Local and almost global solutions for fully-nonlinear Schrödinger equations on the circle

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Introduction

In this thesis we address two questions concerning the Cauchy problem associated to fully-nonlinear one dimensional Schrödinger equations

$$\mathrm{i}\partial_t u + u_{xx} + P * u = f(u, u_x, u_{xx}) \tag{0.0.1}$$

with periodic boundary conditions $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. We have denoted by $P * u$ the convolution between the potential $P(x)$ and the unknown u defined as

$$P * u = \int_{\mathbb{T}} P(x-y)u(y)dy.$$

1. The first problem we study is related to the *local existence* issues, namely we prove that for any regular enough initial datum there exists a unique *classical* solution of the equation (0.0.1) defined in a certain interval of time, whose (short) length depends on the size of the initial datum. We have this result for two different classes of equations: the *Hamiltonian* (see Hypothesis 0.2.1) and the *parity-preserving* (see Hypothesis 0.2.2) ones. We collect these results in Theorems 0.2.1 and 0.2.2. In Theorem 0.2.1 we state the local-existence result also for *big sized* initial data if the nonlinearity $f(u, u_x, u_{xx})$ satisfies an *ellipticity* hypothesis (see Hypothesis 0.2.3); as a model equation for this result one can consider equation (0.2.10). In Theorem 0.2.2 we have the same result for *small enough* initial data if the nonlinearity $f(u, u_x, u_{xx})$ does not satisfy the aforementioned ellipticity condition (see for instance (0.2.11)). For the local theory the presence of the convolution potential in the equation (0.0.1) is not important and its kernel P may be identically 0. We remark that for these type of equations on $\mathbb{T}$ the local existence problems is quite subtle. The problem has been solved by Poppenberg and Kenig-Ponce-Vega in the case of equations posed on $\mathbb{R}^d$ for $d \geq 1$, by using, among several other things, a local smoothing property enjoyed

by the flow of the linear equation (see (0.2.12)). For the equation posed on $\mathbb{T}$ the smoothing property (0.2.12) is not available. For detailed statements, further comments and related literature see Section 0.2. This part of the thesis is the content of the paper [46].

2. We do not know if the solutions of (0.0.1) exhibited in Theorems 0.2.1 and 0.2.2 are globally defined or not. In this direction, inspired by the work of Berti-Delort [21], in [47] we prove an almost global existence result: in the parity-preserving case if the equation is also *reversible* (see Hypothesis 0.3.1) with respect to the involution $S : u \mapsto \bar{u}$, the solutions exist and stay bounded for very long, but finite, time if some *non-resonance* conditions among linear frequencies are fulfilled. In order to ensure the non-resonance conditions, a particular choice of the kernel P in (0.0.1) will be made. We show that there exists a "large" class of functions P (see (0.3.3) and (0.3.4)) such that for any N in $\mathbb{N}$ and for any initial condition even in the space variable x, regular enough and of size ε sufficiently small (depending on N), the lifespan of the solution is of order ε^{-N}. Moreover the size of solution remains of order ε. For a detailed statement and a description of the techniques used we refer to Section 0.3 of this introduction. This part of the thesis is the content of [47].

We remark that, besides the mathematical interest, nonlinear Schrödinger type equations with non-linearities depending on the derivatives of the solution often appears in the description of phenomena in which the wave packet disperses in media, see for instance Zakharov [90].
The results obtained in this PhD thesis are the content of the papers [46, 47]. I mention also that I have written the paper [56] concerning dispersive estimates for a class of singularly perturbed Schrödinger equations.
Before describing in detail our main results in Sections 0.2 and 0.3, we outline a number of problems, and related literature, regarding dispersive PDEs on compact manifolds.

0.1 Dispersive PDEs on compact manifolds

In recent years the study of non-linear *dispersive* PDEs on compact manifolds has been subject of interest of many authors. A dispersive PDE is an equation for which different frequencies propagates at different group velocities. When

$x \in \mathbb{R}^n$ the wave solutions spread out as they evolve in time, therefore, on long time scales, one expects to see the decay of the L^∞ norm as a consequence of the dispersion. In the case of compact manifolds we are in presence of a much more diverse scenario of the time-dinamics; here the dispersive character of the equation is absent: the solutions of the linear equations do not decay when the time goes to ∞. Being the dynamics very different from the Euclidean case, new techniques have been developed, in particular the dynamical system approach turned out to be very fruitful. From one hand many authors, in analogy with finite dimensional dynamical systems, started to look for invariant manifolds on which the dynamics is "simple", hence, for instance, time *periodic* or *quasi periodic* solutions. In this direction a *KAM theory* for PDEs has been developed and during past years it has been subject of a cascade of successive generalizations. On the other hand in [28] Bourgain asked the following questions:

> *Are there solutions of the cubic nonlinear Schrödinger equation*
>
> $$\mathrm{i}\partial_t u + \Delta u = |u|^2 u \tag{0.1.1}$$
>
> *in* $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$ *such that the Sobolev norm* $\|u(t)\|_{H^s}$*, with* $s > 1$*, grows to* ∞ *when the time* t *goes to* ∞*? If so, how fast may this growth be?*

In finite dimension these problems correspond to so called *Arnol′d diffusion* introduced by Arnol′d in [4].
A dual issue concerning solutions of dispersive equations on compact manifolds is the study of *long time existence* and *stability* of the solutions. The typical approach to such problem is the *Birkhoff normal form* one, which seems to be the most convenient one when dealing with problems with lack of conservation laws. This technique comes from Hamiltonian dynamical systems. In finite dimensions one look for a change of coordinates, defined in a neighborhood of an elliptic point, which puts the Hamiltonian in normal form up to a small remainder. Therefore the normal form gives a very good description of the dynamics in the neighborhood of the elliptic point. The smaller is the remainder the longer is the time for which such approximation holds. In finite dimension the Birkhoff normal form theory is nowadays well understood, in recent years many progresses have been done in the context of PDEs.
Before introducing the precise topic of this thesis we summarise in the following three paragraphs, without trying to be exhaustive, the literature concerning the three problems introduced above.

Periodic and quasi periodic solutions: some literature. The first results about existence of periodic solutions of PDEs goes back to the end of seventies. The results by Rabinowitz [82] and Brezis-Coron-Nirenberg [30] proved the existence of periodic solutions with rational frequency for a one dimensional nonlinear wave equation under Dirichlet boundary conditions. The methods used here are based on global variational methods, the restriction to periodic solutions having rational frequency is considered in order to avoid the presence of *small denominators*. *Small denominators* arise by resonance phenomena between the period and the normal frequencies of the linear PDEs.
The existence of small amplitude periodic and quasi-periodic solutions for a positive measure set of frequencies was proved, in the end of eighties, by generalizing the *KAM theory* to the infinite dimensional setting. This is the content of the pioneering works by Kuksin [66] and Wayne [88], where *parameter-dependent*, bounded perturbations of one dimensional nonlinear Schrödinger and wave equations

$$u_t = -\mathrm{i} u_{xx} + V(x,a)u + 2\varepsilon \frac{\partial}{\partial \bar{u}} \Phi(|u|^2, x), \tag{0.1.2}$$

$$u_{tt} - u_{xx} + v(x)u + \varepsilon u^3 = 0 \tag{0.1.3}$$

with Dirichlet boundary conditions were considered, in (0.1.2) $a \in \mathbb{R}^n$, $n \geq 1$, is a parameter. These results were extended to resonant (namely parameter independent) nonlinear Schrödinger and wave equations by Kuksin-Pöschel [69] and by Pöschel [79]. In the case of periodic solutions, the restriction to Dirichlet boundary condition was removed by Craig-Wayne [36] where they used a Lyapunov-Schmidt reduction method in order to deal with periodic boundary conditions and hence multiple eigenvalues of the linear part. A generalization of the techniques introduced by Craig-Wayne was proposed by Bourgain in [27], where he proved existence of quasi-periodic solutions of nonlinear Schrödinger with a convolution potential

$$\mathrm{i} u_t = \Delta u + M * u + \varepsilon \frac{\partial H}{\partial \bar{u}}$$

posed on the torus $\mathbb{T}^2$, where $H := H(u, \bar{u})$. The case in arbitrary dimension was treated by Bourgain in [29]. The study of equations in presence of the more natural multiplicative potential was done by Berti-Bolle in the papers [18, 17]; the approach of such papers is quite flexible and it has been used by Berti-Corsi-Procesi in [20] to study nonlinear wave and Schrödinger equations on compact Lie groups. For a Nash-Moser approach to KAM theory for autonomous equations we quote the paper by Berti-Bolle [19]. We mention also the remarkable paper by Procesi-Procesi [81] in which a KAM theory is developed for autonomous, resonant nonlinear Schrödinger equations

$$\mathrm{i}u_t - \Delta u = u|u|^2 + \partial_{\bar{u}} G(|u|^2)$$

on $\mathbb{T}^d$ for $d \geq 1$, where $G(a)$ is a real analytic function whose Taylor series start from degree 3. For completely resonant NLS type equations we quote also the paper by Wang [87]. A paradifferential approach to find periodic solutions of semilinear Schrödinger equations on $\mathbb{T}^d$ was introduced by Delort in [38].
All these results are related with *bounded perturbations* of dispersive equations, namely equations whose nonlinearity depends only on the unknown and not on its derivatives. The KAM theory concerning *semilinear unbounded perturbations* of dispersive equations, i.e. equations whose nonlinearity may depend on derivatives of the unknown of order strictly less than the one of the linear operator, has been developed in the 1-dimensional case by Kuksin in [68] for equations of the Korteweg-de Vries type

$$u_t = \frac{\partial}{\partial_x}\big(-u_{xx} + V(x)u + \varepsilon f(u,x)\big), \tag{0.1.4}$$

see also Kappler-Pöschel [61]. In [68] is treated also the case of perturbations of large finite-gap solutions. The same problem has been solved by Berti-Kappler-Montalto in [22] for semilinear perturbations of the dNLS equation. Schrödinger type equations in presence of one derivative in the nonlinearity were also considered by Zhang-Gao-Yuan in [91] and Liu-Yuan [89]. The 1-dimensional nonlinear wave with one derivative in the nonlinearity was treated by Berti-Biasco-Procesi for the Hamiltonian case in [15] and for the reversible case in [16].
The first breakthrough results for *fully nonlinear* PDEs, i.e. equations whose nonlinearity contains derivatives of order less or equal to the one of the linear operator, is due to Plotnikov-Toland in [77] and Iooss-Plotnikov-Toland who studied in [60] the existence of *periodic* solutions for the water-waves. The water waves equations are the Euler's equations for an incompressible fluid, in an irrotational regime, under the action of gravity and/or capillary forces on the boundary surface. Inspired by [60] Baldi studied the same problem on the Kirchhoff equation in [5] and on the Benjamin-Ono in [6] posed on the one dimensional torus $\mathbb{T}$. The first result proving existence of small amplitude, *quasi-periodic* solutions for fully-nonlinear equations is due to Baldi-Berti-Montalto [8, 9] for (fully-nonlinear) perturbations

of the forced Airy equation

$$u_t + u_{xxx} + \varepsilon f(\omega t, x, u, u_x, u_{xx}, u_{xxx}) = 0, \tag{0.1.5}$$

where the nonlinearity is quasi-periodic in time, and for (quasi-linear) perturbations of the autonomous Korteweg-de Vries equation

$$u_t + u_{xxx} - 6uu_x + \mathcal{N}_4(x, u, u_x, u_{xx}, u_{xxx}) = 0, \tag{0.1.6}$$

where $\mathcal{N}_4$ is an Hamiltonian nonlinearity. Then in [48] the same result has been obtained by Feola-Procesi for fully-nonlinear Schrödinger equations and by Giuliani in [50] for a class of generalized Korteweg-de Vries equations. Finally we quote Berti-Montalto [23] about the equations describing gravity-capillary water waves and the more recent paper by Baldi-Berti-Haus-Montalto [7] for gravity water waves.

Growth of the Sobolev norms: some literature. Let us make some short comments concerning the aforementioned questions of Bourgain. First of all it is well known that the cubic Schrödinger equation (0.1.1) admits globally defined classical solutions for initial data in $H^s(\mathbb{T}^2)$ for $s \geq 1$, see for instance [31]. The questions are non trivial only in the case $s > 1$ since the equation is Hamiltonian and the Hamiltonian function, or the energy if one prefers, is constant along the solutions and controls their H^1 norm forcing them to stay bounded. The questions, indeed, are related to the problem of understanding if, and how fast, the energy stored in low Fourier modes may escape to higher modes as the time evolves. If one would ask the same question on the one dimensional torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ one would end up with a negative answer, indeed the cubic Schrödinger equation in dimension one is *completely integrable*, see for instance the book by Grébert-Kappeler [51]; the equation enjoys infinitely many conservation laws which are able to control all the Sobolev norms, therefore the growth cannot occur at any index of regularity s.
The first question of Bourgain, about the existence of an unbounded orbit, is, as far as we know, unanswered. We remark that regarding the cubic Schrödinger equation (0.1.1) the problem has been recently solved by Hani-Pausader-Tzvetkov-Visciglia [54] in the case of $\mathbb{T}^2 \times \mathbb{R}$, where they were able to show the existence of orbits which, slowly, diverge to ∞ as the time evolves. In such paper it is used the particular construction exhibited in the breakthrough paper [34] by Colliander-Keel-Staffilani-Takaoka-Tao. In [34] there is a "partial" positive answer to the first question of Bourgain, indeed here the authors prove that there is an arbitrarily big,

but finite, "norm inflation phenomenon", namely they show that given two constants $\mu \ll 1$ and $\mathscr{C} \gg 1$ there are orbits which grow from μ to $\mathscr{C}$ after a certain amount of time.
The result in [34] was improved in [53] by Guardia-Kaloshin, where they are able to give some estimate on the time that the solution needs to grow. The result of Tao and collaborators has been generalized by Haus-Procesi [55] for the quintic NLS on $\mathbb{T}^2$ and by Guardia-Haus-Procesi [52] for general NLS with analytic nonlinearities. We also mention that the first result of arbitrary big (and finite) growth for large initial conditions was proved by Kuksin in [67].
Presuming that an unbounded orbit exists one could try to answer the second question of Bourgain, many authors are putting a lot of efforts into it. Upper bounds for higher order Sobolev norms have been given by several authors for several models. We quote but a few of them Bourgain: [25, 26], Staffilani [84], Planchon-Tzvetkov-Visciglia [76].
The existence of unbounded orbits has been proven for the cubic Szegő equation

$$\mathrm{i}\partial_t u = \Pi(u|u|^2)$$

on $\mathbb{T}$ by Gérard-Grellier in [49], we have denoted by Π the Szegő projector, i.e. the operator which project a function onto the space spanned by $\{e^{\mathrm{i}jx}\}$ with $j \geq 0$.
The growth of Sobolev norms has been studied also for linear Schrödinger equations with potentials, we quote for instance Maspero [70], Maspero-Robert [71], Bambusi-Grébert-Maspero-Robert [14], Delort [40].

Birkhoff Normal Forms: basic ideas and some literature. The equation (0.0.1), if the potential P has real Fourier coefficients, belongs to the following general class of problems:

$$u_t = Lu + \mathtt{f}(u), \tag{0.1.7}$$

where L is an unbounded linear operator with discrete spectrum made of purely imaginary eigenvalues $\lambda_j \in \mathrm{i}\mathbb{R}$, $\mathtt{f}(u)$ is a non linear function and u belongs to some Sobolev space. In the last years several authors investigated whether there is a "stable behavior" of solutions of small amplitude. By "stable solution" we mean that its Sobolev norms $\|\cdot\|_{H^s}$ remain *bounded* for long times.
This problem is non trivial in the case that the system (0.1.7) does not enjoy conservations laws able to control Sobolev norms with high index s. In such a case the only general fruitful approach seems to be the Birkhoff Normal Form (BNF) procedure. This technique, when dealing with non Hamiltonian systems, is often

called Poincaré or Poincaré-Dulac normal form. Below we briefly describe the basic ideas and the difficulties that arise in implementing such a procedure. According to the local existence theory (at a sufficiently large order of regularity), assuming that the non linearity $\mathtt{f}(u)$ vanishes quadratically at the origin, we deduce that if the size of the initial datum is $\varepsilon \ll 1$ then the corresponding solution may be extended up to a time of magnitude $1/\varepsilon$. The basic idea to prove a longer time of existence using a BNF approach is to reduce the *size* of the non linearity near the origin. In other words one looks for a change of coordinates in order to cancel out, from the non linearity, when possible, all the monomials of homogeneity less than N for some $N \geq 2$. In this way, in the new coordinates system, one would have that $\mathtt{f}(u) \sim u^N$, and hence the lifespan would be of order ε^{-N+1}. In performing such changes of coordinates non trivial problems arise:

(i) *small divisors* appear: the small divisors involve linear combinations of the eigenvalues λ_j, $j \in \mathbb{N}$, of the linear operator L in (0.1.7) of the form

$$\lambda_{j_1} + \cdots + \lambda_{j_\ell} - \lambda_{j_{\ell+1}} - \ldots - \lambda_{j_N} \tag{0.1.8}$$

for $0 \leq \ell \leq N$ with $N \in \mathbb{N}$. One must impose *non-resonance* conditions, i.e. lower bounds on the quantity in (0.1.8) whether possible.

(ii) It is not possible to cancel out *all* the monomials of low degree of homogeneity from the non linearity: the divisors in (0.1.8) vanish in the case that $\ell = N/2$ and

$$\{\lambda_{n_1}, \ldots, \lambda_{n_\ell}\} = \{\lambda_{n_{\ell+1}}, \ldots, \lambda_{n_N}\};$$

however, starting form (0.1.7), one obtains a system of the form

$$u_t = Lu + Z(u) + P(u),$$

where $P(u) \sim u^N$ and the non linear term Z (which is usually called "resonant normal form") commutes with the operator L. Under some algebraic assumptions on the nonlinearity $\mathtt{f}(u)$ the dynamics generated by the resonant term $Z(u)$ is "stable", with this we mean that the term $Z(u)$ does not contribute to the growth of Sobolev norms. The most studied models in literature are the *Hamiltonian* and the *reversible* PDEs.

(iii) Check that the changes of coordinates are well-defined and bounded, on sufficiently regular Sobolev spaces, even if some loss of regularity appears due to the small divisors in (0.1.8);

Concerning *semi-linear* PDEs (i.e. when the non linearity $\mathtt{f}(u)$ does not contain derivatives of u) the long time existence problem has been extensively studied in literature in the case of *Hamiltonian* PDEs. We quote for instance the papers by Bambusi [11], Bambusi-Grebért [13] and by Delort-Szeftel [42, 43]. Regarding BNF theory for *reversible* PDEs we mention [45] by Grebért-Faou. The paper [12] regards long time existence of solutions for the semi-linear Klein-Gordon equation on Zoll manifolds, here are collected all the ideas of the preceding (and aforementioned) literature. The normal form for the completely resonant nonlinear Schrödinger equation on a torus $\mathbb{T}^d$ has been discussed by Procesi-Procesi in [80], in this case as a consequence they are able to prove existence of quasi-periodic solutions. We quote also the paper [44] by Faou-Gauckler-Lubich about the long time stability of *plane waves* for the cubic Schrödinger equation on $\mathbb{T}^d$, and the paper by Maspero-Procesi [72] about the stability of small finite gap solutions for the same equation on $\mathbb{T}^2$.

In the case that the non linearity $\mathtt{f}$ contains derivatives of u if one would follow the strategy used in the semilinear case, one would end up with only formal results in the sense that the change of coordinates would be unbounded. We remark that this loss of derivatives is originated by the presence of derivatives in the nonlinearity and not by the presence of small divisors small divisors problem. In this direction we quote the early paper concerning the *pure-gravity water waves* (WW) equation by Craig-Worfolk [37].

In the case that $\mathtt{f}(u)$ in (0.1.7) contains derivatives of u of order strictly less than the order of L, we quote the paper by Yuan-Zhang [89]. They proved existence for times of magnitude ε^{-N}, for any $N \geq 1$, for an equation of the form (0.2.1) with the particular nonlinearity $f(u, u_x) = -(\mathrm{i}/2\pi)(|u|^2 u)_x$ by exploiting its Hamiltonian structure and by using the Fourier coefficients of the convolution potential in (0.2.1) as parameters.

The first rigorous long time existence result concerning *quasi-linear* equations, i.e. when $\mathtt{f}$ contains derivatives of u of the same order of L has been obtained by Delort. In [39] the author studied quasi-linear Hamiltonian perturbations of the *Klein-Gordon* (KG) equation on the circle, and in [41] the same equation on higher dimensional spheres. Here the author introduces some classes of multilinear maps which define *para-differential* operators (in the case of (KG) operators of order 1) enjoying a *symbolic* calculus. We remark that in such papers the author deeply use the fact that the (KG) has a linear *dispersion law* (i.e. the operator L in this case has order 1).

A new different approach in the case of *super-linear* dispersion law (i.e. L has

order > 1) has been proposed by Berti-Delort in [21], for the *capillary water waves* equation, where they were able to prove existence for times of magnitude ε^{-N} for any $N \in \mathbb{N}$. We remark that the authors of [21] could prove existence for any N in $\mathbb{N}$ for almost all values of the surface tension which is a physical parameter of the equation. We shall follow strategy proposed by Berti-Delort, for an introduction to this method we refer to Section 0.3 of this introduction.
Without the use of parameters we quote the recent work [59] by Ionescu-Pusateri for the $2D$ periodic water waves system where they obtained the time of existence $\varepsilon^{-5/3}$.
We mention the papers by Ionescu-Pusateri [57, 58], Alazard-Delort [85] for the water waves on the Euclidean space where they proved the global existence of solutions by using the dispersive character of the equation combined with normal form techniques.

0.2 Local theory

We first present the local existence result for fully-nonlinear Schrödinger equations. This part of the thesis is the content of [46].

Statement of the main theorem. We study the initial value problem (IVP)

$$\begin{cases} \mathrm{i}\partial_t u + \partial_{xx} u + P * u + f(u, u_x, u_{xx}) = 0, \quad u = u(t,x), \quad x \in \mathbb{T}, \\ u(0,x) = u_0(x) \end{cases} \tag{0.2.1}$$

where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, the nonlinearity f is in $C^\infty(\mathbb{C}^3;\mathbb{C})$ in the *real sense* (i.e. $f(z_1,z_2,z_3)$ is C^∞ as function of $\mathrm{Re}(z_i)$ and $\mathrm{Im}(z_i)$ for $i = 1,2,3$) vanishing at order 2 at the origin, the potential

$$P(x) = \sum_{j\in\mathbb{Z}} \hat{p}(j) \frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}$$

is a function in $C^1(\mathbb{T};\mathbb{C})$ with real Fourier coefficients $\hat{p}(j) \in \mathbb{R}$ for any $j \in \mathbb{Z}$ and $P * u$ denotes the convolution between P and $u = \sum_{j\in\mathbb{Z}} \hat{u}(j)\frac{e^{\mathrm{i}jx}}{\sqrt{2\pi}}$

$$P * u(x) := \int_{\mathbb{T}} P(x-y)u(y)dy = \sum_{j\in\mathbb{Z}} \hat{p}(j)\hat{u}(j)e^{\mathrm{i}jx}. \tag{0.2.2}$$

Our aim is to prove the local existence, uniqueness and regularity of the classical solution of (0.2.1) on Sobolev spaces

$$H^s := \left\{ u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k) \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}} \,:\, \|u\|^2_{H^s} := \sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} |\hat{u}(j)|^2 < \infty \right\}, \tag{0.2.3}$$

where $\langle j\rangle := \sqrt{1+|j|^2}$ for $j \in \mathbb{Z}$, for s large enough.
We have positive results in two cases. The first one is the *Hamiltonian* case. We assume that equation (0.2.1) can be written in the complex Hamiltonian form

$$\partial_t u = \mathrm{i}\nabla_{\bar{u}} \mathcal{H}(u), \tag{0.2.4}$$

with Hamiltonian function

$$\mathcal{H}(u) = \int_{\mathbb{T}} -|u_x|^2 + (P * u)\bar{u} + F(u, u_x)dx, \tag{0.2.5}$$

for some real valued function $F \in \mathbb{C}^\infty(\mathbb{C}^2;\mathbb{R})$ and where $\nabla_{\bar{u}} := (\nabla_{\mathrm{Re}(u)} + \mathrm{i}\nabla_{\mathrm{Im}(u)})/2$ and ∇ denotes the $L^2(\mathbb{T};\mathbb{R})$ gradient. Note that the assumption $\hat{p}(j) \in \mathbb{R}$ implies that the Hamiltonian $\int_{\mathbb{T}} (P * u)\bar{u}dx$ is real valued. We denote by

$$\partial_{z_i} := (\partial_{\mathrm{Re}(z_i)} - \mathrm{i}\partial_{\mathrm{Im}(z_i)})/2; \quad \partial_{\bar{z}_i} := (\partial_{\mathrm{Re}(z_i)} + \mathrm{i}\partial_{\mathrm{Im}(z_i)})/2$$

for $i = 1, 2$ the Wirtinger derivatives. We assume the following.

Hypothesis 0.2.1 (Hamiltonian structure). *We assume that the nonlinearity f in equation* (0.2.1) *has the form*

$$\begin{aligned} f(z_1, z_2, z_3) = (\partial_{\bar{z}_1} F)(z_1, z_2) - \Big(&(\partial_{z_1\bar{z}_2} F)(z_1, z_2) z_2 + \\ &(\partial_{\bar{z}_1\bar{z}_2} F)(z_1, z_2)\bar{z}_2 + (\partial_{z_2\bar{z}_2} F)(z_1, z_2) z_3 + (\partial_{\bar{z}_2\bar{z}_2} F)(z_1, z_2)\bar{z}_3 \Big), \end{aligned} \tag{0.2.6}$$

where F is a real valued C^∞ function (in the real sense) defined on $\mathbb{C}^2$ vanishing at 0 *at order* 3.

Under the hypothesis above equation (0.2.1) is *quasi-linear* in the sense that the non linearity depends linearly on the variable z_3. We remark that Hyp. 0.2.1 implies that the nonlinearity f in (0.2.1) has the Hamiltonian form

$$f(u, u_x, u_{xx}) = (\partial_{\bar{z}_1} F)(u, u_x) - \frac{d}{dx}[(\partial_{\bar{z}_2} F)(u, u_x)].$$

The second case is the *parity preserving* case.

Hypothesis 0.2.2 (Parity preserving structure). *Consider the equation* (0.2.1). *Assume that f is a C^∞ function in the real sense defined on $\mathbb{C}^3$ and that it vanishes at order* 2 *at the origin. Assume P has real Fourier coefficients. Assume moreover that f and P satisfy the following assumptions*

1. **Parity preserving nonlinearity** $f(z_1, z_2, z_3) = f(z_1, -z_2, z_3)$*;*
2. **Schrödinger type** $(\partial_{z_3} f)(z_1, z_2, z_3) \in \mathbb{R}$*;*
3. **Parity preserving potential** $P(x) = \sum_{j\in\mathbb{Z}} \hat{p}(j)e^{\mathrm{i}jx}$ *is such that* $\hat{p}(j) = \hat{p}(-j) \in \mathbb{R}$ *(this means that* $P(x) = P(-x)$*).*

Note that item 1 in Hyp. 0.2.2 implies that if $u(x)$ is even in x then $f(u, u_x, u_{xx})$ is even in x; item 3 implies that if $u(x)$ is even in x so is $P * u$. Therefore the space of functions even in x is invariant for (0.2.1). Item 2 ensures that (0.2.1) is a *Schrödinger-type* equation; note that in this case the equation may be *fully-nonlinear*, i.e. the dependence on the variable z_3 is not necessarily linear.
In order to treat initial data with big size we shall assume also the following *ellipticity condition*.

Hypothesis 0.2.3 (Global ellipticity). *We assume that there exist constants* $\mathtt{c}_1, \mathtt{c}_2 > 0$ *such that the following holds. If f in* (0.2.1) *satisfies Hypothesis 0.2.1 (i.e. has the form* (0.2.6)*) then*

$$\begin{aligned} &1 - \partial_{z_2}\partial_{\bar{z}_2} F(z_1, z_2) \geq \mathtt{c}_1, \\ &\big((1 - \partial_{z_2}\partial_{\bar{z}_2} F)^2 - |\partial_{\bar{z}_2}\partial_{\bar{z}_2} F|^2\big)(z_1, z_2) \geq \mathtt{c}_2 \end{aligned} \tag{0.2.7}$$

for any (z_1, z_2) in $\mathbb{C}^2$. If f in (0.2.1) *satisfies Hypothesis 0.2.2 then*

$$\begin{aligned} &1 + \partial_{z_3} f(z_1, z_2, z_3) \geq \mathtt{c}_1, \\ &\big((1 + \partial_{z_3} f)^2 - |\partial_{\bar{z}_3} f|^2\big)(z_1, z_2, z_3) \geq \mathtt{c}_2 \end{aligned} \tag{0.2.8}$$

for any (z_1, z_2, z_3) in $\mathbb{C}^3$.

The first main result of the thesis is the following.

Theorem 0.2.1 (Local existence). *Consider equation* (0.2.1)*, assume Hypothesis 0.2.1 (respectively Hypothesis 0.2.2) and Hypothesis 0.2.3. Then there exists $s_0 > 0$ such that for any $s \geq s_0$ and for any u_0 in $H^s(\mathbb{T};\mathbb{C})$ (respectively any u_0 even in x in the case of Hyp. 0.2.2) there exists $T > 0$, depending only on $\|u_0\|_{H^s}$, such that* *the equation* (0.2.1) *with initial datum* u_0 *has a unique classical solution* $u(t,x)$ *(resp.* $u(t,x)$ *even in* x*) such that*

$$u(t,x) \in C^0\Big([0,T); H^s(\mathbb{T})\Big) \bigcap C^1\Big([0,T); H^{s-2}(\mathbb{T})\Big).$$

Moreover there is a constant $C > 0$ *depending on* $\|u_0\|_{H^{s_0}}$ *and on* $\|P\|_{C^1}$ *such that*

$$\sup_{t\in[0,T)} \|u(t,\cdot)\|_{H^s} \le C\|u_0\|_{H^s}.$$

We make some comments about Hypotheses 0.2.1, 0.2.2 and 0.2.3. We remark that the class of Hamiltonian equations satisfying Hyp. 0.2.1 is different from the parity preserving one satisfying Hyp. 0.2.2. For instance the equation

$$\partial_t u = \mathrm{i}\Big[(1+|u|^2)u_{xx} + u_x^2\bar{u} + (u-\bar{u})u_x\Big] \tag{0.2.9}$$

has the form (0.2.4) with Hamiltonian function

$$\mathcal{H} = \int_{\mathbb{T}} -|u_x|^2(1+|u|^2) + |u|^2(u_x+\bar{u}_x)dx,$$

but does not have the parity preserving structure (in the sense of Hyp. 0.2.2). On the other hand the equation

$$\partial_t u = \mathrm{i}(1+|u|^2)u_{xx} \tag{0.2.10}$$

has the parity preserving structure but is not Hamiltonian with respect to the symplectic form $(u,v) \mapsto \mathrm{Re}\int_{\mathbb{T}} \mathrm{i}u\bar{v}dx$. To check this fact one can reason as done in the appendix of [91]. Both the examples (0.2.9) and (0.2.10) satisfy the ellipticity Hypothesis 0.2.3. Furthermore there are examples of equations that satisfy Hyp. 0.2.1 or Hyp. 0.2.2 but do not satisfy Hyp. 0.2.3, for instance

$$\partial_t u = \mathrm{i}(1-|u|^2)u_{xx}. \tag{0.2.11}$$

The equation (0.2.11) has the parity preserving structure and it has the form (0.2.1) with $P \equiv 0$ and $f(u,u_x,u_{xx}) = -|u|^2u_{xx}$, therefore such an f violates (0.2.8) for $|u| \ge 1$. Nevertheless we are able to prove local existence for equations with this kind of non-linearity if the size of the initial datum is sufficiently small; indeed, since f in (0.2.1) is a C^∞ function vanishing at the origin, conditions (0.2.8) in the case of Hyp. 0.2.2 and (0.2.7) in the case of Hyp. 0.2.1 are always locally fulfilled for $|u|$ small enough. More precisely we have the following theorem.

Theorem 0.2.2 (Local existence for small data). *Consider equation* (0.2.1) *and assume only Hypothesis 0.2.1 (respectively Hypothesis 0.2.2). Then there exists* $s_0 > 0$ *such that for any* $s \geq s_0$ *there exists* $r_0 > 0$ *such that, for any* $0 \leq r \leq r_0$*, the thesis of Theorem 0.2.1 holds for any initial datum* u_0 *in the ball of radius* r *of* $H^s(\mathbb{T};\mathbb{C})$ *centered at the origin.*

Our method requires a high regularity of the initial datum. We have not been sharp in quantifying the minimal value of s_0 in Theorems 0.2.1 and 0.2.2. The reason for which we need regularity is to perform suitable changes of coordinates and having a symbolic calculus at a sufficient order, which requires smoothness of the functions of the phase space.

We remark that in the case of semi-linear problems in the notion of local well-posedness one usually requires also the regular dependence on the Cauchy data of the solution map. For quasilinear problems in general such solution map is not regular, we refer to the survey article by Tzvetkov [86] where he explores the distinction between semi-linear and quasi-linear well posedness. We also quote the paper about Benjamin-Ono and related equations by Molinet-Saut-Tzvetkov [74]. We did not address the problem of studying the regularity of the solution map.

Differences with the euclidean case and some related literature. The local Cauchy theory for Schrödinger equations has been widely studied in the case $x \in \mathbb{R}^n$, $n \geq 1$. When the non-linearity does not depend on the derivatives the local theory is well understood, for a complete overview we refer to the book [32] by Cazenave.

Let us describe briefly one of the main features of the solutions of linear Schrödinger equation on $\mathbb{R}^n$. The flow $e^{\mathrm{i}t\Delta}$ of the linear Schrödinger equation posed in the Euclidean space enjoys the following local smoothing property

$$\sup_{R\in(0,\infty)} \frac{1}{R}\int_{-\infty}^{\infty}\int_{B_R} |\nabla e^{\mathrm{i}t\Delta}\varphi|^2 dx dt \leq C\left\|\varphi\right\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)}, \tag{0.2.12}$$

where we have denoted by B_R the ball of radius R in $\mathbb{R}^n$, $n \geq 1$, centered in the origin, ∇ is the gradient with respect to the space variable and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$ is the usual homogeneous Sobolev space. This property was proven the first time by Constantin and Saut in [35]. The estimate (0.2.12) implies that if an initial datum φ is in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^n)$, then its evolution $e^{\mathrm{i}t\Delta}$ is in $H^1(\mathbb{R}^n)$ for almost every time t.

The smoothing property (0.2.12) has been used by Kenig-Ponce-Vega in [63] (for small data) and in [64] (for any data) in order to prove the local existence for the following semilinear Schrödinger equation in $\mathbb{R}^n$ with unbounded nonlinearity

$$\partial_t u = \mathrm{i}\mathscr{L} u + P(u, \nabla_x u, \bar{u}, \nabla_x \bar{u}), \tag{0.2.13}$$

where $\mathscr{L} := \sum_{j\leq k} \partial^2_{x_j} - \sum_{j>k} \partial^2_{x_j}$, for some k in $\{1,\dots,n\}$, and P is a polynomial having no constant or linear terms. The main ingredient in their proof was a smoothing estimate of the non-homogeneous problem. We write it down, for simplicity, in the one dimensional case $n = 1$:

$$\begin{aligned}\sup_x \left(\int_{-\infty}^{\infty} \left|\partial_x \left(\int_0^t e^{\mathrm{i}(t-s)\partial_x^2} F(\cdot, s)\, ds\right)\right|^2 dt\right)^{1/2} \\ \leq c \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |F(x,t)|^2\, dt\right)^{1/2} dx.\end{aligned} \tag{0.2.14}$$

Roughly speaking the estimate (0.2.14) tells us that the gain of derivatives in the inhomogeneous case is twice the one obtained for the homogenous problem. By using this property Kenig-Ponce-Vega solved the Cauchy problem for the equation (0.2.13) by proving that the Duhamel operator associated to (0.2.13) defines a contraction for small times in a suitable Banach space.

Concerning quasi-linear equations the smoothing properties of the linear flow are not strong enough to apply directly a standard fixed point argument.

For $x \in \mathbb{R}$ Poppenberg, in the paper [78], considered the fully nonlinear Schrödinger type equation $\mathrm{i}\partial_t u = F(t, x, u, u_x, u_{xx})$; he showed that the IVP associated to this equation is locally in time well posed in $H^\infty(\mathbb{R};\mathbb{C})$ (where $H^\infty(\mathbb{R};\mathbb{C})$ denotes the intersection of all Sobolev spaces $H^s(\mathbb{R};\mathbb{C})$, $s \in \mathbb{R}$) if the function F satisfies some suitable ellipticity hypotheses. In order to overcome *loss of derivatives* introduced by the nonlinearity, it is used a Nash-Moser implicit function theorem.

Kenig, Ponce and Vega studied in [65] the n-dimensional case in the Sobolev spaces $H^s(\mathbb{R}^n;\mathbb{C})$ with s sufficiently large. Here the key ingredient used to prove energy estimates is a Doi's type lemma which involves pseudo-differential calculus for symbols defined on the Euclidean space $\mathbb{R}^n$.

Coming back to the case $x \in \mathbb{T}$ we mention the local existence result [10] by Baldi, Haus and Montalto. In this paper it is shown that if s is big enough and if the size of the initial datum u_0 is sufficiently small, then (0.2.1) is well posed in the Sobolev space $H^s(\mathbb{T})$ if $P = 0$ and f is *Hamiltonian* (in the sense of Hypothesis 0.2.1). The

proof is based on a Nash-Moser-Hörmander implicit function theorem and the required energy estimates are obtained by means of a procedure of reduction to constant coefficients of the equation (a similar reduction was performed by Feola and Procesi in [48] in the case of quasi-periodic in time coefficients).
The property (0.2.12) is not available, due to lack of dispersion, in the case of equations posed on $\mathbb{T}$. Furthermore there are examples of nonlinearities such that the same problem is *well-posed* on $\mathbb{R}$ and *ill-posed* on $\mathbb{T}$. Christ proved in [33] that the following family of problems

$$\begin{cases} \partial_t u + \mathrm{i} u_{xx} + u^{p-1} u_x = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{0.2.15}$$

is ill-posed in all Sobolev spaces $H^s(\mathbb{T})$ for any $s \in \mathbb{R}$ and any integer $p \geq 2$ and it is well-posed in $H^s(\mathbb{R})$ for $p \geq 3$ and s sufficiently large. The ill-posedness of (0.2.15) is very strong. Indeed in [33] it has been shown that its solutions have the following norm inflation phenomenon: for any $\varepsilon > 0$ there exists a solution u of (0.2.15) and a time $t_\varepsilon \in (0, \varepsilon)$ such that

$$\|u_0\|_{H^s} \leq \varepsilon \quad \text{and} \quad \|u(t_\varepsilon)\|_{H^s} > \varepsilon^{-1}.$$

The examples exhibited in [33] somehow justify our hypotheses 0.2.1 or 0.2.2.

0.3 Long time existence

Statement of the main theorem. We study equation (0.2.1) in the parity-preserving setting assuming some extra structure, with respect to the previous section, on the nonlinearity f and on the convolution potential P. We prove a long time existence result for small enough and sufficiently regular initial data.
The nonlinearity f is a polynomial of degree $\bar{q} \geq 2$ defined on $\mathbb{C}^3$ vanishing at order 2 near the origin of the form

$$f(z_0, z_1, z_2) = \sum_{p=2}^{\bar{q}} \sum_{(\alpha,\beta)\in A_p} C_{\alpha,\beta} z_0^{\alpha_0} \bar{z}_0^{\beta_0} z_1^{\alpha_1} \bar{z}_1^{\beta_1} z_2^{\alpha_2} \bar{z}_2^{\beta_2}, \tag{0.3.1}$$

where $C_{\alpha,\beta} \in \mathbb{C}$ and

$$A_p := \{(\alpha, \beta) := (\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2) \in \mathbb{N}^6 \text{ s.t. } \sum_{i=0}^{2} \alpha_i + \beta_i = p\}; \tag{0.3.2}$$

the potential

$$P(x) = (\sqrt{2\pi})^{-1} \sum_{j\in\mathbb{Z}} \hat{p}(j) e^{\mathrm{i}jx} \tag{0.3.3}$$

is a *real* function with *real* Fourier coefficients defined as follows. Fix $M > 0$ and set

$$\hat{p}(j) := \hat{p}_{\vec{m}}(j) = \sum_{k=1}^{M} \frac{m_k}{\langle j\rangle^{2k+1}}, \tag{0.3.4}$$

where $\vec{m} = (m_1, \dots, m_M)$ is a vector in $\mathcal{O} := [-1/2, 1/2]^M$ and $\langle j\rangle = \sqrt{1+|j|^2}$. We shall assume that the polynomial nonlinearity satisfies the following:

Hypothesis 0.3.1. *The function f in* (0.2.1) *and in* (0.3.1) *satisfies the following :*

1. **Parity-preserving**: $f(z_0, z_1, z_2) = f(z_0, -z_1, z_2)$;
2. **Schrödinger-type**: $(\partial_{z_2} f)(z_0, z_1, z_2) \in \mathbb{R}$;
3. **Reversibility-preserving**: $f(z_0, z_1, z_2) = \overline{f(\bar{z}_0, \bar{z}_1, \bar{z}_2)}$,

for any (z_0, z_1, z_2) in $\mathbb{C}^3$.

We have the following result.

Theorem 0.3.1 (Long time existence). *Fix $M \in \mathbb{N}$ and consider equation* (0.2.1). *Assume that f satisfies Hypothesis 0.3.1 and that the Fourier coefficients of $P(x)$ satisfy* (0.3.4). *Then there is a zero Lebesgue measure set $\mathcal{N} \subset \mathcal{O}$ such that for any integer $0 \le N \le M$ and any $\vec{m} \in \mathcal{O} \setminus \mathcal{N}$ there exists $s_0 \in \mathbb{R}$ such that for any $s \ge s_0$ there are constants $r_0 \in (0,1)$, $c_N > 0$ and $C_N > 0$ such that the following holds true. For any $0 < r \le r_0$ and any even function u_0 in the ball of radius r of $H^s(\mathbb{T}; \mathbb{C})$, the equation* (0.2.1) *has a unique classical solution, which is even in $x \in \mathbb{T}$, and*

$$u(t,x) \in C^0\Big([-T_r, T_r]; H^s(\mathbb{T})\Big), \ \text{ with } \ T_r \ge c_N r^{-N}.$$

Moreover one has that

$$\sup_{t\in(-T_r,T_r)} \|u(t,\cdot)\|_{H^s} \le C_N r.$$

Comments on the hypotheses. Since the Fourier coefficients in (0.3.4) decay as $\langle j\rangle^{-3}$ as j goes to ∞, the potential $P(x)$ is a function in H^s for any $s<5/2$ (in particular it is of class $C^1(\mathbb{T};\mathbb{R})$), therefore Theorem 0.2.2 applies and ensures us that in this setting the problem (0.2.1) is well posed.
The motivation for choosing the convolution potential is that it defines a diagonal operator on the Fourier side, therefore it is easier the study of the resonances of the equation. We plan to extend our result also in the more natural setting of a multiplicative potential. In this direction the particular structure of the Fourier coefficients of the convolution potential in (0.3.4) is inspired by the Dirichlet spectrum of $-\partial_{xx}+V(x)$. In fact (see Section 5.3 of [13] and the references therein) for any $\rho\in\mathbb{N}^*$ the eigenvalue λ_j of $-\partial_{xx}+V(x)$ admits an asymptotic expansion of the form

$$\lambda_j\sim j^2+c_0(V)+c_1(V)j^{-2}+\ldots+c_\rho(V)j^{-2-2\rho},$$

where $c_k(V)$, for $k\in\mathbb{N}^*$, are certain multilinear functions of the Fourier coefficients of $V(x)$.
The first two items of Hyp. 0.3.1 are the same as in Hyp. 0.2.2 and they have been commented below such hypothesis. Item 3 of Hyp. 0.3.1, together with the fact that the convolution potential $P(x)$ is real valued, makes the equation (0.2.1) *reversible* with respect to the involution

$$S: u(x)\mapsto \bar{u}(x), \tag{0.3.5}$$

in the sense that it has the form $\partial_t u=X(u)$ with $S\circ X=-X\circ S$. Since f is assumed to be a polynomial function as in (0.3.1), item 3 of the hypothesis is equivalent to require that the coefficients $C_{\alpha,\beta}$ are real. One of the important dynamical consequences of the *reversible* structure of the equation is that if $u(t,x)$ is a solution of the equation with initial condition u_0 then $S(u(-t,x))=\bar{u}(-t,x)$ solves the same equation with initial condition $\bar{u}_0$. This symmetry of the equation is essential for our scope and will play a fundamental role. Let us explain heuristically why the reversible structure is useful when we work in the subspace of even functions. On the function $u(x)=\sum_{n\geq 0}u_n\cos(nx)$ the involution S reads $u_n\mapsto\bar{u}_n$ for $n\geq 0$. Introducing the action-angle variables $u_n:=\sqrt{I_n}e^{\mathrm{i}\theta_n}$ the involution reads $(\theta_n,I_n)\mapsto(-\theta_n,I_n)$. A vectorfield written in action-angle coordinates reads

$$\dot{\theta}=g(\theta,I),\dot{I}=f(\theta,I);$$

therefore it is reversible if $f(\theta,I)$ is odd in θ and if $g(\theta,I)$ is even in θ. According to the "averaging principle" at the principal order the equation for the action variable

I is approximated by $\dot{I} = \fint f(\theta, I)d\theta$, where by $\fint$ we meant the averaged integral in θ, which is null since the function $f(\theta, I)$ is odd in θ. Therefore the reversibility prevents the drift of the action variable I which is the growth of Sobolev norms in the language of PDEs.
We have chosen to study a polynomial nonlinearity in order to avoid extra technicalities.

0.4 Strategy of the proofs

Let us introduce some notation. It is useful for our purposes to work on the product space $H^s \times H^s$, in particular we will often use its subspace

$$\begin{aligned} &\mathbf{H}^s := \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2) := \left(H^s \times H^s\right) \cap \mathfrak{R}, \quad s > 0, \\ &\mathfrak{R} := \{(u^+, u^-) \in L^2(\mathbb{T}; \mathbb{C}) \times L^2(\mathbb{T}; \mathbb{C}) \, : \, u^+ = \overline{u^-}\}, \end{aligned} \tag{0.4.1}$$

endowed with the product topology. On $\mathbf{H}^0$ we define the scalar product

$$(U, V)_{\mathbf{H}^0} := \int_{\mathbb{T}} U \cdot \overline{V} dx. \tag{0.4.2}$$

We introduce also the following subspaces of H^s and of $\mathbf{H}^s$ made of even functions in $x \in \mathbb{T}$

$$H^s_e := \{u \in H^s \, : \, u(x) = u(-x)\}, \qquad \mathbf{H}^s_e \ := (H^s_e \times H^s_e) \cap \mathbf{H}^0. \tag{0.4.3}$$

We define the operators $\lambda[\cdot]$ and $\bar{\lambda}[\cdot]$ by linearity as

$$\begin{aligned} &\lambda[e^{\mathrm{i}jx}] := \lambda_j e^{\mathrm{i}jx}, \qquad \lambda_j := (\mathrm{i}j)^2 + \hat{p}(j), \qquad j \in \mathbb{Z}, \\ &\bar{\lambda}[e^{\mathrm{i}jx}] := \lambda_{-j} e^{\mathrm{i}jx}, \end{aligned} \tag{0.4.4}$$

where $\hat{p}(j)$ are the Fourier coefficients of the potential P in (0.2.2). Let us introduce the following matrices

$$E := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{0.4.5}$$

and set

$$\Lambda U := \begin{pmatrix} \lambda[u] \\ \bar{\lambda}\,[\bar{u}] \end{pmatrix}, \qquad \forall \ U = (u, \bar{u}) \in \mathbf{H}^s. \tag{0.4.6}$$

We denote by $\mathfrak{P}$ the linear operator on $\mathbf{H}^s$ defined by

$$\mathfrak{P}[U] := \begin{pmatrix} P * u \\ \bar{P} * \bar{u} \end{pmatrix}, \quad U = (u, \bar{u}) \in \mathbf{H}^s, \tag{0.4.7}$$

where $P * u$ is defined in (0.2.2). With this formalism we have that the operator Λ in (0.4.6) and (0.4.4) can be written as

$$\Lambda := \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_{xx} \end{pmatrix} + \mathfrak{P}. \tag{0.4.8}$$

It is convenient to rewrite the equation (0.2.1) as the equivalent system

$$\partial_t U = \mathrm{i} E\big(\Lambda U + \mathtt{F}(U)\big), \qquad \mathtt{F}(U) := \begin{pmatrix} f(u, u_x, u_{xx}) \\ \overline{f(u, u_x, u_{xx})} \end{pmatrix}, \tag{0.4.9}$$

where $U = (u, \bar{u})$. For both the results we rewrite (0.4.9) as a paradifferential system. We shall rigorously introduce in Chapter 1 the classes of symbols and operators we need. For simplicity let us say that we shall deal with functions $\mathbb{T} \times \mathbb{R} \ni (x, \xi) \to a(x, \xi)$ with limited smoothness in x satisfying, for some $m \in \mathbb{R}$, the following estimate

$$|\partial_\xi^\beta a(x, \xi)| \leq C_\beta \langle \xi \rangle^{m-\beta}$$

for any $\beta \in \mathbb{N}$, we have used the notation $\langle \xi \rangle := \sqrt{1 + \xi^2}$. This functions have finite regularity in x because they depend on the dynamical variable U which belongs to some Sobolev space $\mathbf{H}^s$, with s positive and finite. We shall regularize such symbols in the following way. Let χ be a $C_0^\infty(\mathbb{R})$ cut-off function with sufficiently small support and equal to 1 close to 0, then we set $a_\chi(x, \xi) = \mathscr{F}_{\hat{x}}^{-1}\big(\hat{a}(\hat{x}, \xi)\chi(\hat{x}/\langle \xi \rangle)\big)$, where we have denoted by $\mathscr{F}$ the Fourier transform. In other words the new symbol $a_\chi(x, \xi)$ is a localization in the Fourier space, and therefore a regularization in the physical space, of the symbol $a(x, \xi)$. Then we can define the Bony-Weyl quantization of the symbol a as follows

$$\mathrm{Op}^{\mathscr{B}W}(a(x, \xi))\varphi = \frac{1}{2\pi} \int e^{\mathrm{i}(x-y)\xi} a_\chi\Big(\frac{x+y}{2}, \xi\Big)\varphi(y)\,dy\,d\xi.$$

With this formalism (0.4.9) is equivalent to the paradifferential system

$$\partial_t U = \mathrm{i} E \mathscr{G}(U)[U] + \mathscr{R}(U)(U), \tag{0.4.10}$$

where $\mathcal{G}(U)[\cdot]$ is equal to $\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))$ and $A(U;x,\xi)$ is a matrix of symbols depending, in a nonlinear way, on the dynamical variable U and $\mathcal{R}(U)[\cdot]$ is a linear smoothing operator. In system (0.4.10) the second equation is equal to the complex conjugate of the first one. More precisely the operator $R(U)$ is bounded between $\mathbf{H}^s$ to $\mathbf{H}^{s+\rho}$ with $\rho \sim s$. The operator $\mathcal{G}(U)$ is a bounded, linear operator from $\mathbf{H}^s$ to $\mathbf{H}^{s-2}$ for any $s \in \mathbb{R}$ even if the unknown function U has finite regularity. This procedure of rewriting the system as in (0.4.10) is called in literature *para-linearization*, see for instance [73, 24]. Let use describe the strategy of the proofs. We start from the local well-posedness result.

Local theory. We describe the proof given in Chapter 2. Since equation (0.2.1) is quasi-linear the proofs of Theorems 0.2.1, 0.2.2 do not rely on direct fixed point arguments using the Duhamel equation; these arguments are used to study the local theory for the semi-linear equations (i.e. when the nonlinearity f in (0.2.1) depends only on u), see for instance [32]. Our approach is based on the following quasi-linear iterative scheme (similar schemes have been used by Kato in [62] and by Alazard-Baldi-Kwan in [1]). We consider the sequence of linear problems

$$\mathcal{A}_0 := \begin{cases} \partial_t U_0 - \mathrm{i}E\partial_{xx}U_0 = 0, \\ U_0(0) = U^{(0)}, \end{cases} \tag{0.4.11}$$

and for $n \geq 1$

$$\mathcal{A}_n := \begin{cases} \partial_t U_n - \mathrm{i}E\mathcal{G}(U_{n-1})[U_n] - \mathcal{R}(U_{n-1}) = 0, \\ U_n(0) = U^{(0)}, \end{cases} \tag{0.4.12}$$

where $U^{(0)}(x) = (u_0(x), \overline{u_0}(x))$ with $u_0(x)$ given in (0.2.1). The goal is to show that there exists $s_0 > 0$ such that for any $s \geq s_0$ the following facts hold:

1. the iterative scheme is well-defined, i.e. there is $T > 0$ such that for any $n \geq 0$ there exists a unique solution U_n of the problem $\mathcal{A}_n$ which belongs to the space $C^0([0,T)];\mathbf{H}^s)$;

2. the sequence $\{U_n\}_{n\geq 0}$ is bounded in $C^0([0,T)];\mathbf{H}^s)$;

3. $\{U_n\}_{n\geq 0}$ is a Cauchy sequence in $C^0([0,T)];\mathbf{H}^{s-2})$.

From these properties the limit function U belongs to the space $L^\infty([0,T);\mathbf{H}^s)$. In the final part of Section 2.3 we show that actually U is a *classical* solution of (0.2.1), namely U solves (0.4.9) and it belongs to $C^0([0,T);\mathbf{H}^s) \cap C^1([0,T);\mathbf{H}^{s-2})$.

Therefore the key point is to obtain energy estimates for the linear problem in V

$$\begin{cases} \partial_t V - \mathrm{i}E\mathscr{G}(U)[V] - \mathscr{R}(U)U = 0, \\ V(0) = U^{(0)}, \end{cases} \tag{0.4.13}$$

where $U = U(t,x)$ is a fixed function defined for $t \in [0,T], T > 0$, regular enough and $\mathscr{R}(U)$ is regarded as a non homogeneous forcing term. Note that the regularity in time and space of the coefficients of operators $\mathscr{G}(U), \mathscr{R}(U)$ depends on the regularity of the function U. Our strategy is to perform a paradifferential change of coordinates $W := \Phi(U)[V]$ such that the system (0.4.13) in the new coordinates reads

$$\begin{cases} \partial_t W - \mathrm{i}E\widetilde{\mathscr{G}}(U)[W] - \widetilde{\mathscr{R}}(U) = 0, \\ W(0) = \Phi(U^{(0)})[U^{(0)}], \end{cases} \tag{0.4.14}$$

where the operator $\widetilde{\mathscr{G}}(U)[\cdot]$ is diagonal, self-adjoint with constant coefficients in $x \in \mathbb{T}$ and $\widetilde{\mathscr{R}}(U)$ is a bounded term. More precisely we show that the operator $\widetilde{\mathscr{G}}(U)[\cdot]$ has the form

$$\begin{aligned} &\widetilde{\mathscr{G}}(U)[\cdot] := \begin{pmatrix} \mathrm{Op}^{\mathscr{B}W}((\mathrm{i}\xi)^2 + m(U;\xi))[\cdot] & 0 \\ 0 & \mathrm{Op}^{\mathscr{B}W}((\mathrm{i}\xi)^2 + m(U;\xi))[\cdot] \end{pmatrix}, \\ &m(U;\xi) := m_2(U)(\mathrm{i}\xi)^2 + m_1(U)(\mathrm{i}\xi) \in \mathbb{R}, \end{aligned} \tag{0.4.15}$$

with $m(U;\xi)$ real valued and independent of $x \in \mathbb{T}$. Since the symbol $m(U;\xi)$ is real valued the linear operator $\mathrm{i}E\widetilde{\mathscr{G}}(U)$ generates a well defined flow on $L^2 \times L^2$, since it has also constant coefficients in x it generates a flow on $H^s \times H^s$ for $s \geq 0$. The self-adjointness of the operator $\widetilde{\mathscr{G}}(U)$ will be a consequence of the Hamiltonian structure Hyp. 0.2.1 or of the parity preserving one Hyp. 0.2.2. In this diagonalization and reduction to constant coefficients procedure we exploit the fact that the dispersion law of our equation is super-linear. Let us make an example which gives the idea of why the super-linear dispersion law simplify the procedure. Consider the following linear equation with variable coefficients

$$\partial_t u = \mathrm{i}u_{xx} + |v|^2 u_x, \quad v := v(t,x), \quad u := u(t,x) \tag{0.4.16}$$

we want to remove the x-dependence of the coefficient in front of u_x. Let $s(v;x)$ a function to be determined and consider the new variable

$$w := e^{s(v;x)}u, \quad u = e^{-s(v;x)}w,$$

therefore the function w solves the following problem

$$w_t = \underbrace{(\partial_t s(v;x))}_{\text{ord } 0} w + e^{s(v;x)}(\mathrm{i}\partial_{xx} + |v|^2\partial_x)e^{-s(v;x)} w. \tag{0.4.17}$$

Expanding the second summand in the r.h.s. of the above equation we find

$$\begin{aligned} & e^{s(v;x)}(\mathrm{i}\partial_{xx} + |v|^2\partial_x)e^{-s(v;x)} = \\ & \quad \partial_{xx} + |v|^2\partial_x + \underbrace{[\mathrm{i}\partial_{xx}, e^{s(v;x)}]}_{\text{ord } 1} + \text{l.o.t.}, \end{aligned} \tag{0.4.18}$$

where by $[\cdot,\cdot]$ we have denoted the commutator between operators and by l.o.t. we meant differential operators of lower orders. Therefore one can choose a suitable function $s(v;x)$ in order to remove the x dependence of the coefficient of the term of order 1 in the new equation. The advantage of having a super-linear dispersion law is in the fact that the commutator term in (0.4.18) is a differential operator of order strictly bigger than the one coming from the conjugation of ∂_t appearing in (0.4.17). If the linear part of the operator was just of order one, like for instance the half-wave equation, it would be no longer possible to reduce in this way the system to constant coefficients since the contribution coming from the conjugation of the time derivative and from the spatial operator would have the same order. As explained before in the introduction, in the paper we cannot proceed exactly as done in this example: we shall perform *para-differential* changes of coordinates in order to preserve the para-differential structure of (0.4.14).

Long time existence. We describe the strategy adopted in Chapter 3. The first step is to rewrite the system (0.4.9) in the para-differential form (0.4.10). We cannot use the paralinearization done with the aim of proving the local existence because we need to adapt it to symbols which admit multilinear expansions. The further feature we need on the matrix $A(U;x,\xi)$, defining the para-differential operator $\mathcal{G}(U)$ in (0.4.10), is that for any $N>1$ it admits an expansion in homogeneous matrices (with respect to U) up to a non-homogeneous one of size $O(\|U\|^N_{\mathbf{H}^s})$. This paralinearization is performed in Section 3.1.
Instead of reducing directly the *size* of the non linearity (as done in [39] for (KG) or formally in [37] for the (WW)) we perform some para-differential reductions in order to conjugate the para-differential term to an other one which is diagonal with constant coefficients in x up to a remainder which is a very regularizing term (i.e. maps H^s to $H^{s+\rho}$ with $\rho \gg 0$). In this procedure it is fundamental that the

symbols of positive order are purely imaginary, in such a way that the associated para-differential operator is skew self-adjoint. This condition is ensured by some algebraic structure of the equation. More precisely in Theorem 3.2.1 we exhibit a nonlinear map $\Phi(U)U$ with the following properties:

(a) for any fixed U in $\mathbf{H}^{s_0}$, s_0 large enough, the map $\Phi(U)[\cdot]$ is a bounded linear map form $\mathbf{H}^s$ to $\mathbf{H}^s$ for any $s \geq 0$;

(b) set $V := \Phi(U)U$, then one has $\|V\|_{\mathbf{H}^s} \sim \|U\|_{\mathbf{H}^s}$;

(c) the function U solves (0.4.10) then $V = \Phi(U)U$ solves a system of the form (see (3.2.2))

$$\partial_t V = \mathrm{i}E(\Lambda V + \mathrm{Op}^{\mathscr{B}W}(L(U;\xi))V + Q(U)U), \tag{0.4.19}$$

for some diagonal and constant coefficients in x matrix of symbol $L(U;\xi)$ and where $Q(U)$ is a ρ-smoothing remainder for some ρ large. The elements of the matrix $L(U;\xi)$ are real valued at the positive orders.

