\Phi_{3,14}$	None
$\Phi_{4,14}$	$\{\frac{9}{2},14,4,10\}, \{\frac{9}{2},14,4,6\}, \{\frac{9}{2},14,12,10\}, \{\frac{9}{2},14,12,6\}, \{\frac{9}{2},\frac{4}{3},\frac{14}{3},4\},$ $\{\frac{9}{2},\frac{4}{3},\frac{14}{3},\frac{8}{3}\}, \{\frac{9}{2},\frac{16}{3},\frac{14}{3},4\}, \{\frac{9}{2},\frac{16}{3},\frac{14}{3},\frac{8}{3}\}$
$\Phi_{6,14}$	$\{\frac{10}{3},14,6,12\}, \{\frac{10}{3},14,6,4\}, \{\frac{10}{3},14,10,12\}, \{\frac{10}{3},14,10,4\}, \{\frac{10}{3},\frac{6}{5},\frac{14}{5},\frac{16}{5}\},$ $\{\frac{10}{3},\frac{6}{5},\frac{14}{5},\frac{8}{5}\}, \{\frac{10}{3},\frac{18}{5},\frac{14}{5},\frac{16}{5}\}, \{\frac{10}{3},\frac{18}{5},\frac{14}{5},\frac{8}{5}\}, \{\frac{16}{9},\frac{4}{3},\frac{8}{3},\frac{14}{3}\}, \{\frac{16}{9},\frac{4}{3},4,\frac{14}{3}\},$ $\{\frac{16}{9},\frac{16}{3},\frac{8}{3},\frac{14}{3}\}, \{\frac{16}{9},\frac{16}{3},4,\frac{14}{3}\}$
$\Phi_{3,18}$	$\{7,18,12,14\}, \{7,18,12,6\}, \{7,18,8,14\}, \{7,18,8,6\}$
$\Phi_{4,18}$	None
$\Phi_{6,18}$	$\{6,\frac{4}{3},\frac{14}{3},\frac{16}{3}\}, \{6,\frac{4}{3},\frac{14}{3},\frac{8}{3}\}, \{6,\frac{4}{3},\frac{10}{3},\frac{16}{3}\}, \{6,\frac{4}{3},\frac{10}{3},\frac{8}{3}\}, \{6,\frac{20}{3},\frac{14}{3},\frac{16}{3}\},$ $\{6,\frac{20}{3},\frac{14}{3},\frac{8}{3}\}, \{6,\frac{20}{3},\frac{10}{3},\frac{16}{3}\}, \{6,\frac{20}{3},\frac{10}{3},\frac{8}{3}\}, \{\frac{20}{17},\frac{18}{17},\frac{24}{17},\frac{30}{17}\}, \{4,18,12,14\},$ $\{4,18,12,6\}, \{4,18,8,14\}, \{4,18,8,6\}, \{16,18,12,14\}, \{16,18,12,6\},$ $\{16,18,8,14\}, \{16,18,8,6\}, \{\frac{10}{7},\frac{8}{7},\frac{18}{7},\frac{20}{7}\}, \{\frac{10}{7},\frac{8}{7},\frac{18}{7},\frac{12}{7}\}, \{\frac{14}{11},\frac{12}{11},\frac{18}{11},\frac{24}{11}\},$ $\{\frac{14}{11},\frac{12}{11},\frac{18}{11},\frac{16}{11}\}, \{\frac{10}{7},\frac{24}{7},\frac{18}{7},\frac{20}{7}\}, \{\frac{10}{7},\frac{24}{7},\frac{18}{7},\frac{12}{7}\}, \{\frac{8}{5},\frac{6}{5},\frac{16}{5},\frac{18}{5}\}, \{\frac{16}{13},\frac{14}{13},\frac{24}{13},\frac{18}{13}\},$ $\{\frac{8}{5},\frac{6}{5},\frac{12}{5},\frac{18}{5}\}, \{\frac{16}{13},\frac{14}{13},\frac{20}{13},\frac{18}{13}\}, \{\frac{16}{13},\frac{30}{13},\frac{24}{13},\frac{18}{13}\}, \{\frac{16}{13},\frac{30}{13},\frac{20}{13},\frac{18}{13}\}, \{4,\frac{6}{5},\frac{16}{5},\frac{18}{5}\},$ $\{4,\frac{6}{5},\frac{12}{5},\frac{18}{5}\}$

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Table 17 – continued from previous page

C.P.	strongly regular principal 4-tuples
$\Phi_{5,8}$	None
$\Phi_{5,10}$	$\left\{\frac{5}{4},\frac{7}{4},\frac{15}{8},\frac{9}{8}\right\},\left\{\frac{5}{4},\frac{3}{2},\frac{15}{8},\frac{9}{8}\right\},\left\{5,4,\frac{5}{2},\frac{3}{2}\right\},\left\{5,4,\frac{9}{2},\frac{3}{2}\right\},\left\{5,3,\frac{5}{2},\frac{3}{2}\right\},\left\{5,3,\frac{9}{2},\frac{3}{2}\right\},$ $\left\{\frac{3}{2},\frac{5}{2},\frac{7}{4},\frac{5}{4}\right\},\left\{\frac{4}{3},\frac{5}{3},\frac{3}{2},\frac{5}{2}\right\},\left\{\frac{4}{3},\frac{5}{3},\frac{3}{2},\frac{7}{6}\right\},\left\{3,\frac{5}{2},\frac{7}{4},\frac{5}{4}\right\},\left\{\frac{7}{3},\frac{5}{3},\frac{3}{2},\frac{5}{2}\right\},\left\{\frac{7}{3},\frac{5}{3},\frac{3}{2},\frac{7}{6}\right\},$ $\left\{\frac{5}{3},3,\frac{10}{3},4\right\},\left\{\frac{5}{3},3,\frac{10}{3},\frac{4}{3}\right\},\left\{\frac{5}{3},\frac{7}{3},\frac{10}{3},4\right\},\left\{\frac{5}{3},\frac{7}{3},\frac{10}{3},\frac{4}{3}\right\},\{3,7,4,10\},\{3,7,8,10\},$ $\{3,5,4,10\},\{3,5,8,10\},\{9,7,4,10\},\{9,7,8,10\},\{9,5,4,10\},\{9,5,8,10\}$
$\Phi_{5,12}$	None
$\Phi_{8,10}$	None
$\Phi_{8,12}$	None
$\Phi_{10,12}$	None
$\Phi_{3,4,5}$	$\left\{3,\frac{7}{4},\frac{8}{5},\frac{14}{5}\right\},\left\{\frac{8}{3},\frac{9}{4},5,4\right\},\left\{\frac{8}{3},\frac{9}{4},5,3\right\},\left\{\frac{14}{9},\frac{9}{4},\frac{4}{3},\frac{5}{3}\right\},\left\{\frac{14}{9},\frac{9}{4},\frac{7}{3},\frac{5}{3}\right\}$
$\Phi_{3,5,6}$	$\left\{3,\frac{8}{5},\frac{14}{5},\frac{3}{2}\right\},\left\{3,\frac{8}{5},\frac{14}{5},\frac{7}{2}\right\}$
$\Phi_{4,5,6}$	None
$\Phi_{3,4,8}$	$\left\{\frac{7}{3},4,\frac{3}{2},\frac{5}{2}\right\},\left\{\frac{7}{3},4,\frac{3}{2},\frac{7}{2}\right\},\left\{\frac{7}{3},4,\frac{9}{2},\frac{5}{2}\right\},\left\{\frac{7}{3},4,\frac{9}{2},\frac{7}{2}\right\}$
$\Phi_{3,6,8}$	None
$\Phi_{4,6,8}$	$\left\{4,\frac{5}{3},\frac{3}{2},\frac{5}{2}\right\},\left\{4,\frac{5}{3},\frac{3}{2},\frac{7}{2}\right\},\left\{4,\frac{5}{3},\frac{9}{2},\frac{5}{2}\right\},\left\{4,\frac{5}{3},\frac{9}{2},\frac{7}{2}\right\},\left\{3,\frac{7}{3},8,4\right\},\left\{3,\frac{7}{3},8,6\right\},$ $\left\{7,\frac{7}{3},8,4\right\},\left\{7,\frac{7}{3},8,6\right\},\left\{\frac{7}{5},\frac{7}{3},\frac{6}{5},\frac{8}{5}\right\},\left\{\frac{7}{5},\frac{7}{3},\frac{12}{5},\frac{8}{5}\right\}$
$\Phi_{3,4,10}$	None
$\Phi_{3,6,10}$	$\left\{3,6,\frac{14}{5},\frac{8}{5}\right\},\left\{5,6,\frac{14}{5},\frac{8}{5}\right\}$
$\Phi_{4,6,10}$	$\left\{\frac{5}{2},6,\frac{14}{5},\frac{8}{5}\right\},\left\{\frac{7}{2},\frac{14}{9},\frac{10}{3},4\right\},\left\{\frac{7}{2},\frac{14}{9},\frac{10}{3},\frac{4}{3}\right\},\left\{\frac{7}{2},\frac{8}{3},4,10\right\},\left\{\frac{7}{2},\frac{8}{3},8,10\right\}$
$\Phi_{3,4,12}$	$\left\{3,\frac{7}{4},\frac{9}{4},\frac{15}{4}\right\},\left\{\frac{7}{3},4,\frac{8}{3},\frac{14}{3}\right\},\left\{\frac{7}{3},4,\frac{10}{3},\frac{4}{3}\right\},\left\{\frac{9}{5},\frac{8}{5},\frac{12}{5},\frac{6}{5}\right\},\left\{\frac{9}{5},\frac{14}{5},\frac{12}{5},\frac{6}{5}\right\},$ $\left\{\frac{15}{7},\frac{10}{7},\frac{12}{7},\frac{18}{7}\right\},\left\{\frac{15}{7},\frac{16}{7},\frac{12}{7},\frac{18}{7}\right\},\{5,4,6,12\},\left\{\frac{15}{11},\frac{14}{11},\frac{18}{11},\frac{12}{11}\right\},\{5,10,6,12\},$ $\left\{\frac{15}{11},\frac{20}{11},\frac{18}{11},\frac{12}{11}\right\},\{9,4,6,12\},\{9,10,6,12\}$
$\Phi_{3,6,12}$	$\left\{\frac{3}{2},\frac{5}{4},\frac{15}{8},\frac{9}{8}\right\},\left\{\frac{3}{2},\frac{9}{4},\frac{15}{8},\frac{9}{8}\right\},\left\{3,\frac{3}{2},\frac{9}{4},\frac{15}{4}\right\},\left\{3,\frac{7}{2},\frac{9}{4},\frac{15}{4}\right\},\left\{3,6,\frac{9}{2},\frac{3}{2}\right\},$ $\left\{5,6,\frac{9}{2},\frac{3}{2}\right\},\left\{\frac{9}{5},\frac{7}{5},\frac{12}{5},\frac{6}{5}\right\},\left\{\frac{9}{5},3,\frac{12}{5},\frac{6}{5}\right\},\left\{\frac{15}{7},\frac{9}{7},\frac{12}{7},\frac{18}{7}\right\},\{5,3,6,12\},$ $\{9,3,6,12\}$
$\Phi_{4,6,12}$	$\left\{4,\frac{5}{3},\frac{14}{3},\frac{10}{3}\right\},\left\{4,\frac{5}{3},\frac{14}{3},\frac{8}{3}\right\},\left\{4,\frac{5}{3},\frac{8}{3},\frac{4}{3}\right\},\left\{4,\frac{5}{3},\frac{10}{3},\frac{4}{3}\right\},\left\{\frac{5}{2},6,\frac{7}{2},\frac{3}{2}\right\},\left\{\frac{5}{2},6,\frac{9}{2},\frac{3}{2}\right\},$

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C.P.	strongly regular principal 4-tuples
	$\{4,3,12,8\}, \left\{\frac{10}{7},\frac{9}{7},\frac{12}{7},\frac{8}{7}\right\}, \left\{\frac{14}{5},3,\frac{12}{5},\frac{6}{5}\right\}, \{10,3,12,8\}, \{10,3,12,6\},$ $\left\{\frac{8}{5},\frac{7}{5},\frac{12}{5},\frac{6}{5}\right\}, \left\{\frac{16}{7},\frac{9}{7},\frac{12}{7},\frac{8}{7}\right\}, \{4,3,12,6\}, \left\{\frac{8}{5},3,\frac{12}{5},\frac{6}{5}\right\}, \left\{\frac{14}{5},\frac{7}{5},\frac{12}{5},\frac{6}{5}\right\},$ $\left\{\frac{8}{5},\frac{7}{5},\frac{16}{5},\frac{12}{5}\right\}, \left\{\frac{14}{5},\frac{7}{5},\frac{16}{5},\frac{12}{5}\right\}, \left\{\frac{16}{7},\frac{9}{7},\frac{18}{7},\frac{12}{7}\right\}, \left\{\frac{14}{5},3,\frac{16}{5},\frac{12}{5}\right\}, \left\{\frac{10}{7},\frac{9}{7},\frac{18}{7},\frac{12}{7}\right\},$ $\left\{\frac{8}{5},3,\frac{16}{5},\frac{12}{5}\right\}$

Part V

Categories and their physical meaning

13 Mathematical background

The classification problem for $4d$ $\mathcal{N}=2$ SCFTs has been (partially) solved: we are now interested in exploring the physical content of a generic $4d$ $\mathcal{N}=2$ SCFT. In order to do this in general – or at least for a very large class of examples – we have to change the language: we shall use now category theory rather than special Kähler geometry. Eventually, in the concluding section, we shall reunite the two languages explicitly for some classes of examples (see section 18). A deeper connection between the two languages is still work in progress.
In this section we recall the basic definitions of DG categories [160], cluster categories [6], stability conditions for Abelian and triangulated categories [44] and show some concrete examples. We then specialize these definitions to the Ginzburg algebra [132] Γ associated to a BPS quiver with (super)potential (Q,W) [72].

Some readers may prefer to skip this section and refer back to it when looking for definitions and/or details on some mathematical tool used in the main body of this thesis.

13.1 Differential graded categories

The main reference for this section is [160]. Let k be a commutative ring,[85] for example a field or the ring of integers $\mathbb{Z}$. We will write $\otimes$ for the tensor product over k.

Definition 15. A *k-algebra* is a k-module A endowed with a k-linear associative multiplication $A \otimes_k A \to A$ admitting a two-sided unit $1 \in A$.

For example, a $\mathbb{Z}$-algebra is just a (possibly non-commutative) ring. A k-category $\mathcal{A}$ is a "k-algebra with several objects". Thus, it is the datum of a class of objects $\mathrm{obj}(\mathcal{A})$, of a k-module $\mathcal{A}(X,Y)$ for all objects X,Y of $\mathcal{A}$, and of k-linear associative composition maps

$$\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z), \qquad (f,g) \mapsto fg$$

admitting units $1_X \in \mathcal{A}(X,X)$. For example, we can interpret k-algebras as k-categories with only one object. The category $\mathsf{mod}\, A$ of finitely generated right A-modules over a k-algebra A is an example of a k-category with many objects. It is also an example of a k-linear category (i.e. a k-category which admits all finite direct sums).

Definition 16. A *graded k-module* is a k-module V together with a decomposition indexed by the positive and the negative integers:

$$V = \bigoplus_{p\in\mathbb{Z}} V^p.$$

The shifted module $V[1]$ is defined by $V[1]^p = V^{p+1}, p \in \mathbb{Z}$. A morphism $f : V \to V$ of graded k-modules of degree n is a k-linear morphism such that $f(V^p) \subset V^{p+n}$ for all $p \in \mathbb{Z}$.

Definition 17. The tensor product $V \otimes W$ of two graded k-modules V and W is the graded k-module with components

$$(V \otimes W)^n = \bigoplus_{p+q=n} V^p \otimes W^q, \ n \in \mathbb{Z}.$$

[85] In all our physical applications k will be the (algebraically closed) field of complex numbers $\mathbb{C}$.

The tensor product $f \otimes g$ of two maps $f : V \to V$ and $g : W \to W$ of graded k-modules is defined using the Koszul sign rule: we have

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(w)$$

if g is of degree p and v belongs to V^q.

Definition 18. A *graded k-algebra* is a graded k-module A endowed with a multiplication morphism $A \otimes A \to A$ which is graded of degree 0, associative and admits a unit $1 \in A^0$.

An "ordinary" k-algebra may be identified with a graded k-algebra concentrated in degree 0.

Definition 19. A *differential graded* (=DG) *k-module* is a $\mathbb{Z}$-graded k-module V endowed with a differential d_V, i.e. a map $d_V : V \to V$ of degree 1 such that $d_V^2 = 0$. Equivalently, V is a complex of k-modules. The shifted DG module $V[1]$ is the shifted graded module endowed with the differential $-d_V$.

The tensor product of two DG k-modules is the graded module $V \otimes W$ endowed with the differential $d_V \otimes 1_W + 1_V \otimes d_W$.

Definition 20. A *differential graded k-algebra A* is a DG k-module endowed with a multiplication morphism $A \otimes A \to A$ graded of degree 0 and associative. Moreover, the differential satisfies the graded Leibniz rule:

$$d(ab) = (da)b + (-1)^{\deg(a)} a(db), \quad \forall\, a, b \in A \text{ and } a \text{ homogeneous.}$$

The cohomology of a DG algebra is defined as $H^*(A) := \ker d / \operatorname{im} d$. Let $\mathsf{mod}\, A$ denote the category of finitely generated DG modules over the DG algebra A.

Definition 21. The *derived category* $D(A) := D(\mathsf{mod}\, A)$ is the localization of the category $\mathsf{mod}\, A$ with respect to the class of quasi-isomorphisms.

Thus, the objects of $D(A)$ are the DG modules and its morphisms are obtained from morphisms of DG modules by formally inverting all quasi-isomorphisms. The bounded derived category of $\mathsf{mod}\, A$, denoted $D^b A$, is the triangulated subcategory of $D(A)$ whose objects are quasi-isomorphic to objects with bounded cohomology.

Definition 22. The *perfect derived category* of a DG algebra A, $\mathfrak{Per}\, A$, is the smallest full triangulated subcategory of $D(A)$ containing A which is stable under taking shifts, extensions and direct summands.

13.2 Quivers and mutations

In this section we follow [165]. Let k be an algebraically closed field.

Definition 23. A (finite) *quiver* Q is a (finite) oriented graph (possibly with loops and 2-cycles). We denote its set of vertices by Q_0 and its set of arrows by Q_1. For an arrow a of Q, let $s(a)$ denote its source node and $t(a)$ denote its target node. The lazy path corresponding to a vertex i will be denoted by e_i.

Definition 24. The *path algebra* $k\hat{Q}$ is the associative unital algebra whose elements are finite compositions of arrows of Q, where the composition of $a, b \in Q_1$ is denoted ab and it is nonzero iff $s(b) = t(a)$. The complete path algebra kQ is the completion of the path algebra with respect to the ideal I generated by the arrows of Q.

Let I be the ideal of kQ generated by the arrows of Q. A potential W on Q is an element of the closure of the space generated by all non trivial cyclic paths of Q. We say two potentials are *cyclically equivalent* if their difference is in the closure of the space generated by all differences $a_1...a_s - a_2...a_s a_1$, where $a_1...a_s$ is a cycle.

Definition 25. Let u, p and v be nontrivial paths of Q such that $c = upv$ is a nontrivial cycle. For the path p of Q, we define

$$\partial_p : kQ \to kQ$$

as the unique continuous linear map which takes a cycle c to the sum $\sum_{c=upv} vu$ taken over all decompositions of the cycle c (where u and v are possibly lazy paths).

Obviously two cyclically equivalent potentials have the same image under ∂_p. If $p = a$ is an arrow of Q, we call ∂_a the cyclic derivative with respect to a. Let W be a potential on Q such that W is in I^2 and no two cyclically equivalent cyclic paths appear in the decomposition of W. Then the pair (Q, W) is called a *quiver with potential.*

Definition 26. Two quivers with potential (Q, W) and (Q', W') are *right-equivalent* if Q and Q' have the same set of vertices and there exists an algebra isomorphism $\phi : kQ \to kQ'$ whose restriction on vertices is the identity map and $\phi(W)$ and W' are cyclically equivalent. Such an isomorphism ϕ is called a right-equivalence.

Definition 27. The *Jacobian algebra* of a quiver with potential (Q, W), denoted by $J(Q, W)$, is the quotient of the complete path algebra kQ by the closure of the ideal generated by $\partial_a W$, where a runs over all arrows of Q:

$$J(Q, W) := kQ / \langle \partial_a W \rangle .$$

We say that the quiver with potential (Q, W) is *Jacobi-finite* if the Jacobian algebra $J(Q, W)$ is finite-dimensional over k.

It is clear that two right-equivalent quivers with potential have isomorphic Jacobian algebras. A quiver with potential is called *trivial* if its potential is a linear combination of cycles of length 2 and its Jacobian algebra is the product of copies of the base field k.

13.2.1 Quiver mutations

Let (Q, W) be a quiver with potential. Let $i \in Q_0$ a vertex. Assume the following conditions:

- the quiver Q has no loops;
- the quiver Q does not have 2-cycles at i;

We define a new quiver with potential $\tilde{\mu}_i(Q, W) = (Q', W')$ as follows. The new quiver Q' is obtained from Q by

1. For each arrow β with target i and each arrow α with source i, add a new arrow $[\alpha\beta]$ from the source of β to the target of α.

2. Replace each arrow α with source or target i with an arrow α^* in the opposite direction.

If we represent the quiver with its exchange matrix B_{ij}, i.e. the matrix such that

$$B_{ij} = \#\{\text{ arrows from } i \text{ to } j\} - \#\{\text{ arrows from } j \text{ to } i\} \tag{13.1}$$

then the transformation that B_{ij} undergoes is

$$B'_{ij} = \begin{cases} -B_{ij}, & i = k \text{ or } j = k \\ B_{ij} + \max[-B_{ik}, 0]\, B_{kj} + B_{ik} \max[B_{kj}, 0] & \text{otherwise.} \end{cases}$$

The new potential W' is the sum of two potentials W'_1 and W'_2. The potential W'_1 is obtained from W by replacing each composition $\alpha\beta$ by $[\alpha\beta]$, where β is an arrow with target i. The potential W'_2 is given by [94]

$$W'_2 = \sum_{\alpha,\beta} [\alpha\beta]\beta^*\alpha^*,$$

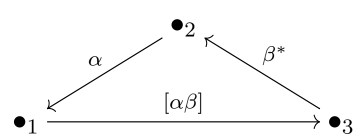
the sum ranging over all pairs of arrows α and β such that β ends at i and α starts at i.

Definition 28. Let I be the ideal in kQ generated by all arrows. Then, a quiver with potential is called *reduced* if $\partial_a W$ is contained in I^2 for all arrows a of Q.

One shows that all quivers with potential (Q, W) are right-equivalent to the direct sum of a reduced quiver with potential and a trivial one.[86]

We can now give the definition of the mutated quiver: we define $\mu_i(Q, W)$ as the reduced part of $\tilde{\mu}_i(Q, W)$, and call μ_i the mutation at the vertex i. An example will clarify all these concepts.

Example 42 (A_3 quiver)**.** Consider the quiver A_3 given by $Q: \bullet_1 \overset{\alpha}{\leftarrow} \bullet_2 \overset{\beta}{\rightarrow} \bullet_3$ with $W = 0$. Let us consider the quiver $\mu_1(Q): \bullet_1 \overset{\alpha^*}{\rightarrow} \bullet_2 \overset{\beta}{\rightarrow} \bullet_3$ with $W = 0$. Now apply the mutation at vertex 2: we get



$\mu_2(\mu_1(Q))$

with potential $W = \alpha\beta^*[\alpha\beta]$.

We conclude this subsection with the following

Theorem 7 ([165])**.**

1. *The right-equivalence class of $\tilde{\mu}_i(Q, W)$ is determined by the right-equivalence class of (Q, W).*

[86] In terms of the corresponding SQM system, the process of replacing the pair (Q, W) by its reduced part $(Q_{\text{red.}}, W_{\text{red.}})$ corresponds to integrate away the massive Higgs bifundamentals.

2. The quiver with potential $\tilde{\mu}_i^2(Q,W)$ is right-equivalent to the direct sum of (Q,W) with a trivial quiver with potential.

3. The correspondence μ_i acts as an involution on the right equivalence classes of reduced quivers with potential.

13.3 Cluster algebras

We follow [217]. Let Q be a 2-acyclic quiver with vertices $1,2,...,n$, and let $F=\Theta(x_1,...,x_n)$ be the function field in n indeterminates over Θ. Consider the pair $(\vec{x},Q)$, where $\vec{x}=\{x_1,...,x_n\}$. The cluster algebra $C(\vec{x},Q)$ will be defined to be a subring of F.

The pair $(\vec{x},Q)$ consisting of a transcendence basis $\vec{x}$ for F over the rational numbers Θ, together with a quiver with n vertices, is called a *seed*. For $i=1,...,n$ we define a mutation μ_i taking the seed $(\vec{x},Q)$ to a new seed $(\vec{x}',Q')$, where $Q'=\mu_i(Q)$ as discussed in 13.2, and $\vec{x}'$ is obtained from $\vec{x}$ by replacing x_i by a new element x_i' in F. Here x_i' is defined by

$$x_ix_i'=m_1+m_2,$$

where m_1 is a monomial in the variables $x_1,...,x_n$, where the power of x_j is the number of arrows from j to i in Q, and m_2 is the monomial where the power of x_j is the number of arrows from i to j. (If there is no arrow from j to i, then $m_1=1$, and if there is no arrow from i to j, then $m_2=1$.) Note that while in the new seed the quiver Q' only depends on the quiver Q, then x' depends on both x and Q. We have

$$\mu_i^2(\vec{x},Q)=(\vec{x},Q).$$

The procedure to get the full cluster algebra is iterative. We perform this mutation operation for all $i=1,...,n$, then we perform it on the new seeds and so on. Either we get new seeds or we get back one of the seeds already computed. The n-element subsets $\vec{x},\vec{x}',\vec{x}'',...$ occurring are by definition the *clusters*, the elements in the clusters are the *cluster variables*, and the *seeds* are all pairs $(\vec{x}',Q')$ occurring in the iterative procedure. The corresponding cluster algebra $C(\vec{x},Q)$, which as an algebra only depends on Q, is the subring of F generated by the cluster variables.

Example 43. Let Q be the quiver $1\to 2\to 3$ and $\vec{x}=\{x_1,x_2,x_3\}$, where x_1,x_2,x_3 are indeterminates, and $F=\Theta(x_1,x_2,x_3)$. We have $\mu_1(\vec{x},Q)=(x',Q')$, where $Q'=\mu_1(Q)$ is the quiver $1\leftarrow 2\to 3$ and $\vec{x}'=\{x_1',x_2,x_3\}$, where $x_1x_1'=1+x_2$, so that $x_1'=\frac{1+x_2}{x_1}$. And so on. The clusters are:

$$\{x_1,x_2,x_3\},\{\frac{1+x_2}{x_1},x_2,x_3\},\{x_1,\frac{x_1+x_3}{x_2},x_3\},\{x_1,x_2,\frac{1+x_2}{x_3}\},$$

$$\{\frac{1+x_2}{x_1},\frac{x_1+(1+x_2)x_3}{x_1x_2},x_3\},\{\frac{1+x_2}{x_1},x_2,\frac{1+x_2}{x_3}\},\{\frac{x_1+(1+x_2)x_3}{x_1x_2},\frac{x_1+x_3}{x_2},x_3\},$$

$$\{x_1,\frac{x_1+x_3}{x_2},\frac{(1+x_2)x_1+x_3}{x_2x_3}\},\{x_1,\frac{(1+x_2)x_1+x_3}{x_2x_3},\frac{1+x_2}{x_3}\},$$

$$\{\frac{1+x_2}{x_1},\frac{x_1+(1+x_2)x_3}{x_1x_2},\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3}\},\{\frac{1+x_2}{x_1},\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3},\frac{1+x_2}{x_3}\},$$

$$\{\frac{x_1+(1+x_2)x_3}{x_1x_2},\frac{x_1+x_3}{x_2},\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3}\},$$

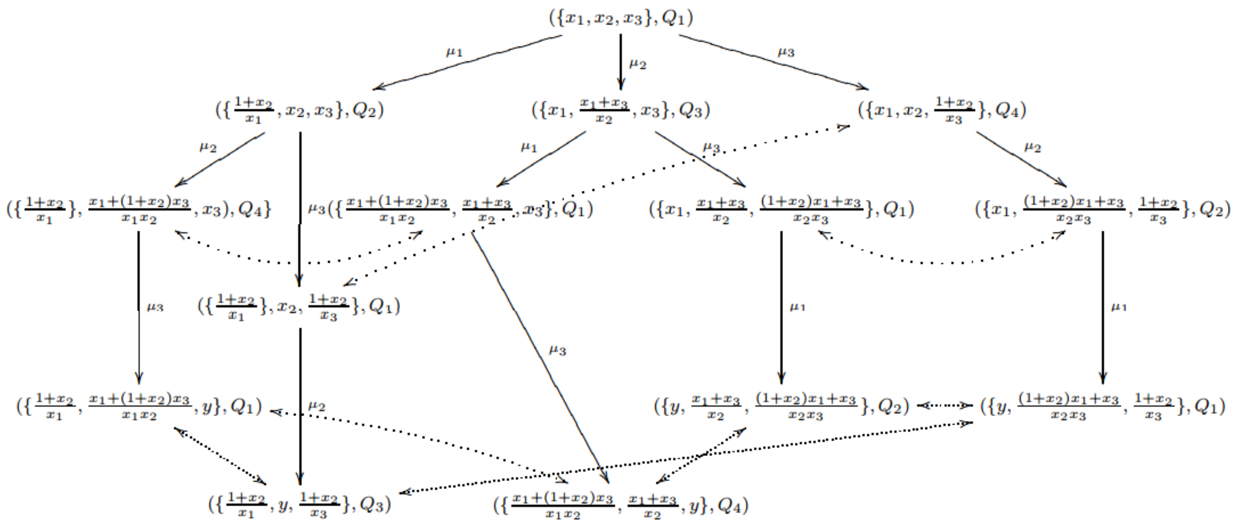


Figure 3: This figure represent the CEG of the A_3 cluster algebra. The dotted arrows are identifications up to permutations of the variables. The plain arrows represent mutations.

$$\{\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3}, \frac{x_1+x_3}{x_2}, \frac{(1+x_2)x_1+x_3}{x_2x_3}\},$$
$$\{\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3}, \frac{(1+x_2)x_1+x_3}{x_2x_3}, \frac{1+x_2}{x_3}\},$$

and the cluster variables are:

$$x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{1+x_2}{x_3}, \frac{x_1+(1+x_2)x_3}{x_1x_2}, \frac{(1+x_2)x_1+x_3}{x_2x_3}, \frac{(1+x_2)x_1+(1+x_2)x_3}{x_1x_2x_3}.$$

13.3.1 The cluster exchange graph (CEG)

If Q' is a quiver mutation equivalent to Q, then the cluster algebras $C(Q')$ and $C(Q)$ are isomorphic. The n-regular connected graph whose vertices are the seeds of $C(\vec{x}, Q)$ (up to simultaneous renumbering of rows, columns and variables) and whose edges connect the seeds related by a single mutation is called *cluster exchange graph* (=CEG). The CEG for **Example 43** is represented in figure 3.

13.4 Ginzburg DG algebras

Given a quiver Q with potential W, we can associate to it the Jacobian algebra $J(Q, W) := kQ/\langle \partial W \rangle$, where kQ is the quiver path algebra (see section 13.2). It is also possible to extend the path algebra kQ to a DG algebra: the Ginzburg algebra.

Definition 29. (Ginzburg [132]). Let (Q, W) be a quiver with potential. Let $\hat{Q}$ be the graded quiver with the same set of vertices as Q and whose arrows are:

- the arrows of Q (of degree 0);

- an arrow $a^* : j \to i$ of degree -1 for each arrow $a : i \to j$ of Q;
- a loop $t_i : i \to i$ of degree -2 for each vertex $i \in Q_0$.

The completed Ginzburg DG algebra $\Gamma(Q, W)$ is the DG algebra whose underlying graded algebra is the completion[87] of the graded path algebra $k\hat{Q}$. The differential of $\Gamma(Q, W)$ is the unique continuous linear endomorphism homogeneous of degree 1 which satisfies the Leibniz rule (i.e. $d(uv) = (du)v + (-1)^p\, udv$ for all homogeneous u of degree p and all v), and takes the following values on the arrows of $\hat{Q}$:

$$\begin{aligned} d(a) &= 0 \\ d(a^*) &= \partial_a W, & \forall a \in Q_1; \\ d(t_i) &= e_i \left(\sum_{a \in Q_1} [a, a^*] \right) e_i, & \forall i \in Q_0. \end{aligned}$$

We shall write $\Gamma(Q, W)$ simply as Γ, unless we wish to stress its dependence on (Q, W).

From the definition of Γ and d, one sees that $H^0\Gamma \cong J(Q, W)$.

To the DG algebra Γ we associate three important triangle categories which we are now going to define and analyze in detail.

13.5 The bounded and perfect derived categories

The DG-category $\mathsf{mod}\,\Gamma$ is the category whose objects are finitely generated graded Γ-modules and the morphisms spaces have the structure of DG modules (cfr. section 13.1). The derived category $D\Gamma := D(\mathsf{mod}\,\Gamma)$ [160, 161] is the localization of $\mathsf{mod}\,\Gamma$ at quasi-isomorphisms (the cohomology structure is given by the differential d of the Ginzburg algebra). Thus, the objects of $D\Gamma$ are DG modules. There are two fundamental subcategories associated to $D\Gamma$:

- The bounded derived category $D^b\Gamma$: it is the full subcategory of $D\Gamma$ such that its objects are graded modules M for which, given a certain $N > 0$, $H^n(M) = 0$ for all $|n| > N$. This category is 3-CY (see below).
- The perfect derived category $\mathfrak{Per}\,\Gamma$, i.e. the smallest full triangulated subcategory of $D\Gamma$ which contains Γ and is closed under extensions, shifts in degree and taking direct summands.

Both $\mathfrak{Per}\,\Gamma$ and $D^b\Gamma$ are triangulated subcategories of $D\Gamma$ and in particular, $\mathfrak{Per}\,\Gamma \supset D^b\Gamma$ as a full subcategory (as explained in [162]). Furthermore,[88] the category $D^b\Gamma$ has finite-dimensional morphism spaces (even its graded morphism spaces are of finite total dimension) and is 3-Calabi-Yau (3-CY), by which we mean that we have bifunctorial isomorphisms[89]

$$D\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X[3]), \tag{13.2}$$

[87] The completion is taken with respect to the I-adic topology, where I is the ideal of the path algebra generated by all arrows of the quiver.
[88] See [162] for more details.
[89] More generally, we say that a triangle category is ℓ-CY (for $\ell \in \mathbb{N}$) iff we have the bifunctorial isomorphism $D\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(Y, X[\ell])$.

where D is the duality functor $\mathrm{Hom}_k(-,k)$ and [1] the shift functor. The simple $J(Q,W)$-modules S_i can be viewed as Γ-modules via the canonical morphism

$$\Gamma \to H^0(\Gamma).$$

Example 44 (A_2 quiver)**.** Consider the A_2 quiver $1 \to 2$. The following is a graded indecomposable Γ-module:

$$t_1^* \circlearrowleft (k[-1] \oplus k[-3]) \underset{a^*}{\overset{a}{\rightleftharpoons}} k \circlearrowright t_2^*,$$

where $a = 0$, $a^* : k \overset{1}{\mapsto} k[-1]$, $t_2^* = 0$, and $t_1^* : k[-1] \overset{1}{\mapsto} k[-3]$. This object can be generated from $S_1[-1]$, $S_1[-3]$ and S_2 by successive extensions. Moreover, the modules $S_i, i = 1, 2$ and their shifts are enough to generate[90] all (homologically finite) graded modules.

13.5.1 Seidel-Thomas twists and braid group actions

Simple Γ-modules S_i become 3-spherical objects in $D^b\Gamma$ (hence also in $D\Gamma$), that is,

$$\mathrm{Hom}(S, S[j]) \cong k(\delta_{j,0} + \delta_{j,3}).$$

They yield the Seidel-Thomas [222, 229] twist functors T_{S_i}. These are autoequivalences of $D\Gamma$ such that each object X fits into a triangle

$$\mathrm{Hom}_D^\bullet(S_i, X) \otimes_k S_i \to X \to T_{S_i}(X) \to . \tag{13.3}$$

By construction, T_{S_i} restricts to an autoequivalence of the subcategory $D^b\Gamma \subset D\Gamma$. From the explicit realization of T_{S_i} as a cone in $D\Gamma$, eqn.(13.3), it is also clear that it restricts to an autoequivalence of $\mathfrak{Per}\,\Gamma$.

As shown in [229], the twist functors give rise to a (weak) action on $D\Gamma$ of the braid group associated with Q, i.e. the group with generators $\sigma_i, i \in Q_0$, and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

if i and j are not linked by an arrow in Q and

$$\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$$

if there is exactly one arrow between i and j (no relation if there are two or more arrows).

Definition 30. We write $\mathrm{Sph}(D^b\Gamma) \subset \mathrm{Aut}(D^b\Gamma)$ for the subgroup of autoequivalences generated by the Seidel-Thomas twists associated to all simple objects $S_i \in D^b\Gamma$.

13.5.2 The natural *t*-structure and the canonical heart

The category $D\Gamma$ admits a natural ***t***-structure whose truncation functors are those of the natural ***t***-structure on the category of complexes of vector spaces (because Γ is concentrated in degrees

[90] In the triangulated category $\mathcal{T}$, a set of objects $S_i \in \mathcal{T}$ is a generating set if all objects of $\mathcal{T}$ can be obtained from the generating set via an iterated cone construction.

≤ 0). Thus, we have an induced natural $\boldsymbol{t}$-structure on $D^b\Gamma$. Its heart $\mathcal{A}$ is canonically equivalent to the category $\mathsf{nil}\, J(Q,W)$ of nilpotent modules[91] [162]. In particular, the inclusion of $\mathcal{A}$ into $D^b\Gamma$ induces an isomorphism of Grothendieck groups

$$K_0(\mathcal{A}) \cong K_0(D^b\Gamma) \cong \bigoplus_i \mathbb{Z}[S_i].$$

The skew-symmetric form. Notice that the lattice $K_0(D^b\Gamma)$ carries the canonical Euler form defined by

$$\langle X, Y\rangle = \sum_{i=0}^{3} (-1)^i \dim \mathrm{Hom}_{D(\Gamma)}(X, Y[i]). \tag{13.4}$$

It is skew-symmetric thanks to the 3-Calabi-Yau property (13.2). Indeed it follows from the Calabi-Yau property and from the fact that $\mathrm{Ext}^i_{\mathcal{A}}(L,M) = \mathrm{Hom}_{D^b\Gamma}(L, M[i])$ for $i=0$ and $i=1$ (but not $i>1$ in general) that for two objects L and M of $\mathcal{A} \subset D^b\Gamma$, we have

$$\langle L, M\rangle = \dim \mathrm{Hom}(L,M) - \dim \mathrm{Ext}^1(L,M) + \dim \mathrm{Ext}^1(M,L) - \dim \mathrm{Hom}(M,L).$$

Since the dimension of $\mathrm{Ext}^1(S_i, S_j)$ equals the number of arrows in Q from j to i (Gabriel theorem [14]), we obtain that the matrix of $\langle -,-\rangle$ in the basis of the simples of $\mathcal{A}$ has its (i,j)-coefficient equal to the number of arrows from i to j minus the number of arrows from j to i in Q, that is, (cfr. eqn.(13.1))

$$\langle S_i, S_j\rangle = B_{ij}. \tag{13.5}$$

13.5.3 Mutations at category level

The main reference for this subsection is [162]. Let k be a vertex of the quiver Q not lying on a 2-cycle and let (Q',W') be the mutation of (Q,W) at k. Let Γ' be the Ginzburg algebra associated with (Q',W'). Let $\mathcal{A}'$ be the canonical heart in $D^b\Gamma'$. There are two canonical equivalences

$$D\Gamma' \to D\Gamma$$

given by functors $\Phi^{\pm}$ related by

$$T_{S_k} \circ \Phi^- \to \Phi^+.$$

where, again, T_{S_k} is the Seidel-Thomas twist generated by the spherical object S_k. If we put $P_i = \Gamma e_i, i \in Q_0$, and similarly for Γ', then both Φ^+ and Φ^- send P'_i to P_i for $i \neq k$; the images of P'_k under the two functors fit into triangles

$$P_k \to \bigoplus_{k\to i} P_i \to \Phi^-(P'_k) \to \tag{13.6}$$

and

$$\Phi^+(P'_k) \to \bigoplus_{j\to k} P_j \to P_k. \tag{13.7}$$

[91] If (Q,W) is Jacobi-finite (as in our applications), $\mathsf{nil}\, J(Q,W) \equiv \mathsf{mod}\, J(Q,W)$.

The functors $\Phi^{\pm}$ send $\mathcal{A}'$ onto the hearts $\mu_k^{\pm}(\mathcal{A})$ of two new $\boldsymbol{t}$-structures. These can be described in terms of $\mathcal{A}$ and the subcategory[92] $\mathsf{add}\, S_k$ as follows: Let $S_k^{\perp}$ be the right orthogonal subcategory of S_k in $\mathcal{A}$[93]. Then $\mu_k^{+}(\mathcal{A})$ is formed by the objects X of $D^b\Gamma$ such that the object $H^0(X)$ belongs to $S_k^{\perp}$, the object $H^1(X)$ belongs to $\mathsf{add}\, S_k$ and $H^p(X)$ vanishes for all $p \neq 0, 1$. Similarly, the subcategory $\mu_k^{-}(\mathcal{A})$ is formed by the objects X such that the object $H^0(X)$ belongs to the left orthogonal subcategory ${}^{\perp}S_k$, the object $H^{-1}(X)$ belongs to $\mathsf{add}\, S_k$ and $H^p(X)$ vanishes for all $p \neq -1, 0$. The subcategory $\mu_k^{+}(\mathcal{A})$ is the right mutation of $\mathcal{A}$ and $\mu_k^{-}(\mathcal{A})$ is its left mutation. By construction, we have

$$T_{S_k}(\mu_k^{-}(\mathcal{A})) = \mu_k^{+}(\mathcal{A}).$$

Since the categories $\mathcal{A}$ and $\mu^{\pm}(\mathcal{A})$ are hearts of bounded, non degenerate $\boldsymbol{t}$-structures on $D^b\Gamma$, their Grothendieck groups identify canonically with that of $D^b\Gamma$. They are endowed with canonical basis given by the simples. Those of $\mathcal{A}$ identify with the simples $S_i, i \in Q_0$, of $\mathsf{nil}\, J(Q, W)$. The simples of $\mu_k^{+}(\mathcal{A})$ are $S_k[-1]$, the simples S_i of $\mathcal{A}$ such that $\mathrm{Ext}^1(S_k, S_i)$ vanishes and the objects $T_{S_k}(S_i)$ where $\mathrm{Ext}^1(S_k, S_i)$ is of dimension ≥ 1. By applying $T_{S_k}^{-1}$ to these objects we obtain the simples of $\mu_k^{-}(\mathcal{A})$.

We saw that $D^b\Gamma \subset \mathfrak{Per}\,\Gamma$ as a full subcategory [165]: what is then the meaning of the Verdier quotient [202] of these two triangulated categories?

13.6 The cluster category

The next result is the main step in the construction of new 2-CY categories with cluster-tilting object which generalize the acyclic cluster categories introduced by Buan-Marsh-Reineke-Todorov to categorify the cluster agebras of Fomin and Zelevinski.

Theorem 8 (Thm 2.1 of [5])**.** *Let A be a DG-algebra with the following properties:*

1. *A is homologically smooth (i.e. $A \in \mathfrak{Per}(A \otimes A^{op})$),*
2. *$H^p(A) = 0$ for all $p \geq 1$,*
3. *$H^0(A)$ is finite dimensional as a k-vector space,*
4. *A is bimodule 3-CY, i.e.*

$$\mathrm{Hom}_{D(A)}(X, Y) \cong D\mathrm{Hom}_{D(A)}(Y, X[3]), \tag{13.8}$$

for any $X \in D(A)$ and $Y \in D^b(A)$.

Then the triangulated category

$$\mathcal{C}(A) = \mathfrak{Per}\, A / D^b A$$

is Hom-finite, 2-CY, i.e.

$$\mathrm{Hom}_{\mathcal{C}(A)}(X, Y) \cong D\mathrm{Hom}_{\mathcal{C}(A)}(Y, X[2]), \qquad X, Y \in \mathcal{C}(A).$$

[92] Here and below, given a (collection of) object(s) $\mathcal{O}$ of a linear category $\mathfrak{L}$, by $\mathsf{add}\,\mathcal{O}$ we mean the *additive closure of $\mathcal{O}$ in $\mathfrak{L}$,* that is, the full subcategory over the direct summands of finite direct sums of copies of $\mathcal{O}$.

[93] Its objects are those M's with $\mathrm{Hom}(S_k, M) = 0$. It is a full subcategory of $\mathcal{A}$.

and the object A is a cluster-tilting object[94] *with*

$$\mathrm{End}_{\mathcal{C}(A)}(A) \cong H^0(A). \tag{13.9}$$

The category $\mathcal{C}(A)$ is called the *generalized cluster category* and it reduces to the standard cluster category [159] in the acyclic case. It is triangulated since it is the Verdier quotient of triangulated categories.[95]

13.6.1 The case of the Ginzburg algebra of (Q, W)

In particular, we may specialize to the case where $A = \Gamma$, i.e. the Ginzburg algebra of a quiver with potential (Q, W), and write the following sequence:

$$0 \to D^b\Gamma \xrightarrow{\mathsf{s}} \mathfrak{Per}\,\Gamma \xrightarrow{\mathsf{r}} \mathcal{C}(\Gamma) \to 0 \tag{13.10}$$

the above theorem states that this sequence is exact and $\mathsf{r}(\Gamma) = T$, where T is the *canonical* cluster-tilting object[96] of $\mathcal{C}(\Gamma)$. The first map in eqn.(13.10) is the inclusion map: see [165] for details.

Remark 13.6.1. Moreover, an object $M \in \mathfrak{Per}\,\Gamma$ belongs to the subcategory $D^b\Gamma$ if and only if the space $\mathrm{Hom}_{\mathfrak{Per}\,\Gamma}(P, M)$ is finite-dimensional for each $P \in \mathfrak{Per}\,\Gamma$. In particular, this implies that there is a duality between the simple objects $S_i \in D^b\Gamma$ and the projective objects $\Gamma e_i \in \mathfrak{Per}\,\Gamma$

$$\langle \Gamma e_i, S_j \rangle = \delta_{ij}. \tag{13.11}$$

Theorem 9 (Keller [164])**.** *The completed Ginzburg DG algebra $\Gamma(Q, W)$ is homologically smooth and bimodule 3-Calabi-Yau.*

We have already shown that $\Gamma(Q, W)$ is non zero only in negative degrees, and that $H^0(\Gamma(Q, W)) \cong J(Q, W)$. Therefore by the theorem above we get the following

Corollary 13.6.1. *Let (Q, W) be a Jacobi-finite quiver with potential. Then the category*

$$\mathcal{C}(\Gamma(Q, W)) := \mathfrak{Per}\,\Gamma(Q, W)\big/D^b(\Gamma(Q, W))$$

is Hom-finite, 2-Calabi-Yau, and has a canonical cluster-tilting object[97] *whose endomorphism algebra is isomorphic to $J(Q, W)$.*

We shall write $\mathcal{C}(\Gamma(Q, W))$ simply as $\mathcal{C}(\Gamma)$ leaving (Q, W) implicit.

---

[94] See **Definition 31**.

[95] The main references for these categorical facts are [105, 202]. We recall the definition of Verdier quotient of triangle categories:

Lemma. *Let D be a triangulated category. Let $D' \subset D$ be a full triangulated subcategory. Let $S \subset \mathsf{Mor}(D)$ be the subset of morphisms such that there exists a distinguished triangle $(X, Y, Z, f, g, h) \in D$ with Z isomorphic to an object of D'. Then S is a multiplicative system compatible with the triangulated structure on D.*

Definition. Let D be a triangulated category. Let B be a full triangulated subcategory. We define the (Verdier) quotient category D/B by the formula $D/B = S^{-1}D$, where S is the multiplicative system of D associated to B via the previous lemma.

[96] See **Definition 31**.

[97] See **Definition 31**.

13.6.2 The cluster category of a hereditary category

The above structure simplifies in the case of a cluster category arising from a hereditary (Abelian) category $\mathcal{H}$ (with a Serre functor and a tilting object) [183]. Physically this happens for the following list of complete $\mathcal{N}=2$ QFTs [67]: *i)* Argyres-Douglas of type ADE, *ii)* asymptotically-free $SU(2)$ gauge theories coupled to fundamental quarks and/or Argyes-Douglas models of type D, and iii) SCFT $SU(2)$ theories with the same kind of matter. In terms of quiver mutations classes, they correspond (respectively) to ADE Dynkin quivers of the finite, affine, and elliptic type[98] [67]. In all these case we have an hereditary (Abelian) category $\mathcal{H}$, with the Serre functor $S=\tau[1]$ where τ is the Auslander-Reiten translation. That is, in their derived category we have

$$\mathrm{Hom}_{D^b(\mathcal{H})}(X,Y) \cong D\mathrm{Hom}_{D^b(\mathcal{H})}(Y,\tau X[1])$$

τ is an auto-equivalence of $D^b(\mathcal{H})$. The cluster category can be shown to be equivalent to the orbit category [32,159]

$$\mathcal{C}(\mathcal{H}) \cong D^b(\mathcal{H})/\langle\tau^{-1}[1]\rangle^{\mathbb{Z}}. \tag{13.12}$$

For future reference, we list the relevant categories $\mathcal{H}$ (further details may be found in [73]):

- For Argyres-Douglas of type ADE, we have $\mathcal{H}\cong \mathsf{mod}\, k\vec{\mathfrak{g}}$, where $\vec{\mathfrak{g}}$ is a quiver obtained by choosing an orientation to the Dynkin graph of type $\mathfrak{g}\in ADE$ (all orientations being derived-equivalent). τ satisfies the equation (for more refined results see [51,194])

  $$\tau^h = [-2], \tag{13.13}$$

  where h is the Coxeter number of the associated Lie algebra $\mathfrak{g}$;

- for $SU(2)$ gauge theories coupled to Argyres-Douglas systems of types[99] $D_{p_1},\cdots,D_{p_s}$, we have $\mathcal{H}=\mathsf{coh}\,\mathbb{X}(p_1,\dots,p_s)$, the coherent sheaves over a weighted projective line of weights $(p_1,\dots,p_s)$ [131, 183][100]. τ acts by multiplication by the canonical sheaf ω, and hence is periodic iff $\deg\omega=0$; in general, $\deg\omega$ is minus the Euler characteristic of $\mathbb{X}(p_1,\dots,p_s)$, $\chi = 2-\sum_i(1-1/p_i)$. However, τ is always periodic of period $\mathrm{lcm}(p_i)$ when restricted to the zero rank sheaves ('skyskrapers' sheaves).

13.7 Mutation invariance

We have already stated that mutations correspond to Seiberg-like dualities. Therefore, our categorical construction makes sense only if it is invariant by mutations: indeed, we do not want the categories representing the physics to change when we change the mathematical description of the same dynamics.

The following two results give a connection between the DG categories we just analyzed and quivers with potentials linked by mutations.

98 In the elliptic type we are restricted to the four types D_4, E_6, E_7 and E_8, corresponding to the four tubular weighted projective lines [131,172,183]. Elliptic D_4 is $SU(2)$ with $N_f=4$ [67].

99 In our conventions, $p_i=1$ means the empty matter system, while $p_i=2$ is a free quark doublet.

100 For a review of the category of coherent sheaves on weighted projective lines and corresponding cluster categories from a physicist prospective, see [75].

Theorem 10. *Let (Q, W) be a quiver with potential without loops and $i \in Q_0$ not on a 2-cycle in Q. Denote by $\Gamma := \Gamma(Q, W)$ and $\Gamma' := \Gamma(\mu_i(Q, W))$ the completed Ginzburg DG algebras.*

1. [164] *There are triangle equivalences*

$$\begin{array}{ccc} \mathfrak{Per}\,\Gamma & \xrightarrow{\sim} & \mathfrak{Per}\,\Gamma' \\ \uparrow & & \uparrow \\ D^b\Gamma & \xrightarrow{\sim} & D^b\Gamma' \end{array}$$

   Hence we have a triangle equivalence $\mathcal{C}(\Gamma) \cong \mathcal{C}(\Gamma')$.

2. [211] *We have a diagram*

$$\begin{array}{ccc} \mathfrak{Per}\,\Gamma & \xrightarrow{\qquad\sim\qquad} & \mathfrak{Per}\,\Gamma' \\ \downarrow{\scriptstyle H^0} & & \downarrow{\scriptstyle H^0} \\ \mathsf{mod}\,J(Q,W) & \overset{mutation}{\longleftrightarrow} & \mathsf{mod}\,J(\mu_i(Q,W)) \end{array}$$

Definition 31. Let $\mathcal{C}$ be a Hom-finite triangulated category. An object $T \in \mathcal{C}$ is called *cluster-tilting* (or *2-cluster-tilting*) if T is basic (i.e. with pairwise non-isomorphic direct summands) and if we have

$$\mathsf{add}\,T = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(X, T[1]) = 0\} = \{X \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}}(T, X[1]) = 0\}.$$

Note that a cluster-tilting object is maximal rigid (the converse is not always true, see [50]), and that the second equality in the definition always holds when $\mathcal{C}$ is 2-Calabi-Yau.

If there exists a cluster-tilting object in a 2-CY category $\mathcal{C}$, then it is possible to construct others by a recursive process resumed in the following:

Theorem 11 (Iyama-Yoshino [152])**.** *Let $\mathcal{C}$ be a Hom-finite 2-CY triangulated category with a cluster-tilting object T. Let T_i be an indecomposable direct summand of $T \cong T_i \oplus T_0$. Then there exists a unique indecomposable T_i^* non isomorphic to T_i such that $T_0 \oplus T_i^*$ is cluster-tilting. Moreover T_i and T_i^* are linked by the existence of triangles*

$$T_i \xrightarrow{u} B \xrightarrow{v} T_i^* \xrightarrow{w} T_i[1] \quad \text{and} \quad T_i^* \xrightarrow{u'} B' \xrightarrow{v'} T_i \xrightarrow{w'} T_i^*[1]$$

where u and u' are minimal left $\mathsf{add}\,T_0$-approximations and v and v' are minimal right $\mathsf{add}\,T_0$-approximations.

These triangles allow to make a mutation of the cluster-tilting object: they are called IY-mutations.

Proposition 13.7.1 (Keller-Reiten [163])**.** *Let $\mathcal{C}$ be a 2-CY triangulated category with a cluster-tilting object T. Then the functor*

$$F_T = \mathrm{Hom}_{\mathcal{C}}(T, -) \colon \mathcal{C} \to \mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T) \tag{13.14}$$

induces an equivalence

$$\mathcal{C}\big/\mathsf{add}\,T[1] \cong \mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T).$$

If the objects T and T' are linked by an IY-mutation, then the categories $\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T)$ *and* $\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T')$ *are* nearly Morita equivalent, *that is, there exists a simple* $\mathrm{End}_{\mathcal{C}}(T)$*-module S, and a simple* $\mathrm{End}_{\mathcal{C}}(T')$*-module S', and an equivalence of categories*

$$\mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T)\big/\mathsf{add}\,S \cong \mathsf{mod}\,\mathrm{End}_{\mathcal{C}}(T')\big/\mathsf{add}\,S'.$$

Moreover, if X has no direct summands in $\mathsf{add}\,T[1]$*, then $F_T X$ is* projective *(resp.* injective*) if and only if X lies in* $\mathsf{add}\,T$ *(resp. in* $\mathsf{add}\,T[2]$*).*

Thus, from **Theorem 8** and the above **Proposition**, we get that in the Jacobi-finite case, for any cluster-tilting object $T \in \mathcal{C}(\Gamma)$ which is IY-mutation equivalent to the canonical one, we have:

$$\begin{array}{ccc} & \mathcal{C}(\Gamma) & \\ {}^{F_{T'}}\swarrow & & \searrow^{F_T} \\ \mathsf{mod}\,\mathrm{End}_{\mathcal{C}(\Gamma)}(T) & \overset{\textit{mutation}}{\longleftrightarrow} & \mathsf{mod}\,\mathrm{End}_{\mathcal{C}(\Gamma)}(T') \end{array}$$

13.8 Grothendieck groups, skew-symmetric pairing, and the index

13.8.1 Motivations from physics

In a quantum theory there are two distinct notions of 'quantum numbers': the quantities which are conserved in all physical processes and, on the other hand, the numbers which are used to label (i.e. to distinguish) states and operators. If a class of BPS objects is described (in a certain physical set-up) by the triangle category $\mathfrak{T}$, these two notions of 'quantum numbers' get identified as follows:

- **conserved quantities:** numerical invariants of objects $X \in \mathfrak{T}$ which only depend on their Grothendieck class $[X] \in K_0(\mathfrak{T})$.[101] This is the free Abelian group over the isoclasses of objects of $\mathfrak{T}$ modulo the relations given by distinguished triangles of $\mathfrak{T}$;
- **labeling numbers:** correspond to numerical invariants of the objects $X \in \mathfrak{T}$ which are well-defined, that is, depend only on its isoclass (technically, on their class in the split-Grothendieck group).

Of course, conserved quantities are in particular labeling numbers. Depending on the category $\mathfrak{T}$ there may be or not be enough conserved quantities $K_0(\mathfrak{T})$ to label all the relevant BPS objects.

In the categorical approach to the BPS sector of a supersymmetric theory, the basic problem takes the form:

[101] In general, the conserved quantum *numbers* take value in the *numeric* Grothendieck group $K_0(\mathfrak{T})_{\mathrm{num}}$. For the categories we consider in this thesis, the Grothendieck group is a finitely generated Abelian group and the two groups coincide.

Problem 3. *Given a class of BPS objects A in a specified physical set-up, determine the corresponding triangulated category $\mathfrak{T}_A$.*

The Grothendieck group is a very handy tool to solve this **Problem**. Indeed, the BPS objects of A carry certain conserved quantum numbers which satisfy a number of physical consistency requirements. The allowed quantum numbers take value in an Abelian group Ab_A, and the consistency requirements endow the group with some extra mathematical structures. Both the group Ab_A and the extra structures on it are known from physical considerations (we shall review the ones of interest in §. 14). Then suppose we have a putative solution $\mathfrak{T}_A$ of the above problem. We compute its Grothendieck group; if $K_0(\mathfrak{T}_A) \ncong \mathsf{Ab}_A$, we can rule out $\mathfrak{T}_A$ as a solution of the above **Problem**. Even if $K_0(\mathfrak{T}_A) \cong \mathsf{Ab}_A$, but $K_0(\mathfrak{T}_A)$ is not naturally endowed with the required extra structures, we may rule out $\mathfrak{T}_A$. On the other hand, if we find that $K_0(\mathfrak{T}_A) \cong \mathsf{Ab}_A$ and the Grothendieck group is canonically equipped with the physically expected structures, we gain confidence on the proposed solution, especially if the requirements on $K_0(\mathfrak{T}_A)$ are quite restrictive.

Therefore, as a preparation for the discussion of their physical interpretation in section 4, we need to analyze in detail the Grothendieck groups of the three triangle categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, or $\mathcal{C}(\Gamma)$. These categories are related by the functors s, r which, being exact, induce group homomorphisms between the corresponding Grothendieck groups. In all three cases $K_0(\mathfrak{T})$ is a finitely generated Abelian group carrying additional structures; later in the thesis we shall compare this structures with the one required by quantum physics.

13.8.2 The lattice $K_0(D^b\Gamma)$ and the skew-symmetric form

The group $K_0(D^b\Gamma)$ is easy to compute using the following

Proposition 13.8.1 (Keller [162]). *The Abelian category $\mathsf{nil}\,J(Q,W)$ is the heart of a bounded $\boldsymbol{t}$-structure in $D^b\Gamma$.*

Hence, since we assume (Q,W) to be Jacobi-finite, $\mathsf{nil}\,J(Q,W) \cong \mathsf{mod}\,J(Q,W)$ and

$$K_0(D^b\Gamma) \simeq K_0(\mathsf{mod}\,J(Q,W))$$

is isomorphic to the free Abelian group over the isoclasses $[S_i]$ of the simple Jacobian modules S_i, that is, $K_0(D^b\Gamma) \cong \mathbb{Z}^n$ (n being the number of nodes of Q).

$D^b\Gamma$ is 3-CY, and then the lattice $K_0(D^b\Gamma)$ is equipped with an intrinsic skew-symmetric pairing given by the Euler characteristics, see discussion around eqn.(13.4). This pairing has an intepretation in terms of modules of the Jacobian algebra $B \equiv J(Q,W) \cong \mathrm{End}_{\mathcal{C}(\Gamma)}(\Gamma)$.

Proposition 13.8.2 (Palu [208]). *Let $X, Y \in \mathsf{mod}\,B$. Then the form*

$$\langle X, Y\rangle_a = \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) - \dim \mathrm{Hom}(Y,X) + \dim \mathrm{Ext}^1(Y,X)$$

descends to an antisymmetric form on $K_0(\mathsf{mod}\,B)$. Its matrix in the basis of simples $\{S_i\}$ is the exchange matrix B of the quiver Q (cfr. eqn.(13.5)).

In conclusion, for the 3-CY category $D^b\Gamma$, the Grothendieck group is a rank n lattice equipped with a skew-symmetric bilinear form $\langle -, -\rangle$. We shall refer to the radical of this form as the *flavor lattice* $\Lambda_{\text{flav}} = \mathrm{rad}\,\langle -, -\rangle$.

13.8.3 $K_0(\mathfrak{Per}\,\Gamma) \cong K_0(D^b\Gamma)^\vee$

More or less by definition, $K_0(\mathfrak{Per}\,\Gamma)$ is the free Abelian group over the classes $[\Gamma_i]$ of indecomposable summands of Γ. Since the general perfect object has infinite homology, there is no well-defined Euler bilinear form. However, eqn.(13.8) implies that for $X \in \mathfrak{Per}\,\Gamma$, $Y \in D^b\Gamma$,

$$\mathrm{Hom}_{\mathfrak{Per}}(X, Y[k]) = \mathrm{Hom}_{\mathfrak{Per}}(Y, X[k]) = 0 \quad \text{for } k < 0 \text{ or } k > 3$$

and hence we have a Euler pairing

$$K_0(\mathfrak{Per}\,\Gamma) \times K_0(D^b\Gamma) \to \mathbb{Z},$$

under which

$$\langle \Gamma_i, S_j \rangle = -\langle S_j, \Gamma_i \rangle = \delta_{ij}.$$

Thus $[S_i]$ and $[\Gamma_i]$ are dual basis and both Grothendieck groups are free (i.e. lattices) of rank n. Then we have two group isomorphisms

$$\mathbb{Z}^n \to K_0(D^b\Gamma) \qquad (m_1, m_2, \dots, m_n) \longmapsto \bigoplus_{i=1}^{n} m_i[S_i] \tag{13.15}$$

$$K_0(\mathfrak{Per}\,\Gamma) \to \mathbb{Z}^n \qquad [X] \longmapsto \Big(\langle X, S_1\rangle, \langle X, S_2\rangle, \dots, \langle X, S_n\rangle\Big). \tag{13.16}$$

The image of $K_0(D^b\Gamma)$ inside $K_0(\mathfrak{Per}\,\Gamma) \cong Z^n$ is isomorphic to the image of $B \colon \mathbb{Z}^n \to \mathbb{Z}^n$ where B is the exchange matrix of the quiver Q.[102] We have the obvious isomorphism

$$K_0(\mathfrak{Per}\,\Gamma) \cong K_0(\mathsf{add}\,\Gamma).$$

13.8.4 The structure of $K_0(\mathcal{C}(\Gamma))$

From the basic exact sequence of categories (13.10) we get

$$0 \longrightarrow K_0(D^b\Gamma) \xrightarrow{\ s\ } K_0(\mathfrak{Per}\,\Gamma) \xrightarrow{\ r\ } K_0(\mathcal{C}(\Gamma)) \longrightarrow 0$$

hence

$$K_0(\mathcal{C}(\Gamma)) \cong \mathbb{Z}^n / B \cdot \mathbb{Z}^n. \tag{13.17}$$

$K_0(\mathcal{C}(\Gamma))$ is not a free Abelian group (in general) but has a torsion part which we denote as tH (and call the *'t Hooft group*)

$$K_0(\mathcal{C}(\Gamma)) = K_0(\mathcal{C}(\Gamma))_{\text{free}} \oplus \mathsf{tH} \cong \mathbb{Z}^f \oplus \mathsf{A} \oplus \mathsf{A} \tag{13.18}$$

where $f = \mathrm{corank}\, B$ and A is the torsion group

$$\mathsf{A} = \bigoplus_s \mathbb{Z}/d_s\mathbb{Z}, \qquad d_s \mid d_{s+1}$$

[102] Note that this image is invariant under quiver mutation.

where the d_s are the positive integers in the normal form of B [31, 203]

$$B \xrightarrow{\text{normal form}} \overbrace{0 \oplus 0 \oplus \cdots \oplus 0}^{f \text{ summands}} \oplus \begin{bmatrix} 0 & d_1 \\ -d_1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & d_2 \\ -d_2 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & d_\ell \\ -d_\ell & 0 \end{bmatrix} \tag{13.19}$$

13.8.5 The index of a cluster object

Since the rank of the Abelian group $K_0(\mathcal{C}(\Gamma))$ is (in general) smaller than n, the Grothendieck class is not sufficient to label different objects (modulo deformation). We need to introduce other 'labeling quantum numbers' which do the job. This corresponds to the math concept of *index* (or, dually, *coindex*).

Lemma 13.8.1 (Keller-Reiten [163])**.** *For each object $L \in \mathcal{C}(\Gamma)$ there is a triangle*

$$T_1 \to T_0 \to L \to \qquad \textit{with } T_1, T_0 \in \mathsf{add}\,\Gamma.$$

The difference

$$[T_0] - [T_1] \in K_0(\mathsf{add}\,\Gamma)$$

does not depend on the choice of this triangle.

Definition 32. The quantity

$$\mathsf{ind}(L) \equiv [T_0] - [T_1] \in K_0(\mathsf{add}\,\Gamma) \equiv K_0(\mathsf{proj}\,J(Q,W)) \cong K_0(\mathfrak{Per}\,\Gamma) \cong \Lambda^\vee$$

is called the *index* of the object $L \in \mathcal{C}(\Gamma)$.

It is clear from the **Lemma** that the class $[L] \in K_0(\mathcal{C}(\Gamma))$ is the image of $\mathsf{ind}(L)$ under the projection

$$\Lambda^\vee \to \Lambda^\vee / B \cdot \Lambda.$$

As always, we use the canonical cluster-tilting object Γ; the modules $F_\Gamma \Gamma_i \in \mathsf{mod}\,J(Q,W)$; are the indecomposable projective modules (cfr. **Proposition 13.7.1**). We write $S_i \equiv \mathsf{Top}\,F_\Gamma\Gamma_i \in \mathsf{mod}\,J(Q,W)$ for the simple with support at the i-th node.

Lemma 13.8.2 (Palu [207])**.** *Let $X \in \mathcal{C}(\Gamma)$ be indecomposable. Then*

$$\mathsf{ind}\,X = \begin{cases} -[\Gamma_i] & X \cong \Gamma_i[1] \\ \sum_{i=1}^n \langle F_\Gamma X, S_i \rangle\, [\Gamma_i] & \textit{otherwise,} \end{cases}$$

where $\langle -, - \rangle$ is the Euler form in $\mathsf{mod}\,J(Q,W)$.

Remark 13.8.1. The dual notion to the index is the *coindex* [207]. For $X \in \mathcal{C}(\Gamma)$ one has

$$\mathsf{ind}\,X = -\mathsf{coind}\,X[1] \tag{13.20}$$

$$\mathsf{coind}\,X - \mathsf{ind}\,X = \sum_{i=1}^n \langle S_i, F_\Gamma X \rangle_{\mathrm{a}}\, [\Gamma_i] \tag{13.21}$$

$$\mathsf{coind}\,X - \mathsf{ind}\,X \quad \text{depends only on } F_\Gamma X \in \mathsf{mod}\,J(Q,W). \tag{13.22}$$

From (13.21) it is clear that the projections in $K_0(\mathcal{C}(\Gamma))$ of the index and coindex agree.

The precise mathematical statement corresponding to the rough idea that the 'index yields enough quantum numbers to distinguish operator' is the following

Theorem 12 (Dehy-Keller [86]). *Two rigid objects of $\mathcal{C}(\Gamma)$ are isomorphic if and only if their indices are equal.*

Remark 13.8.2. We shall show in §.14.2 how this is related to UV completeness of the corresponding QFT.

13.9 Periodic subcategories, the normalized Euler and Tits forms

We have seen that the group $K_0(D^b\Gamma)$ has an extra structure namely a skew-symmetric pairing. It is natural to look for additional structures on the group $K_0(\mathcal{C}(\Gamma))$. The argument around (13.4) implies that the Euler form of the 2-CY category $\mathcal{C}(\Gamma)$ *if defined* is symmetric:

$$\begin{aligned}\langle X, Y\rangle_{\mathcal{C}(\Gamma)} &\equiv \sum_{k\in\mathbb{Z}}(-1)^k \dim \mathrm{Hom}_{\mathcal{C}(\Gamma)}(X, Y[k]) = \\ &= \sum_{k\in\mathbb{Z}}(-1)^{2-k} \dim \mathrm{Hom}_{\mathcal{C}(\Gamma)}(Y, X[2-k]) = \langle Y, X\rangle_{\mathcal{C}(\Gamma)}.\end{aligned}$$

However the sum in the RHS is typically not defined, since it is *not true* (in general) that $\mathrm{Hom}_{\mathcal{C}(\Gamma)}(X, Y[k]) = 0$ for $k \ll 0$. In order to remediate this, we introduce an alternative concept.

Definition 33. We say that a full subcategory $\mathcal{F}(p) \subset \mathcal{C}(\Gamma)$, closed under shifts, direct sums and summands, is *p-periodic* ($p \in \mathbb{N}$) iff the functor $[p]$ restricts to an equivalence in $\mathcal{F}(p)$, and $\mathcal{F}(p)$ is *maximal* with respect to these properties. Note that we do not require p to be the minimal period.

Lemma 13.9.1. *A p-periodic sub-category, $\mathcal{F}(p) \subset \mathcal{C}(\Gamma)$, is triangulated and 2-CY*[103] *and the inclusion functor $\mathcal{F}(p) \xrightarrow{\mathsf{p}} \mathcal{C}(\Gamma)$ is exact.*

Proof. Since $\mathcal{F}(p)$ is closed under shifts, direct sums, and summands in $\mathcal{C}(\Gamma)$, it suffices to verify that $X, Y \in \mathcal{F}(p)$ implies $Z \in \mathcal{F}(p)$ for all triangles $X \to Y \to Z \to$ in $\mathcal{C}(\Gamma)$. Applying $[p]$ to the triangle, one gets $Z[p] \simeq Z$. □

Definition 34. Let $\mathcal{F}(p) \subset \mathcal{C}(\Gamma)$ be p-periodic. We define the *normalized Euler form as*

$$\langle\!\langle X, Y\rangle\!\rangle = \langle\!\langle Y, X\rangle\!\rangle = \frac{1}{p}\sum_{k=0}^{p-1}(-1)^k \dim \mathrm{Hom}_{\mathcal{C}(\Gamma)}(X, Y[k]), \qquad X, Y \in \mathcal{F}(p). \tag{13.23}$$

Note that it is independent of the chosen p as long as $Y[p] \cong Y$.

Remark 13.9.1. If p is odd, $\langle\!\langle -, -\rangle\!\rangle \equiv 0$.

[103] $\mathcal{F}(p)$ is linear, Hom-finite, and 2-CY. However, it is not necessarily a generalized cluster category since it may or may not have a tilting object. The prime examples of such a category without a tilting object are the *cluster tubes*, see [31, 32]. Sometimes the term 'cluster categories' is extended also to such categories.

Proposition 13.9.1. *The normalized Euler form $\langle\langle -, - \rangle\rangle$ induces a symmetric form on the group*

$$K_0(\mathcal{F}(p))/K_0(\mathcal{F}(p))_{torsion},$$

which we call the Tits form *of $\mathcal{F}(p)$.*

Remark 13.9.2. We shall see in §. 15.3.3 the physical meaning of the periodic sub-categories and their Tits form.

13.9.1 Example: cluster category of the projective line of weights (2,2,2,2)

As an example of Tits form in the sense of the above **Proposition**, we consider the cluster category (see §.13.6.2)

$$\mathcal{C} = D^b(\mathcal{H})/\langle \tau^{-1}[1] \rangle^{\mathbb{Z}}, \qquad \text{where } \mathcal{H} = \mathsf{coh}\,\mathbb{X}(2,2,2,2)$$

which corresponds to $SU(2)$ SQCD with $N_f = 4$ [73, 75]. We may think of this cluster category as having the same objects as $\mathsf{coh}\,\mathbb{X}(2,2,2,2)$ and extra arrows [32]. In this case $\deg \omega = 0$, and hence the category $\mathcal{C}$ is triangulated and periodic of period $p = 2$ in the sense of **Definition 33**, so $\mathcal{F}(2)$ is the full cluster category $\mathcal{C}$. We write $\mathcal{O}$ for the structure sheaf and $\mathcal{S}_{i,0}$ for the unique simple sheaf with support at the i-th special point such that $\mathrm{Hom}_{\mathsf{coh}\,\mathbb{X}}(\mathcal{O}, \mathcal{S}_{i,0}) \cong k$. The cluster Grothedieck group $K_0(\mathcal{C})$ is generated by $[\mathcal{O}]$ and $[\mathcal{S}_{i,0}]$ ($i = 1,2,3,4$) subjected to the relation [31]

$$2[\mathcal{O}] = \sum_{i=1}^{4} [\mathcal{S}_{i,0}]. \tag{13.24}$$

Thus we may identify

$$K_0(\mathcal{C}) \cong \left\{ (w_1, w_2, w_3, w_4) \in \left(\tfrac{1}{2}\mathbb{Z}\right)^2 \;\middle|\; w_i = w_j \mod 1 \right\} \equiv \Gamma_{\text{weight},\,\mathfrak{spin}(8)}.$$

by writing a class as $\sum_i w_i [\mathcal{S}_{i,0}]$. The Tits pairing is

$$\langle\langle [\mathcal{S}_{i,0}], [\mathcal{S}_{j,0}] \rangle\rangle = \delta_{i,j},$$

Then $K_0(\mathcal{C})$ equipped with this pairing is isomorphic to the $\mathfrak{spin}(8)$ weight lattice equipped with its standard inner product (valued in $\frac{1}{2}\mathbb{Z}$) dual to the even one given on the root lattice by the Cartan matrix. We remark that a class in $K_0(\mathcal{C})$ is a *spinorial* $\mathfrak{spin}(8)$ weight iff it is of the form $k[\mathcal{O}] + \sum_i m_i [\mathcal{S}_{i,0}]$ ($m_i \in \mathbb{Z}$) with k *odd.* The physical meaning of this statement and eqn.(13.24) will be clear in §. 15.3.4.

13.10 Stability conditions for Abelian and triangulated categories

We start with the Abelian category case, since it all boils down to it. The main reference for this part is [44]. Let $\mathcal{A}$ be an Abelian category and $K_0(\mathcal{A})$ its Grothendieck group.

Definition 35. A *Bridgeland stability condition* on an Abelian category $\mathcal{A}$ is a group homomorphism

$$Z : K_0(\mathcal{A}) \to \mathbb{C},$$

satisfying certain properties:[104]

1. $Z(\mathcal{A}) \subset \overline{\mathbb{H}} \setminus \mathbb{R}_{>0}$, the closed upper half plane minus the positive reals;

2. If $Z(E) = 0$, then $E = 0$. This allows to define the map

$$\arg Z(-) \colon K_0(\mathcal{A}) \setminus \{0\} \to (0, \pi];$$

3. The Harder-Narasimhan (HN) property. Every object $E \in \mathcal{A}$ admits a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E,$$

   such that, for each i:

   - E_{i+1}/E_i is Z-semistable;[105]
   - $\arg Z(E_{i+1}/E_i) > \arg Z(E_{i+2}/E_{i+1})$.

We also have the following

Definition 36. An object $E \in \mathcal{A}$ is called Z-*stable* if for all nonzero proper subobjects $E_0 \subset E$,

$$\arg Z(E_0) < \arg Z(E).$$

If $\leq$ replaces $<$, then we get the definition of Z-semistable.

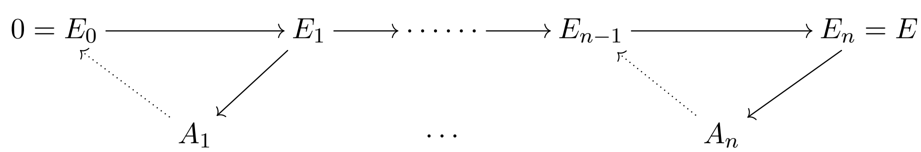
We are now going to give the corresponding definitions for the triangulated categories. The definition is more involved since there is no concept of subobject.

Definition 37. A *slicing* $\mathcal{P}$ of a triangulated category Δ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying

1. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;

2. For all $\phi_1 > \phi_2$ we have $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$;

3. For each $0 \neq E \in \Delta$ there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers and a sequence of exact triangles



   with $A_i \in \mathcal{P}(\phi_i)$ (which we call the Harder-Narasimhan filtration of E).

Remark 13.10.1. We call the objects in $\mathcal{P}(\phi)$ semistable of phase ϕ.

And finally, the definition of stability conditions in a triangulated category.

[104] If $[X] \in K_0(\mathcal{A})$ is the class of $X \in \mathcal{A}$, we write simply $Z(X)$ for $Z([X])$.
[105] See below **Definition 36** of semistability of objects in an abelian category.