The function V solving (0.4.19) satisfies

$$\partial_t \|V(t)\|^2_{\mathbf{H}^s} \leq C\|U(t)\|_{\mathbf{H}^{s_0}}\|V(t)\|^2_{\mathbf{H}^s},$$

therefore, as a consequence of Theorem 3.2.1, we have obtained

$$\|U(t)\|^2_{\mathbf{H}^s} \leq C\|U(0)\|^2_{\mathbf{H}^s} + C\int_0^t \|U(\tau)\|_{\mathbf{H}^{s_0}}\|U(\tau)\|^2_{\mathbf{H}^s} d\tau, \quad s \geq s_0 \gg 1. \tag{0.4.20}$$

The type of changes of coordinates we use in Section 3.2 are inspired to those used in Section 2.2. There are two key differences. For proving the local existence we are only interested in giving some energy estimates on the solution. Here the situation is more complicated and we need further information in order to obtain a much longer time of existence. First of all in Theorem 3.2.1 we take into account that our operators and symbols admit multilinear expansions. This justify our definition of operators and symbols in Definitions 1.1.3 and 1.1.12. On the contrary in Chapter 2 we use classes more similar to the non homogeneous classes defined in Definitions 1.1.2 and 1.1.10. The second fundamental difference is that the final system in (3.2.2) is diagonal, constant coefficients in $x \in \mathbb{T}$, up to terms which are ρ-smoothing operators with ρ arbitrary large. We remark that in Chapter 2 we

only need *bounded* remainders. In this part we exploit the super-linear dispersion law of the equation as explained at the end of the preceding paragraph.
In Section 3.3 we give the proof of Theorem 0.3.1. Notice that the r.h.s. in (0.4.20) is linear in $\|U(t)\|_{\mathbf{H}^{s_0}}$ since both the matrix of symbols $L(U;\xi)$ and the matrix of operators $Q(U)$ are $O(\|U\|_{H^{s_0}})$. The aim of Sec. 3.3 is to prove an estimate of the form

$$\|U(t)\|^2_{\mathbf{H}^s} \leq C\|U(0)\|^2_{\mathbf{H}^s} + C\int_0^t \|U(\tau)\|^N_{\mathbf{H}^{s_0}}\|U(\tau)\|^2_{\mathbf{H}^s}d\tau, \quad s \geq s_0 \gg 1, \quad N > 2. \tag{0.4.21}$$

In order to obtain (0.4.21) we proceed in two steps. In the first one a Birkhoff normal form (BNF) procedure is used in order to reduce the size of the paradifferential term $\mathrm{Op}^{\mathscr{B}W}(\mathrm{Im}\, L(U;\xi))V$ in (0.4.19), the term coming from the real part generate, since there is the matrix iE in front in (0.4.19), a skew-selfadjoint operator and hence does not contribute to the energy estimate. In implementing such procedure, as said in the third paragraph of Section 0.1, small divisors appear. When trying to eliminate the term of homogeneity p of $\mathrm{Op}^{\mathscr{B}W}(\mathrm{Im}\, L(U;\xi))V$, the small divisors appearing are linear combinations of the eigenvalues λ_j in (0.4.4), for instance of the form

$$\lambda_{n_1} - \lambda_{n_2} + \lambda_{n_3} - \ldots - \lambda_{n_p}. \tag{0.4.22}$$

The lower bound we are able to impose, see Subsection 3.3.1, on these small divisors is

$$|\lambda_{n_1} - \lambda_{n_2} + \lambda_{n_3} - \ldots - \lambda_{n_p}| \geq c \max(\langle n_1\rangle, \ldots, \langle n_p\rangle)^{-N_0}, \tag{0.4.23}$$

for some $N_0 > 0$ and some $c > 0$, we have denoted by $\langle x\rangle$ the Japanese bracket $\sqrt{1+x^2}$. The bound (0.4.23) is substantially weaker from the one imposed in the preceding works [11, 12, 13] where the r.h.s. of (0.4.23) is replaced by

$$\max_3(\langle n_1\rangle, \ldots, \langle n_p\rangle)^{-N_0}.$$

The latter condition is essential when dealing with semilinear PDEs, where an estimate involving $\max(\langle n_1\rangle, \ldots, \langle n_p\rangle)$ instead of $\max_3(\langle n_1\rangle, \ldots, \langle n_p\rangle)$ would produce a loss of derivatives in the transformation. In our context this loss of derivatives affects only the coefficients of the equations which are low frequencies thanks to the paradifferential structure; therefore we may afford to loose a large number of derivatives on these coefficients since we are working with very smooth functions. This BNF is performed in Subsection 3.3.2.

The second step is the reduction in size of the smoothing remainder. Since we do not have a good knowledge of the algebraic structure of the remainder we construct some modified energies by means of, again, a BNF-type procedure. More precisely we look for an energy $E_s(U)$ such that $E_s(U) \sim \|U(t,\cdot)\|_{H^s}^2$ and

$$E_s(U(t,\cdot)) \leq E_s(U(0,\cdot)) + \left| \int_0^t \|U(\tau,\cdot)\|_{H^s}^{N+2} \right|.$$

Also here small divisors of the form (0.4.22) appear. We can use the same bound (0.4.23) by exploiting the fact that the remainders are very smoothing operators. This part is the content of subsection 3.3.3.

We conclude this introduction with a short summary of each chapter of the thesis:

- In Chapter 1 we develop a paradifferential calculus which will be used systematically in the rest of the thesis;
- In Chapter 2 we prove a local existence theorem for Hamiltonian and parity preserving equations;
- In Chapter 3 we prove a long time existence theorem in the parity preserving case, if the equation (0.2.1) is also reversible with respect to the involution $S : u \mapsto \bar{u}$.

Chapter 1

Para-differential calculus

In this chapter we develop a para-differential calculus following the ideas in [21]. In the first section we introduce several spaces of symbols and operators defined in a neighborhood of the origin. Our classes depend on some extra function U and the constants in the definitions will depend explicitly on it. More precisely our symbols and operators are polynomial in U up to a degree of homogeneity $N-1$ plus a non-homogeneous term which vanishes as $O(\|U\|^N)$ as U goes to 0. We define a para-differential quantization of such symbols and we prove that they enjoy a *symbolic* calculus. Furthermore, following [21], we prove a para-composition theorem (in the sense of Alinhac [2]). The differences between our classes and those in [21] depend only on the extra function U: in their case it is a function of time and space (x,t) which is of class C^k, w.r.t. the variable t, with values in $H^{s-\frac{3}{2}k}$ for any $0 \le k \le K$ (K big enough) and it has zero mean, in our case it can have non zero mean and it is a function of class C^k, w.r.t. the variable t, with values in H^{s-2k} for any $0 \le k \le K$ (K big enough).
In the second section we introduce classes of symbols and operators defined also far away from zero. The definitions are very similar to the ones given in the first section concerning the non-homogeneous terms in the polynomial expansions. The main difference is the following: we require that these symbols, and operators, are "small" when the function U is "small" without quantifying such smallness with the same precision given in the first section. On one hand this lack of precision is somehow needed since we shall work also in the case that the function U is big, on the other we do not need to quantify how small the symbols and operators are since these classes will be used only to prove the *local* well-posedness theorem for (0.2.1). We prove a para-composition theorem for a class of strictly

monotone symbols.

1.1 Multilinear para-differential calculus

1.1.1 Spaces of Smoothing operators

We introduce some notation. If $K \in \mathbb{N}$, I is an interval of $\mathbb{R}$ containing the origin and $s \in \mathbb{R}^+$ we denote by $C_*^K(I, H^s(\mathbb{T}, \mathbb{C}^2))$ (respectively $C_*^K(I, H^s(\mathbb{T}; \mathbb{C})))$, the space of continuous functions U of $t \in I$ with values in $H^s(\mathbb{T}, \mathbb{C}^2)$ (resp. $H^s(\mathbb{T}; \mathbb{C})$), which are K-times differentiable and such that the k−th derivative is continuous with values in $H^{s-2k}(\mathbb{T}, \mathbb{C}^2)$ (resp. $H^{s-2k}(\mathbb{T}; \mathbb{C})$) for any $0 \le k \le K$. We endow the space $C_*^K(I, H^s(\mathbb{T}; \mathbb{C}^2))$ (resp. $C_*^K(I, H^s(\mathbb{T}; \mathbb{C})))$ with the norm

$$\sup_{t \in I} \|U(t, \cdot)\|_{K,s}, \quad \text{where} \quad \|U(t, \cdot)\|_{K,s} := \sum_{k=0}^{K} \left\| \partial_t^k U(t, \cdot) \right\|_{H^{s-2k}}. \tag{1.1.1}$$

We denote by $C_{*\mathbb{R}}^K(I, H^s(\mathbb{T}, \mathbb{C}^2))$, sometimes with $C_{*\mathbb{R}}^K(I; \mathbf{H}^s)$, the “real” subspace of $C_*^K(I, H^s(\mathbb{T}, \mathbb{C}^2))$ made of the functions of t with values in $\mathbf{H}^s(\mathbb{T}; \mathbb{C}^2)$ (see (0.4.1)). Recalling (0.4.3) we shall denote by $C_*^K(I; H_e^s(\mathbb{T}; \mathbb{C}^2))$ (resp. $C_*^K(I; H_e^s(\mathbb{T}; \mathbb{C})))$ the subspace of $C_*^K(I, H^s(\mathbb{T}, \mathbb{C}^2))$ (resp. $C_*^K(I; H^s(\mathbb{T}; \mathbb{C})))$ made of the functions of t with values in $H_e^s(\mathbb{T}; \mathbb{C}^2)$ (resp. $H_e^s(\mathbb{T}; \mathbb{C})$). Analogously $C_{*\mathbb{R}}^K(I, \mathbf{H}_e^s(\mathbb{T}; \mathbb{C}^2))$ denotes the subspace of $C_{*\mathbb{R}}^K(I, \mathbf{H}^s(\mathbb{T}; \mathbb{C}^2))$ made of those functions which are even in x. Moreover if $r \in \mathbb{R}^+$ we set

$$B_s^K(I, r) := \left\{ U \in C_*^K(I, H^s(\mathbb{T}; \mathbb{C}^2)) : \sup_{t \in I} \|U(t, \cdot)\|_{K,s} < r \right\}. \tag{1.1.2}$$

For $n \in \mathbb{N}^*$ we denote by Π_n the orthogonal projector from $L^2(\mathbb{T}; \mathbb{C}^2)$ (or $L^2(\mathbb{T}, \mathbb{C})$) to the subspace spanned by $\{e^{\mathrm{i}nx}, e^{-\mathrm{i}nx}\}$ i.e.

$$(\Pi_n u)(x) = \hat{u}(n) \frac{e^{\mathrm{i}nx}}{\sqrt{2\pi}} + \hat{u}(-n) \frac{e^{-\mathrm{i}nx}}{\sqrt{2\pi}}, \tag{1.1.3}$$

while in the case $n = 0$ we define the mean

$$\Pi_0 u = \frac{1}{\sqrt{2\pi}} \hat{u}(0) = \frac{1}{2\pi} \int_{\mathbb{T}} u(x) dx.$$

If $\mathscr{U} = (U_1, \dots, U_p)$ is a p-tuple of functions, $\vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, we set

$$\Pi_{\vec{n}} \mathscr{U} := (\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p). \tag{1.1.4}$$

Given $(n_1,\dots,n_{p+1})\in\mathbb{N}^{p+1}$ we denote by

$$\max_2(\langle n_1\rangle,\dots,\langle n_{p+1}\rangle),$$

the second largest among the numbers $\langle n_1\rangle,\dots,\langle n_{p+1}\rangle$. The following is the definition of a class of multilinear smoothing operators.

Definition 1.1.1 (**p−homogeneous smoothing operator**). *Let $p\in\mathbb{N}$, $\rho\in\mathbb{R}$ with $\rho\geq 0$. We denote by $\widetilde{\mathcal{R}}^{-\rho}_{p}$ the space of $(p+1)$-linear maps from $(C^\infty(\mathbb{T};\mathbb{C}^2))^p \times C^\infty(\mathbb{T};\mathbb{C})$ to $C^\infty(\mathbb{T};\mathbb{C})$ symmetric in $(U_1,\dots,U_p)$, of the form*

$$(U_1,\dots,U_{p+1})\to R(U_1,\dots,U_p)U_{p+1}, \tag{1.1.5}$$

that satisfy the following. There is $\mu\geq 0$, $C>0$ such that

$$\|\Pi_{n_0}R(\Pi_{\vec{n}}\mathcal{U})\Pi_{n_{p+1}}U_{p+1}\|_{L^2}\leq C\frac{\max_2(\langle n_1\rangle,\dots,\langle n_{p+1}\rangle)^{\rho+\mu}}{\max(\langle n_1\rangle,\dots,\langle n_{p+1}\rangle)^{\rho}}\prod_{j=1}^{p+1}\|\Pi_{n_j}U_j\|_{L^2}. \tag{1.1.6}$$

for any $\mathcal{U}=(U_1,\dots,U_p)\in(C^\infty(\mathbb{T};\mathbb{C}^2))^p$, any $U_{p+1}\in C^\infty(\mathbb{T};\mathbb{C})$, any $\vec{n}=(n_1,\dots,n_p)\in\mathbb{N}^p$, any $n_0,n_{p+1}\in\mathbb{N}$. Moreover, if

$$\Pi_{n_0}R(\Pi_{n_1}U_1,\dots,\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\neq 0, \tag{1.1.7}$$

then there is a choice of signs $\sigma_0,\dots,\sigma_{p+1}\in\{-1,1\}$ such that $\sum_{j=0}^{p+1}\sigma_j n_j=0$.

We shall need also a class of non-homogeneous smoothing operators.

Definition 1.1.2 (**Non-homogeneous smoothing operators**). *Let $K'\leq K\in\mathbb{N}$, $N\in\mathbb{N}$ with $N\geq 1$, $\rho\in\mathbb{R}$ with $\rho\geq 0$ and $r>0$. We define the class of remainders $\mathcal{R}^{-\rho}_{K,K',N}[r]$ as the space of maps $(V,u)\mapsto R(V)u$ defined on $B^K_{s_0}(I,r)\times C^K_*(I,H^{s_0}(\mathbb{T},\mathbb{C}))$ which are linear in the variable u and such that the following holds true. For any $s\geq s_0$ there exist a constant $C>0$ and $r(s)\in]0,r[$ such that for any $V\in B^K_{s_0}(I,r)\cap C^K_*(I,H^s(\mathbb{T},\mathbb{C}^2))$, any $u\in C^K_*(I,H^s(\mathbb{T},\mathbb{C}))$, any $0\leq k\leq K-K'$ and any $t\in I$ the following estimate holds true*

$$\begin{aligned}&\left\|\partial_t^k\left(R(V)u\right)(t,\cdot)\right\|_{H^{s-2k+\rho}}\leq\\ &\sum_{k'+k''=k}C\Big[\|u\|_{k'',s}\|V\|^N_{k'+K',s_0}+\|u\|_{k'',s_0}\|V\|^{N-1}_{k'+K',s_0}\|V\|_{k'+K',s}\Big].\end{aligned}\tag{1.1.8}$$

We will often use the following general class.

Definition 1.1.3 (**Smoothing operator**). *Let $p, N \in \mathbb{N}$, with $p \le N$, $N \ge 1$, $K, K' \in \mathbb{N}$ with $K' \le K$ and $\rho \in \mathbb{R}$, $\rho \ge 0$. We denote by $\Sigma\mathscr{R}^{-\rho}_{K,K',p}[r,N]$ the space of maps $(V,t,U) \to R(V,t)U$ that may be written as*

$$R(V;t)U = \sum_{q=p}^{N-1} R_q(V,\dots,V)U + R_N(V;t)U, \tag{1.1.9}$$

for some $R_q \in \widetilde{\mathscr{R}}^{-\rho}_q$, $q = p,\dots,N-1$ and R_N belongs to $\mathscr{R}^{-\rho}_{K,K',N}[r]$.

The following is a subclass of the previous class made of those operators which are autonomous, i.e. they depend on the variable t only through the function U.

Definition 1.1.4 (**Autonomous smoothing operator**). *We define, according to the notation of Definition 1.1.2, the class of autonomous non-homogeneous smoothing operator $\mathscr{R}^{-\rho}_{K,0,N}[r,\mathrm{aut}]$ as the subspace of $\mathscr{R}^{-\rho}_{K,0,N}[r]$ made of those maps $(U,V) \to R(U)V$ satisfying estimates (1.1.8) with $K' = 0$, the time dependence being only through $U = U(t)$. In the same way, we denote by $\Sigma\mathscr{R}^{-\rho}_{K,0,p}[r,N,\mathrm{aut}]$ the space of maps $(U,V) \to R(U,V)$ of the form (1.1.9) with $K' = 0$ and where the last term belongs to $\mathscr{R}^{-\rho}_{K,0,N}[r,\mathrm{aut}]$.*

Remark 1.1.1. *We remark that if R is in $\widetilde{R}^{-\rho}_p$, $p \ge N$, then $(V,U) \to R(V,\dots,V)U$ is in $\mathscr{R}^{-\rho}_{K,0,N}[r,\mathrm{aut}]$. To see this we argue as follows. Define the discrete convolution between sequences $\mathbf{a} := (a_j)_{j\in\mathbb{Z}}$ and $\mathbf{b} := (b_j)_{j\in\mathbb{Z}}$ as*

$$(\mathbf{a} \star \mathbf{b})_j := \sum_{k\in\mathbb{Z}} a_{k-j} b_j,$$

furthermore denote by $|\cdot|_{\ell^p}$ the usual ℓ^p-norm defined as $|\mathbf{a}|_{\ell^p} := \big(\sum_{k\in\mathbb{Z}} |a_j|^p\big)^{1/p}$ for $p \in [1,\infty)$. Then one has

$$\Big\|\partial_t^k (R(V)U)\Big\|_{\mathbf{H}^{s+\rho-2k}} = \Big|\langle n_0\rangle^{s+\rho-2k} \Big\|\partial_t^k \Pi_{n_0} R(V)U\Big\|_{L^2}\Big|_{\ell^2}. \tag{1.1.10}$$

By multi-linearity of R, conditions (1.1.6) and (1.1.7) we obtain

$$\begin{aligned}
&\langle n_0\rangle^{s+\rho-2k} \Big\|\partial_t^k \Pi_{n_0} R(V)U\Big\|_{L^2} \le \\
&\qquad \le C\langle n_0\rangle^{s+\rho-2k} \sum_{\substack{n_1+\dots+n_{p+1}=n_0 \\ k_1+\dots+k_{p+1}=k}} \frac{\max_2\{\langle n_1\rangle,\dots,\langle n_{p+1}\rangle\}^{\mu+\rho}}{\max\{\langle n_1\rangle,\dots,\langle n_{p+1}\rangle\}^{\rho}} \prod_{j=1}^{p} \Big\|\partial_t^{k_j} \Pi_{n_j} V\Big\|_{L^2} \times \\
&\qquad\quad \times \Big\|\partial_t^{k_{p+1}} \Pi_{n_{p+1}} U\Big\|_{L^2}.
\end{aligned}$$

Let us suppose for simplicity that $\langle n_1\rangle \geq \ldots \geq \langle n_{p+1}\rangle$. From the fact that $\langle n_0\rangle \leq (p+1)\langle n_1\rangle$ we can bound the r.h.s. of the above equation by

$$C \sum_{\substack{n_1+\ldots+n_{p+1}=n_0 \\ k_1+\ldots+k_{p+1}=k}} \langle n_1\rangle^{s-2k} \langle n_2\rangle^{\mu+\rho} \prod_{j=1}^{p} \left\| \partial_t^{k_j} \Pi_{n_j} V \right\|_{L^2} \left\| \partial_t^{k_{p+1}} \Pi_{n_{p+1}} U \right\|_{L^2} =$$
$$C \sum_{\substack{n_1+\ldots+n_{p+1}=n_0 \\ k_1+\ldots+k_{p+1}=k}} \langle n_1\rangle^{s-2k_1} \left\| \partial_t^{k_1} \Pi_{n_1} V \right\|_{L^2} \langle n_2\rangle^{\mu+\rho-2k_2} \left\| \partial_t^{k_2} \Pi_{n_2} V \right\|_{L^2} \times$$
$$\prod_{j=3}^{p} \left\langle n_j \right\rangle^{-2k_j} \left\| \partial_t^{k_j} \Pi_{n_j} V \right\|_{L^2} \left\langle n_{p+1} \right\rangle^{s-2k_{p+1}} \left\| \partial_t^{k_{p+1}} \Pi_{n_{p+1}} U \right\|_{L^2}.$$

Therefore we can bound, by using the Young inequality for convolution of sequences, the r.h.s. of (1.1.10) *by*

$$\left| \langle n_1\rangle^{s-2k_1} \left\| \partial_t^{k_1} \Pi_{n_1} V \right\|_{L^2} \right|_{\ell^2} \left| \langle n_2\rangle^{\mu+\rho-2k_2} \left\| \partial_t^{k_1} \Pi_{n_1} V \right\|_{L^2} \right|_{\ell^1} \times$$
$$\times \prod_{j=3}^{p} \left| \left\langle n_j \right\rangle^{2k_j} \left\| \partial_t^{k_j} \Pi_{n_j} V \right\|_{L^2} \right|_{\ell^1} \times \left| \left\langle n_{p+1} \right\rangle^{2k_{p+1}} \left\| \partial_t^{k_{p+1}} \Pi_{n_{p+1}} U \right\|_{L^2} \right|_{\ell^1}.$$

By choosing $s_0 > \min\{\rho+\mu+1/2, 2K\}$, summing over $0 \leq k \leq K$ one gets by Cauchy-Schwartz the (1.1.8).

Remark 1.1.2. *Let $R_1(U)$ be a smoothing operator in $\Sigma\mathscr{R}^{-\rho_1}_{K,K',p_1}[r,N]$ and $R_2(U)$ in $\Sigma\mathscr{R}^{-\rho_2}_{K,K',p_2}[r,N]$, then the operator $R_1(U) \circ R_2(U)[\cdot]$ is in $\Sigma\mathscr{R}^{-\rho}_{K,K',p_1+p_2}[r,N]$, where $\rho = \min(\rho_1,\rho_2)$.*

1.1.2 Spaces of Maps

In the following, sometimes, we shall treat operators without having to keep track of the number of lost derivatives in a very precise way. We introduce some further classes.

Definition 1.1.5 ($p-$**homogeneous maps).** *Let $p \in \mathbb{N}$, $m \in \mathbb{R}$ with $m \geq 0$. We denote by $\widetilde{\mathscr{M}}^m_p$ the space of $(p+1)$-linear maps from the space $(C^\infty(\mathbb{T};\mathbb{C}^2))^p \times C^\infty(\mathbb{T};\mathbb{C})$ to the space $C^\infty(\mathbb{T};\mathbb{C})$ symmetric in $(U_1,\ldots,U_p)$, of the form*

$$(U_1,\ldots,U_{p+1}) \to M(U_1,\ldots,U_p)U_{p+1}, \tag{1.1.11}$$

that satisfy the following. There is $\mu \geq 0$, $C > 0$, *such that for any* $\mathscr{U} = (U_1, \dots, U_p) \in (C^\infty(\mathbb{T};\mathbb{C}^2))^p$, *any* $U_{p+1} \in C^\infty(\mathbb{T};\mathbb{C})$, *any* $\vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p$, *any* $n_0, n_{p+1} \in \mathbb{N}$

$$\|\Pi_{n_0} M(\Pi_{\vec{n}}\mathscr{U})\Pi_{n_{p+1}} U_{p+1}\|_{L^2} \leq C(\langle n_0\rangle + \langle n_1\rangle + \ldots + \langle n_{p+1}\rangle)^m \prod_{j=1}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}. \tag{1.1.12}$$

Moreover, if

$$\Pi_{n_0} M(\Pi_{n_1} U_1, \dots, \Pi_{n_p} U_p)\Pi_{n_{p+1}} U_{p+1} \neq 0, \tag{1.1.13}$$

then there is a choice of signs $\sigma_0, \dots, \sigma_{p+1} \in \{-1, 1\}$ *such that* $\sum_{j=0}^{p+1} \sigma_j n_j = 0$. *When* $p = 0$ *the conditions above mean that* M *is a linear map on* $C^\infty(\mathbb{T};\mathbb{C})$ *into itself.*

Definition 1.1.6 (**Non-homogeneous maps).** *Let* $K' \leq K \in \mathbb{N}$, $N \in \mathbb{N}$ *with* $N \geq 1$, $m \in \mathbb{R}$ *with* $m \geq 0$ *and* $r > 0$. *We define the class* $\mathscr{M}^m_{K,K',N}[r]$ *as the space of maps* $(V, u) \mapsto M(V)u$ *defined on* $B^K_{s_0}(I, r) \times C^K_*(I, H^{s_0}(\mathbb{T}, \mathbb{C}))$ *for some* $s_0 > 0$, *which are linear in the variable* u *and such that the following holds true. For any* $s \geq s_0$ *there exist a constant* $C > 0$ *and* $r(s) \in]0, r[$ *such that for any* $V \in B^K_{s_0}(I, r) \cap C^K_*(I, H^s(\mathbb{T}, \mathbb{C}^2))$, *any* $u \in C^K_*(I, H^s(\mathbb{T}, \mathbb{C}))$, *any* $0 \leq k \leq K - K'$ *and any* $t \in I$ *the following estimate holds true*

$$\begin{aligned} &\left\| \partial_t^k (M(V)u)(t,\cdot) \right\|_{H^{s-2k-m}} \leq \\ &\sum_{k'+k''=k} C\Big[\|u\|_{k'',s} \|V\|^N_{k'+K',s_0} + \|u\|_{k'',s_0} \|V\|^{N-1}_{k'+K',s_0} \|V\|_{k'+K',s} \Big]. \end{aligned} \tag{1.1.14}$$

Definition 1.1.7 (**Maps).** *Let* $p, N \in \mathbb{N}$, *with* $p \leq N$, $N \geq 1$, $K, K' \in \mathbb{N}$ *with* $K' \leq K$ *and* $\rho \in \mathbb{R}$, $m \geq 0$. *We denote by* $\Sigma\mathscr{M}^m_{K,K',p}[r, N]$ *the space of maps* $(V, t, U) \to M(V, t)U$ *that may be written as*

$$M(V;t)U = \sum_{q=p}^{N-1} M_q(V, \dots, V)U + M_N(V;t)U, \tag{1.1.15}$$

for some $M_q \in \widetilde{\mathscr{M}}^m_q$, $q = p, \dots, N-1$ *and* M_N *belongs to* $\mathscr{M}^m_{K,K',N}[r]$. *Finally we set* $\widetilde{\mathscr{M}}_p := \cup_{m\geq 0}\widetilde{\mathscr{M}}^m_p$, $\mathscr{M}_{K,K',p}[r] := \cup_{m\geq 0}\mathscr{M}^m_{K,K',p}[r]$ *and* $\Sigma\mathscr{M}_{K,K',p}[r, N] := \cup_{m\geq 0}\Sigma\mathscr{M}^m_{K,K',p}[r]$.

Definition 1.1.8 (**Autonomous maps).** *We define, according to the notation of Definition 1.1.6, the class of autonomous non-homogeneous smoothing operator*

$\mathcal{M}^m_{K,0,N}[r,\mathrm{aut}]$ *as the subspace of* $\mathcal{M}^m_{K,0,N}[r]$ *made of those maps* $(U,V)\to M(U)V$ *satisfying estimates* (1.1.8) *with* $K'=0$*, the time dependence being only through* $U=U(t)$*. In the same way, we denote by* $\Sigma\mathcal{M}^m_{K,0,p}[r,N,\mathrm{aut}]$ *the space of maps* $(U,V)\to M(U,V)$ *of the form* (1.1.9) *with* $K'=0$ *and where the last term belongs to* $\mathcal{M}^m_{K,0,N}[r,\mathrm{aut}]$.

Remark 1.1.3. *We remark that if M is in* $\widetilde{\mathcal{M}}^m_p$*,* $p\geq N$*, then* $(V,U)\to M(V,\ldots,V)U$ *is in* $\mathcal{M}^m_{K,0,N}[r,\mathrm{aut}]$*. The proof of this fact is very similar to the one regarding the smoothing remainders in Remark 1.1.1.*

1.1.3 Spaces of Symbols

We give the definition of a class of multilinear symbols.

Definition 1.1.9 (p**-homogeneous symbols).** *Let* $m\in\mathbb{R}$*,* $p\in\mathbb{N}$*. We denote by* $\widetilde{\Gamma}^m_p$ *the space of symmetric p-linear maps from* $(C^\infty(\mathbb{T};\mathbb{C}^2))^p$ *to the space of* C^∞ *functions in* $(x,\xi)\in\mathbb{T}\times\mathbb{R}$

$$\mathcal{U}\to((x,\xi)\to a(\mathcal{U};x,\xi))$$

satisfying the following. There is $\mu>0$ *and for any* $\alpha,\beta\in\mathbb{N}$ *there is* $C>0$ *such that*

$$|\partial_x^\alpha\partial_\xi^\beta a(\Pi_{\vec{n}}\mathcal{U};x,\xi)|\leq C\langle\vec{n}\rangle^{\mu+\alpha}\langle\xi\rangle^{m-\beta}\prod_{j=1}^p\|\Pi_{n_j}U_j\|_{L^2},\tag{1.1.16}$$

for any $\mathcal{U}=(U_1,\ldots,U_p)$ *in* $(C^\infty(\mathbb{T};\mathbb{C}^2))^p$*, and* $\vec{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$*, where* $\langle\vec{n}\rangle:=\sqrt{1+|n_1|^2+\ldots+|n_p|^2}$*. Moreover we assume that, if for some* $(n_0,\ldots,n_p)\in\mathbb{N}^{p+1}$*,*

$$\Pi_{n_0}a(\Pi_{n_1}U_1,\ldots,\Pi_{n_p}U_p;\cdot)\neq 0,\tag{1.1.17}$$

then there exists a choice of signs $\sigma_0,\ldots,\sigma_p\in\{-1,1\}$ *such that* $\sum_{j=0}^p\sigma_jn_j=0$*. For* $p=0$ *we denote by* $\widetilde{\Gamma}^0_0$ *the space of constant coefficients symbols* $\xi\mapsto a(\xi)$ *which satisfy the* (1.1.16) *with* $\alpha=0$ *and the r.h.s. replaced by* $C\langle\xi\rangle^{m-\beta}$.

Remark 1.1.4. *In the sequel we shall consider functions* $\mathcal{U}=(U_1,\ldots,U_p)$ *which depends also on time t, so that the above definition are functions of* (t,x,ξ) *that we denote by* $a(\mathcal{U};t,x,\xi)$.

Remark 1.1.5. *Let $K \in \mathbb{N}$, $\sigma_0 > 2K + \mu + 1/2$ and $\sigma \geq \sigma_0$. Consider a function $U(t,x)$ in the space $C_*^K(I; H^\sigma(\mathbb{T}, \mathbb{C}^2))$, then for any $0 \leq k \leq K$, $0 \geq \alpha \leq \sigma - \sigma_0$ and $\beta \in \mathbb{N}$ the function $a(U, \ldots, U; x, \xi)$ satisfies*

$$\left|\partial_t^k \partial_x^\alpha \partial_x^\beta a(U, \ldots, U; x, \xi)\right| \leq C \langle\xi\rangle^{m-\beta} \|U\|_{k,\sigma_0+\alpha} \|U\|_{k,\sigma_0}^{p-1}. \tag{1.1.18}$$

To see this let us develop the derivative with respect to the variable t

$$\partial_t^k \partial_x^\alpha \partial_x^\beta a(U, \ldots, U; x, \xi) = \sum_{\substack{k_1+\ldots+k_p=k \\ n_1,\ldots,n_p \in \mathbb{N}}} C_{k_1,\ldots,k_p} a(\partial_t^{k_1} \Pi_{n_1} U, \ldots, \partial_t^{k_p} \Pi_{n_p} U; x, \xi),$$

where $C_{k_1,\ldots,k_p}$ are suitable binomial coefficients. Thanks to the latter formula, condition (1.1.16) *and supposing, for simplicity, that $n_1 \geq \ldots \geq n_p$ we deduce that*

$$\begin{aligned}
&\left|\partial_t^k \partial_x^\alpha \partial_x^\beta a(U, \ldots, U; x, \xi)\right| \leq \\
&C \sum_{\substack{k_1+\ldots+k_p=k \\ n_1,\ldots,n_p \in \mathbb{N}}} \langle\xi\rangle^{m-\beta} \langle n_1\rangle^{\mu+\alpha} \prod_{j=1}^{p} \left\|\partial_t^{k_j} \Pi_{n_j} U\right\|_{L^2} = \\
&C \sum_{\substack{k_1+\ldots+k_p=k \\ n_1,\ldots,n_p \in \mathbb{N}}} \langle\xi\rangle^{m-\beta} \langle n_1\rangle^{\mu+\alpha} \prod_{j=1}^{p} \langle n_j\rangle^{-\sigma_0+2k_j} \prod_{j=1}^{p} \langle n_j\rangle^{\sigma_0-2k_j} \left\|\partial_t^{k_j} \Pi_{n_j} U\right\|_{L^2}.
\end{aligned}$$

From the latter inequality it is easy to obtain the condition (1.1.18) *by using the Cauchy-Schwartz inequality and the fact that $\mu - \sigma_0 + 2k < -1/2$.*

Remark 1.1.6. *We note that, if $a \in \widetilde{\Gamma}_p^m$ and $b \in \widetilde{\Gamma}_q^{m'}$ then $ab \in \widetilde{\Gamma}_{p+q}^{m+m'}$, and $\partial_x a \in \widetilde{\Gamma}_p^m$ while $\partial_\xi a \in \widetilde{\Gamma}_p^{m-1}$.*

Remark 1.1.7. *We have that the function*

$$\mathtt{p}(\xi) := \sum_{k=1}^{M} \frac{m_k}{\langle\xi\rangle^{2k+1}}, \quad \xi \in \mathbb{R}, \tag{1.1.19}$$

belongs to the class $\widetilde{\Gamma}_0^{-3}$.

We shall need also a class of non-homogeneous nonlinear symbols.

Definition 1.1.10 (Non-homogeneous Symbols). *Let $m \in \mathbb{R}$, $p \in \mathbb{N}$, $p \geq 1$, $K' \leq K$ in $\mathbb{N}$, $r > 0$. We denote by $\Gamma^m_{K,K',p}[r]$ the space of functions $(U;t,x,\xi) \mapsto a(U;t,x,\xi)$, defined for $U \in B^K_{\sigma_0}(I,r)$, for some large enough σ_0, with complex values such that for any $0 \leq k \leq K-K'$, any $\sigma \geq \sigma_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{\sigma_0}(I,r(\sigma)) \cap C^{k+K'}_*(I,H^\sigma(\mathbb{T};\mathbb{C}^2))$ and any $\alpha,\beta \in \mathbb{N}$, with $\alpha \leq \sigma - \sigma_0$*

$$\left|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U;t,x,\xi)\right| \leq C\langle\xi\rangle^{m-\beta} \|U\|^{p-1}_{k+K',\sigma_0} \|U\|_{k+K',\sigma}. \tag{1.1.20}$$

Remark 1.1.8. *We note that if $a \in \Gamma^m_{K,K',p}[r]$ with $K'+1 \leq K$, then $\partial_t a \in \Gamma^m_{K,K'+1,p}[r]$. Moreover if $a \in \Gamma^m_{K,K',p}[r]$ then $\partial_x a \in \Gamma^m_{K,K',p}[r]$ and $\partial_\xi a \in \Gamma^{m-1}_{K,K',p}[r]$. Finally if $a \in \Gamma^m_{K,K',p}[r]$ and $b \in \Gamma^{m'}_{K,K',q}[r]$ then $ab \in \Gamma^{m+m'}_{K,K',p+q}[r]$.*

The following is a subclass of the class defined in 1.1.10 made of those symbols which depend on the variable t only through the function U.

Definition 1.1.11 (Autonomous non-homogeneous Symbols). *We denote by $\Gamma^m_{K,0,p}\,[r,\mathrm{aut}]$ the subspace of $\Gamma^m_{K,0,p}[r]$ made of the non-homogeneous symbols $(U,x,\xi) \to a(U;x,\xi)$ that satisfy estimate* (1.1.20) *with $K' = 0$, the time dependence being only through $U = U(t)$.*

Remark 1.1.9. *By using* (1.1.18)*, we deduce that a symbol $a(\mathcal{U};\cdot)$ of $\widetilde{\Gamma}^m_p$ defines, by restriction to the diagonal, the symbol $a(U,\ldots,U;\cdot)$ in $\Gamma^m_{K,0,p}[r,\mathrm{aut}]$ for any $r > 0$.*

The following is the general class of symbols we shall deal with.

Definition 1.1.12 (Symbols). *Let $m \in \mathbb{R}$, $p \in \mathbb{N}$, $K,K' \in \mathbb{N}$ with $K' \leq K$, $r > 0$ and $N \in \mathbb{N}$ with $p \leq N$. One denotes by $\Sigma\Gamma^m_{K,K',p}[r,N]$ the space of functions $(U,t,x,\xi) \to a(U;t,x,\xi)$ such that there are homogeneous symbols $a_q \in \widetilde{\Gamma}^m_q$ for $q = p,\ldots,N-1$ and a non-homogeneous symbol $a_N \in \Gamma^m_{K,K',N}[r]$ such that*

$$a(U;t,x,\xi) = \sum_{q=p}^{N-1} a_q(U,\ldots,U;x,\xi) + a_N(U;t,x,\xi). \tag{1.1.21}$$

We set $\Sigma\Gamma^{-\infty}_{K,K',p}[r,N] = \cap_{m\in\mathbb{R}}\Sigma\Gamma^m_{K,K',p}[r,N]$.
We define the subclasses of autonomous symbols $\Sigma\Gamma^m_{K,K',p}[r,N,\mathrm{aut}]$ by (1.1.21) *where a_N is in the class $\Gamma^m_{K,0,N}[r,\mathrm{aut}]$ of Definition 1.1.11. Finally we set $\Sigma\Gamma^{-\infty}_{K,K',p}[r,N,\mathrm{aut}] = \cap_{m\in\mathbb{R}}\Sigma\Gamma^m_{K,K',p}[r,N,\mathrm{aut}]$.*

We also introduce the following class of "functions", i.e. those bounded symbols which are independent of the variable ξ.

Definition 1.1.13 (Functions). *Fix* $N \in \mathbb{N}$, $p \in \mathbb{N}$ *with* $p \le N$, $K, K' \in \mathbb{N}$ *with* $K' \le K$, $r > 0$. *We denote by* $\widetilde{\mathscr{F}}_p$ *(resp.* $\mathscr{F}_{K,K',p}[r]$*, resp.* $\mathscr{F}_{K,K',p}[r,\mathrm{aut}]$*, resp.* $\Sigma\mathscr{F}_p^q[r,N]$*, resp.* $\Sigma\mathscr{F}_{K,K',p}[r,N,\mathrm{aut}]$*) the subclass of* $\widetilde{\Gamma}_p^0$ *(resp.* $\Gamma_p^0[r]$*, resp.* $\Gamma_p^0[r,\mathrm{aut}]$*, resp.* $\Sigma\Gamma_p^{0,q}[r,N]$*, resp.* $\Sigma\Gamma_p^0[r,N,\mathrm{aut}]$*) made of those symbols which are independent of* ξ.

1.1.4 Quantization of symbols

Given a smooth symbol $(x,\xi) \to a(x,\xi)$, we define, for any $\sigma \in [0,1]$, the quantization of the symbol a as the operator acting on functions u as

$$\mathrm{Op}_\sigma(a(x,\xi))u = \frac{1}{2\pi}\int_{\mathbb{R}\times\mathbb{R}} e^{\mathrm{i}(x-y)\xi} a(\sigma x + (1-\sigma)y,\xi)u(y)\,dy\,d\xi. \tag{1.1.22}$$

This definition is meaningful in particular if $u \in C^\infty(\mathbb{T})$ (identifying u to a 2π-periodic function). By decomposing u in Fourier series as $\sum_{j\in\mathbb{Z}} \hat{u}(j)(1/\sqrt{2\pi})e^{\mathrm{i}jx}$, we may calculate the oscillatory integral in (1.1.22) obtaining for any $\sigma \in [0,1]$

$$\mathrm{Op}_\sigma(a)u := \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j,(1-\sigma)k+\sigma j\big)\hat{u}(j)\Big)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}, \tag{1.1.23}$$

where $\hat{a}(k,\xi)$ is the k^{th}–Fourier coefficient of the 2π–periodic function $x \mapsto a(x,\xi)$. For convenience in the paper we shall use two particular quantizations:

Standard quantization. We define the standard quantization by specifying the formula (1.1.23) for $\sigma = 1$:

$$\mathrm{Op}(a)u := \mathrm{Op}_1(a)u = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j,j\big)\hat{u}(j)\Big)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}; \tag{1.1.24}$$

Weyl quantization. We define the Weyl quantization by specifying the formula (1.1.23) for $\sigma = \frac{1}{2}$:

$$\mathrm{Op}^W(a)u := \mathrm{Op}_{\frac{1}{2}}(a)u = \frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} \hat{a}\big(k-j,\frac{k+j}{2}\big)\hat{u}(j)\Big)\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}. \tag{1.1.25}$$

Moreover the above formulas allow to transform the symbols between different quantizations, in particular we have

$$\mathrm{Op}(a) = \mathrm{Op}^W(b), \qquad \text{where} \quad \hat{b}(j,\xi) = \hat{a}(j,\xi - \frac{j}{2}). \tag{1.1.26}$$

We want to define a *para-differential* quantization. First we give the following definition.

Definition 1.1.14 (Admissible cut-off functions). *Fix $p \in \mathbb{N}$ with $p \geq 1$. We say that $\chi_p \in C^\infty(\mathbb{R}^p \times \mathbb{R};\mathbb{R})$ and $\chi \in C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$ are admissible cut-off functions if they are even with respect to each of their arguments and there exists $\delta > 0$ such that*

$$\begin{aligned}
&\operatorname{supp}\chi_p \subset \left\{(\xi',\xi) \in \mathbb{R}^p \times \mathbb{R}; |\xi'| \leq \delta\langle\xi\rangle\right\}, \qquad \chi_p(\xi',\xi) \equiv 1 \text{ for } |\xi'| \leq \frac{\delta}{2}\langle\xi\rangle,\\
&\operatorname{supp}\chi \subset \left\{(\xi',\xi) \in \mathbb{R} \times \mathbb{R}; |\xi'| \leq \delta\langle\xi\rangle\right\}, \qquad \chi(\xi',\xi) \equiv 1 \text{ for } |\xi'| \leq \frac{\delta}{2}\langle\xi\rangle.
\end{aligned}$$

We assume moreover that for any derivation indices α and β

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi_p(\xi',\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-|\beta|}, \ \forall \alpha \in \mathbb{N}, \beta \in \mathbb{N}^p,\\
&|\partial_\xi^\alpha \partial_{\xi'}^\beta \chi(\xi',\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{-\alpha-\beta}, \ \forall \alpha, \beta \in \mathbb{N}.
\end{aligned}$$

An example of function satisfying the condition above, and that will be extensively used in the rest of the paper, is $\chi(\xi',\xi) := \widetilde{\chi}(\xi'/\langle\xi\rangle)$, where $\widetilde{\chi}$ is a function in $C_0^\infty(\mathbb{R};\mathbb{R})$ having a small enough support and equal to one in a neighborhood of zero. For any $a \in C^\infty(\mathbb{T})$ we shall use the following notation

$$(\chi(D)a)(x) = \sum_{j\in\mathbb{Z}} \chi(j)\Pi_j a. \tag{1.1.27}$$

Definition 1.1.15 (The Bony quantization). *Let χ be an admissible cut-off function according to Definition 1.1.14. If a is a symbol in $\widetilde{\Gamma}_p^m$ and b is in $\Gamma_{K,K',p}^m[r]$, we set, using notation* (1.1.4)*,*

$$\begin{aligned}
a_\chi(\mathcal{U};x,\xi) &= \sum_{\vec{n}\in\mathbb{N}^p} \chi_p(\vec{n},\xi)\, a(\Pi_{\vec{n}}\mathcal{U};x,\xi),\\
b_\chi(U;t,x,\xi) &= \frac{1}{2\pi}\int_{\mathbb{T}} \chi\left(\eta,\xi\right)\hat{b}(U;t,\eta,\xi)e^{\mathrm{i}\eta x}\,d\eta.
\end{aligned} \tag{1.1.28}$$

We define the Bony quantization as

$$\begin{aligned} \mathrm{Op}^{\mathscr{B}}(a(\mathscr{U};\cdot)) &= \mathrm{Op}(a_\chi(\mathscr{U};\cdot)), \\ \mathrm{Op}^{\mathscr{B}}(b(U;t,\cdot)) &= \mathrm{Op}(b_\chi(U;t,\cdot)). \end{aligned} \tag{1.1.29}$$

and the Bony-Weyl quantization as

$$\begin{aligned} \mathrm{Op}^{\mathscr{B}W}(a(\mathscr{U};\cdot)) &= \mathrm{Op}^W(a_\chi(\mathscr{U};\cdot)), \\ \mathrm{Op}^{\mathscr{B}W}(b(U;t,\cdot)) &= \mathrm{Op}^W(b_\chi(U;t,\cdot)). \end{aligned} \tag{1.1.30}$$

Finally, if a is a symbol in the class $\Sigma\Gamma^m_{K,K',p}[r,N]$, that we decompose as in (1.1.21)*, we define its Bony quantization as*

$$\mathrm{Op}^{\mathscr{B}}(a(U;t,\cdot)) = \sum_{q=p}^{N-1} \mathrm{Op}^{\mathscr{B}}(a_q(U,\dots,U;\cdot)) + \mathrm{Op}^{\mathscr{B}}(a_N(U;t,\cdot)), \tag{1.1.31}$$

and its Bony-Weyl quantization as

$$\mathrm{Op}^{\mathscr{B}W}(a(U;t,\cdot)) = \sum_{q=p}^{N-1} \mathrm{Op}^{\mathscr{B}W}(a_q(U,\dots,U;\cdot)) + \mathrm{Op}^{\mathscr{B}W}(a_N(U;t,\cdot)). \tag{1.1.32}$$

For symbols belonging to the autonomous subclass $\Sigma\Gamma^m_{K,0,p}[r,N,\mathrm{aut}]$ we shall not write the time dependence in (1.1.31) *and* (1.1.32)*.*

Remark 1.1.10. *Let $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$. We note that*

$$\begin{aligned} \overline{\mathrm{Op}^{\mathscr{B}}(a(U;t,x,\xi)[v]} &= \mathrm{Op}^{\mathscr{B}}(\overline{a^\vee(U;t,x,\xi)})[\bar v], \\ \overline{\mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi)[v]} &= \mathrm{Op}^{\mathscr{B}W}(\overline{a^\vee(U;t,x,\xi)})[\bar v], \end{aligned} \tag{1.1.33}$$

where

$$a^\vee(U;t,x,\xi) := a(U;t,x,-\xi). \tag{1.1.34}$$

Moreover if we define the operator $A(U,t)[\cdot] := \mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi))[\cdot]$ we have that $A^*(U,t)$, its adjoint operator w.r.t. the $L^2(\mathbb{T};\mathbb{C})$ scalar product, can be written as

$$A^*(U,t)[v] = \mathrm{Op}^{\mathscr{B}W}\Big(\overline{a(U;t,x,\xi)}\Big)[v]. \tag{1.1.35}$$

Remark 1.1.11. *Let us define*

$$\mathtt{l}(\xi) := (\mathrm{i}\xi)^2 + \mathtt{p}(\xi), \tag{1.1.36}$$

with $\mathtt{p}(\xi)$ *defined in* (1.1.19). *By Remark 1.1.7 we have that* $\mathtt{l}(\xi)$ *belongs to* $\widetilde{\Gamma}^2_0$. *Moreover we note that the operator* λ *defined in* (0.4.4) *can be written as*

$$\lambda[\cdot] = \mathrm{Op}(\mathtt{l}(\xi))[\cdot] \tag{1.1.37}$$

Remark 1.1.12. *By formula* (1.1.35) *one has that a para-differential operator* $\mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi))[\cdot]$ *is self-adjoint, w.r.t. the* $L^2(\mathbb{T};\mathbb{C})$ *scalar product, if and only if the symbol* $a(U;t,x,\xi)$ *is real valued for any* $x \in \mathbb{T}$, $\xi \in \mathbb{R}$.

Proposition 1.1.1 (Action of para-differential operator). *One has the following.* (i) *Let* $m \in \mathbb{R}$, $p \in \mathbb{N}$. *There is* $\sigma > 0$ *such that for any symbol* $a \in \widetilde{\Gamma}^m_p$, *the map*

$$(U_1,\ldots,U_{p+1}) \to \mathrm{Op}^{\mathscr{B}W}(a(U_1,\ldots,U_p;\cdot))U_{p+1}, \tag{1.1.38}$$

extends, for any $s \in \mathbb{R}$, *as a continuous* $(p+1)$*-linear map*

$$\big(H^\sigma(\mathbb{T};\mathbb{C}^2)\big)^p \times H^s(\mathbb{T};\mathbb{C}) \to H^{s-m}(\mathbb{T};\mathbb{C}). \tag{1.1.39}$$

Moreover, there is a constant $C > 0$, *depending only on* s *and on* (1.1.16) *with* $\alpha = \beta = 0$, *such that*

$$\|\mathrm{Op}^{\mathscr{B}W}(a(\mathcal{U};\cdot))U_{p+1}\|_{H^{s-m}} \le C \prod_{j=1}^{p} \|U_j\|_{H^\sigma}\|U_{p+1}\|_{H^s}, \tag{1.1.40}$$

where $\mathcal{U} = (U_1,\ldots,U_p)$. *Finally, if for some* $(n_0,\ldots,n_{p+1}) \in \mathbb{N}^{p+2}$,

$$\Pi_{n_0}\mathrm{Op}^{\mathscr{B}W}(a(\Pi_{\vec{n}}\mathcal{U};\cdot))\Pi_{n_{p+1}}U_{p+1} \neq 0, \tag{1.1.41}$$

with $\vec{n} = (n_1,\ldots,n_p) \in \mathbb{N}^p$, *then there is a choice of signs* $\sigma_j \in \{-1,1\}$, $j = 0,\ldots,p+1$, *such that* $\sum_{j=0}^{p+1}\sigma_j n_j = 0$ *and the indices satisfy*

$$n_0 \sim n_{p+1}, \quad n_j \le C\delta n_0, \quad n_j \le C\delta n_{p+1}, \quad j = 1,\ldots,p. \tag{1.1.42}$$

(ii) *Let* $r > 0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$, $p \ge 1$, $K' \le K \in \mathbb{N}$, $a \in \Gamma^m_{K,K',p}[r]$. *There is* $\sigma > 0$ *such that for any* $U \in B^K_\sigma(I,r)$, *the operator* $\mathrm{Op}^{\mathscr{B}W}(a(U;t,\cdot))$ *extends, for any* $s \in \mathbb{R}$, *as a bounded linear operator*

$$C^{K-K'}_*(I,H^s(\mathbb{T};\mathbb{C})) \to C^{K-K'}_*(I,H^{s-m}(\mathbb{T};\mathbb{C})). \tag{1.1.43}$$

Moreover, there is a constant $C>0$, depending only on s,r and (1.1.20) *with* $0\leq\alpha\leq 2$, $\beta=0$, *such that, for any $t\in I$, any $0\leq k\leq K-K'$,*

$$\|\mathrm{Op}^{\mathscr{B}W}(\partial_t^k a(U;t,\cdot))\|_{\mathscr{L}(H^s,H^{s-m})}\leq C\|U\|_{k+K',\sigma}^p, \tag{1.1.44}$$

so that

$$\|\mathrm{Op}^{\mathscr{B}W}(a(U;t,\cdot))V(t)\|_{K-K',s-m}\leq C\|U\|_{K,\sigma}^p\|V\|_{K-K',s}. \tag{1.1.45}$$

Proof. We prove item (i). Let χ_p be an admissible cut-off function as in Definition 1.1.14, we define the symbol

$$b(x,\xi):=a_{\chi_p}(\mathscr{U};x,\xi)=\sum_{n\in\mathbb{N}^p}\chi_p(n,\xi)a(\Pi_n\mathscr{U};x,\xi),$$

where we have set $\mathscr{U}:=(U_1,\dots,U_p)$, $n:=(n_1,\dots,n_p)$ and $\Pi_n\mathscr{U}$ is defined in (1.1.4). Then, by Definition 1.1.15, we have

$$\begin{aligned}&\mathrm{Op}^{\mathscr{B}W}(a(\mathscr{U};x,\xi))u=\mathrm{Op}^W(b(x,\xi))u=\\&\frac{1}{\sqrt{2\pi}}\sum_{k\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}}\hat{b}\big(k-n',\frac{k+n'}{2}\big)\hat{u}(n')\frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}.\end{aligned} \tag{1.1.46}$$

We need to estimate the Fourier coefficient $\hat{b}(\ell,\xi)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{T}}b(x,\xi)e^{\mathrm{i}\ell x}dx$. By using (1.1.16) we get

$$\begin{aligned}|\hat{b}(\ell,\xi)|&\leq C\Big|\int_{\mathbb{T}}b(x,\xi)e^{-\mathrm{i}\ell x}dx\Big|\\&\leq C\int_{\mathbb{T}}\Big|\sum_{n\in\mathbb{N}^p}\chi_p(n,\xi)a(\Pi_n\mathscr{U};x,\xi)\Big|dx\\&\leq C\sum_{n_1,\dots,n_p\in\mathbb{N}}\max(\langle n_1\rangle,\dots,\langle n_p\rangle)^{\mu}\langle\xi\rangle^m\prod_{j=1}^p\Big\|\Pi_{n_j}U_j\Big\|_{L^2}.\end{aligned} \tag{1.1.47}$$

Note that, thanks to the autonomous condition satisfied by the symbol $a(\mathscr{U};x,\xi)$, we have that $\hat{b}(\ell,\xi)$ is different from zero only if $\sum_{j=1}^p\epsilon_j n_j=\pm\ell$ for some choice of signs $\epsilon_j\in\{\pm1\}$. Therefore the sum in the r.h.s. of the above inequality is restricted to the set of indices $n_1,\dots,n_p$ satisfying such a property. Note that this fact implies that the sum in the r.h.s. of (1.1.46) is restricted to the set of indices such that $\pm(k-n')=\sum_{j=1}^p n_j$, therefore the (1.1.41) holds true only if there is a choice of signs $\sigma_j \in \{\pm 1\}$ such that $\sum_{j=0}^{p+1} \sigma_j n_j = 0$. Let us suppose for simplicity that $\langle n_1 \rangle \geq \langle n_2 \rangle \geq \ldots \geq \langle n_p \rangle$; we fix $\sigma > \mu + 1$ and we continue the chain of inequalities (1.1.47) as follows

$$\begin{aligned} \ldots &\leq C \langle \xi \rangle^m \prod_{j=1}^{p} \|U_j\|_\sigma \sum_{n_1,\ldots,n_p \in \mathbb{N}} \langle n_1 \rangle^\mu \prod_{j=1}^{p} \langle n_j \rangle^{-\sigma} \\ &\leq C \langle \xi \rangle^m \prod_{j=1}^{p} \|U_j\|_\sigma c_\ell, \end{aligned}$$

where c_ℓ is a suitable sequence in ℓ^1 and where we have used in the last passage that $\sum_{j=1}^{p} \epsilon_j n_j = \pm \ell$. Therefore using this estimate of the Fourier coefficient $\hat{b}(\ell,\xi)$ and equation (1.1.46) we get

$$\begin{aligned} &\left\| \mathrm{Op}^W(b(x,\xi)) U_{p+1} \right\|_{H^{s-m}}^2 \\ \leq &C \sum_{k \in \mathbb{Z}} \Big| \langle k \rangle^{s-m} \sum_{n' \in \mathbb{Z}} \left\langle \frac{k+n'}{2} \right\rangle^m \prod_{j=1}^{p} \|U_j\|_{H^\sigma} c_{k-n'} \hat{U}_{p+1}(n') \Big|^2 \\ \leq &C \prod_{j=1}^{p} \|U_j\|_{H^\sigma}^2 \sum_{k \in \mathbb{Z}} \Big| \sum_{n' \in \mathbb{Z}} \langle k \rangle^s c_{k-n'} \hat{U}_{p+1}(n') \Big|^2, \end{aligned} \tag{1.1.48}$$

In the last passage we have used that $k \sim n'$ which follows from the fact that the sum is restricted to the set of indices such that $|k - n'| \leq \langle (k+n')/2 \rangle$. Since the sequence c_ℓ is in ℓ^1 and $\langle k \rangle^s \hat{U}_{p+1}(k)$ is in ℓ^2, from (1.1.48) one deduces the (1.1.40) by using the Young inequality for convolution of sequences. Moreover one has $|n| = \sum_{j=1}^{p} |n_j| \leq \langle (k+n')/2 \rangle$, from which the relation (1.1.42) easily follows. This concludes the proof of (i).
The proof of (ii) is very similar. One has to set $b(x,\xi) := \partial_t^k a(U; t, x, \xi)$ for $0 \leq k \leq K - K'$, the condition (1.1.20) with $\beta = 0$ and $\alpha = 2$ provides the estimate

$$\left| \hat{b}(\ell,\xi) \right| \leq \frac{C}{\langle \ell \rangle^2} \langle \xi \rangle^m \|U\|_{K'+k,\sigma}^p,$$

from which one proves (1.1.44) and (1.1.45) as done before. □

Remark 1.1.13. *In the above proof, we did not use any ξ derivatives of the symbol a. The statement of the proposition above applies when $a \in \Gamma^m_{K,K',p}[r,N]$ satisfies* (1.1.20) *for just $|\alpha| \leq 2$ and $\beta = 0$.*

Remark 1.1.14. *We have the following inclusions.*

- *Let* $a \in \Sigma\Gamma^{m}_{K,K',p}[r,N]$ *for* $p \geq 1$. *By Proposition 1.1.1 we have that the map* $(V,U) \to \mathrm{Op}^{\mathscr{B}W}(a(V;t,\cdot))U$ *defined by* (1.1.32) *is in* $\Sigma\mathscr{M}^{m'}_{K,K',p}[r,N]$ *for some* $m' \geq m$.
- *If* $a \in \Sigma\Gamma^{m}_{K,K',p}[r,N]$ *with* $m \leq 0$ *and* $p \geq 1$, *then the map (linear in* U*)* $(V,U) \to \mathrm{Op}^{\mathscr{B}W}(a(V;t,\cdot))U$ *is in* $\Sigma\mathscr{R}^{m}_{K,K',p}[r,N]$.
- *Any smoothing operator* $R \in \Sigma\mathscr{R}^{-\rho}_{K,K',p}[r,N]$ *defines an element of the class* $\Sigma\mathscr{M}^{m}_{K,K',p}[r,N]$ *for some* $m \geq 0$.