Definition 38. A stability condition on a triangulated category Δ is a pair $(Z, \mathcal{P})$ where $Z : K_0(\Delta) \to \mathbb{C}$ is a group homomorphism (called *central charge*) and $\mathcal{P}$ is a slicing, so that for every $0 \neq E \in \mathcal{P}(\phi)$ we have

$$Z(E) = m(E)\, e^{i\pi\phi}$$

for some $m(E) \in \mathbb{R} > 0$.

Indeed, the following proposition shows that to some extent (once we identify a $\boldsymbol{t}$-structure), stability is intrinsically defined. It also describes how stability conditions are actually constructed:

Proposition 13.10.1 ([44]). *Giving a stability condition $(Z, \mathcal{P})$ on a triangulated category Δ is equivalent to giving a heart $\mathcal{A}$ of a bounded $\boldsymbol{t}$-structure with a stability function $Z_{\mathcal{A}} : K_0(\mathcal{A}) \to \mathbb{C}$ such that $(Z_{\mathcal{A}}, \mathcal{A})$ have the Harder-Narasimhan property, i.e. any object in $\mathcal{A}$ has a HN-filtration by $Z_{\mathcal{A}}$-stable objects.*

We will focus on how to obtain a stability condition from the datum $(Z_{\mathcal{A}}, \mathcal{A})$, as this is how stability conditions are actually constructed:

Proof. If $\mathcal{A}$ is the heart of a bounded $\boldsymbol{t}$-structure on Δ, then we have $K_0(\Delta) = K_0(\mathcal{A})$, so clearly Z and $Z_{\mathcal{A}}$ determine each other. Given $(Z_{\mathcal{A}}, \mathcal{A})$, we define $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$ to be the $Z_{\mathcal{A}}$-semistable objects in $\mathcal{A}$ of phase $\phi(E) = \phi$. This is extended to all real numbers by $\mathcal{P}(\phi+n) = \mathcal{P}(\phi)[n] \subset \mathcal{A}[n]$ for $\phi \in (0, 1]$ and $0 \neq n \in \mathbb{Z}$. The compatibility condition

$$\frac{1}{\pi} \arg Z(E) = \phi$$

is satisfied by construction, so we just need show that $\mathcal{P}$ satisfies the remaining properties in our definition of slicing. The Hom-vanishing condition in definition 37 follows from the definition of heart of a bounded $\boldsymbol{t}$-structure. Finally, given $E \in \Delta$, its filtration by cohomology objects $A_i \in \mathcal{A}[k_i]$, and the HN-filtrations $0 \to A_{i1} \to A_{i2} \to \cdots \to A_{im_i} = A_i$ given by the HN-property inside $\mathcal{A}$ can be combined into a HN-filtration of E: it begins with

$$0 \to F_1 = A_{11}[k_1] \to F_2 = A_{12}[k_1] \to \cdots \to F_{m_1} = A_1[k_1] = E_1,$$

i.e. with the HN-filtration of A_1. Then the following filtration steps $F_{m_{1+i}}$ are an extensions of $A_{2i}[k_2]$ by E_1 that can be constructed as the cone of the composition $A_{2i}[k_2] \to A_2[k_2] \to^{[1]} E_1$ (the octahedral axiom shows that these have the same filtration quotients as $0 \to A_{21}[k_2] \to A_{22}[k_2] \cdots$); continuing this way we obtain a filtration of E as desired. Conversely, given the stability condition, we set $\mathcal{A} = \mathcal{P}((0, 1])$ as before; by the compatibility condition, the central charge $Z(E)$ of any $\mathcal{P}$-semistable object E lies in $\overline{\mathbb{H}} \setminus \mathbb{R}_{>0}$; since any object in $\mathcal{A}$ is an extension of semistable ones, this follows for all objects in $\mathcal{A}$ by the additivity. Finally, it is fairly straightforward to show that Z-semistable objects in $\mathcal{A}$ are exactly the semistable objects with respect to $\mathcal{P}$. □

14 Some physical preliminaries

In the next section we shall relate the various triangle categories introduced in the previous section to the BPS objects of a $4d$ $\mathcal{N} = 2$ QFT as described from two different points of view: *i)* the microscopic UV description (i.e. in terms of a UV complete Lagrangian description or a UV fixed-point SCFT), and *ii)* the effective Seiberg-Witten IR description. Before doing that, we discuss some general properties of these physical systems. As discussed in §. 13.8.1, the categories $\mathfrak{T}_A$ which describe the BPS objects should enjoy the categorical versions of these physical properties in order to be valid solutions to the **Problem** in §. 13.8.1.

14.1 IR viewpoint

14.1.1 IR conserved charges

The Seiberg-Witten theory [228] describes, in a quantum exact way, the low-energy physics of our $4d$ $\mathcal{N}=2$ model in a given vacuum u along its Coulomb branch. Assuming u and the mass deformations to be generic, the effective theory is an Abelian gauge theory $U(1)^r$ coupled to states carrying both electric and magnetic charges. The flavor group is also Abelian $U(1)^f$, so that the IR conserved charges consist of r electric, r magnetic, and f flavor charges. In a non-trivial theory the gauge group is compact, and the flavor group is always compact, so these charges are quantized. Then the conserved charges take value in a lattice Λ (a free Abelian group) of rank

$$n = 2r + f.$$

The lattice Λ is equipped with an extra structure, namely a skew-symmetric quadratic form

$$\langle -,-\rangle \colon \Lambda\times\Lambda\to\mathbb{Z},$$

given by the Dirac electro-magnetic pairing. The radical of this form,

$$\Lambda_{\text{flav}} = \text{rad}\,\langle -,-\rangle \equiv \Big\{\lambda\in\Lambda\;\Big|\;\langle\mu,\lambda\rangle=0\;\;\forall\,\mu\in\Lambda\Big\}\subset\Lambda,$$

is the lattice of flavor charges and has rank f. The effective theory has another bosonic complex-valued conserved charge, namely the central charge of the $4d$ $\mathcal{N}=2$ superalgebra $Z:=\epsilon^{\alpha\beta}\epsilon_{AB}\{Q^A_\alpha,Q^B_\beta\}$. Z is not an independent charge but a linear combination of the charges in Λ with complex coefficients which depend on all IR data, and in particular on the vacuum u. Hence, for a given u, the susy central charge is a linear map (group homomorphism)

$$Z_u\colon \Lambda\to\mathbb{C}.$$

Any given state of charge $\lambda\in\Lambda$ has mass greater than or equal to $|Z_u(\lambda)|$. BPS states are the ones which saturate this bound.

In the case of a $4d$ $\mathcal{N}=2$ with a UV Lagrangian formulation, r and f are the ranks of the (non-Abelian) gauge G and flavor F groups, respectively. At extreme weak coupling, the IR electric and flavor charges are the weights under the respective maximal tori.

14.1.2 The IR landscape vs. the swampland

The IR $\mathcal{N}=2$ theories we consider are not generic Abelian gauge theories with electric and magnetic charged matter. They belong to the *landscape* (as opposed to the *swampland*), that is, they have a well defined UV completion. Such theories have special properties.

One property which seems to be true in the landscape, is that there are "enough" conserved IR charges to label all BPS states, so we don't need extra quantum numbers to distinguish the BPS objects in the IR description. This condition is certainly not sufficient to distinguish the landscape from the swampland, but it plays a special role in our discussion.

To support the suggestion that being UV complete is related to Λ being large enough to label IR objects, we mention a simple fact.

Fact. *Let the UV theory consists of a $\mathcal{N}=2$ gauge theory with semi-simple gauge group G and quark half-hypermultiplets in the (reducible) quaternionic representation $\boldsymbol{H}$. Assume that the beta-functions of all simple factor of G are non-positive. In the IR theory along the Coulomb branch, consider the BPS hypermultiplets h_i with zero magnetic charge and write $[h_i]$ for their IR charges in Λ. Then*

$$[h_i]=[h_j] \text{ and } h_i \neq h_j \quad \Rightarrow \quad [h_i] \in \Lambda_{\mathit{flav}}.$$

That is, the charges in Λ are enough to distinguish (zero magnetic charge) hypermultiplets unless they carry only flavor charge (i.e. are electrically neutral).

Remark 14.1.1. The hypermultiplets with purely flavor charge (called "everywhere light" since their mass is independent of the Coulomb branch parameters) just decouple in the IR, so in a sense they are no part of the IR picture.

To show the above fact, just list for all possible gauge group all representations compatible with non-positivity of the beta-function. Check, using Weyl formula, that the multiplicities of all weights for these representations is 1 except for the zero weight.

14.2 UV line operators and the 't Hooft group

14.2.1 't Hooft theory of quantum phases of gauge theories

We start by recalling the classical arguments by 't Hooft on the quantum phases of a $4d$ gauge theory [144–147]. The basic order operator in a gauge theory is the Wilson line associated to a (real) curve C in space time and a representation $\boldsymbol{R}$ of the gauge group G,

$$W_{\boldsymbol{R}}(C)=\mathrm{tr}_{\boldsymbol{R}}\, e^{-\int_C A}. \tag{14.1}$$

Here C is either a closed loop or is stretched out to infinity. In the second case we don't take the trace and hence the operator depends on a choice of a weight w of the representation $\boldsymbol{R}$ modulo che action of Weyl group. In the $\mathcal{N}=2$ case, the Wilson line (14.1) is replaced by its half-BPS counterpart [127] which, to preserve half supersymmetries should be stretched along a straight line L; we still denote this operator as $W_w(L)$.[106]

What are the quantum numbers carried by $W_w(L)$? This class of UV line operators is *labelled* by (the Weyl orbit of) the weight w, so gauge weights are useful quantum numbers. However, these numbers do not correspond to conserved quantities in a general gauge theory. For instance, consider pure (super-)Yang-Mills theory and let $\boldsymbol{R}$ be the adjoint representation. Since an adjoint Wilson

[106] The half-BPS lines are also parametrized by an angle ϑ which specifies which susy subalgebra leaves them invariant. We suppress ϑ from the notation.

line may terminate at the location of a colored particle transforming in the adjoint representation, a gluon (gluino) particle-antiparticle pair may be dynamically created out of the vacuum, breaking the line, see figure 4.

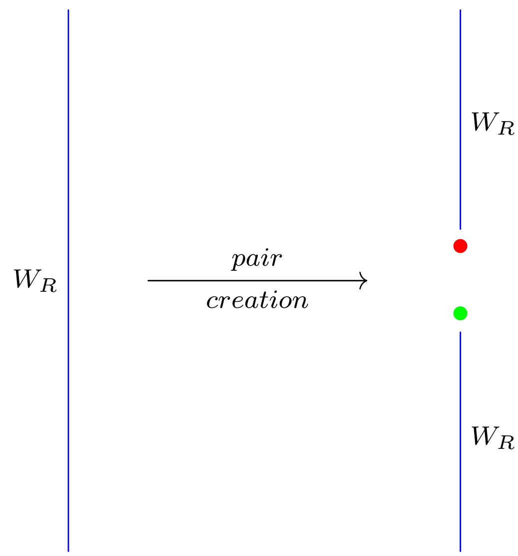


Figure 4: LEFT: an electric flux tube line created by an adjoint Wilson line. RIGHT: the adjoint flux line is broken by the creation of a gluon-antigluon pair out of the vacuum.

If our gauge theory is in the confined phase, breaking the line L is energetically favorable, so the line label w does not correspond to a conserved quantity. On the contrary, a Wilson line in the *fundamental* representation cannot break in pure $SU(N)$ (S)YM, since there is no dynamical particle which can be created out of the vacuum where it can terminate. The obstruction to breaking the line is the center $\boldsymbol{Z}(SU(N)) \cong \mathbb{Z}_N$ of the gauge group under which all *local* degrees of freedom are inert while the fundamental Wilson line is charged. Stated differently, the gluons may screen all color degrees of freedom of a physical state but the center of the gauge group. The conclusion is that the conserved quantum numbers of the line operators $W_{\boldsymbol{R}}(L)$ consist of the representation $\boldsymbol{R}$ seen as a representation of the center of the gauge group, $\boldsymbol{Z}(G)$, which take value in the dual group $\boldsymbol{Z}(G)^\vee \cong \boldsymbol{Z}(G)$. On the other hand, in $SU(N)$ (S)QCD we have quarks transforming in the fundamental representation; hence a quark-antiquark pair may be created to break a fundamental Wilson line. Then, in presence of fundamental matter, Wilson lines do not carry any conserved quantum number. In general, the conserved quantum numbers of the Wilson lines of a gauge theory with gauge group G take value in the finite Abelian group $\pi_1(G_{\text{eff}})^\vee \cong \pi_1(G_{\text{eff}})$, where G_{eff} is the quotient group of G which acts *effectively* on the microscopic UV degrees of freedom.

For clarity of presentation, the above discussion was in the confined phase. This is not the case of the $\mathcal{N}=2$ theory which we assume to be realized in its Coulomb phase. In the physically realized phase the line $W_w(L)$ may be stable; then its labeling quantum number w becomes an *emergent* conserved quantity of the IR description (see §.14.2.3). However, from the UV perspective, the only strictly conserved quantum numbers are still the (multiplicative) characters of $\pi_1(G_{\text{eff}})$ which take value in the group

$$\pi_1(G_{\text{eff}})^\vee \equiv \text{Hom}(\pi_1(G), U(1)) \cong \pi_1(G_{\text{eff}}).$$

More generally, we may have Wilson-'t Hooft lines [144–147] which carry both electric and magnetic weights. Their multiplicative conserved quantum numbers take value in the (Abelian) *'t Hooft group*

$$\mathsf{tH} = \pi_1(G_{\text{eff}})^\vee \oplus \pi_1(G_{\text{eff}}),$$

equipped with the canonical skew-symmetric bilinear pairing (the *Weil pairing*)[107]

$$\mathsf{tH} \times \mathsf{tH} \to \boldsymbol{\mu}, \qquad (x,y)\times(x',y') \mapsto x(y')\,x'(y)^{-1}.$$

The 't Hooft multiplicative quantum numbers of a line operator, written additively, are just its electric/magnetic weights (w_e, w_m) modulo the weight lattice of G_{eff}.

The best way to understand the proper UV conserved quantum numbers of line operators is to consider the different sectors in which we may decompose the microscopic path integral of the theory which preserve the symmetries of a line operator stretched in the 3-direction in space (that is, rotations in the orthogonal plane and translations). In a $4d$ gauge theory quantized on a periodic 3-box of size L we may defined the 't Hooft twisted path integral [145] (see [146, 147] for nice reviews)

$$e^{-\beta\, F(\vec{e},\vec{m},\theta,\mu_s,\beta)} \equiv \mathrm{Tr}_{\vec{e},\vec{m}}\Big[e^{-\beta H + i\theta\nu+\mu_s F_s}\Big], \qquad \vec{e}\in\big(\pi_1(G_{\text{eff}})^\vee\big)^3,\ \vec{m}\in\pi_1(G_{\text{eff}})^3,$$

where $\vec{e}$, $\vec{m}$ are 't Hooft (multiplicative) electric and magnetic fluxes, θ is the instanton angle, and μ_s are chemical potentials in the Cartan algebra of the flavor group F. Imposing rotational invariance in the $1-2$ plane and taking the Fourier transform with respect to the μ_s we remain (at fixed θ) with the quantum numbers

$$(e_3, m_3, w) \in \pi_1(G_{\text{eff}})^\vee \oplus \pi_1(G_{\text{eff}}) \oplus \big(\text{weight lattice of } F\big). \tag{14.2}$$

We shall call the vector (e_3, m_3, w) the *'t Hooft charge* and the group in the RHS the *extended 't Hooft group.*

We stress that the structure of the Weil pairing is required in order to relate the Euclidean path integral in given topological sectors to the free energy $F(\vec{e},\vec{m},\theta,\mu_s,\beta)$ with fixed non-abelian fluxes [145].

Remark 14.2.1. The boundary condition on the Euclidean box which corresponds to a given 't Hooft charge does no break any supercharges, that is, we do not need to specify a BPS angle ϑ to define it.

14.2.2 Non-Abelian enhancement of the flavor group in the UV

Consider a UV complete $\mathcal{N}=2$ gauge theory. In the IR theory the flavor group is (generically) Abelian of rank f. In the UV the masses are irrelevant and the flavor group enhances from the Abelian group $U(1)^f$ to some possibly non-Abelian rank f Lie group F. The free part of the 't Hooft group (14.2) is then the weight lattice of F. This weight lattice is equipped with a quadratic form

[107] As always, $\boldsymbol{\mu}$ denotes the group of roots of unity. The name 'Weil pairing' is due to its analogy with the Weil pairing in the torsion group of a polarized Abelian variety which arises in exactly the same way.

dual to the Cartan form on the root lattice. From the quadratic form we recover the non-Abelian Lie group F. In conclusion:

Fact. *The UV conserved quantities are encoded in the extended 't Hooft group, a finitely generated Abelian group of the form*

$$\pi_1(G_{\mathit{eff}})^\vee \oplus \pi_1(G_{\mathit{eff}}) \oplus \Gamma_{\mathit{flav,weight}}, \tag{14.3}$$

whose free part has rank f. The extended 't Hooft group (14.3) *is equipped with two additional structures:* i) *the Weil pairing on the torsion part,* ii) *the dual Cartan symmetric form on the free part. Moreover,* iii) *the UV lines carry an adjoint action of the half quantum monodromy* $\mathbb{K}$ (see §. 14.3) *which acts on the 't Hooft group as* -1.

Finer structures on the 't Hooft group. The 't Hooft group (14.3) detects the global topology of the gauge group G_{eff}; it also detects the topology of the flavor group F, e.g. it distinguishes between the flavor groups $SO(N)$ and $\text{Spin}(N)$, since they have different weight lattices

$$\big[\,\Gamma_{\mathfrak{spin}(N)} : \Gamma_{\mathfrak{so}(N)}\,\big] = 2.$$

But there even finer informations on the flavor symmetry which we should be able to recover from the relevant categories. To illustrate the issue, consider $SU(2)$ SQCD with N_f fundamental hypers. In the perturbative sector (states of zero magnetic charge) the flavor group is $SO(2N_f)$, but non-perturbatively it gets enhanced to $\text{Spin}(2N_f)$. More precisely, states of *odd* (resp. *even*) magnetic charge are in spinorial (resp. tensorial) representations of the flavor group $\text{Spin}(2N_f)$. This is due to the zero modes of the Fermi fields in the magnetic monopole background [228], which is turn are predicted by the Atiyah-Singer index theorem. The index theorem is an integrated version of the axial anomaly, so the correlation between magnetic charge and flavor representations should emerge from the same aspect of the category which expresses the $U(1)_R$ anomaly (and the β-function).

14.2.3 The effective 'charge' of a UV line operator

We have two kinds of quantum numbers: conserved quantities and labeling numbers. In the IR we expect (see §.14.1.2) that conserved quantities are (typically) sufficient to label BPS objects. However, the UV group of eqn.(14.2) is too small to distinguish inequivalent BPS line operators.

We may introduce a different notion of 'charge' for UV operators which takes value in a rank $n = 2r + f$ lattice. This notion, albeit referred to UV objects, depends on a IR choice, e.g. the choice of a vacuum u. Suppose that in this vacuum we have n species of stable lines L_i ($i = 1, \dots, n$) which are preserved by the the same susy sub-algebra preserving L and carry emergent IR quantum numbers $[L_i]$ which are $\mathbb{Q}$-linearly independent. We may consider the BPS state $|\{n_i\}\rangle$ in which we have a configuration of parallel stable lines with n_1 of type L_1, n_2 of type L_2, and so on. Suppose that for our BPS line operator L

$$\big\langle \{n_i'\} \big|\, L \,\big| \{n_i\} \big\rangle \neq 0 \tag{14.4}$$

It would be tempting to assign to the operator L the 'charge'

$$\sum_i (n_i' - n_i)[L_i] \in \bigoplus_i \mathbb{Z}[L_i].$$

Such a charge would be well-defined on UV operators provided two conditions are satisfied: *i)* for all L we can find a pair of states $|\{n_i\}\rangle$, $|\{n'_i\}\rangle$ such that eqn.(14.4) holds, and moreover *ii)* we can show that $n'_i - n_i$ does not depend on the chosen $|\{n_i\}\rangle$, $|\{n'_i\}\rangle$. The attentive reader may notice that this procedure is an exact parallel to the definition of the index of a cluster object (**Definition 32**). However the i-th 'charge' $n'_i - n_i$ is PCT-odd only if the lines L, L_i carry 'mutually local charge', that is, have trivial braiding; the projection of the 'charge' so defined in the 't Hooft group (14.3) is, of course, independent of all choices. This follows from the fact that the action of PCT on the UV lines is given by the half quantum monodromy (see §. 14.3) which does not act as -1 on the present 'effective' charges; of course, it acts as -1 on the 't Hooft charges as it should.

14.3 The quantum monodromy

There is one more crucial structure on the UV BPS operators, namely the quantum monodromy [65, 69]. Let us consider first the case in which the UV fixed point is a good regular SCFT. At the UV fixed point the $U(1)_r$ R-symmetry is restored. Let $e^{2\pi i r}$ be the operator implementing a $U(1)_r$ rotation by 2π (it acts on the supercharges as -1). $e^{2\pi i r}$ acts on a chiral primary operator of the UV SCFT as multiplication by $e^{2\pi i \Delta}$, where Δ is the scaling dimension of the chiral operator. Suppose that for all chiral operators $\Delta \in \mathbb{N}$, then $e^{2\pi i r} = (-1)^F$ acts as 1 on all UV observables. More generally, if all $\Delta \in m\mathbb{N}$ for some integer m, the operator $(e^{2\pi i r})^m$ acts as 1 on observables [65, 69].

If the theory is asymptotically-free, meaning that the UV fixed point is approached with logarithmic deviations from scaling, the above relations get also corrected, in a way that may be described rather explicitly, see [65] .

Now suppose we deform the SCFT by relevant operators to flow to the original $\mathcal{N} = 2$ theory. We claim that, although the Abelian R-charge r is no longer conserved, $e^{2\pi i r}$ remain a symmetry in this set up[108] [69]. This is obvious when the dimensions Δ are integral, since $e^{2\pi i r}$ commutes with the deforming operator. The quantum monodromy $\mathbb{M}$ is the operator induced in the massive theory from $e^{2\pi i r}$ in this way [65, 69]. It is well defined only up to conjugacy,[109] and may be written as a Kontsevitch-Soibelmann (KS) product of BPS factors ordered according to their phase[110] [65, 69]

$$\mathbb{M} = \prod_{\lambda \in \mathrm{BPS}}^{\circlearrowleft} \Psi(q^{s_\lambda} X_\lambda; q)^{(-1)^{2s_\lambda}}. \tag{14.5}$$

The KS wall-crossing formula [170, 171] is simply the statement that the conjugacy class of $\mathbb{M}$, being an UV datum, is independent of the particular massive deformation as well as of the particular BPS chamber we use to compute it (see [65, 69]).

We may also define the half-monodromy $\mathbb{K}$, such that $\mathbb{K}^2 = \mathbb{M}$ [69]. The effect of the adjoint action of $\mathbb{K}$ on a line operator L is to produce its PCT-conjugate. Then $\mathbb{K}$ inverts the 't Hooft charges.

[108] For the corresponding discussion in 2d, see [68].

[109] When the UV fixed point SCFT is non degenerated, the operator $\mathbb{M}$ is semisimple, and its conjugacy class is encoded in its spectrum, that is, the spectrum of dimensions of chiral operators $\Delta \mod 1$.

[110] In eqn.(14.5) we use the notations of [69]: the product is over the BPS stable states of charge $\lambda \in \Lambda$ and spin s_λ taken in the clockwise order in their phase $\arg Z_u(\lambda)$; $\psi(z; q) = \prod_{n \geq 0}(1 - q^{n+1/2} z)^{-1}$ is the quantum dilogarithm, and the X_λ are quantum torus operators, i.e. they satisfy the algebra $X_\lambda X_{\lambda'} = q^{\langle \lambda, \lambda' \rangle/2} X_{\lambda+\lambda'}$ with $\langle -, - \rangle$ the Dirac pairing.

We summarize this subsection in the following

Fact. *If our $\mathcal{N}=2$ has a regular UV fixed-point SCFT and the dimension of all chiral operators satisfy $\Delta \in m\mathbb{N}$ for a certain integer m, then $\mathbb{K}^{2m}$ acts as the identity on the line operators. $\mathbb{K}$ acts as -1 on the 't Hooft charges.*

15 Physical meaning of the categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, $\mathcal{C}(\Gamma)$

We start this section by reviewing as quivers with (super)potentials arise in the description of the BPS sector of a (large class of) $4d$ $\mathcal{N}=2$ theories, see [3, 4, 67, 72, 73, 91].

15.1 $\mathcal{N}=2$ BPS spectra and quivers

We consider the IR physics of a $4d$ $\mathcal{N}=2$ model at a generic vacuum u along its Coulomb branch. We have the IR structures described in §. 14.1.1: a charge lattice Λ of rank $n=2r+f$, equipped with an integral skew-symmetric form given by the Dirac electro-magnetic pairing, and a complex linear form given by the $\mathcal{N}=2$ central charge:

$$\langle -, - \rangle : \Lambda \times \Lambda \to \mathbb{Z}, \qquad Z_u \colon \Lambda \to \mathbb{C}.$$

A $4d$ $\mathcal{N}=2$ model has a BPS quiver at u iff there exists a set of n hypermultiplets, stable in the vacuum u, such that [4]: *i)* their charges $e_i \in \Lambda$ generate Λ, i.e. $\Lambda \cong \oplus_i \mathbb{Z} e_i$, and *ii)* the charge of each BPS states (stable in u), $\lambda \in \Lambda$, satisfies

$$\lambda \in \Lambda_+ \quad \text{or} \quad -\lambda \in \Lambda_+,$$

where $\Lambda_+ = \oplus_i \mathbb{Z}_+ e_i$ is the convex cone of 'particles' [111]. The BPS quiver Q is encoded in the skew-symmetric $n \times n$ exchange matrix

$$B_{ij} := \langle e_i, e_j \rangle, \qquad i,j = 1, \cdots, n. \tag{15.1}$$

The nodes of Q are in one-to-one correspondence with the generators $\{e_i\}$ of Λ. If $B_{ij} \geq 0$ then there are $|B_{ij}|$ arrows from node i to node j; viceversa for $B_{ij} < 0$.

To find the spectrum of particles with given charge $\lambda = \sum_i m_i e_i \in \Lambda_+$ we may study the effective theory on their world-line. This is a SQM model with four supercharges [4, 93], corresponding to the subalgebra of $4d$ susy which preserves the world-line. A particle is BPS in the $4d$ sense iff it is invariant under 4 supersymmetries, that is, if it is a susy vacuum state of the world-line SQM. The 4-supercharge SQM is based on the quiver Q defined in eqn.(15.1) [4, 93]: to the i–th node there correspond a $1d$ $U(m_i)$ gauge multiplet, while to an arrow $i \to j$ a $1d$ chiral multiplet in the $(\overline{\boldsymbol{m}}_i, \boldsymbol{m}_j)$ bifundamental representation of the groups at its two ends. To each oriented cycle in Q there is associated a single-trace gauge invariant chiral operator, namely the trace of the product of the Higgs fields along the cycle. The (gauge invariant) superpotential of the SQM is a complex linear combination of such operators associated to cycles of Q [4]. Since we are interested only in the susy vacua, we are free to integrate out all fields entering quadratically in the superpotential.

[111] As contrasted with 'antiparticles' whose charges belong to $-\Lambda_+$.

We remain with a SQM system described by a *reduced* quiver with (super)potential (Q,W) in the sense of section 2.

Then the solutions of the SQM F-term equations are exactly the modules X of the Jacobian algebra[112] $J(Q,W)$ with dimension vector $\dim X = \lambda \in \Lambda$.

The D-term equation is traded for the stability condition [4]. Given the central charge $Z_u(-)$, we can choose a phase $\theta \in [0,2\pi)$ such that $Z_u(\Lambda_+)$ lies inside[113] $\mathbb{H}_\theta := e^{i\theta}\mathbb{H}$. Given a module $X \in \mathsf{mod}\, J(Q,W)$, we define its stability function as $\zeta(X) := e^{-i\theta} Z_u(X) \in \mathbb{H}$. The module X is stable iff

$$\arg \zeta(Y) < \arg \zeta(X), \quad \forall Y \subset X \text{ proper submodule.}$$

A stable module X is always a *brick,* i.e. $\mathrm{End}_{\mathsf{mod}\, J(Q,W)} X \cong \mathbb{C}$ [73].

Keeping into account gauge equivalence, the SQM classical vacuum space is the compact Kähler variety [4]

$$M_\lambda := \Big\{ X \in \mathsf{mod}\, J(Q,W) \;\Big|\; X \text{ stable, } \dim X = \lambda \Big\} \Big/ \prod_i GL(m_i, \mathbb{C}), \tag{15.2}$$

that is, the space of isoclasses of stable Jacobian modules of the given dimension λ. The space of SQM quantum vacua is then $H^*(M_\lambda, \mathbb{C})$ which carries a representation $\boldsymbol{R}$ of $SU(2)$ by hard Lefschetz [4, 139], whose maximal spin is $\dim M_\lambda/2$; the space-time spin content of the charge λ BPS particle is[114]

$$\Big(\mathbf{0} \oplus \mathbf{2}\Big) \otimes \boldsymbol{R}.$$

For example, the charge λ BPS states consist of a (half) hypermultiplet iff the corresponding moduli space is a point, i.e. if the module X is rigid.

The splitting between particles and antiparticles is conventional: different choices lead to different pairs (Q,W). However all these (Q,W) should lead to *equivalent* SQM quiver models. Indeed, distinct pairs are related by a chain of $1d$ Seiberg dualities [227]. The Seiberg dualities act on (Q,W) as the quiver mutations described in section 2. Indeed, the authors of [94] modeled their construction on Seiberg's original work [227].

The conclusion of this subsection is that to a (continuous family of) $4d$ $\mathcal{N}=2$ QFT (with the quiver property) there is associated a full *mutation-class* of quivers with potentials (Q,W). All (Q,W) known to arise from consistent QFTs are Jacobi-finite, and we assume this condition throughout.

Using the mathematical constructions reviewed in §. 2, to such an $\mathcal{N}=2$ theory we naturally associate the three triangle categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, and $\mathcal{C}(\Gamma)$, together with the functors s, r relating them. We stress that the association is *intrinsic,* in the sense that the categories are independent of the choice of (Q,W) in the mutation-class modulo triangle equivalence (cfr. **Theorem 10**). Our next task is to give a physical interpretation to these three naturally defined categories. We start from the simpler one, $D^b\Gamma$.

[112] From now on the ground field k is taken to be $\mathbb{C}$.
[113] $\mathbb{H}$ denotes the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} \,|\, \mathrm{Im}\, z > 0\}$.
[114] The Cartan generator of $SU(2)_R$ acting on a BPS particle described by a (p,q)-harmonic form on M_λ is $(p-q)$; however, it is conjectured that only trivial representations of $SU(2)_R$ appear [92, 127].

15.2 Stable objects of $D^b\Gamma$ and BPS states

Let Γ be the Ginzburg algebra associated to the pair (Q, W). Keller proved [162] that the Abelian category $\mathsf{mod}\, J(Q,W)$ is the heart of a bounded $\boldsymbol{t}$-structure in $D^b\Gamma$. In particular, its Grothendieck group is

$$K_0(D^b\Gamma) \cong K_0\big(\mathsf{mod}\, J(Q,W)\big) \equiv \Lambda, \tag{15.3}$$

that is the lattice of the IR conserved charges (§. 14.1.1). Thus, given a stability condition on the Abelian category $\mathsf{mod}\, J(Q,W)$, we can extend it to the entire triangular category $D^b\Gamma$. In particular, since the semi-stable objects of $D^b\Gamma$ are the elements of $\mathcal{P}(\phi)$ (cfr. the proof of **Proposition 13.10.1**), we have two possibilities:

- $\phi \in (0,1]$, then the only semistable objects are the semistable objects of $\mathsf{mod}\, J(Q,W)$ in the sense of "Abelian category stability" plus the zero object of $D^b\Gamma$;
- $\phi \notin (0,1]$, then the only semistable objects are the shifts of the semistable objects of $\mathsf{mod}\, J(Q,W)$ in the sense of "Abelian category stability".

In other words, a generic object $E \in D^b\Gamma$ is *unstable if it has a nontrivial HN filtration.* Thus, up to shift $[n]$, the only possible semistable objects in $D^b\Gamma$ are those objects belonging to the heart $\mathsf{mod}\, J(Q,W)$ that are "Abelian"-stable in it. We have already seen that the category $\mathsf{mod}\, J(Q,W)$ describes the BPS spectrum of our $4d$ $\mathcal{N}=2$ QFT: by what we just concluded, the isoclasses of stable objects X of $D^b\Gamma$ with Grothendieck class $[X] = \lambda \in \Lambda$ are parametrized, up to even shifts[115], by the Kähler manifolds $M_\lambda \cong M_{-\lambda}$ in eqn.(15.2) whose cohomology yields the BPS states.

The category $\mathcal{P}(\phi)$ is an Abelian category in its own right. The stable objects with BPS phase $e^{i\pi\phi}$ are the simple objects in this category; in particular they are bricks in $\mathcal{P}(\phi)$ hence bricks in $\mathsf{mod}\, J(Q,W)$, that is,

$$X \text{ stable} \quad \Rightarrow \quad \mathrm{End}_{\mathsf{mod}\, J(Q,W)}(X) \cong \mathbb{C}.$$

15.3 Grothendieck groups vs. physical charges

When the triangle category $\mathcal{T}$ describes a class of BPS objects, the Abelian group $K_0(\mathcal{T})$ is identified with the conserved quantum numbers carried by those objects. In particular, the group $K_0(\mathcal{T})$ should carry all the additional structures required by the physics of the corresponding BPS objects, as described in §. 14.

Let us pause a while to discuss the Grothendieck groups of the three triangle categories $K_0(\mathcal{T})$, where $\mathcal{T} = D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, or $\mathcal{C}(\Gamma)$, and check that they indeed possess all properties and additional structures as required by their proposed physical interpretation.

15.3.1 $K_0(D^b\Gamma)$

Since $D^b\Gamma$ describes BPS particles, $K_0(D^b\Gamma)$ is just the IR charge lattice Λ, see eqn.(15.3). Physically, the charge lattice carries the structure of a skew-symmetric integral bilinear form, namely the Dirac electromagnetic pairing. This matches with the fact that, since $D^b\Gamma$ is 3-CY, its Euler form (13.4) is skew-symmetric and is identified with the Dirac pairing (compare eqn.(15.1) and the last

[115] Since the shift by [1] acts on the BPS states as PCT, it is quite natural to identify the BPS states associated to stable objects differing by even shifts.

part of **Proposition 13.8.2**). We stress that the pairing is intrinsic (independent of all choices) as it should be on physical grounds.

15.3.2 $K_0(\mathcal{C}(\Gamma))$: structure

The structure of the group $K_0(\mathcal{C}(\Gamma))$ was described in §. 13.8.4. We have

$$K_0(\mathcal{C}(\Gamma)) = \mathbb{Z}^f \oplus \mathsf{A}^\vee \oplus \mathsf{A} \tag{15.4}$$

where A is the torsion group[116]

$$\mathsf{A} = \bigoplus_s \mathbb{Z}/d_s\mathbb{Z}, \qquad d_s \mid d_{s+1}$$

where the d_s are the positive integers appearing in the normal form of B, see eqn.(13.19).

The physical meaning of the Grothendieck group (15.4) is easily understood by considering the case of pure $\mathcal{N}=2$ super-Yang-Mills with gauge group G. Then one shows [76]

$$\mathsf{A} = \boldsymbol{Z}(G) \equiv \text{the center of the (simply-connected) gauge group } G$$

that is

$$K_0(\mathcal{C}(\Gamma_{\mathrm{SYM},G})) \cong \boldsymbol{Z}(G)^\vee \oplus \boldsymbol{Z}(G).$$

This is exactly the group of multiplicative quantum numbers labeling the UV Wilson-'t Hooft line operators in the pure SYM case [144], as reviewed in §. 14.2.1. This strongly suggests the identification of the cluster Grothendieck group $K_0(\mathcal{C}(\Gamma))$ with the group of additive and multiplicative quantum numbers carried by the UV line operators.

This is confirmed by of the example of $\mathcal{N}=2$ SQCD with (semi-simple) gauge group G and quark hypermultiplets in a (generally reducible) representation $\boldsymbol{R}$. One finds [76]

$$K_0(\mathcal{C}(\Gamma_{\mathrm{SQCD}})) \cong \mathbb{Z}^{\operatorname{rank} F} \oplus \pi_1(G_{\mathrm{eff}})^\vee \oplus \pi_1(G_{\mathrm{eff}}),$$

where F is the flavor group and G_{eff} is the quotient of G acting effectively on the UV degrees of freedom. Again, this corresponds to the UV extended 't Hooft group as defined in §. 14.2.1. More generally one has:

Fact. *In all $\mathcal{N}=2$ theories with a Lagrangian formulation (and a BPS quiver) we have*

$$K_0(\mathcal{C}(\Gamma)) \cong \big(\textit{the extended 't Hooft group of §. 14.2.1}\big).$$

This is already strong evidence that the cluster category $\mathcal{C}(\Gamma)$ describes UV line operators. For $\mathcal{N}=2$ theories without a Lagrangian, we adopt the above **Fact** as the definition of the extended 't Hooft group.

From **Fact 14.2.2** we know that the physical 't Hooft group has three additional structures. Let us show that all three structures are naturally present in $K_0(\mathcal{C}(\Gamma))$.

[116] Of course, $\mathsf{A}^\vee \cong \mathsf{A}$; however it is natural to distinguish the group and its dual.

15.3.3 $K_0(\mathcal{C}(\Gamma))$: action of half-monodromy and periodic subcategories

There is a natural candidate for the half-monodromy: on $X \in \mathcal{C}(\Gamma)$, $\mathbb{K}$ acts as $X \mapsto X[1]$ and hence the full monodromy as $X \mapsto X[2]$. Then $\mathbb{K}$ acts on $K_0(\mathcal{C}(\Gamma))$ as -1, as required. Let us check that this action has the correct physical properties e.g. the right periodicity as described in **Fact 14.3**.

Example 45 (Periodicity for Argyres-Douglas models)**.** We use the notations of §. 13.6.2. We know that the quantum monodromy $\mathbb{M}$ has a periodicity[117] equal to (a divisor of) $h+2$ [69], corresponding to the fact that the dimension of the chiral operators $\Delta \in \frac{1}{h+2}\mathbb{N}$. Indeed, from the explicit description of the cluster category, eqn.(13.12), we have $\mathcal{C}(\mathfrak{g}) = D^b(\mathsf{mod}\,\mathbb{C}\mathfrak{g})/\langle \tau^{-1}[1]\rangle^{\mathbb{Z}}$, so that $\tau \cong [1]$ in $\mathcal{C}(\mathfrak{g})$. Hence,

$$\mathbb{M}^{h+2} \equiv [h+2] \cong \tau^h[2] = \mathrm{Id},$$

where we used eqn.(13.13).

Under the identification $\mathbb{K} \leftrightarrow [1]$, we may rephrase **Fact 14.3** in the form:

Fact. *Let $\mathcal{C}(\Gamma)$ be the cluster category associated to a $\mathcal{N}=2$ theory with a* regular *UV fixed-point SCFT such that all chiral operators have dimensions $\Delta \in m\mathbb{N}$. Then $\mathcal{C}(\Gamma)$ is periodic with minimal period $p \mid 2m$. If the theory has flavor charges, p is even (more in general: p is even unless the 't Hooft group is a vector space over $\mathbb{F}_2$). In particular, for $\mathcal{N}=2$ theories with a regular UV fixed-point the cluster Tits form $\langle\!\langle [X],[Y]\rangle\!\rangle$ is well-defined.*

Asymptotically-free theories. It remain to discuss the asymptotically-free theories. The associate cluster categories $\mathcal{C}(\Gamma)$ are not periodic. However, from the properties of the 't Hooft group, we expect that, whenever our theory has a non-trivial flavor symmetry, $\mathcal{C}(\Gamma)$ still contains a periodic sub-cluster category of even period. We give an informal argument corroborating this idea which may be checked in several explicit examples.

Sending all non-exactly marginal couplings to zero, our asymptotically-free theory reduces to a decoupled system of free glue and UV regular matter SCFTs. Categorically, this means that cluster category of each matter SCFT, $\mathcal{C}_{\text{mat}}$ embeds as an additive sub-category in $\mathcal{C}(\Gamma)$ closed under shifts (by PCT). The embedding functor ι is not exact (in general), so we take the triangular hull of the full subcategory over the objects in its image $\mathsf{Hu}_\triangle\big((\iota\,\mathcal{C}_{\text{mat}})_{\text{full}}\big) \subset \mathcal{C}(\Gamma)$. If the model has non-trivial flavor, at least one matter subsector has non trivial flavor, and the corresponding category $\mathcal{C}_{\text{matter}}$ is periodic of even period p. Its objects satisfy $X[p] \cong X$ and this property is preserved by ι. The triangle category $\mathsf{Hu}_\triangle\big((\iota\,\mathcal{C}_{\text{mat}})_{\text{full}}\big)$ is generated by these periodic objects and hence is again periodic of period p. Then set

$$\mathcal{F}(p) = \mathsf{Hu}_\triangle\big((\iota\,\mathcal{C}_{\text{mat}})_{\text{full}}\big).$$

Again, the flavor Tits form is well defined.

The above discussion shows that the presence of a p-periodic subcategory $\mathcal{F}(p) \subset \mathcal{C}$ is related to the presence of a sector in the $\mathcal{N}=2$ theory described by susy protected operators of dimension

$$\Delta = \tfrac{2}{p}\mathbb{N}.$$

Let us present some simple examples.

[117] For the relation of this fact with the Y-systems, see [65] .

Example 46 (Pure $SU(2)$ SYM)**.** The cluster category $\mathcal{C}_{SU(2)}$ is not periodic; this is a manifestation of the fact that the β-function of the theory is non zero [73]. However, let us focus on the perturbative ($\equiv$ zero magnetic charge) sector in the $g_{\rm YM} \to 0$ limit. The chiral algebra is generated by a single operator of dimension $\Delta = 2$, namely $\mathrm{tr}(\phi^2)$. Hence we expect that the zero-magnetic charge sector is described by a subcategory of $\mathcal{C}_{SU(2)}$ which is 1-periodic. Indeed, this is correct, $\mathcal{F}(1)$ being a $\mathbb{P}^1$ family of homogenous cluster tubes.

Example 47 ($SU(2)$ SYM coupled to D_p Argyres-Douglas)**.** In this case the matter is an Argyres-Douglas theory of type D_p; the matter half quantum monodromy $\mathbb{K}_{\rm matter}$ has order $(h(D_p)+2)/\gcd(2,h(D_p)) = p$ as we may read from the spectrum of chiral ring dimensions of the Argyres-Douglas model [68]. Thus the matter corresponds to a periodic subcategory $\mathcal{F}(p) \subset \mathcal{C}$. This category is a cluster tube of period p [31, 32]. See also [73].

Remark 15.3.1. Equivalently, we may understand that the presence of a non-trivial flavor group implies the existence of a 2-periodic subcategory $\mathcal{F}(p) \subset \mathcal{C}$ by the fact that the corresponding conserved super-currents have canonical dimension 1 which cannot be corrected by RG.

15.3.4 $K_0(\mathcal{C}(\Gamma))$: non-Abelian enhancement of flavor

As discussed in §. 14.2.2, the IR flavor symmetry $U(1)^f$ gets enhanced in the UV to a non-Abelian group F. The identification of $K_0(\mathcal{C}(\Gamma))$ with the extended 't Hooft group requires, in particular, that its free part is equipped with the correct dual Cartan form for the flavor group F.

In §. 13.9 we defined a Tits form associated to (a periodic subcategory of) $\mathcal{C}(\Gamma)$. This is a symmetric form on the free part of the Grothendieck group, and is the natural candidate for the dual Cartan form of the physical flavor group F. Let us check in a couple of examples that this identification yields the correct flavor group: the cluster category knows the actual non-Abelian group.

Example 48 ($SU(2)$ with $N_f \geq 1$ fundamentals)**.** We use the same notations[118] as in §. 13.9.1. The cluster category is

$$\mathcal{C}_{N_f} = D^b\big(\mathsf{coh}\,\mathbb{X}(\overbrace{2,\cdots,2}^{N_f\ 2's})\big)\big/\langle \tau^{-1}[1]\rangle^{\mathbb{Z}}.$$

For $N_f \neq 4$ this category is not periodic since the canonical sheaf has non-zero degree (in the physical language: the β-function is non-zero). We are in the situation discussed at the end of §. 15.3.3, and the present example is also an illustration of that issue.

The 2-periodic triangle 2-CY subcategory $\mathcal{F}(2) \xrightarrow{\mathsf{j}} \mathcal{C}(N_f)$ is given by the orbit category of the derived category of finite-length sheaves. It consists of a $\mathbb{P}^1$ family of cluster tubes; in $\mathbb{P}^1$ there are N_f special points whose cluster tubes have period 2. Let $\mathcal{S}_{i,k}$, $k \in \mathbb{Z}/2\mathbb{Z}$, be the simples in the i-th special cluster tube, satisfying

$$\mathcal{S}_{i,k}[1] \cong \tau\mathcal{S}_{i,k} \cong \mathcal{S}_{i,k+1}$$

and let $\mathcal{S}_z$ be the simple over the regular point $z \in \mathbb{P}^1$, $\tau\mathcal{S}_z \cong \mathcal{S}_z$. Thus $[\mathcal{S}_z] = 0$ and $K_0(\mathcal{F}(2))$ is generated by the $[\mathcal{S}_{i,0}]$ ($i = 1, \dots, N_f$). The image of $K_0(\mathcal{F}(2))$ in $K_0(\mathcal{C}(N_f))$ has index 2; indeed

[118] However we often write simply $\mathbb{X}$ instead of $\mathbb{X}(p_1,\dots,p_s)$ leaving the weights implicit.

in $K_0(\mathcal{C}(N_f))$ we have an extra generator $[\mathcal{O}]$ and a relation [31]

$$2[\mathcal{O}] = \sum_{i=1}^{N_f} [\mathcal{S}_{i,0}] \tag{15.5}$$

Then as in §. 13.9.1 (for the special case $N_f = 4$) we have

$$K_0(\mathcal{C}(N_f)) \cong \left\{ (w_1, \cdots, w_{N_f}) \in \left(\tfrac{1}{2}\mathbb{Z}\right)^{N_f} \;\middle|\; w_i = w_j \mod 1 \right\} \equiv \Gamma_{\text{weight}, \mathfrak{spin}(2N_f)}$$

with

$$\langle\!\langle [\mathcal{S}_{i,0}], [\mathcal{S}_{j,0}] \rangle\!\rangle = \delta_{i,j},$$

that is, $K_0(\mathcal{C}(N_f))$ is the $\mathfrak{spin}(2N_f)$ weight lattice equipped with the dual Cartan pairing which is the correct physical extended ' t Hooft group for this model which has $\pi_1(G_{\text{eff}}) = 1$ and $F = \text{Spin}(2N_f)$, as expected.

Remark 15.3.2 (Spin(8) triality)**.** The case of $N_f = 4$ was already presented in §. 13.9.1. In that case $\deg \mathcal{K} = 0$ (i.e. $\beta = 0$), the theory is UV superconformal, and the cluster category is periodic. The correlation between magnetic charge and Spin(8) representation becomes the fact that the modular group $PSL(2, \mathbb{Z})$ acts on the flavor by triality [228], see [75] for details from the cluster category viewpoint.

15.3.5 Example 48: Finer flavor structures, $U(1)_r$ anomaly, Witten effect

The cluster category contains even more detailed information on the UV flavor physics of the corresponding $\mathcal{N} = 2$ QFT. Let us illustrate the finer flavor structures in the case of $SU(2)$ SYM coupled to N_f flavors[119] (**Example 48**).

Note that the sublattice $K_0(\mathcal{F}(2)) \subset K_0(\mathcal{C}(N_f))$ is the weight lattice of $SO(2N_f)$; since $\mathcal{F}(2)$ is the cluster sub-category of the 'perturbative' (zero magnetic charge) sector, we recover the finer flavor structures mentioned at the end of §. 14.2.2. In facts, eqn.(15.5) is the image in the Grothendieck group of the equation which is the categorical expression of the $U(1)_r$ anomaly [73]. Indeed, in the language of coherent sheaves, the $U(1)_r$ anomaly is measured by the non-triviality of the canonical sheaf $\mathcal{K}$ (think of a (1,1) σ-model: $\mathcal{K}$ trivial means the target space is Calabi-Yau, which is the condition of no anomaly). The coefficient of the β-function, b, is (twice) its degree,[120] $\deg \mathcal{K} = -\chi(\mathbb{X})$ [73]. As a preparation to the examples of §. 6, we briefly digress to recall how this comes about.

β-function and Witten effect. The AR translation τ acts on $\mathsf{coh}\,\mathbb{X}$ as multiplication by the canonical sheaf [75, 131, 183]

$$\tau \colon \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{K} \equiv \mathcal{A} \otimes \mathcal{O}(\vec{\omega}). \tag{15.6}$$

[119] Or, more generally, to several Argyres-Douglas systems of type D.
[120] Notice that $\deg \mathcal{K} = 0$ does not mean that $\mathcal{K}$ is trivial but only that it is a torsion sheaf in the sense that $\mathcal{K}^m \cong \mathcal{O}$ for some integer m.

Hence the $U(1)_R$ anomaly and β-function may be read from the action of τ on the derived category $D^b\mathsf{coh}\,\mathbb{X}$ which we may identify as the IR category of BPS particles.[121] Now, in the cluster category of a weighted projective line, $\mathcal{C}(\mathsf{coh}\,\mathbb{X}) \equiv D^b(\mathsf{coh}\,\mathbb{X})/\langle\tau^{-1}[1]\rangle$, one has $\tau \cong [1]$, while $[1]$ acts in the UV as the half-monodromy, that is, as a UV $U(1)_r$ rotation by π. In the normalization of ref. [228] (see their eqn.(4.3)), the complexified $SU(2)$ Yang-Mills coupling at weak coupling, $a \to \infty$, is

$$\frac{\theta}{\pi} + \frac{8\pi i}{g^2} = -\frac{b}{\pi i}\log a + \cdots ,$$

Under a $U(1)_r$ rotation by π, $a \to e^{\pi i}a$, the vacuum angle shifts as $\theta \to \theta - b\pi$. Since a dyon of magnetic charge m carries an electric charge $m\,\theta/2\pi \bmod 1$ (the Witten effect [246]), under the action of τ the IR electric/magnetic charges (e,m) should undergo the flow

$$\tau\colon (e,m) \to (e - mb/2, m). \tag{15.7}$$

For an object of $D^b(\mathsf{coh}\,\mathbb{X})$ the magnetic (electric) charge correspond to its rank (degree); then comparing eqns.(15.6),(15.7) we get $b = -2\deg\mathcal{K} = 2\,\chi(\mathbb{X})$.

Finer flavor structures (§. 14.2.2). The Grothendieck group of $\mathsf{coh}\,\mathbb{X}(2,\dots,2)$ is generated by $[\mathcal{O}]$, $[\mathcal{S}_0]$, $[\mathcal{S}_{i,j}]$ $(i = 1,\dots,N_f,\ j \in \mathbb{Z}/2\mathbb{Z})$ subjected to the relations $[\mathcal{S}_0] = [\mathcal{S}_{i,0}] + [\mathcal{S}_{i,1}]\ \forall\, i$, see **Proposition 2.1** of [31]. The action of τ in $K_0(\mathsf{coh}\,\mathbb{X})$ is

$$[\tau\mathcal{S}_{i,j}] = [\mathcal{S}_{i,j+1}], \qquad [\tau\mathcal{O}] - [\mathcal{O}] = (N_f - 2)[\mathcal{S}_0] - \sum_{i=1}^{N_f}[\mathcal{S}_{i,0}]. \tag{15.8}$$

The difference $[\tau\mathcal{O}]-[\mathcal{O}]$ measures the non-triviality of the canonical sheaf, that is, the β-function/$U(1)_r$ anomaly. In the cluster category, for all sheaf $[\tau\mathcal{A}] = -[\mathcal{A}]$, so that $[\mathcal{S}_{i,0}] = 0$ and the second eqn.(15.8) reduces to (15.5). Hence, as suggested by the physical arguments at the end of §. 14.2.2, the non-perturbative flavor enhancement $SO(2N_f) \to \mathrm{Spin}(2N_f)$ follows from the counting of the Fermi zero-modes implied by the axial anomaly.

15.3.6 $K_0(\mathcal{C}(\Gamma))_{\textbf{torsion}}$: the Weil pairing

Let $X \in \mathcal{C}(\Gamma)$ The projection

$$\langle S_i, F_\Gamma X\rangle \in \mathbb{Z}^n/B\mathbb{Z}^n,$$

depends only on $[X]$. Rewrite the integral vector $\langle S_i, F_\Gamma X\rangle$ in the $\mathbb{Z}$-basis where B takes the normal form (13.19)

$$\big(\langle S_1, F_\Gamma X\rangle, \cdots, \langle S_n, F_\Gamma X\rangle\big) \xrightarrow{\text{normal form basis}} (w_1, w_2, \cdots, w_f, u_{1,1}, u_{2,1}, \cdots, u_{1,s}, u_{2,s}, \cdots)$$

[121] Indeed, for $N_f \leq 3$, the triangle category $D^b\mathsf{coh}\,\mathbb{X}$ admits $\mathsf{mod}\,\mathbb{C}\hat{\mathfrak{g}}$ as the core of a $\boldsymbol{t}$-structure (here $\hat{\mathfrak{g}}$ is an acyclic affine quiver in the mutation class of the model [67]; see also **Example 51**.

and see its class as an element of $(\mathbb{Q}^2/\mathbb{Z}^2)^r$

$$(w_1, w_2, \cdots, w_f, u_{1,1}, u_{2,1}, \cdots, u_{1,s}, u_{2,s}, \cdots) \mapsto \left(\frac{u_{1,1}}{d_1}, \frac{u_{2,1}}{d_1}, \cdots, \frac{u_{1,s}}{d_s}, \frac{u_{2,s}}{d_s} \cdots\right) \in (\mathbb{Q}^2/\mathbb{Z}^2)^r.$$

The skew-symmetric matrix B then defines a skew-symmetric pairing

$$2\pi i \sum_{s=1}^{r} \frac{\epsilon^{ab}\, u_{a,s}\, u'_{b,s}}{d_s} \in 2\pi i\, \mathbb{Q}/\mathbb{Z}.$$

The exponential of this expression is the canonical Weil pairing. Let us check one example.

Example 49 (Pure $SU(2)$)**.** The basis $[P_1]$, $[P_2]$ is canonical. Then the Weil pairing is

$$(Z/2\mathbb{Z})^2 \times (Z/2\mathbb{Z})^2 \ni (e, m) \times (e', m') \mapsto (-1)^{em' - me'}.$$

15.4 The cluster category as the UV line operators

We have seen that for a $\mathcal{N} = 2$ theory (with quiver property) the Grothendieck group $K_0(\mathcal{C}(\Gamma))$ is the extended 't Hooft group of additive and multiplicative conserved quantum numbers of the UV line operators and that this group is naturally endowed with all the structures required by physics, including the finer ones.

This amazing correspondence makes almost inevitable the identification of the the cluster category $\mathcal{C}(\Gamma)$ of the mutation-class of quivers with (super)potentials associated to a $4d$ $\mathcal{N} = 2$ model with the triangle category describing its UV BPS line *operators*. This identification has been pointed out by several authors working from different points of view [68, 84, 127]. In particular, the structure of the mutations of the Y–seeds in the cluster algebras lead to the Kontsevich-Soibelman wall crossing formula [170] (see [65, 68] for details). This is just the action of the shift [1] on the cluster category which implements the quantum half monodromy $\mathbb{K}$ (cfr. §. 15.3.3).

In section 18 below we check explicitly this identification by relating the geometrical description of the cluster category of a surface as given in the mathematical literature with the WKB analysis of line operators by GMN [126, 127].

For BPS line operators we also had a notion of 'charge' which is useful to distinguish them, see §. 14.2.3. We already mentioned there that both the definition and the properties of this 'charge' have a precise correspondent in the mathematical notion of the *index* of a cluster object. Now we may identify these two quantities. Note that, while the 't Hooft charge is invariant under quantum monodromy (i.e. under the shift [2]), the index is not. This is the effect of non-trivial wall-crossing and, essentially, measures it [68].

In §. 13.8.5 we saw that the index is fine enough to distinguish rigid objects of the cluster category. This is reminiscent of our discussion in §.14.1.2 about a (necessary) condition for UV completeness.

In section §. 19.1, building over refs. [84, 127], we discuss how the interpretation of the cluster category $\mathcal{C}(\Gamma)$ as describing UV BPS line operators $L_{\mathrm{ind}\, X}(\zeta)$ (labeled by the index of the corresponding cluster object X and the phase ζ of the preserved supersymmetry) leads to concrete expressions for their vacuum expectation values in the vacuum u

$$\langle L_{\mathrm{ind}\,X}(\zeta)\rangle_u.$$

15.5 The perfect derived category $\mathfrak{Per}\,\Gamma$

To complete the understanding of the web of categories and functors describing the BPS physics of a $4d$ $\mathcal{N}=2$ theory, it remains to discuss the physical meaning of the perfect category $\mathfrak{Per}\,\Gamma$. To the best of our knowledge, an interpretation of the perfect category of a Ginzburg DG algebra has not appeared before in the physics literature.

We may extract some properties of the BPS objected described by the perfect category already from its Grothendieck group $K_0(\mathfrak{Per}\,\Gamma)$ and the basic sequence of functors

$$0 \to D^b\Gamma \xrightarrow{\mathsf{s}} \mathfrak{Per}\,\Gamma \xrightarrow{\mathsf{r}} \mathcal{C}(\Gamma) \to 0. \tag{15.9}$$

The Grothendieck group $K_0(\mathfrak{Per}\,\Gamma)$ is isomorphic to the IR charge lattice Λ, so $\mathfrak{Per}\,\Gamma$ is a category of IR BPS objects whose existence (i.e. "stability") depends on the particular vacuum u. $\mathfrak{Per}\,\Gamma$ yields the description of these physical objects from the viewpoint of the Seiberg-Witten low-energy effective Abelian theory. This is already clear from the fact that $\mathfrak{Per}\,\Gamma$ contains the category describing the IR BPS particles i.e. $D^b\Gamma$; BPS particles then form part of the physics described by $\mathfrak{Per}\,\Gamma$. A general object in $\mathfrak{Per}\,\Gamma \setminus D^b\Gamma$ differs from an object in the category $D^b\Gamma$ in one crucial aspect: its total homology has infinite dimension, so (typically) infinite SUSY central charge and hence infinite energy. Then $\mathfrak{Per}\,\Gamma$ is naturally interpreted as the category yielding the IR description of half-BPS *branes* of some kind. They may have infinite energy just because their volume may be infinite. Although their central charge is not well defined, its phase is: it is just the angle θ corresponding to the subalgebra of supersymmetries under which the brane is invariant.

On the other hand, the RG functor r in (15.9) associates to each IR object in $\mathcal{O} \in \mathfrak{Per}\,\Gamma \setminus D^b\Gamma$ a non-trivial UV line operator $\mathsf{r}(\mathcal{O})$. This suggests a heuristic physical picture: let $\mathcal{O} \in \mathfrak{Per}\,\Gamma \setminus D^b\Gamma$ describe a BPS brane which is stable in the Coulomb vacuum u; this brane should be identified with the "state" obtained by acting with the UV line operator $\mathsf{r}(\mathcal{O})$ on the vacuum u as seen in the low-energy Seiberg-Witten effective Abelian theory.

In order to make this proposal explicit, in the next section we shall consider a particular class of examples, namely the class $\mathcal{S}[A_1]$ theories [123, 126]. In this case all three categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$ and $\mathcal{C}(\Gamma)$ are explicitly understood both from the Representation-Theoretical side (in terms of string/band modules [10]) as well as in terms of the geometry of curves on the Gaiotto surface C. In this setting BPS objects are also well understood from the physical side since WKB is exact in the BPS sector.

Comparing the mathematical definition of the various triangle categories associated to a class $\mathcal{S}[A_1]$ model, and the physical description of the BPS objects, we shall check that the above interpretation of $\mathfrak{Per}\,\Gamma$ is correct.

15.5.1 "Calibrations" of perfect categories

To complete the story we need to introduce a notion of "calibration" on the objects of $\mathfrak{Per}\,\Gamma$ which restricts in the full subcategory $D^b\Gamma$ to the Bridgeland notion of stability. The specification of a "calibration" requires the datum of the Coulomb vacuum u and a phase $\theta = \pi\phi \in \mathbb{R}$. Given an u (corresponding to specifying a central charge Z), the ϕ-calibrated objects form a full additive subcategory of $\mathfrak{Per}\,\Gamma$, $\mathcal{K}(\phi)$, such that

$$\mathcal{P}(\phi) \subset \mathcal{K}(\phi) \subset \mathfrak{Per}\,\Gamma, \qquad \forall\, \phi \in \mathbb{R}.$$

We use the term "calibration" instead of "stability" since it is quite a different notion with respect to Bridgeland stability (in a sense, it has "opposite" properties), and it does not correspond to the physical idea of stability. These aspects are already clear from the fact that the central charge Z is not defined for general objects in $\mathfrak{Per}\,\Gamma$.

In the special case of the perfect categories arising from class $\mathcal{S}[A_1]$ QFTs, where everything is explicit and geometric, the calibration condition may be expressed in terms of flows of quadratic differentials, see §. 18.

We leave a more precise discussion of calibrations for perfect categories to future work. Here we limit ourselves to make some observations we learn from the class $\mathcal{S}[A_1]$ example.

Definition 39. A phase $\pi\phi \in \mathbb{R}$ is called a *BPS phase* if the slice $\mathcal{P}(\phi) \subset D^b\Gamma$ contains non-zero objects. A phase $\pi\phi$ is *generic* if it is not a BPS phase nor an accumulation point of BPS phases.

Fact. *In a class $\mathcal{S}[A_1]$ theory, assume there is no BPS phase in the range $[\pi\phi, \pi\phi']$. Then*

$$\mathcal{K}(\phi) \cong \mathcal{K}(\phi').$$

Moreover, let $\pi\phi$ be a generic *phase. Then the ϕ-calibrated category $\mathcal{K}(\phi) \subset \mathfrak{Per}\,\Gamma$ has the form*

$$\mathcal{K}(\phi) \cong \mathsf{add}\,\mathcal{T}_\phi$$

for an object $\mathcal{T}_\phi \in \mathfrak{Per}\,\Gamma$ such that

$$\mathsf{r}(\mathcal{T}_\phi) \in \mathcal{C}(\Gamma) \ \textit{is cluster-tilting.}$$

In other words, the generic $\mathcal{T}_\phi$ is a silting object of $\mathfrak{Per}\,\Gamma$.

We conjecture that something like the above **Fact** holds for general $4d$ $\mathcal{N} = 2$ theories.

Part VI

Homological S-duality

16 Cluster automorphisms and S-duality

16.1 Generalities

A duality between two supersymmetric theories induces a (triangle) equivalence between the triangle categories describing its BPS objects. The celebrate example is mirror symmetry between IIA and IIB string theories compactified on a pair of mirror Calabi-Yau 3-folds, $\mathcal{M}$, $\mathcal{M}^\vee$. At the level of the corresponding categories of BPS branes, mirror symmetry duality induces *homological mirror symmetry*, that is the equivalences of triangle categories [155, 169]

$$D^b(\mathsf{Coh}\,\mathcal{M}) \cong D^b(\mathsf{Fuk}\,\mathcal{M}^\vee), \qquad D^b(\mathsf{Coh}\,\mathcal{M}^\vee) \cong D^b(\mathsf{Fuk}\,\mathcal{M}).$$

In fact, since to a supersymmetric theory $\mathcal{T}$ we associate a family of triangle categories, $\{\mathfrak{T}_{(a)}\}_{a\in I}$, depending on the class of BPS objects and the physical picture (e.g. IR versus UV), a dual pair of theories $\mathcal{T}$, $\mathcal{T}^\vee$, yields a *family* of equivalences of categories labeled by the index set I

$$\mathfrak{T}_{(a)} \xrightarrow{\mathsf{d}_{(a)}} \mathfrak{T}^\vee_{(a)} \qquad a \in I.$$

The several categories associated to the theory, $\{\mathfrak{T}_{(a)}\}_{a\in I}$, are related by physical compatibility functors having the schematic form

$$\mathfrak{T}_{(a)} \xrightarrow{\mathsf{c}_{(a,b)}} \mathfrak{T}_{(b)} \qquad a, b \in I$$

(e.g. the 'inverse RG flow' functor r in eqn.(4.1)). Physical consistency of the duality then require that we have commutative diagrams of functors of the form

$$\begin{array}{ccc} \mathfrak{T}_{(a)} & \xrightarrow{\mathsf{d}_{(a)}} & \mathfrak{T}^\vee_{(a)} \\ {\scriptstyle \mathsf{c}_{(a,b)}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{c}^\vee_{(a,b)}} \\ \mathfrak{T}_{(b)} & \xrightarrow{\mathsf{d}_{(b)}} & \mathfrak{T}^\vee_{(b)} \end{array}$$

The philosophy of the present review is that the dualities are better understood in terms of such diagrams of exact functors between the relevant triangle categories. This idea may be applied to all kinds of dualities; here we are particularly interested in *auto-dualities*, that is, dualities of the theory with itself. The prime examples of auto-dualities is S-duality in $\mathcal{N}=2^*$ SYM and Gaiotto's $\mathcal{N}=2$ generalized S-dualities [123] . One of the motivation of this thesis is to use categorical methods to compute the group $\mathbb{S}$ of S-dualities which generalize the $PSL(2,\mathbb{Z})$ group for $\mathcal{N}=2^*$ as well as the results by Gaiotto.

An auto-duality induces a family of exact functors $\mathsf{d}_{(a)}\colon \mathfrak{T}_{(a)} \to \mathfrak{T}_{(a)}$, one for each BPS category $\mathfrak{T}_{(a)}$, such that:

a) for all $a \in I$, $\mathsf{d}_{(a)}$ is an*autoequivalence* of the triangle category $\mathfrak{T}_{(a)}$;

b) the $\{\mathsf{d}_{(a)}\}$ satisfy physical consistency conditions in the form of commutative diagrams

$$\begin{array}{ccc} \mathfrak{T}_{(a)} & \xrightarrow{\mathsf{d}_{(a)}} & \mathfrak{T}_{(a)} \\ \mathsf{c}_{(a,b)}\downarrow & & \downarrow \mathsf{c}_{(a,b)} \\ \mathfrak{T}_{(b)} & \xrightarrow{\mathsf{d}_{(b)}} & \mathfrak{T}_{(b)} \end{array} \tag{16.1}$$

Definition 40. 1) The group $\mathfrak{S}$ of *generalized auto-dualities* is the group of families $\mathsf{d}_{(a)}$ of autoequivalences satisfying eqn.(16.1) *modulo* its subgroup acting trivially on the physical observables. 2) The group $\mathbb{S}$ of *(generalized) S-dualities* is the quotient group of $\mathfrak{S}$ which acts effectively on the (UV) microscopic local degrees of freedom of the theory.

Remark 16.1.1. With our definition of the S-duality group, the Weyl group of the flavor group is always part of the duality group $\mathbb{S}$. It action on the free part of the cluster Grothendieck group is the natural one on the weight lattice.

Example 50. With this definition, the group $\mathbb{S}$ for $SU(2)$ SQCD with $N_f = 4$ is [75]

$$\mathbb{S}_{SU(2),\, N_f=4} = SL(2,\mathbb{Z}) \rtimes \text{Weyl}(SO(8)).$$

Remark 16.1.2. We shall see in **Example 54** that with this definition the S-duality group of a class $\mathcal{S}[A_1]$ theory is the tagged mapping class group of its Gaiotto surface, in agreement with the geometric picture in [123] , see also [107].

16.2 Specializing to $\mathcal{N} = 2$ in $4d$

We specialize the discussion to the case of a $4d$ $\mathcal{N} = 2$ theory having the BPS quiver property. Such a theory is associated to a mutation-class of quivers with potential, hence to the three categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, $\mathcal{C}(\Gamma)$, discussed in the previous sections. They are related by the compatibility functors s, r as in the exact sequence (4.1).

Applying **Definition 40** to the present set-up, we are lead to consider the diagram of triangle functors

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D^b\Gamma & \xrightarrow{\mathsf{s}} & \mathfrak{Per}\,\Gamma & \xrightarrow{\mathsf{r}} & \mathcal{C}(\Gamma) & \longrightarrow & 0 \\ & & d_D\downarrow & & d_{\mathfrak{P}}\downarrow & & d_{\mathcal{C}}\downarrow & & \\ 0 & \longrightarrow & D^b\Gamma & \xrightarrow{\mathsf{s}} & \mathfrak{Per}\,\Gamma & \xrightarrow{\mathsf{r}} & \mathcal{C}(\Gamma) & \longrightarrow & 0 \end{array}$$

having exact rows and commuting squares, where

$$d_D \in \text{Aut}\, D^b\Gamma, \quad d_{\mathfrak{P}} \in \text{Aut}\, \mathfrak{Per}\,\Gamma, \quad d_{\mathcal{C}} \in \text{Aut}\, \mathcal{C}(\Gamma).$$

The group $\mathfrak{S}$ is the group of such triples $(d_D, d_{\mathfrak{P}}, d_{\mathcal{C}})$ modulo the subgroup which acts trivially on the observables. The S-duality group $\mathbb{S}$ is the image of $\mathfrak{S}$ under the homomorphism

$$r\colon \mathfrak{S} \to \text{Aut}\,\mathcal{C}(\Gamma)/\text{Aut}\,\mathcal{C}(\Gamma)_{\text{trivial}}, \qquad (d_D, d_{\mathfrak{P}}, d_{\mathcal{C}}) \mapsto d_{\mathcal{C}}. \tag{16.2}$$

16.2.1 The trivial subgroup $(\mathrm{Aut}\, D^b\Gamma)_0$

We start by characterizing the subgroup $(\mathrm{Aut}\, D^b\Gamma)_0 \subset \mathrm{Aut}\, D^b\Gamma$ of 'trivial' auto-equivalences, i.e. the ones which leave the physical observables invariant. Since the Grothendieck group is identified with the IR charge lattice Λ, and charge is an observable, $(\mathrm{Aut}\, D^b\Gamma)_0$ is a subgroup of the kernel $\mathrm{Aut}\, D^b\Gamma \to \mathrm{Aut}\, K_0(D^b\Gamma)$. Next all $\varrho \in (\mathrm{Aut}\, D^b\Gamma)_0$ should leave invariant the stability condition, that is the slicing $\mathcal{P}(\phi)$, and hence the canonical heart $\mathsf{mod}\, J(Q,W)$ of $D^b\Gamma$. Since ϱ acts trivially on the Grothendieck group, it should fix all simples S_i. Hence the projection $\mathrm{Aut}\, D^b\Gamma \to \mathrm{Aut}\, D^b\Gamma/(\mathrm{Aut}\, D^b\Gamma)_0$ factors through the quotient group

$$\mathrm{Autph}\, D^b\Gamma := \mathrm{Aut}\, D^b\Gamma \Big/ \Big\{\text{autoequivalences preserving the simples } S_i \text{ (element-wise)}\Big\}.$$

An equivalence in the kernel of the projection $\mathrm{Aut}\, D^b\Gamma \to \mathrm{Autph}\, D^b\Gamma$ preserves (Q,W), the central charge Z, and the Grothendieck class λ. Hence it maps stable objects of charge λ into stable objects of charge λ. Comparing with eqn.(15.2), we see that the net effect of an autoequivalence in the kernel is to produce an automorphism of projective varieties $M_\lambda \to M_\lambda$ for each λ. Since the BPS states are the SUSY vacua of the $1d$ sigma-model with target space M_λ, this is just a change of variables in the SQM path integral, which leave invariant all physical observables[122]. Since the auto-duality groups are defined modulo transformations acting trivially on the observables, $\mathrm{Auteq}\, D^b$ is the proper auto-duality group $\mathcal{S}_{\mathrm{IR}}$ at the level of the BPS category $D^b\Gamma$.

The automorphisms of the quiver extend to automorphisms of $D^b\Gamma$; let $\mathrm{Aut}(Q)$ be the group of quiver automorphisms modulo the ones which fix the nodes. Clearly,

$$\mathrm{Autph}\, D^b\Gamma = \mathrm{Auteq}\, D^b\Gamma \rtimes \mathrm{Aut}(Q),$$

where

$$\mathrm{Auteq}\, D^b\Gamma := \mathrm{Aut}\, D^b\Gamma \Big/ \Big\{\text{autoequivalences preserving the simples } S_i \text{ (as a set)}\Big\}.$$

16.2.2 The duality groups $\mathfrak{S}$ and $\mathbb{S}$

With the notation of section 13.6, Bridgeland in [45] and Goncharov in [133] showed that the following sequence

$$0 \to \mathrm{Sph}\, D^b\Gamma \to \mathrm{Auteq}\, D^b\Gamma \to \mathrm{Aut}_Q(CEG) \to 0 \tag{16.3}$$

is exact. Here CEG stands for the cluster exchange graph (cfr. §. 13.3.1): the clusters of the cluster algebra C_Γ are the vertices of the CEG and the edges are single mutations connecting two seeds; $\mathrm{Aut}_Q(CEG)$ is the graph automorphism group that sends the quiver to itself up to relabeling of the vertices. By construction this graph is connected.

[122] The simplest example of such a negligible equivalence is the case of pure $SU(2)$ whose quiver is the Kronecker quiver, $\mathsf{Kr} = \bullet \rightrightarrows \bullet$. The stable representations associated to the W boson are the simples in the homogeneous tube which form a $\mathbb{P}^1$ family (i.e. $M_{\text{W boson}} \equiv \mathbb{P}^1$) since the W boson belongs to a *vector* superfield. Then a negligible auto-equivalence is just a projective automorphism of $\mathbb{P}^1$.

Theorem 13 (Goncharov [133], see also [162]). *One has*

$$\mathrm{Aut}_Q(CEG) \subset \mathrm{Aut}\,\mathcal{C}(\Gamma),$$

i.e. the graph automorphisms (see [13]*) are a subgroup of the autoequivalences of the cluster category.*

Note that $\mathrm{Auteq}\, D^b\Gamma \equiv \mathrm{Auteq}\,\mathfrak{Per}\,\Gamma$, the quotient group of $\mathrm{Aut}\,\mathfrak{Per}\,\Gamma$ by the subgroup fixing the Γ_i (as a set). Indeed, all autoequivalences of $\mathfrak{Per}\,\Gamma$ preserve the subcategory $D^b\Gamma$ and hence restrict to autoequivalences of the bounded category; an autoequivalence $\varrho \in \mathrm{Aut}\,\mathfrak{Per}\,\Gamma$ which does not preserve the Γ_i's restricts to an element $\bar{\varrho} \in \mathrm{Aut}\, D^b\Gamma$ which does not preserve the S_i's. Hence the restriction homomorphism

$$\mathrm{Auteq}\,\mathfrak{Per}\,\Gamma \to \mathrm{Auteq}\, D^b\Gamma,$$

is injective. On the other hand, from eqn.(16.3) we see that all autoequivalences in $\mathrm{Auteq}\, D^b\Gamma$ extend to autoequivalences in $\mathrm{Auteq}\,\mathfrak{Per}\,\Gamma$: indeed, the objects which are spherical in the subcategory $D^b\Gamma$ remain spherical and 3-CY in the larger category $\mathfrak{Per}\,\Gamma$ (cfr. eqn.(13.8)), so the auto-equivalences is $\mathrm{Sph}\, D^b\Gamma$ extend to $\mathfrak{Per}\,\Gamma$; the autoequivalences in $\mathrm{Aut}_Q(CEG)$ are induced by quiver mutations, and hence induce auto-equivalences of $\mathfrak{Per}\,\Gamma$.

Comparing with our discussion around eqn.(16.2) we conclude:

Corollary 16.2.1. *For a 4d $\mathcal{N}=2$ theory with the BPS quiver property*

$$\mathfrak{S} \cong \mathrm{Auteq}\, D^b\Gamma \rtimes \mathrm{Aut}(Q), \tag{16.4}$$

$$\mathbb{S} \cong \mathrm{Aut}_Q(CEG) \rtimes \mathrm{Aut}(Q). \tag{16.5}$$

16.2.3 Example: the group $\mathbb{S}$ for $SU(2)$ $\mathcal{N}=2^*$

The mutation class of $SU(2)$ $\mathcal{N}=2^*$ contains a single quiver, the Markoff one

$$Q_{\mathrm{Mar}} \equiv \begin{array}{ccc} \bullet_1 & \Longrightarrow & \bullet_2 \\ \Uparrow & \swarrow\!\!\swarrow & \\ \bullet_3 & & \end{array}$$

which is the quiver associated to the once punctured torus [67,117]. Clearly $\mathrm{Aut}(Q_{\mathrm{Mar}}) \cong \mathbb{Z}_3$, while all mutations leave Q_{Mar} invariant up to a permutation of the nodes. Consider the covering graph $\widetilde{CEG}$ of CEG where we do not mod out the permutations of the nodes. Then $\widetilde{CEG}$ is the trivalent tree whose edges are decorated by $\{1,2,3\}$, the number attached to an edge corresponding to the nodes which gets mutated along that edge. One can check that there are no identifications between the nodes of this tree.