Remark 1.1.15. *From Proposition* (1.1.1) *we deduce that the Bony-Weyl quantization of a symbol is unique up to smoothing remainders. More precisely consider two admissible cut off functions* $\chi^{(1)}_p$ *and* $\chi^{(2)}_p$ *according to Def. 1.1.14 with* $\delta_1 > 0$ *and* $\delta_2 > 0$. *Define* $\chi_p := \chi^{(1)}_p - \chi^{(2)}_p$ *and for* a *in* $\widetilde{\Gamma}^{m}_{p}$ *set*

$$R(\mathscr{U}) := \mathrm{Op}^{W}\Big(\sum_{n\in\mathbb{N}^p} \chi_p(n,\xi)a(\Pi_n\mathscr{U};\cdot)\Big). \tag{1.1.49}$$

Then, by applying (1.1.40) *with* $s = m$, *we get*

$$\Big\|\Pi_{n_0}R(\Pi_n\mathscr{U})\Pi_{n_{p+1}}U_{p+1}\Big\|_{L^2} \leq C\langle n_1\rangle^{\sigma}\cdots\langle n_p\rangle^{\sigma}\langle n_{p+1}\rangle^{m}\prod_{j=1}^{p+1}\Big\|\Pi_{n_j}U_j\Big\|_{L^2}.$$

The l.h.s. of the equation above is non zero only if $\delta_1\langle n_{p+1}\rangle \leq |n| \leq \delta_2\langle n_{p+1}\rangle$. *As a consequence we deduce the equivalence*

$$\max_2(\langle n_1\rangle,\ldots,\langle n_{p+1}\rangle) \sim \max(\langle n_1\rangle,\ldots,\langle n_{p+1}\rangle)$$

and hence the operator R *belongs to* $\widetilde{\mathscr{R}}^{-\rho}_{p}$.
A similar statement holds for the non homogeneous case.

In the following we shall deal with operators defined on the product space $H^s \times H^s$. We have the following definition.

Definition 1.1.16 (**Matrices of operators).** *Let* $\rho, m \in \mathbb{R}$, $\rho \geq 0$, $K' \leq K \in \mathbb{N}$, $r > 0$, $N \in \mathbb{N}$, $p \in \mathbb{N}$ *with* $p \geq 1$. *We denote by* $\Sigma\mathscr{R}^{-\rho}_{K,K',p}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$ *the space of* 2×2 *matrices whose entries are smoothing operators in the class* $\Sigma\mathscr{R}^{-\rho}_{K,K',p}[r,N]$. *Analogously we denote by* $\Sigma\mathscr{M}^{m}_{K,K',p}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$ *the space of* 2×2 *matrices whose entries are maps in the class* $\Sigma\mathscr{M}^{m}_{K,K',p}[r,N]$. *We also set* $\Sigma\mathscr{M}_{K,K',p}[r,N] \otimes \mathscr{M}_2(\mathbb{C}) = \cup_{m\in\mathbb{R}}\Sigma\mathscr{M}^{m}_{K,K',p}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$.

Definition 1.1.17 (Matrices of symbols). *Let $m \in \mathbb{R}$, $K' \le K \in \mathbb{N}$, $p, N \in \mathbb{N}$. We denote by $\Sigma\Gamma^{m}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ the space 2×2 matrices whose entries are symbols in the class $\Sigma\Gamma^{m}_{K,K',p}[r,N]$.*

We have the following result.

Lemma 1.1.1. *Let $\rho, m \in \mathbb{R}$, $m \ge 0$, $\rho \ge 0$, $K' \le K \in \mathbb{N}$, $r > 0$, $N \in \mathbb{N}$, $p \in \mathbb{N}$, $p \ge 1$ and consider $R \in \Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, $M \in \Sigma\mathcal{M}^{m}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ and $A \in \Sigma\Gamma^{m}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. There is $\sigma > 0$ such that*

$$R : B^{K}_{s}(I,r) \times C^{K-K'}_{*}(I, H^{s}(\mathbb{T};\mathbb{C}^2)) \to C^{K-K'}_{*}(I, H^{s+\rho}(\mathbb{T};\mathbb{C}^2)); \tag{1.1.50}$$

$$M : B^{K}_{s}(I,r) \times C^{K-K'}_{*}(I, H^{s}(\mathbb{T};\mathbb{C}^2)) \to C^{K-K'}_{*}(I, H^{s-m}(\mathbb{T};\mathbb{C}^2)), \tag{1.1.51}$$

and

$$\mathrm{Op}^{\mathcal{B}W}(A(U;t,\cdot)) : B^{K}_{\sigma}(I,r) \times C^{K-K'}_{*}(I, H^{s}(\mathbb{T};\mathbb{C}^2)) \to C^{K-K'}_{*}(I, H^{s-m}(\mathbb{T};\mathbb{C}^2)). \tag{1.1.52}$$

Proof. The (1.1.50) follows by Definition 1.1.3 (see bound (1.1.8)). The (1.1.51) follows by Definition 1.1.7 (see bound (1.1.14)). The (1.1.52) follows by Proposition 1.1.1. □

1.1.5 Symbolic calculus and Compositions theorems

We define the following differential operator

$$\sigma(D_x, D_\xi, D_y, D_\eta) = D_\xi D_y - D_x D_\eta, \tag{1.1.53}$$

where $D_x := \frac{1}{\mathrm{i}}\partial_x$ and D_ξ, D_y, D_η are similarly defined.
Let $K' \le K, \rho, p, q$ be in $\mathbb{N}$, $m, m' \in \mathbb{R}$, $r > 0$ and consider $a \in \widetilde{\Gamma}^{m}_{p}$ and $b \in \widetilde{\Gamma}^{m'}_{q}$. Set

$$\begin{aligned} &\mathcal{U} := (\mathcal{U}', \mathcal{U}''), \quad \mathcal{U}' := (U_1, \ldots, U_p), \quad \mathcal{U}'' := (U_{p+1}, \ldots, U_{p+1}), \\ &U_j \in H^{s}(\mathbb{T};\mathbb{C}^2), \quad j = 1, \ldots, p+q. \end{aligned} \tag{1.1.54}$$

We define the asymptotic expansion (up to order ρ) of the composition symbol as follows:

$$\begin{aligned} &(a\#b)_{\rho}(\mathcal{U}; x, \xi) := \\ &\sum_{k=0}^{\rho} \frac{1}{k!}\left(\frac{\mathrm{i}}{2}\sigma(D_x, D_\xi, D_y, D_\eta)\right)^{k} \Big[a(\mathcal{U}'; x, \xi) b(\mathcal{U}''; y, \eta)\Big]_{|x=y, \, \xi=\eta} \end{aligned} \tag{1.1.55}$$

modulo symbols in $\widetilde{\Gamma}^{m+m'-\rho}_{p+q}$.

Consider $a \in \Gamma^{m}_{K,K',p}[r]$ and $b \in \Gamma^{m'}_{K,K',q}[r]$. For U in $B^{K}_{\sigma}(I,r)$ we define, for $\rho < \sigma - \sigma_0$,

$$(a\#b)_{\rho}(U;t,x,\xi) := \sum_{k=0}^{\rho} \frac{1}{k!}\Big(\frac{\mathrm{i}}{2}\sigma(D_x,D_\xi,D_y,D_\eta)\Big)^{k}\Big[a(U;t,x,\xi)b(U;t,y,\eta)\Big]_{|_{\substack{x=y\\ \xi=\eta}}}, \tag{1.1.56}$$

modulo symbols in $\Gamma^{m+m'-\rho}_{K,K',p+q}[r]$.

Remark 1.1.16.

- *By Remark 1.1.6 one can note that the symbol $(a\#b)_\rho$ in* (1.1.55) *belongs to the class $\widetilde{\Gamma}^{m+m'}_{p+q}$ with the exponent μ in* (1.1.16) *large as function of ρ. This reflect the fact that the larger is ρ the smoother the functions U_j in the coefficient must be.*
- *Similarly by Remark 1.1.8 one can note that the symbol $(a\#b)_\rho$ in* (1.1.56) *belongs to the class $\Gamma^{m+m'}_{p+q}[r]$ (with σ_0 in Def. 1.1.10 large as function of ρ).*

The following proposition ensures that $(a\#b)_\rho$ is the symbol of the composition up to smoothing remainders.

Proposition 1.1.2 (Composition of Bony-Weyl operators). *Let $K' \le K, \rho, p, q$ be in $\mathbb{N}$, $m, m' \in \mathbb{R}$, $r > 0$.*
(i) Consider $a \in \widetilde{\Gamma}^{m}_{p}$ and $b \in \widetilde{\Gamma}^{m'}_{q}$. Then (recalling the notation in (1.1.54)*) one has that*

$$\mathrm{Op}^{\mathscr{B}W}(a(\mathscr{U};x,\xi)) \circ \mathrm{Op}^{\mathscr{B}W}(b(\mathscr{U}'';x,\xi)) - \mathrm{Op}^{\mathscr{B}W}\big((a\#b)_\rho(\mathscr{U};x,\xi)\big) \tag{1.1.57}$$

belongs to the class of smoothing remainder $\widetilde{\mathscr{R}}^{-\rho+m+m'}_{p+q}$.
(ii) Consider $a \in \Gamma^{m}_{K,K',p}[r]$ and $b \in \Gamma^{m'}_{K,K',q}[r]$. Then one has that

$$\mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi)) \circ \mathrm{Op}^{\mathscr{B}W}(b(U;t,x,\xi)) - \mathrm{Op}^{\mathscr{B}W}\big((a\#b)_\rho(U;t,x,\xi)\big) \tag{1.1.58}$$

belongs to the class of non-homogeneous smoothing remainders $\mathscr{R}^{-\rho+m+m'}_{K,K',p+q}[r]$. If a and b are symbols in the autonomous classes of Definition 1.1.11 then the symbol $(a\#b)_\rho(U;t,x,\xi)$ belongs to $\Gamma^{m+m'}_{K,K',p+q}[r,\mathrm{aut}]$ and (1.1.58) *is an autonomous smoothing remainder in $\mathscr{R}^{-\rho+m+m'}_{K,K',p+q}[r,\mathrm{aut}]$.*

The proof of this important proposition needs a couple of lemmas. In order to treat at the same time conditions (i) and (ii) we introduce the following notation. Let $a(x,\xi)$ and $b(x,\xi)$ be two tempered distribution in x which depend smoothly on ξ. Assume that their x-Fourier transforms, $\hat{a}(\eta,\xi)$ and $\hat{b}(\eta,\xi)$, are supported on the set $\{|\eta| \le \delta \langle\xi\rangle\}$ for some $0<\delta \ll 1$. Define the integral

$$a\#b(x,\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}x(\eta^*+\xi^*)}\hat{a}(\eta^*,\xi+\frac{\xi^*}{2})\hat{b}(\xi^*,\xi-\frac{\eta^*}{2})d\xi^* d\eta^*, \tag{1.1.59}$$

which makes sense since $\hat{a}$ and $\hat{b}$ are tempered distribution with compact support in (ξ^*,η^*) acting on the smooth function $e^{\mathrm{i}x(\xi^*+\eta^*)}$. Let us assume moreover that for some $\rho\in\mathbb{N}$, any $0\le\alpha\le\rho$, any $\beta\in\mathbb{N}$ there are constants $M_{\alpha,\beta}(\cdot)$ such that

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right| &\le M_{\alpha,\beta}(a)\langle\xi\rangle^{m-\beta},\\ \left|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)\right| &\le M_{\alpha,\beta}(b)\langle\xi\rangle^{m-\beta}.\end{aligned} \tag{1.1.60}$$

Then $a\#b$ is also given by the *oscillatory integral*

$$a\#b(x,\xi) = \frac{1}{\pi^2}\int_{\mathbb{R}^4} e^{-2\mathrm{i}\sigma(x^*,\xi^*,y^*,\eta^*)}a(x+x^*,\xi+\xi^*)b(x+y^*,\xi+\eta^*)dx^*d\xi^*dy^*d\eta^*. \tag{1.1.61}$$

To prove that the latter integral is well defined one just need to check that, thanks to (1.1.60) and to the fact that $\hat{a}$ and $\hat{b}$ are supported on $\{|\eta|\le\xi\}$, the functions a and b are *amplitudes* in the sense of Definition A.0.1 and hence apply Theorem A.0.5. To check that (1.1.59) is equal to (1.1.61) we proceed in the following way. By writing explicitly the Fourier transforms $\hat{a}$ and $\hat{b}$ in (1.1.59) and using the Lemma A.0.3 we obtain that (1.1.59) may be rewritten as

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^4} e^{\mathrm{i}[\eta^*(x-x^*)+\xi^*(x-y^*)]}a(x^*,\xi+\frac{\xi^*}{2})b(y^*,\xi-\frac{\eta}{2})dx^*dy^*d\xi^*d\eta^*,$$

which is equal to (1.1.61) after a linear change of variable (which may be performed thanks to Lemma A.0.1).

We are now ready to state and prove two lemmas. In the first one (Lemma 1.1.2) we show that $\mathrm{Op}^{\mathscr{B}W}(a)\circ\mathrm{Op}^{\mathscr{B}W}(b)$ may be written as $\mathrm{Op}^{W}(c)$ where $c=a_\chi\#b_\chi$. In the second one (Lemma 1.1.3) we provide an asymptotic expansion for the symbol $a_\chi\#b_\chi$.

Lemma 1.1.2. *Let a and b as in (i) of Proposition 1.1.2 (resp. as in (ii) of the same proposition) then*

$$\mathrm{Op}^{\mathscr{B}W}(a)\circ\mathrm{Op}^{\mathscr{B}W}(b)=\mathrm{Op}^{W}(c),$$

where $c(\mathcal{U};x,\xi)=a_{\chi_p}(\mathcal{U}';x,\xi)\# b_{\chi_q}(\mathcal{U}'';x,\xi)$, and a_{χ_p} and b_{χ_q} are defined in (1.1.28) *(resp. where*

$$c(U;x,\xi)=a_{\chi}(U;x,\xi)\# b_{\chi}(U;x,\xi)$$

and a_{χ} and b_{χ} are defined in (1.1.28)*).*

Proof. Let $a(\mathcal{U}';x,\xi)$ be a symbol in $\widetilde{\Gamma}^{m}_{p}$ and let us denote it, for simplicity, by $a(x,\xi)$. We preliminarily rewrite $\mathrm{Op}^{\mathscr{B}W}(a(x,\xi))$ in an alternative way.
By using (1.1.30) and formula (1.1.22) with $\sigma=1/2$ we have that, for any smooth function v, the following holds

$$\begin{aligned}\mathrm{Op}^{\mathscr{B}W}(a(x,\xi))v&=\mathrm{Op}^{W}(a_{\chi_p}(x,\xi))v\\&=\frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}e^{\mathrm{i}(x-y)\xi}a_{\chi_p}\Big(\frac{x+y}{2},\xi\Big)v(y)dyd\xi\\&=\int_{\mathbb{R}}K_a(x,y)v(y)dy,\end{aligned}$$

where $k_a(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{\mathrm{i}(x-y)\xi}a_{\chi_p}(\frac{x+y}{2},\xi)d\xi$. The last integral makes sense thanks to Theorem A.0.5 and, in the previous calculation, we have used Lemma A.0.3. We note that the kernel $K_a(x,y)$ is such that

$$K_a(x+t/2,x-t/2)=\frac{1}{2\pi}\int_{\mathbb{R}}e^{\mathrm{i}t\xi}a_{\chi_p}(x,\xi)d\xi=\mathscr{F}^{-1}_{\xi}(a_{\chi_p})(x,t),$$

therefore

$$\begin{aligned}a_{\chi_p}(x,\xi)&=\mathscr{F}_t(K_a(x+t/2,x-t/2))\\&=\int_{\mathbb{R}}e^{-\mathrm{i}t\xi}K_a(x+t/2,x-t/2)dt.\end{aligned}$$

Having this formula in hand we obtain, for $b(x,\xi):=b(\mathcal{U}'';x,\xi)\in\widetilde{\Gamma}^{m'}_{q}$, that

$$\mathrm{Op}^{\mathscr{B}W}(a(x,\xi))\circ\mathrm{Op}^{\mathscr{B}W}(b(x,\xi))v(x)=\int_{\mathbb{R}}K(x,y)v(y)dy,$$

where the kernel is given by

$$K(x,y) = \int_{\mathbb{R}} K_a(x,z) K_b(z,y)\, dz.$$

Hence, by using Lemma A.0.3, we have

$$\mathrm{Op}^{\mathscr{B}W}(a(x,\xi)) \circ \mathrm{Op}^{\mathscr{B}W}(b(x,\xi)) = \mathrm{Op}^{W}(c(x,\xi)),$$

with $c(x,\xi)$ equal to

$$\frac{1}{4\pi^2}\int_{\mathbb{R}^4} e^{-\mathrm{i}t\xi} e^{\mathrm{i}(x+t/2-z)s} a_{\chi_p}\Big(\frac{x+t/2+z}{2}, s\Big) \times \\ \times e^{\mathrm{i}(z-x+t/2)\tau} b_{\chi_q}\Big(\frac{z+x-t/2}{2}, \tau\Big) d\tau\, ds\, dz\, dt.$$

Performing the linear change of variable

$$\begin{cases} z = x + x^* + y^* \\ t = 2x^* - 2y^* \\ s = \xi + \xi^* \\ \tau = \xi + \eta^*, \end{cases}$$

which is possible thanks to Lemma A.0.1, one gets that $c(x,\xi)$ is equal to (1.1.61) with a and b replaced by a_{χ_p} and b_{χ_q}.
In the case of symbols $a \in \Gamma^{m}_{K,K',p}[r]$ and $b \in \Gamma^{m'}_{K,K',q}[r]$ the proof is the same. □

Lemma 1.1.3. *Let $a(x,\xi)$ and $b(x,\xi)$ satisfying conditions* (1.1.60) *for some $\rho \in \mathbb{N}$. Suppose moreover that $\hat{a}(\eta,\xi)$ and $\hat{b}(\eta,\xi)$ are supported for $|\eta| \le \delta \langle \xi \rangle$ for a small enough $\delta > 0$. Then for any $\ell \in \mathbb{N}$ with $\ell \le \rho$ we have*

$$\left|\partial_x^{\alpha}(a\#b - (a\#b)_{\rho-\ell})(x,\xi)\right| \le C_{\rho,\ell} \langle \xi \rangle^{m+m'-(\rho-\ell)}, \tag{1.1.62}$$

where

$$C_{\rho,\ell} = K_{\rho,\ell} \sum_{\alpha'+\alpha''=\alpha} \sum_{\beta'+\beta''=\beta} M_{\alpha',\beta'}(a) M_{\alpha'',\beta''}(b), \tag{1.1.63}$$

for some universal constants $K_{\rho,\ell}$.

Proof. It is sufficient to prove the lemma in the case that $\ell = 0$ since $\partial_x^\ell a\#b = (\partial_x^{\ell'} a)\#(\partial_x^{\ell''} b)$ with $\ell' + \ell'' = \ell$ and a similar formula holds for $\partial_x^\ell(a\#b)$. Define for $\tau \in [0,1]$, $X^* = (x^*, \xi^*)$ and $Y^* = (y^*, \eta^*)$ in $\mathbb{R}^2$ the function

$$c(\tau, x, \xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} [a(x', \xi') b(y', \eta')]|_{\substack{x' = x + \sqrt{\tau} x^*, y' = y + \sqrt{\tau} y^* \\ \xi' = \xi + \sqrt{\tau}\xi^*, \eta' = \eta + \sqrt{\tau}\eta^*}} e^{-2\mathrm{i}\sigma(X^*, Y^*)} dX^* dY^*.$$

Note that $c(1, x, \xi) = a\#b(x, \xi)$ and that moreover

$$\partial_\tau^k c(\tau, x, \xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} \Big(\frac{\mathrm{i}}{2}\sigma(D_{x'}, D_{\xi'}, D_{y'}, D_{\eta'})\Big)^k [a(x', \xi') b(y', \eta')] \\ \big((x, \xi) + \sqrt{\tau} X^*, (x, \xi) + \sqrt{\tau} Y^*\big) e^{-2\mathrm{i}\sigma(X^*, Y^*)} dX^* dY^*,$$

hence by Taylor expanding in $\tau = 1$ we get

$$a\#b(x, \xi) - (a\#b)_\rho(x, \xi) = \frac{1}{(\rho - 1)!} \int_0^1 c^{(\rho)}(\tau, x, \xi)(1 - \tau)^{\rho - 1} d\tau,$$

where

$$c^{(\rho)}(\tau, x, \xi) = \frac{1}{\pi^2} \int_{\mathbb{R}^4} e^{-\mathrm{i}\sigma(X^*, Y^*)} e\big((x, \xi) + \sqrt{\tau} X^*, (x, \xi) + \sqrt{\tau} Y^*\big) dX^* dY^*, \quad (1.1.64)$$

and

$$e(x', \xi', y', \eta') = \Big(\frac{\mathrm{i}}{2}\sigma(D_{x'}, D_{\xi'}, D_{y'}, D_{\eta'})\Big)^\rho [a(x', \xi') b(y', \eta')].$$

Moreover, since a and b satisfy (1.1.60), we have

$$\Big|\partial_{\xi'}^{\gamma'} \partial_{\eta'}^{\gamma''} e(x', \xi', y', \eta')\Big| \le \\ \le \sum_{\substack{\alpha' + \alpha'' = \rho \\ \beta' + \beta'' = \rho}} M_{\alpha', \beta' + \gamma'}(a) M_{\alpha'', \beta'' + \gamma''}(b) \langle \xi' \rangle^{m - \beta' - \gamma'} \langle \eta' \rangle^{m' - \beta'' - \gamma''}. \quad (1.1.65)$$

The symbol $e(x', \xi', y', \eta')$ may be written as a linear combination of elements of the form $a_1(x', \xi') b_1(y', \eta')$, where a_1 and b_1 are expressed in terms of derivatives of a and b. Therefore, arguing as done below the statement of Prop. 1.1.2 in order to transform (1.1.61) into (1.1.59), we can represent $c^{(\rho)}$ as sum of integrals of the form

$$\int_{\mathbb{R}^2} e^{\mathrm{i}x(\xi^* + \eta^*)} \hat{a}_1(\eta^*, \xi + \tau\frac{\xi^*}{2}) \hat{b}_1(\xi^*, \xi - \tau\frac{\eta^*}{2}) d\xi^* d\eta^*. \quad (1.1.66)$$

Thanks to the support condition on $\hat{a}$ and $\hat{b}$ we may insert into the integral (1.1.66) a cut-off function $\chi((\xi^*,\eta^*)/\langle\xi\rangle)$ where χ is a C^∞ function with small enough support and equal to 1 near the origin. Expressing the Fourier transforms in (1.1.66) we get an integral of the form (1.1.64) with moreover the cut-off function $\chi((\xi^*,\eta^*)/\langle\xi\rangle)$ inside the integral. Now consider the vector-fields $L_1[\cdot] := (1+(2x^*)^2)^{-1}(1+\partial_{\xi^*}[\cdot])$ and $L_2[\cdot] := (1+(2y^*)^2)^{-1}(1+\partial_{\eta^*}[\cdot])$. The function $e^{-2\mathrm{i}(X^*,Y^*)}$ is a fixed point for such vector-fields, therefore using them two times in the integral (1.1.64) and integrating by parts, using (1.1.65) we bound by $C\langle\xi\rangle^{m+m'-\rho}$ for some constant $C>0$, from which it is easy to conclude. □

We are now in position to give the proof of Proposition 1.1.2.

proof of Prop. 1.1.2. We start by proving (i). By Lemma 1.1.2 we know that

$$\mathrm{Op}^W\big(a_{\chi_p}(\mathscr{U}';x,\xi)\#a_{\chi_q}(\mathscr{U}'';x,\xi)\big)-\mathrm{Op}^{\mathscr{B}W}\big((a\#b)_\rho(\mathscr{U};x,\xi)\big)=\mathrm{Op}^W\big(r(\mathscr{U};x,\xi)\big),$$

with $r(\mathscr{U};x,\xi)=r_1(\mathscr{U};x,\xi)+r_2(\mathscr{U};x,\xi)$ and

$$\begin{aligned} r_1(\mathscr{U};x,\xi)&=a_{\chi_p}(\mathscr{U}';x,\xi)\#a_{\chi_q}(\mathscr{U}'';x,\xi)-\big(a_{\chi_p}\#a_{\chi_q}\big)_\rho(\mathscr{U};x,\xi),\\ r_2(\mathscr{U};x,\xi)&=\big(a_{\chi_p}\#a_{\chi_q}\big)_\rho(\mathscr{U};x,\xi)-(a\#b)_{\chi_{p+q},\rho}(\mathscr{U};x,\xi). \end{aligned}$$

Let us study the first summand. By (1.1.62), (1.1.63) and (1.1.16) we have the bound

$$|r_1(\Pi_n\mathscr{U};x,\xi)|\le C|n|^{\mu+\rho}\langle\xi\rangle^{m+m'-\rho}\prod_{j=1}^{p+q}\left\|\Pi_{n_j}U_j\right\|_{L^2}. \tag{1.1.67}$$

Note that, if the cut-off functions χ_p and χ_q are chosen with small enough support (i.e. δ small enough in Def. 1.1.14), then the symbol r_1 satisfies the spectral condition $\mathrm{supp}(\widehat{r}_1(\eta,\xi))\subset\big\{|\eta|\ll\langle\xi\rangle\big\}$, hence $\mathrm{Op}^W(r_1)$ coincides with $\mathrm{Op}^{\mathscr{B}W}(r_1)$. Therefore by (1.1.40), in Prop. 1.1.1, applied with $s=m+m'-\rho$, up to changing the definition of μ, we have

$$\begin{aligned} &\left\|\mathrm{Op}^{\mathscr{B}W}(r_1(\Pi_n\mathscr{U};\cdot,\xi))\Pi_{n_{p+q+1}}U_{p+q+1}\right\|_{L^2}\\ &\qquad\le C|n|^{\mu+\rho}n_{p+q+1}^{m+m'-\rho}\prod_{j=1}^{p+q+1}\left\|\Pi_{n_j}U_j\right\|_{L^2}. \end{aligned}\tag{1.1.68}$$

Furthermore we have that $\mathrm{Op}^{\mathscr{B}W}(r_1(\Pi_n\mathscr{U};\cdot,\xi))\Pi_{n_{p+q+1}}U_{p+q+1}$ is different from 0 only if there exist a choice of signs $\sigma_j\in\{\pm1\}$ such that $\sum_{j=1}^{p+q+1}\sigma_jn_j=0$. This fact is true since $\mathrm{Op}^{\mathscr{B}W}(a)$ and $\mathrm{Op}^{\mathscr{B}W}(b)$ satisfy this property, hence so does their composition, together with $\mathrm{Op}^{W}\big((a_{\chi_p}\#b_{\chi_q})_\rho\big)$. Since, thanks to the discussion above, $|n| \ll \langle n_{p+q+1}\rangle$ we have $\max(\langle n_1\rangle,\ldots,\langle n_{p+q+1}\rangle) = \langle n_{p+q+1}\rangle$ and

$$\max_2(\langle n_1\rangle,\ldots,\langle n_{p+q+1}\rangle) \sim \max(\langle n_1\rangle,\ldots,\langle n_{p+q+1}\rangle),$$

which, together with (1.1.68) proves a bound of the form (1.1.5) for the operator $\mathrm{Op}^{\mathscr{B}W}(r_1(\Pi_n\mathscr{U};\cdot,\xi))\Pi_{n_{p+q+1}}U_{p+q+1}$. Therefore it is a smoothing operator.
Lets study the term $\mathrm{Op}^{W}(r_2(\mathscr{U};\xi))$. The symbol $r_2(\mathscr{U};\xi)$ is a combination of terms of the form

$$\begin{aligned}&(\frac{\mathrm{i}}{2}\sigma(D_x,D_\xi,D_y,D_\eta))^\ell\big[a(\Pi_{n'}\mathscr{U}';x,\xi)b(\Pi_{n''}\mathscr{U}'';y,\eta)\times\\ &\qquad\chi_p(n',\xi)\chi_q(n'',\eta)\big]|_{\substack{x=y\\ \xi=\eta}}-\\ &-\chi_{p+q}(n',n'',\xi)(\frac{\mathrm{i}}{2}\sigma(D_x,D_\xi,D_y,D_\eta))^\ell\big[a(\Pi_{n'}\mathscr{U}';x,\xi)b(\Pi_{n''}\mathscr{U}'';y,\eta)\big]|_{\substack{x=y\\ \xi=\eta}}.\end{aligned}$$

Thanks to the properties of the cut-off functions χ_p, χ_q and χ_{p+q} the symbols above are supported for

$$\delta_1\langle\xi\rangle \le |(n',n'')| \le \delta_2\langle\xi\rangle. \tag{1.1.69}$$

Moreover, since $a \in \widetilde{\Gamma}^m_p$ and $b \in \widetilde{\Gamma}^{m'}_q$, we have

$$|r_2(\Pi_n\mathscr{U};x,\xi)| \le C|n|^\sigma\langle\xi\rangle^{m+m'-\rho}\prod_{j=1}^{p+q}\left\|\Pi_{n_j}U_j\right\|_{L^2}$$

with $\sigma \sim \rho$. Therefore, reasoning as done for $r_1(\mathscr{U};x,\xi)$, we deduce that

$$\begin{aligned}&\left\|\mathrm{Op}^{\mathscr{B}W}(r_2(\Pi_n(\mathscr{U};x,\xi))\Pi_{n_{p+q+1}}U_{p+q+1})\right\|_{L^2} \le\\ &C|n|^{\sigma'+\rho}\langle n_{p+q+1}\rangle^{m+m'-\rho}\prod_{j=1}^{p+q+1}\left\|\Pi_{n_j}U_j\right\|_{L^2},\end{aligned}$$

from which we conclude the proof since (1.1.69) implies that

$$\max(\langle n_1\rangle,\ldots,\langle n_{p+q+1}\rangle) \sim \max_2(\langle n_1\rangle,\ldots,\langle n_{p+q+1}\rangle).$$

We now prove item (ii) of the statement. We can rewrite, thanks to Lemma 1.1.2, the expression (1.1.58) as $\mathrm{Op}^W(r(U;t,x,\xi))$, where $r(U;t,x,\xi) = r_1(U;t,x,\xi) + r_2(U;t,x,\xi)$ and

$$r_1(U;t,x,\xi) = a_\chi(U;t,x,\xi)\#b_\chi(U;t,x,\xi) - (a_\chi\#b_\chi)_\rho(U;t,x,\xi),$$
$$r_2(U;t,x,\xi) = (a_\chi\#b_\chi)_\rho(U;t,x,\xi) - (a\#b)_{\rho,\chi}(U;t,x,\xi).$$

Let us study the contribution coming from r_1, the other one is similar. The symbol a is in the class $\Gamma^m_{K,K',p}[r]$, b is in $\Gamma^{m'}_{K,K',q}[r]$ so they verify condition (1.1.18) up to level $0 \le \alpha \le \rho$ if σ is chosen big enough. So, thanks to (1.1.62) and (1.1.63) we have

$$\left|\partial_t^k \partial_x^\ell r_1(U;t,x,\xi)\right| \le C\langle\xi\rangle^{m+m'-(\rho-\ell)} \|U(t)\|^{p+q}_{K'+k,\sigma}$$

for $0 \le \ell \le 2$. This is enough, thanks to Remark 1.1.13, to obtain the following

$$\left\|\mathrm{Op}^W(\partial_t^k r_1(U;t,x,\xi))\right\|_{\mathcal{L}(H^s,H^{s-m-m'+\rho-2})} \le C\|U(t)\|^{p+q}_\sigma$$

for any $0 \le k \le K - K'$. Thanks to the condition on the support, introduced by the cut-off function χ, we have that $\mathrm{Op}^{\mathcal{B}W}(\partial_t^k r_1(U;t,x,\xi)) = \mathrm{Op}^W(\partial_t^k r_1(U;t,x,\xi))$, so that

$$\left\|\partial_t^k\Big(\mathrm{Op}^{\mathcal{B}W}(r_1(U;t,\cdot,\xi))V\Big)\right\|_{H^{s-m-m'+\rho-2-2k}} \le C\sum_{k'+k''=k}\|U(t)\|^{p+q}_{K'+k,\sigma}\|V(t)\|_{k'',s},$$

for any $0 \le k \le K - K'$, which is an estimate of the form (1.1.8). This concludes the proof. □

Consider now symbols $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$, $b \in \Sigma\Gamma^{m'}_{K,K',q}[r,N]$. By definition (see Def. 1.1.12) we have

$$\begin{aligned}
a(U;t;x,\xi) &= \sum_{k=p}^{N-1} a_k(U,\dots,U;x,\xi) + a_N(U;t,x,\xi),\\
b(U;t;x,\xi) &= \sum_{k'=p}^{N-1} b_{k'}(U,\dots,U;x,\xi) + b_N(U;t,x,\xi),\\
a_k &\in \widetilde{\Gamma}^m_k, \quad a_N \in \Gamma^m_{K,K',N}[r], \quad b_{k'} \in \widetilde{\Gamma}^{m'}_{k'}, \quad b_N \in \Gamma^{m'}_{K,K',N}[r].
\end{aligned} \tag{1.1.70}$$

We set also

$$\begin{aligned}
c_{k''}(\mathcal{U};x,\xi) &:= \sum_{k+k'=k''}(a_k\#b_{k'})_\rho(\mathcal{U};x,\xi), \quad k'' = p+q,\dots,N-1,\\
c_N(U;t,x,\xi) &:= \sum_{k+k'\ge N}(a_k\#b_{k'})_\rho(U;t,x,\xi),
\end{aligned} \tag{1.1.71}$$

where the factors a_k and $b_{k'}$, for $k,k' \le N-1$, have to be considered as elements of $\Gamma^{m}_{K,0,k}[r]$ and $\Gamma^{m'}_{K,0,k'}[r]$ respectively according to Remark 1.1.9. We define the composition symbol $(a\#b)_{\rho,N} \in \Sigma\Gamma^{m+m'}_{K,K',p+q}[r,N]$ as

$$
\begin{aligned}
(a\#b)_{\rho,N}(U;t,x,\xi) &:= (a\#b)_{\rho}(U;t,x,\xi) \\
&:= \sum_{k''=p+q}^{N-1} c_{k''}(U,\dots,U;x,\xi) + c_N(U;t,x,\xi).
\end{aligned} \tag{1.1.72}
$$

We collect in the following proposition all the results concerning compositions between Bony-Weyl para-differential operators, smoothing remainders and maps.

Proposition 1.1.3 (Compositions). *Let $m,m',m'' \in \mathbb{R}$, $K,K',N,p_1,p_2,p_3,p_4,\rho \in \mathbb{N}$ with $K' \le K$, $p_1+p_2 < N$, $\rho \ge 0$ and $r>0$. Let $a \in \Sigma\Gamma^{m}_{K,K',p_1}[r,N]$, $b \in \Sigma\Gamma^{m'}_{K,K',p_2}[r,N]$, $R \in \Sigma\mathscr{R}^{-\rho}_{K,K',p_3}[r,N]$ and $M \in \Sigma\mathscr{M}^{m''}_{K,K',p_4}[r,N]$. Then the following holds.*
(i) There exists a smoothing operator R_1 in the class $\Sigma\mathscr{R}^{-\rho}_{K,K',p_1+p_2}[r,N]$ such that

$$
\begin{aligned}
&\mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi)) \circ \mathrm{Op}^{\mathscr{B}W}(b(U;t,x,\xi)) = \\
&\qquad \mathrm{Op}^{\mathscr{B}W}\big((a\#b)_{\rho,N}(U;t,x,\xi)\big) + R_1(U;t)
\end{aligned} \tag{1.1.73}
$$

(ii) One has that the compositions operators

$$
R(U;t)\circ \mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi)), \quad \mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi))\circ R(U;t), \tag{1.1.74}
$$

are smoothing operators in the class $\Sigma\mathscr{R}^{-\rho+m}_{K,K',p_1+p_3}[r,N]$.
(iii) Assume $\rho > m''$. One has that the compositions operators

$$
R(U;t)\circ M(U;t), \quad M(U;t)\circ R(U;t), \tag{1.1.75}
$$

are smoothing operators in the class $\Sigma\mathscr{R}^{-\rho+m''}_{K,K',p_3+p_4}[r,N]$.
(iv) Let $R_2(U,W;t)[\cdot]$ be a smoothing operator of $\Sigma\mathscr{R}^{-\rho}_{K,K',p_3}[r,N]$ depending linearly on W, i.e.

$$
R(U,W;t)[\cdot] = \sum_{q=p_3}^{N-1} R_q(U,\dots,U,W)[\cdot] + R_N(U,W;t)[\cdot],
$$

where $R_q \in \widetilde{\mathscr{R}}^{-\rho}_q$ *and* R_N *satisfies for any* $0 \le k \le K - K'$ *(instead of* (1.1.8)*) the following*

$$
\begin{aligned}
&\|\partial_t^k R_N(U,W;t)V(t,\cdot)\|_{H^{s-2k}} \\
&\quad \le C \sum_{k'+k''=k} \Big(\|U\|^{N-1}_{k'+K',\sigma}\|W\|_{k'+K',\sigma}\|V\|_{k'',s} + \|U\|^{N-1}_{k'+K',\sigma}\|W\|_{k'+K',s}\|V\|_{k'',\sigma} \\
&\qquad\qquad + \|U\|^{N-2}_{k'+K',\sigma}\|U\|_{k'+K',s}\|W\|_{k'+K',\sigma}\|V\|_{k'',\sigma}\Big).
\end{aligned}
$$

Then one has that $R(U,M(U;t)W;t)$ *belongs to* $\Sigma\mathscr{R}^{-\rho+m''}_{K,K',p_3+p_4}[r,N]$.
(v) *Let* c *be in* $\widetilde{\Gamma}^m_p$, $p \in \mathbb{N}$. *Then*

$$
U \to c(U,\dots,U,M(U;t)U;t,x,\xi) \tag{1.1.76}
$$

is in $\Sigma\Gamma^m_{K,K',p+p_4}[r,N]$. *If the symbol* c *is independent of* ξ *(i.e.* c *is in* $\widetilde{\mathscr{F}}_p$*), so is the symbol in* (1.1.76) *(thus it is a function in* $\Sigma\mathscr{F}_{K,K',p+p_4}[r,N]$*). Moreover if* c *is a symbol in* $\Gamma^m_{K,K',N}[r]$ *then the symbol in* (1.1.76) *is in* $\Gamma^m_{K,K',N}[r]$.
All the statements of the proposition have their counterpart for autonomous classes.

Proof. The item (i) is an immediate corollary of Proposition 1.1.2.
Let us prove (ii). We shall give the proof for the term $\mathrm{Op}^{\mathscr{B}W}(a(U;t,\cdot,\xi)) \circ R(U;t)$, the other one is very similar. Decomposing $a = \sum_{q'=p_1}^N a_q$ and $R = \sum_{q''=p_3}^N R_q$ we need to prove the following facts

- $\mathrm{Op}^{\mathscr{B}W}(a_{q'}(U_1,\dots,U_{q'};\xi)) \circ R_{q''}(U_{q'+1},\dots,U_{q'+q''})$ is in $\widetilde{\mathscr{R}}^{-\rho+m}_{q'+q''}$ for $p_1+p_3 \le q'+q'' \le N-1$;
- $\mathrm{Op}^{\mathscr{B}W}(a_{q'}(U,\dots,U;\xi)) \circ R_{q''}(U,\dots,U)$ is in $\mathscr{R}^{-\rho+m}_{K,K',N}[r]$ for $q'+q'' \ge N$;
- $\mathrm{Op}^{\mathscr{B}W}(a_{q'}(U,\dots,U;\xi)) \circ R_N(U;t)$ is in $\mathscr{R}^{-\rho+m}_{K,K',N}[r]$ for any $p_1 \le q' \le N-1$;
- $\mathrm{Op}^{\mathscr{B}W}(a_N(U;t,\cdot,\xi)) \circ R_N(U;t)$ is in $\mathscr{R}^{-\rho+m}_{K,K',N}[r]$;
- $\mathrm{Op}^{\mathscr{B}W}(a_N(U;t,\cdot,\xi)) \circ R_{q''}(U,\dots,U)$ is in $\mathscr{R}^{-\rho+m}_{K,K',N}[r]$ for any $p_3 \le q'' \le N-1$.

Consider the term in the first item and replace U_j by $\Pi_{n_j}U_j$ for any $j = 1,\dots,q'+q''$. We have to estimate the L^2 norm of

$$
\begin{aligned}
&\sum_{n_0'} \mathrm{Op}^{\mathscr{B}W}(\Pi_{n_1}U_1,\dots,\Pi_{n_{q'}}U_{q'};n_0') \times \\
&\times \Big[R(\Pi_{n_{q'+1}}U_{q'+1},\dots \Pi_{n_{q'+q''}}U_{q'+q''})[\Pi_{n_{q'+q''+1}}U_{q'+q''+1})]\Big].
\end{aligned}
$$

By using (1.1.16) and (1.1.6) we can bound the L^2 norm of the term above by

$$\begin{aligned}\sum_{n_0'\in\mathbb{N}}\langle n_0'\rangle^m \max(\langle n_1\rangle,\ldots,\langle n_{q'}\rangle)^{\mu_1}\frac{\max_2(\langle n_{q'+1}\rangle,\ldots,\langle n_{q'+q''+1}\rangle)^{\mu_2+\rho}}{\max_2(\langle n_{q'+1}\rangle,\ldots,\langle n_{q'+q''+1}\rangle)^{\rho}}\times\\ \times\prod_{j=1}^{q'+q''}\left\|\Pi_{n_j}U_j\right\|_{L^2}.\end{aligned}\tag{1.1.77}$$

Moreover the indexes have to satisfy the following

$$\sigma_0 n_0=\sum_{j=1}^{q'}\sigma_j n_j+\sigma n_0',\quad n_0'=\sum_{j=q'+1}^{q'+q''+1}\sigma_j n_j,\tag{1.1.78}$$

for some choice of signs σ, σ_j in $\{\pm1\}$ for $j=0,\ldots,q'+q''+1$. Thanks to (1.1.78) the sum in (1.1.77) disappears since n_0' is uniquely determined once fixed n_0, ..., $n_{q'+q''+1}$. Furthermore from the second term in (1.1.78) we deduce that $\langle n_0'\rangle\le C\max(\langle n_{q'+1}\rangle,\ldots,\langle n_{q'+q''+1}\rangle)$, moreover, thanks to the spectral condition satisfied by $\mathrm{Op}^{\mathscr{B}W}(a)$, we must have $\max(\langle n_1\rangle,\ldots,\langle n_{q'}\rangle)\ll\langle n_0'\rangle\sim\langle n_0\rangle$. Therefore the following relations hold true

$$\begin{aligned}&\max(\langle n_1\rangle,\ldots,\langle n_{q'+q''+1}\rangle)\sim\max(\langle n_{q'+1}\rangle,\ldots,\langle n_{q'+q'+1}\rangle),\\ &\max_2(\langle n_1\rangle,\ldots,\langle n_{q'+q''+1}\rangle)\sim\max(\langle n_1\rangle,\ldots,\langle n_{q'}\rangle),\end{aligned}$$

hence from (1.1.77) we conclude.

The second item can be treated as follows. One could prove, exactly in the same way as done above, that $\mathrm{Op}^{\mathscr{B}W}(a_{q'}(U_1,\ldots,U_{q'};\xi))\circ R_{q''}(U_{q'+1},\ldots,U_{q'+q''})$ actually is in $\widetilde{\mathscr{R}}^{-\rho+m}_{q'+q''}$ for $q'+q''\ge N$. Once proved this one concludes by Remark 1.1.1.

The reasoning for the remaining items is similar, for instance we explain how to deal with the third item. One first has to embed the symbol $a_{q'}(U,\ldots,U;\xi)$ in the class $\Gamma^m_{K,0,q'}[r,\mathrm{aut}]$ by using Remark 1.1.9, then the thesis just follows by Liebniz rule and a combination of estimates (1.1.45) and (1.1.8).

We now prove item (iii) of the Proposition. By definition we can decompose $R(U;t)=\sum_{q'=p_3}^N R_{q'}(U,\ldots,U;t)$ and $R(U;t)=\sum_{q'=p_3}^N R_{q'}(U,\ldots,U;t)$, we argue as done for the item (ii), hence the only non trivial fact to prove is that

$$R(U_1,\ldots,U_{q'})\circ M(U_{q'+1},\ldots,U_{q'+q''})\in\widetilde{\mathscr{R}}^{-\rho+m''}_{q'+q''}$$

in the case that $p_2 + p_4 \le q' + q'' \le N-1$. We need to estimate the L^2-norm of the quantity

$$\begin{aligned}\sum_{n_0' \in \mathbb{N}} &\Pi_{n_0} R(\Pi_{n_1} U_1, \dots, \Pi_{n_{q'}} U_{q'}) \Pi_{n_0'} \\ &\Pi_{n_0'} M(\Pi_{n_{q'+1}} U_{q'+1}, \dots, \Pi_{n_{q'+q''}} U_{q'+q''}) \Pi_{n_{q'+q''+1}} U_{q'+q''+1}.\end{aligned} \tag{1.1.79}$$

As before, the (1.1.79) is different from 0 only if (1.1.78) holds true. Therefore, by using (1.1.5) and (1.1.12), we can bound the L^2-norm of (1.1.79) by

$$\begin{aligned}\sum_{n_0' \in \mathbb{N}} &\frac{\max_2(\langle n_1 \rangle, \dots, \langle n_{q'} \rangle, \langle n_0' \rangle)^{\mu+\rho}}{\max(\langle n_1 \rangle, \dots, \langle n_{q'} \rangle, \langle n_0' \rangle)^{\rho}} \\ &\times \max(\langle n_0' \rangle, \langle n_{q'+1} \rangle, \dots, \langle n_{q'+q''+1} \rangle)^{m''}.\end{aligned} \tag{1.1.80}$$

Since the indexes satisfy (1.1.78) we can remove the sum. Moreover we may assume that there exists one index, let's say n_j, among $n_1, \dots, n_{q'+q''+1}$ larger than the others, indeed if it is not the case we have that $\max_2(\langle n_1 \rangle, \dots, \langle n_{q'+q''+1} \rangle)$ is equivalent to $\max(\langle n_1 \rangle, \dots, \langle n_{q'+q''+1} \rangle)$ and therefore the thesis follows. If j belongs to $\{q'+1, \dots, q'+q''+1\}$ then by (1.1.78) we must have $n_j \sim n_0'$, therefore from (1.1.80) we prove a bound of the form (1.1.6) with ρ replaced by $\rho - m''$. In the other case, i.e. when $j \in \{1, \dots, q'\}$, the reasoning is similar.
The proof of (iv) follows just by the previous item and a combination of Liebniz rule, (1.1.8) and (1.1.14).
Item (v) follows reasoning as above and using (1.1.16), (1.1.17), (1.1.12), (1.1.13), (1.1.14), (1.1.20). □

Lemma 1.1.4. *Let $C(U; t, \cdot) \in \Sigma\Gamma^m_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ for some $m \in \mathbb{R}$, $K' \le K-1$, $0 \le p \le N$ and assume that U is a solution of an equation*

$$\partial_t U = \mathrm{i} E \widetilde{M}(U; t) U, \tag{1.1.81}$$

for some $\widetilde{M} \in \Sigma\mathcal{M}_{K,1,0}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. Then the the symbol $\partial_t C(U; t, \cdot)$ belongs to $\Sigma\Gamma^m_{K,K'+1,p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. Decompose the symbol $C(U; t, x, \xi)$ as

$$C(U; t, x, \xi) = \sum_{q=p}^{N-1} C_q(U, \dots, U; x, \xi) + C_N(U; t, x, \xi).$$

The derivative with respect to t of the non homogeneous term $D_t C_N(U;t,x,\xi)$ is in the space $\Sigma\Gamma^m_{K,K'+1,N}[r]\otimes\mathcal{M}_2(\mathbb{C})$ by Remark 1.1.8. The derivative of the homogeneous part is a sum of terms of the form $C(U,\dots,D_tU,\dots,U;x,\xi)$. We conclude the proof by using equation (1.1.81) and item (v) of Proposition 1.1.3. □

1.1.6 Para-composition

In this subsection we define the para-composition operator (in the sense of Alinhac [2]) associated to a diffeomorphism of $\mathbb{T}$ of the form $x\mapsto x+\beta(x)$. The two main properties of a para-composition operator are the following: on one hand it has to be a bounded operator between any Sobolev space even if the function $\beta(x)$ has only finite regularity, on the other it has to conjugate a para-differential operator into another one up to a smoothing remainder. We give below the alternative definition, following [21], of the para-composition using flows.
First of all we need the following lemma.

Lemma 1.1.5. *Let $\beta(U;t,\cdot)$ be a real valued function in $\Sigma\mathcal{F}_{K,K',1}[r,N]$ with U in the ball of center 0 and radius r of the space $C^K_{*\mathbb{R}}(I;H^\sigma(\mathbb{T};\mathbb{C}^2))$, consider the map*

$$\Phi_U : x\mapsto x+\beta(U;t,x). \tag{1.1.82}$$

Then Φ_U is a diffeomorphism of $\mathbb{T}$ is r if small enough. Moreover its inverse may be written as $\Phi_U^{-1}: y\mapsto y+\gamma(U;t,y)$ where $\gamma(U;t,\cdot)$ is a real valued function in $\Sigma\mathcal{F}_{K,K',1}[r,N]$.

Proof. By Definition 1.1.13 and (1.1.20) we have that a function $\beta(U;t,x)$ in $\mathcal{F}_{K,K',1}[r]$ satisfies

$$|\partial_t^k\partial_x^\alpha\beta(U;t,x)|\leq C\,\|U(t,\cdot)\|_{K'+k,\sigma},\quad 0\leq k\leq K-K',\quad \alpha\leq\sigma-\sigma_0, \tag{1.1.83}$$

if $\sigma\geq\sigma_0\gg 1$ and $U\in B^K_{\sigma_0}(I,r(\sigma))\cap C_*^{k+K'}(I;H^\sigma(\mathbb{T};\mathbb{C}^2))$. For the moment we just look for a function γ which satisfies the estimate (1.1.83) for $k\leq K-K'$, $0\leq\alpha\leq 1$ and such that $(\mathrm{Id}+\beta)\circ(\mathrm{Id}+\gamma)=\mathrm{Id}$. In order to do this it is enough to note that on the space of functions satisfying the estimate (1.1.83) for $k\leq K-K'$, $0\leq\alpha\leq 1$, the operator $F_\beta(\gamma):=-\beta\circ(\mathrm{Id}+\gamma)$ defines a contraction if β satisfies (1.1.83) for $0\leq k\leq K-K'$, $0\leq\alpha\leq 2$, $\sigma=\sigma_0+2$ and if $\|U(t,\cdot)\|_{\sigma,k}$ is small enough. Then one proves that γ belongs to $\mathcal{F}_{K,K',1}[r]$ by using the fact that $\partial_x\gamma$ may be expressed from $\partial_x\beta$.

We now prove that if $\beta(U;t,x)$ belongs $\Sigma\mathscr{F}_{K,K',1}[r,N]$ then so does $\gamma(U;t,x)$. We have

$$\beta(U;t,x) = \sum_{p=1}^{N-1} \beta_p(U,\dots,U;x) + \beta_N(U;t,x),$$

for some multilinear functions β_p in $\widetilde{\mathscr{F}}_p$ and β_N in $\mathscr{F}_{K,K',N}[r]$. Therefore the function $\gamma(U;t,x)$ found before satisfies

$$\begin{aligned}\gamma(U;t,x) &= -\beta(U;t,x+\gamma(U;t,x))\\ &- \sum_{p=1}^{N-1} \beta_p(U,\dots,U;x+\gamma(U;t,x)) - \beta_N(U;t,x+\gamma(U;t,x)).\end{aligned}$$

The thesis follows by Taylor expanding the r.h.s. of above equation. □

With the aim of simplifying the notation we set $\beta(x) := \beta(U;x)$, $\gamma(y) := \gamma(U;x)$ and we define the following quantities

$$\begin{aligned} B(\tau;x,\xi) = B(\tau,U;x,\xi) &:= -\mathrm{i}b(\tau;x)(\mathrm{i}\xi),\\ b(\tau;x) &:= \frac{\beta(x)}{(1+\tau\beta_x(x))}. \end{aligned} \tag{1.1.84}$$

Then we define the para-composition operator associated to the diffeomorphism (1.1.82) as $\Omega_{B(U)}(1)$, where $\Omega_{B(U)}(\tau)$ is the flow of the linear paradifferential equation

$$\begin{cases} \dfrac{d}{d\tau}\Omega_{B(U)}(\tau) = \mathrm{i}\mathrm{Op}^{\mathscr{B}W}(B(\tau;U,\xi))\Omega_{B(U)}(\tau)\\ \Omega_{B(U)}(0) = \mathrm{id}. \end{cases} \tag{1.1.85}$$

We state here a lemma which asserts that the problem (1.1.85) is well posed and whose solution is a one parameter family of bounded operators on H^s, which is one of the main properties of a para-composition operator.

Lemma 1.1.6. *Let $0 \le K' \le K$ be in $\mathbb{N}$, $r > 0$ and $\beta(U;x) \in \mathscr{F}_{K,K',1}[r]$ for U in the space $C^K_{*\mathbb{R}}(I,\mathbf{H}^s)$. The system (1.1.85) has a unique solution defined for $\tau \in [-1,1]$. Moreover for any s in $\mathbb{R}$ there exists a constant $C_s > 0$ such that for any U in $B^K_{s_0}(I,r)$ and any W in H^s*

$$C_s^{-1}\|W\|_{H^s} \le \|\Omega_{B(U)}(\tau)W\|_{H^s} \le C_s\|W\|_{H^s}, \quad \forall\, \tau \in [-1,1], \;\; W \in H^s, \tag{1.1.86}$$

and

$$\|\Omega_{B(U)}(\tau)W\|_{K-K',s} \le (1 + C\|U\|_{K,s_0})\|W\|_{K-K',s}, \tag{1.1.87}$$

where $C > 0$ is a constant depending only on s.

Proof. First of all note that the operator ${\rm Op}^{\mathscr{B}W}(B(\tau,U;x,\xi))$ is self-adjoint since the symbol B is real, this is crucial for the well posedness of (1.1.85) since the symbol B has order one. Let χ be a $C_0^\infty(\mathbb{R})$ cut-off function having small enough support a and being equal to 1 near the origin. For $\lambda>1$ consider the symbols $B_\lambda(\tau,U;x,\xi):=\chi(\xi/\lambda)B(\tau,U;x,\xi)$ and let $W_\lambda(\tau)$ be the solution of the Banach space ODE

$$\begin{cases} \dfrac{d}{d\tau}W_\lambda(\tau)={\rm iOp}^{\mathscr{B}W}(B_\lambda(\tau,U;x,\xi))W_\lambda(\tau)\\ W_\lambda(0)=W\in H^s(\mathbb{T})\end{cases}. \tag{1.1.88}$$

Set $\Lambda^s(\xi):=(1+\xi^2)^{s/2}$ and define $\Lambda^s(D)={\rm Op}(\Lambda^s(\xi))$ with $D=(1/{\rm i})\partial_x$. Thanks to Propositions 1.1.2, 1.1.1 we have

$$\|[\Lambda^s(D),{\rm Op}^{\mathscr{B}W}(B_\lambda)]W_\lambda\|_{L^2}\leq C\|W_\lambda\|_{H^s}\|U\|_{\mathbf{H}^{s_0}}, \tag{1.1.89}$$

for some constant $C>0$. Since the operator ${\rm Op}^{\mathscr{B}W}(B_\lambda)-({\rm Op}^{\mathscr{B}W}(B_\lambda))^*$ is uniformly bounded in λ on L^2, the equation (1.1.88) implies

$$\frac{d}{d\tau}\|\Lambda^s W_\lambda\|_{L^2}^2\leq\|W_\lambda\|_{H^s}^2\|U\|_{\mathbf{H}^{s_0}}. \tag{1.1.90}$$

The inequality (1.1.90) implies that the family of functions W_λ is equi-bounded (with respect to λ) in the space $C^0([-1,1];H^s)$. Similarly we have

$$\|\frac{d}{d\tau}W_\lambda(\tau)\|_{H^{s-1}}\leq O(\|U\|_{\mathbf{H}^{s_0}})\|W_\lambda\|_{H^s}\leq O(\|U\|_{\mathbf{H}^{s_0}})\|W\|_{H^s}, \tag{1.1.91}$$

hence W_λ is uniformly in λ bounded in the space $Lip([-1,1,];H^{s-1})$. Writing $s'=\mu(s-1)+(1-\mu)s$ for $\mu\in(0,1]$ and using the log-convexity of the Sobolev norm we deduce that W_λ is uniformly in λ bounded in the space $C^{0,\mu}([-1,1];H^{s-\mu})$ for any $0<\mu\leq 1$. Hence, as a consequence of Ascoli theorem, $W_\lambda(\tau)$ converges, up to subsequences, to $W(\tau)$ in $C^{0,\mu}H^{s-\mu}$ for any $0<\mu\leq1$ as λ goes to infinity. We show that $W(\tau)$ belongs to $C^0H^{s'}\cap C^1H^{s'-1}$ and solves the equation (1.1.88) with B_λ replaced by B and the initial data $W\in H^{s'}$. The r.h.s. of (1.1.88) converges strongly to ${\rm Op}^{\mathscr{B}W}(B)W(\tau)$ in $C^0H^{s'-1}$ for any $s'<s$ and for each fixed τ. Indeed

$$\begin{aligned}\|{\rm Op}^{\mathscr{B}W}(B_\lambda)W_\lambda(\tau)-{\rm Op}^{\mathscr{B}W}(B)W(\tau)\|_{C^0H^{s'-1}}\leq&\\ \|{\rm Op}^{\mathscr{B}W}(B_\lambda)(W_\lambda(\tau)-W(\tau))\|_{C^0H^{s'-1}}&\\ +\|{\rm Op}^{\mathscr{B}W}(B_\lambda-B)W(\tau)\|_{C^0H^{s'-1}}.&\end{aligned} \tag{1.1.92}$$

The first summand tends to zero since $\|W_\lambda(\tau)-W(\tau)\|_{C^0H^{s'}}$ goes to zero as $\lambda\to\infty$. The second summand goes to zero thanks to Lebesgue Dominated Convergence Theorem. On the other hand $\|\frac{d}{d\tau}W_\lambda(\tau)\|_{C^0H^{s'-1}}$ is uniformly bounded, hence $\frac{d}{d\tau}W_\lambda(\tau)\overset{*}{\rightharpoonup}\frac{d}{d\tau}W(\tau)$ in $C^0H^{s'-1}$. Hence $W(\tau)$ solves the equation and $W(\tau)\in C^0H^{s'}\cap C^1H^{s'-1}$. The final step is to show that actually $W(\tau)$ belongs to $C^0H^s\cap C^1H^{s-1}$. First note that by (1.1.91) we have $\frac{d}{d\tau}W_\lambda\overset{*}{\rightharpoonup}W(\tau)$ in $L^\infty H^{s-1}$ and hence $W(\tau)$ belongs to the space $L^\infty([-1,1];H^s)\cap Lip([-1,1,];H^{s-1})$. In order to prove that $W(\tau)$ is strongly continuous in H^s we show that $W(\tau)$ is weak continuos in H^s and moreover that the function $\tau\to\|W(\tau)\|_{H^s}$ is continuos. Consider a sequence τ_n converging to τ as $n\to\infty$. Let $\phi\in H^{-s}$ and $\phi_\varepsilon\in C_0^\infty$ such that $\|\phi-\phi_\varepsilon\|_{H^{-s}}\le\varepsilon$. We have

$$\begin{aligned}&\left|\int_{\mathbb{T}}(W(\tau_n)-W(\tau))\phi dx\right|\le\\&\left|\int_{\mathbb{T}}(W(\tau_n)-W(\tau))\phi_\varepsilon dx\right|+\left|\int_{\mathbb{T}}(W(\tau_n)-W(\tau))(\phi-\phi_\varepsilon)dx\right|\le\\&\|W(\tau_n)-W(\tau)\|_{H^{s'}}\|\phi_\varepsilon\|_{H^{-s'}}+\|W(\tau_n)-W(\tau)\|_{H^s}\|\phi-\phi_\varepsilon\|_{H^{-s}}\le\\&C\varepsilon+2\|u\|_{L^\infty H^s}\varepsilon\end{aligned}\tag{1.1.93}$$

for n sufficiently large. This means that $W(\tau)$ is weakly continuous in H^s.
On the other hand, since $\mathrm{Op}^{\mathcal{B}W}(B)$ is self-adjoint, we have

$$\frac{d}{d\tau}\|W(\tau)\|_{H^s}=2\mathrm{Re}\langle \mathrm{iOp}^{\mathcal{B}W}(B)W(\tau),W(\tau)\rangle_{H^s},\tag{1.1.94}$$

which implies, using the symbolic calculus , that $\|W(\tau)\|^2_{H^s}$ is Lipschitz continuous in τ. Thus $\|W(\tau)\|^2_{H^s}\to\|W(\tau')\|^2_{H^s}$ as $\tau\to\tau'$. Therefore the flow of the system (1.1.85) is well defined on H^s by setting $\Omega_{B(U)}(\tau)W:=W(\tau)$ for any initial data W. It remains to prove (1.1.87). By setting $V(\tau):=\partial_\tau(\Omega_{B(U)}(\tau)W)$ we have that V satisfies the following equation:

$$\partial_\tau V(\tau)=\mathrm{iOp}^{\mathcal{B}W}(B(U))V(\tau)+\mathrm{iOp}^{\mathcal{B}W}(\partial_\tau B(U))\Omega_{B(U)}W.\tag{1.1.95}$$

Hence we obtain the estimate of $\|V(\tau)\|_{H^{s-1}}$ using the Duhamel formula and the estimate of the flow $\Omega_{B(U)}(\tau)$. Repeating the same argument for the higher order derivatives ∂_τ^k for $k\le K-K'$ we obtain (1.1.87). □

We now prove the other fundamental property of the para-composition operator. Given a symbol a in the space $\Sigma\Gamma^m_{K,K',p}[r,N]$, consider the conjugate with the para-composition $\Omega_{B(U)}(\tau)$

$$A(U,\tau) := \Omega_{B(U)}(\tau)\circ \mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi))\circ \big(\Omega_{B(U)}(\tau)\big)^{-1}. \tag{1.1.96}$$

We prove that $A(U,\tau)$ is still a para-differential operator up to a smoothing reminder. In order to do this we note that (1.1.96) solves the Heisenberg equation

$$\begin{cases} \partial_\tau A(U,\tau) = \mathrm{i}\big[\mathrm{Op}^{\mathscr{B}W}(B(U,\tau;x,\xi)), A(U,\tau)\big] \\ A(U,0) = \mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi)). \end{cases} \tag{1.1.97}$$

We shall prove that equation (1.1.97) admits an approximate solution of the form

$$A(U,\tau) = \mathrm{Op}^{\mathscr{B}W}(a_0(U,\tau;t,x,\xi) + a_1(U,\tau;t,x,\xi)+\dots)$$

with

$$a_0(U,\tau;t,x,\xi)\in \Sigma\Gamma^m_{K,K',p}[r,N], \quad a_1(U,\tau;t,x,\xi)\in \Sigma\Gamma^{m-2}_{K,K',p}[r,N],\dots$$

which is enough to show that $A(U,\tau)$ is still a para-differential operator (up to a smoothing reminder).
In the next lemma we show that the classes of symbols introduced in Subsection 1.1.3 are stable under changes of coordinates.

Lemma 1.1.7. *Let a be a symbol in* $\Sigma\Gamma^m_{K,K',p}[r,N]$, *consider two functions b and c in the class* $\Sigma\mathscr{F}_{K,K',1}[r,N]$. *Then*

$$a(V;x+b(V;t,x),\xi(1+c(V;t,x))) \tag{1.1.98}$$

is still a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$. *In the case that a does not depend on* ξ, *i.e. belongs to* $\Sigma\mathscr{F}_{K,K',p}[r,N]$, *then* $a(V;x+b(V;t,x))$ *is in the class* $\Sigma\mathscr{F}_{K,K',p}[r,N]$.

Proof. It follows by Taylor expansion. More precisely decompose

$$a(V;\cdot) = \sum_{q=p}^{N-1} a_q(V,\dots,V;\cdot) + a_N(V;\cdot),$$
$$b(V;\cdot) = \sum_{q'=p}^{N-1} b_{q'}(V,\dots,V;\cdot) + b_N(V;\cdot),$$
$$c(V;\cdot) = \sum_{q''=p}^{N-1} c_{q''}(V,\dots,V;\cdot) + b_N(V;\cdot),$$

where a_q is in $\widetilde{\Gamma}^m_q$ for $q = p,\dots,N-1$, a_N is in $\Gamma^m_{K,K',N}[r]$, $b_{q'}$ (resp. $c_{q''}$) are in $\widetilde{\mathscr{F}}_{q'}$ (resp. in $\widetilde{\mathscr{F}}_{q''}$) for $q' \le N-1$ (resp. $q'' \le N-1$), b_N and c_N are in $\mathscr{F}_{K,K',N}[r]$. Then one Taylor expand the function $a(V;t,x,\xi)$ with respect to the couple (x,ξ) at order N and rewrite (1.1.98) as sum of terms of the following type:

- multilinear terms of the form

$$\begin{aligned}&\partial_x^\alpha\partial_\xi^\beta a_q(\Pi_{n_0}\mathcal{V}_0;x,\xi)b_{q'_1}(\Pi_{n'_1}\mathcal{V}'_1;x)\dots b_{q'_\alpha}(\Pi_{n'_\alpha}\mathcal{V}'_\alpha;x)\\ &\quad\times\xi^\beta c_{q''_1}(\Pi_{n''_1}\mathcal{V}''_1;x)\dots c_{q''_\beta}(\Pi_{n''_\beta}\mathcal{V}''_\beta;x),\end{aligned} \tag{1.1.99}$$

  where $\alpha+\beta \le N-1$ and $\mathcal{V}_0 := (V_{0,1},\dots,V_{0,q})$, $\mathcal{V}'_\ell := (V'_{\ell,1},\dots,V'_{0,q_\ell})$ and $\mathcal{V}''_\ell := (V''_{\ell,1},\dots,V''_{0,q_\ell})$. These are multilinear symbols in $\widetilde{\Gamma}^m_{\tilde{q}}$ for $\tilde{q} \ge q$. Therefore the terms such that $\tilde{q} \le N-1$ contribute to the multilinear part, the ones such that $\tilde{q} \ge N$ contribute to the non-homogeneous term of degree N.

- term of the expansion of the same type of (1.1.99) but such that at least on among q, q'_ℓ, q''_ℓ is equal to N. These terms contribute to the non homogeneous part of degree N.

- terms coming from the integral reminder in Taylor formula of the form

$$\begin{aligned}&\int_0^1 (1-\lambda)^{N-1}(\partial_x^\alpha\partial_\xi^\beta a)(V;t,x+\lambda b(V;t,x),\xi(1+\lambda c(V;t,x)))d\lambda\\ &\quad\times b(V;t,x)^\alpha(\xi c(V;t,x))^\beta\end{aligned}$$

  for $\alpha+\beta = N$. These terms contribute to the non homogeneous part of degree N.

□

We need the following lemma in which we give an expansion in decreasing orders of the commutator between two para-differential operators.

Lemma 1.1.8. *Consider* $a \in \Sigma\Gamma^m_{K,K',p}[r,N]$ *and* $B \in \Sigma\Gamma^1_{K,K',1}[r,N]$. *Then*

$$\begin{aligned}&\Big[\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}B(U;t,x,\xi)),\mathrm{Op}^{\mathscr{B}W}(a(U;t,x,\xi))\Big] = \mathrm{Op}^{\mathscr{B}W}(\{B(U;t,x,\xi),a(U;t,x,\xi)\})\\ &\quad+\mathrm{Op}^{\mathscr{B}W}(r_{-3,\rho}(B,a))+R,\end{aligned}$$

where R is a smoothing remainder in $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$, the symbol $r_{-3,\rho}(B,a)$ is in $\Sigma\Gamma^{m-2}_{K,K',p+1}[r,N]$, $\{B(U;t,x,\xi),a(U;t,x,\xi)\}$ is in $\Sigma\Gamma^m_{K,K',p+1}[r,N]$ and by $\{\cdot,\cdot\}$ we have denoted the standard Poisson brackets between functions.

Proof. It follows from Proposition 1.1.2 and formulas (1.1.55) and (1.1.56). □

Having Lemma 1.1.8 in hand it is evident that the first order approximation of the solution of equation (1.1.97) is

$$A_0(\tau,U) = \mathrm{Op}^{\mathscr{B}W}(a_0(\tau,U;t,x,\xi)), \tag{1.1.100}$$

where a_0 solves the following transport equation

$$\begin{cases} \partial_\tau a_0(\tau,U;t,x,\xi) = \{B(U;t,x,\xi), a_0(\tau,U;t,x,\xi)\} \\ a_0(0,U;t,x,\xi) = a(U;t,x,\xi). \end{cases} \tag{1.1.101}$$

In the next lemma we solve equation (1.1.101) in the next lemma by using the method of characteristics.

Lemma 1.1.9. *The solution of* (1.1.101) *is*

$$a_0(U;\tau;t,\xi) = a(U;\Phi^{\tau,0}(x,\xi)), \tag{1.1.102}$$

where

$$\Phi^{\tau,0}(x,\xi) = \Big(x+\tau\beta(U;x), \xi(1+\partial_y\gamma(U;\tau,y))_{|y=x+\tau\beta(U;x)}\Big)$$

are the solutions of the characteristic system

$$\begin{cases} \frac{d}{ds}x(s) = -b(s,x(s)) \\ \frac{d}{ds}\xi(s) = b_x(s,x(s))\xi(s). \end{cases} \tag{1.1.103}$$

Moreover $a_0(\tau,\cdot)$ belongs to $\Sigma\Gamma^m_{K,K',p}[r,N]$.

Proof. The map $s \mapsto a_0(s,x(s),\xi(s))$ is constant along the flow of (1.1.103). Therefore, denoting by $\Phi^{\tau_0,\tau}(x,\xi)$ the solution with initial condition $\Phi^{\tau_0,\tau_0}(x,\xi) = (x,\xi)$, the solution of the transport equation (1.1.101) is $a_0(\tau,x,\xi) = a(\Phi^{\tau,0}(x,\xi))$.
The solution of (1.1.103) are directly given by the path of diffeomorphism

$$y = x+\tau\beta(x), \quad x = y+\gamma(\tau,y),$$

indeed by formulas (1.1.84) $(x(\tau),\xi(\tau))$ defined by

$$x(\tau) := y+\gamma(\tau,y), \quad \xi(\tau) := \xi_0(1+\partial_x\beta)(x(\tau)) = \frac{\xi_0}{1+\partial_y\gamma(\tau,y)}$$

is the solution of (1.1.103) with initial condition $x(0) = y$ and $\xi(0) = \xi_0$.
The symbol (1.1.102) is in $\Sigma\Gamma^m_{K,K',p}[r,N]$ with estimates uniform in τ thanks to Lemma 1.1.98. □

Let us quantify how approximatively the operator $A_0(\tau, U)$ solves the equation (1.1.97). By using Lemmata 1.1.9 and 1.1.8

$$
\begin{aligned}
\partial_\tau A_0(U,\tau) &= \mathrm{Op}^{\mathscr{B}W}(\partial_\tau a_0(\tau,U;\cdot)) \\
&= \mathrm{Op}^{\mathscr{B}W}(\{B(\tau,U;\cdot), a_0(\tau,U,\cdot)\}) \\
&= \Big[\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}B(\tau,U;\cdot), \mathrm{Op}^{\mathscr{B}W}(a_0(\tau,U;\cdot)))\Big] - \\
&\quad - \mathrm{Op}^{\mathscr{B}W}(r_{-3,\rho}(B,a_0)) - R,
\end{aligned}
$$

where $r_{-3,\rho}(B,a_0) \in \Sigma\Gamma^{m-2}_{K,K',p}[r,N]$ and $R \in \Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p}[r,N]$. Therefore A_0 is a solution of (1.1.97) up to an operator of order $m-2$.
Let $a_1(\tau, U; t, x, \xi)$ be the solution of the following non-homogeneous transport equation

$$
\begin{cases}
\partial_\tau a_1(\tau,U;\cdot) = \{B(\tau,U;\cdot), a_1(\tau,U;\cdot) + r_{-3,\rho}(B,a_0)\} \\
a_1(0,U;\cdot) = 0,
\end{cases}
\tag{1.1.104}
$$

we define $A_1(\tau,U) := \mathrm{Op}^{\mathscr{B}W}(a_0(\tau,U;\cdot) + a_1(\tau,U;\cdot))$. Let us see that $A_1(\tau,U)$ is a better approximation of the solution of (1.1.97). We have

$$
\begin{aligned}
\partial_\tau A_1(\tau,U) &= \mathrm{Op}^{\mathscr{B}W}(\partial_\tau a_0(\tau,U;\cdot)) + \mathrm{Op}^{\mathscr{B}W}(\partial_\tau a_1(\tau,U;\cdot)) \\
&= \mathrm{Op}^{\mathscr{B}W}(\{B(\tau,U;\cdot), B(\tau,U;\cdot)\}) + \mathrm{Op}^{\mathscr{B}W}(\{B(\tau,U;\cdot), a_1(\tau,U;\cdot)\}) + \\
&\quad + \mathrm{Op}^{\mathscr{B}W}(r_{-3,\rho}(B,a_0)) \\
&= \Big[\mathrm{i}\mathrm{Op}^{\mathscr{B}W}(B(\tau,U;\cdot)), A_1(\tau,U)\Big] + R' - \mathrm{Op}^{\mathscr{B}W}(r_{-3,\rho}(B,a_1)),
\end{aligned}
\tag{1.1.105}
$$

where we have used (1.1.104) and, again, Lemmata 1.1.8 and 1.1.9. It turns out that $r_{-3,\rho}(B,a_1)$ and R' are lower orders than $m-2$, we shall prove this fact below. In the following lemma we solve the equation (1.1.104).

Lemma 1.1.10. *The solution of* (1.1.104) *is*

$$
a_1(\tau,x,\xi) = \int_0^\tau r_{-3,\rho}(B(s,\cdot), a_0(s,\cdot)\Phi^{\tau,s}(x,\xi))\,ds,
\tag{1.1.106}
$$

where $\Phi^{\tau,s}$ *solves* (1.1.9) *with initial condition* $\Phi^{\tau,\tau}(x,\xi) = (x,\xi)$. *Moreover the symbol* (1.1.106) *is in the space* $\Sigma\Gamma^{m-2}_{K,K',p+1}[r,N]$ *with estimates uniform in* $\tau \in [-1,1]$.

Proof. Let $\Phi^{0,s}(x_0,\xi_0) := (x(s),\xi(s))$ be the solution of (1.1.103) with initial condition $\Phi^{0,0}(x_0,\xi_0) = (x_0,\xi_0)$. By taking the derivative of the function $a_1(s,x(s),\xi(s))$, where a_1 solves (1.1.104), we obtain

$$\frac{d}{ds}a_1(s,x(s),\xi(s)) = r_{-3,\rho}(B,a_0)(x(s),\xi(s)).$$

Set $(x,\xi) = \Phi^{0,\tau}(x_0,\xi_0)$, then, by integrating with respect to the variable s and using that $a_1(0,x_0,\xi_0) = 0$, we get the (1.1.106). Moreover the symbol (1.1.106) is in $\Sigma\Gamma^{m-2}_{K,K',p+1}[r,N]$ thanks to Lemmata 1.1.8 and 1.1.98. □

We now quantify how approximatively the operator $A_1(\tau,U)$ solves (1.1.97), i.e. we estimate the lower order terms in (1.1.105). Lemma 1.1.10 implies that the symbol $a_1(\tau,U;\cdot)$ is in $\Sigma\Gamma^{m-2}_{K,K',p+1}[r,N]$ with estimates uniform in $\tau \in [-1,1]$. Therefore, by applying Lemma 1.1.8 with m replaced by $m-2$ and p by $p+1$ we deduce that the symbol $r_{-3,\rho}(B,a_1)$ is in the class $\Sigma\Gamma^{m-4}_{K,K',p+2}[r,N]$. The remainder R' is the sum of a remainder in the class $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$ and another one in $\Sigma\mathscr{R}^{-\rho+m-1}_{K,K',p+2}[r,N]$, therefore it belongs to $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$.
Repeating $\ell \sim \rho/2$ times the above reasoning, until the paradifferential term may be incorporated in the smoothing remainder, we obtain an approximate solution of (1.1.97) of the form

$$A_\ell(\tau,U) := \mathrm{Op}^{\mathscr{B}W}\big(a_0(\tau,U;\cdot)+\ldots+a_\ell(\tau,U;\cdot)\big). \tag{1.1.107}$$

Such operator is the solution of

$$\begin{cases} \partial_\tau A_\ell(\tau,U) = \mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}(B(\tau,U;\cdot)), A_\ell(\tau,U)\Big] + R(\tau) \\ A_\ell(0,U) = \mathrm{Op}^{\mathscr{B}W}\big(a(U;\cdot)\big), \end{cases} \tag{1.1.108}$$

where $R(\tau)$ is a smoothing remainder in $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$.
Let us estimate the difference between $A(\tau,U)$, solution of (1.1.97), and $A_\ell(\tau,U)$ solution of (1.1.108). We have

$$\begin{aligned} &A_\ell(\tau,U) - A(\tau,U) \\ =&A_\ell(\tau,U)\Omega_B(U)(\tau)\big(\Omega_B(U)(\tau)\big)^{-1} - \Omega_B(U)(\tau)\mathrm{Op}^{\mathscr{B}W}\big(a(U;\cdot)\big)\big(\Omega_B(U)(\tau)\big)^{-1} \\ :=&V(\tau)\big(\Omega_B(U)(\tau)\big)^{-1}. \end{aligned}$$

Therefore $V(\tau)$ solves the problem

$$\begin{cases} \partial_\tau V(\tau) = \mathrm{i}\mathrm{Op}^{\mathscr{B}W}(B(\tau,U))V(\tau) + R(\tau)\Omega_B(U)(\tau) \\ V(0) = 0, \end{cases}$$

and hence, by Duhamel principle

$$V(\tau) = \Omega_B(U)(\tau)\int_0^\tau \big(\Omega_B(U)(\tau')\big)^{-1} R(\tau')\Omega_B(U)(\tau')d\tau' \tag{1.1.109}$$

We shall prove that $V(\tau)$ is a smoothing reminder in the class $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$. We are now in position to state the main theorem of this sub-section.

Theorem 1.1.1 (Para-composition)**.** *Let p and $N \in \mathbb{N}$ with $p \le N-1$, $K' \le K$ in $\mathbb{N}$ and $r > 0$. Let U be a function in the ball of center 0 and radius r of $C^K_{*\mathbb{R}}(I;\mathbf{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ with r small enough and σ sufficiently large. Consider $\beta(U;\cdot)$ a function in $\Sigma\mathscr{F}_{K,K',1}[r,N]$ and the diffeomorphism $\Phi_U : x \mapsto x + \beta(U;t,x)$. Let $a(U;\cdot)$ be a symbol in $\Sigma\Gamma^m_{K,K',p}[r,N]$. Then the flow of (1.1.85) is well defined for $|\tau| \le 1$ and for any ρ large enough there is a symbol $a_\Phi \in \Sigma\Gamma^m_{K,K',p}[r,N]$ such that*

$$\begin{aligned} &\Omega_{B(U)}(1)\mathrm{Op}^{\mathscr{B}W}\big(a(U;\cdot)\big)\big(\Omega_{B(U)}(1)\big)^{-1} \\ &\qquad = \mathrm{Op}^{\mathscr{B}W}\big(a_\Phi(U;\cdot)\big) + R(U;t) \end{aligned} \tag{1.1.110}$$

with R in $\Sigma\mathscr{R}^{-\rho+m}_{K,K',p+1}[r,N]$. Moreover we have the expansion

$$a_\Phi(U;\cdot) = a^0_\phi(U;\cdot) + a^1_\phi(U;\cdot), \tag{1.1.111}$$

where

$$a^0_\phi(U;\cdot) = a(U;\Phi_U(t,x),\xi\partial_y\Phi_U^{-1}(t,y)_{|_{y=\Phi_U(t,x)}})$$

is in $\Sigma\Gamma^m_{K,K',p}[r,N]$, while $a^1_\phi(U;\cdot)$ is in $\Sigma\Gamma^{m-2}_{K,K',p+1}[r,N]$. Finally we have $a_\Phi \equiv 1$ in the case that $a \equiv 1$.
The operator

$$\Phi^\star_U := \Omega_{B(U)}(1) \tag{1.1.112}$$

is by definition the para-composition operator associated to the diffeomorphism Φ_U. Furthermore there are multilinear maps M_j in $\widetilde{\mathscr{M}}_p$ for $j = 1,\dots,N-1$ and a map M_N in $\mathscr{M}_{K,K',N}[r]$ such that

$$\Phi^\star_U W = W + \sum_{j=1}^{N-1} M_j(U,\dots,U)W + M_N(U;t)W. \tag{1.1.113}$$

Proof. Recalling equation (1.1.107), Lemmata 1.1.8 and 1.1.10, we set

$$a^1_\Phi(U;\cdot) = a_1(1,U;\cdot) + \ldots + a_\ell(1,U;\cdot) \in \Sigma^{m-2}_{K,K',p+1}[r,N].$$

We know that $a^0_\Phi(U;\cdot)$ is given by (1.1.102) with $\tau = 1$. Therefore it remains to prove that the operator in (1.1.109), recall that $R(\tau')$ is in the class $\Sigma\mathscr{R}^{-\rho+m+1}_{K,K',p+1}[r,N]$, with $\tau = 1$ is a smoothing remainder. We need an expansion of the flow (1.1.85) in homogeneous operators. By applying iteratively the fundamental calculus theorem and using equation (1.1.85), we deduce that

$$\begin{aligned}&\Omega_{B(U)}(\tau) - \mathrm{Id} = \Omega_{B(U)}(\tau) - \Omega_{B(U)}(0) = \\ &\sum_{\ell=1}^{N-(p+1)} \int_{\{0<\tau_\ell<\ldots<\tau_1<\tau\}} \prod_{j=1}^{\ell} \mathrm{iOp}^{\mathscr{B}W}\big(B(\tau_j,U;\cdot)\big) d\tau_\ell \cdots d\tau_1 + \\ &\int_{\{0<\tau_{N-p}<\ldots<\tau_1<\tau\}} \prod_{j=1}^{N-p} \mathrm{iOp}^{\mathscr{B}W}\big(B(\tau_j,U;\cdot)\big)\Omega_{B(U)}(\tau_{N-p}) d\tau_1 \cdots d\tau_{N-p},\end{aligned} \tag{1.1.114}$$

where the product of operators has to be understood as the composition of operators. Let $I(U)$ the second summand in the r.h.s. of (1.1.114). Since the symbol $B(\tau,U;\cdot)$ is in $\Sigma\Gamma^1_{K,K',1}[r,N]$, by using Propositions 1.1.2 and 1.1.1 we deduce that for any $0 \le k \le K - K'$

$$\left\| \partial_t^k I(U) W \right\|_{\mathbf{H}^{s-2k-(N-p)}} \le \sum_{k'+k''=k} \|U\|^{N-p}_{K'+k',s_0} \|W\|_{k'',s}. \tag{1.1.115}$$

After performing a similar expansion for the inverse flow $[\Omega(\tau')]^{-1}$ one plugs such expansions in (1.1.109). The terms containing at least one remainder $I(U)$ (or the respective one coming from the expansion of the inverse flow) are in $\mathscr{R}^{-\rho+m+1+N}_{K,K',N}[r]$ thanks to (1.1.115). The other terms may be written as composition between para-differential operators and smoothing remainders, so that Proposition 1.1.3 implies that the operator in (1.1.109) is in the space $\Sigma\mathscr{R}^{-\rho+m+1+N}_{K,K',p+1}[r,N]$. The thesis follows by renaming ρ as $\rho - N - 1$. The formula (1.1.113) is just a consequence of the above reasoning and Remark 1.1.14. □

In the last proposition of this subsection we study the conjugation of the composition operator $\Omega_{B(U)}(\tau) \circ \partial_t \circ [\Omega_{B(U)}(\tau)]^{-1}$.

Proposition 1.1.4. *Use assumptions and notation of Theorem 1.1.1. If U is a solution of* (1.1.81) *in Lemma 1.1.4 then*

$$\begin{aligned}\Omega_{B(U)}(\tau)\circ\partial_t\circ\Omega^{-1}_{B(U)} &= \partial_t+\Omega_{B(U)}(\tau)\circ(\partial_t\Omega^{-1}_{B(U)}(\tau))\\ &= \partial_t+\mathrm{Op}^{\mathscr{B}W}\big(e(U;t,x,\xi)\big)+R(U;t),\end{aligned}\tag{1.1.116}$$

where

$$e(U;t,x,\xi)=e_1(U;t,x)(\mathrm{i}\xi)+e_0(U;t,x,\xi),\tag{1.1.117}$$

with $e_1(U;t,x)\in\Sigma\mathscr{F}_{K,K'+1,1}[r,N]$*,* $e_0(U;t,x,\xi)$ *is in* $\Sigma\Gamma^{-1}_{K,K'+1,1}[r,N]$ *and* $R(U;t)$ *is a smoothing remainder belonging to* $\Sigma\mathscr{R}^{-\rho}_{K,K'+1,1}[r,N]$.
Moreover $\mathrm{Re}(e)\in\Sigma\Gamma^{-1}_{K,K'+1,1}[r,N]$.

Proof. The function $\psi(\tau):=\Omega_{B(U)}(\tau)\circ\partial_t\circ\Omega_{B(U)}(\tau)^{-1}$ solves the Heinsenberg equation

$$\begin{cases}\partial_\tau\psi(\tau)=\mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}(B(\tau,U;\cdot)),\psi(\tau)\Big]\\ \psi(0)=\partial_t.\end{cases}\tag{1.1.118}$$

We write $\psi(\tau)=\partial_t+Q(\tau)$, where $Q(\tau)=\Omega_{B(U)}(\tau)\circ[\partial_t\Omega_{B(U)}(\tau)]$, then $Q(\tau)$ solves

$$\begin{cases}\partial_\tau Q(\tau)=\mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}(B(\tau,U;\cdot)),Q(\tau)\Big]-\mathrm{i}\mathrm{Op}^{\mathscr{B}W}(\partial_t B(\tau,U;\cdot))\\ Q(0)=0.\end{cases}\tag{1.1.119}$$

We analyse equation (1.1.119) in decreasing orders as done in the proof of Theorem 1.1.1. We look for a solution of the form

$$Q(\tau)=\mathrm{Op}^{\mathscr{B}W}(q_0+q_1+\ldots),$$

up to a smoothing remainder. The principal symbol q_0 has to solve the equation

$$\begin{cases}\partial_\tau q_0(\tau)=\mathrm{i}\big\{B(\tau,U;\cdot)),q_0(\tau)\big\}-\mathrm{i}\partial_t B(\tau,U;\cdot)\\ q_0(0)=0.\end{cases}\tag{1.1.120}$$

By Lemma 1.1.10 the solution of (1.1.120) is

$$q_0(\tau,U;x,\xi)=-\mathrm{i}\int_0^\tau\partial_t B(\tau,U;\Phi^{\tau,s}(x,\xi))ds.\tag{1.1.121}$$

Recalling (1.1.84) and Lemma 1.1.7 the symbol $B(\tau,U;\Phi^{\tau,s}(x,\xi))$ is in $\Sigma\Gamma^1_{K,K',1}[r,N]$ with estimates uniform in $|\tau|\le 1$ and $|s|\le 1$. Lemma 1.1.4 implies that the term

$\partial_t B(\tau, U; \Phi^{\tau,s}(x,\xi))$ is in $\Sigma\Gamma^1_{K,K'+1,1}[r,N]$ with estimates uniform in τ and s as before. We deduce that the symbol in (1.1.121) is in $\Sigma\Gamma^1_{K,K'+1,1}[r,N]$. Note that since B in (1.1.84) is equal to a function in $\Sigma\mathscr{F}_{K,K',1}[r,N]$ times iξ then the symbol $q_0(\tau,U;x,\xi)$ is equal to a function in $\Sigma\mathscr{F}_{K,K'+1,1}[r,N]$ times iξ.
Equations (1.1.119) and (1.1.120) together with Lemma 1.1.8 imply that

$$\begin{aligned}\partial_\tau \mathrm{Op}^{\mathscr{B}W}(q_0(\tau,\cdot)) = &\\ &\Big[\mathrm{iOp}^{\mathscr{B}W}(B(\tau,U;\cdot)), \mathrm{Op}^{\mathscr{B}W}(q_0(\tau,\cdot))\Big]\\ &-\mathrm{iOp}^{\mathscr{B}W}(\partial_t B(\tau,U;\cdot))\\ &-\mathrm{Op}^{\mathscr{B}W}(r_{-3,\rho}(B,q_0(\tau))) - R(\tau),\end{aligned}$$

where $r_{-3,\rho}(B,q_0(\tau))$ is in $\Sigma\Gamma^{-1}_{K,K'+1,2}[r,N]$ and $R(\tau)$ in $\Sigma\mathscr{R}^{-\rho+2}_{K,K'+1,2}[r,N]$ are given by Lemma 1.1.8. Therefore one can continue the proof exactly as done for Theorem 1.1.1. □

1.2 Non-homogeneous para-differential calculus

In this section we give the definitions of non-homogeneous symbols and operators which are defined also far away from the origin.

Definition 1.2.1 (Symbols). *Let $m \in \mathbb{R}$, $K' \le K$ in $\mathbb{N}$, $r > 0$. We denote by $\Gamma^m_{K,K'}[r]$ the space of functions $(U;t,x,\xi) \mapsto a(U;t,x,\xi)$, defined for $U \in B^K_{\sigma_0}(I,r)$, for some large enough σ_0, with complex values such that for any $0 \le k \le K-K'$, any $\sigma \ge \sigma_0$, there are $C > 0$, $0 < r(\sigma) < r$ and for any $U \in B^K_{\sigma_0}(I,r(\sigma)) \cap C^{k+K'}_{*\mathbb{R}}(I,\mathbf{H}^\sigma)$ and any $\alpha,\beta \in \mathbb{N}$, with $\alpha \le \sigma - \sigma_0$*

$$\Big|\partial_t^k \partial_x^\alpha \partial_\xi^\beta a(U;t,x,\xi)\Big| \le C \|U\|_{k+K',\sigma} \langle\xi\rangle^{m-\beta}, \tag{1.2.1}$$

for some constant $C = C(\sigma, \|U\|_{k+K',\sigma_0})$ depending only on σ and $\|U\|_{k+K',\sigma_0}$.

Throughout this section the time t is treated as a parameter, we shall write, for instance, $a(U;x,\xi)$ instead of $a(U;t,x,\xi)$ as done in the preceding section. On the other hand we continue to emphasize the x-dependence of symbols, we shall denote by $a(U;\xi)$ only those symbols which are independent on x. We need the following lemma.

Lemma 1.2.1. *Let $a \in \Gamma^m_{K,K'}[r]$ and $U \in B^K_{\sigma_0}(I,r)$ for some σ_0. One has that*

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m} \| a(U;\cdot,\xi)\|_{K-K',s} \leq C \|U\|_{K,s+\sigma_0+1} . \tag{1.2.2}$$

for $s \geq 0$.

Proof. Assume that $s \in \mathbb{N}$. We have

$$\begin{aligned} \|a(U;x,\xi)\|_{K-K',s} &\leq C_1 \sum_{k=0}^{K-K'} \sum_{j=0}^{s-2k} \|\partial_t^k \partial_x^j a(U;\cdot,\xi)\|_{L^\infty} \\ &\leq C_2 \langle\xi\rangle^m \sum_{k=0}^{K-K'} \|U\|_{k+K',s+\sigma_0}, \end{aligned} \tag{1.2.3}$$

with $C_1, C_2 > 0$ depend only on s, K and $\|U\|_{k+K',\sigma_0}$, and where we used formula (1.2.1) with $\sigma = s + \sigma_0$. Equation (1.2.3) implies (1.2.2) for $s \in \mathbb{N}$. The general case $s \in \mathbb{R}_+$, follows by using the log-convexity of the Sobolev norm by writing $s = [s]\tau + (1-\tau)(1+[s])$ where $[s]$ is the integer part of s and $\tau \in [0,1]$. □

We define the following special subspace of $\Gamma^0_{K,K'}[r]$ made of those symbols which are independent of ξ.

Definition 1.2.2 (Functions). *Let $K' \leq K$ in $\mathbb{N}$, $r > 0$. We denote by $\mathcal{F}_{K,K'}[r]$ the subspace of $\Gamma^0_{K,K'}[r]$ made of those symbols which are independent of ξ.*

Remark 1.2.1 (Regularized symbols). *Fix $m \in \mathbb{R}$, $p, K, K' \in \mathbb{N}$, $K' \leq K$ and $r > 0$. Consider $a \in \Gamma^m_{K,K'}[r]$ and χ in $C^\infty(\mathbb{R}\times\mathbb{R};\mathbb{R})$ an admissible cut-off function according to Definition 1.1.14. Then the function*

$$a_\chi(U;x,\xi) := \sum_{n\in\mathbb{Z}} \chi(n,\xi)\, \Pi_n a(U;x,\xi) \tag{1.2.4}$$

belongs to $\Gamma^m_{K,K'}[r]$.

We define the Bony and the Bony-Weyl quantizations as done in the previous section. Consider an admissible cut-off function χ and a symbol a belonging to the class $\Gamma^m_{K,K'}[r]$, we set

$$\mathrm{Op}^{\mathcal{B}}(a(U;x,j))[v] := \mathrm{Op}(a_\chi(U;x,j))[v], \tag{1.2.5}$$

where a_χ is defined in (1.2.4). Analogously we define the Bony-Weyl quantization

$$\mathrm{Op}^{\mathscr{B}W}(b(U;x,j))[v] := \mathrm{Op}^{W}(b_\chi(U;x,j))[v]. \tag{1.2.6}$$

The definition of the operators $\mathrm{Op}^{\mathscr{B}}(b)$ and $\mathrm{Op}^{\mathscr{B}W}(b)$ is independent of the choice of the cut-off function χ modulo smoothing operators that we define now.

Definition 1.2.3 (**Smoothing remainders).** *Let $K' \le K \in \mathbb{N}$, $\rho \ge 0$ and $r > 0$. We define the class of remainders $\mathscr{R}^{-\rho}_{K,K'}[r]$ as the space of maps $(V,u) \mapsto R(V)u$ defined on $B^K_{s_0}(I,r) \times C^K_{*\mathbb{R}}(I, H^{s_0}(\mathbb{T},\mathbb{C}))$ which are linear in the variable u and such that the following holds true. For any $s \ge s_0$ there exists a constant $C > 0$ and $r(s) \in]0,r[$ such that for any $V \in B^K_{s_0}(I,r) \cap C^K_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}^2))$, any $u \in C^K_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}))$, any $0 \le k \le K - K'$ and any $t \in I$ the following estimate holds true*

$$\begin{aligned}&\left\|\partial_t^k (R(V)u)(t,\cdot)\right\|_{H^{s-2k+\rho}} \le \\ &\le \sum_{k'+k''=k} C\Big[\, \|u\|_{k'',s}\, \|V\|_{k'+K',s_0} + \|u\|_{k'',s_0}\, \|V\|_{k'+K',s}\Big],\end{aligned} \tag{1.2.7}$$

where $C = C(s, \|V\|_{k'+K',s_0})$ is a constant depending only on s and $\|V\|_{k'+K',s_0}$.

Now we state a proposition describing the action of paradifferential operators defined in (1.2.5) and in (1.2.6).

Proposition 1.2.1 (**Action of paradifferential operators).** *Let $r > 0$, $m \in \mathbb{R}$, $K' \le K \in \mathbb{N}$ and consider a symbol $a \in \Gamma^m_{K,K'}[r]$. There exists $s_0 > 0$ such that for any $U \in B^K_{s_0}(I,r)$, the operator $\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))$ extends, for any $s \in \mathbb{R}$, as a bounded operator from the space $C^{K-K'}_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}))$ to $C^{K-K'}_{*\mathbb{R}}(I, H^{s-m}(\mathbb{T},\mathbb{C}))$. Moreover there is a constant $C > 0$ depending on s and on the constant in* (1.2.1) *such that*

$$\|\mathrm{Op}^{\mathscr{B}W}(\partial_t^k a(U;x,\cdot))\|_{\mathscr{L}(H^s,H^{s-m})} \le C\|U\|_{k+K',s_0}, \tag{1.2.8}$$

for $k \le K - K'$, so that

$$\left\|\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))(v)\right\|_{K-K',s-m} \le C\,\|U\|_{K,s_0}\,\|v\|_{K-K',s}, \tag{1.2.9}$$

for any $v \in C^{K-K'}_{*\mathbb{R}}(I, H^s(\mathbb{T},\mathbb{C}))$.

Proof. The proof is very similar to the one of Proposition 1.1.1. □

Remark 1.2.2. *In analogy with Remark 1.1.15 we have the following. Consider* χ_1 *and* χ_2 *admissible cut-off functions. Fix* $m \in \mathbb{R}$, $r > 0$, $K' \le K \in \mathbb{N}$. *Then for* $a \in \Gamma^m_{K,K'}[r]$, *we have* $\mathrm{Op}(a_{\chi_1} - a_{\chi_2}) \in \mathscr{R}^{-\rho}_{K,K'}[r]$ *for any* $\rho \in \mathbb{N}$.

Remark 1.2.3. *Actually the estimates* (1.2.8) *and* (1.2.9) *follow by*

$$\left\| \mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))(v) \right\|_{K,s-m} \le C_1 \sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m} \| a(U;\cdot,\xi)\|_{K-K',s_0} \|v\|_{K-K',s},$$

where $C_1 > 0$ *is some constant depending only on* s, s_0 *and Remark 1.2.1.*

Remark 1.2.4. *We note that, as in Remark 1.1.13 regarding Prop. 1.1.1, the Proposition 1.2.1 applies if a satisfies* (1.2.1) *with* $|\alpha| \le 2$ *and* $\beta = 0$. *Moreover, by following the same proof, one can show that*

$$\|\mathrm{Op}^W(\partial_t^k a_\chi(U;x,\cdot))\|_{\mathscr{L}(H^s,H^{s-m})} \le C\|U\|_{k+K',s_0}, \tag{1.2.10}$$

if $\chi(\eta,\xi)$ *is supported for* $|\eta| \le \delta\langle\xi\rangle$ *for* $\delta > 0$ *small. Note that this is slightly different from the Definition 1.1.14 of admissible cut-off function since we are not requiring that* $\chi \equiv 1$ *for* $|\eta| \le \frac{\delta}{2}\langle\xi\rangle$.

Remark 1.2.5. *Note that, if* $m < 0$, *and* $a \in \Gamma^m_{K,K'}[r]$, *then estimate* (1.2.8) *implies that the operator* $\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))$ *belongs to the class of smoothing operators* $\mathscr{R}^m_{K,K'}[r]$.

Proposition 1.2.2 (Composition of Bony-Weyl operators). *Let a be a symbol in* $\Gamma^m_{K,K'}[r]$ *and b a symbol in* $\Gamma^{m'}_{K,K'}[r]$, *if* $U \in B^K_{s_0}(I,r)$ *with* s_0 *large enough then*

$$\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi)) \circ \mathrm{Op}^{\mathscr{B}W}(b(U;x,\xi)) - \mathrm{Op}^{\mathscr{B}W}((a\sharp b)_\rho(U;x,\xi)) \tag{1.2.11}$$

belongs to the class $\mathscr{R}^{-\rho+m+m'}_{K,K'}[r]$.

Proof. The proof is the same as the one done for item (ii) of Proposition 1.1.2. □

In the following we will need to compose smoothing operators and paradifferential ones, the next proposition asserts that the outcome is another smoothing operator.

Proposition 1.2.3. *Let a be a symbol in* $\Gamma^m_{K,K'}[r]$ *with* $m \ge 0$ *and R be a smoothing operator in* $\mathscr{R}^{-\rho}_{K,K'}[r]$. *If U belongs to* $B^K_{s_0}(I,r)$ *with* s_0 *large enough, then the composition operators*

$$\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi)) \circ R(U)[\cdot], \quad R(U) \circ \mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))[\cdot]$$

belong to the class $\mathscr{R}^{-\rho+m}_{K,K'}[r]$.

Proof. The proof is the same of the one done for item (ii) of Prop. 1.1.3. □

We can compose smoothing operators with smoothing operators as well.

Proposition 1.2.4. *Let R_1 be a smoothing operator in $\mathscr{R}^{-\rho_1}_{K,K'}[r]$ and R_2 in $\mathscr{R}^{-\rho_2}_{K,K'}[r]$. If U belongs to $B^K_{s_0}[I,r]$ with s_0 large enough, then the operator $R_1(U)\circ R_2(U)[\cdot]$ belongs to the class $\mathscr{R}^{-\rho}_{K,K'}[r]$, where $\rho=\min(\rho_1,\rho_2)$.*

We need also the following.

Lemma 1.2.2. *Fix $K,K'\in\mathbb{N}$, $K'\le K$ and $r>0$. Let $\{c_i\}_{i\in\mathbb{N}}$ a sequence in $\mathscr{F}_{K,K'}[r]$ such that for any $i\in\mathbb{N}$*

$$\left|\partial_t^k\partial_x^\alpha c_i(U;x)\right|\le M_i\,\|U\|_{k+K',s_0}\,, \tag{1.2.12}$$

for any $0\le k\le K-K'$ and $|\alpha|\le 2$ and for some $s_0>0$ big enough. Then for any $s\ge s_0$ and any $0\le k\le K-K'$ there exists a constant $C>0$ (independent of n) such that for any $n\in\mathbb{N}$

$$\left\|\partial_t^k\left[\mathrm{Op}^{\mathscr{B}W}\Big(\prod_{i=1}^n c_i(U;x)\Big)h\right]\right\|_{H^{s-2k}}\le C^n\prod_{i=1}^n M_i\sum_{k_1+k_2=k}\|U\|^n_{k_1+K',s_0}\|h\|_{k_2,s}\,, \tag{1.2.13}$$

for any $h\in C^{K-K'}_{*\mathbb{R}}(I,H^s(\mathbb{T};\mathbb{C}))$. Moreover there exists $\widetilde{C}$ such that

$$\|\mathrm{Op}^{\mathscr{B}W}\Big(\prod_{i=1}^n c_i\Big)h\|_{K-K',s}\le\widetilde{C}^n\prod_{i=1}^n M_i\|U\|^n_{K,s_0}\|h\|_{K-K',s}, \tag{1.2.14}$$

for any $h\in C^{K-K'}_{*\mathbb{R}}(I,H^s(\mathbb{T};\mathbb{C}))$.

Proof. Let χ an admissible cut-off function and set $b(U;x,\xi):=(\prod_{i=1}^n c_i(U;x))_\chi$. By Liebniz rule and interpolation one can prove that

$$|\partial_t^k\partial_x^\alpha\partial_\xi^\beta b(U;x,\xi)|\le C^n\,\|U\|^n_{k+K',s_0}\prod_{i=1}^n M_i \tag{1.2.15}$$

for any $0\le k\le K-K'$, $\alpha\le 2$, any $\xi\in\mathbb{R}$ and where the constant C is independent of n. Denoting by $\widehat{b}(U;\ell,\xi)=\widehat{b}(\ell,\xi)$ the ℓ^{th} Fourier coefficient of the function $b(U;x,\xi)$, from (1.2.15) with $\alpha=2$ one deduces the following decay estimate

$$|\partial_t^k\widehat{b}(\ell,\xi)|\le C^n\,\|U\|^n_{k+K',s_0}\prod_{i=1}^n M_i\langle\ell\rangle^{-2}. \tag{1.2.16}$$

With this setting one has

$$\begin{aligned}\mathrm{Op}^{\mathscr{B}W}\Big(\prod_{i=1}^{n} c_i(U;x)\Big)h &= \mathrm{Op}^{W}(b(U;x,\xi))h\\ &= \frac{1}{2\pi}\sum_{\ell\in\mathbb{Z}}\Bigg(\sum_{n'\in\mathbb{Z}}\widehat{b}\Big(\ell-n',\frac{\ell+n'}{2}\Big)\widehat{h}(n')\Bigg)e^{\mathrm{i}\ell x},\end{aligned}$$

where the sum is restricted to the set of indices such that $|\ell-n'|\leq\delta\frac{|\ell+n'|}{2}$ with $0<\delta<1$ (which implies that $\ell\sim n'$). Let $0\leq k\leq K-K'$, one has

$$\begin{aligned}&\Bigg\|\partial_t^k\Bigg[\mathrm{Op}^{\mathscr{B}W}\Big(\prod_{i=1}^{n} c_i(U;x)\Big)h\Bigg]\Bigg\|^2_{H^{s-2k}}\\ \leq& C^n\sum_{k_1+k_2=k}\sum_{\ell\in\mathbb{Z}}\langle\ell\rangle^{2(s-2k)}\Bigg|\sum_{n'\in\mathbb{Z}}\partial_t^{k_1}\Big(\widehat{b}\Big(\ell-n',\frac{\ell+n'}{2}\Big)\Big)\partial_t^{k_2}\Big(\widehat{h}(n')\Big)\Bigg|^2\\ \leq& C^n\prod_{i=1}^{n}M_i^2\sum_{k_1+k_2=k}\|U\|^{2n}_{k_1+K',s_0}\sum_{\ell\in\mathbb{Z}}\Bigg(\sum_{n'\in\mathbb{Z}}\langle\ell-n'\rangle^{-2}\langle n'\rangle^{s-2k}\Big|\partial_t^{k_2}\widehat{h}(n')\Big|\Bigg)^2,\end{aligned}$$

where in the last passage we have used (1.2.16) and that $\ell\sim n'$. By using Young inequality for sequences one can continue the chain of inequalities above and finally obtain the (1.2.13). The estimate (1.2.14) follows summing over $0\leq k\leq K-K'$. □

Proposition 1.2.5. *Fix $K,K'\in\mathbb{N}$, $K'\leq K$ and $r>0$. Let $\{c_i\}_{i\in\mathbb{N}}$ a sequence in $\mathscr{F}_{K,K'}[r]$ satisfying the hypotheses of Lemma 1.2.2. Then the operator*

$$Q^{(n)}_{c_1,\ldots,c_n} := \mathrm{Op}^{\mathscr{B}W}(c_1)\circ\cdots\circ\mathrm{Op}^{\mathscr{B}W}(c_n)-\mathrm{Op}^{\mathscr{B}W}(c_1\cdots c_n) \tag{1.2.17}$$

belongs to the class $\mathscr{R}^{-\rho}_{K,K'}[r]$ for any $\rho\geq 0$. More precisely there exists $s_0>0$ such that for any $s\geq s_0$ the following holds. For any $0\leq k\leq K-K'$ and any $\rho\geq 0$ there exists a constant $\mathtt{C}>0$ (depending on $\|U\|_{K,s_0}$, s,s_0,ρ,k and independent of n) such that

$$\begin{aligned}&\Big\|\partial_t^k\big(Q^{(n)}_{c_1,\ldots,c_n}[h]\big)\Big\|_{s+\rho-2k}\leq\\ &\mathtt{C}^n\mathtt{M}\sum_{k_1+k_2=k}\Big(\|U\|^n_{K'+k_1,s_0}\|h\|_{k_2,s}+\|U\|^{n-1}_{K'+k_1,s_0}\|h\|_{k_2,s_0}\|U\|_{K'+k_1,s}\Big),\end{aligned} \tag{1.2.18}$$

for any $n\geq 1$, any h in $C^K_{*\mathbb{R}}(I,H^s(\mathbb{T},\mathbb{C}))$, any $U\in C^K_{*\mathbb{R}}(I,\mathbf{H}^s)\cap B^K_s(I,r)$ and where $\mathtt{M}=M_1\cdots M_n$ (see (1.2.12)).

Proof. We proceed by induction. For $n=1$ is trivial. Let us study the case $n=2$. Since c_1, c_2 belong to $\mathscr{F}_{K,K'}[r]$, then $c_1 \cdot c_2 = (c_1 \sharp c_2)_\rho$ for any $\rho > 0$. Then by Lemma 1.1.2 there exists an admissible cut-off function χ such that

$$\begin{aligned} &\mathrm{Op}^{\mathscr{B}W}(c_1) \circ \mathrm{Op}^{\mathscr{B}W}(c_2) - \mathrm{Op}^{\mathscr{B}W}(c_1 \cdot c_2) \\ &\quad = \mathrm{Op}^{\mathscr{B}W}(c_1) \circ \mathrm{Op}^{\mathscr{B}W}(c_2) - \mathrm{Op}^{\mathscr{B}W}((c_1 \sharp c_2)_\rho) \\ &\quad = \mathrm{Op}^{W}((c_1)_\chi \sharp (c_2)_\chi) - \mathrm{Op}^{W}((c_1 \sharp c_2)_{\rho,\chi}) = \mathrm{Op}^{W}(r_1) + \mathrm{Op}^{W}(r_2), \end{aligned} \tag{1.2.19}$$

where

$$\begin{aligned} r_1(x,\xi) &= (c_1)_\chi \sharp (c_2)_\chi - ((c_1)_\chi \sharp (c_2)_\chi)_\rho, \\ r_2(x,\xi) &= ((c_1)_\chi \sharp (c_2)_\chi)_\rho - (c_1 \sharp c_2)_{\rho,\chi}. \end{aligned} \tag{1.2.20}$$

Then, by Lemma 1.1.3 and (1.2.12), one has that r_1 satisfies the bound

$$|\partial_t^k \partial_x^\ell r_1(U;x,\xi)| \le \widetilde{C} M_1 M_2 \langle \xi \rangle^{-\rho+\ell} \|U\|^2_{k+K',s_0} \tag{1.2.21}$$

for any $|\ell| \le 2$ and some universal constant $\widetilde{C} > 0$ depending only on s, s_0, ρ. Therefore Proposition 1.2.1 and Remark 1.2.4 imply that

$$\left\| \mathrm{Op}^{W}(\partial_t^k r_1(U;x,\cdot)) \right\|_{\mathscr{L}(H^s, H^{s+\rho-2})} \le \widetilde{C} M_1 M_2 \|U\|^2_{k+K',s_0}, \tag{1.2.22}$$

for $\widetilde{C} > 0$ possibly larger than the one in (1.2.21), but still depending only on k, s, s_0, ρ. From the bound (1.2.22) one deduces the estimate (1.2.18) for some $\mathtt{C} \ge 2\widetilde{C}$. One can argue in the same way to estimate the term $\mathrm{Op}^{W}(r_2)$ in (1.2.19). Assume now that (1.2.18) holds for $j \le n-1$ for $n \ge 3$. We have that

$$\mathrm{Op}^{\mathscr{B}W}(c_1) \circ \cdots \circ \mathrm{Op}^{\mathscr{B}W}(c_n) = \big(\mathrm{Op}^{\mathscr{B}W}(c_1 \cdots c_{n-1}) + Q_{n-1}\big) \circ \mathrm{Op}^{\mathscr{B}W}(c_n), \tag{1.2.23}$$

where Q_{n-1} satisfies condition (1.2.18). For the term $\mathrm{Op}^{\mathscr{B}W}(c_1 \cdots c_{n-1}) \circ \mathrm{Op}^{\mathscr{B}W}(c_n)$ one has to argue as done in the case $n=2$.

Consider the term $Q_{n-1} \circ \mathrm{Op}^{\mathscr{B}W}(c_n)$ and let $C > 0$ be the universal constant given by Lemma 1.2.2.

Using the inductive hypothesis on Q_{n-1} and estimate (1.2.13) in Lemma 1.2.2 (in the case $n = 1$) we have

$$
\begin{aligned}
&\|\partial_t^k\big(Q_{n-1}\circ \mathrm{Op}^{\mathscr{B}W}(c_n)h\big)\|_{s+\rho-2k}\\
&\le \mathtt{K}\mathtt{C}^{n-1}M_1\cdots M_{n-1}\sum_{k_1+k_2=k}\sum_{j_1+j_2=k_2} CM_n\|U\|^{n-1}_{K'+k_1,s_0}\|U\|_{K'+j_1,s_0}\|h\|_{j_2,s}\\
&+\mathtt{K}\mathtt{C}^{n-1}M_1\cdots M_{n-1}\sum_{k_1+k_2=k}\sum_{j_1+j_2=k_2} CM_n\|U\|^{n-2}_{K'+k_1,s_0}\|U\|_{K'+k_1,s}\|U\|_{K'+j_1,s_0}\|h\|_{j_2,s_0}\\
&\le \mathtt{K}\mathtt{M}\mathtt{C}^{n-1}C\sum_{k_1=0}^{k}\sum_{j_1=0}^{k-k_1}\|U\|^n_{K'+k_1+j_1,s_0}\|h\|_{k-k_1-j_1,s}\\
&+\mathtt{K}\mathtt{M}\mathtt{C}^{n-1}C\sum_{k_1=0}^{k}\sum_{j_1=0}^{k-k_1}\|U\|^{n-1}_{K'+k_1+j_1,s_0}\|U\|_{K'+k_1+j_1,s}\|h\|_{k-k_1-j_1,s_0}\\
&\le \mathtt{K}\mathtt{M}\mathtt{C}^{n-1}C\sum_{m=0}^{k}(\|U\|^n_{K'+m,s_0}\|h\|_{k-m,s}+\|U\|^{n-1}_{K'+m,s_0}\|U\|_{K'+m,s}\|h\|_{k-m,s_0})(m+1),
\end{aligned}
$$

for constant $\mathtt{K}$ depending only on k. This implies (1.2.18) by choosing $\mathtt{C} > (k+1)C\mathtt{K}$. □

Definition 1.2.4 (**Matrices**). *We denote by* $\Gamma^m_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C})$ *the matrices* $A(U;x,\xi)$ *of the form* (1.3.30) *whose components are symbols in the class* $\Gamma^m_{K,K'}[r]$. *The space* $\mathscr{F}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C})$ *is defined similarly. We denote by* $\mathscr{R}^{-\rho}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C})$ *the operators* $R(U)$ *of the form* (1.3.21) *whose components are smoothing operators in the class* $\mathscr{R}^{-\rho}_{K,K'}[r]$.

Corollary 1.2.2. *Fix* $K, K'\in\mathbb{N}$, $K'\le K$ *and* $r>0$. *Let* $s(U;x)$ *and* $z(U;x)$ *be symbols in the class* $\mathscr{F}_{K,K'}[r]$. *Consider the following two matrices*

$$
\begin{aligned}
S(U;x) &:= \begin{pmatrix} s(U;x) & 0\\ 0 & \overline{s(U;x)}\end{pmatrix}\in\mathscr{F}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C}),\\
Z(U;x) &:= \begin{pmatrix} 0 & z(U;x)\\ \overline{z(U;x)} & 0\end{pmatrix}\in\mathscr{F}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C}).
\end{aligned}
\tag{1.2.24}
$$

Then one has the following

$$
\begin{aligned}
&\exp\Big\{\mathrm{Op}^{\mathscr{B}W}(S(U;x))\Big\}-\mathrm{Op}^{\mathscr{B}W}(\{\exp S(U;x)\})\in\mathscr{R}^{-\rho}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C}),\\
&\exp\Big\{\mathrm{Op}^{\mathscr{B}W}(Z(U;x))\Big\}-\mathrm{Op}^{\mathscr{B}W}(\{\exp Z(U;x)\})\in\mathscr{R}^{-\rho}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C}),
\end{aligned}
$$

for any $\rho\ge 0$.

Proof. Let us prove the result for the matrix $S(U;x)$.
Since $s(U;x)$ belongs to $\mathscr{F}_{K,K'}[r]$ then there exists $s_0>0$ such that if $U\in B^K_{s_0}(I,r)$, then there is a constant $\mathtt{N}>0$ such that

$$\left|\partial_t^k\partial_x^\alpha s(U;x)\right|\leq \mathtt{N}\,\|U\|_{k+K',s_0}\,,$$

for any $0\leq k\leq K-K'$ and $|\alpha|\leq 2$. By definition one has

$$\begin{aligned}\exp\Big(\mathrm{Op}^{\mathscr{B}W}(S(U;x))\Big)&=\sum_{n=0}^{\infty}\frac{\big(\mathrm{Op}^{\mathscr{B}W}(S(U;x))\big)^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\big(\mathrm{Op}^{\mathscr{B}W}(s(U;x))\big)^n & 0\\ 0 & \big(\mathrm{Op}^{\mathscr{B}W}\overline{(s(U;x))}\big)^n\end{pmatrix},\end{aligned}$$

on the other hand

$$\begin{aligned}\mathrm{Op}^{\mathscr{B}W}\Big(\exp\big(S(U;x)\big)\Big)&=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix}\big[s(U;x)\big]^n & 0\\ 0 & \big[\overline{s(U;x)}\big]^n\end{pmatrix}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix}\mathrm{Op}^{\mathscr{B}W}\big(\big[s(U;x)\big]^n\big) & 0\\ 0 & \mathrm{Op}^{\mathscr{B}W}\big(\big[\overline{s(U;x)}\big]^n\big)\end{pmatrix}.\end{aligned}$$

We argue component-wise. Let h be a function in $C^K_{*\mathbb{R}}(I,H^s(\mathbb{T},\mathbb{C}))$, then using Proposition 1.2.5, one has

$$\begin{aligned}&\left\|\sum_{n=0}^{\infty}\frac{1}{n!}\partial_t^k\Big(\big[\mathrm{Op}^{\mathscr{B}W}(s(U;x))\big]^n[h]-\mathrm{Op}^{\mathscr{B}W}\big(s(U;x)^n\big)[h]\Big)\right\|_{s+\rho-2k}\leq\\&\sum_{n=1}^{\infty}\frac{\mathtt{C}^n\mathtt{N}^n}{n!}\sum_{k_1+k_2=k}\Big(\|U\|^n_{K'+k_1,s_0}\|h\|_{k_2,s}+\|U\|^{n-1}_{K'+k_1,s_0}\|h\|_{k_2,s_0}\|U\|_{K'+k_1,s}\Big)\leq\\&\sum_{k_1+k_2=k}\big(\|U\|_{K'+k_1,s_0}\|h\|_{k_2,s}+\|U\|_{K'+k_1,s}\|h\|_{k_2,s_0}\big)\sum_{n=1}^{\infty}\frac{\mathtt{C}^n\mathtt{N}^n}{n!}\|U\|^{n-1}_{K'+k_1,s_0}.\end{aligned}$$

Therefore we have proved the (1.2.7) with constant

$$C=\sum_{n=1}^{\infty}\frac{\mathtt{C}^n\mathtt{N}^n}{n!}\|U\|^{n-1}_{K'+k_1,s_0}=\frac{\exp(\mathtt{C}\mathtt{N}\|U\|_{K'+k_1,s_0})-1}{\|U\|_{K'+k_1,s_0}}.$$

For the other non zero component of the matrix the argument is the same.