One may compare this ($\{1,2,3\}$-decorated) trivalent tree with the ($\{1,2,3\}$-decorated) standard triangulation of the upper half-plane $\mathbb{H}$ given by the reflections of the geodesic triangle of vertices $0, 1, \infty$ (see ref. [115]). One labels the nodes of a triangle of the standard triangulation by elements of $\{1,2,3\}$, and then extends (uniquely) the numeration to all other vertices so that the vertices of each triangle get different labels. The sides of a triangle are numbered as their opposite vertex. The dual of this decorated triangulation is our decorated trivalent graph $\widetilde{CEG}$, see figure 5. The

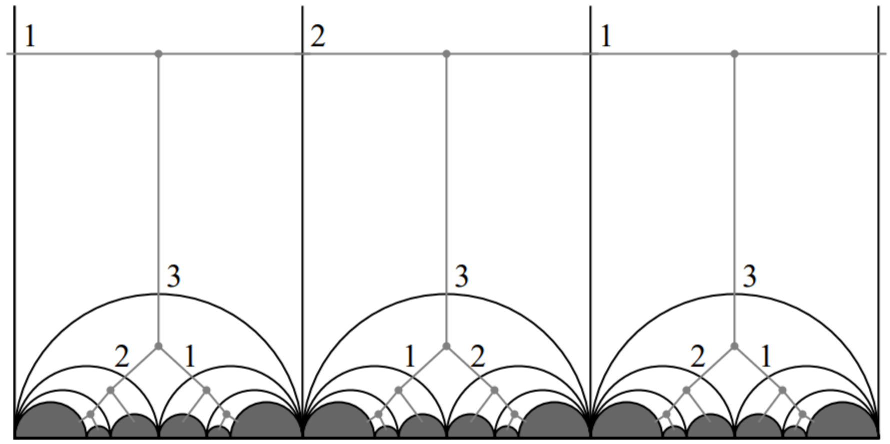


Figure 5: The modular triangulation of the upper half plane and its dual graph $\widetilde{CEG}$. The picture is reproduced from [115].

arithmetic subgroup of the hyperbolic isometry group, $PGL(2,\mathbb{Z}) \subset PGL(2,\mathbb{R})$ preserves the standard triangulation of $\mathbb{H}$ while permuting the decorations $\{1,2,3\}$. Since permutations are valid S-dualities, we get

$$\mathbb{S} \cong PGL(2,\mathbb{Z}) \cong PSL(2,\mathbb{Z}) \rtimes \mathbb{Z}_2$$

where the extra $\mathbb{Z}_2$ may be identified with the Weyl group of the flavor $SU(2)$. Thus we recover as S-duality group in the usual sense ($\equiv$ the kernel of $\mathbb{S} \to \text{Weyl}(F)$) the modular group $PSL(2,\mathbb{Z})$ [228]. In the case of $SU(2)$ $\mathcal{N}=2^*$ we have

$$K_0(\mathcal{C}_{\text{Mar}}) \equiv \text{coker}\, B \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z},$$

as expected for a quark in the adjoint representation (since $\pi_1(G_{\text{eff}}) = \mathbb{Z}_2$), with the free part the weight lattice of $SU(2)_{\text{flav}}$. Hence the flavor Weyl group acts on $K_0(\mathcal{C}_{\text{Mar}})$ as -1, that is, as the cluster auto-equivalence [1]. Notice that the cluster category is 2-periodic, as expected for a UV SCFT with integral dimensions Δ.

16.3 Relation to duality walls and $3d$ mirrors

The UV S-duality group $\mathbb{S}$ has a clear interpretation: it is the usual S-duality group of the $\mathcal{N}=2$ theory (twisted by the flavor Weyl group). What about its IR counterpart $\mathfrak{S}$?

For Argyres-Douglas models we can put forward a precise physical interpretation based on the findings of [70]. Similar statements should hold in general.

Given an element of the S-duality group, $\sigma \in \mathbb{S}$ we may construct a half-BPS duality wall in the $4d$ theory [97, 98, 240]: just take the theory for $x_3 < 0$ to be the image through σ of the theory for $x_3 > 0$ and adjust the field profiles along the hyperplane $x_3 = 0$ in such a way that the resulting Janus configuration is $\frac{1}{2}$-BPS. It is a domain wall interpolating between two dual $\mathcal{N}=2$ theories in complementary half-spaces. On the wall live suitable $3d$ degrees of freedom interacting with the bulk $4d$ fields on both sides [97, 98, 240]. In this construction we may use a UV duality as well as an IR one [98]. Hence we expect to get duality walls for all elements of $\mathfrak{S}$. An element $\mathfrak{s} \in \mathfrak{S}$ acts non-trivially on the central charge Z so, in general, as we go from $x_3 = -\infty$ to $x_3 = +\infty$ we induce a non-trivial flow of the central charge Z in the space of stability functions. If $\lim_{x_3 \to \pm\infty} Z$ is such that all the bulk degrees of freedom get an infinite mass and decouple, we remain with a pure $3d$ $\mathcal{N} = 2$ theory on the wall. Of course this may happen only for special choices of $\mathfrak{s}$. Thus we may use (suitable) $4d$ dualities to engineer $3d$ $\mathcal{N} = 2$ QFTs.

The engineering of $3d$ $\mathcal{N} = 2$ theories as a domain wall in a $4d$ $\mathcal{N} = 2$ QFT, by central-charge flow in the normal direction, is precisely the set-up of ref. [70]. In that paper one started from a $4d$ Argyres-Douglas of type $\mathfrak{g} \in ADE$. The Z-flow along the x_3-axis was such that asymptotic behaviors as $x_3 \to -\infty$ and $x_3 \to +\infty$ were related in the UV by the action of the quantum half-monodromy $\mathbb{K}$, that is, in the categorical language by the shift $[1] \in \mathbb{S}$. Two choices of IR duality elements, $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$, which produce the half-monodromy in the UV, differ by an element of the spherical twist group (cfr. eqn.(16.3))

$$\mathfrak{s}' \mathfrak{s}^{-1} \in \mathrm{Sph}\, D^b.$$

The arguments at the end of §. 13.5.1 imply that for Argyres-Douglas of type $\mathfrak{g}$ the group $\mathrm{Sph}\, D^b$ is isomorphic to the Artin braid group of type $\mathfrak{g}$, $\mathcal{B}_{\mathfrak{g}}$.

As the title of ref. [70] implies, the explicit engineering of a $3d$ $\mathcal{N} = 2$ theory along those lines requires a specification of a braid, i.e. of an element of $\mathcal{B}_{\mathfrak{g}}$. More precisely, in §.5.3.2 of ref. [70] is given an explicit map (for $\mathfrak{g} = A_r$)

$$\big(\text{a braid in } \mathcal{B}_{\mathfrak{g}}\big) \longleftrightarrow \big(\text{a } 3d\ \mathcal{N} = 2 \text{ Lagrangian}\big).$$

So the Lagrangian description/Z-flow engineering of the $3d$ theories are in one-to-one correspondence with the $\mathfrak{s} \in \mathfrak{S}$ such that $r(\mathfrak{s}) = [1]$. It is natural to think of the $3d$ Lagrangian theory associated to $\mathfrak{s} \in \mathfrak{S}$ as the duality wall associated to the IR duality $\mathfrak{s}$. Distinct $\mathfrak{s}$ lead to $3d$ theories which superficially look quite different. However, in this context, *$3d$ mirror symmetry* is precisely the statement that two theories defined by different IR dualities $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$ which induce the same UV duality, $r(\mathfrak{s}') = r(\mathfrak{s})$ produce equivalent $3d$ QFTs. From this viewpoint $3d$ mirror symmetry is a bit tautological, since the condition $r(\mathfrak{s}') = r(\mathfrak{s})$ just says that the two $3d$ theories have the same description in terms of $4d$ microscopic degrees of freedom.

16.4 S-duality for Argyres-Douglas and $SU(2)$ gauge theories

When (Q, W) is in the mutation-class of an ADE Dynkin graph (corresponding to an Argyres-Douglas model [3,4]) or of an $\widehat{A}\widehat{D}\widehat{E}$ acyclic affine quiver (corresponding to $SU(2)$ SYM coupled to matter such that the YM coupling is asymptotically-free [3]) to get $\mathbb{S}$ we can equivalently study the automorphism of the transjective component of the AR quiver associated to the cluster category $\mathcal{C}(\Gamma)$: the inclusion above is due to the fact that we only consider the transjective component:

Theorem 14 (See [13])**.** *Let C be an acyclic cluster algebra and Γ_{tr} the transjective component of the Auslander-Reiten quiver of the associated cluster category $\mathcal{C}(\Gamma)$. Then $\mathrm{Aut}^+ C$ is the quotient of the group $\mathrm{Aut}\,\Gamma_{\mathrm{tr}}$ of the quiver automorphisms of Γ_{tr}, modulo the stabilizer $\mathrm{Stab}(\Gamma_{\mathrm{tr}})_0$ of the points*

Q	$\mathrm{Aut}_Q(CEG)$	Q	$\mathrm{Aut}_Q(CEG)$
$A_{n>1}$	$\mathbb{Z}_{n+3}$	D_4	$\mathbb{Z}_4 \times S_3$
$D_{n>4}$	$\mathbb{Z}_n \times \mathbb{Z}_2$	E_6	$\mathbb{Z}_{14}$
E_7	$\mathbb{Z}_{10}$	E_8	$\mathbb{Z}_{16}$
$\hat{A}_{p,q}$	$H_{p,q}$	$\hat{A}_{p,p>1,1}$	$H_{p,p} \rtimes \mathbb{Z}_2$
$\hat{D}_4$	$\mathbb{Z} \times S_4$	$\hat{A}_{1,1}$	$\mathbb{Z}$
$\hat{D}_{n>4}$	G	$\hat{E}_6$	$\mathbb{Z} \times S_3$
$\hat{E}_7$	$\mathbb{Z} \times \mathbb{Z}_2$	$\hat{E}_8$	$\mathbb{Z}$

Table 18: S-duality groups for $\mathcal{N} = 2$ theory with an acyclic quiver.

of this component. Moreover, if $\Gamma_{\rm tr} \cong \mathbb{Z}\Delta$, *where* Δ *is a tree or of type* $\hat{A}$ *then*

$$\mathrm{Aut}\, C = \mathrm{Aut}^{+} C \rtimes \mathbb{Z}_2$$

and this semidirect product is not direct.

In order to understand why this is the relevant component, we first recall that the Auslander-Reiten quiver of a cluster-tilted algebra always has a unique component containing local slices, which coincides with the whole Auslander-Reiten quiver whenever the cluster-tilted algebra is representation-finite. This component is called the *transjective component* and an indecomposable module lying in it is called a transjective module. With this terminology, the main result is:

Theorem 15 (See [11])**.** *Let* C *be a cluster-tilted algebra and* M, N *be indecomposable transjective* C*-modules. Then* M *is isomorphic to* N *if and only if* M *and* N *have the same dimension vector.*

Therefore, since the dimension vector is the physical charge, we focus our attention to this class of autoequivalences. The classification results are summarized in table 18 where

$$H_{p,q} := \langle r, s | r^p = s^q,\ sr = rs \rangle$$
$$G = \big\langle \tau, \sigma, \rho_1, \rho_n | \rho_1^2 = \rho_n^2 = 1, \tau\rho_1 = \rho_1\tau, \tau\rho_n = \rho_n\tau, \tau\sigma = \sigma\tau, \sigma^2 = \tau^{n-3}, \rho_1\sigma = \sigma\rho_n, \sigma\rho_1 = \rho_n\sigma \big\rangle$$

Example 51 ($SU(2)$ with $N_f \leq 3$)**.** $SU(2)$ SQCD with $N_f = 0, 1, 2, 3$ correspond, respectively, to the following four affine $\mathcal{N} = 2$ theories [67]

$$\hat{A}_{1,1}, \quad \hat{A}_{2,1}, \quad \hat{A}_{2,2}, \quad \hat{D}_4.$$

A part for the flavor Weyl group $\mathrm{Weyl}(\mathfrak{spin}(2N_f))$ (cfr. **Example 48**) we get a duality group $\mathbb{Z}$ generated by the shift [1]. As discussed around eqn.(15.7), this is equivalent to the shift of the Yang-Mills angle θ

$$\theta \to \theta - 4\pi + N_f \pi.$$

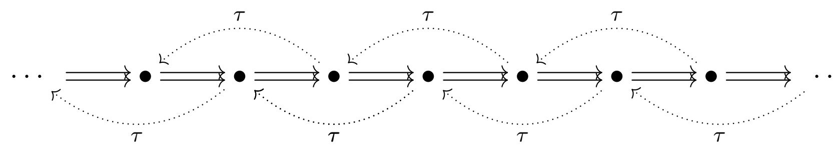


Figure 6: The translation quiver $\mathbb{Z}\hat{A}_{1,1}$ ($\equiv$ the AR quiver of the transjective component of the cluster category for pure $SU(2)$). Dotted arrows stands for the action of the AR translation τ. Clearly τ is the translation to the left by 2 nodes. The auto-equivalence ξ is translation to the left by 1 node: $\xi^2 = \tau$.

The case $N_f = 0$ is special; physically one expects that the shift of θ by -2π should also be a valid S-duality. This shift should correspond to an auto-equivalence ξ of the $N_f = 0$ cluster category with $\xi^2 = \tau$. Indeed, this is what one obtains from the automorphism of the transjective component see figure 6. Alternatively, we may see the cluster category of pure $SU(2)$ as the category of coherent sheaves on $\mathbb{P}^1$ endowed with extra *odd* morphisms [32]. In this language τ acts as the tensor product with the canonical bundle $\tau\colon \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{K}$ (cfr. eqn.(15.6)). Let $\mathcal{L}$ be the unique *spin structure* on $\mathbb{P}^1$; we have the obvious auto-equivalence $\xi\colon \mathcal{A} \mapsto \mathcal{A} \otimes \mathcal{L}$. From $\mathcal{L}^2 = \mathcal{K}$ we see that $\xi^2 = \tau$.

17 Computer algorithm to determine the S-duality group

The identification of the S-duality group with $\mathrm{Aut}_Q(CEG)$ yield a combinatoric characterization of S-dualities which leads to an algorithm to search S-dualities for an arbitrary $\mathcal{N} = 2$ model having a BPS quiver. This algorithm is similar in spirit to the *mutation algorithm* to find the BPS spectrum [3] but in a sense more efficient. The algorithm may be easily implemented on a computer; if the ranks of the gauge and flavor groups are not too big (say < 10), running the procedure on a laptop typically produces the generators of the duality group in a matter of minutes.

17.1 The algorithm

The group $\mathrm{Aut}_Q(CEG)$ may be defined in terms of the transformations under quiver mutations of the d-vectors which specify the denominators of the generic cluster variables [108]. The actions of the elementary quiver mutation at the k–th node, μ_k, on the exchange matrix B and the d-vector d_i are

$$\mu_k(B)_{ij} = \begin{cases} -B_{ij}, & i = k \text{ or } j = k \\ B_{ij} + \max[-B_{ik}, 0]\, B_{kj} + B_{ik}\, \max[B_{kj}, 0] & \text{otherwise.} \end{cases} \tag{17.1}$$

$$\mu_k(d)_l = \begin{cases} d_l, & l \neq k \\ -d_k + \max\Big[\sum_i \max\big[B_{ik}, 0\big] d_i, \sum_i \max\big[-B_{ik}, 0\big] d_i\Big] & l = k \end{cases} \tag{17.2}$$

A quiver mutation $\mu = \mu_{k_s}\mu_{k_{s-1}} \cdots \mu_{k_1}$ is the composition of a finite sequence of elementary quiver mutations $\mu_{k_1}, \mu_{k_2}, \cdots, \mu_{k_s}$. We write Mut for the set of all quiver mutations. $\mathrm{Aut}_Q(CEG)$ is the group of quiver mutations which leave invariant the quiver Q up to a permutation π of its

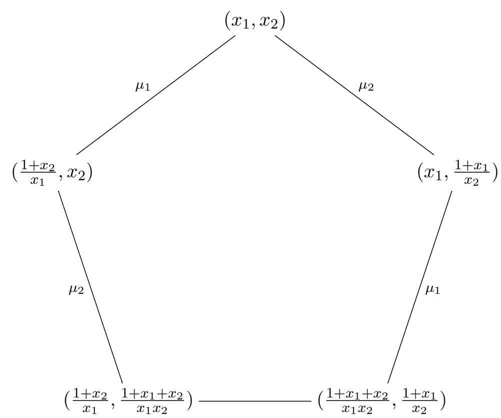


Figure 7: The CEG of the A_2 Argyres-Douglas theory.

nodes, modulo the ones which leave the d-vector invariant up to π:

$$\mathrm{Aut}_Q(CEG) = \frac{\Big\{\mu \in \mathsf{Mut} \;\Big|\; \exists\, \pi \in S_n\colon \mu(B)_{i,j} = B_{\pi(i),\pi(j)}\Big\}}{\Big\{\mu \in \mathsf{Mut} \;\Big|\; \exists\, \pi \in S_n\colon \mu(B)_{i,j} = B_{\pi(i),\pi(j)} \text{ and } \mu(d)_i = d_{\pi(i)}\Big\}}, \tag{17.3}$$

while $\mathbb{S} = \mathrm{Aut}_Q(CEG) \rtimes \mathrm{Aut}(Q)$.

Example 52 (A_2 cluster automorphisms)**.** Consider the quiver $\bullet_1 \to \bullet_2$. The CEG is the pentagon in figure 7: every vertex is associated to a quiver of the form $\bullet_1 \to \bullet_2$ or $\bullet_2 \to \bullet_1$. Thus, in this case every sequence of mutations gives rise to a cluster automorphism. For example, consider μ_1: the quiver nodes get permuted under $\pi = (1\ 2)$. We explicitly check – for example using Keller applet[123] – that

$$(\mu_{\text{source}})^5 = \mu_1\mu_2\mu_1\mu_2\mu_1 = 1$$

since $(\mu_{\text{source}})^5$ leaves the d-vectors invariant. From figure 7 one sees that $\mathbb{Z}_5$ is indeed the full automorphism group of the CEG of A_2. This result is coherent with the analysis leading to table 18, as well as with the tagged mapping class group of the associated Gaiotto surface, see **Example 66**.

The explicit expression (17.3) of the S-duality group is the basis of a computer search for S-dualities. Schematically: let the computer generate a finite sequence of nodes of Q, $k_1, \cdots, k_s$, then construct the corresponding mutation $\mu_{k_s}\mu_{k_{s-1}}\cdots\mu_{k_1} = \mu$, and check whether it leaves the exchange matrix B invariant up to a permutation π. If the answer is *yes,* let the machine check whether $\mu(d)_i \neq d_{\pi(i)}$. If the answer is again *yes* the computer has discovered a non-trivial S-duality and prints it. Then the computer generates another sequence and go cyclically through the same steps again and again. After running the procedure for some time t, we get a print-out with a list $\mathcal{L}_t$ of non-trivial S-dualities of our $\mathcal{N} = 2$ theory. A Mathematica Code performing this routine is presented in Appendix D.

If the S-duality group is finite (and not too huge) $\mathcal{L}_t$ will contain the full list of S-dualities. However, the most interesting S-duality groups are *infinite*, and the computer cannot find all its

[123]See `https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/`.

elements in finite time. This is not a fundamental problem for the automatic computation of the S-duality group. Indeed, the S-duality groups, while often infinite, are expected to be *finitely generated,* and in fact finitely presented. If this is the case, we need only that the finite list $\mathcal{L}_t$ produced by the computer contains a complete set of generators of $\mathbb{S}$. Taking various products of these generators, and checking which products act trivially on the d-vectors, we may find the finitely many relations. The method works better if we have some physical hint on what the generators and relations may be.

Of course, the duality group obtained from the computer search is *a priori* only a subgroup of the actual $\mathbb{S}$ because there is always the possibility of further generators of the group which are outside our range of search. However, pragmatically, running the procedure for enough time, the group one gets is the full one at a high confidence level.

17.2 Sample determinations of S-duality groups

We present a sample of the results obtained by running our Mathematica Code.

Example 53 ($SU(2)$ $\mathcal{N}=2^*$ again)**.** The CEG automorphism group for this model was already described in §. 16.2.3. Recall that $PSL(2,\mathbb{Z})$ is the quotient of the braid group over three strands, $\mathcal{B}_3$ by its center $\boldsymbol{Z}(\mathcal{B}_3)$

$$PSL(2,\mathbb{Z}) \cong \mathcal{B}_3/\boldsymbol{Z}(\mathcal{B}_3).$$

Running our algorithm for a short time returns a list of dualities which in particular contains the two standard generators of the braid group $\sigma_1, \sigma_2 \in \mathcal{B}_3$, which correspond to the following sequences of elementary quiver mutations:

$$\sigma_1 := \mu_1\mu_2, \qquad \text{and} \qquad \sigma_2 := \mu_1\mu_3, \text{ with permutation } \pi = (1\,3\,2). \tag{17.4}$$

One easily checks the braid relation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \quad \text{up to permutation,}$$

as well as that the generator of the center $\boldsymbol{Z}(\mathcal{B}_3)$, $(\sigma_2\sigma_1)^3$, acts trivially on the cluster category: indeed, it sends the initial dimension vector $\vec{d} = -Id_{3\times 3}$ to itself. From eqn.(17.4) we conclude that the two S-dualities σ_1, σ_2 generate a $PSL(2,\mathbb{Z})$ duality (sub)group. In facts, $\mathbb{S}/PSL(2,\mathbb{Z}) \cong \mathbb{Z}_2$ where the class of the non-trivial $\mathbb{Z}_2$ element may be represented (say) by μ_1. Indeed the map

$$\mathbb{S} \to \mathbb{Z}_2 \equiv \mathrm{Weyl}(F_{\mathrm{flav}})$$

send the mutation μ to $(-1)^{\ell(\mu)}$, where the *length* $\ell(\mu)$ of $\mu \equiv \mu_{k_s}\mu_{k_{s-1}}\cdots\mu_{k_1}$ is s (length is well defined mod 2).

Example 54 ($SU(2)$ with $N_f=4$)**.** We use the quiver in figure 8 where for future reference we also draw the corresponding ideal triangulation of the sphere with 4 punctures [67]. The following two even-length sequences of mutations leave the quiver invariant:

$$S = \mu_2\mu_3\mu_2\mu_0\mu_2\mu_5\mu_3\mu_0,$$
$$T = \mu_5\mu_2\mu_0\mu_3\mu_5\mu_3\mu_4\mu_2\mu_4\mu_1\mu_4\mu_2\mu_4\mu_5\mu_1\mu_2.$$

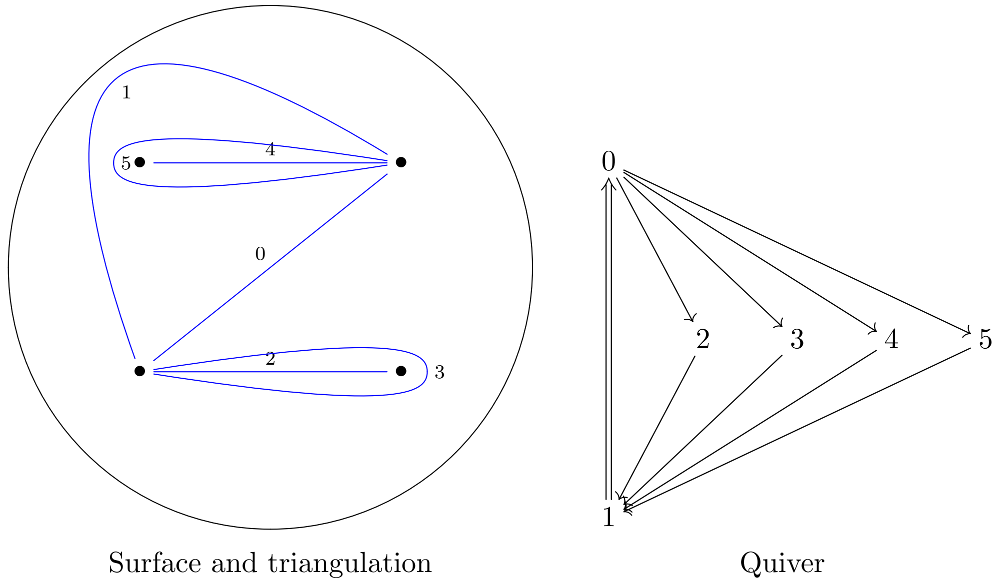


Figure 8: The Gaiotto surface (S, M) of the theory $SU(2)$, $N_f = 4$ and its associated quiver.

These sequences of mutations satisfy the following relations:

$$S^4 = 1, \quad (ST)^6 = 1, \quad T \text{ has infinite order.}$$

Moreover, T and S commute with S^2 and $(ST)^3$. Write $\mathbb{Z}_2 \times \mathbb{Z}_2$ for the subgroup generated by S^2 and $(ST)^3$. Then we have

$$1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \langle\, S, T\,\rangle \to PSL(2, \mathbb{Z}) \to 1.$$

Again this shows that the duality sub-group $\langle\, S, T\,\rangle$ generated by S and T is equal to the the mapping class group of the sphere with four punctures (cfr. **Proposition 2.7** of [113]). In fact one has $\mathrm{Aut}_Q(CEG)/\langle\, S, T\,\rangle \cong \mathbb{Z}_2$; geometrically (see next section) the extra $\mathbb{Z}_2$ arises because for class $\mathcal{S}[A_1]$ theories $\mathrm{Aut}_Q(CEG)$ is the *tagged* mapping class group of the corresponding Gaiotto surface (Bridgeland theorem [45]); the extra $\mathbb{Z}_2$ is just the change in tagging. This extra $\mathbb{Z}_2$ is also detected by the computer program which turns out dualities of order 12 and 8 which are not contained in $\langle\, S, T\,\rangle$ but in its $\mathbb{Z}_2$ extension. Taking into account the S_4 automorphism of the quiver, we recover $PSL(2, \mathbb{Z}) \ltimes \mathrm{Weyl}(\mathfrak{spin}(8))$ with the proper triality action of the modular group on the flavor weights [228]. For an alternative discussion of the S-duality group of this model as the automorphism group of the corresponding cluster category, see ref. [75].

Example 55 (E_6 Minahan-Nemeschanski)**.** This SCFT is the T_3 theory, that is, the Gaiotto theory obtained by compactifying the $6d$ $(2, 0)$ SCFT of type A_2 on a sphere with three maximal punctures [123] . Since the three-punctured sphere is rigid, geometrically we expect a finite S-duality group. The homological methods of [51] confirm this expectation. The computer search produced a list of group elements of order $2, 3, 4, 5, 6, 8, 9, 10, 12$ and 18. Since, with our definition, the S-duality group should contain the Weyl group of E_6, we may compare this list with the list of orders of elements of Weyl(E_6),

$$\{2,3,4,5,6,8,9,10,12\}.$$

We see that the two lists coincide, except for 18. Thus the S-duality group is slightly larger than the Weyl group, possibly just Weyl$(E_6)\rtimes\mathbb{Z}_2$, where $\mathbb{Z}_2$ is the automorphism of the Dynkin diagram. Notice that this is the largest group which may act on the free part of the cluster Grothendieck group (since it should act by isometries of the Tits form).

Example 56 (Generic $T_{\mathfrak{g}}$ theories)**.** By the same argument as in the previous **Example**, we expect the S-duality group to be finite for all $T_{\mathfrak{g}}$ ($\mathfrak{g}\in ADE$) theories. We performed a few sample computer searches getting agreement with the expectation.

Example 57 (E_7 Minahan-Nemeschanski)**.** The computer search for this example produced a list of group elements of order $2,3,4,5,6,7,8,9,10,12,14,15,18$ and 30. Since our S-duality group contains the Weyl group of the flavor E_7, we compare this list with the list of orders of elements of Weyl(E_7), $\{2,3,4,5,6,7,8,9,10,12,14,15,18,30\}$. We see that the two lists coincide. It is reasonable to believe that the full S-duality group coincides with Weyl(E_7). This is also the largest group preserving the flavor Tits form.

17.3 Asymptotic-free examples

As an appetizer, let us consider a $\mathcal{N}=2$ gauge theory with a gauge group of the form $SU(2)^k$ coupled to (half-)hypermultiplets in some representation of the gauge group so that all Yang-Mills couplings g_i ($i=1,\dots,k$) have strictly negative β-functions. As discussed in §. 15.3, the fact that the theory is asymptotically-free means that its cluster category $\mathcal{C}$ is not periodic. However, its Coulomb branch is parametrized by k operators whose dimension in the UV limit $g_i\to 0$ becomes $\Delta=2$. As in **Example 46**, this implies the existence of a 1-periodic sub-category $\mathcal{F}(1)\subset\mathcal{C}$. Iff all YM couplings g_i are strictly asymptotically-free, the category $\mathcal{F}(1)$ consists of k copies of the 1-periodic sub-category of pure $SU(2)$, **Example 46**. In such an asymptotic-free theory the S-duality group is bound to be 'small' since all auto-equivalence σ of the cluster category should preserve the 1-periodic sub-category $\mathcal{F}(1)$; therefore, up to (possibly) permutations of the various $SU(2)$ gauge factors, σ should restrict to a subgroup of autoequivalences of the periodic category $\mathcal{F}(1)_{\text{pure}}$ of pure $SU(2)$ SYM. As we saw in **Example 51**, the S-dualities corresponding to shifts of the Yang-Mills angle θ preserve[124] the subcategory $\mathcal{F}(1)_{\text{pure}}$. Thus besides shifts of the various theta angles, permutations of identical subsectors, and flavor Weyl groups/Dynkin graph automorphism, we do not expect additional S-dualities in these models. Let us check this expectaction against the computer search for dualities in a tricky example.

Example 58 ($SU(2)^3$ with $\frac{1}{2}(\mathbf{2},\mathbf{2},\mathbf{2})$)**.** A quiver for this model is given in figure 9. In this case the cluster Grothendieck group $K_0(\mathcal{C}_{\text{pris}})$ is pure torsion, since a single half-hyper carries no flavor charge. The three $SU(2)$ gauge couplings g_i are asymptotically-free and the cluster category $\mathcal{C}_{\text{pris}}$ is *not* periodic but it contains the 1-periodic subcategory $\mathcal{F}(1)\subset\mathcal{C}_{\text{pris}}$ described above[125]. The

[124] Physically this is obvious. Mathematically, consider e.g. the shift shift $\theta\to\theta-4\pi+N_f\pi$ in $SU(2)$ with N_f flavors. It corresponds to the auto-equivalence $\mathcal{A}\mapsto\mathcal{A}[1]$, which acts trivially on the 1-periodic subcategory.

[125] Notice that there is no periodic sub-category associated to the quark sector; this is related to the absence of conserved flavor currents in this model.

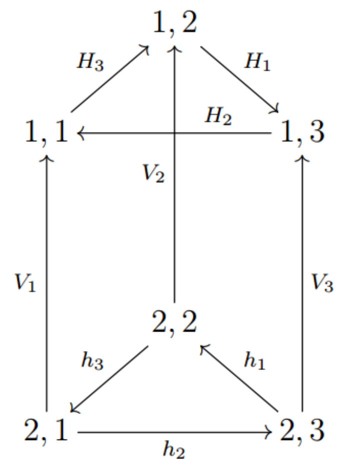


Figure 9: A quiver Q_{pris} for the gauge theory with $G_{\text{gauge}} = SU(2)^3$ coupled to a half-hyper in the three-fundamental. The superpotential for Q_{pris} is $W_{\text{pris}} = \text{Tr}(H_1H_2H_3) + \text{Tr}(h_1h_2h_3)$.

S-duality group is then expected to consists of permutations of the three $SU(2)$'s and the three independent shifts of the Yang-Mills angles $\theta_i \to \theta_i - 2\pi$, that is, $\mathbb{S} = S_3 \ltimes \mathbb{Z}^3$.

The computer algorithm produced the following three commuting generators of the cluster automorphism group of infinite

$$\theta_1 = \mu_{23}\mu_{22}\mu_{11}, \qquad \theta_2 = \mu_{21}\mu_{23}\mu_{12}, \qquad \theta_3 = \mu_{22}\mu_{21}\mu_{13}.$$

These three generators are identified with the three θ-shifts.

Remark 17.3.1. Since the model is of class $\mathcal{S}[A_1]$ (with irregular poles), the S-duality group may also be computed geometrically (see section 7). The computer result is of course consistent with geometry: each θ_i translation correspond to a twist around one of the three holes on the sphere: their order is clearly infinite and the three twists commute with one another.

17.4 Q-systems as groups of S-duality

The above discussion may be generalized to all $\mathcal{N} = 2$ QFTs having a weakly coupled Lagrangian formulation. If the gauge group G is a product of k simple factors G_i, we expect the S-duality group to contain a universal subgroup $\mathbb{Z}^k$ consisting of shifts $\theta_i \to \theta_i - b_i\pi$, with b_i the β-function coefficient of the i-th YM coupling. One may run the algorithm and find the universal subgroup; however, just because it is *universal,* its description in terms of quiver mutations also has a universal form which is easy to describe.

We begin with an example.

Example 59 (Pure SYM: simply-laced gauge group)**.** If the gauge group G is simply-laced, the

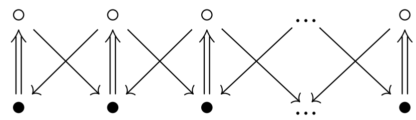


Figure 10: The BPS quiver for pure SYM theory with gauge group $SU(N)$.

exchange matrix of its quiver may be put in the form [4, 69, 73][126]

$$B = \left(\begin{array}{c|c} 0 & C \\ \hline -C & 0 \end{array}\right) \equiv C \otimes i\sigma_2, \qquad C \equiv \text{the Cartan matrix of } G.$$

For instance, the quiver for $SU(N)$ SYM is represented in figure 10. These quivers are *bipartite*: we may color the nodes black and white so that a node is linked only to nodes of the opposite color. Quiver mutations at nodes of the same color commute, so the product

$$\nu = \prod_{i \text{ white}} \mu_i$$

is well-defined. Moreover interchanging (black) $\leftrightarrow$ (white) yields the opposite quiver Q^{opp} which is isomorphic to Q *via* the node permutation $\pi = \mathbf{1} \otimes \sigma_1$. The effect of the canonical mutation ν on the quiver is to invert all arrows i.e. it gives back the same quiver up to the involution π. Thus ν corresponds to an universal duality of pure SYM. One checks that it has infinite order, i.e. generates a subgroup of S-dualities isomorphic to $\mathbb{Z}$.

This sub-group $\mathbb{Z}$ of S-dualities has different physical interpretations/applications in statistical physics [95, 96, 158] as well as in the context of the Thermodynamical Bethe Ansatz [65] . Indeed, consider its index 2 subgroup generated by the square of ν

$$\nu^2 = \prod_{j \text{ black}} \mu_j \prod_{i \text{ white}} \mu_i.$$

The repeated application of the S-duality ν^2 generates a recursion relation for the cluster variables which is known as the *Q-system of type G*. It has deep relation with the theory of quantum groups; moreover it generates a linear recursion relation of finite length and has many other "magical" properties [65, 95, 96].

We claim that the duality ν^2 corresponds to a shift of θ. Indeed, the cluster category in this case is the triangular hull of the orbit category of $D^b(\mathsf{mod}\, \mathbb{C}\hat{A}_{1,1} \otimes \mathbb{C}G)$ and ν^2 corresponds to the auto-equivalence $\tau \otimes \mathrm{Id}$ [51]. Comparing the action of $\tau \otimes \mathrm{Id}$ in the covering category with the Witten effect (along the lines of §.15.3.5) one gets the claim.

Example 60 (SYM with non-simply laced gauge group)**.** The authors of ref. [96] defined Q-systems also for non-simply laced Lie groups. To a simple Lie group G one associates a quiver and a mutation ν^2 which generates a group $\mathbb{Z}$ which has all the required "magic" properties. In ref. [71] it was

[126] In particular, coker $B = \boldsymbol{Z}(G)^\vee \oplus \boldsymbol{Z}(G)$ is the correct 't Hooft group for pure SYM.

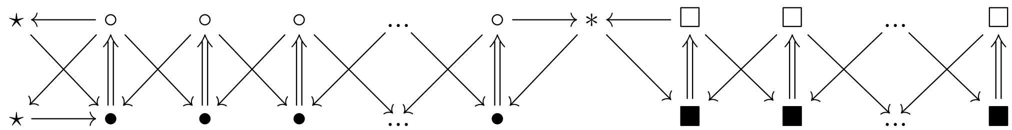


Figure 11: The BPS quiver of $SU(N)\times SU(M)$ SQCD with two quarks ($\star$ nodes) in the fundamental representation of $SU(N)$ and one (the $*$ node) in the bifundamental representation. The number of $\circ$ (resp. $\square$) is N (resp. M).

shown that the non-simply-laced Q-system does give the quiver description of the BPS sectors of the corresponding SYM theories. The Q-system group is again the group of S-dualities corresponding to θ-shifts.

Example 61 (General $\mathcal{N}=2$ SQCD models)**.** We may consider the general Lagrangian case in which the gauge group is a product of simple Lie groups, $\prod_j G_j$ and we have hypermultiplets in some representation of the gauge group. The quivers for such a theory may be found in refs. [4, 71, 72]. For instance figure 11 shows the quiver for $SU(M)\times SU(N)$ gauge theory coupled to 2 flavors of quark in the $(\boldsymbol{N},\boldsymbol{1})$ and a quark bifundamental in the $(\bar{\boldsymbol{N}},\boldsymbol{M})$. It is easy to check that the two canonical mutations of the subquivers associated to the two simple factors of the gauge group

$$\nu_\circ=\prod_{i=\circ}\mu_i,\qquad \nu_\square=\prod_{i=\square}\mu_i,$$

leave the quiver invariant up to the permutation $\circ\leftrightarrow\bullet$ and, respectively, $\square\leftrightarrow\blacksquare$. The construction extends straightforwardly to any number of gauge factors G_j and all matter representations. The conclusion is that we have a canonical $\mathbb{Z}$ subgroup of the S-duality group per simple factor of the gauge group. It corresponds to shifts of the corresponding θ-angle. If the matter is such that the β-function vanishes, the full cluster category becomes periodic, and we typically get a larger S-duality group.

One can convince himself that the sequence of mutations μ does not change the quiver and that its order, in all the above cases, is infinite, as it is for the shift is the θ_j's.

18 Class $\mathcal{S}$ QFTs: Surfaces, triangulations, and categories

In this section we focus on a special class of $\mathcal{N}=2$ theories: the Gaiotto $\mathcal{S}[A_1]$ models [123] . We study them for two reasons: first of all they are interesting for their own sake, and second for these theories the three categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$, and $\mathcal{C}(\Gamma)$ have geometric constructions, directly related to the WKB analysis of [126, 127]. Comparing the categorical description with the results of refs. [126, 127] we check the correctness of our physical interpretation of the various categories and functors.

Class $\mathcal{S}[A_1]$ theories are obtained by the compactification of the $6d$ $(2,0)$ SCFT of type A_1 over a complex curve C having regular and irregular punctures [126]. If there is at least one puncture, these theory have the quiver property [67], and their quivers with superpotentials are constructed in terms of an ideal triangulation of C [173]. In the geometrical setting of Gaiotto curves, we can interpret the categories defined in section 13.5 as categories of (real) curves on the spectral cover of the Gaiotto curve C.

When only irregular punctures are present, the quiver with potential arising from these theories [173] has a Jacobian algebra which is *gentle* [10, 74], and hence all triangle categories associated to its BPS sector, eqn.(4.1), have a simple explicit description.[127] When only regular punctures are present, the $\mathcal{N} = 2$ theory has a Lagrangian formulation (which is weakly coupled in some corner of its moduli space) and is UV superconformal. In particular the corresponding cluster category is 2-periodic, as the arguments of §. 15.3.3 imply.