In order to simplify the notation, set $z(U;x) = z$ and $\overline{z(U;x)} = \overline{z}$, therefore for the matrix $Z(U;x)$, by definition, one has

$$\mathrm{Op}^{\mathscr{B}W}\big(\exp(Z(U;x))\big) = \mathrm{Op}^{\mathscr{B}W}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} |z|^{2n} & |z|^{2n+1}z \\ |z|^{2n+1}\overline{z} & |z|^{2n}\end{pmatrix}\right).$$

On the other hand, by setting $A^n_{z,\bar{z}} = \big(\mathrm{Op}^{\mathscr{B}W}(z)\circ\mathrm{Op}^{\mathscr{B}W}(\bar{z})\big)^n$ and $B^n_{z,\bar{z}} = A^n_{z,\bar{z}}\circ \mathrm{Op}^{\mathscr{B}W}(z)$, one has

$$\exp\Big(\mathrm{Op}^{\mathscr{B}W}(Z(U;x))\Big) = \sum_{n=0}^{\infty}\frac{1}{n!}\begin{pmatrix} \frac{A^n_{z,\bar{z}}}{B^n_{z,\bar{z}}} & \frac{B^n_{z,\bar{z}}}{A^n_{z,\bar{z}}}\end{pmatrix}.$$

Therefore one can study each component of the matrix $\exp\big(\mathrm{Op}^{\mathscr{B}W}(Z(U;x))\big) - \mathrm{Op}^{\mathscr{B}W}\big(\exp Z(U;x)\big)$ in the same way as done in the case of the matrix $S(U,x)$. □

1.2.1 Para-composition 2

Consider a real symbol $\beta(U;x)$ in the class $\mathscr{F}_{K,K'}[r]$ and the map (1.1.82).
We have the following.

Lemma 1.2.3. *Let $0 \le K' \le K$ be in $\mathbb{N}$, $r > 0$ and $\beta(U;x) \in \mathscr{F}_{K,K'}[r]$ for U in the space $C^K_{*\mathbb{R}}(I,\mathbf{H}^{s_0})$. If s_0 is sufficiently large and β is 2π-periodic in x and satisfies*

$$1+\beta_x(U;x) \ge \Theta > 0, \quad x \in \mathbb{R}, \tag{1.2.25}$$

for some constant Θ depending on $\sup_{t\in I}\|U(t)\|_{\mathbf{H}^{s_0}}$, then the map Φ_U in (1.1.82) is a diffeomorphism of $\mathbb{T}$ to itself, and its inverse may be written as

$$(\Phi_U)^{-1} : y \mapsto y+\gamma(U;y) \tag{1.2.26}$$

for γ in $\mathscr{F}_{K,K'}[r]$.

Proof. Under condition (1.2.25) there exists $\gamma(U;y)$ such that

$$x+\beta(U;x)+\gamma(U;x+\beta(U;x)) = x, \quad x\in\mathbb{R}. \tag{1.2.27}$$

One can prove the bound (1.2.1) on the function $\gamma(U;y)$ by differentiating in x equation (1.2.27) and using that $\beta(U;x)$ is a symbol in $\mathscr{F}_{K,K'}[r]$. □

Remark 1.2.6. *The Lemma above is very similar to Lemma 1.1.5. In such a lemma we used a smallness assumption on U to prove the result. Here this assumption is replaced by* (1.2.25) *in order to treat big sized initial conditions.*

Remark 1.2.7. *By Lemma 1.2.3 one has that* $x \mapsto x+\tau\beta(U;x)$ *is a diffeomorphism of* $\mathbb{T}$ *for any* $\tau \in [0,1]$. *Indeed*

$$1+\tau\beta_x(U;x) = 1-\tau+\tau(1+\beta_x(U;x)) \geq (1-\tau)+\tau\Theta \geq \min\{1,\Theta\} > 0,$$

for any $\tau \in [0,1]$. *Hence the* (1.2.25) *holds true with* $\mathtt{c} = \min\{1,\Theta\}$ *and Lemma 1.2.3 applies.*

Lemma 1.2.4. *Let a be a symbol in* $\Sigma\Gamma^m_{K,K'}[r]$, *consider two functions b and c in the class* $\Sigma\mathcal{F}_{K,K'}[r]$. *Then*

$$a(V;x+b(V;t,x),\xi(1+c(V;t,x)))$$

is still a symbol in $\Sigma\Gamma^m_{K,K'}[r]$. *In the case that a does not depend on* ξ, *i.e. belongs to* $\Sigma\mathcal{F}_{K,K'}[r]$, *then* $a(V;x+b(V;t,x))$ *is in the class* $\Sigma\mathcal{F}_{K,K'}[r]$.

Proof. One proceeds analogously as done for Lemma 1.1.7. □

We define the symbol $B(U;\tau,x,\xi)$ from the function $\beta(U;t,x)$ as done in (1.1.84) and then the para-composition operator associated to the diffeomorphism (1.1.82) is defined as $\Omega_{B(U)}(1)$, where $\Omega_{B(U)}(\tau)$ is the flow of (1.1.85). The well-posedness issues for such a flow can be analysed as in Lemma 1.1.6, indeed in such a Lemma no assumption on the smallness of U are made.
In the following we state a theorem similar to Theorem 1.1.1 in which study how symbols $a(U;x,\xi)$ changes under conjugation through the flow $\Omega_{B(U)}(\tau)$ introduced in Lemma 1.1.6. We do not write down the proof since it is similar to the one given in Subsection 1.1.6 for Theorem 1.1.1. The key ingredient for such a proof is that $x \mapsto x+\tau\beta(U;x)$ is a path of diffeomorphism for $\tau \in [0,1]$. In Subsection 1.1.6 this fact is achieved by using the smallness of r, here it is implied by Remark 1.2.7.

Theorem 1.2.3 (Para-composition 2)**.** *Let* $N \in \mathbb{N}$, $K' \leq K$ *in* $\mathbb{N}$ *and* $r > 0$. *Let U be a function in the ball of center* 0 *and radius r of* $C^K_{*\mathbb{R}}(I;\mathbf{H}^\sigma(\mathbb{T};\mathbb{C}^2))$ *with* σ *sufficiently large. Consider* $\beta(U;\cdot)$ *a function in* $\Sigma\mathcal{F}_{K,K'}[r]$ *satisfying* (1.2.25) *and the diffeomorphism* $\Phi_U : x \mapsto x+\beta(U;t,x)$. *Let* $a(U;\cdot)$ *be a symbol in* $\Sigma\Gamma^m_{K,K'}[r]$.

Then the flow of (1.1.85) *is well defined for* $|\tau| \leq 1$ *and for any* ρ *large enough there is a symbol* $a_\Phi \in \Sigma\Gamma^m_{K,K'}[r]$ *such that*

$$\begin{aligned}\Omega_{B(U)}(1)\mathrm{Op}^{\mathscr{B}W}\big(a(U;\cdot)\big)\big(\Omega_{B(U)}(1)\big)^{-1}\\ = \mathrm{Op}^{\mathscr{B}W}\big(a_\Phi(U;\cdot)\big) + R(U;t)\end{aligned} \tag{1.2.28}$$

with R *in* $\Sigma\mathscr{R}^{-\rho+m}_{K,K'}[r]$*. Moreover we have the expansion*

$$a_\Phi(U;\cdot) = a^0_\phi(U;\cdot) + a^1_\phi(U;\cdot), \tag{1.2.29}$$

where

$$a^0_\phi(U;\cdot) = a(U;\Phi_U(t,x),\xi\partial_y\Phi_U^{-1}(t,y)_{|_{y=\Phi_U(t,x)}})$$

is in $\Sigma\Gamma^m_{K,K'}[r]$*, while* $a^1_\phi(U;\cdot)$ *is in* $\Sigma\Gamma^{m-2}_{K,K'}[r]$*. Finally we have* $a_\Phi \equiv 1$ *in the case that* $a \equiv 1$.

Proposition 1.2.6. *Use assumptions and notation of Theorem 1.2.3. If U is a solution of a system of the form*

$$\partial_t U = \mathrm{Op}^{\mathscr{B}W}(A(U))U + Q(U)U,$$

for some matrices of symbols $A(U)$ *in* $\Gamma^m_{K,K'}[r] \otimes \mathscr{M}_2(\mathbb{C})$ *and some remainder* $Q(U)U$ *in* $\mathscr{R}^0_{K,K'}[r] \otimes \mathscr{M}_2(\mathbb{C})$ *for some* m *and* r *positive. Then*

$$\begin{aligned}\Omega_{B(U)}(\tau)\circ\partial_t\circ\Omega^{-1}_{B(U)} &= \partial_t + \Omega_{B(U)}(\tau)\circ(\partial_t\Omega^{-1}_{B(U)}(\tau))\\ &= \partial_t + \mathrm{Op}^{\mathscr{B}W}\big(e(U;t,x,\xi)\big) + R(U;t),\end{aligned} \tag{1.2.30}$$

where

$$e(U;t,x,\xi) = e_1(U;t,x)(\mathrm{i}\xi) + e_0(U;t,x,\xi), \tag{1.2.31}$$

with $e_1(U;t,x) \in \mathscr{F}_{K,K'+1}[r]$*,* $e_0(U;t,x,\xi)$ *is in* $\Gamma^{-1}_{K,K'+1}[r]$ *and* $R(U;t)$ *is a smoothing remainder belonging to* $\mathscr{R}^{-1}_{K,K'+1}[r,N]$*.*
Moreover $\mathrm{Re}(e) = 0$*.*

Proof. One proceeds as done for Lemma 1.1.4. □

1.3 Algebraic properties

1.3.1 Parity, reality and reversibility properties

In this Section we analyse the parity, reality and the reversibility structure for para-differential and smoothing operators. Denote by S the linear involution, i.e. $S^2 = \mathbb{1}$,

$$S : \mathbb{C}^2 \to \mathbb{C}^2, \qquad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.3.1}$$

For any $U \in B^K_\sigma(I, r)$ we set

$$U_S(t) := (SU)(-t). \tag{1.3.2}$$

Note that $U \in L^2(\mathbb{T}; \mathbb{C}^2)$ belongs to the subspace $\mathfrak{R}$ (see (0.4.1)) if and only if

$$(SU)(x) = \overline{U}(x). \tag{1.3.3}$$

We have the following definitions.

Definition 1.3.1. *Let $m \in \mathbb{R}$, $p, N, K, K' \in \mathbb{N}$ with $p \leq N$, $K' \leq K$ and $r > 0$, $\rho \geq 0$ and consider a matrix $A(U; t, x, \xi) \in \Sigma\Gamma^m_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ (or in $\Sigma\Gamma^m_{K,K'}[r] \otimes \mathcal{M}_2(\mathbb{C})$) where U satisfies* (1.3.3).

- **Reality preserving matrices of symbols.** *We say that a matrix $A(U; t, x, \xi)$ is* reality *preserving if*

$$\overline{A(U; t, x, -\xi)} = SA(U; t, x, \xi)S. \tag{1.3.4}$$

- **Anti-reality preserving matrices of symbols.** *We say that $A(U; t, x, \xi)$ is* anti-reality *preserving if*

$$\overline{A(U; t, x, -\xi)} = -SA(U; t, x, \xi)S. \tag{1.3.5}$$

- **Reversible and reversibility preserving matrices of symbols.** *We say that $A(U; t, x, \xi)$ is reversible if*

$$-SA(U; -t, x, \xi) = A(U_S; t, x, \xi)S. \tag{1.3.6}$$

  We say that $A(U; t, x, \xi)$ is reversibility preserving if

$$SA(U; -t, x, \xi) = A(U_S; t, x, \xi)S. \tag{1.3.7}$$

- **Parity preserving matrices of symbols.** *We say that* $A(U;t,x,\xi)$ *is parity preserving if*

$$A(U;t,x,\xi) = A(U;t,-x,-\xi). \tag{1.3.8}$$

- **(R,R,P)-matrices.** *We say that* $A(U;t,x,\xi)$ *is a (R,R,P)-matrix if it is a reality, reversibility and parity preserving matrix of symbols.*

Remark 1.3.1. *Consider* $A(U;t,x,\xi)$ *in* $\Sigma\Gamma^{m}_{K,K',p}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ *(or in* $\Gamma^{m}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C})$*). If* $A(U;t,x,\xi)$ *is reality preserving, i.e. satisfies* (1.3.4)*, then it has the form*

$$A(U;t,x,\xi) := \begin{pmatrix} a(U;t,x,\xi) & b(U;t,x,\xi) \\ \overline{b(U;t,x,-\xi)} & \overline{a(U;t,x,-\xi)} \end{pmatrix}. \tag{1.3.9}$$

One can note also the following facts:

- *if* $A(U;t,x,\xi)$ *satisfy one among the properties* (1.3.6), (1.3.7), (1.3.8) *and it is invertible, then* $A(U;t,x,\xi)^{-1}$ *satisfies the same property;*

- *if the matrix* $A(U;t,x,\xi)$ *is reversibility preserving then* $\mathrm{i}EA(U;t,x,\xi)$ *is reversible.*

- *if the matrix* $A(U;t,x,\xi)$ *is reality preserving then the matrix of symbols* $\mathrm{i}A(U;t,x,\xi)$ *is anti-reality preserving.*

- *the matrix* $A(U;t,x,\xi)$ *is reversibility preserving if and only if its symbols verify the following*

$$\begin{aligned} \overline{b(U;-t,x,\xi)} &= b(U_S;t,x,\xi), \\ \overline{a(U;-t,x,\xi)} &= a(U_S;t,x,\xi); \end{aligned} \tag{1.3.10}$$

  furthermore note that in the case that the symbols are autonomous (i.e. when the dependence of time is through the function $U(t,x)$*) the conditions above reads*

$$\begin{aligned} \overline{b(U;x,\xi)} &= b(SU;x,\xi), \\ \overline{a(U;x,\xi)} &= a(SU;x,\xi). \end{aligned} \tag{1.3.11}$$

Remark 1.3.2. *Recalling* (0.4.6) *and Remark 1.1.11 we write*

$$\Lambda[\cdot] := \begin{pmatrix} \mathrm{Op}^{\mathscr{B}W}(\mathtt{l}(\xi))[\cdot] & 0 \\ 0 & \mathrm{Op}^{\mathscr{B}W}(\mathtt{l}(\xi))[\cdot] \end{pmatrix}. \tag{1.3.12}$$

In particular, since the symbol $\mathtt{l}(\xi)$ *is real and even in* ξ *one has that the operator* Λ *is reality, parity and reversibility preserving.*

Definition 1.3.2. *Fix* $\rho > 0$, $m \in \mathbb{R}$, $p \in \mathbb{N}$ *and let* $A_p \in \widetilde{\Gamma}^m_p \otimes \mathcal{M}_2(\mathbb{C})$ *and let* U_j, *with* $j = 1, \dots, p$, *be functions satisfying* $SU_j = \overline{U}_j$ *(see* (1.3.1)*). We say that* A_p *is reversibility preserving if*

$$A_p(SU_1, \dots, SU_p; x, \xi)S = SA_p(U_1, \dots, U_p; x, \xi). \tag{1.3.13}$$

We say that A_p *is reversible if*

$$A_p(SU_1, \dots, SU_p; x, \xi)S = -SA_p(U_1, \dots, U_p; x, \xi). \tag{1.3.14}$$

We have the following Lemma.

Lemma 1.3.1. *Let* $A \in \Sigma\Gamma^m_{K,K',p}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$ *and write*

$$A(U; t, x, \xi) = \sum_{q=p}^{N-1} A_q(U, \dots, U; x, \xi) + A_N(U; t, x, \xi),$$

with $A_q \in \widetilde{\Gamma}^m_q \otimes \mathcal{M}_2(\mathbb{C})$, $q = p, \dots, N-1$ *and* $A_N \in \Sigma\Gamma^m_{K,K',N}[r, N] \otimes \mathcal{M}_2(\mathbb{C})$.
(i) If A_q *satisfies* (1.3.13) *(resp.* (1.3.14)*), for* $q = p, \dots, N-1$ *and* A_N *satisfies* (1.3.7) *(resp.* (1.3.6)*), then* A *satisfies* (1.3.7) *(resp.* (1.3.6)*).*
(ii) If A *satisfies* (1.3.7) *(resp.* (1.3.6)*) then there are matrices of symbols* $A'_q \in \widetilde{\Gamma}^m_q \otimes \mathcal{M}_2(\mathbb{C})$, *satisfying* (1.3.13) *(resp.* (1.3.14)*), and a matrix* $A'_N(U; t, x, \xi) \in \Gamma^m_{K,K',N}$ *satisfying* (1.3.7) *(resp* (1.3.6)*) such that for any* U *we have*

$$A(U; t, x, \xi) = \sum_{q=p}^{N-1} A'_q(U, \dots, U; x, \xi) + A'_N(U; t, x, \xi).$$

Proof. Let us prove the lemma for the conditions (1.3.13) and (1.3.7), the other case is similar. Let us assume that for any $q = p, \dots, N-1$ the symbol A_q satisfies (1.3.13). Then we have

$$\begin{aligned} A_q(U, \dots, U; -t, x, \xi)S &= A_q(U(-t), \dots, U(-t); x, \xi)S \\ &= A_q(SU_S(t), \dots, SU_S(t); x, \xi)S \\ &= -SA_q(U_S(t), \dots, U_S(t); x, \xi) \\ &= -SA_q(U_S, \dots, U_S; t, x, \xi), \end{aligned}$$

therefore, since $A_N(U; t, x, \xi)$ satisfies (1.3.7) the matrix $A(U; t, x, \xi)$ satisfies (1.3.7).

Conversely assume that $A(U;t,x,\xi)$ satisfies (1.3.7), then each $A_q(U,\dots,U;t,x,\xi)$ satisfies (1.3.7). Define the new symbol

$$A'_q(U_1,\dots,U_q;x,\xi) := \frac{1}{2}\Big[A_q(U_1,\dots,U_q;t,x,\xi) - SA_q(SU_1,\dots,SU_q;t,x,\xi)\Big].$$

The symbol A'_q when computed on the diagonal is equal to A_q and satisfies (1.3.13) by construction. One constructs the matrix A'_N from A_N in the same way. □

We now give the definition of general linear operators satisfying the reversibility, parity and reality properties.

Definition 1.3.3. *Let $M(U;t)$ be a linear operator with U satisfying* (1.3.3).

- **Reality preserving maps.** *We say that the map $M(U;t)$ is* reality *preserving if*
$$\overline{M(U;t)[V]} = SM(U;t)[S\overline{V}]. \tag{1.3.15}$$
- **Anti-reality condition.** *We say that the map $M(U;t)$ satisfies the* anti-reality *condition if*
$$\overline{M(U;t)[V]} = -SM(U;t)[S\overline{V}]. \tag{1.3.16}$$
- **Reversible maps.** *We say that the map $M(U;t)$ is* reversible *w.r.t. the involution* (1.3.1) *if one has*
$$-SM(U;-t) = M(U_S;t)S. \tag{1.3.17}$$
- **Reversibility preserving maps.** *We say that the map $M(U;t)$ is* reversibility preserving *if*
$$SM(U;-t) = M(U_S;t)S. \tag{1.3.18}$$
- **Parity preserving maps.** *We say that $M(U;t)$ is* parity preserving *if*
$$M(U;t)\circ\tau = \tau\circ M(U;t), \tag{1.3.19}$$
where τ is the map acting on functions $\tau V(x) = V(-x)$.
- **(R,R,P)-maps/ operators.** *We say that $M(U;t)$ is a (R,R,P)-map (resp. (R,R,P)-operator) if it is a reality, reversibility and parity preserving map (resp. operator).*

Definition 1.3.4. *Let A be a linear operator. We define the operator $\overline{A}$ as*

$$\overline{A}[h] := \overline{A[\bar{h}]}. \tag{1.3.20}$$

Remark 1.3.3. *Let $\mathfrak{F}$ be a reality preserving matrix of operators satisfying* (1.3.15)*. Then there are linear operators $A(U;t)$ and $B(U;t)$ such that*

$$\mathfrak{F}(U;t)[\cdot] := \begin{pmatrix} A(U;t)[\cdot] & B(U;t)[\cdot] \\ \overline{B}(U;t)[\cdot] & \overline{A}(U;t)[\cdot]. \end{pmatrix}, \tag{1.3.21}$$

Remark 1.3.4. *If a matrix of symbols $A(U;t,x,\xi)$ is reality, reversibility and parity preserving (resp. reversible, reality and parity preserving) according to Def. 1.3.1 , then the operator $\mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))[\cdot]$ is a reality, reversibility and parity preserving operator according to Def. 1.3.3.*

Remark 1.3.5. *An important class of* parity preserving *maps according to Definition 1.3.3 is the following. Consider a matrix of symbols $C(U;t,x,\xi)$ satisfying* (1.3.8)*, with U even in x, and the system*

$$\begin{cases} \partial_\tau \Phi^\tau(U;t)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(C(U;t,x;\xi))\Phi^\tau(U;t)[\cdot], \\ \Phi^0(U) = \mathbb{1}. \end{cases}$$

If the flow $\Phi^\tau(U;t)$ is well defined for $\tau \in [0,1]$, then it defines a family of parity preserving *maps according to Def. 1.3.3.*

We have the following lemmata.

Lemma 1.3.2. *Let $p,N,K,K' \in \mathbb{N}$ with $p \le N$, $K' \le K$ and $r > 0$. Let $M \in \Sigma\mathscr{M}_{K,K',p}[r,N]$. If M is decomposed as in* (1.1.15) *as a sum*

$$M(V;t)U = \sum_{q=p}^{N-1} M_q(V,\dots,V)U + M_N(V;t)U, \tag{1.3.22}$$

in terms of homogeneous operators $M_q, q = p,\dots,N-1$, and if M satisfies the reversibility condition (1.3.17)*, respectively reversibility preserving* (1.3.18)*, we may assume that M_q, $q = p,\dots,N-1$ satisfy the reversibility property*

$$M_q(SU_1,\dots,SU_q)S = -SM_q(U_1,\dots,U_q), \tag{1.3.23}$$

respectively the reversibility preserving property

$$M_q(SU_1,\dots,SU_q)S = SM_q(U_1,\dots,U_q). \tag{1.3.24}$$

Proof. It is similar to the proof of Lemma 1.3.1. □

All the algebraic properties of compositions are collected in the following lemma. We omit its proof since it is trivial.

Lemma 1.3.3. *Composition of an operator satisfying the anti-reality property* (1.3.16) *(resp. the reversibility property* (1.3.17)*) with one or several operators satisfying the reality property* (1.3.15) *(resp. the reversibility preserving property* (1.3.18)*) still satisfies the anti-reality property* (1.3.16) *(resp. reversibility property* (1.3.17)*). Composition of operators which are parity preserving is as well the parity preserving. Composition of operators satisfying the reality property* (1.3.15) *satisfy the reality property* (1.3.15) *as well.*

We prove a lemma which asserts that a (R,R,P) operator which is the sum of a para-differential operator and a smoothing remainder may be rewritten as the sum of a (R,R,P) para-differential operator and a (R,R,P) smoothing remainder.

Lemma 1.3.4. *Fix* $\rho, r > 0$, $K \geq K' > 0$ *in* $\mathbb{N}$, m', m'' *in* $\mathbb{N}$. *Let*

$$A(U;t,x,\xi) = \sum_{j=-m'}^{m''} A_j(U;t,x,\xi)$$

be a matrix of symbols such that $A_j(U;t,x,\xi)$ *is in* $\Sigma\Gamma^{j}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *for any* $j = -m', \ldots, m''$ *and* $R(U;t)$ *a matrix of operators in* $\Sigma\mathcal{R}^{-\rho}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$. *If the sum* $\mathrm{Op}^{\mathcal{B}W}(A(U;t,x,\xi)) + R(U;t)$ *is a (R,R,P) operator, then there exist (R,R,P) matrices* $\widetilde{A}_j(U;t,x,\xi)$ *in* $\Sigma\Gamma^{j}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *for any* $j = -m', \ldots, m''$ *and a (R,R,P) smoothing remainder* $\widetilde{R}(U;t)$ *such that the following facts hold true:*

(i) one has that

$$\mathrm{Op}^{\mathcal{B}W}(A(U;t,x,\xi)) + R(U;t) = \mathrm{Op}^{\mathcal{B}W}(\widetilde{A}(U;t,x,\xi)) + \widetilde{R}(U;t)$$

with $\widetilde{A}(U;t,x,\xi) = \sum_{j=-m'}^{m''} \widetilde{A}_j(U;t,x,\xi)$;

(ii) if one component of a matrix $A_j(U;t,x,\xi)$ *is real valued, then the corresponding component in the matrix* $\widetilde{A}_j(U;t,x,\xi)$ *is real valued;*

(iii) if, for $j \geq 0$, *the matrix* $A_j(U;t,x,\xi)$ *has the form* $B_j(U;t,x)(\mathrm{i}\xi)^j$, *for some* $B_j(U;t,x) \in \Sigma\mathcal{F}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, *then the corresponding matrix* $\widetilde{A}_j(U;t,x,\xi)$ *is equal to* $\widetilde{B}_j(U;t,x)(\mathrm{i}\xi)^j$ *where* $\widetilde{B}_j(U;t,x)$ *belongs to* $\Sigma\mathcal{F}_{K,K',p}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$.

Proof. We show how to construct the reversibility preserving operator associated to the one of the hypothesis, the parity and reality preserving construction is similar. Since the sum $\mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))+R(U;t)$ is a (R,R,P) operator we obtain

$$\begin{aligned}&\mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))+R(U;t)=\\&\frac{1}{2}\Big(\mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))+R(U;t)-\mathrm{Op}^{\mathscr{B}W}(SA(U_S;-t,x,\xi))S-SR(U_S;t)S\Big)\end{aligned}\tag{1.3.25}$$

therefore it is sufficient to define $\widetilde{A}(U;t,x,\xi):=\frac{1}{2}(A(U;t,x,\xi)-SA(U_S;-t,x,\xi))S$ and $\frac{1}{2}(\widetilde{R}(U;t):=R(U;t)-SR(U_S;t))$. Consequently each term $\widetilde{A}_j(U;t,x,\xi)$ may be chosen equal to $\frac{1}{2}(A_j(U;t,x,\xi)-SA_j(U_S;-t,x,\xi))$ for $j=-m',\dots,m''$. Items $(ii),(iii)$ can be deduced by (1.3.25). □

Lemma 1.3.5. *Assume the hypotheses of Lemma 1.1.4. If $\widetilde{M}(U;t)$ is a (R,R,P) map (see Def. 1.3.3) and $C(U;t,x,\xi)$ is a (R,R,P) symbol (see Def. 1.3.1) then the symbol $\partial_t C(U;t,\cdot)$ is reality preserving, parity preserving and reversible, i.e. satisfies respectively the* (1.3.4), (1.3.8) *and* (1.3.6).

Proof. Assume that $C(U;t,x,\xi)$ is a non-homogeneous symbol in $\Gamma^m_{K,K',p}[r]$. By differentiating in t the relation

$$SC(U;-t,x,\xi)=C(U_S;t,x,\xi)$$

one gets that $(\partial_t C)(U;t,x,\xi)$ is reversible. Assume now that $C\in\widetilde{\Gamma}^m_p$. Since $C(U;x,\xi)$ is reversibility preserving then

$$C(U_S,\dots,U_S;t,x,\xi)S=SC(U,\dots,U;-t,x,\xi).\tag{1.3.26}$$

Hence differentiating in t we get

$$\begin{aligned}&\sum_{j=1}^{p}C(U_S,\dots,-\underbrace{(\partial_t U)_S}_{j-th},\dots,U_S;t,x,\xi)S=\\&-\sum_{j=1}^{p}SC(U,\dots,\underbrace{\mathrm{i}E\widetilde{M}(U,t)U}_{j-th},\dots,U;-t,x,\xi).\end{aligned}\tag{1.3.27}$$

Using that $\widetilde{M}(U;t)$ is reversibility preserving we have

$$(\partial_t U)_S=S(\mathrm{i}E\widetilde{M}(U;\cdot)U)(-t)=-\mathrm{i}ES\widetilde{M}(U;-t)U(-t)=-\mathrm{i}E\widetilde{M}(U_S;t)U_S(t),$$

which implies, together with (1.3.27) the (1.3.6) for $(\partial_t C)(U;t,x,\xi)$. The (1.3.4) and (1.3.8) follow by using the definitions. □

Lemma 1.3.6. *Consider the system* (1.1.85). *If* $\mathrm{Op}^{\mathscr{B}W}(B(\tau,U;t,x,\xi))$ *satisfies* (1.3.16), (1.3.19) *and* (1.3.18), *then the flow* $\Omega_{B(U)}(\tau)$ *satisfies for any fixed* τ (1.3.15), (1.3.19) *and* (1.3.18).

Proof. It follows immediately by definition. □

1.3.2 Hamiltonian and parity preserving vector fields.

We now study self-adjoint matrices para-differential operators. We shall restrict to the case that such matrices are reality preserving, i.e. matrices of the form (1.3.21). Consider an operator $\mathfrak{F}$ of the form (1.3.21) and denote by $\mathfrak{F}^*$ its adjoint with respect to the scalar product (0.4.2)

$$(\mathfrak{F}U,V)_{\mathbf{H}^0}=(U,\mathfrak{F}^*V)_{\mathbf{H}^0},\quad \forall\, U,V\in\mathbf{H}^s.$$

One can check that

$$\mathfrak{F}^*:=\begin{pmatrix}A^*(U;t) & \overline{B}^*(U;t)\\ B^*(U;t) & \overline{A}^*(U;t)\end{pmatrix},\tag{1.3.28}$$

where A^* and B^* are respectively the adjoints of the operators A and B with respect to the complex scalar product on $L^2(\mathbb{T};\mathbb{C})$

$$(u,v)_{L^2}:=\int_{\mathbb{T}}u\cdot\bar{v}\,dx,\quad u,v\in L^2(\mathbb{T};\mathbb{C}),$$

and $\overline{A}$, $\overline{B}$ are defined Definition 1.3.4.

Definition 1.3.5 (Self-adjointness). *Let* $\mathfrak{F}$ *be a reality preserving linear operator of the form* (1.3.21). *We say that* $\mathfrak{F}$ *is* self-adjoint *if* $A,A^*,B,B^*:H^s\to H^{s'}$, *for some* $s,s'\in\mathbb{R}$ *and*

$$A^*=A,\quad \overline{B}=B^*.\tag{1.3.29}$$

We consider paradifferential operators of the form:

$$\begin{aligned}\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi)) &:= \mathrm{Op}^{\mathscr{B}W}\begin{pmatrix}a(U;x,\xi) & b(U;x,\xi)\\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)}\end{pmatrix}\\ &:= \begin{pmatrix}\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi)) & \mathrm{Op}^{\mathscr{B}W}(b(U;x,\xi))\\ \mathrm{Op}^{\mathscr{B}W}\overline{(b(U;x,-\xi))} & \mathrm{Op}^{\mathscr{B}W}\overline{(a(U;x,-\xi))}\end{pmatrix},\end{aligned}\tag{1.3.30}$$

where a and b are symbols in $\Gamma^m_{K,K'}[r]$ and U is a function belonging to $B^K_{s_0}(I,r)$ for some s_0 large enough. Note that the matrix of operators in (1.3.30) is of the form (1.3.21). Moreover it is self-adjoint if and only if

$$a(U;x,\xi) = \overline{a(U;x,\xi)}, \quad b(U;x,-\xi) = b(U;x,\xi), \tag{1.3.31}$$

indeed conditions (1.3.29) on these operators read

$$\begin{aligned} \left(\mathrm{Op}^{\mathscr{B}W}(a(U;x,\xi))\right)^* &= \mathrm{Op}^{\mathscr{B}W}\left(\overline{a(U;x,\xi)}\right), \\ \overline{\mathrm{Op}^{\mathscr{B}W}(b(U;x,\xi))} &= \mathrm{Op}^{\mathscr{B}W}\left(\overline{b(U;x,-\xi)}\right). \end{aligned} \tag{1.3.32}$$

Let $\mathfrak{F}$ be a reality preserving, self-adjoint (or parity preserving respectively) operator as in (1.3.21) and consider the linear system

$$\partial_t U = \mathrm{i}E\mathfrak{F}U, \tag{1.3.33}$$

on $\mathbf{H}^s$ where E is given in (0.4.5). We want to analyse how the properties of the system (1.3.33) change under the conjugation through maps

$$\Phi : \mathbf{H}^s \to \mathbf{H}^s,$$

which are reality preserving. We have the following lemma.

Lemma 1.3.7. *Let $\mathscr{X} : \mathbf{H}^s \to \mathbf{H}^{s-m}$, for some $m \in \mathbb{R}$ and $s > 0$ be a reality preserving, self-adjoint operator according to Definitions 1.3.3, 1.3.5 and assume that its flow*

$$\partial_\tau \Phi^\tau = \mathrm{i}E\mathscr{X}\Phi^\tau, \qquad \Phi^0 = \mathbb{1}, \tag{1.3.34}$$

satisfies the following. The map Φ^τ is a continuous function in $\tau \in [0,1]$ with values in the space of bounded linear operators from $\mathbf{H}^s$ to $\mathbf{H}^s$ and $\partial_\tau\Phi^\tau$ is continuous as well in $\tau \in [0,1]$ with values in the space of bounded linear operators from $\mathbf{H}^s$ to $\mathbf{H}^{s-m}$.
Then the map Φ^τ satisfies the condition

$$(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau = -\mathrm{i}E. \tag{1.3.35}$$

Proof. First we note that the adjoint operator $(\Phi^\tau)^*$ satisfies the equation $\partial_\tau(\Phi^\tau)^* = (\Phi^\tau)^*\mathscr{X}(-\mathrm{i}E)$. Therefore one can note that

$$\partial_\tau\left[(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau\right] = 0,$$

which implies $(\Phi^\tau)^*(-\mathrm{i}E)\Phi^\tau = (\Phi^0)^*(-\mathrm{i}E)\Phi^0 = -\mathrm{i}E$. □

Lemma 1.3.8. *Consider a reality preserving, self-adjoint linear operator $\mathfrak{F}$ (i.e. it satisfies* (1.3.21) *and* (1.3.29)*) and a reality preserving map Φ. Assume that Φ satisfies condition* (1.3.35) *and consider the system*

$$\partial_t W = \mathrm{i}E\mathfrak{F}W, \qquad W \in \mathbf{H}^s. \tag{1.3.36}$$

By setting $V = \Phi W$ one has that the system (1.3.36) *reads*

$$\partial_t V = \mathrm{i}E\mathscr{Y}V, \tag{1.3.37}$$

$$\mathscr{Y} := -\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} - \mathrm{i}E(\partial_t\Phi)\Phi^{-1}, \tag{1.3.38}$$

and $\mathscr{Y}$ is self-adjoint, i.e. it satisfies conditions (1.3.29)*.*

Proof. One applies the changes of coordinates and one gets the form in (1.3.38). We prove that separately each term of $\mathscr{Y}$ is self-adjoint. Note that by (1.3.35) one has that $(-\mathrm{i}E)\Phi = (\Phi^*)^{-1}(-\mathrm{i}E)$, hence $-\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} = (\Phi^*)^{-1}\mathfrak{F}\Phi^{-1}$. Then

$$\Big((\Phi^*)^{-1}\mathfrak{F}\Phi^{-1}\Big)^* = (\Phi^{-1})^*\mathfrak{F}[(\Phi^*)^{-1}]^*, \tag{1.3.39}$$

since $\mathfrak{F}$ is self-adjoint. Moreover we have that $(\Phi^{-1})^* = (\Phi^*)^{-1}$. Indeed again by (1.3.35) one has that

$$\Phi^{-1} = (\mathrm{i}E)\Phi^*(-\mathrm{i}E), \qquad (\Phi^{-1})^* = (\mathrm{i}E)\Phi(-\mathrm{i}E), \quad \Phi^* = (-\mathrm{i}E)\Phi^{-1}(\mathrm{i}E)$$

Hence one has

$$(\Phi^{-1})^*\Phi^* = (\mathrm{i}E)\Phi(-\mathrm{i}E)(-\mathrm{i}E)\Phi^{-1}(\mathrm{i}E) = -(\mathrm{i}E)(\mathrm{i}E) = \mathbb{1}. \tag{1.3.40}$$

Then by (1.3.39) we conclude that $(-\mathrm{i}E)\Phi\mathrm{i}E\Phi^{-1}$ is self-adjoint. Let us study the second term of (1.3.38). First note that

$$\partial_t[\Phi^*] = -(\Phi^*)(-\mathrm{i}E)(\partial_t\Phi)\Phi^{-1}(\mathrm{i}E), \qquad (\partial_t\Phi)^* = \Phi^*(\mathrm{i}E)(\partial_t(\Phi^*))^*\Phi^{-1}(\mathrm{i}E) \tag{1.3.41}$$

then

$$\Big((-\mathrm{i}E)(\partial_t\Phi)(\Phi^{-1})\Big)^* = (\Phi^{-1})^*(\partial_t\Phi)^*(\mathrm{i}E) = (-\mathrm{i}E)(\partial_t(\Phi^*))^*\Phi^{-1}. \tag{1.3.42}$$

By (1.3.41) we have $\partial_t(\Phi^*) = (\partial_t\Phi)^*$, hence we get the result. □

Lemma 1.3.9. *Consider a reality and parity preserving linear operator $\mathfrak{F}$ according to Def. 1.3.3 and a map Φ as in* (1.3.21) *which is parity preserving. Consider the system*

$$\partial_t W = \mathrm{i}E\mathfrak{F}W, \qquad W \in \mathbf{H}^s. \tag{1.3.43}$$

By setting $V = \Phi W$ one has that the system (1.3.36) *reads*

$$\partial_t V = \mathrm{i}E\mathcal{Y}V, \tag{1.3.44}$$

$$\mathcal{Y} := -\mathrm{i}E\Phi(\mathrm{i}E)\mathfrak{F}\Phi^{-1} - \mathrm{i}E(\partial_t\Phi)\Phi^{-1}, \tag{1.3.45}$$

and $\mathcal{Y}$ is reality preserving and parity preserving.

Proof. It follows straightforward by the Definition 1.3.3. □

Chapter 2

Local well-posedness

In this chapter we give the proof of Theorems 0.2.1 and 0.2.2. This part of the thesis is the content of the paper [46]. Throughout this chapter we shall use classes of symbols and operators introduced in Section 1.2. According to notation of section 1.2 the time t is treated as a parameter, we shall write, for instance, $a(U;x,\xi)$ instead of $a(U;t,x,\xi)$ even if the symbol a depends on the time t, on the other hand emphasize the x-dependence: we shall denote by $a(U;\xi)$ only those symbols which are independent of x.

2.1 Paralinearization of the equation

In this section we give a paradifferential formulation of the equation (0.2.1), in order to do this we need to "double" the variables. We consider a system of equations for the variables (u^+,u^-) in $H^s\times H^s$ which is equivalent to (0.2.1) if $u^+=\bar{u}^-$. More precisely we give the following definition.

Definition 2.1.1. *Let f be the $C^{\infty}(\mathbb{C}^3;\mathbb{C})$ function in the equation* (0.2.1). *We define the "vector" NLS as*

$$\begin{aligned}
&\partial_t U = \mathrm{i}E\left[\Lambda U + \mathtt{F}(U,U_x,U_{xx})\right], \quad U\in H^s\times H^s,\\
&\mathtt{F}(U,U_x,U_{xx}) := \begin{pmatrix} f_1(U,U_x,U_{xx})\\ f_2(U,U_x,U_{xx})\end{pmatrix},
\end{aligned} \tag{2.1.1}$$

where

$$\mathtt{F}(Z_1,Z_2,Z_3)=\begin{pmatrix} f_1(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)\\ f_2(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)\end{pmatrix},\quad Z_i=\begin{pmatrix} z_i^+\\ z_i^-\end{pmatrix},\quad i=1,2,3,$$

extends $(f,\overline{f})$ *in the following sense. The functions* f_i *for* $i=1,2$ *are* C^∞ *on* $\mathbb{C}^6$ *(in the real sense). Moreover one has the following:*

$$\begin{pmatrix} f_1(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \\ f_2(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \end{pmatrix} = \begin{pmatrix} f(z_1,z_2,z_3) \\ \overline{f(z_1,z_2,z_3)} \end{pmatrix}, \tag{2.1.2}$$

and

$$\begin{aligned} &\partial_{z_3^+} f_1 = \partial_{z_3^-} f_2, \quad \partial_{z_i^+} f_1 = \overline{\partial_{z_i^-} f_2}, \ i=1,2, \ \partial_{z_i^-} f_1 = \overline{\partial_{z_i^+} f_2}, \quad i=1,2,3 \\ &\partial_{\overline{z_i^+}} f_1 = \partial_{\overline{z_i^+}} f_2 = \partial_{\overline{z_i^-}} f_1 = \partial_{\overline{z_i^-}} f_2 = 0 \end{aligned} \tag{2.1.3}$$

where $\partial_{\overline{z_j^\sigma}} = \partial_{\mathrm{Re}\, z_j^\sigma} + \mathrm{i}\partial_{\mathrm{Im}\, z_j^\sigma}$, $\sigma=\pm$.

Remark 2.1.1. *In the case that* f *has the form*

$$f(z_1,z_2,z_3) = C z_1^{\alpha_1}\bar{z}_1^{\beta_1} z_2^{\alpha_2}\bar{z}_2^{\beta_2}$$

for some $C\in\mathbb{C}$, $\alpha_i,\beta_i\in\mathbb{N}$ *for* $i=1,2$*, a possible extension is the following:*

$$\begin{aligned} f_1(z_1^+,z_1^-,z_2^+,z_2^-) &= C(z_1^+)^{\alpha_1}(z_1^-)^{\beta_1}(z_2^+)^{\alpha_2}(z_2^-)^{\beta_2}, \\ f_2(z_1^+,z_1^-,z_2^+,z_2^-) &= \overline{C}(z_1^-)^{\alpha_1}(z_1^+)^{\beta_1}(z_2^-)^{\alpha_2}(z_2^+)^{\beta_2}. \end{aligned}$$

Remark 2.1.2. *Using* (2.1.2) *one deduces the following relations between the derivatives of* f *and* f_j *with* $j=1,2$*:*

$$\begin{aligned} \partial_{z_i} f(z_1,z_2,z_3) &= (\partial_{z_i^+} f_1)(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \\ \partial_{\bar{z}_i} f(z_1,z_2,z_3) &= (\partial_{z_i^-} f_1)(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \\ \overline{\partial_{\bar{z}_i} f(z_1,z_2,z_3)} &= (\partial_{z_i^+} f_2)(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \\ \overline{\partial_{z_i} f(z_1,z_2,z_3)} &= (\partial_{z_i^-} f_2)(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3). \end{aligned} \tag{2.1.4}$$

In the rest of the paper we shall use the following notation. Given a function $g(z_1^+,z_1^-,z_2^+,z_2^-,z_3^+,z_3^-)$ defined on $\mathbb{C}^6$ which is differentiable in the real sense, we shall write for $i=0,1,2$

$$\begin{aligned} (\partial_{\partial_x^i u} g)(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx}) &:= (\partial_{z_{i+1}^+} g)(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx}), \\ (\partial_{\overline{\partial_x^i u}} g)(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx}) &:= (\partial_{z_{i+1}^-} g)(u,\bar{u},u_x,\bar{u}_x,u_{xx},\bar{u}_{xx}). \end{aligned} \tag{2.1.5}$$

By Definition 2.1.1 one has that equation (0.2.1) is equivalent to the system (2.1.1) on the subspace $\mathbf{H}^s$.

We state the Bony paralinearization lemma, which is adapted to our case from Lemma 2.4.5 of [21].

Lemma 2.1.1 (Bony paralinearization of the composition operator). *Let f be a complex-valued function of class C^∞ in the real sense defined in a ball centered at 0 of radius $r>0$, in $\mathbb{C}^6$, vanishing at 0 at order 2. There exists a 1×2 matrix of symbols $q\in\Gamma^2_{K,0}[r]$ and a 1×2 matrix of smoothing operators $Q(U)\in\mathscr{R}^{-\rho}_{K,0}[r]$, for any ρ, such that*

$$f(U,U_x,U_{xx}) = \mathrm{Op}^{\mathscr{B}W}(q(U,U_x,U_{xx};x,\xi))[U] + Q(U)U. \tag{2.1.6}$$

Moreover the symbol $q(U;x,\xi)$ has the form

$$q(U;x,\xi) := d_2(U;x)(\mathrm{i}\xi)^2 + d_1(U;x)(\mathrm{i}\xi) + d_0(U;x), \tag{2.1.7}$$

where $d_j(U;x)$ are 1×2 matrices of symbols in $\mathscr{F}_{K,0}[r]$, for $j=0,1,2$.

Proof. By the paralinearization formula of Bony, we know that

$$f(U,U_x,U_{xx}) = T_{D_U f}U + T_{D_{U_x} f}U_x + T_{D_{U_{xx}} f}U_{xx} + R_0(U)U, \tag{2.1.8}$$

where $R_0(U)$ satisfies estimates (1.2.7) and where

$$\begin{aligned}
T_{D_U f}U &= \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_U(U;x,\xi)]U(y)\,dyd\xi,\\
T_{D_{U_x} f}U_x &= \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_{U_x}(U;x,\xi)]U(y)\,dyd\xi,\\
T_{D_{U_{xx}} f}U_{xx} &= \frac{1}{2\pi}\int e^{\mathrm{i}(x-y)\xi}\chi(\langle\xi\rangle^{-1}D)[c_{U_{xx}}(U;x,\xi)]U(y)\,dyd\xi,
\end{aligned}$$

with

$$\begin{aligned}
c_U(U;x,\xi) &= D_U f,\\
c_{U_x}(U;x,\xi) &= D_{U_x} f(\mathrm{i}\xi),\\
c_{U_{xx}}(U;x,\xi) &= D_{U_{xx}} f(\mathrm{i}\xi)^2,
\end{aligned} \tag{2.1.9}$$

for some $\chi\in C_0^\infty(\mathbb{R})$ with small enough support and equal to 1 close to 0. Using (1.1.26) we define the x-periodic function $b_i(U;x,\xi)$, for $i=0,1,2$, through its Fourier coefficients

$$\hat{b}_i(U;n,\xi) := \hat{c}_{U_i}(U;n,\xi-n/2) \tag{2.1.10}$$

where $U_i := \partial_x^i U$. In the same way we define the function $d_i(U;x,\xi)$, for $i=0,1,2$, as

$$\hat{d}_i(U;n,\xi) := \chi\left(n\langle\xi-n/2\rangle^{-1}\right)\hat{c}_{U_i}(U;n,\xi-n/2). \tag{2.1.11}$$

We have that $T_{D_U f}U = \mathrm{Op}^W(d_0(U,\xi))U$. We observe the following

$$\hat{d}_0(U;n,\xi) = \chi\left(n\langle\xi\rangle^{-1}\right)\widehat{D_U f}(n) + \left(\chi\left(n\langle\xi - n/2\rangle^{-1}\right) - n\langle\xi\rangle^{-1}\right)\widehat{D_U f}(n) \quad (2.1.12)$$

therefore if the support of χ is small enough, thanks to Lemma 1.2.2, we obtained

$$T_{D_U f}U = \mathrm{Op}^{\mathscr{B}W}(b_0(U;x,\xi))U + R_1(U)U, \quad (2.1.13)$$

for some smoothing reminder $R_1(U)$. Reasoning in the same way we get

$$\begin{aligned} T_{D_{U_x} f}U_x &= \mathrm{Op}^{\mathscr{B}W}\big(b_1(U;\xi)\big)U + R_2(U)U \\ T_{D_{U_{xx}} f}U_{xx} &= \mathrm{Op}^{\mathscr{B}W}\big(b_2(U;\xi))\big)U + R_3(U)U. \end{aligned} \quad (2.1.14)$$

The theorem is proved defining $Q(U) = \sum_{k=0}^{3} R_k(U)$ and $q(U;x,\xi) = b_2(U;\xi) + b_1(U;\xi) + b_0(U;\xi)$. Note that the symbol q satisfies conditions (2.1.7) by (2.1.9) and formula (1.1.26). □

We have the following Proposition.

Proposition 2.1.1 (Paralinearization of the system). *There are a matrix $A(U;x,\xi)$ in $\Gamma^2_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$ and a smoothing operator R in $\mathscr{R}^{-\rho}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$, for any $K, r > 0$ and $\rho \geq 0$ such that the system* (2.1.1) *is equivalent to*

$$\partial_t U := \mathrm{i}E\Big[\Lambda U + \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[U] + R(U)[U]\Big], \quad (2.1.15)$$

on the subspace $\mathscr{U}$ (see (0.4.1) *and Def. 2.1.1) and where Λ is defined in* (0.4.6) *and* (0.4.4)*. Moreover the operator $R(U)[\cdot]$ satisfies* (1.3.15)*, the matrix A has the form* (1.3.30)*, i.e.*

$$A(U;x,\xi) := \begin{pmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)} \end{pmatrix} \in \Gamma^2_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C}) \quad (2.1.16)$$

with a, b in $\Gamma^2_{K,0}[r]$. In particular we have that

$$A(U;x,\xi) = A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi) + A_0(U;x), \quad (2.1.17)$$

where $A_i \in \mathscr{F}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$ for $i = 0,1,2$.

Proof. The functions f_1, f_2 in (2.1.1) satisfy the hypotheses of Lemma 2.1.1 for any $r > 0$. Hence the result follows by setting $q(U;x,\xi) =: (a(U;x,\xi), b(U;x,\xi))$. □

In the following we study some properties of the system in (2.1.15).
We first prove some lemmata which translate the Hamiltonian Hyp. 0.2.1, parity-preserving Hyp. 0.2.2 and global ellipticity Hyp. 0.2.3 in the paradifferential setting.

Lemma 2.1.2 (Hamiltonian structure). *Assume that f in* (0.2.1) *satisfies Hypothesis 0.2.1. Consider the matrix $A(U;x,\xi)$ in* (2.1.16) *given by Proposition 2.1.1. Then the term*

$$A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi)$$

in (2.1.17) *satisfies conditions* (1.3.31). *More explicitly one has*

$$A_2(U;x) := \begin{pmatrix} a_2(U;x) & b_2(U;x) \\ \overline{b_2(U;x)} & a_2(U;x) \end{pmatrix}, \quad A_1(U;x) := \begin{pmatrix} a_1(U;x) & 0 \\ 0 & \overline{a_1(U;x)} \end{pmatrix}, \quad (2.1.18)$$

with $a_2, a_1, b_2 \in \mathscr{F}_{K,0}[r]$ and $a_2 \in \mathbb{R}$.

Proof. Recalling the notation introduced in (2.1.5) we shall write

$$\partial_{\partial_x^i u} f := \partial_{z_{i+1}^+} f_1, \quad \partial_{\overline{\partial_x^i u}} f := \partial_{z_{i+1}^-} f_1, \quad i = 0,1,2, \quad (2.1.19)$$

when restricted to the real subspace $\mathscr{U}$ (see (0.4.1)). Using conditions (2.1.2), (2.1.3) and (2.1.4) one has that

$$\begin{pmatrix} f(u,u_x,u_{xx}) \\ \overline{f(u,u_x,u_{xx})} \end{pmatrix} = \begin{pmatrix} f_1(U,U_x,U_{xx}) \\ f_2(U,U_x,U_{xx}) \end{pmatrix}$$
$$= \mathrm{Op}^{\mathscr{B}}\left[\begin{pmatrix} \partial_{u_{xx}} f & \partial_{\bar{u}_{xx}} f \\ \overline{\partial_{\bar{u}_{xx}} f} & \overline{\partial_{u_{xx}} f} \end{pmatrix}(\mathrm{i}\xi)^2\right] U + \mathrm{Op}^{\mathscr{B}}\left[\begin{pmatrix} \partial_{u_x} f & \partial_{\bar{u}_x} f \\ \overline{\partial_{\bar{u}_x} f} & \overline{\partial_{u_x} f} \end{pmatrix}(\mathrm{i}\xi)\right] U + R(U)[U] \quad (2.1.20)$$

where $R(U)$ belongs to $\mathscr{R}^0_{K,0}[r]$. By Hypothesis 0.2.1 we have that

$$\begin{aligned} \partial_{u_{xx}} f &= -\partial_{u_x \bar{u}_x} F, \\ \partial_{\bar{u}_{xx}} f &= -\partial_{\bar{u}_x \bar{u}_x} F, \\ \partial_{u_x} f &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\partial_{u_x \bar{u}_x} F\right] - \partial_{u \bar{u}_x} F + \partial_{u_x \bar{u}} F, \\ \partial_{\bar{u}_x} f &= -\frac{\mathrm{d}}{\mathrm{d}x}\left[\partial_{\bar{u}_x \bar{u}_x} F\right]. \end{aligned} \quad (2.1.21)$$

We now pass to the Weyl quantization in the following way. Set

$$c(x,\xi) = \partial_{u_{xx}} f(x)(\mathrm{i}\xi)^2 + \partial_{u_x} f(x)(\mathrm{i}\xi).$$

Passing to the Fourier side we have that

$$
\begin{aligned}
&\widehat{c}(j,\xi-\frac{j}{2}) = \\
&\widehat{(\partial_{u_{xx}}f)}(j)(\mathrm{i}\xi)^2 + \Big[\widehat{(\partial_{u_x}f)}(j) - (\mathrm{i}j)\widehat{(\partial_{u_{xx}}f)}(j)\Big](\mathrm{i}\xi) \\
&+\Big[\frac{(\mathrm{i}j)^2}{4}\widehat{(\partial_{u_{xx}}f)}(j) - \frac{(\mathrm{i}j)}{2}\widehat{(\partial_{u_x}f)}(j)\Big],
\end{aligned}
$$

therefore by using formula (1.1.26) we have that $\mathrm{Op}^{\mathscr{B}}(c(x,\xi)) = \mathrm{Op}^{\mathscr{B}W}(a(x,\xi))$, where

$$
a(x,\xi) = \partial_{u_{xx}}f(x)(\mathrm{i}\xi)^2 + [\partial_{u_x}f(x) - \frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_{xx}}f)](\mathrm{i}\xi) + \frac{1}{4}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\partial_{u_{xx}}f) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(\partial_{u_x}f).
$$

Using the relations in (2.1.21) we obtain a matrix A as in (2.1.18), and in particular we have

$$
a_2(U;x) = -\partial_{u_x\bar{u}_x}F, \quad a_1(U;x) = -\partial_{u\bar{u}_x}F + \partial_{u_x\bar{u}}F, \quad b_2(U;x) = -\partial_{\bar{u}_x\bar{u}_x}F. \tag{2.1.22}
$$

Since F is real then a_2 is real, while a_1 is purely imaginary. This implies conditions (1.3.31). □

Lemma 2.1.3 (**Parity preserving structure**). *Assume that f in* (0.2.1) *satisfies Hypothesis 0.2.2. Consider the matrix $A(U;x,\xi)$ in* (2.1.16) *given by Proposition 2.1.1. One has that $A(U;x,\xi)$ has the form* (2.1.17) *where*

$$
\begin{aligned}
&A_2(U;x) := \begin{pmatrix} a_2(U;x) & b_2(U;x) \\ \overline{b_2(U;x)} & \overline{a_2(U;x)} \end{pmatrix}, \quad A_1(U;x) := \begin{pmatrix} a_1(U;x) & b_1(U;x) \\ \overline{b_1(U;x)} & \overline{a_1(U;x)} \end{pmatrix}, \\
&A_0(U;x) := \begin{pmatrix} a_0(U;x) & b_0(U;x) \\ \overline{b_0(U;x)} & \overline{a_0(U;x)} \end{pmatrix},
\end{aligned}
\tag{2.1.23}
$$

with $a_2, b_2, a_1, b_1, a_0, b_0 \in \mathscr{F}_{K,0}[r]$ such that, for U even in x, the following holds:

$$
a_2(U;x) = a_2(U;-x), \quad b_2(U;x) = b_2(U;-x), \tag{2.1.24a}
$$

$$
a_1(U;x) = -a_1(U;-x), \quad b_1(U;x) = -b_1(U;-x), \tag{2.1.24b}
$$

$$
a_0(U;x) = a_0(U;-x), \quad b_0(U;x) = b_0(U;-x), \quad U \in \mathbf{H}^s_e, \tag{2.1.24c}
$$

and

$$
a_2(U;x) \in \mathbb{R}. \tag{2.1.25}
$$

The matrix $R(U)$ in (2.1.15) *is parity preserving according to Definition 1.3.3.*

Proof. Using the same notation introduced in the proof of Lemma 2.1.2 (recall (2.1.4)) we have that formula (2.1.20) holds. Under the Hypothesis 0.2.2 one has that the functions $\partial_u f, \partial_{\bar{u}} f, \partial_{u_{xx}} f, \partial_{\bar{u}_{xx}} f$ are *even* in x while $\partial_{u_x} f, \partial_{\bar{u}_x} f$ are *odd* in x. Passing to the Weyl quantization by formula (1.1.26) we get

$$
\begin{aligned}
&a_2(U;x) = \partial_{u_{xx}} f, \\
&a_1(U;x) = \partial_{u_x} f - \partial_x(\partial_{u_{xx}} f), \\
&a_0(U;x) = \partial_u f + \frac{1}{4}\partial_x^2(\partial_{u_{xx}} f) - \frac{1}{2}\partial_x(\partial_{u_x} f), \\
&b_2(U;x) = \partial_{\bar{u}_{xx}} f, \\
&b_1(U;x) = \partial_{\bar{u}_x} f - \partial_x(\partial_{\bar{u}_{xx}} f), \\
&b_0(U;x) = \partial_{\bar{u}} f + \frac{1}{4}\partial_x^2(\partial_{\bar{u}_{xx}} f) - \frac{1}{2}\partial_x(\partial_{\bar{u}_x} f)
\end{aligned}
\tag{2.1.26}
$$

which imply conditions (2.1.24), while (2.1.25) is implied by item 2 of Hypothesis 0.2.2. The term R is parity preserving by difference. □

Lemma 2.1.4 (Global ellipticity). *Assume that f in* (0.2.1) *satisfies Hyp. 0.2.1 (respectively Hyp. 0.2.2). If f satisfies also Hyp. 0.2.3 then the matrix $A_2(U;x)$ in* (2.1.18) *(resp. in* (2.1.23)*) is such that*

$$
\begin{aligned}
&1 + a_2(U;x) \geq \mathtt{c}_1 \\
&(1 + a_2(U;x))^2 - |b_2(U;x)|^2 \geq \mathtt{c}_2 > 0,
\end{aligned}
\tag{2.1.27}
$$

where $\mathtt{c}_1$ and $\mathtt{c}_2$ are the constants given in (0.2.7) *and* (0.2.8).

Proof. It follows from (2.1.22) in the case of Hyp. 0.2.1 and from (2.1.26) in the case of Hyp. 0.2.2. □

Lemma 2.1.5 (Lipschitz estimates). *Fix $r > 0$, $K > 0$ and consider the matrices A and R given in Proposition 2.1.1. Then there exists $s_0 > 0$ such that for any $s \geq s_0$ the following holds true. For any $U, V \in C^K_{*\mathbb{R}}(I; \mathbf{H}^s) \cap B^K_{s_0}(I, r)$ there are constants $C_1 > 0$ and $C_2 > 0$, depending on s, $\|U\|_{K,s_0}$ and $\|V\|_{K,s_0}$, such that for any $H \in C^K_{*\mathbb{R}}(I; \mathbf{H}^s)$ one has*

$$
\|\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[H] - \mathrm{Op}^{\mathscr{B}W}(A(V;x,\xi))[H]\|_{K,s-2} \leq C_1 \|H\|_{K,s} \|U - V\|_{K,s_0} \tag{2.1.28}
$$

$$
\|R(U)[U] - R(V)[V]\|_{K,s+\rho} \leq C_2(\|U\|_{K,s} + \|V\|_{K,s})\|U - V\|_{K,s}, \tag{2.1.29}
$$

for any $\rho \geq 0$.

Proof. We prove bound (2.1.28) on each component of the matrix A in (2.1.16) in the case that f satisfies Hyp. 0.2.2. The Hamiltonian case of Hyp. 0.2.1 follows by using the same arguments. From the proof of Lemma 2.1.3 we know that the symbol $a(U;x,\xi)$ of the matrix in (2.1.16) is such that $a(U;x,\xi) = a_2(U;x)(\mathrm{i}\xi)^2 + a_1(U;x)(\mathrm{i}\xi) + a_0(U;x)$ where $a_i(U;x)$ for $i = 0,1,2$ are given in (2.1.26).
By Remark 1.2.3 there exists $s_0 > 0$ such that for any $s \geq s_0$ one has

$$\begin{aligned} &\|\mathrm{Op}^{\mathscr{B}W}\big((a_2(U;x) - a_2(V;x))(\mathrm{i}\xi)^2\big)h\|_{K,s-2} \leq \\ &C \sup_{\xi} \langle\xi\rangle^{-2} \|(a_2(U;x) - a_2(V;x))(\mathrm{i}\xi)^2\big)\|_{K,s_0} \|h\|_{K,s}. \end{aligned} \tag{2.1.30}$$

with C depending on s, s_0. Let $U, V \in C^K_{*\mathbb{R}}(I;\mathbf{H}^s) \cap B^K_{s_0+2}(I,r)$, by Lagrange theorem, recalling the relations in (2.1.4), (2.1.5) and (2.1.19), one has that

$$\begin{aligned} &\big(a_2(U;x) - a_2(V;x)\big)(\mathrm{i}\xi)^2 \\ &= \big((\partial_{u_{xx}} f_1)(U, U_x, U_{xx}) - (\partial_{u_{xx}} f_1)(V, V_x, V_{xx})\big)(\mathrm{i}\xi)^2 \\ &= (\partial_U \partial_{u_{xx}} f_1)(W^{(0)}, U_x, U_{xx})(U - V)(\mathrm{i}\xi)^2 + \\ &+ (\partial_{U_x} \partial_{u_{xx}} f_1)(V, W^{(1)}, U_{xx})(U_x - V_x)(\mathrm{i}\xi)^2 + \\ &+ (\partial_{U_{xx}} \partial_{u_{xx}} f_1)(V, V_x, W^{(2)})(U_{xx} - V_{xx})(\mathrm{i}\xi)^2 \end{aligned} \tag{2.1.31}$$

where $W^{(j)} = \partial_x^j V + t_j(\partial_x^j U - \partial_x^j V)$, for some $t_j \in [0,1]$ and $j = 0,1,2$. Hence, for instance, the first summand of (2.1.31) can be estimated as follows

$$\begin{aligned} &\sup_{\xi} \langle\xi\rangle^{-2} \|(\partial_U \partial_{u_{xx}} f_1)(W^{(0)}, U_x, U_{xx})(U - V)(\mathrm{i}\xi)^2\|_{K,s_0} \\ &\qquad \leq C_1 \|U - V\|_{K,s_0} \sup_{U,V \in B^K_{s_0+2}(I,r)} \|(\partial_U \partial_{u_{xx}} f_1)(W^{(0)}, U_x, U_{xx})\|_{K,s_0} \\ &\qquad \leq C_2 \|U - V\|_{K,s_0}, \end{aligned} \tag{2.1.32}$$

where C_1 depends on s_0 and C_2 depends only on s_0 and $\|U\|_{K,s_0+2}, \|V\|_{K,s_0+2}$ and where we have used a Moser type estimates on composition operators on H^s since f_1 belongs to $C^\infty(\mathbb{C}^6;\mathbb{C})$. We refer the reader to Lemma $A.50$ of [48] for a complete statement (see also [6], [75]). The other terms in the r.h.s. of (2.1.31) can be treated in the same way. Hence from (2.1.30) and the discussion above we have obtained

$$\|\mathrm{Op}^{\mathscr{B}W}\big((a_2(U;x) - a_2(V;x))(\mathrm{i}\xi)^2\big)h\|_{K,s-2} \leq C\|U - V\|_{K,s_0+2}\|h\|_{K,s}, \tag{2.1.33}$$

with C depending on s and $\|U\|_{K,s_0+2}, \|V\|_{K,s_0+2}$. One has to argue exactly as done above for the lower order terms $a_1(U;x)(\mathrm{i}\xi)$ and $a_0(U;x)$ of $a(U;x,\xi)$. In the same way one is able to prove the estimate

$$\|\mathrm{Op}^{\mathscr{B}W}\big((b(U;x,\xi)-b(V;x,\xi))\big)\bar{h}\|_{K,s-2} \leq C\|U-V\|_{K,s_0+2}\|\bar{h}\|_{K,s}. \tag{2.1.34}$$

Thus the (2.1.28) is proved renaming s_0 as s_0+2.
In order to prove (2.1.29) we show that the operator $\mathrm{d}_U(R(U)U)[\cdot]$ belongs to the class $\mathscr{R}^{-\rho}_{K,K'}[r]\otimes\mathscr{M}_2(\mathbb{C})$ for any $\rho\geq 0$ (where $d_U(R(U)U)[\cdot]$ denotes the differential of $R(U)[U]$ w.r.t. the variable U). We recall that the operator R in (2.1.15) is of the form

$$R(U)[\cdot] := \begin{pmatrix} Q(U)[\cdot] \\ \overline{Q(U)[\cdot]} \end{pmatrix},$$

where $Q(U)[\cdot]$ is the 1×2 matrix of smoothing operators in (2.1.6) with f given in (0.2.1). We claim that $\mathrm{d}_U(Q(U)U)[\cdot]$ is 1×2 matrix of smoothing operators in $\mathscr{R}^{-\rho}_{K,0}[r]$. By Lemma 2.1.1 we know that $Q(U)[\cdot] = R_0(U) + \sum_{j=1}^{3} R_j(U)$, where R_0 is 1×2 matrix of smoothing operators coming from the Bony paralinearization formula (see (2.1.8)), while R_j, for $j=1,2,3$, are the 1×2 matrices of smoothing operators in (2.1.13) and (2.1.14).
One can prove the claim for the terms R_j, $j=1,2,3$, by arguing as done in the proof of (2.1.28). Indeed we know the explicit paradifferential structure of these remainders. For instance, by (2.1.10), (2.1.11), (2.1.12) and (2.1.13) we have that

$$R_1(U)[\cdot] := \mathrm{op}\Big(k(x,\xi)\Big)[\cdot], \tag{2.1.35}$$

where $k(x,\xi) = \sum_{j\in\mathbb{Z}} \hat{k}(j,\xi)e^{\mathrm{i}jx}$ and

$$k(j,\xi) = \big(\chi\left(n\langle\xi - n/2\rangle^{-1}\right) - \chi(n\langle\xi\rangle^{-1})\big)\widehat{D_U f}(n)$$

(see formula (2.1.12)). The remainders R_2, R_3 have similar expressions. We reduced to prove the claim for the term R_0. Recalling (2.1.9) we set

$$c(U;x,\xi) := c_U(U;x,\xi) + c_{U_x}(U;x,\xi) + c_{U_{xx}}(U;x,\xi).$$

Using this notation, formula (2.1.8) reads

$$f(u,u_x,u_{xx}) = f_1(U,U_x,U_{xx}) = \mathrm{Op}^{\mathscr{B}}(c(U;x,\xi))U + R_0(U)U. \tag{2.1.36}$$

Differentiating (2.1.36) we get

$$
\begin{aligned}
&\mathrm{d}_U(f_1(U,U_x,U_{xx}))[H] = \\
&\quad \mathrm{Op}^{\mathscr{B}}(c(U;x,\xi))[H] + \mathrm{Op}^{\mathscr{B}}(\partial_U c(U;x,\xi)\cdot H)[U] + \mathrm{d}_U(R_0(U)[U])[H].
\end{aligned}
\tag{2.1.37}
$$

The l.h.s. of (2.1.37) is nothing but

$$
\begin{aligned}
&\partial_U f_1(U,U_x,U_{xx})\cdot H + \partial_{U_x} f_1(U,U_x,U_{xx})\cdot H_x \\
&\quad + \partial_{U_{xx}} f_1(U,U_x,U_{xx})\cdot H_{xx} =: G(U,H).
\end{aligned}
$$

By applying the Bony paralinearization formula to $G(U,H)$ (as a function of the six variables $U, U_x, U_{xx}, H, H_x, H_{xx}$) we get

$$
\begin{aligned}
&G(U,H) = \\
&\mathrm{Op}^{\mathscr{B}}(\partial_U G(U,H))[U] + \mathrm{Op}^{\mathscr{B}}(\partial_{U_x} G(U,H))[U_x] \\
&+ \mathrm{Op}^{\mathscr{B}}(\partial_{U_{xx}} G(U,H))[U_{xx}] + \mathrm{Op}^{\mathscr{B}}(\partial_H G(U,H))[H] \\
&+ \mathrm{Op}^{\mathscr{B}}(\partial_{H_x} G(U,H))[H_x] + \mathrm{Op}^{\mathscr{B}}(\partial_{H_{xx}} G(U,H))[H_{xx}] \\
&+ R_4(U)[H],
\end{aligned}
\tag{2.1.38}
$$

where $R_4(U)[\cdot]$ satisfies estimates (1.2.7) for any $\rho \geq 0$. By (2.1.9) and (2.1.38) we have that (2.1.37) reads

$$
\mathrm{d}_U(R_0(U)U)[H] = R_4(U)[H]. \tag{2.1.39}
$$

Therefore $\mathrm{d}_U(R_0(U)U)[\cdot]$ is a 1×2 matrix of operators in the class $R^{-\rho}_{K,0}[r]$ for any $\rho \geq 0$. □

2.2 Regularization

We consider the system

$$
\begin{aligned}
&\partial_t V = \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[V] + R_1^{(0)}(U)[V] + R_2^{(0)}(U)[U]\Big], \\
&\qquad\qquad U \in B^K_{s_0}(I,r) \cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),
\end{aligned}
\tag{2.2.1}
$$

for some s_0 large, $s \geq s_0$ and where Λ is defined in (0.4.4). The operators $R_1^{(0)}(U)$ and $R_2^{(0)}(U)$ are in the class $\mathscr{R}^{-\rho}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$ for some $\rho \geq 0$ and they are reality preserving. The matrix $A(U;x,\xi)$ satisfies the following.

Constraint 2.2.4. *The matrix $A(U;x,\xi)$ belongs to $\Gamma^2_{K,0}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and has the following properties:*

- *$A(U;x,\xi)$ is* reality preserving, *i.e. has the form* (1.3.30)*;*
- *the components of $A(U;x,\xi)$ have the form*

$$\begin{aligned} a(U;x,\xi) &= a_2(U;x)(\mathrm{i}\xi)^2 + a_1(U;x)(\mathrm{i}\xi),\\ b(U;x,\xi) &= b_2(U;x)(\mathrm{i}\xi)^2 + b_1(U;x)(\mathrm{i}\xi), \end{aligned} \tag{2.2.2}$$

  for some $a_i(U;x), b_i(U;x)$ belonging to $\mathcal{F}_{K,0}[r]$ for $i=1,2$.

In addition to Constraint 2.2.4 we assume that the matrix A satisfies one the following two Hypotheses:

Hypothesis 2.2.1 (Self-adjoint). *The operator* $\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))$ *is self-adjoint according to Definition 1.3.5, i.e. the matrix $A(U;x,\xi)$ satisfies conditions* (1.3.31).

Hypothesis 2.2.2 (Parity preserving). *The operator* $\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))$ *is parity preserving according to Definition 1.3.1, i.e. the matrix $A(U;x,\xi)$ satisfies the conditions*

$$A(U;x,\xi) = A(U;-x,-\xi), \qquad a_2(U;x)\in\mathbb{R}. \tag{2.2.3}$$

The function P in (0.2.2) *is such that $\hat{p}(j)=\hat{p}(-j)$ for $j\in\mathbb{Z}$.*

Finally we need the following *ellipticity condition*.

Hypothesis 2.2.3 (Ellipticity). *There exist $\mathtt{c}_1,\mathtt{c}_2>0$ such that components of the matrix $A(U;x,\xi)$ satisfy the condition*

$$\begin{aligned} &1+a_2(U;x)\geq \mathtt{c}_1,\\ &(1+a_2(U;x))^2-|b_2(U;x)|^2\geq \mathtt{c}_2>0, \end{aligned} \tag{2.2.4}$$

for any $U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))$.

The goal of this section is to transform the linear paradifferential system (2.2.1) into a constant coefficient one up to bounded remainder.
The following result is the core of our analysis.

Theorem 2.2.1 (Regularization). *Fix $K\in\mathbb{N}$ with $K\geq 4$, $r>0$. Consider the system* (2.2.1)*. There exists $s_0>0$ such that for any $s\geq s_0$ the following holds. Fix U in $B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))$ (resp. $U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,\mathbf{H}^s_e(\mathbb{T},\mathbb{C}^2))$) and assume that the system* (2.2.1) *has the following structure:*

- *the operators* $R_1^{(0)}$, $R_2^{(0)}$ *belong to the class* $\mathscr{R}^{-\rho}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$*;*
- *the matrix* $A(U;x,\xi)$ *satisfies Constraint 2.2.4,*
- *the matrix* $A(U;x,\xi)$ *satisfies Hypothesis 2.2.1 (resp. together with P satisfy Hyp. 2.2.2)*
- *the matrix* $A(U;x,\xi)$ *satisfies Hypothesis 2.2.3.*

Then there exists an invertible map (resp. an invertible and parity preserving map)

$$\Phi=\Phi(U):C^{K-4}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2))\to C^{K-4}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),$$

with

$$\|(\Phi(U))^{\pm1}V\|_{K-4,s}\le\|V\|_{K-4,s}(1+C\|U\|_{K,s_0}),\tag{2.2.5}$$

for a constant $C>0$ *depending on s,* $\|U\|_{K,s_0}$ *and* $\|P\|_{C^1}$ *such that the following holds. There exist operators* $R_1(U),R_2(U)$ *in* $\mathscr{R}^0_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$*, and a diagonal matrix* $L(U)$ *in* $\Gamma^2_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$ *of the form* (1.3.30) *satisfying condition* (1.3.31) *and independent of* $x\in\mathbb{T}$*, such that by setting* $W=\Phi(U)V$ *the system* (2.2.1) *reads*

$$\partial_tW=\mathrm{i}E\Big[\Lambda W+\mathrm{Op}^{\mathscr{B}W}(L(U;\xi))[W]+R_1(U)[W]+R_2(U)[U]\Big].\tag{2.2.6}$$

Remark 2.2.1. *Note that, under the Hypothesis 2.2.2, if the term* $R_1^{(0)}(U)[V]+R_2^{(0)}(U)[U]$ *in* (2.2.1) *is* parity preserving*, according to Definition 1.3.3, then the flow of the system* (2.2.1) *preserves the subspace of even functions. Since the map* $\Phi(U)$ *in Theorem 2.2.1 is* parity preserving*, then Lemma 1.3.9 implies that also the flow of the system* (2.2.6) *preserves the same subspace.*

The proof of Theorem 2.2.1 is divided into four steps which are performed in the remaining part of the section. We first explain our strategy and set some notation. We consider the system (2.2.1)

$$\begin{aligned}V_t=\mathscr{L}^{(0)}(U)[V]:=&\\ \mathrm{i}E\Big[\Lambda V+\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[V]+R_1^{(0)}(U)[V]+R_2^{(0)}(U)[U]\Big].&\end{aligned}\tag{2.2.7}$$

The idea is to construct several maps

$$\Phi_i[\cdot]:=\Phi_i(U)[\cdot]:C^{K-(i-1)}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T}))\to C^{K-(i-1)}_{*\mathbb{R}}(I,\mathbf{H}^s(\mathbb{T})),$$

for $i = 1, \dots, 4$ which conjugate the system $\mathscr{L}^{(i)}(U)$ to $\mathscr{L}^{(i+1)}(U)$, with $\mathscr{L}^{(0)}(U)$ in (2.2.7) and

$$\begin{aligned}&\mathscr{L}^{(i)}(U)[\cdot] := \\ &\mathrm{i}E\Big[\Lambda + \mathrm{Op}^{\mathscr{B}W}(L^{(i)}(U;\xi))[\cdot] + \mathrm{Op}^{\mathscr{B}W}(A^{(i)}(U;x,\xi))[\cdot] \\ &\quad + R_1^{(i)}[\cdot] + R_2^{(i)}(U)[U]\Big],\end{aligned} \tag{2.2.8}$$

where $R_1^{(i)}$ and $R_2^{(i)}$ belong to $\mathscr{R}^0_{K,i}[r] \otimes \mathscr{M}_2(\mathbb{C})$, $L^{(i)}$ belong to $\Gamma^2_{K,i}[r] \otimes \mathscr{M}_2(\mathbb{C})$ and moreover they are diagonal, self-adjoint and independent of $x \in \mathbb{T}$ and finally $A^{(i)}$ are in $\Gamma^2_{K,i}[r] \otimes \mathscr{M}_2(\mathbb{C})$. As we will see, the idea is to look for Φ_i in such a way $A^{(i+1)}$ is actually a matrix with symbols of order less or equal than the order of $A^{(i)}$.

We now prove a lemma in which we study the conjugate of the convolution operator.

Lemma 2.2.1. *Let Q_1, Q_2 operators in the class $\mathscr{R}^0_{K,K'}[r] \otimes \mathscr{M}_2(\mathbb{C})$ and $P : \mathbb{T} \to \mathbb{R}$ a continuous function. Consider the operator $\mathfrak{P}$ defined in* (0.4.7). *Then there exists R belonging to $\mathscr{R}^0_{K,K'}[r] \otimes \mathscr{M}_2(\mathbb{C})$ such that*

$$(\mathbb{1} + Q_1(U)) \circ \mathfrak{P} \circ (\mathbb{1} + Q_2(U))[\cdot] = \mathfrak{P}[\cdot] + R(U)[\cdot]. \tag{2.2.9}$$

Moreover if P is even in x and the operators $Q_1(U)$ and $Q_2(U)$ are parity-preserving then the operator $R(U)$ is parity preserving according to Definition 1.3.1.

Proof. By linearity it is enough to show that the terms

$$Q_1(U) \circ \mathfrak{P} \circ (\mathbb{1} + Q_2(U))[h], \quad (\mathbb{1} + Q_1(U)) \circ \mathfrak{P} \circ Q_2(U)[h], \quad Q_1(U) \circ \mathfrak{P} \circ Q_2(U)[h]$$

belong to $\mathscr{R}^0_{K,K'}[r] \otimes \mathscr{M}_2(\mathbb{C})$. Note that, for any $0 \leq k \leq K - K'$,

$$\|\partial_t^k (P * h)\|_{H^{s-2k}} \leq C \|\partial_t^k h\|_{H^{s-2k}}, \tag{2.2.10}$$

for some $C > 0$ depending only on $\|P\|_{L^\infty}$. The (2.2.10) and the estimate (1.2.7) on Q_1 and Q_2 imply the thesis. If P is even in x then the convolution operator with kernel P is a parity preserving operator according to Definition 1.3.3. Therefore if in addiction $Q_1(U)$ and $Q_2(U)$ are parity preserving so is $R(U)$. □

2.2.1 Diagonalization of the second order operator

Consider the system (2.2.1) and assume the Hypothesis of Theorem 2.2.1. The matrix $A(U;x,\xi)$ satisfies conditions (2.2.2), therefore it can be written as

$$A(U;x,\xi) := A_2(U;x)(\mathrm{i}\xi)^2 + A_1(U;x)(\mathrm{i}\xi), \tag{2.2.11}$$

with $A_i(U;x)$ belonging to $\mathscr{F}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$ and satisfying either Hyp. 2.2.1 or Hyp. 2.2.2. In this Section, by exploiting the structure of the matrix $A_2(U;x)$, we show that it is possible to diagonalize the matrix $E(\mathbb{1}+A_2)$ through a change of coordinates which is a multiplication operator. We have the following lemma.

Lemma 2.2.2. *Under the Hypotheses of Theorem 2.2.1 there exists $s_0>0$ such that for any $s\geq s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_1 = \Phi_1(U) : C^K_{*\mathbb{R}}(I,\mathbf{H}^s) \to C^K_{*\mathbb{R}}(I,\mathbf{H}^s),$$

with

$$\|(\Phi_1(U))^{\pm 1}V\|_{K,s} \leq \|V\|_{K,s}(1+C\|U\|_{K,s_0}) \tag{2.2.12}$$

where C depends only on s and $\|U\|_{K,s_0}$ such that the following holds. There exists a matrix $A^{(1)}(U;x,\xi)$ satisfying Constraint 2.2.4 and Hyp. 2.2.1 (resp. Hyp. 2.2.2) of the form

$$\begin{aligned}
A^{(1)}(U;x,\xi) &:= A_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;x)(\mathrm{i}\xi),\\
A_2^{(1)}(U;x) &:= \begin{pmatrix} a_2^{(1)}(U;x) & 0 \\ 0 & a_2^{(1)}(U;x)\end{pmatrix} \in \mathscr{F}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C}),\\
A_1^{(1)}(U;x) &:= \begin{pmatrix} a_1^{(1)}(U;x) & b_1^{(1)}(U;x) \\ \overline{b_1^{(1)}(U;x,)} & \overline{a_1^{(1)}(U;x)}\end{pmatrix} \in \mathscr{F}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})
\end{aligned} \tag{2.2.13}$$

and operators $R_1^{(1)}(U)$, $R_2^{(1)}(U)$ in $\mathscr{R}^0_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$ such that by setting $V_1=\Phi(U)V$ the system (2.2.1) *reads*

$$\partial_t V_1 = \mathrm{i}E\Big[\Lambda V_1 + \mathrm{Op}^{\mathscr{B}W}(A^{(1)}(U;x,\xi))[V_1] + R_1^{(1)}(U)[V_1] + R_2^{(1)}(U)[U]\Big]. \tag{2.2.14}$$

Moreover there exists a constant $\mathtt{k}>0$ such that

$$1 + a_2^{(1)}(U;x) \geq \mathtt{k}. \tag{2.2.15}$$

Proof. Let us consider a symbol $z(U;x)$ in the class $\mathscr{F}_{K,0}[r]$ and set

$$Z(U;x) := \begin{pmatrix} 0 & z(U;x) \\ \overline{z(U;x)} & 0 \end{pmatrix} \in \mathscr{F}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C}). \tag{2.2.16}$$

Let $\Phi_1^\tau(U)[\cdot]$ the solution at time $\tau \in [0,1]$ of the system

$$\begin{cases} \partial_\tau \Phi_1^\tau(U)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(Z(U;x))\Phi_1^\tau(U)[\cdot], \\ \Phi_1^0(U)[\cdot] = \mathbb{1}[\cdot]. \end{cases} \tag{2.2.17}$$

Since $\mathrm{Op}^{\mathscr{B}W}(Z(U;x))$ is a bounded operator on $\mathbf{H}^s$, by standard theory of Banach space ODE we have that the flow Φ_1^τ is well defined, moreover by Proposition 1.2.1 one gets

$$\begin{aligned} \partial_\tau \|\Phi_1^\tau(U)V\|_{\mathbf{H}^s}^2 &\leq \|\Phi_1^\tau(U)V\|_{\mathbf{H}^s}\|\mathrm{Op}^{\mathscr{B}W}(Z(U;x))\Phi_1^\tau(U)V\|_{\mathbf{H}^s} \\ &\leq \|\Phi_1^\tau(U)V\|_{\mathbf{H}^s}^2 C\|U\|_{\mathbf{H}^{s_0}}, \end{aligned} \tag{2.2.18}$$

hence one obtains

$$\|\Phi_1^\tau(U)[V]\|_{\mathbf{H}^s} \leq \|V\|_{\mathbf{H}^s}(1 + C\|U\|_{\mathbf{H}^{s_0}}), \tag{2.2.19}$$

where $C > 0$ depends only on $\|U\|_{\mathbf{H}^{s_0}}$. The latter estimate implies (2.2.12) for $K = 0$. By differentiating in t the equation (2.2.17) we note that

$$\partial_\tau\partial_t\Phi_1^\tau(U)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(Z(U;x))\partial_t\Phi_1^\tau(U)[\cdot] + \mathrm{Op}^{\mathscr{B}W}(\partial_t Z(U;x))\Phi_1^\tau(U)[\cdot]. \tag{2.2.20}$$

Now note that, since Z belongs to the class $\mathscr{F}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$, one has that $\partial_t Z$ is in $\mathscr{F}_{K,1}[r] \otimes \mathscr{M}_2(\mathbb{C})$. By performing an energy type estimate as in (2.2.18) one obtains

$$\|\Phi_1^\tau(U)[V]\|_{C^1\mathbf{H}^s} \leq \|V\|_{C^1\mathbf{H}^s}(1 + C\|U\|_{C^1\mathbf{H}^{s_0}}),$$

which implies (2.2.12) with $K = 1$. Iterating K times the reasoning above one gets the bound (2.2.12). By using Corollary 1.2.2 one gets that

$$\Phi_1^\tau(U)[\cdot] = \exp\{\tau\mathrm{Op}^{\mathscr{B}W}(Z(U;x))\}[\cdot] = \mathrm{Op}^{\mathscr{B}W}(\exp\{\tau Z(U;x)\})[\cdot] + Q_1^\tau(U)[\cdot], \tag{2.2.21}$$

with Q_1^τ belonging to $\mathscr{R}_{K,0}^{-\rho}[r] \otimes \mathscr{M}_2(\mathbb{C})$ for any $\rho > 0$ and any $\tau \in [0,1]$. We now set $\Phi_1(U)[\cdot] := \Phi_1^\tau(U)[\cdot]_{|_{\tau=1}}$. In particular we have

$$\begin{aligned} &\Phi_1(U)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(C(U;x))[\cdot] + Q_1^1(U)[\cdot], \\ &C(U;x) := \exp\{Z(U;x)\} := \begin{pmatrix} c_1(U;x) & c_2(U;x) \\ \overline{c_2(U;x)} & \overline{c_1(U;x)} \end{pmatrix}, \\ &C(U;x) - \mathbb{1} \in \mathscr{F}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C}), \end{aligned} \tag{2.2.22}$$

where

$$c_1(U;x) := \cosh(|z(U;x)|), \qquad c_2(U;x) := \frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|). \tag{2.2.23}$$

Note that the function $c_2(U;x)$ above is not singular indeed

$$\begin{aligned} c_2(U;x) &= \frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|) = \frac{z(U;x)}{|z(U;x)|}\sum_{k=0}^{\infty}\frac{(|z|(U;x))^{2k+1}}{(2k+1)!} \\ &= z(U;x)\sum_{k=0}^{\infty}\frac{\big(z(U;x)\overline{z(U;x)}\big)^k}{(2k+1)!}. \end{aligned}$$

We note moreover that for any $x\in\mathbb{T}$ one has $\det(C(U;x)) = 1$, hence its inverse $C^{-1}(U;x)$ is well defined. In particular, by Propositions 1.2.2 and 1.2.3, we note that

$$\mathrm{Op}^{\mathscr{B}W}(C^{-1}(U;x))\circ\Phi_1 = \mathbb{1}+\tilde{Q}(U), \qquad \tilde{Q}\in\mathscr{R}^{-\rho}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C}), \tag{2.2.24}$$

for any $\rho>0$, since the expansion of $(C^{-1}(U;x)\sharp C(U;x))_\rho$ (see formula (1.2.11)) is equal to $C^{-1}(U;x)C(U;x)$ for any ρ. This implies that

$$(\Phi_1(U))^{-1}[\cdot] = \mathrm{Op}^{\mathscr{B}W}(C^{-1}(U;x))[\cdot] + Q_2(U)[\cdot], \tag{2.2.25}$$

for some $Q_2(U)$ in the class $\mathscr{R}^{-\rho}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$ for any $\rho>0$. By setting $V_1 := \Phi_1(U)[V]$ the system (2.2.1) in the new coordinates reads

$$\begin{aligned} (V_1)_t &= \Phi_1(U)\Big(\mathrm{i}E(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi)))\Phi_1^{-1}(U)\Big)V_1 + (\partial_t\Phi_1(U))\Phi_1^{-1}(U)V_1 + \\ &\quad + \Phi_1(U)(\mathrm{i}E)R_1^{(0)}(U)\Phi_1^{-1}(U)[V_1] + \Phi_1(U)(\mathrm{i}E)R_2^{(0)}(U)[U] \\ &= \mathrm{i}\Phi_1(U)\Big[E\mathfrak{P}[\Phi_1^{-1}(U)[V_1]]\Big] + \mathrm{i}\Phi_1(U)E\mathrm{Op}^{\mathscr{B}W}\big((\mathbb{1}+A_2(U;x))(\mathrm{i}\xi)^2\big)\Phi_1^{-1}(U)[V_1] + \\ &\quad + \mathrm{i}\Phi_1(U)E\mathrm{Op}^{\mathscr{B}W}\big(A_1(U;x)(\mathrm{i}\xi)\big)\Phi_1^{-1}(U)[V_1] + (\partial_t\Phi_1)\Phi_1^{-1}(U)V_1 + \\ &\quad + \Phi_1(U)(\mathrm{i}E)R_1^{(0)}(U)\Phi_1^{-1}(U)[V_1] + \Phi_1(U)(\mathrm{i}E)R_2^{(0)}(U)[U], \end{aligned} \tag{2.2.26}$$

where $\mathfrak{P}$ is defined in (0.4.7). We have that

$$\Phi_1(U)\circ E = E\circ\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} c_1(U;x) & -c_2(U;x) \\ -\overline{c_2(U;x)} & c_1(U;x)\end{pmatrix},$$

up to remainders in $\mathscr{R}^{-\rho}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$, where $c_i(U;x)$, $i=1,2$, are defined in (2.2.23). Since the matrix $C(U;x)-\mathbb{1}\in\mathscr{F}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$ (see (2.2.22)) then by Lemma 2.2.1 one has that

$$\Phi_1(U)\circ E\mathfrak{P}\circ\Phi_1^{-1}(U)[V_1]]=E\mathfrak{P}[V_1]+Q_3(U)[V_1],$$

where $Q_3(U)$ belongs to $\mathscr{R}^0_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$. The term $(\partial_t\Phi_1)$ is $\mathrm{Op}^{\mathscr{B}W}(\partial_t C(U;x))$ plus a remainder in the class $\mathscr{R}^0_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$. Note that, since $(C(U;x)-\mathbb{1})$ belongs to the class $\Gamma^0_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$, one has that $\partial_t C(U;x)$ is in $\Gamma^0_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$. Therefore, by the composition Propositions 1.2.2 and 1.2.3, Remark 1.2.5, and using the discussion above we have that, there exist operators $R_1^{(1)},R_2^{(1)}$ belonging to $\mathscr{R}^0_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$ such that

$$\begin{aligned}(V_1)_t=\,&\mathrm{i}E\mathfrak{P}V_1+\mathrm{i}\mathrm{Op}^{\mathscr{B}W}\big(C(U;x)E(\mathbb{1}+A_2(U;x))C^{-1}(U;x)(\mathrm{i}\xi)^2)\big)V_1\\&+\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(A_1^{(1)}(U;x)(\mathrm{i}\xi))V_1+\\&+\mathrm{i}E\Big(R_1^{(1)}(U)[V_1]+R_2^{(1)}(U)[U]\Big)\end{aligned}\tag{2.2.27}$$

where

$$\begin{aligned}A_1^{(1)}(U;x):=\,&EC(U;x)E(\mathbb{1}+A_2(U;x))\partial_x C^{-1}(U;x)\\&-(\partial_x(C)(U;x))E(\mathbb{1}+A_2(U;x))C^{-1}(U;x)\\&+EC(U;x)A_1(U;x)C^{-1}(U;x),\end{aligned}\tag{2.2.28}$$

with $A_1(U;x),A_2(U;x)$ defined in (2.2.11). Our aim is to find a symbol $z(U;x)$ such that the matrix of symbols $C(U;x)E(\mathbb{1}+A_2(U;x))C^{-1}(U;x)$ is diagonal. We reason as follows. One can note that the eigenvalues of $E(\mathbb{1}+A_2(U;x))$ are

$$\lambda^{\pm}:=\pm\sqrt{(1+a_2(U;x))^2-|b_2(U;x)|^2}.$$

We define the symbols

$$\begin{aligned}&\lambda_2^{(1)}(U;x):=\lambda^+,\\&a_2^{(1)}(U;x):=\lambda_2^{(1)}(U;x)-1\in\mathscr{F}_{K,0}[r].\end{aligned}\tag{2.2.29}$$

The symbol $\lambda_2^{(1)}(U;x)$ is well defined and satisfies (2.2.15) thanks to Hypothesis 2.2.3. The matrix of the normalised eigenvectors associated to the eigenvalues of

$E(\mathbb{1}+A_2(U;x))$ is

$$
\begin{aligned}
S(U;x) &:= \begin{pmatrix} s_1(U;x) & s_2(U;x) \\ \overline{s_2(U;x)} & s_1(U;x) \end{pmatrix}, \\
s_1(U;x) &:= \frac{1+a_2(U;x)+\lambda_2^{(1)}(U;x)}{\sqrt{2\lambda_2^{(1)}(U;x)\big(1+a_2(U;x)+\lambda_2^{(1)}(U;x)\big)}}, \\
s_2(U;x) &:= \frac{-b_2(U;x)}{\sqrt{2\lambda_2^{(1)}(U;x)\big(1+a_2(U;x)+\lambda_2^{(1)}(U;x)\big)}}.
\end{aligned} \tag{2.2.30}
$$

Note that $1+a_2(U;x)+\lambda_2^{(1)}(U;x) \geq \mathtt{c}_1+\sqrt{\mathtt{c}_2} > 0$ by (2.2.4). Therefore one can check that $S(U;x)-\mathbb{1} \in \mathscr{F}_{K,0}[r] \otimes \mathscr{M}_2(\mathbb{C})$. Therefore the matrix S is invertible and one has

$$
S^{-1}(U;x)\big[E(\mathbb{1}+A_2(U;x))\big]S(U;x) = E\begin{pmatrix} 1+a_2^{(1)}(U;x) & 0 \\ 0 & 1+a_2^{(1)}(U;x) \end{pmatrix}. \tag{2.2.31}
$$

We choose $z(U;x)$ in such a way that $C^{-1}(U;x) := S(U;x)$. Therefore we have to solve the following equations

$$
\cosh(|z(U;x)|) = s_1(U;x), \qquad \frac{z(U;x)}{|z(U;x)|}\sinh(|z(U;x)|) = -s_2(U;x). \tag{2.2.32}
$$

Concerning the first one we note that s_1 satisfies

$$
(s_1(U;x))^2 - 1 = \frac{|b_2(U;x)|^2}{2\lambda_2^{(1)}(U;x)(1+a_2(U;x)+\lambda_2^{(1)}(U;x))} \geq 0,
$$

indeed we remind that $1+a_2(U;x)+\lambda_2^{(1)}(U;x) \geq \mathtt{c}_1+\sqrt{\mathtt{c}_2} > 0$ by (2.2.4), therefore

$$
|z(U;x)| := \operatorname{arccosh}(s_1(U;x)) = \ln\Big(s_1(U;x)+\sqrt{(s_1(U;x))^2-1}\Big),
$$

is well-defined. For the second equation one observes that the function

$$
\frac{\sinh(|z(U;x)|)}{|z(U;x)|} = 1 + \sum_{k\geq 0} \frac{(z(U;x)\bar{z}(U;x))^k}{(2k+1)!} \geq 1,
$$

hence we set

$$
z(U;x) := s_2(U;x)\frac{|z(U;x)|}{\sinh(|z(U;x)|)}. \tag{2.2.33}
$$

We set

$$A^{(1)}(U;x,\xi) := A_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;x)(\mathrm{i}\xi),$$
$$A_2^{(1)}(U;x) := \begin{pmatrix} a_2^{(1)}(U;x) & 0 \\ 0 & a_2^{(1)}(U;x) \end{pmatrix} \tag{2.2.34}$$

where $a_2^{(1)}(U;x)$ is defined in (2.2.29) and $A_1^{(1)}(U;x)$ is defined in (2.2.28). Equation (2.2.31), together with (2.2.27) and (2.2.34) implies that (2.2.14) holds. By construction one has that the matrix $A^{(1)}(U;x,\xi)$ satisfies Constraint 2.2.4. It remains to show that $A^{(1)}(U;x,\xi)$ satisfies either Hyp. 2.2.1 or Hyp 2.2.2.
If $A(U;x,\xi)$ satisfies Hyp. 2.2.2 then we have that $a_2^{(1)}(U;x)$ in (2.2.29) is real. Moreover by construction $S(U;x)$ in (2.2.30) is even in x, therefore by Remark 1.3.5 we have that the map $\Phi_1(U)$ in (2.2.21) is parity preserving according to Definition 1.3.3. This implies that the matrix $A^{(1)}(U;x,\xi)$ satisfies Hyp. 2.2.2. Let us consider the case when $A(U;x,\xi)$ satisfies Hyp. 2.2.1. One can check, by an explicit computation, that the map $\Phi_1(U)$ in (2.2.21), is such that

$$\Phi_1^*(U)(-\mathrm{i}E)\Phi_1(U) = (-\mathrm{i}E) + \tilde{R}(U), \tag{2.2.35}$$

for some smoothing operators $\tilde{R}(U)$ belonging to $\mathscr{R}^{-\rho}_{K,0}[r]\otimes\mathscr{M}_2(\mathbb{C})$. In other words, up to a $\rho-$smoothing operator, the map $\Phi_1(U)$ satisfies conditions (1.3.35). By following essentially word by word the proof of Lemma 1.3.8 one obtains that, up to a smoothing operator in the class $\mathscr{R}^{-\rho}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$, the operator $\mathrm{Op}^{\mathscr{B}W}(A^{(1)}(U;x,\xi))$ in (2.2.14) is self-adjoint. This implies that the matrix $A^{(1)}(U;x,\xi)$ satisfies Hyp. 2.2.1. This concludes the proof. □

2.2.2 Diagonalization of the first order operator

In the previous Section we conjugated system (2.2.1) to (2.2.14), where the matrix $A^{(1)}(U;x,\xi)$ has the form

$$A^{(1)}(U;x,\xi) = A_2^{(1)}(U;x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;x)(\mathrm{i}\xi), \tag{2.2.36}$$

with $A_i^{(1)}(U;x)$ belonging to $\mathscr{F}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$ and where $A_2^{(1)}(U;x)$ is diagonal. In this Section we show that, since the matrices $A_i^{(1)}(U;x)$ satisfy Hyp. 2.2.1 (respectively Hyp. 2.2.2), it is possible to diagonalize also the term $A_1^{(1)}(U;x)$ through a change of coordinates which is the identity plus a smoothing term. This is the result of the following lemma.

Lemma 2.2.3. *If the matrix* $A^{(1)}(U;x,\xi)$ *in* (2.2.14) *satisfies Hypothesis 2.2.1 (resp. together with P satisfy Hyp. 2.2.2) then there exists* $s_0>0$ *(possibly larger than the one in Lemma 2.2.2) such that for any* $s\geq s_0$ *there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_2=\Phi_2(U):C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s)\to C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s),$$

with

$$\|(\Phi_2(U))^{\pm1}V\|_{K-1,s}\leq\|V\|_{K-1,s}(1+C\|U\|_{K,s_0}) \tag{2.2.37}$$

where $C>0$ *depends only on* s *and* $\|U\|_{K,s_0}$ *such that the following holds. There exists a matrix* $A^{(2)}(U;x,\xi)$ *satisfying Constraint 2.2.4 and Hyp. 2.2.1 (resp. Hyp. 2.2.2) of the form*

$$\begin{aligned}
A^{(2)}(U;x,\xi)&:=A_2^{(2)}(U;x)(\mathrm{i}\xi)^2+A_1^{(2)}(U;x)(\mathrm{i}\xi),\\
A_2^{(2)}(U;x)&:=A_2^{(1)}(U;x);\\
A_1^{(2)}(U;x)&:=\begin{pmatrix}a_1^{(2)}(U;x) & 0\\ 0 & \overline{a_1^{(2)}(U;x)}\end{pmatrix}\in\mathcal{F}_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C}),
\end{aligned}\tag{2.2.38}$$

and operators $R_1^{(2)}(U),R_2^{(2)}(U)$ *in* $\mathcal{R}^0_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$*, such that by setting* $V_2=\Phi_2(U)V_1$ *the system* (2.2.14) *reads*

$$\partial_t V_2=\mathrm{i}E\Big[\Lambda V_2+\mathrm{Op}^{\mathcal{B}W}(A^{(2)}(U;x,\xi))[V_2]+R_1^{(2)}(U)[V_2]+R_2^{(2)}(U)[U]\Big]. \tag{2.2.39}$$

Proof. We recall that by Lemma 2.2.2 we have that

$$A^{(1)}(U;x,\xi):=\begin{pmatrix}a^{(1)}(U;x,\xi) & b^{(1)}(U;x,\xi)\\ \overline{b^{(1)}(U;x,-\xi)} & \overline{a^{(1)}(U;x,-\xi)}\end{pmatrix}.$$

Moreover by (2.2.13) we can write

$$\begin{aligned}
a^{(1)}(U;x,\xi)&=a_2^{(1)}(U;x)(\mathrm{i}\xi)^2+a_1^{(1)}(U;x)(\mathrm{i}\xi),\\
b^{(1)}(U;x,\xi)&=b_1^{(1)}(U;x)(\mathrm{i}\xi),
\end{aligned}$$

with $a_2^{(1)}(U;x),a_1^{(1)}(U;x),b_1^{(1)}(U;x)\in\mathcal{F}_{K,1}[r]$. In the case that $A^{(1)}(U;x,\xi)$ satisfies Hyp. 2.2.1, we can note that $b_1(U;x)\equiv0$. Hence it is enough to choose $\Phi_2(U)\equiv\mathbb{1}$ to obtain the thesis. On the other hand, assume that $A^{(1)}(U;x,\xi)$ satisfies Hyp. 2.2.2 we reason as follows.

Let us consider a symbol $d(U;x,\xi)$ in the class $\Gamma^{-1}_{K,1}[r]$ and define

$$D(U;x,\xi) := \begin{pmatrix} 0 & d(U;x,\xi) \\ \overline{d(U;x,-\xi)} & 0 \end{pmatrix} \in \Gamma^{-1}_{K,1}[r]\otimes \mathcal{M}_2(\mathbb{C}). \tag{2.2.40}$$

Let $\Phi_2^{\tau}(U)[\cdot]$ be the flow of the system

$$\begin{cases} \partial_{\tau}\Phi_2^{\tau}(U) = \mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi))\Phi_2^{\tau}(U) \\ \Phi_2^{0}(U) = \mathbb{1}. \end{cases} \tag{2.2.41}$$

Reasoning as done for the system (2.2.17) one has that there exists a unique family of invertible bounded operators on $\mathbf{H}^s$ satisfying with

$$\|(\Phi_2^{\tau}(U))^{\pm 1}V\|_{K-1,s} \leq \|V\|_{K-1,s}(1+C\|U\|_{K,s_0}) \tag{2.2.42}$$

for $C>0$ depending on s and $\|U\|_{K,s_0}$ for $\tau\in[0,1]$.
The operator $W^{\tau}(U)[\cdot] := \Phi_2^{\tau}(U)[\cdot] - (\mathbb{1}+\tau\mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi)))$ solves the following system:

$$\begin{cases} \partial_{\tau}W^{\tau}(U) = \mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi))W^{\tau}(U) + \tau\mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi))\circ\mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi)) \\ W^{0}(U) = 0. \end{cases} \tag{2.2.43}$$

Therefore, by Duhamel formula, one can check that $W^{\tau}(U)$ is a smoothing operator in the class $\mathscr{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ for any $\tau\in[0,1]$. We set $\Phi_2(U)[\cdot] := \Phi_2^{\tau}(U)[\cdot]_{|_{\tau=1}}$, by the discussion above we have that there exists $Q(U)$ in $\mathscr{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that

$$\Phi_2(U)[\cdot] = \mathbb{1} + \mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi)) + Q(U).$$

Since $\Phi_2^{-1}(U)$ exists, by symbolic calculus, it is easy to check that there exists $\tilde{Q}(U)$ in $\mathscr{R}^{-2}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ such that

$$\Phi_2^{-1}(U)[\cdot] = \mathbb{1} - \mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi)) + \tilde{Q}(U).$$

We set $V_2 := \Phi_2(U)[V_1]$, therefore the system (2.2.14) in the new coordinates reads

$$\begin{aligned} (V_2)_t = {} & \Phi_2(U)\mathrm{i}E\Big(\Lambda + \mathrm{Op}^{\mathscr{B}W}(A^{(1)}(U;x,\xi)) + R_1^{(1)}(U)\Big)(\Phi_2(U))^{-1}[V_2] + \\ & + \Phi_2(U)\mathrm{i}ER_2^{(1)}(U)[U] + \mathrm{Op}^{\mathscr{B}W}(\partial_t\Phi_2(U))(\Phi_2(U))^{-1}[V_2]. \end{aligned} \tag{2.2.44}$$

The summand $\Phi_2(U)\mathrm{i}ER_2^{(1)}(U)[\cdot]$ belongs to the class $\mathscr{R}^{0}_{K,1}[r]\otimes\mathcal{M}_2(\mathbb{C})$ by composition Propositions. Since $\partial_t D(U;x,\xi)$ belongs to $\Gamma^{-1}_{K,2}[r]\otimes\mathcal{M}_2(\mathbb{C})$ and $\partial_t Q$ is in

$\mathscr{R}^{-2}_{K,2}[r]\otimes\mathscr{M}_2(\mathbb{C})$ then the last summand in (2.2.44) belongs to $\mathscr{R}^{0}_{K,2}[r]\otimes\mathscr{M}_2(\mathbb{C})$. We now study the first summand. First we note that $\Phi_2(U)\mathrm{i}ER_1^{(1)}(U)\Phi_2^{-1}(U)$ is a bounded remainder in $\mathscr{R}^{0}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$. It remains to study the term

$$
\begin{aligned}
&\mathrm{i}\Phi_2(U)\Big[E\mathfrak{P}\big(\Phi_2^{-1}(U)[V_2]\big)\Big]\\
&+\mathrm{i}\Phi_2(U)\Big[\mathrm{Op}^{\mathscr{B}W}\big(E(\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2+EA_1^{(1)}(U;x)(\mathrm{i}\xi)\big)\Big]\Phi_2^{-1}(U)[V_2],
\end{aligned}
$$

where $\mathfrak{P}$ is defined in (0.4.7). The first term is equal to $\mathrm{i}E(\mathfrak{P}V_2)$ up to a bounded term in $\mathscr{R}^{0}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$ by Lemma 2.2.1. The second is equal to

$$
\begin{aligned}
&\mathrm{i}\mathrm{Op}^{\mathscr{B}W}\big(E(\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2+EA_1^{(1)}(U;x)(\mathrm{i}\xi)\big)+\\
&+\Big[\mathrm{Op}^{\mathscr{B}W}(D(U;x,\xi)),\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\big((\mathbb{1}+A_2^{(1)}(U;x))(\mathrm{i}\xi)^2\big)\Big]
\end{aligned}
\tag{2.2.45}
$$

modulo bounded terms in $\mathscr{R}^{0}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$. By using formula (1.1.56) one get that the commutator above is equal to $\mathrm{Op}^{\mathscr{B}W}(M(U;x,\xi))$ with

$$
\begin{aligned}
M(U;x,\xi)&:=\begin{pmatrix}0 & m(U;x,\xi)\\ \overline{m(U;x,-\xi)} & 0\end{pmatrix},\\
m(U;x,\xi)&:=-2d(U;x,\xi)(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2,
\end{aligned}
\tag{2.2.46}
$$

up to terms in $\mathscr{R}^{0}_{K,1}[r]\otimes\mathscr{M}_2(\mathbb{C})$. Therefore the system obtained after the change of coordinates reads

$$
(V_2)_t=\mathrm{i}E\Big[\Lambda V_2+\mathrm{Op}^{\mathscr{B}W}(A^{(2)}(U;x,\xi))[V_2]+Q_1(U)[V_2]+Q_2(U)[U]\Big],\tag{2.2.47}
$$

where $Q_1(U)$ and $Q_2(U)$ are bounded terms in $\mathscr{R}^{0}_{K,2}[r]\otimes\mathscr{M}_2(\mathbb{C})$ and the new matrix $A^{(2)}(U;x,\xi))$ is

$$
\begin{pmatrix}a_2^{(1)}(U;x) & 0\\ 0 & \overline{a_2^{(1)}(U;x)}\end{pmatrix}(\mathrm{i}\xi)^2+\begin{pmatrix}a_1^{(1)}(U;x) & b_1^{(1)}(U;x)\\ \overline{b_1^{(1)}(U;x)} & \overline{a_1^{(1)}(U;x)}\end{pmatrix}(\mathrm{i}\xi)+M(U;x,\xi).\tag{2.2.48}
$$

Hence the elements on the diagonal are not affected by the change of coordinates, now our aim is to choose $d(U;x,\xi)$ in such a way that the symbol

$$
b_1(U;x)(\mathrm{i}\xi)+m(U;x,\xi)=b_1(U;x)(\mathrm{i}\xi)-2d(U;x,\xi)(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2,\tag{2.2.49}
$$

belongs to $\Gamma^{0}_{K,2}[r]$. We split the symbol in (2.2.49) in low-high frequencies: let $\varphi(\xi)$ a function in $C_0^{\infty}(\mathbb{R};\mathbb{R})$ such that $\mathrm{supp}(\varphi)\subset[-1,1]$ and $\varphi\equiv 1$ on $[-1/2,1/2]$.

Trivially one has that $\varphi(\xi)(b_1(U;x)(\mathrm{i}\xi)+m(U;x,\xi))$ is a symbol in $\Gamma^0_{K,1}[r]$, so it is enough to solve the equation

$$\big(1-\varphi(\xi)\big)\Big[b_1(U;x)(\mathrm{i}\xi)-2d(U;x,\xi)\Big(1+a_2^{(1)}(U;x))(\mathrm{i}\xi)^2\Big)\Big]=0. \tag{2.2.50}$$

So we should choose the symbol d as

$$\begin{aligned} d(U;x,\xi) &= \left(\frac{b_1^{(1)}(U;x)}{2(1+a_2^{(1)}(U;x))}\right)\cdot\gamma(\xi) \\ \gamma(\xi) &= \begin{cases} \dfrac{1}{\mathrm{i}\xi} & \text{if } |\xi|\geq\dfrac{1}{2} \\ \text{odd continuation of class } \mathrm{C}^{\infty} & \text{if } |\xi|\in[0,\dfrac{1}{2}). \end{cases} \end{aligned} \tag{2.2.51}$$

Clearly the symbol $d(U;x,\xi)$ in (2.2.51) belongs to $\Gamma^{-1}_{K,1}[r]$, hence the map $\Phi_2(U)$ in (2.2.40) is well defined and estimate (2.2.37) holds. It is evident that, after the choice of the symbol in (2.2.51), the matrix $A^{(2)}(U;x,\xi)$ is

$$\begin{pmatrix} a_2^{(1)}(U;x) & 0 \\ 0 & \overline{a_2^{(1)}(U;x)} \end{pmatrix}(\mathrm{i}\xi)^2+\begin{pmatrix} a_1^{(1)}(U;x) & 0 \\ 0 & \overline{a_1^{(1)}(U;x).} \end{pmatrix}(\mathrm{i}\xi) \tag{2.2.52}$$

The symbol $d(U;x,\xi)$ is equal to $d(U;-x,-\xi)$ because $b_1^{(1)}(U;x)$ is odd in x and $a_2^{(1)}(U;x)$ is even in x, therefore, by Remark 1.3.5 the map $\Phi_2(U)$ is *parity preserving*. □

2.2.3 Reduction to constant coefficients 1: paracomposition

Consider the diagonal matrix of functions $A_2^{(2)}(U;x)\in\mathscr{F}_{K,2}[r]\otimes\mathscr{M}_2(\mathbb{C})$ defined in (2.2.38). In this section we shall reduce the operator $\mathrm{Op}^{\mathscr{B}W}(A_2^{(2)}(U;x)(\mathrm{i}\xi)^2)$ to a constant coefficient one, up to bounded terms (see (2.2.57)).
We now study how the convolution operator $P*$ changes under the flow $\Omega_{B(U)}(\tau)$ introduced in Lemma 1.1.6.

Lemma 2.2.4. *Let $P:\mathbb{T}\to\mathbb{R}$ be a C^1 function, let us define $P_*[h]=P*h$ for $h\in H^s$, where $*$ denote the convolution between functions, and set $\Phi(U)[\cdot]:=\Omega_{B(U)}(\tau)_{|\tau=1}$. There exists R belonging to $\mathscr{R}^0_{K,K'}[r]$ such that*

$$\Phi(U)\circ P_*\circ\Phi^{-1}(U)[\cdot]=P_*[\cdot]+R(U)[\cdot]. \tag{2.2.53}$$

Moreover if $P(x)$ is even in x and $\Phi(U)$ is parity preserving according to Definition 1.3.3 then the remainder $R(U)$ in (2.2.53) *is parity preserving.*

Proof. Using equation (1.1.85) and estimate (1.2.8) one has that, for $0 \le k \le K - K'$, the following holds true

$$\|\partial_t^k\big(\Phi^{\pm 1}(U) - \mathrm{Id}\big)h\|_{H^{s-1-2k}} \le \sum_{k_1+k_2=k} C\|U\|_{K'+k_1,s_0}\|h\|_{k_2,s} \tag{2.2.54}$$

where $C > 0$ depends only on $\|U\|_{K,s_0}$ and Id is the identity map on H^s. Therefore we can write

$$\begin{aligned}&\Phi(U)\Big[P * \big[\Phi^{-1}(U)h\big]\Big]\\ &= P * h + \Big((\Phi(U) - \mathrm{Id})(P * h)\Big) + \Phi\Big[P * \Big((\Phi^{-1}(U) - \mathrm{Id})h\Big)\Big].\end{aligned} \tag{2.2.55}$$

Using estimate (2.2.54) and the fact that the function P is of class $C^1(\mathbb{T})$ we can estimate the last two summands in the r.h.s. of (2.2.55) as follows

$$\begin{aligned}&\Big\|\partial_t^k(\Phi(U) - \mathrm{Id})(P * h)\Big\|_{H^{s-2k}}\\ &\le \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|P * h\|_{k_2,s+1} \le \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|h\|_{k_2,s}\end{aligned}$$

and

$$\begin{aligned}&\Big\|\partial_t^k\Big(\Phi(U)\left[P * \big((\Phi^{-1}(U) - \mathrm{Id})h\big)\right]\Big)\Big\|_{s-2k}\\ &\le \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\Big\|(\Phi^{-1}(U) - \mathrm{Id})h\Big\|_{k_2,s-1}\\ &\le \sum_{k_1+k_2=k} C\,\|U\|_{K'+k_1,s_0}\,\|h\|_{k_2,s}\,,\end{aligned}$$

for $0 \le k \le K - K'$ and where C is a constant depending on $\|P\|_{C^1}$ and $\|U\|_{K,s_0}$. Hence they belong to the class $\mathscr{R}^0_{K,K'}[r]$. Finally if $P(x)$ is even in x then the operator P_* is parity preserving according to Definition 1.3.3, therefore if in addiction $\Phi(U)$ is parity preserving so must be $R(U)$ in (2.2.53). □

We are now in position to prove the following.