18.1 UV and IR descriptions

The main reference for this part is [123] .

In the deep UV a class $\mathcal{S}[A_1]$ $\mathcal{N} = 2$ theory is described by the Gaiotto curve C, namely a complex curve of genus g with a number of punctures $x_i \in C$. Punctures are of two kinds: *regular punctures* (called simply *punctures*) and *irregular* ones (called *boundaries*). The i–th boundary carries a positive integer $k_i \geq 1$ (the number of its *cilia*); sometimes it is convenient to regard regular punctures as boundaries with $k_i = 0$. Iff $k_i \leq 2$ for all i, the $\mathcal{N} = 2$ theory is a Lagrangian model with gauge group[128]

$$\mathcal{G} = SU(2)^m, \qquad m = 3g - 3 + p + 2b \quad \text{where} \begin{cases} p = \#\{\text{regular punctures}\} \\ b = \#\{\text{boundaries}\}. \end{cases} \tag{18.1}$$

If $b = 0$ the theory is superconformal in the UV, and the space of exactly marginal coupling coincides with the moduli space of genus g curves with p punctures, $\mathcal{M}_{g,p}$, whose complex dimension is m $\equiv$ the rank of the gauge group $\mathcal{G}$. Instead, if $b \geq 1$ (and $m \geq 2$), b out of the m $SU(2)$ factors in the Yang-Mills group $\mathcal{G}$ have *asymptotically free* couplings; these b YM couplings go to zero in the extreme UV, so that the UV marginal couplings are again equal in number to the complex deformations $\mathcal{M}_{g,p+b}$ of C.

If some of the boundaries have $k_i \geq 3$, we have a gauge theory with the same gauge group $SU(2)^m$ coupled to "matter" consisting, besides free quarks (in the fundamental, bi-fundamental, and three-fundamental of $\mathcal{G}$), in an Argyres-Douglas SCFT of type D_{k_i} for each boundary[129] [66] . The space of exactly marginal deformations is as before.

The IR description of the model is given by the Seiberg-Witten curve Σ which, for class $\mathcal{S}[A_1]$, is a double cover of C. More precisely, one considers in the total space of the $\mathbb{P}^1$-bundle

$$\mathbb{P}(K_C \oplus \mathcal{O}_C) \to C \qquad \text{where} \begin{cases} K_C & \text{canonical line bundle on } C \\ \mathcal{O}_C & \text{trivial line bundle on } C \end{cases}$$

[127] In facts, there is a systematic procedure, called *gentling* in ref. [74] which allow to reduce the general class $\mathcal{S}[A_1]$ model to one having a gente Jacobian algebra.

[128] When $g = 0$, the theory is defined only if $b \geq 1$ or $b = 0$ and $p \geq 3$; in case $p = 0$, $b = 1$ we require $k \geq 4$; when $g = 1$ we need $p + b \geq 1$. Except for the case $p = 0$, $b = 1$, corresponding to Argyres-Douglas of type A, m in eqn.(18.1) is ≥ 0. $m = 0$ only for Argyres-Douglas of type D [66] .

[129] Argyres-Douglas of type D_1 is the empty theory and the one of type D_2 a fundamental quark doublet.

the curve

$$\Sigma \equiv \Big\{ y^2 = \phi_2(x)\, z^2 \;\Big|\; (y,z) \text{ homogeneous coordinates in the fiber} \Big\} \to C,$$

where $\phi_2(x)$ is a quadratic differential on C with poles of degree at most $k_i + 2$ at x_i. The Seiberg-Witten differential is the tautological one

$$\lambda = \frac{y}{z}\, dx$$

whose periods in Σ yield the $\mathcal{N} = 2$ central charges of the BPS states.

The dimension of the space of IR deformations is then[130] [131]

$$s = \dim H^0(C, PK_C^2) \equiv 3g - 3 + 2p + 2b + \sum_i k_i$$
$$\text{where } P = \sum_i (k_i + 2)[x_i],$$

so that the total space of parameters, UV+IR, has dimension

$$n = m + s = 6g - 6 + 3p + 3b + \sum_i k_i. \tag{18.2}$$

There are two kinds of IR deformations, *normalizable* and *unnormalizable* ones. The unnormalizable ones correspond to deformations of the Lagrangian, while the normalizable ones to moduli space of vacua (that is, Coulomb branch parameters); their dimensions are[132]

$$s_{\text{nor}} \equiv \dim\,(\text{Coulomb branch}) = 3g - 3 + p + b + \sum_i \left(k_i - \left[\frac{k_i}{2} \right] \right),$$
$$s_{\text{un-nor}} = s - s_{\text{nor}}.$$

The double cover $\Sigma \to C$ is branched over the zeros $w_a \in C$ of the quadratic differential $\phi_2(x)$. Their number is

$$t = 4g - 4 + 2p + 2b + \sum_i k_i,$$

but their positions are constrained by the condition that the divisor $\sum_a [w_a]$ is linear equivalent to PK_C^2, so that their positions depend on only $t - g$ parameters; $\phi_2(x)$ depends on one more parameter

$$s = t - g + 1$$

since the positions of its zeros fix $\phi_2(x)$ only up to an overall scale (that is, up to the overall normalization of the Seiberg-Witten differential, which is the overall mass scale).

Therefore, up to the overall mass scale, giving the cover $\Sigma \to C$ is equivalent to specifying the

[130] This formula holds under the condition $\dim \mathcal{M}_{g,p+b} = 3g - 3 + p + b \geq 0$.
[131] Here the asymptotically-free gauge couplings are counted as IR deformations.
[132] As written, these equations hold even if the condition in footnote 130 is not satisfied. Notice that we count also the dimensions of the internal Coulomb branches of the matter Argyres-Douglas systems.

zeros $w_a \in C$ of the quadratic differential. Indeed, double covers are fixed, up to isomorphism, by their branching points. We shall refer to the points $w_a \in C$ as *decorated points* on the Gaiotto curve C.

In summary: the UV description of a class $\mathcal{S}[A_1]$ amounts to giving the datum of the Gaiotto curve C, that is, a complex structure of a genus g Riemann surface and a number of *marked* points $x_i \in C$ together with a non-negative integer k_i at each marked point. To get the IR description we have to specify, in addition, the *decorated* points $w_a \in C$ (whose divisor is constrained to be linear equivalent to PK_C^2). We may equivalently state this result in the form:

Principle. *In theories of class $\mathcal{S}[A_1]$, to go from the IR to the UV description we simply delete (i.e. forget) the decorated points of C.*

We shall see in §.18.7 below that this 'forget the decoration' prescription is precisely the map denoted r in the exact sequence of triangle categories of eqn.(4.1).

18.2 BPS states

In class $\mathcal{S}[A_1]$ $\mathcal{N}=2$ theories, the natural BPS objects are described by (real) curves η on the Seiberg-Witten curve Σ which are *calibrated* by the Seiberg-Witten differential [228]

$$\lambda = \frac{y}{z}\, dx \equiv \sqrt{\phi_2(x)}\, dx,$$

that is, they are required to satisfy the condition (we set $\phi \equiv \phi_2\,(dx)^2$),

$$\sqrt{\phi}\Big|_\eta = e^{i\theta}\, dt, \qquad \text{here } t \in \mathbb{R}, \tag{18.3}$$

for some real constant θ, and are *maximal* with respect to this condition. Being maximal, η may terminate only at marked or decorated points.[133] BPS particles have finite mass, i.e. they correspond to calibrated arcs η with $|Z(\eta)| < \infty$ where the central charge of the would be BPS state η, is

$$Z(\eta) = \int_\eta \lambda. \tag{18.4}$$

In this case, the parameter θ in eqn.(18.3) is given by $\theta = \arg Z(\eta)$. Arcs η associated to BPS particles may end only at decorations. All other maximal calibrated arcs have infinite mass and are interpreted as BPS *branes* [231].

There are two possibilities for BPS particles:

- they are closed arcs connecting zeros of ϕ. These calibrated arcs are rigid and hence correspond to BPS hypermultiplets;
- they are loops. Such calibrated arcs form $\mathbb{P}^1$-families and give rise to BPS vector multiplets.

Conserved charges. From eqn.(18.4) we see that the central charge of an arc η factors through its homology class $\eta \in H_1(\Sigma, \mathbb{Z})$. More precisely, since the Seiberg-Witten differential λ is *odd*

[133] We call marked/decorated points in Σ the pre-images of marked/decorated points on C.

under the covering group $\mathbb{Z}_2 \cong \mathsf{Gal}(\Sigma/C)$, Z factors through the free Abelian group

$$\Lambda \equiv H_1(\Sigma, \mathbb{Z})_{\text{odd}}, \tag{18.5}$$

$$\operatorname{rank} \Lambda = 2\big(g(\Sigma) - g(C)\big) + \#\big\{k_i \text{ even}\big\}. \tag{18.6}$$

Applying the Riemann-Hurwitz formula[134] to the cover $\Sigma \to C$, we see that the rank of Λ is equal to the number n of UV+IR deformations, see eqn.(18.2). In turn n is the number of conserved charges (electric, magnetic, flavor) of the IR theory. Hence the group homomorphism

$$Z \colon \Lambda \to \mathbb{C}, \qquad [\eta] \mapsto \int_\eta \lambda,$$

is the map which associates to the IR charge $\gamma \in \Lambda$ of a state of the $\mathcal{N} = 2$ theory the corresponding central charge $Z(\gamma)$. An arc with homology $[\eta] \in \Lambda$ then has 'mass'

$$\int_\eta |\lambda| \geq |Z(\eta)|$$

with equality iff and only if it is calibrated, that is, BPS.

To get the corresponding UV statements, we apply to these results our **RG principle**, that is, we forget the decorations. BPS particles then disappear (as they should form the UV perspective), while BPS branes project to arcs on the Gaiotto curve C. Several IR branes project to the same arc on C. The arcs on C have the interpretation of UV BPS *line operators*, and the branes which project to it are the objects they create in the given vacuum (specified by the cover $\Sigma \to C$) which may be dressed (screened) in various ways by BPS states, so that the IR-to-UV correspondence is many-to-one in the line sector.

The UV conserved charges is the projection of Λ; over $\mathbb{Q}$ all electric/magnetic charges are projected out by the oddness condition, and we remain with just the flavor lattice. However over $\mathbb{Z}$ the story is more interesting and we get [67]

$$\Lambda_{\text{UV}} \cong \mathbb{Z}^{\#\{k_i \text{ even}\}} \oplus 2\text{-torsion}. \tag{18.7}$$

Comparing with our discussion in the **Introduction**, we see that Λ and Λ_{UV} should be identified with the Grothedieck group of the triangle categories $D^b\Gamma$ and $\mathcal{C}(\Gamma)$, respectively. We shall check these identifications below.

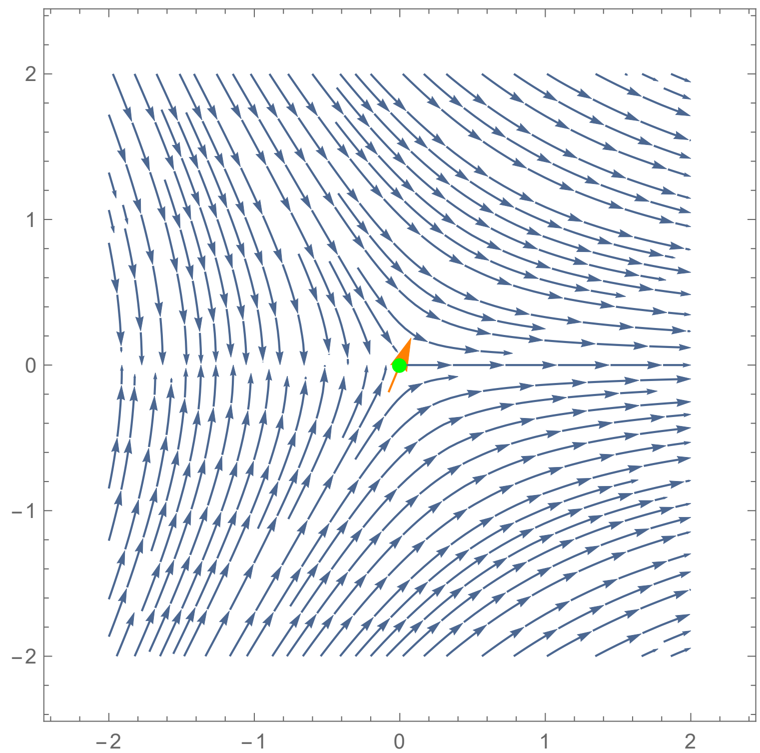
18.3 Quadratic differentials

We want to study the BPS equations to find the BPS spectrum of the theory. We start by analyzing the local behavior of the flow of the quadratic differential. The quadratic differential near a zero can be locally analyzed in a coordinate patch where $\phi \sim w$; thus we have to solve the following equation:

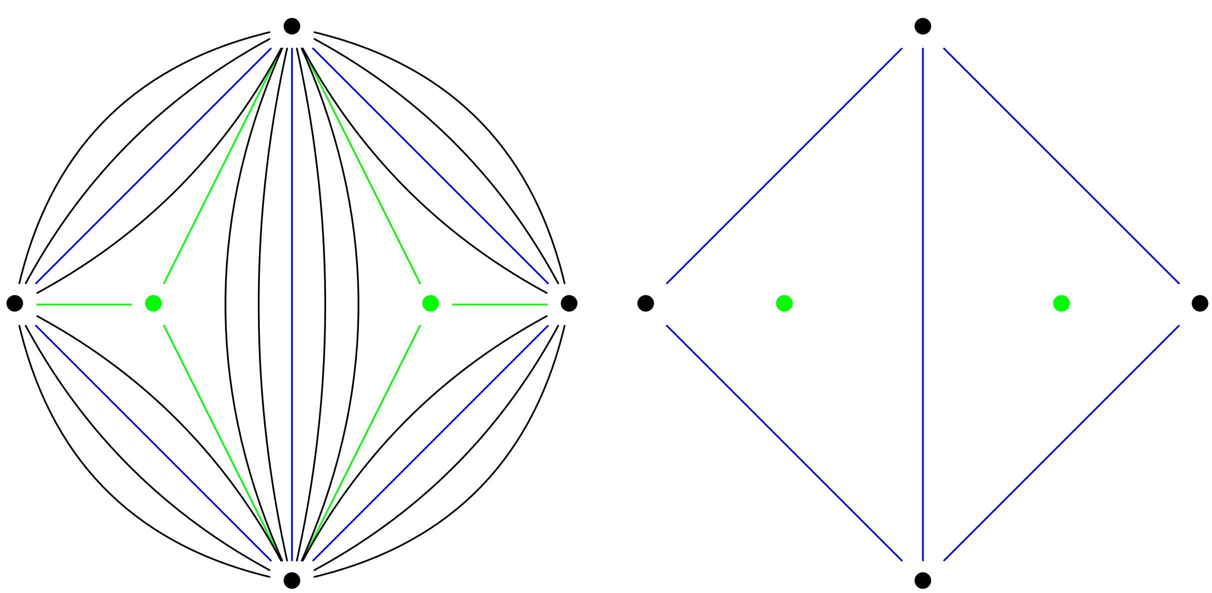
$$\sqrt{w}\,\frac{dw}{dt} = e^{i\theta},$$

[134] Compare eqns.(6.26)-(6.28) in ref. [67].

which gives $w(t) = (\frac{3}{2}te^{i\theta} + w_0^{3/2})^{2/3}$. We plot here the solution:

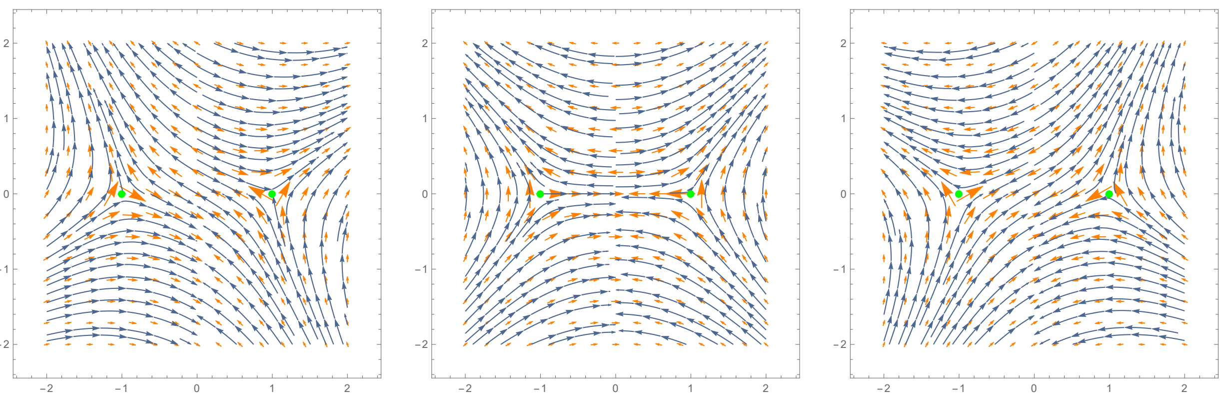


The three straight trajectories, which all start at the zero of ϕ, end at infinity in the poles of ϕ, i.e. the marked points on the boundaries of C. Since all zeros of ϕ are simple by hypothesis, it is possible to associate a triangulation to a quadratic differential by selecting the flow lines connecting the marked points as shown in the following figure:



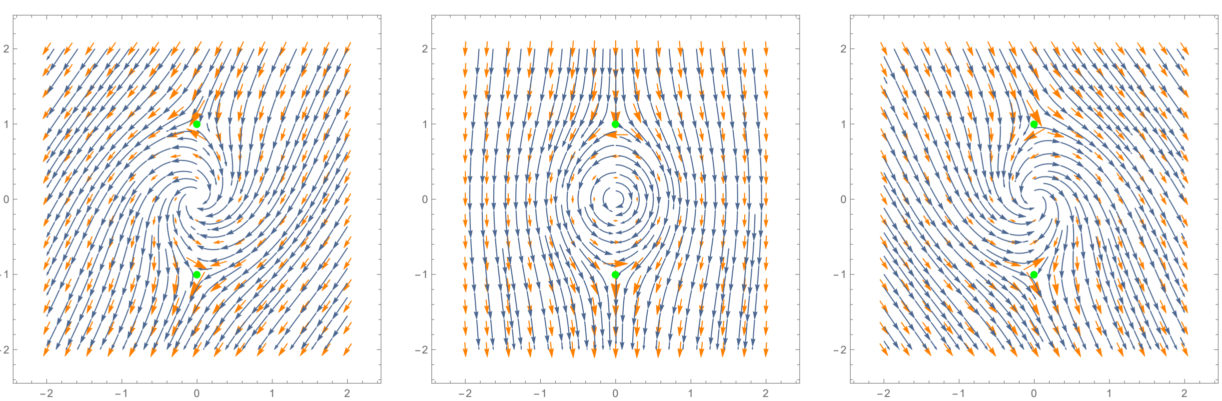
Flow lines Triangulation

Moreover, by construction, it is clear that to each triangle we associate a zero of ϕ. These are the decorating points Δ of section 18.7. If we make θ vary, we deform the triangulation up to a point in which the triangulation jumps: at that value of $\theta = \theta_c$, two zeros of ϕ are connected by a curve η: this curve is the stable BPS state. From what we have just stated, it will be clear that the closed arcs of section 18.7 will correspond to BPS states. Before and after the critical value of $\theta = \theta_c$, the triangulation undergoes a *flip*. Flips of the triangulation correspond to mutations at the level of BPS quiver (see section 18.7 on how to associate a quiver to a triangulation). This topic is develop in full details in [10, 45, 214].
Moreover, the second class of BPS objects, i.e. loops representing vector-multiplets BPS states, appears in one-parameter families and behave as in example 67; the map X of section 18.7 will allow us to write the corresponding graded module $X(a) \in D^b(\Gamma)$.



$\theta < \theta_c$ $\qquad$ $\theta = \theta_c$ $\qquad$ $\theta > \theta_c$

Figure 12: For $\theta = \theta_c$ we have an arc corresponding to a BPS hypermultiplet: it is the solution connecting the two zero of the differential.



$\theta < \theta_c$ $\qquad$ $\theta = \theta_c$ $\qquad$ $\theta > \theta_c$

Figure 13: The 1-parameter family of BPS curves corresponding to a vector multiplet appears for $\theta = \theta_c$.

What about the curves connecting punctures or marked points (but not zeros of the quadratic differential)?[135] To answer this question, we have to take a detour into line operators.

Remark 18.3.1. We point out that the space of quadratic differential is isomorphic to the Coulomb branch of the theory. In a recent paper of Bridgeland [45] it is stated that the space of stability conditions of $D^b\Gamma$ satisfy the following equation:

$$\mathrm{Stab}^0(D^b\Gamma)/\mathrm{Sph}(D^b\Gamma) \cong \mathrm{Quad}(S, M).$$

Thus, the Coulomb branch is isomorphic to the space of stability conditions, up to a spherical twist. Indeed, a stability condition is a pair $(Z, \mathcal{P})$, where Z is the stability function (i.e. the central charge) and the category $\mathcal{P}$ is the category of stable objects, i.e. stable BPS states (see section 13.10 for more details).

18.4 Geometric interpretation of defect operators

The main reference for this section is [2] and also [127]. There it is explained how to use M-theory to construct vertex operators, line operators and surface operators by intersecting an M5 brane with an M2 one. In particular,

- a *vertex operator* corresponds to a point in the physical space: it means that the remaining two dimensions of the M2 brane are wrapped on the Gaiotto surface S, forming a co-dimension 0 object.
- a *line operator* corresponds to a one-dimensional object in the physical space: the remaining one dimension of the M2 brane is wrapped around the surface S as a 1-cycle (i.e. non self-intersecting closed loop) or as an arc connecting two punctures or marked points.
- a *surface operator* has two spacial dimensions and thus it is represented by a puncture over the surface C.

These operators are physically and geometrically related. A vertex operator, since it is a sub-variety of codimension 0 over the complex curve S, can be interpreted as connecting line operators (i.e the loops at the boundary of the sub-variety representing the vertex operator). Moreover, a line operator γ can be interpreted as acting on a surface operator x and transporting the point corresponding to the surface operator around the curve S; the action on its dual Liouville field ([2]) is the monodromy action along the line associated to the line operator γ. Finally, as explained in [128], when we describe the curve S via a quadratic differential ϕ, we can interpret the poles of ϕ as surface operators and the arcs connecting marked points and the closed loops correspond to Verlinde line operators[136]. If we consider two curves γ_1, γ_2 that intersect at some point, the line operators $L(\gamma_1, \zeta), L(\gamma_2, \zeta)$ corresponding to these curves do not commute (see section 19.1). In the case of self-intersecting arcs, we get more general Verlinde operators: as line operators, they

[135] These are the objects in the perfect derived category $\mathfrak{Per}\,\Gamma$

[136] Traditionally, they are defined by composing a sequence of elementary operations on conformal blocks, each corresponding to a map between spaces of conformal blocks which may differ in the number or type of insertions. Roughly speaking, one inserts an identity operator into the original conformal block, splits into two conjugate chiral operators ϕ_a and $\bar{\phi}_a$, transports ϕ_a along γ and then fuses the operators ϕ_a and $\bar{\phi}_a$ back to the identity channel.

can be decomposed into a linear combination of non self-intersecting line operators by splitting the intersection in all possible pairs.

18.5 Punctures and tagged arcs

The main reference, for this short section, is [126]. In there it is pointed out how the arcs over the Gaiotto curve C are tagged arcs: at each singularity (extrema of the arc) we have to choose the eigenvalue of the monodromy operator around that singular point. In particular, in section 8 of [126], we discover that for irregular singularity, the tagging is not necessary, since it is equal to an overall rotation of the marked points around that boundary component. For punctures, on the other hand, it is not the case: we have to specify a $\mathbb{Z}_2$-tagging. This boils down to a tagged triangulation, as explained in [126, 214]. Therefore, when considering surfaces with punctures, we have to consider tagged arcs and not simple arcs. This observation will be important when we discover that the S-duality group is the tagged mapping class group of the Gaiotto surface C (see section 16).

18.6 Ideal triangulations surfaces and quivers with potential

Let C be the Gaiotto curve of a class $\mathcal{S}[A_1]$ model. The invariant of the family of QFTs obtained by continuous deformations of it is the topological type of C. More precisely, we define the underlying topological surface S of C by the following steps: *i)* forget the complex structure, and *ii)* replace each irregular punture with a boundary component ∂S_i with $k_i \geq 1$ marked points (ordinary punctures on C remain punctures on S). S is then the invariant datum which describes the continuous family of $\mathcal{S}[A_1]$ theories.

An *ideal triangulation* of S is a maximal set of pairwise non-isotopic arcs ending in punctures and marked points which do not intersect (except at the end points) and are not homotopic to a boundary arc.[137] All ideal triangulations have the same number of arcs [117]

$$n = 6g - 6 + 3p + 3b + \sum_i k_i.$$

Note that is the same number as the number of UV+IR deformations, eqn.(18.2), as well as the number of IR conserved charges rank Λ, eqn.(18.6).

To an ideal triangulation T of the surface of S we associate a quiver with superpotential (Q, W). The association $S \leftrightarrow (Q, W)$ is intrinsic in the following sense:

Proposition 18.6.1 (Labardini-Fragoso [173])**.** *Let (Q, W) be the quiver with potential associated to an ideal triangulation T of the surface S. A quiver with potential (Q', W') is* mutation equivalent *to (Q, W) if and only if it*[138] *arises from an ideal triangulation T' of the* same *surface S.*

In view of **Corollary 13.6.1**, an important result is:

Proposition 18.6.2 (Labardini-Fragoso [173])**.** *The quiver with superpotential of an ideal triangulation is always Jacobi-finite.*

[137] A boundary arc is the part of a boundary component between two adjacent marked points.

[138] This is slightly imprecise since, in presence of regular punctures W contains free parameters [173]. The statement in the text refers to the full family of allowed W's.

Thus an ideal triangulation T defines a Jacobi-finite Ginzburg DG algebra $\Gamma \equiv \Gamma(Q, W)$ and therefore also the three triangle categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$ and $\mathcal{C}(\Gamma)$ described in §§. 2.5, 2.6. Then

Corollary 18.6.1. *Up to isomorphism, the three triangle categories $D^b\Gamma$, $\mathfrak{Per}\,\Gamma$ and $\mathcal{C}(\Gamma)$ are independent of the chosen triangulation T, and hence are intrinsic properties of the topological surface S.*

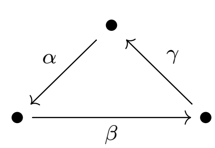
It remains to describe the quiver with potential (Q, W) associated to the ideal triangulation T. The nodes of Q are in one-to-one correspondence with the arcs γ_i of T (their number being equal to the number of IR charges, as required for the BPS quiver of any $\mathcal{N} = 2$ theory). Giving the quiver, is equivalent to specifying its exchange matrix:

Definition 41. For any triangle D in $T = \{\gamma_i\}_{i=1}^n$ which is not self-folded, we define a matrix $B^D = (b^D)_{ij}$,$1 \leq i \leq n, 1 \leq j \leq n$ as follows.

- $b^D_{ij} = 1$ and $b^D_{ji} = -1$ in each of the following cases:
  1. γ_i and γ_j are sides of D with γ_j following γ_i in the clockwise order;
  2. γ_j is a radius in a self-folded triangle enclosed by a loop γ_l, and γ_i and γ_l are sides of D with γ_l following γ_i in the clockwise order;
  3. γ_i is a radius in a self-folded triangle enclosed by a loop γ_i, and γ_l and γ_j are sides of D with γ_j following γ_l in the clockwise order;
- $b^D_{ij} = 0$ otherwise.

Then define the matrix $B^T := (b_{ij}), 1 \leq i \leq n, 1 \leq j \leq n$ by $b_{ij} = \sum_D b^D_{ij}$, where the sum is taken over all triangles in T that are not self-folded. The matrix B^T is a skew-symmetric matrix whose incidence graph is the quiver Q associated to the triangulation.

The superpotential. The superpotential W is the sum of two parts. The first one is a sum over all internal triangles of T (that is, triangles having no side on a boundary component). The full subquiver over the three nodes of Q associated with an internal triangle of T has the form



Such a triangle contributes a term $\gamma\beta\alpha$ to W. The second part of W is a sum over the regular punctures. Let $\gamma_1, \gamma_2 \cdots, \gamma_n$ be the set of arcs ending at the puncture p taken in the clockwise order. The full subquiver of Q over the nodes corresponding to this set of arcs: it is an oriented n-cycle. The contribution to W from the puncture p is λ_p times the associated n-cycle, where $\lambda_p \neq 0$ is a complex coefficient.

No regular puncture: gentle algebras. Suppose S has no regular puncture. Since an arc of T belongs to two triangles (which may be internal or not),

$$\text{at a node of } Q \text{ end (start) at most 2 arrows.} \tag{18.8}$$

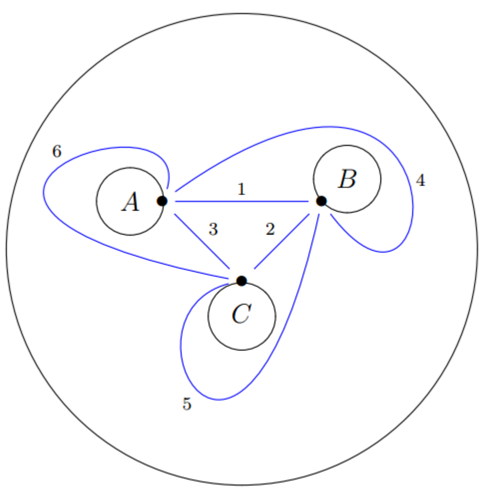
The superpotential is a sum over the internal triangles $\sum_i \gamma_i\beta_i\alpha_i$ and the Jacobi relations are of the form

$$\text{the arrows } \alpha,\ \beta \text{ arise from the same internal triangle } \implies \alpha\beta = 0. \tag{18.9}$$

A finite-dimensional algebra whose quiver satisfies (18.8) and whose relations have the form (18.9) is called a *gentle* algebra [10], a special case of a string algebra. Thus, in absence of regular punctures, the Jacobian algebra $J(Q, W)$ is gentle. Indecomposable modules of a gentle algebra may be explicitly constructed in terms of string and band modules [10] (for a review in the physics literature see [73]). A gentle algebra is then automatically *tame*; in particular, the BPS particles are either hypermultiplets or vector multiplets, higher spin BPS particles being excluded.

How to reduce the general case of a class $\mathcal{S}[A_1]$ theory to this gentle situation is explained in ref. [74].

Example 62. We give here an example of the quiver associated to an ideal triangulation T, whose incidence matrix is B^T:



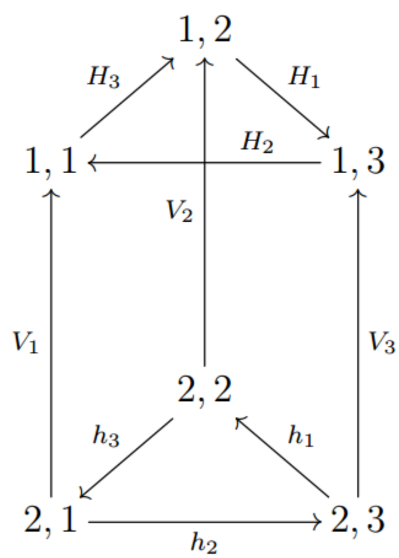


Figure 14: LEFT: the sphere with three holes and one marked point per each boundary component. RIGHT: the quiver associated to the triangulation T drawn on the surface on the right. Its adjacency matrix is the matrix B^T.

18.7 Geometric representation of categories

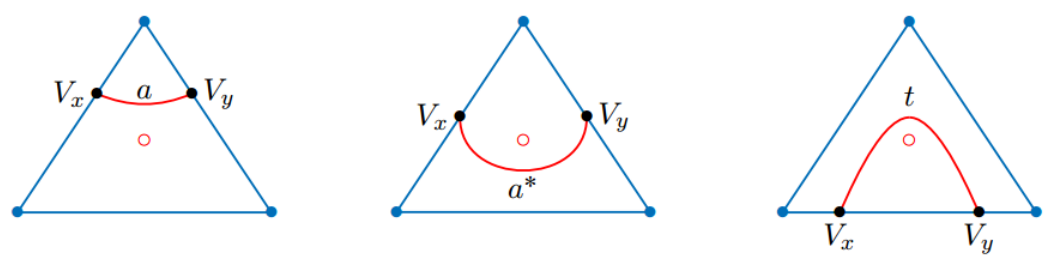
The main reference is [213]. There is a precise dictionary between curves over a decorated marked surface and the objects in the category $D\Gamma$.

Then, let Δ be the set of *decorated points*: to each triangle of T, choose a point in the interior of that triangle. With respect to a quadratic differential ϕ of section 18.3, the decorated points correspond to the simple zeros of ϕ. The marked points, on the other hand, correspond to poles of ϕ of order $m_i + 2$. Let S_Δ be the surface S with the decorated points. The basic correspondence between geometry and categories is, on the one hand, between objects in $D(\Gamma)$ and curves over S_Δ and on the other hand between morphisms in $D(\Gamma)$ and intersections between curves. Here follow the complete dictionary:

1. Recall that an object $S \in D^b(\Gamma)$ is spherical iff

$$\mathrm{Hom}_{D^b\Gamma}(S, S[j]) \cong k(\delta_{j,0} + \delta_{j,3}).$$

A spherical object in the category $D^b(\Gamma)$ corresponds to a simple[139] closed[140] arc (CA) between 2 points in Δ. In particular, simple objects are elements of the dual triangulation and are all spherical. These are some of the BPS hypermultiplets. We can describe these curves as elements of the relative homology with $\mathbb{Z}$-coefficients $H_1(S, \Delta, \mathbb{Z})$ over the curve S and the set of points Δ: indeed, the operation of "sum" is well defined and it corresponds to the relation that defines the Grothendieck group $K_0(D^b\Gamma)$. We now describe the map that associates a graded module (or equivalently a complex) to a closed arc. Consider a closed arc γ that is in minimal position with respect to the triangulation: every time γ intersects the triangulation, we add to the complex a simple shifted module $S_i[j]$ and we connect it to the complex with a graded arrow: the grading of the map depends on how the curve and the decorated points are related. In particular, the grading – corresponding to the Ginzburg algebra grading – is defined in the following figure[141]:



$a : k_x \to k_y$, $\deg a = 0$ $\quad$ $a^* : k_y \to k_x$, $\deg a^* = -1$ $\quad$ $t : k_x \to k_y$, $\deg t = -2$

We call X both the map $X : CA(S_\Delta) \to D^b(\Gamma)$ and $X : OA(S_\Delta) \to \mathfrak{Per}\,\Gamma$, where $CA(S_\Delta)$ are the arcs of the decorated surface S_Δ connecting at most two points in Δ and $OA(S_\Delta)$ are the arcs connecting marked points. Notice that for the open arc (OA) case, we also have to take into account the tagging at the punctures. In particular, the situation is the following:

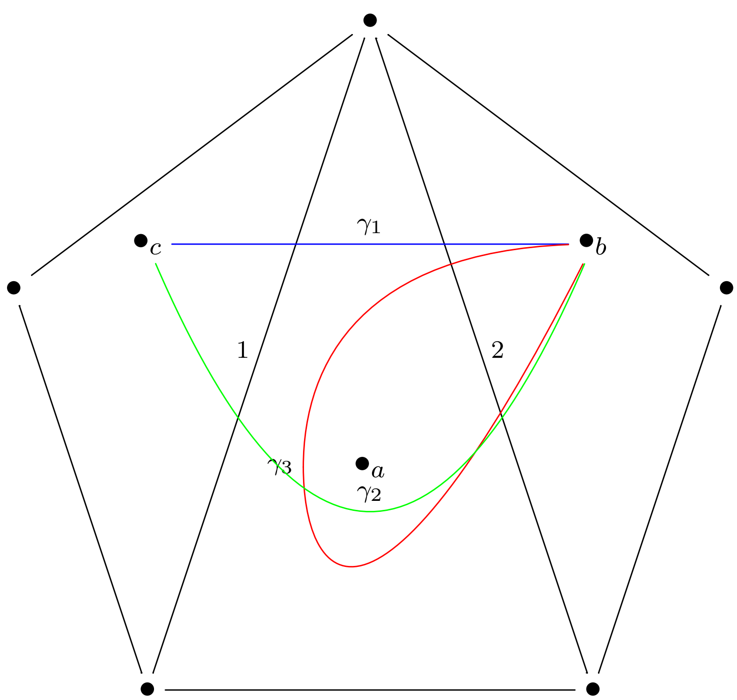
- For an open curve ending on a puncture inside a monogon: if the curve is not tagged,

[139] A simple arc does not have self intersections.
[140] A closed arc starts and ends in Δ.
[141] The figure is taken from [213].

then it intersects only the monogon boundary; if the curve is tagged, then the curve intersect the ray inside the monogon.

- For an open curve ending on a puncture that is not inside a monogon, then the untagged curve does intersect the curve it would intersect as if it were untagged; if the curve is tagged, then it is as if the curve made a little loop around the puncture and so changes the intersection.

Example 63. Let us consider the quiver A_2 again. Then, consider the curves γ_1, γ_2 and γ_3 as in the picture.



By applying the rules above we get:

$$X(\gamma_1): \quad S_1 \overset{a=1}{\to} S_2 \cong k \overset{1}{\to} k$$

$$X(\gamma_2): \quad S_1[-1] \overset{a^*=1}{\leftarrow} S_2 \cong k[-1] \overset{1}{\leftarrow} k$$

$$X(\gamma_3): \quad S_2 \overset{t_1=1}{\to} S_2[-2] \cong 0 \overset{0}{\to} k \oplus k[-2] \circlearrowleft 1$$

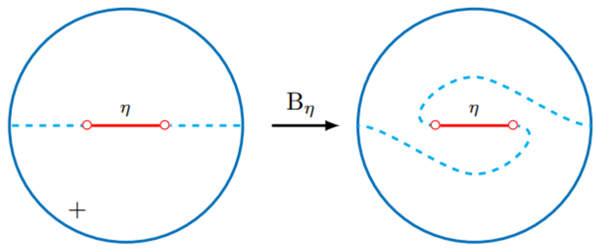
This map also works for the graded modules corresponding to closed loops: both those starting at the zeros of the quadratic differential and those not. As we will see in the example 67 of the Kronecker quiver, only certain loops are in the category $D^b\Gamma$. All other possible loops belong to $\mathfrak{Per}\,\Gamma$. Moreover, even in the case with punctures, the algorithm to get (graded) modules form the curves is the same (thanks to proposition 4.4 of [214]).

2. Since every simple object is a spherical objects in $D^b\Gamma$ – in particular, since $D^b\Gamma$ is 3-CY, the simple objects are 3-spherical – we can define the Thomas-Seidel twist T_S associated to such

a spherical object S by the following triangle

$$\mathrm{Hom}^{\bullet}_{D^b\Gamma}(X,S)\otimes S \to X \to T_S(X) \to .$$

These twists are autoequivalences of $D\Gamma$. Geometrically, these spherical twists correspond to *braid twists* associated to the simple closed arc γ_S: let $BT(S_\Delta)$ denote the full group generated by all the braid twists. The action of the braid twist is like in figure



Moreover, being T^* the dual triangulation of the triangulation T,

$$CA(S_\Delta) = BT(S_\Delta)\cdot T^*$$

as shown in [213].