Lemma 2.2.5. *If the matrix $A^{(2)}(U;x,\xi)$ in* (2.2.39) *satisfies Hyp. 2.2.1 (resp. together with P satisfy Hyp. 2.2.2) then there exists $s_0 > 0$ (possibly larger than the one in Lemma 2.2.3) such that for any $s \ge s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_3 = \Phi_3(U) : C^{K-2}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T},\mathbb{C}^2)) \to C^{K-2}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T},\mathbb{C}^2)),$$

with

$$\|(\Phi_3(U))^{\pm 1} V\|_{K-2,s} \leq \|V\|_{K-2,s}(1 + C\|U\|_{K,s_0}) \tag{2.2.56}$$

where $C > 0$ depends only on s and $\|U\|_{K,s_0}$ such that the following holds. There exists a matrix $A^{(3)}(U;x,\xi)$ satisfying Constraint 2.2.4 and Hyp. 2.2.1 (resp. Hyp. 2.2.2) of the form

$$\begin{aligned}
A^{(3)}(U;x,\xi) &:= A_2^{(3)}(U)(\mathrm{i}\xi)^2 + A_1^{(3)}(U;x)(\mathrm{i}\xi), \\
A_2^{(3)}(U) &:= \begin{pmatrix} a_2^{(3)}(U) & 0 \\ 0 & a_2^{(3)}(U) \end{pmatrix}, \quad a_2^{(3)} \in \mathscr{F}_{K,3}[r], \quad \text{independent of } x \in \mathbb{T}, \\
A_1^{(3)}(U;x) &:= \begin{pmatrix} a_1^{(3)}(U;x) & 0 \\ 0 & \overline{a_1^{(3)}(U;x)} \end{pmatrix} \in \mathscr{F}_{K,3}[r] \otimes \mathscr{M}_2(\mathbb{C}),
\end{aligned} \tag{2.2.57}$$

and operators $R_1^{(3)}(U), R_2^{(3)}(U)$ in $\mathscr{R}^0_{K,3}[r] \otimes \mathscr{M}_2(\mathbb{C})$, such that by setting $V_3 = \Phi_3(U)V_2$ the system (2.2.39) *reads*

$$\partial_t V_3 = \mathrm{i}E\Big[\Lambda V_3 + \mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;x,\xi))[V_3] + R_1^{(3)}(U)[V_3] + R_2^{(3)}(U)[U]\Big]. \tag{2.2.58}$$

Proof. Let $\beta(U;x)$ be a real symbol in $\mathscr{F}_{K,2}[r]$ to be chosen later such that condition (1.2.25) holds. Set moreover $\gamma(U;x)$ the symbol such that (1.2.27) holds. Consider accordingly to the hypotheses of Lemma 1.1.6 the system

$$\dot{W} = \mathrm{i}EMW, \quad W(0) = \mathbb{1}, \quad M := \mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} B(\tau,x,\xi) & 0 \\ 0 & \overline{B(\tau,x,-\xi)} \end{pmatrix}, \tag{2.2.59}$$

where B is defined in (1.1.84). Note that $\overline{B(\tau,x,-\xi)} = -B(\tau,x,\xi)$. By Lemma 1.1.6 the flow exists and is bounded on $\mathbf{H}^s(\mathbb{T},\mathbb{C}^2)$ and moreover (2.2.56) holds. We want to conjugate the system (2.2.39) through the map $\Phi_3(U)[\cdot] = W(1)[\cdot]$. Set $V_3 = \Phi_3(U)V_2$. The system in the new coordinates reads

$$\begin{aligned}
\frac{d}{dt}V_3 = \Phi_3(U)\Big[&\mathrm{i}E(\mathfrak{P}[\Phi_3^{-1}(U)V_3)] \\
&+ (\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3] \\
&+ \Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}((\mathbb{1} + A_2^{(2)}(U;x))(\mathrm{i}\xi)^2)\Big]\Phi_3^{-1}(U)[V_3] \\
&+ \Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(A_1^{(2)}(U;x)(\mathrm{i}\xi))\Big]\Phi_3^{-1}(U)[V_3] \\
&+ \Phi_3(U)\Big[\mathrm{i}ER_1^{(2)}(U)\Big]\Phi_3^{-1}(U)[V_3] + \Phi_3(U)\mathrm{i}ER_2^{(2)}(U)[U],
\end{aligned} \tag{2.2.60}$$

where $\mathfrak{P}$ is defined in (0.4.7). We now discuss each term in (2.2.60). The first one, by Lemma 2.2.4, is equal to $\mathrm{i}E(\mathfrak{P}V_3)$ up to a bounded remainder in the class $\mathscr{R}^0_{K,2}[r]\otimes\mathscr{M}_2(\mathbb{C})$. The last two terms also belongs to the latter class because the map Φ_3 is a bounded operator on $\mathbf{H}^s$. For the term $(\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3]$ we apply Lemma 1.2.6 and we obtain that

$$(\partial_t\Phi_3(U))\Phi_3^{-1}(U)[V_3]=\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} e(U;x)(\mathrm{i}\xi) & 0\\ 0 & \overline{e(U;x)(\mathrm{i}\xi)}\end{pmatrix}[V_3]+\tilde{R}(U)[V_3], \tag{2.2.61}$$

where $\tilde{R}$ belongs to $\mathscr{R}^{-1}_{K,3}[r]\otimes\mathscr{M}_2(\mathbb{C})$ and $e(U;x)$ is a symbol in $\mathscr{F}_{K,3}[r]\otimes\mathscr{M}_2(\mathbb{C})$ such that $\mathrm{Re}(e(U;x))=0$. It remains to study the conjugate of the paradifferential terms in (2.2.60). We note that

$$\begin{aligned}&\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}((\mathbb{1}+A_2^{(2)}(U;x))(\mathrm{i}\xi)^2)\Big]\Phi_3^{-1}(U)[V_3]\\ &\quad+\Phi_3(U)\Big[\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(A_1^{(2)}(U;x)(\mathrm{i}\xi))\Big]\Phi_3^{-1}(U)[V_3]\\ &\quad=\begin{pmatrix} T & 0\\ 0 & \overline{T}\end{pmatrix}\end{aligned}$$

where T is the operator

$$T=\Omega_{B(U)}(1)\mathrm{Op}^{\mathscr{B}W}\Big((1+a_2^{(2)}(U;x))(\mathrm{i}\xi)^2+a_1^{(2)}(U;x)(\mathrm{i}\xi)\Big)\Omega^{-1}_{B(U)}(1). \tag{2.2.62}$$

The Theorem 1.2.3 guarantees that

$$T=\mathrm{Op}^{\mathscr{B}W}(\tilde{a}_2^{(3)}(U;x,\xi)+a_1^{(3)}(U;x)(\mathrm{i}\xi))[\cdot] \tag{2.2.63}$$

up to a bounded term in $\mathscr{R}^0_{K,3}[r]$ and where

$$\begin{aligned}\tilde{a}_2^{(3)}(U;x,\xi)&=\big(1+a_2^{(2)}(U;y)\big)\big(1+\gamma_y(1,y)\big)^2_{|_{y=x+\beta(x)}}(\mathrm{i}\xi)^2,\\ a_1^{(3)}(U;x)&=a_1^{(2)}(U;y)\big(1+\gamma_y(1,y)\big)_{|_{y=x+\beta(x)}}.\end{aligned} \tag{2.2.64}$$

Here $\gamma(1,x)=\gamma(\tau,x)_{|_{\tau=1}}=\gamma(U;\tau,x)_{|_{\tau=1}}$ with

$$y=x+\tau\beta(U;x)\ \Leftrightarrow\ x=y+\gamma(\tau,y),\ \ \tau\in[0,1],$$

where $x+\tau\beta(U;x)$ is the path of diffeomorphism given by Remark 1.2.7.
By Lemma 1.2.4 one has that the new symbols $\tilde{a}_2^{(3)}(U;x,\xi)$, $a_1^{(3)}(U;x)$ defined in (2.2.64) belong to the class $\Gamma^2_{K,3}[r]$ and $\mathscr{F}_{K,3}[r]$ respectively. At this point we want

to choose the symbol $\beta(x)$ in such a way that $\tilde{a}_2^{(3)}(U;x,\xi)$ does not depend on x. One can proceed as follows. Let $a_2^{(3)}(U)$ a x-independent function to be chosen later, one would like to solve the equation

$$\big(1+a_2^{(2)}(U;y)\big)\big(1+\gamma_y(1,y)\big)^2_{|_{y=x+\beta(x)}}(\mathrm{i}\xi)^2=(1+a_2^{(3)}(U))(\mathrm{i}\xi)^2. \tag{2.2.65}$$

The solution of this equation is given by

$$\gamma(U;1,y)=\partial_y^{-1}\left(\sqrt{\frac{1+a_2^{(3)}(U)}{1+a_2^{(2)}(U;y)}}-1\right). \tag{2.2.66}$$

In principle this solution is just formal because the operator ∂_y^{-1} is defined only for function with zero mean, therefore we have to choose $a_2^{(3)}(U)$ in such a way that

$$\int_{\mathbb{T}}\left(\sqrt{\frac{1+a_2^{(3)}(U)}{1+a_2^{(2)}(U;y)}}-1\right)dx=0, \tag{2.2.67}$$

which means

$$1+a_2^{(3)}(U):=\left[2\pi\left(\int_{\mathbb{T}}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}dy\right)^{-1}\right]^2. \tag{2.2.68}$$

Note that everything is well defined thanks to the positivity of $1+a_2^{(2)}$. Indeed $a_2^{(2)}=a_2^{(1)}$ by (2.2.38), and $a_2^{(1)}$ satisfies (2.2.15). Indeed every denominator in (2.2.66), (2.2.67) and in (2.2.68) stays far away from 0. Note that $\gamma(U;y)$ belongs to $\mathscr{F}_{K,2}[r]$ and so does $\beta(U;x)$ by Lemma 1.2.3. By using (1.2.27) one can deduce that

$$1+\beta_x(U;x)=\frac{1}{1+\gamma_y(U;1,y)} \tag{2.2.69}$$

where

$$1+\gamma_y(U;1,y)=2\pi\left(\int_{\mathbb{T}}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}dy\right)^{-1}\frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}, \tag{2.2.70}$$

thanks to (2.2.66) and (2.2.68). Since the matrix $A_2^{(2)}$ satisfies Hypothesis 2.2.3 one has that there exists a universal constant $\mathtt{c} > 0$ such that $1 + a_2^{(2)}(U; y) \geq \mathtt{c}$. Therefore one has

$$\begin{aligned} 1 + \beta_x(U;x) = \frac{1}{1+\gamma_y(U;1,y)} &\geq \sqrt{\mathtt{c}}\frac{1}{2\pi}\int_{\mathbb{T}} \frac{1}{\sqrt{1+a_2^{(2)}(U;y)}}dy \\ &\geq \frac{1}{2\pi}\frac{\sqrt{\mathtt{c}}}{1+C\|U\|_{0,s_0}} := \Theta > 0, \end{aligned}$$

for some C depending only on $\|U\|_{K,s_0}$, where we used the fact that $a_2^{(2)}(U; y)$ belongs to the class $\mathscr{F}_{K,2}[r]$ (see Def. 1.2.2). This implies that $\beta(U;x)$ satisfies condition (1.2.25). We have written system (2.2.39) in the form (2.2.58) with matrices defined in (2.2.57).
It remains to show that the new matrix $A^{(3)}(U;x,\xi)$ satisfies either Hyp. 2.2.1 or 2.2.2. If $A^{(2)}(U;x,\xi)$ is selfadjoint, i.e. satisfies Hypothesis 2.2.1, then one has that the matrix $A^{(3)}(U;x,\xi)$ is selfadjoint as well thanks to the fact that the map $W(1)$ satisfies the hypotheses (condition (1.3.35)) of of Lemma 1.3.8, by using Lemma 1.3.7. In the case that $A^{(2)}(U;x,\xi)$ is parity preserving, i.e. satisfies Hypothesis 2.2.2, then $A^{(3)}(U;\xi)$ has the same properties for the following reasons. The symbols $\beta(U;x)$ and $\gamma(U;x)$ are odd in x if the function U is even in x. Hence the flow map $W(1)$ defined by equation (2.2.59) is parity preserving. Moreover the matrix $A^{(3)}(U;x,\xi)$ satisfies Hypothesis 2.2.2 by explicit computation. □

2.2.4 Reduction to constant coefficients 2: first order terms

Lemmata 2.2.2, 2.2.3, 2.2.5 guarantee that one can conjugate the system (2.2.1) to the system (2.2.58) in which the matrix $A^{(3)}(U;x,\xi)$ (see (2.2.57)) has the form

$$A^{(3)}(U;x,\xi) = A_2^{(3)}(U)(\mathrm{i}\xi)^2 + A_1^{(3)}(U;x)(\mathrm{i}\xi), \tag{2.2.71}$$

where the matrices $A_2^{(3)}(U), A_1^{(3)}(U;x)$ are diagonal and belong to $\mathscr{F}_{K,3}[r] \otimes \mathscr{M}_2(\mathbb{C})$, for $i = 1,2$. Moreover $A_2^{(3)}(U)$ does not depend on $x \in \mathbb{T}$. In this Section we show how to eliminate the x dependence of the symbol $A_1^{(3)}(U;x)$ in (2.2.57). We prove the following.

Lemma 2.2.6. *If the matrix* $A^{(3)}(U;x,\xi)$ *in* (2.2.58) *satisfies Hyp. 2.2.1 (resp. together with P satisfy Hyp. 2.2.2) then there exists* $s_0 > 0$ *(possibly larger than* *the one in Lemma 2.2.5) such that for any $s \geq s_0$ there exists an invertible map (resp. an invertible and parity preserving map)*

$$\Phi_4 = \Phi_4(U) : C^{K-3}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2)) \to C^{K-3}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T}, \mathbb{C}^2)),$$

with

$$\|(\Phi_4(U))^{\pm 1} V\|_{K-3,s} \leq \|V\|_{K-3,s}(1 + C\|U\|_{K,s_0}) \tag{2.2.72}$$

where $C > 0$ depends only on s and $\|U\|_{K,s_0}$ such that the following holds. Then there exists a matrix $A^{(4)}(U;\xi)$ independent of $x \in \mathbb{T}$ of the form

$$A^{(4)}(U;\xi) := \begin{pmatrix} a_2^{(3)}(U) & 0 \\ 0 & \overline{a_2^{(3)}(U)} \end{pmatrix} (\mathrm{i}\xi)^2 + \begin{pmatrix} a_1^{(4)}(U) & 0 \\ 0 & \overline{a_1^{(4)}(U)} \end{pmatrix} (\mathrm{i}\xi), \tag{2.2.73}$$

where $a_2^{(3)}(U)$ is defined in (2.2.57) *and $a_1^{(4)}(U)$ is a symbol in $\mathscr{F}_{K,4}[r]$, independent of x, which is purely imaginary in the case of Hyp. 2.2.1 (resp. is equal to* 0*). There are operators $R_1^{(4)}(U), R_2^{(4)}(U)$ in $\mathscr{R}^0_{K,4}[r] \otimes \mathscr{M}_2(\mathbb{C})$, such that by setting $V_4 = \Phi_4(U)V_3$ the system* (2.2.58) *reads*

$$\partial_t V_4 = \mathrm{i}E\Big[\Lambda V_4 + \mathrm{Op}^{\mathscr{B}W}(A^{(4)}(U;\xi))[V_4] + R_1^{(4)}(U)[V_4] + R_2^{(4)}(U)[U]\Big]. \tag{2.2.74}$$

Proof. Consider a symbol $s(U;x)$ in the class $\mathscr{F}_{K,3}[r]$ and define

$$S(U;x) := \begin{pmatrix} s(U;x) & 0 \\ 0 & \overline{s(U;x)} \end{pmatrix}.$$

Let $\Phi_4^\tau(U)[\cdot]$ be the flow of the system

$$\begin{cases} \partial_\tau \Phi_4^\tau(U)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(S(U;x))\Phi_4^\tau(U)[\cdot] \\ \Phi_4^0(U)[\cdot] = \mathbb{1}. \end{cases} \tag{2.2.75}$$

Again one can reason as done for the system (2.2.17) to check that there exists a unique family of invertible bounded operators on $\mathbf{H}^s$ satisfying

$$\|(\Phi_4^\tau(U))^{\pm 1} V\|_{K-3,s} \leq \|V\|_{K-3,s}(1 + C\|U\|_{K,s_0}) \tag{2.2.76}$$

for $C > 0$ depending on s and $\|U\|_{K,s_0}$ for $\tau \in [0,1]$. We set

$$\Phi_4(U)[\cdot] = \Phi_4^\tau(U)[\cdot]_{|_{\tau=1}} = \exp\{\mathrm{Op}^{\mathscr{B}W}(S(U;x))\}. \tag{2.2.77}$$

By Corollary 1.2.2 we get that there exists $Q(U)$ in the class of smoothing remainder $\mathscr{R}^{-\rho}_{K,3}[r]\otimes\mathscr{M}_2(\mathbb{C})$ for any $\rho>0$ such that

$$\Phi_4(U)[\cdot] := \mathrm{Op}^{\mathscr{B}W}(\exp\{S(U;x)\})[\cdot] + Q(U)[\cdot]. \tag{2.2.78}$$

Since $\Phi_4^{-1}(U)$ exists, by symbolic calculus, it is easy to check that there exists $\tilde{Q}(U)$ in $\mathscr{R}^{-\rho}_{K,3}[r]\otimes\mathscr{M}_2(\mathbb{C})$ such that

$$\Phi_4^{-1}(U)[\cdot] = \mathrm{Op}^{\mathscr{B}W}(\exp\{-S(U;x)\})[\cdot] + \tilde{Q}(U)[\cdot].$$

We set $G(U;x)=\exp\{S(U;x)\}$ and $V_4=\Phi_4(U)V_3$. Then the system (2.2.58) becomes

$$\begin{aligned}(V_4)_t = {}& \Phi_4(U)\mathrm{i}E\Big(\Lambda + \mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;x,\xi)) + R_1^{(3)}(U)\Big)(\Phi_4(U))^{-1}[V_4]+\\ &+\Phi_4(U)\mathrm{i}ER_2^{(3)}(U)[U] + \mathrm{Op}^{\mathscr{B}W}(\partial_t G(U;x,\xi))(\Phi_4(U))^{-1}[V_4].\end{aligned} \tag{2.2.79}$$

Recalling that $\Lambda = \mathfrak{P} + \frac{d^2}{dx^2}$ (see (0.4.8)) we note that by Lemma 2.2.1 the term $\mathrm{i}\Phi_4(U)\big[E\mathfrak{P}\big(\Phi_4^{-1}(U)[V_4]\big]$ is equal to $\mathrm{i}E\mathfrak{P}V_4$ up to a remainder in $\mathscr{R}^0_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$. Secondly we note that the operator

$$\begin{aligned}\hat{Q}(U)[\cdot] := {}& \Phi_4(U)\mathrm{i}ER_1^{(3)}(U)\Phi_4^{-1}(U)[\cdot]\\ &+\Phi_4(U)\mathrm{i}ER_2^{(3)}(U)[U] + \mathrm{Op}^{\mathscr{B}W}(\partial_t G(U;x))\circ\Phi_4^{-1}(U)[\cdot]\end{aligned} \tag{2.2.80}$$

belongs to the class of operators $\mathscr{R}^0_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$. This follows by applying Propositions 1.2.2, 1.2.3, Remark 1.2.5 and the fact that $\partial_t G(U;x)$ is a matrix in $\mathscr{F}_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$. It remains to study the term

$$\Phi_4(U)\mathrm{i}E\Big(\mathrm{Op}^{\mathscr{B}W}\big((\mathbb{1}+A_2^{(3)}(U))(\mathrm{i}\xi)^2\big) + \mathrm{Op}^{\mathscr{B}W}(A_1^{(3)}(U;x)(\mathrm{i}\xi))\Big)(\Phi_4(U))^{-1}. \tag{2.2.81}$$

By using formula (1.1.56) and Remark 1.2.5 one gets that, up to remainder in $\mathscr{R}^0_{K,4}[r]\otimes\mathscr{M}_2(\mathbb{C})$, the term in (2.2.81) is equal to

$$\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\big((\mathbb{1}+A_2^{(3)}(U))(\mathrm{i}\xi)^2\big) + \mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} r(U;x)(\mathrm{i}\xi) & 0\\ 0 & \overline{r(U;x)}(\mathrm{i}\xi)\end{pmatrix} \tag{2.2.82}$$

where

$$r(U;x) := a_1^{(3)}(U;x) + 2(1+a_2^{(3)}(U))\partial_x s(U;x). \tag{2.2.83}$$

We look for a symbol $s(U;x)$ such that, the term of order one has constant coefficient in x. This requires to solve the equation

$$a_1^{(3)}(U;x) + 2(1 + a_2^{(3)}(U))\partial_x s(U;x) = a_1^{(4)}(U), \tag{2.2.84}$$

for some symbol $a_1^{(4)}(U)$ constant in x to be chosen. Equation (2.2.84) is equivalent to

$$\partial_x s(U;x) = \frac{-a_1^{(3)}(U;x) + a_1^{(4)}(U)}{2(1 + a_2^{(3)}(U))}. \tag{2.2.85}$$

We choose the constant $a_1^{(4)}(U)$ as

$$a_1^{(4)}(U) := \frac{1}{2\pi}\int_{\mathbb{T}} a_1^{(3)}(U;x)dx, \tag{2.2.86}$$

so that the r.h.s. of (2.2.85) has zero average, hence the solution of (2.2.85) is given by

$$s(U;x) := \partial_x^{-1}\left(\frac{-a_1^{(3)}(U;x) + a_1^{(4)}(U)}{2(1 + a_2^{(3)}(U))}\right). \tag{2.2.87}$$

It is easy to check that $s(U;x)$ belongs to $\mathscr{F}_{K,4}[r]$. Using equation (2.2.83) we get (2.2.74) with $A^{(4)}(U;\xi)$ as in (2.2.73).
It remains to prove that the constant $a_1^{(4)}(U)$ in (2.2.86) is purely imaginary. On one hand, if $A^{(3)}(U;x,\xi)$ satisfies Hyp. 2.2.1, we note the following. The coefficient $a_1^{(3)}(U;x)$ must be purely imaginary hence the constant $a_1^{(4)}(U)$ in (2.2.86) is purely imaginary.
On the other hand, if $A^{(3)}(U;x,\xi)$ satisfies Hyp. 2.2.2, we note that the function $a_1^{(3)}(U;x)$ is *odd* in x. This means that the constants $a_1^{(4)}(U)$ in (2.2.86) is zero. Moreover the symbol $s(U;x)$ in (2.2.87) is even in x, hence the map $\Phi_4(U)$ in (2.2.75) is parity preserving according to Def. 1.3.3 thanks to Remark 1.3.5. This concludes the proof. □

Proof of Theorem 2.2.1. It is enough to choose $\Phi(U) := \Phi_4(U)\circ\cdots\circ\Phi_1(U)$. The estimates (2.2.5) follow by collecting the bounds (2.2.12), (2.2.37), (2.2.56) and (2.2.72). We define the matrix of symbols $L(U;\xi)$ as

$$L(U;\xi) := \begin{pmatrix} \mathtt{m}(U,\xi) & 0 \\ 0 & \mathtt{m}(U,-\xi) \end{pmatrix}, \quad \mathtt{m}(U,\xi) := a_2^{(3)}(U)(\mathrm{i}\xi)^2 + a_1^{(4)}(U)(\mathrm{i}\xi) \tag{2.2.88}$$

where the coefficients $a_2^{(3)}(U), a_1^{(4)}(U)$ are x-independent (see (2.2.73)). One concludes the proof by setting $R_1(U) := R_1^{(4)}(U)$ and $R_2(U) := R_2^{(4)}(U)$. □

An important consequence of Theorem 2.2.1 is that system (2.2.1) admits a regular and unique solution. More precisely we have the following.

Proposition 2.2.1. *Let s_0 given by Theorem 2.2.1 with $K = 4$. For any $s \geq s_0 + 2$ let $U = U(t,x)$ be a function in $B^4_s([0,T),\theta)$ for some $T > 0$, $r > 0$, $\theta \geq r$ with $\|U(0,\cdot)\|_{\mathbf{H}^s} \leq r$ and consider the system*

$$\begin{cases} \partial_t V = \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[V] + R_1^{(0)}(U)[V] + R_2^{(0)}(U)[U]\Big], \\ V(0,x) = U(0,x) \in \mathbf{H}^s, \end{cases} \tag{2.2.89}$$

where the matrix $A(U;x,\xi)$, the operators $R_1^{(0)}(U)$ and $R_2^{(0)}(U)$ satisfy the hypotheses of Theorem 2.2.1. Then the following holds true.

- (i) *There exists a unique solution $\psi_U(t)U(0,x)$ of the system* (2.2.89) *defined for any $t \in [0,T)$ such that*

$$\begin{aligned} &\left\|\psi_U(t)U(0,x)\right\|_{4,s} \leq \\ &\quad \mathtt{C}\,\|U(0,x)\|_{\mathbf{H}^s}(1 + t\mathtt{C}\,\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}} + t\mathtt{C}\,\|U\|_{4,s}\,e^{t\mathtt{C}\|U\|_{4,s}} + \mathtt{C}, \end{aligned} \tag{2.2.90}$$

  where $\mathtt{C}$ is constant depending on s, r, $\sup_{t\in[0,T]}\|U\|_{4,s-2}$ and $\|P\|_{C^1}$.

- (ii) *In the case that U is even in x, the matrix $A(U;x,\xi)$ and the operator Λ satisfy Hyp. 2.2.2, the operator $R_1^{(0)}(U)[\cdot]$ is parity preserving according to Def. 1.3.3 and $R_2^{(0)}(U)[U]$ is even in x, then the solution $\Psi_U(t)U(0,x)$ is even in $x \in \mathbb{T}$.*

Proof. We apply to system (2.2.89) Theorem 2.2.1 defining $W = \Phi(U)V$. The system in the new coordinates reads

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda W + \mathrm{Op}^{\mathscr{B}W}(L(U;\xi))W + R_1(U)W + R_2(U)[U]\Big] = 0 \\ W(0,x) = \Phi(U(0,x))U(0,x) := W^{(0)}(x), \end{cases} \tag{2.2.91}$$

where $L(U;\xi)$ is a diagonal, self-adjoint and constant coefficient in x matrix in $\Gamma^2_{4,4}[r] \otimes \mathscr{M}_2(\mathbb{C})$, $R_1(U), R_2(U)$ are in $\mathscr{R}^0_{4,4}[r] \otimes \mathscr{M}_2(\mathbb{C})$. Therefore the solution of the linear problem

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda W + \mathrm{Op}^{\mathscr{B}W}(L(U;\xi))\Big]W = 0 \\ W(0,x) = W^{(0)}(x), \end{cases} \tag{2.2.92}$$

is well defined as long as U is well defined, moreover it is an isometry of $\mathbf{H}^s$. We denote by ψ_L^t the flow at time t of such equation. Then one can define the operator

$$\begin{aligned} T_{W^{(0)}}(W)(t,x) &= \psi_L^t\big(W^{(0)}(x)\big) \\ &+ \psi_L^t \int_0^t (\psi_L^s)^{-1} \mathrm{i} E\Big(R_1(U(s,x))W(s,x) + R_2(U(s,x))U(s,x)\Big) ds. \end{aligned} \tag{2.2.93}$$

Thanks to (2.2.5) and by the hypothesis on $U(0,x)$ one has that $\|W^{(0)}\|_{\mathbf{H}^s} \leq (1 + cr)r$ for some constant $c > 0$ depending only on r. In order to construct a fixed point for the operator $T_{W^{(0)}}(W)$ in (2.2.93) we consider the sequence of approximations defined as follows:

$$\begin{cases} W_0(t,x) = \psi_L^t W^{(0)}(x), \\ W_n(t,x) = T_{W^{(0)}}(W_{n-1})(t,x), \qquad n \geq 1, \end{cases}$$

for $t \in [0,T)$. For the rest of the proof we will denote by $\mathtt{C}$ any constant depending on r, s, $\sup_{t\in[0,T)} \|U(t,\cdot)\|_{4,s-2}$ and $\|P\|_{C^1}$. Using estimates (1.2.7) one gets for $n \geq 1$

$$\|(W_{n+1} - W_n)(t,\cdot)\|_{\mathbf{H}^s} \leq \mathtt{C}\,\|U(t,\cdot)\|_{\mathbf{H}^s} \int_0^t \|(W_n - W_{n-1})(\tau,\cdot)\|_{\mathbf{H}^s} d\tau.$$

Arguing by induction over n, one deduces

$$\|(W_{n+1} - W_n)(t,\cdot)\|_{\mathbf{H}^s} \leq \frac{(\mathtt{C}\,\|U(t,\cdot)\|_{\mathbf{H}^s})^n t^n}{n!} \|(W_1 - W_0)(t,\cdot)\|_{\mathbf{H}^s}, \tag{2.2.94}$$

which implies that $W(t,x) = \sum_{n=1}^{\infty}(W_{n+1} - W_n)(t,x) + W_0(t,x)$ is a fixed point of the operator in (2.2.93) belonging to the space $C^0_{*\mathbb{R}}([0,T);\mathbf{H}^s(\mathbb{T};\mathbb{C}^2))$. Therefore by Duhamel principle the function W is the unique solution of the problem (2.2.91). Moreover, by using (1.2.7), we have that the following inequality holds true

$$\|W_1(t,\cdot) - W_0(t,\cdot)\|_{\mathbf{H}^s} \leq t\mathtt{C}\big(\|U\|_{\mathbf{H}^s} \|W^{(0)}\|_{\mathbf{H}^s} + \|U\|_{\mathbf{H}^{s-2}} \|U\|_{\mathbf{H}^s}\big),$$

from which, together with estimates (2.2.94), one deduces that

$$\begin{aligned} \|W(t,\cdot)\|_{\mathbf{H}^s} &\leq \sum_{n=0}^{\infty} \|(W_{n+1} - W_n)(t,\cdot)\|_{\mathbf{H}^s} + \|W^{(0)}\|_{\mathbf{H}^s} \\ &\leq \|W^{(0)}\|_{\mathbf{H}^s}\Big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} \sum_{n=0}^{\infty} \frac{(t\mathtt{C}\,\|U\|_{\mathbf{H}^s})^n}{n!}\Big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} \sum_{n=0}^{\infty} \frac{(t\mathtt{C}\,\|U\|_{\mathbf{H}^s})^n}{n!} \\ &= \|W^{(0)}\|_{\mathbf{H}^s}\big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}} \\ &\leq \mathtt{C}\,\|U^{(0)}\|_{\mathbf{H}^s}\big(1 + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\big) + t\mathtt{C}\,\|U\|_{\mathbf{H}^s} e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}} \end{aligned}$$

Applying the inverse transformation $V = \Phi^{-1}(U)W$ and using (2.2.5) we find a solution V of the problem (2.2.89) such that

$$\|V\|_{\mathbf{H}^s} \leq \mathtt{C}\left\|U^{(0)}\right\|_{\mathbf{H}^s}\left(1+t\mathtt{C}\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}\|U\|_{\mathbf{H}^s}}\right)+t\mathtt{C}\|U\|_{\mathbf{H}^s}e^{t\mathtt{C}.\|U\|_{\mathbf{H}^s}}$$

We now prove a similar estimate for $\partial_t V$. More precisely one has

$$\begin{aligned}\|\partial_t V\|_{\mathbf{H}^{s-2}} &\leq \|\Lambda V+\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))V\|_{\mathbf{H}^{s-2}}+\|R_1^{(0)}(U)V\|_{\mathbf{H}^{s-2}}+\|R_2^{(0)}(U)U\|_{\mathbf{H}^{s-2}}\\ &\leq \mathtt{C}\|V\|_{\mathbf{H}^s}+\mathtt{C}\|V\|_{\mathbf{H}^{s-2}}+\mathtt{C}\\ &\leq \mathtt{C}\|U(0,x)\|_{\mathbf{H}^s}(1+t\mathtt{C}\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}}+t\mathtt{C}\|U\|_{4,s}e^{t\mathtt{C}\|U\|_{4,s}}+\mathtt{C},\end{aligned} \tag{2.2.95}$$

where we used estimates (1.2.9) and (1.2.7). By differentiating equation (2.2.89) and arguing as done in (2.2.95) one can bound the terms $\|\partial_t^k V\|_{\mathbf{H}^{s-2k}}$, for $2 \leq k \leq 4$, and hence obtain the (2.2.90).
In the case that U is even in x, $\Lambda, A(U;x,\xi)$ satisfy Hyp. 2.2.2, $R_1(U)[\cdot]$ is parity preserving according to Def. 1.3.3 and $R_2^{(0)}(U)[U]$ is even in x we have, by Theorem 2.2.1, that the map $\Phi(U)$ is parity- preserving. Hence the flow of the system (2.2.91) preserves the subspace of even functions. This follows by Lemma 1.3.9. Hence the solution of (2.2.89) defined as $V = \Phi^{-1}(U)W$ is even in x. This concludes the proof. □

Remark 2.2.2. *In the notation of Prop. 2.2.1 the following holds true.*

- *If* $R_2^{(0)} \equiv 0$ *in* (2.2.89)*, then the estimate* (2.2.90) *may be improved as follows:*

$$\left\|\psi_U(t)U(0,x)\right\|_{4,s} \leq \mathtt{C}\|U(0,x)\|_{\mathbf{H}^s}(1+t\mathtt{C}\|U\|_{4,s})e^{t\mathtt{C}\|U\|_{4,s}}. \tag{2.2.96}$$

  This follows straightforward from the proof of Prop. 2.2.1.

- *If* $R_2^{(0)} \equiv R_1^{(0)} \equiv 0$ *then the flow* $\psi_U(t)$ *of* (2.2.89) *is invertible and* $(\psi_U(t))^{-1}U(0,x)$ *satisfies an estimate similar to* (2.2.96)*. To see this one proceed as follows. Let* $\Phi(U)[\cdot]$ *the map given by Theorem 2.2.1 and set* $\Gamma(t) := \Phi(U)\psi_U(t)$*. Thanks to Theorem 2.2.1,* $\Gamma(t)$ *is the flow of the linear para-differential equation*

$$\begin{cases}\partial_t\Gamma(t) = \mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(L(U;\xi))\Gamma(t)+R(U)\Gamma(t),\\ \Gamma(0)=\mathrm{Id},\end{cases}$$

  where $R(U)$ *is a remainder in* $\mathscr{R}^0_{K,4}[r]$ *and* $\mathrm{Op}^{\mathscr{B}W}(L(U;\xi))$ *is diagonal, self-adjoint and constant coefficients in* x*. Then, if* $\psi_L(t)$ *is the flow generated* *by* $\mathrm{i}\mathrm{Op}^{\mathscr{B}W}(L(U;\xi))$ *(which exists and is an isometry of* $\mathbf{H}^s$*), we have that* $\Gamma(t)=\psi_L(t)\circ F(t)$*, where* $F(t)$ *solves the Banach space ODE*

$$\begin{cases}\partial_t F(t) = \big((\psi_L(t))^{-1}R(U)\psi_L(t)\big)F(t),\\ F(0)=\mathrm{Id}.\end{cases}$$

To see this one has to use the fact that the operators $\mathrm{i}\mathrm{Op}^{\mathscr{B}W}(L(U;\xi))$ *and* $\psi_L(t)$ *commutes. Standard theory of Banach spaces ODE implies that* $F(t)$ *exists and is invertible, therefore* $\psi_U(t)$ *is invertible as well and* $(\psi_U(t))^{-1}=(F(t))^{-1}\circ(\psi_L(t))^{-1}\circ\Phi(U)$*. To deduce the estimate satisfied by* $(\psi_U(t))^{-1}$ *one has to use* (1.2.7) *to control the contribution coming from* $R(U)$*, the fact that* $\psi_L(t)$ *is an isometry and* (2.2.5).

2.3 Proofs of the local existence theorems

In this Section we prove Theorem 0.2.1. By previous discussions we know that (0.2.1) is equivalent to the system (2.1.15) (see Proposition 2.1.1). Our method relies on an iterative scheme. Namely we introduce the following sequence of linear problems. Let $U^{(0)}\in\mathbf{H}^s$ such that $\|U^{(0)}\|_{\mathbf{H}^s}\le r$ for some $r>0$. For $n=0$ we set

$$\mathscr{A}_0 := \begin{cases}\partial_t U_0 - \mathrm{i}E\Lambda U_0 = 0,\\ U_0(0)=U^{(0)}.\end{cases} \tag{2.3.1}$$

The solution of this problem exists and is unique, defined for any $t\in\mathbb{R}$ by standard linear theory, it is a group of isometries of $\mathbf{H}^s$ (its k-th derivative is a group of isometries of $\mathbf{H}^{s-2k}$) and hence satisfies $\|U_0\|_{4,s}\le r$ for any $t\in\mathbb{R}$.
For $n\ge 1$, assuming $U_{n-1}\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I,H^s(\mathbb{T},\mathbb{C}^2))$ for some $s_0,K>0$ and $s\ge s_0$, we define the Cauchy problem

$$\mathscr{A}_n := \begin{cases}\partial_t U_n = \mathrm{i}E\Big[\Lambda U_n + \mathrm{Op}^{\mathscr{B}W}(A(U_{n-1};x,\xi))U_n + R(U_{n-1})[U_{n-1}]\Big],\\ U_n(0)=U^{(0)},\end{cases} \tag{2.3.2}$$

where the matrix of symbols $A(U;x,\xi)$ and the operator $R(U)$ are defined in Proposition 2.1.1 (see (2.1.15)).
One has to show that each problem $\mathscr{A}_n$ admits a unique solution U_n defined for $t\in I$. We use Proposition 2.2.1 in order to prove the following lemma.

Lemma 2.3.1. *Let f be a C^∞ function from $\mathbb{C}^3$ in $\mathbb{C}$ satisfying Hyp. 0.2.1 (resp. Hyp. 0.2.2). Let $r > 0$ and consider $U^{(0)}$ in the ball of radius r of $\mathbf{H}^s$ (resp. of $\mathbf{H}^s_e$) centered at the origin. Consider the operators $\Lambda, R(U)$ and the matrix of symbols $A(U; x, \xi)$ given by Proposition 2.1.1 with $K = 4$, $\rho = 0$. If f satisfies Hyp. 0.2.3, or r is sufficiently small, then there exists $s_0 > 0$ such that for all $s \geq s_0$ the following holds. There exists a time T and a constant θ, both of them depending on r and s, such that for any $n \geq 0$ one has:*

$(S1)_n$ *for $0 \leq m \leq n$ there exists a function U_m in*

$$U_m \in B^4_s([0,T), \theta), \tag{2.3.3}$$

which is the unique solution of the problem $\mathscr{A}_m$; in the case of parity preserving Hypothesis 0.2.2 the functions U_m for $0 \leq m \leq n$ are even in $x \in \mathbb{T}$;

$(S2)_n$ *for $0 \leq m \leq n$ one has*

$$\|U_m - U_{m-1}\|_{4,s'} \leq 2^{-m} r, \quad s_0 \leq s' \leq s - 2, \tag{2.3.4}$$

where $U_{-1} := 0$.

Proof. We argue by induction. The $(S1)_0$ and $(S2)_0$ are true thanks to the discussion following the equation (2.3.1). Suppose that $(S1)_{n-1}$,$(S2)_{n-1}$ hold with a constant $\theta = \theta(s, r, \|P\|_{C^1}) \gg 1$ and a time $T = T(s, r, \|P\|_{C^1}, \theta) \ll 1$. We show that $(S1)_n$,$(S2)_n$ hold with the same constant θ and T.
The Hypothesis 0.2.1, together with Lemma 2.1.2 (resp. Hyp. 0.2.2 together with Lemma 2.1.3) implies that the matrix $A(U; x, \xi)$ satisfies Hyp. 2.2.1 (resp. Hyp. 2.2.2) and Constraint 2.2.4. The Hypothesis 0.2.3, together with Lemma 2.1.4, (or r small enough) implies that $A(U; x, \xi)$ satisfies also the Hypothesis 2.2.3. Therefore the hypotheses of Theorem 2.2.1 are fulfilled. In particular, in the case of Hyp. 0.2.2, Lemma 2.1.3 guarantees also that the matrix of operators $R(U)[\cdot]$ is parity preserving according to Def. 1.3.3.
Moreover by (2.3.3), we have that $\|U_{n-1}\|_{4,s} \leq \theta$, hence the hypotheses of Proposition 2.2.1 are fulfilled by system (2.3.2) with $R_1^{(0)} = 0$, $R_2^{(0)} = R$, $U \rightsquigarrow U_{n-1}$ and $V \rightsquigarrow U_n$ in (2.2.89). We note that, by $(S2)_{n-1}$, one has that the constant $\mathtt{C}$ in (2.2.90) does not depend on θ, but it depend only on $r > 0$. Indeed (2.3.4) implies

$$\|U_{n-1}\|_{4,s-2} \leq \sum_{m=0}^{n-1} \|U_m - U_{m-1}\|_{4,s-2} \leq r \sum_{m=0}^{n-1} \frac{1}{2^m} \leq 2r, \quad \forall\, t \in [0,T]. \tag{2.3.5}$$

Proposition 2.2.1 provides a solution $U_n(t)$ defined for $t \in [0,T]$. By (2.2.90) one has that

$$\|U_n(t)\|_{4,s} \leq \\ \mathtt{C}\left\|U^{(0)}\right\|_{\mathbf{H}^s}(1+t\mathtt{C}\|U_{n-1}\|_{4,s})e^{t\mathtt{C}\|U_{n-1}\|_{4,s}} + t\mathtt{C}\|U_{n-1}\|_{4,s}\,e^{t\mathtt{C}\|U_{n-1}\|_{4,s}} + \mathtt{C}, \tag{2.3.6}$$

where $\mathtt{C}$ is a constant depending on $\|U_{n-1}\|_{4,s-2}$, r, s and $\|P\|_{C^1}$, hence, thanks to (2.3.5), it depends only on r, s, $\|P\|_{C^1}$. We deduce that, if

$$T\mathtt{C}\theta \ll 1, \qquad \theta > \mathtt{C}r2e + e + \mathtt{C}, \tag{2.3.7}$$

then $\|U_n\|_{4,s} \leq \theta$. If $A(U_{n-1};x,\xi)$ and Λ satisfy Hyp. 2.2.2, $R(U_{n-1})$ is parity preserving then the solution U_n is even in $x \in \mathbb{T}$. Indeed by the inductive hypothesis U_{n-1} is even, hence item (ii) of Proposition 2.2.1 applies. This proves $(S1)_n$.
Let us check $(S2)_n$. Setting $V_n = U_n - U_{n-1}$ we have that

$$\begin{cases} \partial_t V_n - \mathrm{i}E\Big[\Lambda V_n + \mathrm{Op}^{\mathscr{B}W}(A(U_{n-1};x,\xi))V_n + f_n\Big] = 0, \\ V_n(0) = 0, \end{cases} \tag{2.3.8}$$

where

$$f_n := \mathrm{Op}^{\mathscr{B}W}\Big(A(U_{n-1};x,\xi) - A(U_{n-2};x,\xi)\Big)U_{n-1} + R(U_{n-1})U_{n-1} - R(U_{n-2})U_{n-2}. \tag{2.3.9}$$

Note that, by (2.1.28), (2.1.29), we have

$$\begin{aligned} \|f_n\|_{4,s'} &\leq \left\|\mathrm{Op}^{\mathscr{B}W}\Big(A(U_{n-1};x,\xi) - A(U_{n-2};x,\xi)\Big)U_{n-1}\right\|_{4,s'} \\ &\quad + \|R(U_{n-1})U_{n-1} - R(U_{n-2})U_{n-2}\|_{4,s'} \\ &\leq C\Big[\,\|V_{n-1}\|_{4,s_0}\|U_{n-1}\|_{4,s'+2} + (\|U_{n-1}\|_{4,s'} + \|U_{n-2}\|_{4,s'})\,\|V_{n-1}\|_{4,s'}\Big] \\ &\leq C\Big(\,\|U_{n-1}\|_{4,s'+2} + \|U_{n-2}\|_{4,s'+2}\Big)\|V_{n-1}\|_{4,s'}\,, \end{aligned} \tag{2.3.10}$$

where $C > 0$ depends only on s, $\|U_{n-1}\|_{4,s_0}, \|U_{n-2}\|_{4,s_0}$. Recalling the estimate (2.3.5) we can conclude that the constant C in (2.3.10) depends only on s, r.
The system (2.3.8) with $f_n = 0$ has the form (2.2.89) with $R_2^{(0)} = 0$ and $R_1^{(0)} = 0$. Let $\psi_{U_{n-1}}(t)$ be the flow of system (2.3.8) with $f_n = 0$, which is given by Proposition 2.2.1. The Duhamel formulation of (2.3.8) is

$$V_n(t) = \psi_{U_{n-1}}(t)\int_0^t (\psi_{U_{n-1}}(\tau))^{-1}\mathrm{i}Ef_n(\tau)d\tau. \tag{2.3.11}$$

Then using the inductive hypothesis (2.3.3), inequality (2.2.96) and the second item of Remark 2.2.2 we get

$$\|V_n\|_{4,s'} \leq \theta \mathtt{K}_1 T \|V_{n-1}\|_{4,s'}, \quad \forall\, t \in [0,T], \tag{2.3.12}$$

where $\mathtt{K}_1 > 0$ is a constant depending r, s and $\|P\|_{C^1}$. If $\mathtt{K}_1\theta T \leq 1/2$ then we have $\|V_n\|_{4,s'} \leq 2^{-n} r$ for any $t \in [0,T)$ which is the $(S2)_n$. □

We are now in position to prove Theorem 0.2.1.

Proof of Theorem 0.2.1. Consider the equation (0.2.1). By Lemma 2.1.1 we know that (0.2.1) is equivalent to the system (2.1.15). Since f satisfies Hyp. 0.2.1 (resp. Hyp 0.2.2) and Hyp. 0.2.3, then Lemmata 2.1.2 (resp 2.1.3) and 2.1.4 imply that the matrix $A(U;x,\xi)$ satisfies Constraint 2.2.4 and Hypothesis 2.2.1 (resp. Hypothesis 2.2.2 and $R(U)$ is parity preserving according to Definition 1.3.3). According to this setting consider the problem $\mathscr{A}_n$ in (2.3.2).
By Lemma 2.3.1 we know that the sequence U_n defined by (2.3.2) converges strongly to a function U in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'})$ for any $s' \leq s-2$ and, up to subsequences,

$$\begin{aligned} &U_n(t) \rightharpoonup U(t), \ \text{ in } \ \mathbf{H}^s, \\ &\partial_t U_n(t) \rightharpoonup \partial_t U(t), \ \text{ in } \ \mathbf{H}^{s-2}, \end{aligned} \tag{2.3.13}$$

for any $t \in [0,T)$, moreover the function U is in $L^\infty([0,T),\mathbf{H}^s) \cap \mathrm{Lip}([0,T),\mathbf{H}^{s-2})$. In order to prove that U solves (2.1.15) it is enough to show that

$$\left\| \mathrm{Op}^{\mathscr{B}W}(A(U_{n-1};x,\xi))]U_n + R(U_{n-1})[U_{n-1}] - \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))]U - R(U)[U] \right\|_{\mathbf{H}^{s-2}}$$

goes to 0 as n goes to ∞. Using (2.1.28) and (1.2.9) we obtain

$$\begin{aligned} &\|\mathrm{Op}^{\mathscr{B}W}(A(U_{n-1};x,\xi)U_n - \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi)U)\|_{\mathbf{H}^{s-2}} \leq \\ &\|\mathrm{Op}^{\mathscr{B}W}(A(U_{n-1};x,\xi) - A(U;x,\xi))U_n\|_{\mathbf{H}^{s-2}} + \|\mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))(U-U_n)\|_{\mathbf{H}^{s-2}} \leq \\ &C\Big(\|U-U_n\|_{\mathbf{H}^{s-2}} \|U\|_{\mathbf{H}^{s_0}} + \|U-U_{n-1}\|_{\mathbf{H}^{s_0}} \|U_n\|_{\mathbf{H}^s} \Big), \end{aligned}$$

which tends to 0 since $s-2 \geq s'$. In order to show that $R(U_{n-1})[U_{n-1}]$ tends to $R(U)[U]$ in $\mathbf{H}^{s-2}$ it is enough to use (2.1.29). Using the equation (2.1.15) and the discussion above the solution U has the following regularity:

$$\begin{aligned} U \in B^4_{s'}([0,T);\theta) \cap L^\infty([0,T),\mathbf{H}^s) \cap \mathrm{Lip}([0,T),\mathbf{H}^{s-2}), \quad \forall\, s_0 \leq s' \leq s-2, \\ \|U\|_{L^\infty([0,T),\mathbf{H}^s)} \leq \theta, \end{aligned} \tag{2.3.14}$$

where θ and s_0 are given by Lemma 2.3.1. We show that U actually belongs to $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$. Let us consider the problem

$$\begin{cases} \partial_t V - \mathrm{i}E\Big[\Lambda V + \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))V + R(U)[U]\Big] = 0, \\ V(0) = U^{(0)}, \quad U^{(0)} \in \mathbf{H}^s, \end{cases} \tag{2.3.15}$$

where the matrices A and R are defined in Proposition 2.1.1 (see (2.1.15)) and U is defined in (2.3.13) (hence satisfies (2.3.14)). Theorem 2.2.1 applies to system (2.3.15) and provides a map

$$\Phi(U)[\cdot] : C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'}(\mathbb{T},\mathbb{C}^2)) \to C^0_{*\mathbb{R}}([0,T),\mathbf{H}^{s'}(\mathbb{T},\mathbb{C}^2)), \tag{2.3.16}$$

which satisfies (2.2.5) with $K=4$ and s' as in (2.3.14). One has that the function $W := \Phi(U)[U]$ solves, the problem

$$\begin{cases} \partial_t W - \mathrm{i}E\Big[\Lambda + \mathrm{Op}^{\mathscr{B}W}(L(U;\xi))\Big]W + R_2(U)[U] + R_1(U)W = 0 \\ W(0) = \Phi(U^{(0)})U^{(0)} := W^{(0)}, \end{cases} \tag{2.3.17}$$

where $L(U)$ is a diagonal, self-adjoint and constant coefficient in x matrix of symbols in $\Gamma^2_{K,4}[\theta]$, and $R_1(U), R_2(U)$ are matrices of bounded operators (see eq. (2.2.6)). We prove that W is weakly-continuous in time with values in $\mathbf{H}^s$. First of all note that $U \in C^0([0,T);\mathbf{H}^{s'})$ with s' given in (2.3.14), therefore W belongs to the same space thanks to (2.3.16). Moreover W is in $L^\infty([0,T),\mathbf{H}^s)$ (again by (2.3.14) and (2.3.16)). Consider a sequence τ_n converging to τ as $n\to\infty$. Let $\phi \in \mathbf{H}^{-s}$ and $\phi_\varepsilon \in C^\infty_0(\mathbb{T};\mathbb{C}^2)$ such that $\|\phi-\phi_\varepsilon\|_{\mathbf{H}^{-s}} \le \varepsilon$. Then we have

$$\begin{aligned} &\Big|\int_{\mathbb{T}} \big(W(\tau_n) - W(\tau)\big)\phi dx\Big| \\ &\le \Big|\int_{\mathbb{T}} \big(W(\tau_n) - W(\tau)\big)\phi_\varepsilon dx\Big| + \Big|\int_{\mathbb{T}} (W(\tau_n) - W(\tau))(\phi-\phi_\varepsilon) dx\Big| \\ &\le \|W(\tau_n) - W(\tau)\|_{\mathbf{H}^{s'}} \|\phi_\varepsilon\|_{\mathbf{H}^{-s'}} + \|W(\tau_n) - W(\tau)\|_{\mathbf{H}^s}\|\phi-\phi_\varepsilon\|_{\mathbf{H}^{-s}} \\ &\le C\varepsilon + 2\|W\|_{L^\infty\mathbf{H}^s}\varepsilon \end{aligned} \tag{2.3.18}$$

for n sufficiently large and where $s' \le s-2$ as above.
Therefore W is weakly continuous in time with values in $\mathbf{H}^s$. In order to prove that W is in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$, we show that the map $t \mapsto \|W(t)\|_{\mathbf{H}^s}$ is continuous on $[0,T)$. We introduce, for $0<\epsilon\le 1$, the Friedrichs mollifier $J_\epsilon := (1-\epsilon\partial_{xx})^{-1}$ and

the Fourier multiplier $\Lambda^s := (1-\partial_{xx})^{s/2}$. Using the equation (2.3.17) and estimates (1.2.7) one gets

$$\frac{d}{dt}\left\|\Lambda^s J_\epsilon W(t)\right\|^2_{\mathbf{H}^0} \leq C\Big[\,\|U(t)\|^2_{\mathbf{H}^s}\|W(t)\|_{\mathbf{H}^s} + \|W(t)\|^2_{\mathbf{H}^s}\|U(t)\|_{\mathbf{H}^s}\Big], \tag{2.3.19}$$

where the right hand side is independent of ϵ and the constant C depends on s and $\|U\|_{\mathbf{H}^{s_0}}$. Moreover, since U, W belong to $L^\infty([0,T),\mathbf{H}^s)$, the right hand side of inequality (2.3.19) is bounded from above by a constant independent of t. Therefore the function $t \mapsto \|J_\epsilon W(t)\|_{\mathbf{H}^0}$ is Lipschitz continuous in t, uniformly in ϵ. As $J_\epsilon W(t)$ converges to $W(t)$ in the $\mathbf{H}^s$-norm, the function $t \mapsto \|W(t)\|_{\mathbf{H}^0}$ is Lipschitz continuous as well. Therefore W belongs to $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$ and so does U. To recover the regularity of $\frac{d}{dt}U$ one may use equation (2.1.15).
Let us show the uniqueness. Suppose that there are two solution U and V in $C^0_{*\mathbb{R}}([0,T),\mathbf{H}^s)$ of the problem (2.1.15) . Set $H := U - V$, then H solves the problem

$$\begin{cases} \partial_t H - \mathrm{i}E\Big[\Lambda H + \mathrm{Op}^{\mathscr{B}W}(A(U;x,\xi))[H] + R(U)[H]\Big] + \mathrm{i}EF = 0 \\ H(0) = 0, \end{cases}$$

where

$$F := \mathrm{Op}^{\mathscr{B}W}\big(A(U;x,\xi) - A(V;x,\xi)\big)V + \big(R(U) - R(V)\big)[V].$$

Thanks to estimates (2.1.28) and (2.1.29) we have the bound

$$\|F\|_{\mathbf{H}^{s-2}} \leq C\|H\|_{\mathbf{H}^{s-2}}\Big(\|U\|_{\mathbf{H}^s} + \|V\|_{\mathbf{H}^s}\Big). \tag{2.3.20}$$

By Proposition 2.2.1, using Duhamel principle and (2.3.20), it is easy to show the following:

$$\|H(t)\|_{\mathbf{H}^{s-2}} \leq C(r)\int_0^t \|H(\sigma)\|_{\mathbf{H}^{s-2}}d\sigma.$$

Thus by Gronwall Lemma the solution is equal to zero for almost everywhere time t in $[0,T)$. By continuity one gets the unicity. □

Proof of Theorem 0.2.2. The proof is the same of the one of Theorem 0.2.1, one only has to note that the matrix $A(U;x,\xi)$ satisfies Hypothesis 2.2.3 thanks to the smallness of the initial datum instead of Hyp. 0.2.3. □

Chapter 3

Long time existence

In this chapter we give the proof of Theorem 0.3.1. This is the content of the paper [47]. We shall use classes of symbols and operators introduced in Section 1.1.

3.1 Paralinearization of NLS

The main result of this section is the following.

Theorem 3.1.1 (**Para-linearization of NLS).** *Consider the system* (0.4.9) *under the Hypothesis 0.3.1. For any $N \in \mathbb{N}$, $K \in \mathbb{N}$, $r > 0$ and any $\rho > 0$ there exists a (R,R,P)-matrix of symbols (see Def. 1.3.1) $A(U;t,x,\xi)$ belonging to the class $\Sigma\Gamma^{2}_{K,0,1}[r,N,\mathrm{aut}]\otimes\mathscr{M}_2(\mathbb{C})$ and a (R,R,P)-operator (see Def. 1.3.3) $R(U)[\cdot]$ belonging to $\Sigma\mathscr{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]\otimes\mathscr{M}_2(\mathbb{C})$ such that the system* (0.4.9) *can be written as*

$$\partial_t U = \mathrm{i}E\Big[\Lambda U + \mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))[U] + R(U)[U]\Big], \tag{3.1.1}$$

where E and Λ are defined respectively in (0.4.5) *and* (0.4.6)*. Moreover $A(U;t,x,\xi)$ has the form*

$$\begin{aligned} A(U;t,x,\xi) &:= A_2(U;t,x)(\mathrm{i}\xi)^2 + A_1(U;t,x)(\mathrm{i}\xi) + A_0(U;t,x),\\ A_i(U;t,x) &:= \begin{pmatrix} a_i(U;t,x) & b_i(U;t,x)\\ \overline{b_i(U;t,x)} & \overline{a_i(U;t,x)}\end{pmatrix}, \end{aligned} \tag{3.1.2}$$

where $a_j(U;t,x), b_j(U;t,x) \in \Sigma\mathscr{F}_{K,0,1}[r,N,\mathrm{aut}]$ for $j = 0,1,2$; $a_2(U;t,x)$ is real *for any $x \in \mathbb{T}$.*

Theorem 3.1.1 is a consequence of the para-product formula of Bony which we prove below in our multilinear setting.
For fixed $p \in \mathbb{N}$, $p \geq 2$ for any $u_1, \ldots, u_p$ in $C^{\infty}(\mathbb{T};\mathbb{C})$, define the map M as

$$M : (u_1, \ldots, u_p) \mapsto M(u_1, \ldots, u_p) := \prod_{i=1}^{p} u_i = \sum_{n_1, \ldots, n_p \in \mathbb{N}} \prod_{i=1}^{p} \Pi_{n_i} u_i. \tag{3.1.3}$$

Notice that we can also write

$$M(u_1, \ldots, u_p) = \sum_{n_0 \in \mathbb{N}} \sum_{n_1, \ldots, n_p \in \mathbb{N}} \Pi_{n_0} M(\Pi_{n_1} u_1, \ldots, \Pi_{n_p} u_p). \tag{3.1.4}$$

We remark that the term $\Pi_{n_0} M(\Pi_{n_1} u_1, \ldots, \Pi_{n_p} u_p)$ is different from zero only if there exists a choice of signs $\sigma_j \in \{\pm 1\}$ such that

$$\sum_{j=0}^{p} \sigma_j n_j = 0, \tag{3.1.5}$$

since the map M is just a product of functions.
Fix $0 < \delta < 1$ and consider an admissible cut-off function $\chi_{p-1} : \mathbb{R}^{p-1} \times \mathbb{R} \to \mathbb{R}$ (see Def. 1.1.14). We define a new cut-off function $\Theta : \mathbb{N}^p \to [0,1]$ in the following way: given any $\vec{n} := (n_1, \ldots, n_p) \in \mathbb{N}^p$ we set

$$\begin{aligned} &\Theta(n_1, \ldots, n_p) := 1 - \sum_{i=1}^{p} \chi_{p-1}^{(i)}(\vec{n}), \\ &\chi_{p-1}^{(i)}(\vec{n}) := \chi_{p-1}(\xi', n_i), \quad \xi' := (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_p). \end{aligned} \tag{3.1.6}$$

We use the following notation: for any $u_1, \ldots, u_p \in C^{\infty}(\mathbb{T};\mathbb{C})$ we shall write

$$(u_1, \ldots, \hat{u}_i, \ldots, u_p) = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_p), \quad i = 1, \ldots, p, \tag{3.1.7}$$

similarly for any $U_1, \ldots, U_p \in C^{\infty}(\mathbb{T};\mathbb{C}^2)$ we shall write

$$(U_1, \ldots, \hat{U}_i, \ldots, U_p) = (U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_p). \quad i = 1, \ldots, p. \tag{3.1.8}$$

Using the splitting in (3.1.6) we write

$$
\begin{aligned}
&M(u_1,\dots,u_p) = \sum_{i=1}^{p} M_i(u_1,\dots,u_p) + M^{\Theta}(u_1,\dots,u_p),\\
&M_i(u_1,\dots,u_p) := A^{(i)}(u_1,\dots,\hat{u}_i,\dots,u_p)[u_i] :=\\
&\qquad\qquad = \sum_{n_1,\dots,n_p\in\mathbb{N}} \chi^{(i)}_{p-1}(\vec{n})\Big(\prod_{\substack{j=1\\ j\neq i}}^{p} \Pi_{n_j} u_j\Big)\Pi_{n_i} u_i,\\
&M^{\Theta}(u_1,\dots,u_p) := A^{\Theta}(u_1,\dots,u_{p-1})[u_p] := \sum_{n_1,\dots,n_p\in\mathbb{N}} \Theta(n_1,\dots,n_p)\prod_{j=1}^{p}\Pi_{n_j} u_j.
\end{aligned}
\tag{3.1.9}
$$

In Lemma 3.1.1 we prove that the multilinear operator A^{Θ} in (3.1.9) is a smoothing remainder, in Lemma 3.1.2 we show that $A^{(i)}$ in (3.1.9) is a paradifferential operator acting on the function u_i for any $i = 1,\dots,p$.