3. In general, braid twists of S_Δ correspond to spherical twists of $D^b\Gamma$: $BT(T_{S_\Delta}) = \mathrm{Sph}(D^b(\Gamma))$. Moreover, the quiver representing the braid relations is exactly the quiver Q: to each vertex we associate a twist T_{S_i} and to each single arrow $i \to j$ a braid relation $T_{S_i}T_{S_j}T_{S_i} = T_{S_j}T_{S_i}T_{S_j}$. If there is no arrow between i and j, then $[T_{S_i}, T_{S_j}] = 0$.

4. Rigid and reachable objects in $\mathfrak{Per}\,\Gamma$ correspond to the simple open arcs, i.e. simple curves connecting marked points. The other objects in $\mathfrak{Per}\,\Gamma$ correspond to generic arcs: both those arcs connecting two different punctures or marked points and closed loops encircling decorations or boundaries or punctures. We can describe these curves as elements of the relative homology $H_1(S_\Delta, M, \mathbb{Z})$ over the curve S_Δ (where the points in Δ are topological points in S_Δ) and the set of marked points M: indeed, the operation of "sum" is well defined and it corresponds to the relation that defines the Grothendieck group $K_0(\mathfrak{Per}\,\Gamma)$.

5. Let T be the triangulation of the surface. The arcs of the triangulations are associated to the Γe_i objects in $\mathfrak{Per}\,\Gamma$ and the elements of the dual triangulations are the simple objects in $D^b\Gamma$. This is the geometrical version of the simple-projective duality:

$$\dim \mathrm{Hom}^j(\Gamma e_i, S_k[l]) = \delta_{lj}\,\delta_{ki}.$$

The choice of a heart in $D^b\Gamma$ corresponds to the choice of the simple objects; thus, via the simple-projective duality it also corresponds to the choice of a triangulation T. The relation between the Grothendieck groups of $D^b\Gamma$ and $\mathfrak{Per}\,\Gamma$ is via the Euler form (which corresponds to the intersection form, as pointed out in item 7) and it corresponds to Poincaré duality of the relative homology groups.

6. Flips of the triangulation (forward and backward) correspond to right and left mutations $\mu_i^{\pm}$. Two different flips of the same arc are connected by a braid twist assocaited to that simple closed arc: $T_{S_i} = \mu_i^+(\mu_i^-)^{-1}$.

7. Hom spaces correspond to intersection numbers:[142] the full proof of the following facts can be found in [215]:

$$\dim \operatorname{Hom}(X(CA), X(CA)) = 2\operatorname{Int}(CA, CA)$$

$$\dim(X(OA), X(CA)) = \operatorname{Int}(OA, CA)$$

The intersection numbers between arcs in S_Δ are defined as follows:

- For an open arc γ and any arc η, their intersection number is the geometric intersection number in $S_\Delta - M$:

$$\operatorname{Int}(\gamma, \eta) = \min\left\{|\gamma' \cap \eta' \cap (S_\Delta - M)||\gamma' \cong \gamma, \eta' \cong \eta\right\}.$$

- For two closed arcs α, β in $CA(S_\Delta)$, their intersection number is an half integer in $\frac{1}{2}\mathbb{Z}$ and defined as follows:

$$\operatorname{Int}(\alpha, \beta) = \frac{1}{2}\operatorname{Int}_\Delta(\alpha, \beta) + \operatorname{Int}_{S-\Delta}(\alpha, \beta),$$

where

$$\operatorname{Int}_{S_\Delta - \Delta}(\alpha, \beta) = \min\left\{|\alpha' \cap \beta' \cap S_\Delta - \Delta| \;\middle|\; \alpha \cong \alpha', \beta \cong \beta'\right\}$$

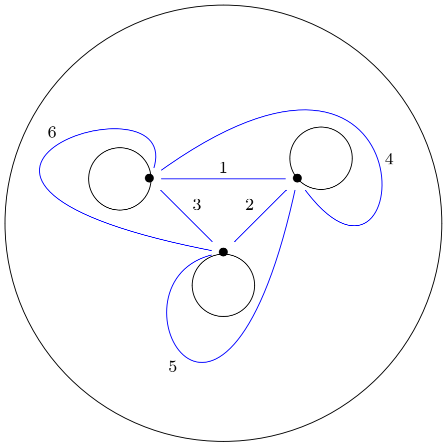
and

$$\operatorname{Int}_\Delta(\alpha, \beta) = \sum_{Z \in \Delta} \left|\left\{t \mid \alpha(t) = Z\right\}\right| \cdot \left|\left\{r \mid \beta(r) = Z\right\}\right|.$$

Let T_0 be a triangulation and η any arc; it is straightforward to see $\operatorname{Int}(T_0, \eta) \geq 2$ for a loop, and the equality holds if and only if η is contained within two triangles of T_0 (in this case, η encircles exactly one decorating point).

Example 64 (Sphere with three holes and one marked point per each boundary component.)**.** In this particular case (the following results do not hold in general), the matrix corresponding to the bilinear form $\operatorname{Int}(-,-)$ can be obtained by taking the Cartan matrix of the quiver with potential (Q, W), i.e. the matrix whose columns are the dimensions of the projective modules, inverting and transposing it. In particular, the incidence matrix for a sphere with three boundary components with $m_i = 1, \forall i \in \{1, 2, 3\}$ – over the basis of simples (corresponding to the edges of the dual triangulation) – is

[142] CA =closed arc, OA =open arc.



$$\mathrm{Int} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The Euler characteristic of $D^b(\Gamma)$, as a bilinear form defined in 13.4, on the other hand, is an antisymmetric integral matrix that is the antisymmetric part of $\mathrm{Int}(-,-)$:

$$\chi(-,-) = \begin{pmatrix} 0 & 1 & -1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

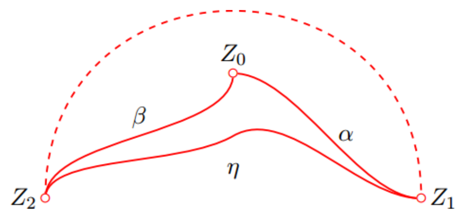
Notice that the skew-symmetric matrix we have just found corresponds to the matrix B^T associated to the ideal triangulation of figure 14.

8. Relations amongst the exchange graphs (EG) of the surface S (to each vertex of this graph we associate a triangulation and to each edge of the graph a flip of the triangulation) and the cluster exchange graph (CEG) of the cluster algebra (to each vertex of the graph we associate a cluster and to each edge a mutation):

$$EG(S) = CEG(\Gamma)$$

$$EG(D^b(\Gamma))/\mathrm{Sph}(D^b(\Gamma)) = CEG(\Gamma)$$

9. A distinguished triangle in $D^b(\Gamma)$ corresponds to a *contractible* triangle in S_Δ whose edges are 3 closed arcs (α, β, η) as in the figure, such that the categorical triangle is $X(\alpha) \to X(\eta) \to X(\beta) \to$.



Exploiting the group structure of the homology group $H_1(S, \Delta, \mathbb{Z})$, we see that we have the following relation:

$$[\alpha] - [\eta] + [\beta] = 0;$$

this is the defining relation of the Grothendieck group $K_0(D^b\Gamma)$.

10. The triangulated structure of $\mathfrak{Per}\,\Gamma$ is less easy to represent geometrically. Before proceeding with an example, we define the left and right mutations in $\mathfrak{Per}\,\Gamma$ starting from the *silting set*.

Definition 42. A *silting set* $\mathbb{P}$ in a category D is an $\text{Ext}^{>0}$-configuration, i.e. a maximal collection of non-isomorphic indecomposables such that $\text{Ext}^i(P, T) = 0$ for any $P, T \in \mathbb{P}$ and integer $i > 0$. The forward mutation μ_P^- at an element $P \in \mathbb{P}$ is another silting set $\mathbb{P}_P^-$, obtained from $\mathbb{P}$ by replacing P with

$$P^- = \text{Cone}\left(P \to \bigoplus_{T \in \mathbb{P}\{P\}} D\text{Hom}_{\text{irr}}(P, T) \otimes T \right), \tag{18.10}$$

where $\text{Hom}_{\text{irr}}(X, Y)$ is the space of irreducible maps $X \to Y$, in the additive subcategory $\mathsf{add} \bigoplus_{T \in \mathbb{P}} T$ of D. The backward mutation μ_P^+ at an element $P \in \mathbb{P}$ is another silting set $\mathbb{P}_P^+$, obtained from $\mathbb{P}$ by replacing P with

$$P^+ = \text{Cone}\left(\bigoplus_{T \in \mathbb{P}\{P\}} \text{Hom}_{\text{irr}}(T, P) \otimes T \to P \right)[-1] \tag{18.11}$$

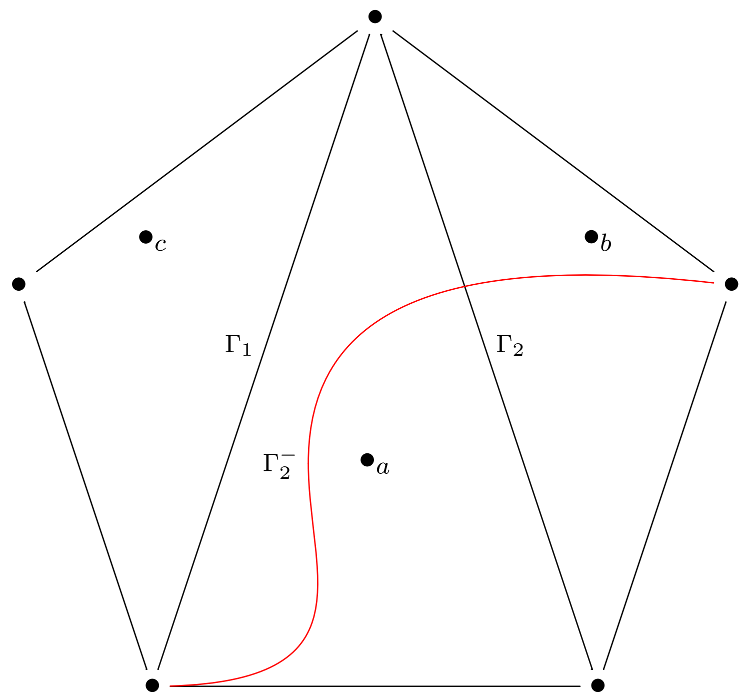
Remark 18.7.1. Notice that equations (13.6) and (13.7) are exactly the same as (18.10) and (18.11): the notation of the latter is more straightforward for the next computations.

We give now an example to show how one can get the triangulated structure of $per\Gamma$ and we relate it to the group structure of $H_1(S_\Delta, M, \mathbb{Z})$.

Example 65 (A_2 example)**.** The silting set for the quiver $A_2 : \bullet_1 \to \bullet_2$ is $\mathbb{P} = \{\Gamma e_1, \Gamma e_2\}$ which we denote as $\{\Gamma_1, \Gamma_2\}$. We apply the left mutation corresponding to a forward flip: it gives the following triangle

$$\Gamma_2 \to \Gamma_1 \to \Gamma_2^- \to .$$

The element Γ_2^- is an infinite complex of the form $S_1 \xrightarrow{t_1} \Gamma_1[-2]$. At a geometrical level it corresponds to the red curve in this figure:



As one can see, these three open arcs do not seem to be geometrically easily related. The next step, which is fundamental for consistency of the geometric representation, is to consider the following triangle:

$$\Gamma_1[-2] \to \Gamma_2^- \to S_1 \to$$

This triangle is exact and moreover $S_1 \in D^b\Gamma$. This implies that in the cluster category, both Γ_2^- and $\Gamma_1[-2]$ map to the same curve. We can exploit the geometric effect of the shift (see item 12) to compute $\Gamma_1[-2]$: it is the green curve in the next figure

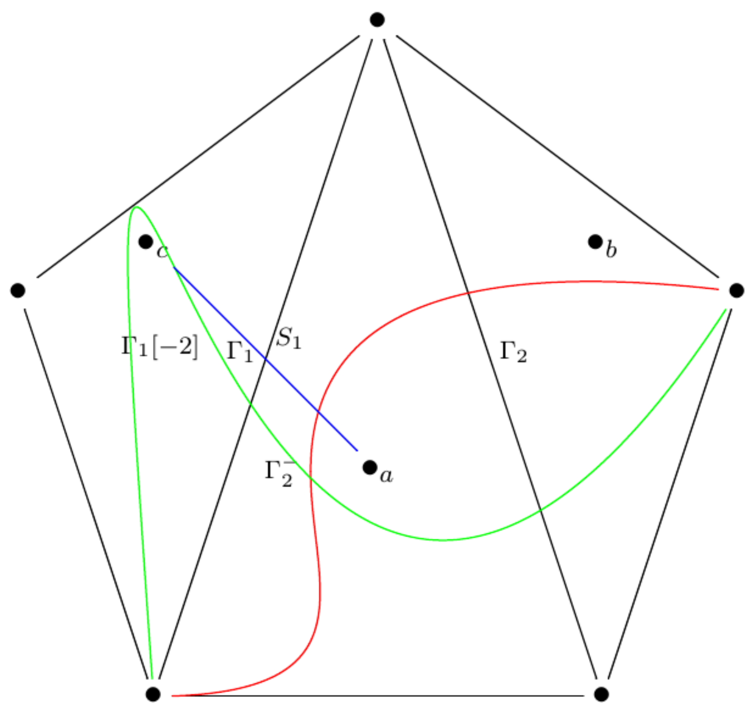
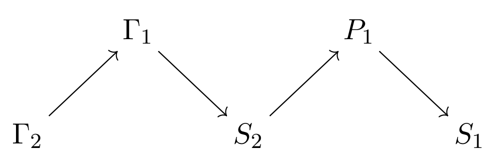
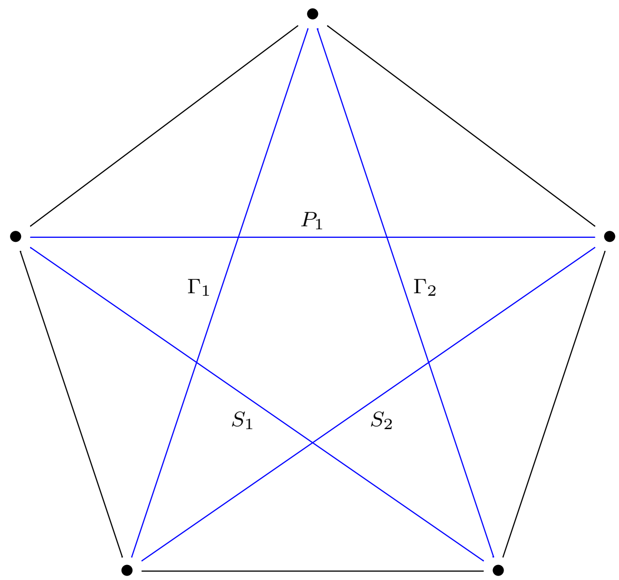


Figure 15: The geometric representation of the distinguished triangle $\Gamma_1[-2] \to \Gamma_2^- \to S_1 \to$ in the perfect category $\mathfrak{Per}\,\Gamma$.

When we map to the cluster category via the forgetful functor (see item 11), we have that both $\Gamma_1[-2]$ and Γ_2^- are sent to the same object: S_2. Indeed, the corresponding cluster category is made of 5 indecomposables which form the following periodic AR diagram



where we can read the corresponding triangles in the cluster category. The geometric picture is



We can now generalize the above results by stating that the triangulated structure is consistent with the group structure of the relative homology group $H_1(S_\Delta, M, \mathbb{Z})$ paired with the relative homology of $H_1(S, \Delta, \mathbb{Z})$: indeed, we see that in $H_1(S_\Delta, M, \mathbb{Z})$ – after choosing opposite direction for the red and green path of figure 15– we have

$$[\Gamma_1[-2]] - [\Gamma_2^-] + [S_1] = 0.$$

The relations defining the Grothendieck $K_0(\mathfrak{Per}\,\Gamma)$, such as

$$[\Gamma_2] - [\Gamma_1] + [\Gamma_2^-] = 0,$$

are less obvious from a homological viewpoint: there is no other way but compute them explicitly when needed.

11. The Amiot quotient $\mathfrak{Per}\,\Gamma/D^b(\Gamma)$ [6] – through which the cluster category is defined – corresponds to the forgetful map $F : S_\Delta \to S$. For a short reminder of triangulated quotients, see section 13.5. In particular, we recover easily the results of [47]: the indecomposable objects of the cluster categories are string modules or band modules. Geometrically a string is an open arc and the procedure to associated a module to it is the same as the one described in item 1 for closed arcs. So indeed, since open curves are indecomposable objects of $\mathfrak{Per}\,\Gamma$, as pointed out in item 4, the following diagram – at least for string modules – commutes:

$$\begin{array}{ccc} \mathcal{O}A(S_\Delta) & \xrightarrow{F} & OA(S) \\ \downarrow X & & \downarrow X \\ \mathfrak{Per}\,\Gamma & \xrightarrow{\pi} & \mathcal{C}(\Gamma) \end{array}$$

We expect no difference in the case of band modules (which correspond to families of loops).

12. In the cluster category $\mathcal{C}(\Gamma)$ and in the perfect derived category $\mathfrak{Per}\,\Gamma$, the shift [1] corresponds

to a global anticlockwise rotation of all the marked points on each boundary component. For punctures, the action of [1] corresponds to a change in the tagging. In $\mathcal{C}(\Gamma)$, it is equivalent to the operation τ (the AR translation), as defined in [47]. In particular we see that in presence of only regular punctures, [2] flips twice the tagging getting back to the original situation, that is, in this case the cluster category is 2-periodic, as expected on physical grounds.

Let us consider here a simple example that will allow us to clarify some aspects.

Example 66 (A_2 again)**.** Let us consider the following curves over a disk with 5 marked points on the boundary and no punctures. The triangulation is made of the black lines 1 and 2 and the boundary arcs:

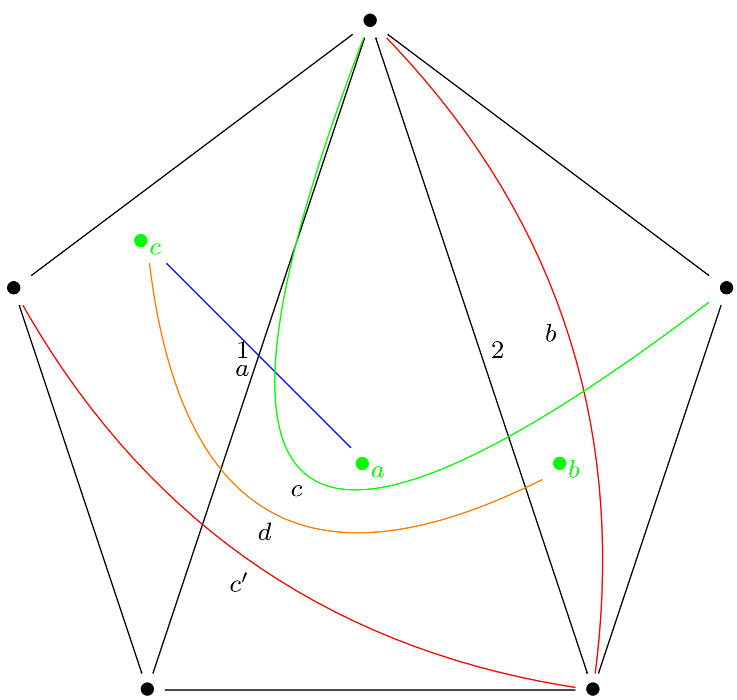


Figure 16: This is the surface corresponding to the quiver $A_2 : 1 \to 2$.

The curves a and d correspond to graded modules in $D^b\Gamma$ as shown in the example 63; indeed, the curve d corresponds to the graded module $S_1 \leftarrow S_2[-1]$ in $D^b\Gamma$. The red curves b and c', on the other hand, cannot be associated to any closed curve, since they cannot intersect any closed curve in minimal position: they are elements only of $\mathfrak{Per}\,\Gamma$ and not of $D^b\Gamma$. When we take the Amiot quotient, the green dots disappear and so do the curves a and d. Moreover, the curve c is homotopic to a boundary arc and b is homotopic to 2. Thus the quotient does what we expect: only the curves in $\mathfrak{Per}\,\Gamma$ that are not in $D^b\Gamma$ do not vanish. The intersection form in this example is

$$\mathrm{Int} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and thus the Euler characteristic is

$$\chi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We can thus construct the Thomas-Seidel twists associated to the simple modules:

$$T_{S_1} = Id - |S_1\rangle \langle S_1| \cdot \chi = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$T_{S_2} = Id - |S_2\rangle \langle S_2| \cdot \chi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

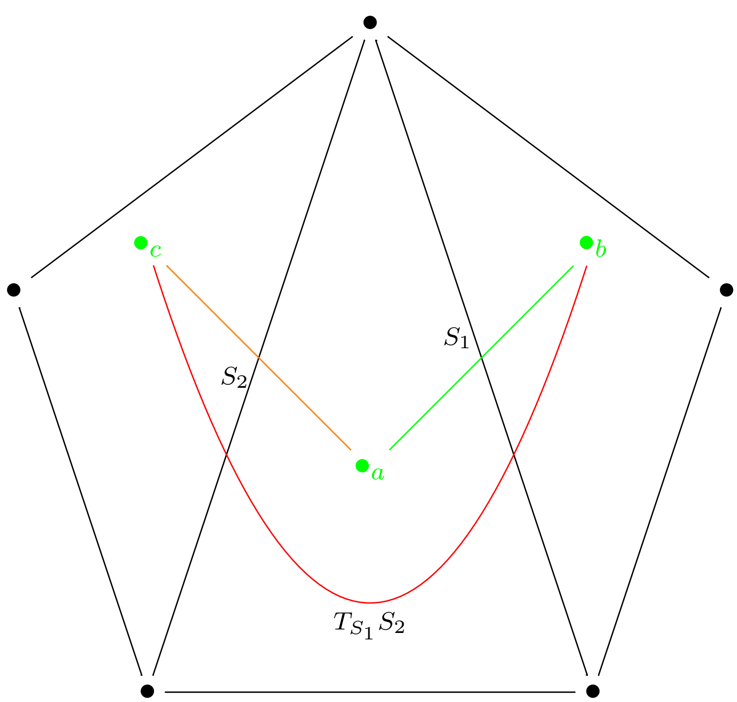
And we can explicitly check the braid relation

$$T_{S_1} \cdot T_{S_2} \cdot T_{S_1} = T_{S_2} \cdot T_{S_1} \cdot T_{S_2}$$

both on $K_0(D^b\Gamma)$ and geometrically. Moreover it is clear from the matrix representation that T_{S_1} and T_{S_2} generate $SL_2(\mathbb{Z})$. We can also act with these twists on the graded modules:

$$T_{S_1} \cdot X(S_2) = X((-1,1))$$

The graded module whose dimension vector is $(-1,1)$ is $k[-1] \overset{a^*=1}{\leftarrow} k$. This is consistent with the geometric picture, as one can verify:



Example 67 (Kronecker quiver)**.** The surface corresponding to the Kronecker quiver is an annulus with one marked point on each boundary component. This theory corresponds to a pure $SU(2)$ SYM. Thus, we shall find closed curves corresponding to BPS vector bosons. They must also appear a one-parameter family. In the following figure, the green loops around the two points represent a band module, the black dots the marked points; the black lines are the flow lines associated to the quadratic differential ϕ described in section 18.3.

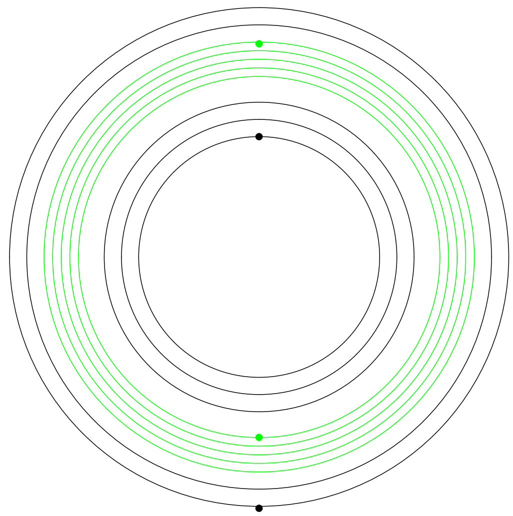


Figure 17: The 1-parameter family of curves corresponding to the vector multiplet of charge $(1,1)$.

The module corresponding to (one of) the green curve can be computed via the map $X : a \mapsto X(a) \in D^b\Gamma$, starting at a generic point along the curve. We get

$$X(a): \quad S_1 \overset{a=1,b=\lambda}{\Longrightarrow} S_2 \;\cong\; k \overset{a=1,b=\lambda}{\Longrightarrow} k.$$

This module is stable (see section 13.10) since it is equivalent to the regular module in the homogeneous tube of $\mathsf{mod}\text{-}k\,\mathsf{Kr}$.

18.8 Summary

We give here a sketchy summary of what we have written so far; recall that, given a quadratic differential $\phi(z)dz \otimes dz$ there are three kind of markings: the zeros of ϕ (decorations), the simple poles of $\sqrt{\phi}$ (punctures) and the irregular singularities of ϕ, which generate the marked points on the boundary segments.

- BPS vector multiplets correspond to loops, not crossing the separating arcs of the flow of ϕ, for $\theta = \arg Z(BPS) = \theta_c$. They belong to the category $D^b\Gamma$ via the maps X.
- BPS hypermultiplets correspond to arcs connecting two zeros of ϕ. They belong to the category $D^b\Gamma$ via the map X and can be identified with elements in the relative homology $H_1(S, \Delta, \mathbb{Z})$, where Δ are the zeros of ϕ.
- Surface operators corresponds to punctures and marked points.

- The objects in the perfect derived category $\mathfrak{Per}\,\Gamma$ are the screening states created by line operators acting on the vacuum. They correspond to arcs connecting marked points and punctures and thus belong to $H_1(S_\Delta, M, \mathbb{Z})$.
- UV line operators correspond to arcs over S connecting marked points and punctures (but not zeros of ϕ). They belong to the cluster category $\mathcal{C}(\Gamma)$ and can also be identified with elements of the relative homology $H_1(S, M, \mathbb{Z})$, where M is the set of markings of the surface S.

With this dictionary in mind, we can now exploit the categorical language to compute physical quantities (such as vacuum expectation values of UV line operators).

Remark 18.8.1. A generalization of these concepts, in particular towards ideal webs and dimer models can be found in [133]. There it is argued that the 3-CY category $D^b\Gamma$ is the physical BPS states category, in accordance with our analysis. Moreover, in the case in which we have a geometrical interpretation via bipartite graphs, the mapping class group of the punctured surface, is a subgroup of the full S-duality group. Here we give a simple example.

Example 68 (Pure $SU(3)$)**.** The pure $SU(3)$ theory can be described as a $S[A_2]$ Gaiotto theory over a cylinder with one full punctures per each boundary [248]. The mapping class group is generated by a single Dehn twist around the cylinder. It acts on the bipartite graph associated to the $SU(3)$ theory and it must be a subgroup of the full S-duality group. It is isomorphic to $\mathbb{Z}$. Since we also have that pure $SU(3)$ theory can be described by the quiver

$$\mathsf{Kr} \boxtimes A_2,$$

We explicitly find – using he techniques of [51] – that a cluster automorphism is given by $\tau_{\mathsf{Kr}} \otimes \tau_{A_2}$, which generates a free group, thus isomorphic to $\mathbb{Z}$.

19 Cluster characters and line operators

19.1 Quick review of line operators

The main reference for this part is [84]. We are going to study, in the following section, the algebra of line operators: we shall discover that this algebra is closely related to the cluster algebra of Fomin and Zelevinski [118]. Recall that an IR line operator (also called *framed BPS state* [127]) is characterize by a central charge ζ and a charge γ. Similarly for a UV line operator. The starting point is to consider the RG flow:

$$RG(\cdot, \alpha, \zeta) : \{\text{UV line operators}\} \to \{\text{IR line operators}\}$$
$$L(\alpha, \zeta) \mapsto \sum_{\gamma\in\Gamma} \bar{\Omega}(\alpha, \zeta, \gamma, u, y)\, L(\gamma, \zeta),$$

where $L(\alpha, \zeta)$ is a supersymmetric UV line operator of UV charge α (see §14.2.3). We can think of it as a supersymmetric Wilson line operator:

$$L(\alpha, \zeta) := \exp\left(i\alpha \int_{\text{time}} A + \frac{1}{2}(\zeta^{-1}\phi + \zeta\bar{\phi})\right).$$

where A is the gauge connection and ϕ and $\bar{\phi}$ are the supersymmetric partners. The idea is that *cluster characters* provide the coefficients $\bar{\Omega}(\alpha,\zeta,\gamma,u,y)$; moreover, the OPE's of line operators can be identified with the *cluster exchange relations*. The physical definition of $\bar{\Omega}(\alpha,\zeta,\gamma,u,y)$ as supersymmetric index is the following. Define the Hilbert space of our system with a line operator in it polarizing the vacuum: $H_{L,\zeta,u} = \bigoplus_{\gamma\in\Gamma_u} H_{L,\zeta,u,\gamma}$, where Γ_u is the charge lattice and u a point in the Coulomb branch. When we restrict only to BPS states we have $H^{BPS}_{L,\zeta,u}$. We now define the following index (i.e. a number that counts the line operators):

$$\bar{\Omega}(\alpha,\zeta,\gamma,u,y) = \mathrm{Tr}_{H_{L,\zeta,u,\gamma}}(y^{2J_3}(-y)^{2I_3}),$$

where the I and J operators are the Cartan generators associated to the unbroken Lorentz symmetry $SO(3)$ and unbroken R-symmetry $SU(2)_R$ by the presence of the line operator which moves along a straight line in the time direction. In particular, if $y=1$:

$$\bar{\Omega}(\alpha,\zeta,\gamma,u,1) = \mathrm{Tr}_{H_{L,\zeta,u,\gamma}}(1^{2J_3}(-1)^{2I_3}) = \sum_m 1^{2m}(-1)^0 = \dim H^{BPS}_{L,\zeta,u,\gamma}.$$

This index corresponds to the Poincaré polynomial of the quiver Grassmannian $Gr_\gamma(\alpha)$, where we interpret the UV line operator L of charge α as the quiver representation of which we compute the cluster character (see §19.2.1 for more details).

19.1.1 Algebra of UV line operators

Let the OPE's of UV line operators be defined as follows

$$L(\alpha,\zeta)L(\alpha',\zeta) = \sum_\beta c(\alpha,\alpha',\beta)L(\beta,\zeta).$$

From now on, let us fix the generic point of the Coulomb branch u. We also define the generating functions for the indexes $\bar{\Omega}(L,\gamma) = \bar{\Omega}(L,\gamma,u=\text{fixed},y=1)$:

$$F(L) = \sum_\gamma \bar{\Omega}(L,\gamma)X_\gamma,$$

where the formal variable X_γ is such that $X_\gamma X_{\gamma'} = X_{\gamma+\gamma'}$. One can check that $F(LL') = F(L)F(L')$. This equality gives a recursive formula to compute $F(LL')$. Furthermore, we can study the wall-crossing of UV line operators via the formula of KS [127]:

$$F^+(L_{\gamma_c}) = F^-(L_{\gamma_c})\prod_\gamma \prod_{m=-M_\gamma}^{M_\gamma}(1+(-1)^m X_\gamma)^{|\langle\gamma,\gamma_c\rangle|a_{m,\gamma}}$$

where γ_c is the charge of the wall we are crossing, and M_γ is the maximal value of the operator $\mathcal{J}_3 = J_2 + I_3$ and the $a_{m,\gamma}$ are the coefficients of the index:

$$\bar{\Omega}(\alpha,\zeta,\gamma,u,y) = \sum_{m=-M_\gamma}^{M_\gamma} a_{m,\gamma}y^m.$$

We can transfer this formula on the newly defined variables to implement the wall-crossing more efficiently:

$$X'_\gamma = X_\gamma \prod_\gamma \prod_{m=-M_\gamma}^{M_\gamma} (1+(-1)^m X_\gamma)^{\langle\gamma,\gamma_c\rangle a_{m,\gamma}}. \tag{19.1}$$

Moreover, the transformation of the indices $\bar{\Omega}(L,\gamma)$ is an automorphism of the OPE algebra. Thus, the algebra obeyed by the generating functionals *is in fact an invariant of the UV sector theory.* The properties of the KS wall-crossing formula of 19.1 and the fact that these generating functionals are invariants of the UV theory (i.e. of the cluster category $C(\Gamma)$), tell us that $F(L)$ are *exactly the cluster character.* The non-commutative generalization is done via the star product [84]:

$$L(\alpha,\zeta) *_y L(\alpha',\zeta) = \sum_\beta c(\alpha,\alpha',\beta,y) L(\beta,\zeta)$$

and $X_\gamma *_y X_{\gamma'} = y^{\langle\gamma,\gamma'\rangle_D} X_{\gamma+\gamma'}$. We define again the generating functions $F(L) = \sum_\gamma \bar{\Omega}(L,\gamma,y) X_\gamma$, and then verify that

$$F(L *_y L') = F(L) *_y F(L').$$

Indeed we discover that the non-commutative version of F behaves exactly like a *quantum cluster character* (usual cluster characters are obtained by setting $y = 1$). We can thus analyze these objects from a purely algebraic point of view and study cluster algebras and cluster characters: we shall do this is the next section.

19.2 Cluster characters

The main references for this section are [102, 207]. We begin by recalling some basic definitions and properties of cluster characters. Let $\mathcal{C} := C(\Gamma)$ be a cluster category.

Definition 43. A cluster character on $\mathcal{C}$ with values in a commutative ring A is a map

$$X : \mathrm{obj}(\mathcal{C}) \to A$$

such that

- for all isomorphic objects L and M, we have $X(L) = X(M)$,
- for all objects L and M of $\mathcal{C}$, we have $X(L \oplus M) = X(L)X(M)$,
- for all objects L and M of $\mathcal{C}$ such that $\dim \mathrm{Ext}^1_{\mathcal{C}}(L,M) = 1$, we have

  $$X(L)X(M) = X(B) + X(B'),$$

  where B and B' are the middle terms of the non-split triangles

  $$L \to B \to M \to \quad \text{and} \quad M \to B' \to L \to$$

  with end terms L and M.[143]

[143] If B' does not exist, then $X(B') = 1$.

Let $T = \bigoplus_i T_i$ be a cluster tilting object, and let $B = \mathrm{Ent}_{\mathcal{C}} T$. The functor

$$F_T : \mathcal{C} \to \mathsf{mod}\text{-}B, \qquad X \mapsto \mathrm{Hom}(T, X)$$

is the projection functor that induces an equivalence between $\mathcal{C}/\mathsf{add}\, T[1] \to \mathsf{mod}\text{-}B$. Then the Caldero-Chapoton map [102],

$$X_?^T : \mathrm{ind}\, \mathcal{C} \to \mathbb{Q}(x_1, ..., x_n)$$

is given by

$$X_M^T = \begin{cases} x_i & \text{if } M \cong \Sigma T_i \\ \sum_e \chi(\mathrm{Gr}_e F_T M) \prod_{i=1}^n x_i^{\langle S_i, e\rangle_D - \langle S_i, FM\rangle} & \text{else,} \end{cases}$$

where the summation is over the isoclasses of submodules[144] of M and S_i are the simple B-modules. Moreover, the Euler form and Dirac form in the formula above, are those of $\mathsf{mod}\text{-}B$. We now recall some properties of quiver Grassmannians[145] and in particular their Euler Poincaré characteristic χ (with respect to the étale cohomology).

Definition 44. Let Λ be a finite dimensional basic $\mathbb{C}$-algebra. For a Λ-module M we define the *F-polynomial* to be the generating function for the Euler characteristic of all possible quiver grassmanians, i.e.

$$F_M := \sum_e \chi(\mathrm{Gr}_e(M)) y^e \in \mathbb{Z}[y_1, ..., y_n]$$

where the sum runs over all possible dimension vectors of submodules of M.

Moreover, we assume that $S_1, ..., S_n$ is a complete system of representatives of the simple Λ-modules, and we identify the classes $[S_i] \in K_0(\Lambda)$ with the natural basis of $\mathbb{Z}^n$.

Proposition 19.2.1. *Let Λ be a finite dimensional basic $\mathbb{C}$-algebra. Then the following holds:*

1. *If*

$$0 \to L \xrightarrow{i} M \xrightarrow{\pi} N \to 0$$

   is an Auslander-Reiten sequence in Λ-mod, then

$$F_L F_N = F_M + y^{\dim N}.$$

2. *For the indecomposable projective Λ-module P_i with top S_i we have*

$$F_{P_i} = F_{\mathrm{rad}\, P_i} + y^{\dim P_i}$$

   for $i = 1, ..., n$.

3. *For the indecomposable injective Λ-module I_j module with socle S_j we have*

$$F_{I_j} = y_j F_{I_j/S_j} + 1$$

[144] Recall that a module N is a submodule of M iff there exists an injective map $N \to M$.
[145] $\mathrm{Gr}_e(FM) := \{N \subset FM \mid \dim N = e\}$, i.e. it is the space of subrepresentations of M with fixed dimension e.

for $j = 1, 2, ..., n$.

The recursive relations of cluster characters and F-polynomials are the key tools to find a computational recipe: the next section is devoted to pointing out this algorithm. All aspects will be clarified in **Example 69**.

19.2.1 Computing cluster characters

The best way to compute cluster characters, is to exploit the results in [12]. The idea is to associate a Laurent polynomial to a path in the quiver. If the algebra is gentle, to a path we can associate a string module: computing the cluster character associated to string modules (up to an overall monomial factor) becomes a simple combinatorics problem. For any locally finite quiver Q, we define a family of matrices with coefficients in $\mathbb{Z}[x_Q] = \mathbb{Z}[x_i | i \in Q_0]$ as follows. For any arrow $\beta \in Q_1$, we set

$$A(\beta) := \begin{bmatrix} x_{t(\beta)} & 0 \\ 1 & x_{s(\beta)} \end{bmatrix} \quad \textit{and} \quad A(\beta^{-1}) := \begin{bmatrix} x_{t(\beta)} & 1 \\ 0 & x_{s(\beta)} \end{bmatrix}.$$

Let $c = c_1 ... c_n$ be a walk of length $n \geq 1$ in Q. For any $i \in \{0, ..., n\}$ we set

$$v_{i+1} = t(c_i)$$

(still with the notation $c_0 = e_{s(c)}$) and

$$V_c(i) := \begin{bmatrix} \prod_{\alpha \in Q_1(v_i,-), \alpha \neq c_i^{\pm 1}, c_{i-1}^{\pm 1}} x_{t(\alpha)} & 0 \\ 0 & \prod_{\alpha \in Q_1(-,v_i), \alpha \neq c_i^{\pm 1}, c_{i-1}^{\pm 1}} x_{s(\alpha)} \end{bmatrix}.$$

We then set

$$L_c = \frac{1}{x_{v_1} ... x_{v_{n+1}}} [1, 1] V_c(1) \prod_{i=1}^{n} A(c_i) V_c(i+1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{L}(x_Q).$$

If $c = e_i$ is a walk of length 0 at a point i, we similarly set

$$V_{e_i}(1) := \begin{bmatrix} \prod_{\alpha \in Q_1(v_i,-)} x_{t(\alpha)} & 0 \\ 0 & \prod_{\alpha \in Q_1(-,v_i)} x_{s(\alpha)} \end{bmatrix}.$$

and

$$L_{e_i} = \frac{1}{x_i} [1, 1] V_{e_i} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{L}(x_Q).$$

In other words, if c is any walk, either of length zero, or of the form $c = c_1 ... c_n$, we have

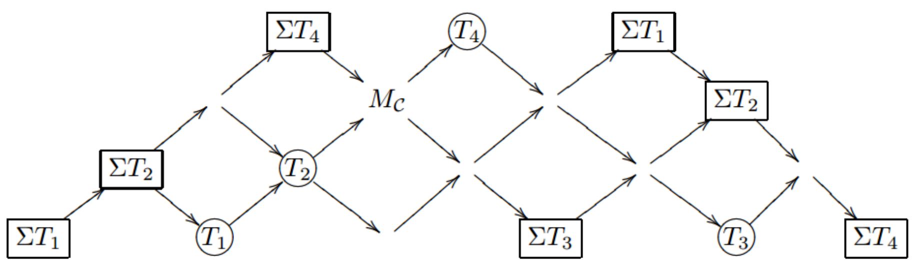
$$L_c = \frac{1}{\prod_{i=0}^{n} x_{t(c_i)}} [1,1] \prod_{i=0}^{n} A(c_i) V_c(i+1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathcal{L}(x_Q).$$

with the convention that $A(c_0)$ is the identity matrix. In general, we have the following result:

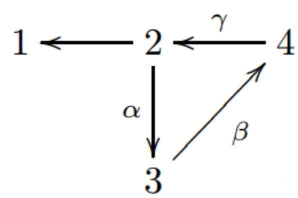
$$X_M = \frac{1}{x^{n_M}} L_c,$$

where M is the string module associated to the path c and the monomial x^{n_M} is the normalization coefficient.

Example 69. Let us consider the cluster category of A_4: its AR quiver is the following



We have also made the choice of tilting objects. The algebra End T is given by the following quiver:[146]



with relations $\beta\alpha = \gamma\beta = \alpha\gamma = 0$. The Dirac form is the following:

$$\langle -, - \rangle_D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix},$$

whereas the Euler form is $\langle a, b \rangle := \dim \mathrm{Hom}_{\mathcal{C}}(a,b) - \dim \mathrm{Hom}_{\mathcal{C}}(a, b[1])$. Consider the B-module $F_T M = (1,1,0,0)$. Its submodules are $0, S_1$ and $F_T M$ itself. The corresponding path is just the

[146] The vertices are the T_i and the arrows $j \to i$ correspond to $\mathrm{Hom}_{\mathcal{C}}(T_i, T_j)$.

arrow $c : 2 \to 1$. By applying the formulas above we get:

$$L_c = \frac{1}{x_1x_2}[1,1]A(c_0)\cdot V_c(1)\cdot A(c)\cdot V_c(2)\begin{bmatrix}1\\1\end{bmatrix} \tag{19.2}$$

$$= \frac{1}{x_1x_2}[1,1]\mathrm{Id}\begin{bmatrix}x_3 & 0\\0 & x_4\end{bmatrix}\begin{bmatrix}x_1 & 1\\0 & x_2\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} \tag{19.3}$$

$$= \frac{x_1x_3 + x_4 + x_2x_4}{x_1x_2}. \tag{19.4}$$

Notice that the denominator is exactly $x^{\dim FM}$: this is a general feature for the decategorification process [108]. Moreover, we know that the Euler characteristic of a point is 1 and thus $\chi(\mathrm{Gr}_0 F_T M) = 1 = \chi(\mathrm{Gr}_{F_T M} F_T M)$. We then exploit the AR sequence

$$0 \to S_1 \to F_T M \to S_2 \to 0$$

and get the recursive relation

$$F_{S_1}F_{S_2} = F_{F_T M} + y_2,$$

which is equivalent to the following polynomial equation:

$$1 + y_1\chi_{S_1} + y_2\chi_{S_2} + y_1y_2\chi_{S_1}\chi_{S_2} = 1 + \chi_{S_1}y_1 + \chi_{FM}y_1y_2 + y_2,$$

which implies that $\chi_{S_1} = \chi_{S_2} = 1$ and this is consistent with the previous result. One can check this and many other computations using appendix C.

Remark 19.2.1. One final remark is needed: we could have computed the cluster characters by a sequence of mutations of the standard seed of the cluster algebra associated to the quiver of $B = \mathrm{End}\,T$. For the non-commutative case, i.e. when

$$x^\alpha x^\beta = q^{\frac{1}{2}\langle\alpha,\beta\rangle_D}\, x^{\alpha+\beta},$$

this procedure is the only one we know to compute quantum cluster characters. From the physics point of view, this is the important quantity: since cluster variables behave like UV line operators, they must satisfy the same non-commutative algebra.

19.3 Cluster characters and vev's of UV line operators

Let us start by recalling how the vacuum expectation values of line operators are computed in [127]. The idea is associate to a loop over a punctured Gaiotto surface a product of matrices. In the case of irregular singularities, since these singularities can be understood as coming from a collision of punctures, loops can get pinched and become *laminations*. Thus to both string modules (associated to *laminations*) and band modules (associated to loops), we can associate a rational function in some shear variables Y_i. Their expression turns out to be equal to cluster characters: for string modules we can use section 19.2.1, whereas for band modules we can use the bangle basis of [198] and the multiplication formula or the Galois covering technique of [74]. We shall now give some detailed examples in which we apply what we just described.

Example 70 (A_2 quiver)**.** The computations of [127] of the VEV's of the UV line operators can be found in section 10.1. They have been made using the "traffic rule". The idea is to follow the lamination path and create a sequence of matrices according to the traffic rule. In the end, one takes the trace of the product of matrices (loop case) or contract the product of matrices with special vectors (open arcs case). For example,

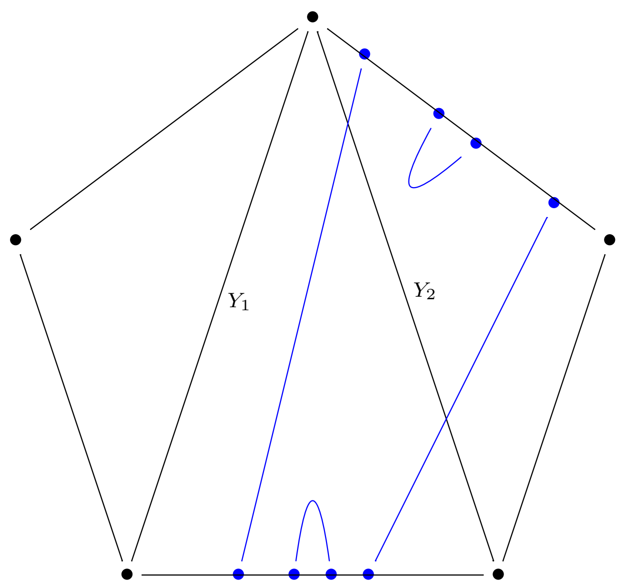


Figure 18: The L_1 lamination of the A_2 theory.

The matrix product is the following:

$$\langle L_1 \rangle = (B_R \cdot R \cdot M_{Y_2} \cdot L \cdot E_R)(B_R \cdot R \cdot M_{Y_2} \cdot L \cdot E_R),$$

where the matrices are

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_X = \begin{pmatrix} \sqrt{X} & 0 \\ 0 & 1/\sqrt{X} \end{pmatrix},$$

and the vectors are

$$B_R = (1\ 0), \quad B_L = (0\ 1), \quad E_L = (1\ 0)^t, \quad E_R = (0\ 1).$$

Then we get:

$$R.M_{Y_2}.L = \begin{pmatrix} \sqrt{Y_2} & \sqrt{Y_2} \\ \sqrt{Y_2} & \sqrt{Y_2} + \frac{1}{\sqrt{Y_2}} \end{pmatrix},$$

and finally

$$\langle L_1 \rangle = \sqrt{Y_2}\sqrt{Y_2} = Y_2.$$

The other four line operators, corresponding to the four remaining indecomposable objects of the cluster category of $\mathcal{C}(\Gamma_{A_2})$ (or equivalently the remaining four cluster variables) are:

$$\langle L_2 \rangle = Y_1 + Y_2 Y_1,\ \langle L_3 \rangle = \frac{1}{Y_2} + \frac{Y_1}{Y_2} + Y_1,\ \langle L_4 \rangle = \frac{1}{Y_2} + \frac{1}{Y_2 Y_1},\ \langle L_5 \rangle = \frac{1}{Y_1}.$$

On the other hand, the cluster characters of A_2 are:

$$x_1,\ x_2,\ \frac{1}{x_2} + \frac{x_1}{x_2},\ \frac{1}{x_1} + \frac{x_2}{x_1}, \frac{1}{x_1 x_2} + \frac{1}{x_1} + \frac{1}{x_2}.$$

The following map $(Y_1, Y_2) \mapsto (\frac{1}{x_2}, x_1)$ transforms one set to the other. This map is the tropicalization map of Fock and Goncharov [116]:

$$Y_i = \prod_j x_j^{B_{ij}}.$$

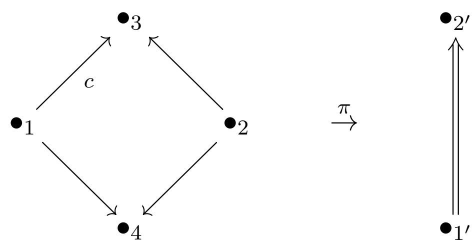
Notice that this result was expected from the general algebraic properties of the line operators algebra and the cluster algebras.

We now proceed to a more interesting example: the pure $SU(2)$ theory. The computations of section 19.2.1 has to be modified a bit: as we will see, it is convenient to exploit the Galois covering techniques of [74].

Example 71 (Kronecker quiver)**.** Let us focus our attention to the non rigid modules, i.e. those belonging to the homogeneous tubes of the AR quiver of the cluster category $C(\Gamma_{Kr})$. There is a $\mathbb{P}^1$ family of these modules and, amongst them, two of them are string modules (those of the form $1 \overset{\lambda=0}{\underset{1}{\rightrightarrows}} 2$). By the theorems of [108], the cluster characters do not depend of the value of the parameter λ and we are thus free to choose the simplest one to compute the character. Geometrically, this family of modules corresponds to loops around the cylinder (see figure 17). With the traffic rule techniques – with the slight modification of taking the trace instead of using the B and E vectors – we compute the VEV of the line operator whose e.m. charge is $(1, 1)$:

$$\langle L_{(1,1)} \rangle = \sqrt{Y_1 Y_2} + \sqrt{\frac{Y_1}{Y_2}} + \frac{1}{\sqrt{Y_1 Y_2}}. \tag{19.5}$$

We can reproduce this result using cluster characters. The only observation is that we cannot simply use section 19.2.1 to compute the character associated to the module $dimM = (1, 1)$: we have a path ambiguity. We thus have to construct a $\mathbb{Z}_2$ Galois cover [74], compute the character on the cover, and then project it down to the Kr quiver. The reason is that the Kronecker quiver has a double arrow and we have to be able to specify the path we follow uniquely. On the $\mathbb{Z}_2$ cover the ambiguity is lifted and the character can be computed. The $\mathbb{Z}_2$ cover is:



where the covering map π sends $1, 2 \mapsto 1'$ and $3, 4 \mapsto 2'$. The character corresponding to the string c with respect to the covering quiver is

$$\frac{1 + x_1 x_2 + x_3 x_4}{x_1 x_3}$$

Therefore, if we identify the cluster variables following the covering map π, we get

$$\frac{1 + x_1'^2 + x_2'^2}{x_1' x_2'}. \tag{19.6}$$

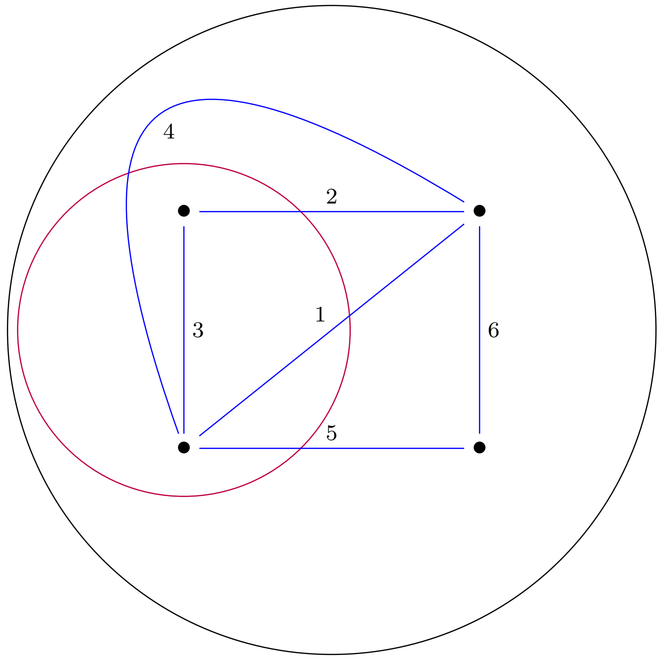
This result is consistent with what we find in literature (e.g. [108]). Also in this case, we find that the tropicalization map [147] sends the rational function 19.5 to 19.6:

$$(Y_1, Y_2) \mapsto (x_2^{-2}, x_1^2).$$

In this final example we show how to compute the cluster character associated to a band module in a more complicated quiver.

Example 72 ($SU(2)$ with $N_f = 4$)**.** We are interested in the module M corresponding to the purple loop in the following figure

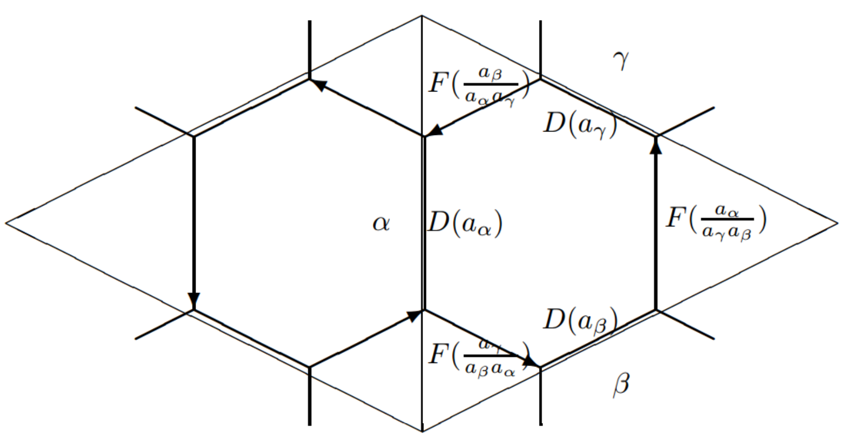
[147] *Id est* $Y_i = \prod_j x_j^{B_{ij}}$.



Using the traffic rule, it is rather straightforward to compute the VEV of the line operator associated to the module M. The result – which is similar to the ones computed in [127] – is:

$$\langle L_M \rangle = \mathrm{Tr}(L \cdot M_{Y_1} \cdot R \cdot M_{Y_2} \cdot R.M_{Y_4} \cdot L \cdot M_{Y_5}) = \frac{1 + Y_4 + Y_2Y_4 + Y_4Y_5 + Y_2Y_4Y_5 + Y_1Y_2Y_4Y_5}{\sqrt{Y_1Y_2Y_4Y_5}} \quad (19.7)$$

The cluster character computation is more involved than the simple application of section 19.2.1: we exploit the techniques of [116]. The idea is similar to the traffic rule: we find a path over the hexagonal graph of [116] that is homotopic to the path considered. Then, to each edge of the hexagonal graph we associate a matrix with the following rule:



The matrices D and F are:

$$D(x) = \begin{pmatrix} 0 & x \\ -\frac{1}{x} & 0 \end{pmatrix}, \quad F\left(\frac{x_\alpha}{x_\beta x_\gamma}\right) = \begin{pmatrix} 1 & 0 \\ \frac{x_\alpha}{x_\beta x_\gamma} & 1 \end{pmatrix}.$$

In our example we find:

$$\begin{aligned} &\mathrm{Tr}\Big(D(x_1)F(\frac{x_3}{x_1x_2})F(\frac{x_3}{x_4x_2})D(x_4)F^{-1}(\frac{x_6}{x_4x_5})F^{-1}(\frac{x_6}{x_1x_5})\Big) \\ &= \frac{x_2x_5{x_1}^2 + x_2x_5{x_4}^2 + x_3x_6x_1^2 + x_3x_6{x_4}^2 + 2x_1x_4x_3x_6}{x_1x_2x_4x_5}. \end{aligned} \tag{19.8}$$

Again we can check that the result (19.7) is the tropicalization of (19.8).

Part VII
Appendices

A 4d/2d correspondence and 4d chiral operators

The 4d/2d correspondence [60] states that — for a certain class of 4d $\mathcal{N} = 2$ models — the exchange matrices B_{ij} of their quivers arise as the BPS counting matrices of 2d $(1,1)$ models with $\hat{c} < 2$. More precisely, for a $\mathcal{N} = 2$ QFT in this class there is a 2d $(1,1)$ theory with n supersymmetric vacua and $0 \leq \hat{c} < 2$ such that

$$B = S^t - S \tag{A.1}$$

where S is the unipotent integral tt^* Stokes matrix of the 2d model [63]. In a suitable basis the matrix S is upper triangular with 1's along the diagonal, and the off–diagonal (generically) integral entries count the 2d BPS states as in [63]. Quiver mutations correspond to 2d wall–crossing. The matrix $\boldsymbol{H} = (S^t)^{-1} S$ is the 2d quantum monodromy with eigenvalues

$$\Big\{ \exp\!\big(2\pi i (q_a - \hat{c}/2)\big) \;:\; q_a \equiv \text{UV } U(1)_R \text{ charges of 2d chiral primaries} \Big\}. \tag{A.2}$$

In particular, only Stokes matrices such that the eigenvalues of $\boldsymbol{H}$ are roots of unity may correspond to unitary 2d QFT.

It follows from 2d PCT that the set $\{q_a\}$ is symmetric under

$$q_a \longleftrightarrow \hat{c} - q_a. \tag{A.3}$$

The 2d theory has always an operator with $q_a = 0$, namely the identity, so $\exp(\pm 2\pi i \hat{c}/2)$ are always eigenvalues of $\boldsymbol{H}$. *A priori* this fixes $\hat{c}/2$ only mod 1, but since $0 \leq \hat{c}/2 < 1$, the value of $\hat{c}$ is uniquely fixed once we know *which* eigenvalue of $\boldsymbol{H}$ is to be identified with $\exp(2\pi i \hat{c}/2)$. Only eigenvalues consistent with the symmetry (A.3) may be identified with $\hat{c}$. $\hat{c}$ is also determined as the fractional CY dimension of the corresponding derived brane category.

4d flavor charges correspond to zero–eigenvectors of B, $(S - S^t)\psi = 0$. Now

$$S\psi = S^t\psi \quad \Longleftrightarrow \quad \boldsymbol{H}\psi \equiv (S^t)^{-1} S\psi = \psi, \tag{A.4}$$

so flavor charges correspond to eigenvectors of the 2d quantum monodromy associated to the eigenvalue $+1$, that is, comparing with eqn.(A.2), to 2d chiral primaries of dimension $\hat{c}/2 \mod 1$. Since 2d unitarity imples $q_a \leq \hat{c} < \hat{c}/2 + 1$, the dimension of the 2d 'flavor' operators $\mathcal{O}_f$ is precisely $\hat{c}/2$. The dual parameters m_f in the 2d action

$$S_0 + \sum_f \left(\int d^2z\, d^2\theta\, m_f\, \mathcal{O}_f + \text{H.c.} \right) \tag{A.5}$$

have a 2d $U(1)_R$ charge $1 - \hat{c}/2$. From the 4d viewpoint the m_f's, being dual to conserved flavor charges, have the dimension of masses; so

$$\text{4d dimension} = \frac{\text{2d } U(1)_R \text{ charge}}{1 - \hat{c}/2}. \tag{A.6}$$

In particular, the dimensions of the operators parametrizing the Coulomb branch are given by the

k numbers

$$\Big\{\Delta_1, \Delta_2, \cdots, \Delta_k\Big\} \equiv \left\{\frac{1-q_a}{1-\hat{c}/2} \ \text{ such that } q_a < \hat{c}/2\right\} \tag{A.7}$$

which are determined by $\boldsymbol{H}$ and the identification of which eigenvalue is identified with $\exp(2\pi i\hat{c}/2)$, up to a few mod 1 ambiguities. Note that for an interacting theory $\Delta_\ell > 1$, as required by 4d unitarity. Since the minimal q_a is always zero, the *largest* dimension of a Coulomb branch operator is given by

$$\Delta_k = \frac{1}{1-\hat{c}/2}. \tag{A.8}$$

B Deferred proofs

B.1 Properties of the dual fractional ideal

Lemma B.1.1. *Let $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ be the decomposition of n in prime factors. We write ζ_i for a primitive $p_i^{r_i}$-root of unity. Then*

$$u \equiv \frac{\Phi_n'(\zeta)}{\prod_{i=1}^s \Phi'_{p_i^{r_i}}(\zeta_i)} \quad \textit{is a unit in } \mathfrak{O}. \tag{B.1}$$

The proof will be given in §.B.2 below.

Lemma B.1.2 ([186])**.** *Let $3 \leq n \neq p^r$, $2p^r$ with p an odd prime. Then in $\mathfrak{O}$ there is a purely imaginary unit.*

Proof. If $n = 0 \mod 4$, $i \in \mathfrak{O}$. If $n \neq 0 \mod 4$, we replace n by the conductor $\breve{n}$ which is an odd integer divisible by two distinct primes. Let ζ be a primitive $\breve{n}$-th root of unity. $(\zeta - \zeta^{-1}) = \zeta(1-\zeta^{-2})$ is an imaginary unit. □

Remark B.1.1. It is easy to check that if $\breve{n} = p^r$ with p an odd prime and the class number of $\mathbb{K}$ is 1, there are no imaginary units.

Lemma B.1.3 ([186])**.** *There exists a unit $\varepsilon \in \mathfrak{O}$ such that*

$$\frac{\varepsilon}{\overline{\Phi_n'(\zeta)}} \equiv \varrho \tag{B.2}$$

is ι-odd (i.e. purely imaginary).

Proof. If n is a prime power $p^r \neq 2$ we have

$$\bar{\varepsilon} = \begin{cases} -i\,\zeta^{-1} & n = 2^r,\ r \geq 2 \\ \zeta^{-1}\zeta^{-(p^{r-1}:2)} & n = p^r,\ p \text{ odd prime} \end{cases} \tag{B.3}$$

where : means division in $(\mathbb{Z}/p^r\mathbb{Z})^\times$. In the general case, $n = \prod_{i=1}^s p_i^{r_i}$ we take ϵ equal to $\bar{u}$ in **Lemma B.1.1** times the product of the ε associated to each prime power in the product. If s is odd ϱ is purely imaginary and we are done. If s is even, ϱ is real, and we multiply it by the imaginary unity in **Lemma B.1.2.** □

Corollary B.1.1. *For all fractional ideal* $\mathfrak{a}$ *of* $\mathbb{K}$ *we have*

$$\mathfrak{a}^* = \varrho/\bar{\mathfrak{a}}, \text{ for a certain } \varrho \in \mathbb{K}^\times \text{ with } \iota(\varrho) = -\varrho. \tag{B.4}$$

If $\breve{n}$ *is not of a power of an odd prime, we may alternatively choose* ϱ *to be real by multiplying it by the appropriate imaginary unit.*

B.2 Proof of Lemma B.1.1

Lemma B.2.1. *Let* $n = p_1^{r_1} \cdots p_s^{n_s}$ *and* ζ *a primitive* n*-th root of unity. We write* $\zeta_{p_i^{r_i}}$ *for a primitive* $p_i^{r_i}$ *root of unity. Then*

$$\Phi_n'(\zeta) = u \prod_i \Phi_{p^{r_i}}'(\zeta_{p_i^{r_i}}) \qquad u \text{ is a unit of } \mathbb{Z}[\zeta] \tag{B.5}$$

Here Φ_n' *denotes the derivative of the polynomial* Φ_n*.*

Proof. We shall use the symbol $\sim$ to mean equality up to multiplication by a unity. If n is a prime power there is nothing to show, so we assume n is divisible by two distinct primes.

We consider first the case of $n = p_1 p_2 \cdots p_s$ square-free (and $s \geq 2$). Then

$$\begin{aligned} \Phi_n'(\zeta) = \prod_{\substack{(a,n)=1 \\ a\neq 1}} (\zeta - \zeta^a) = \zeta^{\phi(n)-1} \prod_{\substack{(a,n)=1 \\ a\neq 1}} (1 - \zeta^{a-1}) \sim \\ \sim \prod_{\substack{(a,n)=1 \\ (a-1,n)=p_1 \ldots \widehat{p_i} \ldots p_s}} (1 - \zeta_i) \sim \prod_i (1 - \zeta_i)^{m_i} \end{aligned} \tag{B.6}$$

where ζ_i is a primitive p^i-root of unity and m_i are certain integers to be determined. To determine the m_i it is enough to compute the norm of the lhs which is the discriminant of the cyclotomic polynomial. Hence

$$\pm \prod_i \frac{p_i^{\phi(n)}}{p_i^{\phi(n)/(p_i-1)}} \sim \prod_i p_i^{m_i \phi(n)/(p_i-1)} \tag{B.7}$$

so $m_i = p_i - 2$. On the other hand $\Phi_{p_i}'(\zeta_i) \sim (1-\zeta_i)^{p_i-2}$, so the **Lemma** is proven for n square-free.
Now let $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ and $q = p_1 p_2 \cdots p_s$ its radical (which is square-free by definition). One has

$$\Phi_n(x) = \Phi_q(x^{n/q}). \tag{B.8}$$

Hence $\Phi_n'(x) = (n/q) x^{n/q-1}\, \Phi_q'(x^{n/q})$ and

$$\Phi_n'(\zeta_n) \sim \frac{n}{q}\, \Phi_q'(\zeta_q) \sim \Big(\prod_i p_i^{r_i - 1} \Big) \prod_i \Phi_{p_i}'(\zeta_{p_i}) \sim \prod_i \Phi_{p_i^{r_i}}'(\zeta_{p_i^{r_i}}). \tag{B.9}$$

□

C Code for cluster characters

This is a short MATHEMATICA code that computes the L_c polynomials of section 19.2.1. Up to an overall normalization factor, the L_c polynomials are the cluster characters. The algorithm follows precisely the procedure described in section 19.2.1.

```
(*set the incidence matrix of the cluster algebra*)
Dirac = {{0, 0, 1, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, -1, 0, 0}, {-1, 0,
   0, 1, 0, 0, 0, 0}, {0, -1, -1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0,
   0, -1, -1}, {0, 1, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 1, -1, 0,
   0}, {-1, 0, 0, 0, 1, 0, 0, 0}}
(*the rows are the dimension vectors of the projectives*)
string = {1, 3, 4, 2};
(*=====================================================*)

Print["Pfaffian: ", Sqrt[Det[Dirac]] ];
AllArrows = Position[-Dirac, 1];
var = Table[ToExpression["x" <> ToString[i]], {i, 1, Length[Dirac]}];
(*indicare come stringa i vertici successivi raggiunti*)

arrows[string_] :=
  Table[{string[[i]], string[[i + 1]]}, {i, 1, Length[string] - 1}];
arrowsandinverse[str_] := Join[arrows[str], Reverse /@ (arrows[str])];
NoStringArrows = Complement[AllArrows, arrowsandinverse[string]];
Amat[ci_] :=
  If[MemberQ[AllArrows,
    ci], {{var[[ ci[[2]] ]], 0}, {1,
     var[[ ci[[1]] ]]}}, {{var[[ ci[[1]] ]], 1}, {0,
     var[[ ci[[2]] ]]}}];
texp[str_, n_] :=
  Plus @@ (SparseArray[# -> 1, Length[Dirac]] & /@ (#[[2]] & /@
       Select[NoStringArrows, #[[1]] == str[[n]] &]));
sexp[str_, n_] :=
  Plus @@ (SparseArray[# -> 1, Length[Dirac]] & /@ (#[[1]] & /@
       Select[NoStringArrows, #[[2]] == str[[n]] &]));
Vmat[str_,
   n_] := {{Times @@ (var^texp[str, n]), 0}, {0,
    Times @@ (var^sexp[str, n])}};
char[str_] :=
 1/(Product[var[[ str[[i]] ]], {i, 1, Length[str]}]) ({1, 1}.Vmat[str,
      1].Dot @@
     Table[Amat[str[[i ;; i + 1]] ].Vmat[str, i + 1], {i, 1,
       Length[str] - 1}].{1, 1})
```

```
charloop[str_] :=
 1/(Product[
     var[[ str[[i]] ]], {i, 1, Length[str]}]) Tr@(Vmat[str, 1].Dot @@
      Table[Amat[str[[i ;; i + 1]] ].Vmat[str, i + 1], {i, 1,
        Length[str] - 1}])
(*example*)
Print["The cluster character corresponding to ", string, " is ",
  If[Length[string] > 1 && string[[1]] == string[[Length[string]]],
   Simplify[charloop[string]],
   Simplify[char[string]] ]];
```

D Code for cluster automorphisms

This short MATHEMATICA script is useful to find generators and relations for the automorphisms of the cluster exchange graph. The formulas used to implement the mutations for the exchange matrix B_{ij} and the dimension vectors d_l (where l is an index that runs over the nodes) are the following:

$$\mu_k(B)_{ij} = \begin{cases} -B_{ij}, & i = k \text{ or } j = k \\ B_{ij} + \max[-B_{ik}, 0]\, B_{kj} + B_{ik}\, \max[B_{kj}, 0] & \text{otherwise.} \end{cases} \tag{D.1}$$

$$\mu_k(d)_l = \begin{cases} d_l, & l \neq k \\ -d_k + \max\Big[\sum_i \max\big[B_{ik}, 0\big] d_i, \sum_i \max\big[-B_{ik}, 0\big] d_i\Big] & l = k \end{cases} \tag{D.2}$$

The procedure of this script is explained in section 17.

```
(*general functions*)
(*mutation b matrix*)
mub[b_, k_] :=
  Table[If[i == k || j == k, -b[[i, j]],
    b[[i, j]] + Max[0, -b[[i, k]]] b[[k, j]] +
     b[[i, k]] Max[0, b[[k, j]]]], {i, 1, Length[b]}, {j, 1,
    Length[b]}];
(*mutation d-vectors*)
mud[d_, {b_, k_}] :=
 Table[If[l != k,
   d[[l]], -d[[k]] +
    Max /@ Transpose[{Sum[
        Max[b[[i, k]], 0] d[[i]], {i, 1, Length[d]}],
       Sum[Max[-b[[i, k]], 0] d[[i]], {i, 1, Length[d]}]}]], {l, 1,
```

```
   Length[d]}]
(*how b transforms after a sequence of mutations*)
mudseqb[seq_, b_] :=
  Thread[List[FoldList[mub, b, seq], Join[seq, {0}]]];
(*how a d-vector transforms after a sequence of mutations*)
mudseq[seqBmenoLast_, b_, d_] := Fold[mud, d, seqBmenoLast];
(*creating the permutation associated to a mutation sequence*)
PermD[seq_, b_, d_] :=
  FindPermutation[
   Plus @@ b + Sqrt[2] (Max /@ Transpose[b]) +
    Sqrt[3] (Min /@ Transpose[b]),
   Plus @@ Last[mudseqb[seq, b]][[1]] +
    Sqrt[2] (Max /@ Transpose[Last[mudseqb[seq, b]][[1]] ]) +
    Sqrt[3] (Min /@ Transpose[Last[mudseqb[seq, b]][[1]] ])];

(*composing different sequences*)
ComposizioneSequenzeConPermutazione[{seq2_, perm2_}, {seq1_,
   perm1_}] := {Join[seq1, PermutationReplace[seq2, perm1]],
  PermutationProduct[perm2, perm1]}
(*checking whether two b matrices are related by a permutation*)
EqualPermb[A_, B_] :=
  Expand[CharacteristicPolynomial[A, z]] ==
    Expand[CharacteristicPolynomial[B, z]] && (Sort[
      Plus @@ A + Sqrt[2] (Max /@ Transpose[A]) +
       Sqrt[3] (Min /@ Transpose[A])] ==
     Sort[Plus @@ B + Sqrt[2] (Max /@ Transpose[B]) +
       Sqrt[3] (Min /@ Transpose[B])]);
(*checking whether two d vectors are related by a permutation*)
EqualPermd[A_, B_] := Sort[A] == Sort[B];



(*checking the order of a sequence*)
OrdineNEW[randseqCONPerm_] :=
 Module[{ord = 0, index = 2, randseq1 = randseqCONPerm,
   randseqtemp = randseqCONPerm, b = b, d = d},
  If[EqualPermd[mudseq[Drop[mudseqb[randseq1[[1]], b], -1], b, d], d],
    ord = 1, ord = 0];
  While[ord == 0 && index <= 45,
   If[EqualPermd[
      mudseq[Drop[
        mudseqb[ComposizioneSequenzeConPermutazione[randseqtemp,
           randseq1][[1]], b], -1], b, d], d], ord = index;
```

```
      randseq1 =
       CompozizioneSequenzeConPermutazione[randseqtemp, randseq1];,
      index++;
      randseq1 =
       ComposizioneSequenzeConPermutazione[randseqtemp, randseq1];];];
   ord]

(*checking the sl2Z relations for S and T generators*)
Relationssl2NEW[{Sconperm_, Tconperm_}] :=
  OrdineNEW[
     ComposizioneSequenzeConPermutazione[Sconperm, Tconperm]] == 6 &&
    EqualPermd[
     mudseq[Drop[
       mudseqb[ComposizioneSequenzeConPermutazione[Sconperm,
          ComposizioneSequenzeConPermutazione[Sconperm, Tconperm]][[
         1]], b], -1], b, d],
     mudseq[
      Drop[mudseqb[
        ComposizioneSequenzeConPermutazione[Tconperm,
          ComposizioneSequenzeConPermutazione[Sconperm, Sconperm]][[
         1]], b], -1], b, d] ] &&
    EqualPermd[
     mudseq[Drop[
       mudseqb[ComposizioneSequenzeConPermutazione[Sconperm,
          ComposizioneSequenzeConPermutazione[Tconperm,
           ComposizioneSequenzeConPermutazione[Tconperm, Tconperm]]][[
         1]], b], -1], b, d],
     mudseq[
      Drop[mudseqb[
        ComposizioneSequenzeConPermutazione[Tconperm,
          ComposizioneSequenzeConPermutazione[Tconperm,
           ComposizioneSequenzeConPermutazione[Tconperm, Sconperm]]][[
         1]], b], -1], b, d] ];

(*Example E7 MN*)
b = {{0, 3, -1, -1, -1, -1, -1, -1, -1}, {-3, 0, 1, 1, 1, 1, 1, 1,
    1}, {1, -1, 0, 0, 0, 0, 0, 0, 0}, {1, -1, 0, 0, 0, 0, 0, 0,
    0}, {1, -1, 0, 0, 0, 0, 0, 0, 0}, {1, -1, 0, 0, 0, 0, 0, 0,
    0}, {1, -1, 0, 0, 0, 0, 0, 0, 0}, {1, -1, 0, 0, 0, 0, 0, 0,
    0}, {1, -1, 0, 0, 0, 0, 0, 0, 0}};
d = -IdentityMatrix[9];
```

```
(*NEW Algorithm to find some generators and their order*)
Print[Dynamic[ii]];
MaxLength = 60;
MAX = 200000000;
ListAutomorph = {};
For[ii = 0, ii < MAX, ii++,
 length = RandomInteger[{1, MaxLength}];
 randseq = RandomInteger[{1, Length@b}, length];

 If[EqualPermb[Last[mudseqb[randseq, b]][[1]], b], index = 2;
  randseq = {randseq, PermD[randseq, b, d]};
  randseqtemp = randseq;
  AppendTo[ListAutomorph, randseq];
  If[EqualPermd[mudseq[Drop[mudseqb[randseq[[1]], b], -1], b, d], d],
   ord = 1, ord = 0];
  While[ord == 0 && index <= 19,
   If[EqualPermd[
      mudseq[Drop[
        mudseqb[ComposizioneSequenzeConPermutazione[randseqtemp,
           randseq][[1]], b], -1], b, d], d], ord = index;
     randseq =
      ComposizioneSequenzeConPermutazione[randseqtemp, randseq];,
     index++;
     randseq =
      ComposizioneSequenzeConPermutazione[randseqtemp, randseq];
     If[! EqualPermb[Last[mudseqb[randseq[[1]], b]][[1]], b],
      Print["Failed: "]; index = 10000;];];];
  If[ord != 1 , Print["Order: ", ord, "  ; Sequence: ", randseqtemp];]
  ];]
```

E Weyl group of E_6

With this short *Mathematica* script, we explicitly construct the Weyl group of E_6 over the basis of simple roots.

```
n = 6;
Projectives = {{1, 1, 1, 0, 0, 0}, {0, 1, 1, 0, 0, 0}, {0, 0, 1, 0, 0,
     0}, {0, 0, 1, 1, 0, 0}, {0, 0, 1, 0, 1, 0}, {0, 0, 1, 0, 1, 1}};
(*Cartan Matrix*)
Cartan = Inverse[Projectives] + Transpose[Inverse[Projectives]];
SimpleRoots = IdentityMatrix[n];
```

```
SimpleWeylGroup =
  Join[Table[
    IdentityMatrix[n] -
     KroneckerProduct[SimpleRoots[[i]], SimpleRoots[[i]]].Cartan, {i,
     1, n}], {IdentityMatrix[6]}];
sr1 = SimpleWeylGroup[[1]];
sr3 = SimpleWeylGroup[[2]];
sr3 = SimpleWeylGroup[[3]];
sr4 = SimpleWeylGroup[[4]];
sr5 = SimpleWeylGroup[[5]];
sr6 = SimpleWeylGroup[[6]];
WeylGroup =
  FixedPoint[
   DeleteDuplicates@
     Partition[
      Partition[Flatten[Outer[Dot, SimpleWeylGroup, #, 1]], 6],
      6] &, {IdentityMatrix[6]}, 36];
Print["Order of the group: "]
Length@WeylGroup
Print["Order of the elements: "]
MatrixOrder[M_, i0_] :=
 Module[{i = i0, Mat = M},
  While[MatrixPower[Mat, i] != IdentityMatrix[n], i++]; i]
DeleteDuplicates[MatrixOrder[#, 1] & /@ WeylGroup]
```

We directly checked that the longest elements has length 36, that the order of the Weyl group is

$$51840 = 2^7 3^4 5$$

and the order of each element belongs to this set:

$$\{1, 3, 2, 5, 4, 6, 12, 8, 10, 9\}.$$

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