Lemma 3.1.1 (**Remainders**)**.** *Let* A^{Θ} *be the operator defined in* (3.1.9)*. There is* Q *in* $\widetilde{\mathscr{R}}^{-\rho}_{p-1}$*, for any* $\rho \geq 0$*, such that for any* $U_i \in C^{\infty}(\mathbb{T};\mathbb{C}^2)$*,* $U_p \in C^{\infty}(\mathbb{T};\mathbb{C})$*, for* $i = 1,\dots,p-1$*, we have*

$$
Q(U_1,\dots,U_{p-1})[U_p] \equiv A^{\Theta}(u_1,\dots,u_{p-1})[U_p], \tag{3.1.10}
$$

where $U_i = (u_i, z_i)^T$*,* $z_i \in C^{\infty}(\mathbb{T};\mathbb{C})$*, for* $i = 1,\dots,p-1$*.*

Proof. Let $U_i \in C^{\infty}(\mathbb{T};\mathbb{C}^2)$ be of the form $U_i = (u_i, z_i)^T$ for $i = 1,\dots,p-1$, consider also $U_p \in C^{\infty}(\mathbb{T};\mathbb{C})$. In order to obtain (3.1.10) it is enough to choose

$$
Q(U_1,\dots,U_{p-1})[U_p] = \sum_{n_1,\dots,n_p\in\mathbb{N}} \Theta(n_1,\dots,n_p)\Big[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\prod_{j=1}^{p-1}\Pi_{n_j} u_j\\ \prod_{j=1}^{p-1}\Pi_{n_j} z_j\end{pmatrix}\Big]\Pi_{n_p} U_p.
\tag{3.1.11}
$$

The "autonomous" condition in (1.1.7) for Q follows from the following fact: from (3.1.9) we have

$$
\Pi_{n_0} Q(\Pi_{n_1} U_1,\dots,\Pi_{n_{p-1}} U_{p-1})[\Pi_{n_p} U_p] = \Pi_{n_0} M^{\Theta}(\Pi_{n_1} u_1,\dots,\Pi_{n_p} u_p),
$$

which is different from zero only if (3.1.5) holds true for a suitable choice of signs $\sigma_j \in \{\pm 1\}$. In order to prove (1.1.6) we need estimate, for any $n_0 \in \mathbb{N}$, the term

$$
\begin{aligned}
&\Big\|\Pi_{n_0} Q(\Pi_{n_1} U_1,\dots,\Pi_{n_{p-1}} U_{p-1})[\Pi_{n_p} U_p]\Big\|_{L^2}\\
&\quad = \Big\|\Pi_{n_0} A^{\Theta}(\Pi_{n_1} u_1,\dots,\Pi_{n_{p-1}} u_{p-1})[\Pi_{n_p} U_p]\Big\|_{L^2}
\end{aligned}
\tag{3.1.12}
$$

for $\sum_{j=1}^{p}\sigma_j n_j = 0$ for some choice of signs $\sigma_j \in \{\pm 1\}$. We note that, if there exists $i = 1,\dots,p$ such that $\chi^{(i)}_{p-1} \equiv 1$, i.e. $\sum_{j\neq i} n_j \leq (\delta/2)n_i$, then we have $\Theta(n_1,\dots,n_p) \equiv 0$. Hence we have the following inclusion:

$$\big\{(n_1,\dots,n_p)\in\mathbb{N}^p\,:\,\Theta(n_1,\dots,n_p)\neq 0\big\}\subseteq\bigcap_{i=1}^{p}\left\{(n_1,\dots,n_p)\in\mathbb{N}^p\,:\,\frac{\delta}{2}n_i<\sum_{j\neq i}n_j\right\}. \tag{3.1.13}$$

This implies that there exists constants $0 < c \leq C$ such that

$$c\max\{\langle n_1\rangle,\dots,\langle n_p\rangle\}\leq \max\nolimits_2\{\langle n_1\rangle,\dots,\langle n_p\rangle\}\leq \max\{\langle n_1\rangle,\dots,\langle n_p\rangle\}. \tag{3.1.14}$$

There exists a constant $K > 0$, depending on p, such that we can bound (3.1.12) by

$$K\prod_{j=1}^{p}\|\Pi_{n_j}u_j\|_{L^2}\overset{(3.1.14)}{\leq}K\frac{\max_2\{\langle n_1\rangle,\dots,\langle n_p\rangle\}^{\mu+\rho}}{\max\{\langle n_1\rangle,\dots,\langle n_p\rangle\}^{\rho}}\prod_{j=1}^{p}\|\Pi_{n_j}u_j\|_{L^2}, \tag{3.1.15}$$

for any $\mu \geq 0$. This is the (1.1.6). □

Lemma 3.1.2 (Para-differential operators). *Let $A^{(i)}$ the operators defined in (3.1.9) for $i = 1,\dots,p$. There are functions $b^{(i)}(U_1,\dots,\hat{U}_i,\dots U_p;x)$ belonging to $\widetilde{\mathscr{F}}_{p-1}$ such that (recalling Definition 1.1.15)*

$$A^{(i)}(u_1,\dots,\hat{u}_i,\dots,u_p)[u_i]=\mathrm{Op}^{\mathscr{B}}(b^{(i)}(U_1,\dots,\hat{U}_i,\dots,U_p;x))\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot U_i\right], \tag{3.1.16}$$

where $U_j = (u_j, z_j)^T$, $z_j \in C^\infty(\mathbb{T};\mathbb{C})$ for $j = 1,\dots,p$.

Proof. We introduce the function

$$a^{(i)}(u_1,\dots,\hat{u}_i,\dots,u_p;x):=\prod_{j\neq i}u_j. \tag{3.1.17}$$

By (3.1.9) and Definition 1.1.15 we can note that

$$A^{(i)}(u_1,\dots,\hat{u}_i,\dots,u_p)[u_i]=\mathrm{Op}^{\mathscr{B}}(a^{(i)}(u_1,\dots,\hat{u}_i,\dots,u_p;x))[u_i]. \tag{3.1.18}$$

For $i = 1,\dots,p$ we set

$$b^{(i)}(U_1,\dots,\hat{U}_i,\dots,U_p;x):=\left[\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}\prod_{j\neq i}u_j\\ \prod_{j\neq i}z_j\end{pmatrix}\right]. \tag{3.1.19}$$

We show that $b^{(i)}$ belongs to the class $\widetilde{\mathscr{F}}_{p-1}$. Let us check condition (1.1.16). By symmetry we study the case $i = p$. We have that

$$
\begin{aligned}
|\partial_x^\alpha b^{(p)}(\Pi_{n_1}U_1,\dots,\Pi_{n_{p-1}}U_{p-1};x)| &\overset{(3.1.19),(3.1.17)}{=} |\partial_x^\alpha a^{(p)}(\Pi_{n_1}u_1,\dots,\Pi_{n_{p-1}}u_{p-1};x)| \\
&\overset{(3.1.17)}{\le} C(\alpha,\beta) \sum_{\substack{s_j\in\mathbb{N},\\ s_1+\ldots+s_{p-1}=\alpha}} \prod_{j=1}^{p-1} |n_j|^{s_j}|\Pi_{n_j}u_j| \\
&\le C(\alpha,\beta)\max\{\langle n_1\rangle,\dots,\langle n_{p-1}\rangle\}^\alpha \prod_{j=1}^{p-1}\|\Pi_{n_j}u_j\|_{L^2}.
\end{aligned}
\tag{3.1.20}
$$

□

Proof of Theorem 3.1.1. Let us consider a single monomial of the non linearity f in (0.3.1), i.e.

$$
C_{\alpha,\beta} z_0^{\alpha_0}\bar{z}_0^{\beta_0} z_1^{\alpha_1}\bar{z}_1^{\beta_1} z_2^{\alpha_2}\bar{z}_2^{\beta_2}, \quad C_{\alpha,\beta}\in\mathbb{R},
$$

with $(\alpha,\beta) := (\alpha_0,\beta_0,\alpha_1,\beta_1,\alpha_2,\beta_2)\in\mathbb{N}^6$ and $\sum_{i=0}^2 \alpha_i+\beta_i = p$ for a fixed $2\le p\le\bar{q}$. Recall that the coefficients $C_{\alpha,\beta}$ of the polynomial in (0.3.1) are real thanks to item 3 of Hypothesis 0.3.1. We start by proving that we can write

$$
\begin{pmatrix} C_{\alpha,\beta}u^{\alpha_0}\bar{u}^{\beta_0}u_x^{\alpha_1}\bar{u}_x^{\beta_1}u_{xx}^{\alpha_2}\bar{u}_{xx}^{\beta_2} \\ C_{\alpha,\beta}\bar{u}^{\alpha_0}u^{\beta_0}\bar{u}_x^{\alpha_1}u_x^{\beta_1}\bar{u}_{xx}^{\alpha_2}u_{xx}^{\beta_2} \end{pmatrix} = \mathrm{Op}^{\mathscr{B}}(B^{\alpha,\beta}(U;x,\xi))[U] + Q_1^{\alpha,\beta}(U)[U], \tag{3.1.21}
$$

where $B^{\alpha,\beta}(U;x,\xi)$ is a matrix of symbols and $Q_1^{\alpha,\beta}(U)$ a matrix of smoothing operator. For any $(\alpha,\beta)\in A_p$ (see (0.3.2)) let M be the multilinear operator defined in (3.1.3) and write

$$
\begin{aligned}
&C_{\alpha,\beta}u^{\alpha_0}\bar{u}^{\beta_0}u_x^{\alpha_1}\bar{u}_x^{\beta_1}u_{xx}^{\alpha_2}\bar{u}_{xx}^{\beta_2} = \\
&\qquad C_{\alpha,\beta}M(\underbrace{u,\dots,u}_{\alpha_0-times},\underbrace{\bar{u},\dots,\bar{u}}_{\beta_0-times},\underbrace{u_x,\dots,u_x}_{\alpha_1-times},\underbrace{\bar{u}_x,\dots,\bar{u}_x}_{\beta_1-times},\underbrace{u_{xx},\dots,u_{xx}}_{\alpha_2-times},\underbrace{\bar{u}_{xx},\dots,\bar{u}_{xx}}_{\beta_2-times}).
\end{aligned}
\tag{3.1.22}
$$

Lemmata 3.1.1, 3.1.2 guarantee that there are multilinear functions $\tilde{b}_j^{\alpha,\beta},\tilde{c}_j^{\alpha,\beta}\in\widetilde{\mathscr{F}}_{p-1}$, $j=0,1,2$ and a multilinear remainder $\tilde{Q}^{\alpha,\beta}\in\widetilde{\mathscr{R}}^{-\rho}_{p-1}$ such that the r.h.s. of (3.1.22) is equal to

$$
\sum_{j=0}^{2}\mathrm{Op}^{\mathscr{B}}\big(b_j^{\alpha,\beta}(U;x)(\mathrm{i}\xi)^j\big)u + \sum_{j=0}^{2}\mathrm{Op}^{\mathscr{B}}\big(c_j^{\alpha,\beta}(U;x)(\mathrm{i}\xi)^j\big)\bar{u} + Q^{\alpha,\beta}(U)[u], \tag{3.1.23}
$$

where $b_j^{\alpha,\beta}(U;x) := \tilde{b}_j^{\alpha,\beta}(U,\dots,U;x)$, $c_j^{\alpha,\beta}(U;x) := \tilde{c}_j^{\alpha,\beta}(U,\dots,U;x)$ and $Q^{\alpha,\beta}(U)[\cdot] := \tilde{Q}^{\alpha,\beta}(U,\dots,U)[\cdot]$. We set

$$\begin{aligned} B^{\alpha,\beta}(U;x,\xi) &:= B_2^{\alpha,\beta}(U;x)(\mathrm{i}\xi)^2 + B_1^{\alpha,\beta}(U;x)(\mathrm{i}\xi) + B_0^{\alpha,\beta}(U;x), \\ B_j^{\alpha,\beta}(U;x) &:= \begin{pmatrix} b_j^{\alpha,\beta}(U;x) & c_j^{\alpha,\beta}(U;x) \\ \overline{c_j^{\alpha,\beta}(U;x)} & \overline{b_j^{\alpha,\beta}(U;x)} \end{pmatrix}, \quad j = 0,1,2. \end{aligned} \tag{3.1.24}$$

The matrix of symbols $B^{\alpha,\beta}(U;x,\xi)$ belongs to $\Sigma\Gamma^2_{K,0,1}[r,N,\mathrm{aut}] \otimes \mathcal{M}_2(\mathbb{C})$, therefore the (3.1.21) follows for some $Q_1^{\alpha,\beta} \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}] \otimes \mathcal{M}_2(\mathbb{C})$ for any $N > 0$ and $\rho > 0$. Notice that, by construction, (see equations (3.1.17), (3.1.19) in Lemma 3.1.2) we have

$$b_2^{\alpha,\beta}(U;x) = \alpha_2 C_{\alpha,\beta} u^{\alpha_0} \bar{u}^{\beta_0} u_x^{\alpha_1} \bar{u}_x^{\beta_1} u_{xx}^{\alpha_2 - 1} \bar{u}_{xx}^{\beta_2}. \tag{3.1.25}$$

By using equations (0.3.1) and (3.1.21), we deduce that

$$\begin{pmatrix} f(u,u_x,u_{xx}) \\ \overline{f(u,u_x,u_{xx})} \end{pmatrix} = \mathrm{Op}^{\mathcal{B}}(B(U;x,\xi))[U] + Q_1(U)[U], \tag{3.1.26}$$

where $B(U;x,\xi) := B_2(U;x)(\mathrm{i}\xi)^2 + B_1(U;x)(\mathrm{i}\xi) + B_0(U;x)$ with

$$B_j(U;x) := \begin{pmatrix} b_j(U;x) & c_j(U;x) \\ \overline{c_j(U;x)} & \overline{b_j(U;x)} \end{pmatrix} := \sum_{p=2}^{\bar{q}} \sum_{\alpha,\beta \in A_p} B_j^{\alpha,\beta}(U;x), \quad j = 0,1,2,$$

and $Q_1(U)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}] \otimes \mathcal{M}_2(\mathbb{C})$. Notice that

$$b_2(U;x) = \sum_{p=2}^{\bar{q}} \sum_{\alpha,\beta \in A_p} b_2^{\alpha,\beta}(U;x) \overset{(3.1.25)}{=} (\partial_{u_{xx}} f)(u,u_x,u_{xx}),$$

hence $b_2(U;x)$ is real thanks to item 2 of Hypothesis 0.3.1. We now pass to the Weyl quantization. By using the formula (1.1.26) one constructs a matrix of symbols $A(U;x,\xi) \in \Sigma\Gamma^2_{K,0,1}[r,N,\mathrm{aut}] \otimes \mathcal{M}_2(\mathbb{C})$ such that

$$\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[\cdot] = \mathrm{Op}^{\mathcal{B}}(B(U;x,\xi))[\cdot],$$

up to smoothing remainders in $\Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}] \otimes \mathcal{M}_2(\mathbb{C})$. Hence we have obtained

$$\begin{pmatrix} f(u,u_x,u_{xx}) \\ \overline{f(u,u_x,u_{xx})} \end{pmatrix} = \mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))[U] + R(U)[U], \tag{3.1.27}$$

for some $R \in \Sigma\mathcal{R}^{-\rho}_{K,0,1}[r,N,\mathrm{aut}]\otimes\mathcal{M}_2(\mathbb{C})$. The matrix $A(U;x,\xi)$ has the form (3.1.2), in particular $a_2(U;x)$ is real valued since $a_2(U;x) = b_2(U;x)$. This is a consequence of the fact that the Weyl and the standard quantizations coincide at the principal order (see (1.1.26)).
It remains to show the reality, parity and reversibility properties of the matrices $A(U;x,\xi)$ and $R(U)$.
Since the function f satisfies Hypothesis 0.3.1, we can write the l.h.s. of (3.1.27) as $M(U)[U]$ for some (R,R,P)-map $M \in \Sigma\mathcal{M}_{K,0,1}[r,N,\mathrm{aut}]\otimes\mathcal{M}_2(\mathbb{C})$ (see Remark 1.1.14). Therefore by Lemma 1.3.4 we may assume that both $A(U;x,\xi)$ and $R(U)$ are respectively (R,R,P)-matrix of symbols and (R,R,P)-matrix of operators. Lemma 1.3.4 guarantees also that the new matrix $A(U;x,\xi)$ has still the form (3.1.2). □

Remark 3.1.1. *The explicit expression of the whole matrix $A(U;x,\xi)$ in* (3.1.2) *of Theorem 3.1.1 can be obtained as follows. First of all, reasoning as done in the proof of Theorem 3.1.1 one obtains*

$$\begin{aligned}&\mathrm{Op}^{\mathcal{B}}\left(\begin{pmatrix}\partial_{z_2} f & \partial_{\bar z_2} f\\ \partial_{\bar z_2} f & \partial_{z_2} f\end{pmatrix}(\mathrm{i}\xi)^2\right)\begin{pmatrix}u\\ \bar u\end{pmatrix}+\\ &+\mathrm{Op}^{\mathcal{B}}\left(\begin{pmatrix}\partial_{z_1} f & \partial_{\bar z_1} f\\ \partial_{\bar z_1} f & \partial_{z_1} f\end{pmatrix}(\mathrm{i}\xi)\right)\begin{pmatrix}u\\ \bar u\end{pmatrix}+\mathrm{Op}^{\mathcal{B}}\left(\begin{pmatrix}\partial_{z_0} f & \partial_{\bar z_0} f\\ \partial_{\bar z_0} f & \partial_{z_0} f\end{pmatrix}\right)\begin{pmatrix}u\\ \bar u\end{pmatrix},\end{aligned}$$

where in the first line we have used the Schrödinger type *hypothesis, see item 2 of Hypothesis 0.3.1. In the above formula we omitted the dependence on (u,u_x,u_{xx}). Then one has to pass to the Weyl quantization by using formula* (1.1.26) *obtaining, up to smoothing remainders, a new paradifferential operator* $\mathrm{Op}^{\mathcal{B}W}(A(U;x,\xi))$ *of the form* (3.1.2) *whose entries are given in* (2.1.26)*. In* (2.1.26) *∂_u has to be to be understood as ∂_{z_0} and similarly the other derivatives.*

3.2 Regularization

We proved in Theorem 3.1.1 that for any $N \in \mathbb{N}$ and any $\rho > 0$ the equation (0.2.1) is equivalent to the system (3.1.1). The key result of this section is the following.

Theorem 3.2.1 (**Regularization**)**.** *Fix $N > 0$, $\rho \gg N$ and $K \gg \rho$. There exist $s_0 > 0$ and $r_0 > 0$ such that for any $s \geq s_0$, $0 < r \leq r_0$ and any $U \in B^K_s(I,r)$ solution even in $x \in \mathbb{T}$ of* (3.1.1) *the following holds.*

There exist two (R,R,P)-maps

$$\Phi(U)[\cdot],\, \Psi(U)[\cdot] : C^{K-K'}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T};\mathbb{C}^2)) \to C^{K-K'}_{*\mathbb{R}}(I, \mathbf{H}^s(\mathbb{T};\mathbb{C}^2)),$$

with $K' := 2\rho + 4$ *satisfying the following:*

(i) there exists a constant C depending on s, r and K such that

$$\begin{aligned} \|\Phi(U)V\|_{K-K',s} &\leq \|V\|_{K-K',s}\big(1 + C\|U\|_{K,s_0}\big) \\ \|\Psi(U)V\|_{K-K',s} &\leq \|V\|_{K-K',s}\big(1 + C\|U\|_{K,s_0}\big) \end{aligned} \tag{3.2.1}$$

for any V in $C^{K-K'}_{*\mathbb{R}}(I, \mathbf{H}^s)$*;*

(ii) $\Phi(U)[\cdot] - \mathbb{1}$ *and* $\Psi(U)[\cdot] - \mathbb{1}$ *belong to the class* $\Sigma\mathcal{M}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$*; moreover* $\Psi(U)[\Phi(U)[\cdot]] - \mathbb{1}$ *is an operator in the class of smoothing remainders* $\Sigma\mathcal{R}^{-\rho}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$*;*

(iii) the function $V = \Phi(U)U$ *solves the system*

$$\partial_t V = \mathrm{i}E\big(\Lambda V + \mathrm{Op}^{\mathscr{B}W}(L(U;t,\xi))V + Q_1(U)V + Q_2(U)U\big), \tag{3.2.2}$$

where Λ *is defined in* (0.4.6)*, the operators* $Q_1(U)[\cdot]$ *and* $Q_2(U)[\cdot]$ *are (R,R,P) smoothing operators in the class* $\Sigma\mathcal{R}^{-\rho+m}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *for some* $m > 0$ *depending on N,* $L(U;t,\xi)$ *is a (R,R,P)-matrix in* $\Sigma\Gamma^2_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *and has the form*

$$\begin{aligned} L(U;t,\xi) &:= \begin{pmatrix} \mathtt{m}(U;t,\xi) & 0 \\ 0 & \overline{\mathtt{m}(U;t,-\xi)} \end{pmatrix}, \\ \mathtt{m}(U;t,\xi) &= \mathtt{m}_2(U;t)(\mathrm{i}\xi)^2 + \mathtt{m}_0(U;t,\xi), \end{aligned} \tag{3.2.3}$$

where $\mathtt{m}_2(U;t)$ *is a real symbol in* $\Sigma\mathscr{F}_{K,K',1}[r,N]$*,* $\mathtt{m}_0(U;t,\xi)$ *is in* $\Sigma\Gamma^0_{K,K',1}[r,N]$ *and both of them are constant in* $x \in \mathbb{T}$*.*

During this sections we shall use the following notation.

Definition 3.2.1. *We define the* commutator *between the operators A and B as* $[A,B]_- = A \circ B - B \circ A$ *and the* anti-commutator *as* $[A,B]_+ = A \circ B + B \circ A$*.*

3.2.1 Diagonalization of the second order operator

The goal of this subsection is to transform the matrix of symbols $E(\mathbb{1}+A_2(U;t,x))(\mathrm{i}\xi)^2$ (where $A_2(U;t,x)$ is defined in (3.1.2)) into a diagonal one up to a smoothing term.

Proposition 3.2.1. *Fix $N>0$, $\rho \gg N$ and $K \gg \rho$, then there exist $s_0>0$, $r_0>0$, such that for any $s \geq s_0$, any $0<r\leq r_0$ and any $U \in B_s^K(I,r)$ solution of* (3.1.1) *the following holds. There exist two (R,R,P)-maps*

$$\Phi_1(U)[\cdot],\Psi_1(U)[\cdot] : C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s) \to C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s),$$

satisfying the following

(i) there exists a constant C depending on s, r and K such that

$$\begin{aligned}\|\Phi_1(U)V\|_{K-1,s} &\leq \|V\|_{K-1,s}\big(1+C\|U\|_{K,s_0}\big)\\ \|\Psi_1(U)V\|_{K-1,s} &\leq \|V\|_{K-1,s}\big(1+C\|U\|_{K,s_0}\big)\end{aligned} \tag{3.2.4}$$

for any V in $C^{K-1}_{*\mathbb{R}}(I,\mathbf{H}^s)$;

(ii) $\Phi_1(U)[\cdot]-\mathbb{1}$ and $\Psi_1(U)[\cdot]-\mathbb{1}$ belong to the class $\Sigma\mathcal{M}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$; $\Psi_1(U)[\Phi_1(U)[\cdot]]-\mathbb{1}$ is a smoothing operator in the class $\Sigma\mathcal{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$;

(iii) the function $V_1=\Phi_1(U)U$ solves the system

$$\partial_t V_1 = \mathrm{i}E\big(\Lambda V_1 + \mathrm{Op}^{\mathcal{B}W}(A^{(1)}(U;t,x,\xi))V_1 + R_1^{(1)}(U)V_1 + R_2^{(1)}(U)U\big), \tag{3.2.5}$$

where Λ is defined in (0.4.6)*, and*

$$A^{(1)}(U;t,x,\xi) = A_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;t,x)(\mathrm{i}\xi) + A_0^{(1)}(U;t,x,\xi)$$

is a (R,R,P) matrix in the class $\Sigma\Gamma^2_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ with $A_j^{(1)}(U;t,x)$ in $\Sigma\mathcal{F}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for $j=1,2$, $A_0^{(1)}(U;t,x,\xi)$ is a matrix of symbols in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and

$$A_2^{(1)}(U;t,x) = \begin{pmatrix} a_2^{(1)}(U;t,x) & 0 \\ 0 & a_2^{(1)}(U;t,x)\end{pmatrix}, \tag{3.2.6}$$

with $a_2^{(1)}(U;t,x)$ real valued, the operators $R_1^{(1)}(U)[\cdot]$ and $R_2^{(1)}(U)[\cdot]$ are (R,R,P) smoothing operators in the class $\Sigma\mathcal{R}^{-\rho+2}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

Proof. The matrix $E(\mathbb{1}+A_2(U;t,x))$ in (3.1.1) and (3.1.2) has eigenvalues

$$\lambda^{\pm}(U;t,x)=\pm\sqrt{(1+a_2(U;t,x))^2-|b_2(U;t,x)|^2},$$

which are real and well defined since U is, by assumption, in $B^K_s(I,r)$ with r small enough. The matrix of eigenfunctions is

$$M(U;t,x)=\frac{1}{2}\begin{pmatrix}1+a_2(U;t,x)+\lambda^+(U;t,x) & -b_2(U;t,x)\\ -\overline{b_2}(U;t,x) & 1+a_2(U;t,x)+\lambda^+(U;t,x)\end{pmatrix},$$

it is invertible with inverse

$$\begin{aligned}&M(U;t,x)^{-1}=\\ &\frac{1}{\det(M(U;t,x))}\begin{pmatrix}1+a_2(U;t,x)+\lambda^+(U;t,x) & b_2(U;t,x)\\ \overline{b_2}(U;t,x) & 1+a_2(U;t,x)+\lambda^+(U;t,x)\end{pmatrix}\\ &=2\begin{pmatrix}\frac{1}{\lambda^+(U;t,x)} & \frac{b_2(U;t,x)}{\lambda^+(U;t,x)(\lambda^+(U;t,x)+a_2(U;t,x))}\\ \frac{\overline{b_2}(U;t,x)}{\lambda^+(U;t,x)(\lambda^+(U;t,x)+a_2(U;t,x))} & \frac{1}{\lambda^+(U;t,x)}\end{pmatrix}.\end{aligned}$$

Therefore one has

$$\begin{aligned}&M(U;t,x)^{-1}E(\mathbb{1}+A_2(U;t,x))M(U;t,x)=\\ &\begin{pmatrix}\lambda^+(U;t,x) & 0\\ 0 & \lambda^-(U;t,x)\end{pmatrix}=E\begin{pmatrix}\lambda^+(U;t,x) & 0\\ 0 & \lambda^+(U;t,x)\end{pmatrix}.\end{aligned} \tag{3.2.7}$$

By using the last item in Remark 1.3.1, since the matrix $E(\mathbb{1}+A_2(U;x))$ is reversibility preserving, we have that the matrix $M(U;t,x)$ (and therefore the matrix $M^{-1}(U;t,x)$ by the first item in Remark 1.3.1) is reversibility preserving. Arguing in the same way one deduces that both the matrices are (R,R,P). In particular the matrix in (3.2.7) is (R,R,P).
Note that by Taylor expanding the function $\sqrt{1+x}$ at $x=0$ one can prove that the matrices $M(U;t,x)-\mathbb{1}$, $M(U;t,x)^{-1}-\mathbb{1}$ and $M(U;t,x)^{-1}E(\mathbb{1}+A_2(U;t,x))$ $M(U;t,x)-E$ belong to the space $\Sigma\mathscr{F}_{K,0,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. We set

$$\begin{aligned}\Phi_1(U;t,x)[\cdot]&:=\mathrm{Op}^{\mathscr{B}W}\left(M(U;t,x)^{-1}\right)[\cdot],\\ \Psi_1(U;t,x)[\cdot]&:=\mathrm{Op}^{\mathscr{B}W}\left(M(U;t,x)\right)[\cdot],\end{aligned}$$

these are (R,R,P) maps according Definition 1.3.3, moreover by using Propositions 1.1.3, 1.1.1 and the discussion above one proves items (i) and (ii) of the statement.

The function $V_1 := \Phi_1(U)U$ solves the equation

$$\begin{aligned}\partial_t V_1 &= \mathrm{Op}^{\mathscr{B}W}(\partial_t(M(U;t,x)^{-1}))U + \mathrm{Op}^{\mathscr{B}W}(M(U;t,x)^{-1})\partial_t U\\ &\overset{(3.1.1)}{=} \mathrm{Op}^{\mathscr{B}W}(\partial_t(M(U;t,x)^{-1}))U+\\ &\quad + \mathrm{Op}^{\mathscr{B}W}(M(U;t,x)^{-1})\mathrm{i}E\Big(\Lambda U + \mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi)U + R(U)U)\Big).\end{aligned} \tag{3.2.8}$$

We know by previous discussions that $U = \Psi_1(U)V_1 + \widetilde{R}(U)U$ for a (R,R,P) smoothing operator $\widetilde{R}(U)$ belonging to $\Sigma\mathscr{R}^{-\rho}_{K,0,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$; plugging this identity in the equation (3.2.8) we get

$$\begin{aligned}&\partial_t V_1 = \mathrm{Op}^{\mathscr{B}W}(\partial_t(M(U;t,x)^{-1}))\mathrm{Op}^{\mathscr{B}W}(M(U;t,x))V_1+\\ &\mathrm{Op}^{\mathscr{B}W}(M(U;t,x)^{-1})\mathrm{i}E\Big(\big(\Lambda + \mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi)\big)\mathrm{Op}^{\mathscr{B}W}(M(U;t,x))V_1\Big)\\ &+\widetilde{\widetilde{R}}(U)U\end{aligned} \tag{3.2.9}$$

where $\partial_t(M(U;t,x)^{-1})$ is a reversible, reality and parity preserving matrix of symbols in the class $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ thanks to Lemma 1.1.4 and

$$\begin{aligned}\widetilde{\widetilde{R}}(U)[U] = &\Big(\mathrm{Op}^{\mathscr{B}W}(\partial_t M(U;t,x)^{-1})\Big)\widetilde{R}(U)U\\ &+\Big(\mathrm{Op}^{\mathscr{B}W}(M(U;t,x)^{-1})\circ\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}E(\Lambda + A(U;t,x,\xi)))\Big)\big[\widetilde{R}(U)U\big]\\ &+\mathrm{Op}^{\mathscr{B}W}(M(U;t,x)^{-1})\mathrm{i}ER(U)U\end{aligned}$$

is a reality, parity preserving and reversible smoothing operator (according to Def. 1.3.3) in the class $\Sigma\mathscr{R}^{-\rho+2}_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ thanks to Proposition 1.1.3 and Remark 1.3.1. Owing to Proposition 1.1.3, the first summand in the r.h.s. of (3.2.9) is equal to

$$\mathrm{Op}^{\mathscr{B}W}\big(\partial_t(M(U;t,x)^{-1})M(U;t,x)\big)V_1 + Q_1(U)V_1,$$

where $Q_1(U)[\cdot]$ is a reversible, parity and reality preserving smoothing operator in the class $\Sigma\mathscr{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and $\partial_t(M(U;t,x)^{-1})M(U;t,x)$ is in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. Recalling that $A(U;t,x,\xi)$ has the form (3.1.2), Λ has the form (0.4.6) (see also Remark 1.3.2), using Proposition 1.1.3 and (3.2.7) we expand the second summand in the r.h.s. of (3.2.9) as follows

$$\begin{aligned}\mathrm{i}E\Lambda V_1 + \mathrm{i}E\mathrm{Op}^{\mathscr{B}W}&\left(\begin{pmatrix}\lambda^+(U;t,x)-1 & 0\\ 0 & \lambda^+(U;t,x)-1\end{pmatrix}(\mathrm{i}\xi)^2\right)V_1\\ &+\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\big(A_1^{(1)}(U;t,x)(\mathrm{i}\xi)\big)V_1\\ &+\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\big(\widetilde{A}_0^{(1)}(U;t,x,\xi)\big)V_1 + Q_2(U)V_1\end{aligned}$$

where $Q_2(U)[\cdot]$ is a smoothing operator in the class $\Sigma\mathscr{R}^{-\rho}_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$, while $\widetilde{A}_0^{(1)}(U;t,x,\xi)$ and $A_1^{(1)}(U;t,x)$ are matrices of symbols respectively in $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and in $\Sigma\mathscr{F}_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. Moreover, by Lemmata 1.3.4 and 1.3.3, the matrices

$$\widetilde{A}_0^{(1)}(U;t,x,\xi),\ A_1^{(1)}(U;t,x)(\mathrm{i}\xi),\ A_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2$$

are (R,R,P) according to Definition 1.3.1 and the operator $Q_2(U)$ is reversible, reality and parity preserving according to Definition 1.3.3. Therefore the theorem is proved by setting

$$\begin{aligned}
a_2^{(1)}(U;t,x) &:= \lambda^+(U;t,x)-1,\\
A_0^{(1)}(U;t,x,\xi) &:= \widetilde{A}_0^{(1)}(U;t,x,\xi)-\mathrm{i}E(\partial_t(M(U;t,x)^{-1})M(U;t,x)),\\
R_2^{(1)}(U) &:= -\mathrm{i}E\widetilde{\widetilde{R}}(U)U,\\
R_1^{(1)}(U) &:= -\mathrm{i}E(Q_1(U)+Q_2(U)).
\end{aligned}$$

□

3.2.2 Diagonalization of lower order operators

Proposition 3.2.2. *There exist $s_0>0$, $r_0>0$, such that for any $s\geq s_0$, any $0<r\leq r_0$ and any $U\in B^K_s(I,r)$ solution of* (3.1.1) *the following holds. There exist two (R,R,P)-maps*

$$\Phi_2(U)[\cdot],\ \Psi_2(U)[\cdot]: C^{K-\rho-2}_{*\mathbb{R}}(I,\mathbf{H}^s)\to C^{K-\rho-2}_{*\mathbb{R}}(I,\mathbf{H}^s),$$

satisfying the following

(i) there exists a constant C depending on s, r and K such that

$$\begin{aligned}
\|\Phi_2(U)V\|_{K-\rho-2,s} &\leq \|V\|_{K-\rho-2,s}\left(1+C\|U\|_{K,s_0}\right)\\
\|\Psi_2(U)V\|_{K-\rho-2,s} &\leq \|V\|_{K-\rho-2,s}\left(1+C\|U\|_{K,s_0}\right)
\end{aligned} \tag{3.2.10}$$

for any V in $C^{K-\rho-2}_{*\mathbb{R}}(I,\mathbf{H}^s)$;

(ii) $\Phi_2(U)[\cdot]-\mathbb{1}$ and $\Psi_2(U)[\cdot]-\mathbb{1}$ belong to the class $\Sigma\mathscr{M}_{K,\rho+2,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$; $\Psi_2(U)[\Phi_2(U)[\cdot]]-\mathbb{1}$ is a smoothing operator in the class $\Sigma\mathscr{R}^{-\rho}_{K,\rho+2,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$;

(iii) the function $V_2 = \Phi_2(U)V_1$ *(where* V_1 *is the solution of* (3.2.5)*) solves the system*

$$\partial_t V_2 = \mathrm{i}E\big(\Lambda V_2 + \mathrm{Op}^{\mathscr{B}W}(A^{(2)}(U;t,x,\xi))V_2 + R_1^{(2)}(U)V_2 + R_2^{(2)}(U)U\big), \quad (3.2.11)$$

where Λ *is defined in* (0.4.6), $A^{(2)}(U;t,x,\xi) = \sum_{j=-(\rho-1)}^{2} A_j^{(2)}(U;t,x,\xi)$ *is a (R,R,P) matrix in the class* $\Sigma\Gamma^2_{K,\rho+2,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$ *with* $A_j^{(2)}(U;t,x,\xi)$ *diagonal matrices in* $\Sigma\Gamma^j_{K,\rho+2,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$ *for* $j = -(\rho-1),\dots,2$ *and*

$$\begin{aligned} A_2^{(2)}(U;t,x,\xi) &= \begin{pmatrix} a_2^{(2)}(U;t,x)(\mathrm{i}\xi)^2 & 0 \\ 0 & a_2^{(2)}(U;t,x)(\mathrm{i}\xi)^2 \end{pmatrix}, \\ A_1^{(2)}(U;t,x,\xi) &= \begin{pmatrix} a_1^{(2)}(U;t,x)(\mathrm{i}\xi) & 0 \\ 0 & \overline{a_1^{(2)}(U;t,x)(\mathrm{i}\xi)} \end{pmatrix}, \\ & a_2^{(2)}(U;t,x) \in \Sigma\mathscr{F}_{K,\rho+2,1}[r,N], \\ & a_1^{(2)}(U;t,x) \in \Sigma\mathscr{F}_{K,\rho+2,1}[r,N] \end{aligned} \quad (3.2.12)$$

with $a_2^{(2)}(U;t,x)$ *real valued, the operators* $R_1^{(2)}(U)[\cdot]$ *and* $R_2^{(2)}(U)[\cdot]$ *are (R,R,P) smoothing operators in the class* $\Sigma\mathscr{R}^{-\rho+2}_{K,\rho+2,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$.

Proof. Consider the following matrix

$$D_1(U;t,x,\xi) := \begin{pmatrix} 0 & d_1(U;t,x,\xi) \\ \overline{d_1(U;t,x,-\xi)} & 0 \end{pmatrix}, \quad (3.2.13)$$

where the symbol $d_1(U;t,x,\xi)$ is in the class $\Sigma\Gamma^{-1}_{K,1,1}[r,N]$. Note that the matrix $(\mathbb{1} + D_1(U;t,x,\xi))(\mathbb{1} - D_1(U;t,x,\xi))$ is equal to the identity modulo a matrix of symbols in $\Sigma\Gamma^{-2}_{K,1,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$. Define the following matrices of symbols

$$\begin{aligned} G_1(U;t,x,\xi) &:= \mathbb{1} - \big(\mathbb{1} + D_1(U;t,x,\xi)\sharp(\mathbb{1} - D_1(U;t,x,\xi))\big)_\rho \in \Sigma\Gamma^{-2}_{K,1,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C}) \\ Q_1(U;t,x,\xi) &:= (\mathbb{1} - D_1) + \big((\mathbb{1} - D_1)\sharp G_1\big)_\rho + \ldots + \big((\mathbb{1} - D_1)\sharp \underbrace{G_1\sharp\ldots\sharp G_1}_{\rho-times}\big)_\rho, \end{aligned} \quad (3.2.14)$$

where in the right hand side of the latter equation we omitted, with abuse of notation, the dependence on U, x and ξ. Then one has

$$
\begin{aligned}
&\big((\mathbb{1}+D_1)\sharp Q_1\big)_\rho = \big((\mathbb{1}+D_1)\sharp(\mathbb{1}-D_1)\big)_\rho +\\
&+\big((\mathbb{1}+D_1)\sharp(\mathbb{1}-D_1)\sharp G_1\big)_\rho + \ldots + \big((\mathbb{1}+D_1)\sharp(\mathbb{1}-D_1)\sharp G_1\sharp\ldots\sharp G_1\big)_\rho\\
&=(\mathbb{1}-G_1)+\big((\mathbb{1}-G_1)\sharp G_1\big)_\rho+\ldots+\big((\mathbb{1}-G_1)\sharp G_1\sharp\ldots\sharp G_1\big)_\rho\\
&=\mathbb{1}-\big(G_1\sharp\ldots\sharp G_1\big)_\rho,
\end{aligned} \tag{3.2.15}
$$

moreover the matrix of symbols $\big(G_1\sharp\ldots\sharp G_1\big)_\rho$ is in the class $\Sigma\Gamma^{-2\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. We set

$$
\begin{aligned}
&\Phi_{2,1}(U)[\cdot] := \mathrm{Op}^{\mathscr{B}W}(\mathbb{1}+D_1(U;t,x,\xi))[\cdot],\\
&\Psi_{2,1}(U)[\cdot] := \mathrm{Op}^{\mathscr{B}W}(Q_1(U;t,x,\xi)).
\end{aligned} \tag{3.2.16}
$$

The previous discussion proves that the maps $\Phi_{2,1}(U)$ and $\Psi_{2,1}(U)$ satisfy the estimates (3.2.10) with $\rho=0$, moreover thanks to Prop. 1.1.3 and Remark 1.1.14 there exists a smoothing remainder $R(U)$ in the class $\Sigma\mathscr{R}^{-2\rho}_{K,1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ such that $(\Psi_{2,1}(U)\circ\Phi_{2,1}(U))V = V + R(U)U$.

The function $V_{2,1} := \Phi_{2,1}(U)V_1$ solves the equation

$$
\begin{aligned}
&\partial_t V_{2,1} =\\
&\mathrm{Op}^{\mathscr{B}W}(\partial_t D_1(U;t,x,\xi))V_{2,1} + \Phi_{2,1}(U)\mathrm{i}E\Big[\mathrm{Op}^{\mathscr{B}W}(A^{(1)}(U;t,x,\xi))\Psi_{2,1}(U)V_{2,1}\\
&+\Lambda\Psi_{2,1}(U)V_{2,1}+R_1^{(1)}(U)\Psi_{2,1}(U)V_{2,1}+R_2^{(1)}(U)U\Big]\\
&-\Big\{\mathrm{Op}^{\mathscr{B}W}(\partial_t D_1(U;t,x,\xi))\\
&\qquad+\Phi_{2,1}(U)\mathrm{i}E\big[\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(1)}(U;t,x,\xi))+R_1^{(1)}(U)\big]\Big\}R(U)U.
\end{aligned} \tag{3.2.17}
$$

Owing to Lemma 1.1.4 the matrix of symbols $\partial_t D_1(U;t,x,\xi)$ is in $\Sigma\Gamma^{-1}_{K,2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. The last summand in the r.h.s. of (3.2.17) is a (R,R,P) smoothing remainder in the class $\Sigma\mathscr{R}^{-2\rho+2}_{K,2,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ thanks to Lemmata 1.1.4, 1.3.3, 1.3.4 and Proposition 1.1.3. Recalling that $A^{(1)}(U;t,x,\xi) = A_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2 + A_1^{(1)}(U;t,x)(\mathrm{i}\xi) + A_0^{(1)}(U;t,x,\xi)$, we have, up to a (R,R,P) smoothing operator in the class $\Sigma\mathscr{R}^{-\rho}_{K,2,1}[r,N]\otimes$

$\mathscr{M}_2(\mathbb{C})$, that

$$
\begin{aligned}
&\mathrm{i}\Phi_{2,1}(U)\mathrm{Op}^{\mathscr{B}W}\Big(E\big(\mathbb{1}+A_2^{(1)}(U;t,x)\big)(\mathrm{i}\xi)^2\Big)\Psi_{2,1}(U) = \\
&\mathrm{i}\mathrm{Op}^{\mathscr{B}W}\Big(E\big(\mathbb{1}+A_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2\big)\Big)+ \\
&+\mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}\big(D_1(U;t,x,\xi)\big),\mathrm{Op}^{\mathscr{B}W}\big(E\big(\mathbb{1}+A_2^{(1)}(U;t,x)\big)(\mathrm{i}\xi)^2\big)\Big]_- \\
&+\mathrm{Op}^{\mathscr{B}W}\big(M_1(U;t,x,\xi)\big),
\end{aligned}
$$

where $M_1(U;t,x,\xi)$ is a (R,R,P) matrix of symbols in $\Sigma\Gamma^0_{K,2,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ (actually $M_1(U;t,x,\xi)$ belongs to $\Sigma\Gamma^0_{K,1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$, but we preferred to embed it in the above larger class in order to simplify the notation; we shall do this simplification systematically), here the commutator $[\cdot,\cdot]_-$ is defined in Definition 3.2.1. The conjugation of the term $A_1^{(1)}(U;t,x)(\mathrm{i}\xi)$ is, up to a (R,R,P) smoothing operator in $\Sigma\mathscr{R}^{-\rho}_{K,2,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$,

$$
\begin{aligned}
&\mathrm{i}\Phi_{2,1}(U)\mathrm{Op}^{\mathscr{B}W}\Big(EA_1^{(1)}(U;t,x)(\mathrm{i}\xi)\Big)\Psi_{2,1}(U) = \\
&\quad\mathrm{i}\mathrm{Op}^{\mathscr{B}W}\Big(EA_1^{(1)}(U;t,x)(\mathrm{i}\xi)\Big)+\mathrm{Op}^{\mathscr{B}W}\Big(M_2(U;t,x,\xi)\Big)
\end{aligned}
$$

for a (R,R,P) matrix of symbols $M_2(U;t,x,\xi)$ in $\Sigma\Gamma^0_{K,2,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. Therefore the matrix of operators of order one is given by

$$
\begin{aligned}
&\mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}\big(D_1(U;t,x,\xi)\big),\mathrm{Op}^{\mathscr{B}W}\big(E\big(\mathbb{1}+A_2^{(1)}(U;t,x)\big)(\mathrm{i}\xi)^2\big)\Big]_- +\mathrm{i}EA_1^{(1)}(U;t,x)(\mathrm{i}\xi) = \\
&\mathrm{i}E\begin{pmatrix}\mathrm{Op}^{\mathscr{B}W}\big(a_1^{(1)}(U;t,x)(\mathrm{i}\xi)\big) & M_+ \\ \overline{M_+} & \mathrm{Op}^{\mathscr{B}W}\Big(\overline{a_1^{(1)}(U;t,x)}(\mathrm{i}\xi)\Big)\end{pmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
&M_+ := \\
&\mathrm{Op}^{\mathscr{B}W}\big(b_1^{(1)}(U;t,x)(\mathrm{i}\xi)\big)-\Big[\mathrm{Op}^{\mathscr{B}W}(d_1(U;t,x,\xi)),\mathrm{Op}^{\mathscr{B}W}\big(1+a_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2\big)\Big]_+,
\end{aligned}
$$

thus our aim is to choose the symbol $d_1(U;t,x,\xi)$ in such a way that M_+ at the principal order is 0. Developing the compositions by means of Proposition 1.1.3 we obtain that, at the level of principal symbol, we need to solve the equation

$$
2d_1(U;t,x,\xi)(1+a_2^{(1)}(U;t,x))(\mathrm{i}\xi)^2 = b_1^{(1)}(U;t,x)(\mathrm{i}\xi).
$$

We choose the symbol $d_1(U;t,x,\xi)$ as follows

$$
\begin{aligned}
d_1(U;t,x,\xi) &:= \left(\frac{b_1^{(1)}(U;t,x)}{2(1+a_2^{(1)}(U;t,x))}\right)\cdot\gamma(\xi),\\
\gamma(\xi) &:= \begin{cases} \frac{1}{\mathrm{i}\xi} & |\xi|\geq 1/2,\\ \text{odd continuation of class } C^\infty & |\xi|\in[0,1/2). \end{cases}
\end{aligned}
\tag{3.2.18}
$$

Note that by Taylor expanding the function $x\mapsto(1+x)^{-1}$ one gets that $d_1(U;t,x,\xi)$ in (3.2.18) is a symbol in the class $\Sigma\Gamma^{-1}_{K,2,1}[r,N]$, therefore by symbolic calculus (Prop. 1.1.3) one has that M_+ is equal to $\mathrm{Op}^{\mathscr{B}W}(\widetilde{b}_0(U;t,x,\xi))+\widetilde{R}(U)$ for a symbol $\widetilde{b}_0(U;t,x,\xi)$ in $\Sigma\Gamma^0_{K,2,1}[r,N]$ and a smoothing operator $\widetilde{R}(U)$ in $\Sigma\mathscr{R}^{-\rho}_{K,2,1}[r,N]$.

The symbol $d_1(U;t,x,\xi)$ defined in (3.2.18) satisfies the equation $\overline{d_1(U;-t,x,\xi)}=d_1(U_S;t,x,\xi)$ since both the symbols $a_2^{(1)}(U;t,x)$ and $b_1^{(1)}(U;t,x)$ fulfil the same condition, therefore by Remark 1.3.1 (see the last item) we deduce that the matrix $D_1(U;t,x,\xi)$ is reversibility preserving. By hypothesis the symbol $b_1^{(1)}(U;t,x)$ is odd in x, $a_2^{(1)}(U;t,x)$ is even and then, since $\gamma(\xi)$ is odd in ξ, we have $d_1(U;t,x,\xi)=d_1(U;t,-x,-\xi)$, which means that the matrix $D_1(U;t,x,\xi)$ is parity preserving. Furthermore $D_1(U;t,x,\xi)$ is reality preserving by construction, therefore we can deduce that such a matrix is a (R,R,P) matrix of symbols. Therefore the function $V_{2,1}$ solves the system

$$
\begin{aligned}
&\partial_t V_{2,1}=\mathrm{i}E\big(\Lambda V_{2,1}+\mathrm{Op}^{\mathscr{B}W}(A^{(2,1)}(U;t,x,\xi))V_{2,1}+R_1^{(2,1)}(U)V_{2,1}+R_2^{(2,1)}(U)U\big),\\
&A^{(2,1)}(U;t,x,\xi)=\\
&\begin{pmatrix} a_2^{(1)}(U;t,x) & 0\\ 0 & a_2^{(1)}(U;t,x)\end{pmatrix}(\mathrm{i}\xi)^2+\begin{pmatrix} a_1^{(1)}(U;t,x) & 0\\ 0 & \overline{a_1^{(1)}(U;t,x)}\end{pmatrix}(\mathrm{i}\xi)+\\
&+\begin{pmatrix} a_0^{(2,1)}(U;t,x,\xi) & b_0^{(2,1)}(U;t,x,\xi)\\ \overline{b_0^{(2,1)}(U;t,x,-\xi)} & \overline{a_1^{(2,1)}(U;t,x,-\xi)}\end{pmatrix}.
\end{aligned}
$$

Suppose now that there exist $j\geq 1$ (R,R,P) maps $\Phi_{2,1}(U),\ldots,\Phi_{2,j}(U)$ such that $V_{2,j}:=\Phi_{2,1}(U)\circ\ldots\circ\Phi_{2,j}(U)[V_1]$ solves the problem

$$
\partial_t V_{2,j}=\mathrm{i}E\big(\Lambda V_{2,j}+\mathrm{Op}^{\mathscr{B}W}(A^{(2,j)}(U;t,x,\xi))V_{2,j}+R_1^{(2,j)}(U)V_{2,j}+R_2^{(2,j)}(U)U\big),
$$

where $R_1^{(2,j)}(U)$ and $R_2^{(2,j)}(U)$ are in the class $\Sigma\mathscr{R}^{-\rho}_{K,j+1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and

$$A^{(2,j)}(U;t,x,\xi)=\sum_{j'=-j}^{2}A^{(2,j)}_{j'}(U;t,x,\xi),$$

$$A^{(2,j)}_{j'}(U;t,x,\xi)=\begin{pmatrix}a^{(2,j)}_{j'}(U;t,x,\xi) & 0\\ 0 & \overline{a^{(2,j)}_{j'}(U;t,x,-\xi)}\end{pmatrix}\in\Sigma\Gamma^{j'}_{K,j+1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C}),$$

for $\quad j'=-j+1,\dots,2$ and

$$a_2^{(2,j)}(U;t,x,\xi)=a_2^{(1)}(U;t,x)(\mathrm{i}\xi)^2\in\mathbb{R}$$

$$A^{(2,j)}_{-j}(U;t,x,\xi)=\begin{pmatrix}a^{(2,j)}_{-j}(U;t,x,\xi) & b^{(2,j)}_{-j}(U;t,x,\xi)\\ \overline{b^{(2,j)}_{-j}(U;t,x,-\xi)} & \overline{a^{(2,j)}_{-j}(U;t,x,-\xi)}\end{pmatrix}\in\Sigma\Gamma^{-j}_{K,j+1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C}).$$

We now explain how to construct a map $\Phi_{j+1}(U)$ which diagonalize the matrix $A^{(2,j)}_{-j}(U;t,x,\xi)$ up to lower order terms. Define

$$\Phi_{2,j+1}(U):=\mathbb{1}+\mathrm{Op}^{\mathscr{B}W}\big(D_{j+1}(U;t,x,\xi)\big);$$
$$D_{j+1}(U;t,x,\xi):=\begin{pmatrix}0 & d_{j+1}(U;t,x,\xi)\\ \overline{d_{j+1}(U;t,x,-\xi)} & 0\end{pmatrix},$$

with $d_{j+1}(U;t,x,\xi)$ a symbol in $\Sigma\Gamma^{-j-2}_{K,2+j,1}[r,N]$. An approximate inverse $\Psi_{2,j+1}(U):=\mathrm{Op}^{\mathscr{B}W}(Q_{j+1}(U))$ can be constructed exactly as done in (3.2.14) and (3.2.15). Reasoning as done above one can prove that the function $V_{2,j+1}:=\Phi_{2,j+1}(U)V_{2,j}$ solves the problem

$$\begin{aligned}&\partial_t V_{2,j+1}=\\ &\mathrm{i}E\Big(\Lambda V_{2,j+1}+\sum_{j'=-j+1}^{2}\mathrm{Op}^{\mathscr{B}W}(A^{(2,j)}_{j'}(U;t,x,\xi))V_{2,j+1}+R_1^{(2,j+1)}(U)V_{2,j+1}+R_2^{(2,j+1)}(U)U\Big)\\ &+\mathrm{i}\Big[\mathrm{Op}^{\mathscr{B}W}\big(D_{j+1}(U;t,x,\xi)\big),E\mathrm{Op}^{\mathscr{B}W}\big(\mathbb{1}+A_2^{(2,j)}(U;t,x,\xi)\big)\Big]_-\\ &+\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\big(A^{(2,j)}_{-j}(U;t,x,\xi)\big).\end{aligned}\tag{3.2.19}$$

Developing the commutator above one obtains that the sum of the last two terms in (3.2.19) is equal to

$$\mathrm{i}E\begin{pmatrix}\mathrm{Op}^{\mathscr{B}W}\big(a^{(2,j)}_{-j}(U;t,x,\xi)\big) & M_{j,+}\\ \overline{M}_{j,+} & \mathrm{Op}^{\mathscr{B}W}\big(\overline{a^{(2,j)}_{-j}(U;t,x,-\xi)}\big),\end{pmatrix}$$

where

$$M_{j,+} := \mathrm{Op}^{\mathscr{B}W}\big(b^{(2,j)}_{-j}(U;t,x,\xi)\big) - \Big[\mathrm{Op}^{\mathscr{B}W}(d_{j+1}(U;t,x,\xi)), \mathrm{Op}^{\mathscr{B}W}\big((1+a_2^{(1)}(U;t,x))(\mathrm{i}\xi)^2\big)\Big]_+,$$

therefore, repeating the same argument used in the case of the symbol of order one, one has to choose

$$d_{j+1}(U;t,x,\xi) := \left(\frac{b^{(2,j)}_{-j}(U;t,x,\xi)}{2\big(1+a_2^{(1)}(U;t,x)\big)}\right)\cdot\gamma(\xi),$$

$$\gamma(\xi) := \begin{cases} \frac{1}{(\mathrm{i}\xi)^2} & |\xi|\geq 1/2, \\ \text{odd continuation of class } C^\infty & |\xi|\in[0,1/2). \end{cases}$$

Therefore we obtain the thesis of the theorem by setting $\Phi_2(U) := \Phi_{2,1}(U)\circ\ldots\circ\Phi_{2,\rho-1}(U)$ and $\Psi_2(U) := \Psi_{2,\rho-1}(U)\circ\ldots\circ\Psi_{2,1}(U)$. □

3.2.3 Reduction to constant coefficients: paracomposition

In this section we shall reduce the operator $\mathrm{Op}^{\mathscr{B}W}(A_2^{(2)}(U;t,x,\xi))$, given in terms of the diagonal matrix (3.2.12), to a constant coefficients one up to smoothing remainders. We shall conjugate the system (3.2.11) under the paracomposition operator $\Phi_U^\star := \Omega_{B(U)}\cdot\mathbb{1}$ defined in Subsection 1.1.6.

Proposition 3.2.3. *In the notation of Prop. 1.1.4 there exists a real valued function $\beta(U;t,x)\in\Sigma\mathscr{F}_{K,\rho+2,1}[r,N]$ such that the function $V_3 := \Phi_U^\star V_2$ (where V_2 is a solution of* (3.2.11)*) solves the following problem*

$$\partial_t V_3 = \mathrm{i}E\big(\Lambda V_3 + \mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;t,x,\xi))V_3 + R_1^{(3)}(U)V_3 + R_2^{(3)}(U)U\big), \tag{3.2.20}$$

where $A^{(3)}(U;t,x,\xi) = \sum_{j=-(\rho-1)}^{2} A_j^{(3)}(U;t,x,\xi)$ is a (R,R,P) matrix in the class $\Sigma\Gamma^2_{K,\rho+3,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ with $A_j^{(3)}(U;t,x,\xi)$ diagonal matrices in $\Sigma\Gamma^j_{K,\rho+3,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ for $j=-(\rho-1),\ldots,2$ and

$$A_2^{(3)}(U;t,\xi) = \begin{pmatrix} a_2^{(3)}(U;t)(\mathrm{i}\xi)^2 & 0 \\ 0 & a_2^{(3)}(U;t)(\mathrm{i}\xi)^2 \end{pmatrix}, \quad a_2^{(3)}(U;t)\in\Sigma\mathscr{F}_{K,\rho+3,1}[r,N]$$
$$A_1^{(3)}(U;t,x,\xi) = \begin{pmatrix} a_1^{(3)}(U;t,x)(\mathrm{i}\xi) & 0 \\ 0 & \overline{a_1^{(3)}(U;t,x)(\mathrm{i}\xi)} \end{pmatrix}, \quad a_1^{(3)}(U;t,x)\in\Sigma\mathscr{F}_{K,\rho+3,1}[r,N] \tag{3.2.21}$$

with $a_2^{(3)}(U;t)$ *real valued and independent of* x*, the operators* $R_1^{(3)}(U)[\cdot]$ *and* $R_2^{(3)}(U)[\cdot]$ *are (R,R,P) smoothing operators in the class* $\Sigma\mathscr{R}^{-\rho+m}_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *for some* $m=m(N)>0$*.*

Proof. The function $V_3 := \Phi_U^{\star}V_2$ solves the following problem

$$\begin{aligned}\partial_t V_3 &= (\partial_t\Phi_U^{\star})(\Phi_U^{\star})^{-1}V_3 + \Phi_U^{\star}\Big(\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(2)}(U;t,x,\xi))\big)\Big)(\Phi_U^{\star})^{-1}V_3\\ &\quad + \Phi_U^{\star}(\mathrm{i}ER_1^{(2)}(U))(\Phi_U^{\star})^{-1}V_3 + \Phi_U^{\star}(\mathrm{i}ER_2^{(2)}(U)U).\end{aligned} \tag{3.2.22}$$

Our aim is to choose $\beta(U;t,x)$ in such a way that the coefficient in front of $(\mathrm{i}\xi)^2$ in the new symbol is constant in $x\in\mathbb{T}$. Recalling that $A^{(2)}(U;t,x,\xi)$ has the form (3.2.12), we have, by Theorem 1.1.1, that the term

$$\Phi_U^{\star}\Big(\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(2)}(U;t,x,\xi))\big)\Big)(\Phi_U^{\star})^{-1}V_3 \tag{3.2.23}$$

in (3.2.22) is equal to

$$\begin{aligned}&\mathrm{i}E\Big[\Lambda+\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} r(U;t,x)(\mathrm{i}\xi)^2 & 0\\ 0 & r(U;t,x)(\mathrm{i}\xi)^2\end{pmatrix}V_3\\ &\quad + \mathrm{Op}^{\mathscr{B}W}(A_1^{+}(U;t,x)(\mathrm{i}\xi))V_3 + \mathrm{Op}^{\mathscr{B}W}(A_0^{+}(U;t,x,\xi))V_3\Big]\end{aligned} \tag{3.2.24}$$

up to smoothing remainders in $\Sigma\mathscr{R}^{-\rho}_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, where

$$r(U;t,x) = (1+a_2^{(2)}(U;t,y))(1+\partial_y\gamma(U;t,y))^2_{|y=x+\beta(U;t,x)} - 1,$$

$A_1^{+}(U;t,x)\in\Sigma\mathscr{F}_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $A_0^{+}(U;t,x,\xi)\in\Sigma\Gamma^0_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ are diagonal matrices of symbols.

We define

$$\gamma(U;t,y) = \partial_y^{-1}\left(\sqrt{\frac{1+a_2^{(3)}(U;t)}{1+a_2^{(2)}(U;t,y)}}-1\right), \tag{3.2.25}$$

where

$$a_2^{(3)}(U;t) := \left[2\pi\left(\int_{\mathbb{T}}\frac{1}{\sqrt{1+a_2^{(2)}(U;t,y)}}dy\right)^{-1}\right]^2 - 1. \tag{3.2.26}$$

Thanks to this choice we have

$$r(U;t,x) \equiv a_2^{(3)}(U;t),$$

moreover the paracomposition operator $\Phi_U^{\star}$ is parity and reversibility preserving, satisfies the anti-reality condition for the following reasons. The real valued function $\gamma(U;t,x)$ in (3.2.25) satisfies $\gamma(U;-t,x) = \gamma(U_S;t,x)$ since $a_2^{(2)}(U;t,x)$ satisfies the same equation, moreover $\gamma(U;t,x)$ is an odd function since is defined as a primitive of the even function $a_2^{(2)}(U;t,x)$. It follows that also the function $\beta(U;t,x)$ satisfies the same properties, therefore the matrix of symbols

$$B(\tau,U;t,x,\xi)\cdot\mathbb{1} = \frac{\beta(U;t,x)}{1+\tau\beta_x(U;t,x)}\xi\cdot\mathbb{1} \tag{3.2.27}$$

is a (R,R,P) matrix in $\Sigma\Gamma^1_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Therefore the operator $\Omega_{B(U)}(1)\cdot\mathbb{1}$ generated by $\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}B(U;t,x,\xi)\cdot\mathbb{1})$ is parity and reversibility preserving and it satisfies the anti-reality condition (1.3.16) by Lemma 1.3.6. Thanks to this the term in (3.2.23) is a parity and reality preserving and reversible vector field, therefore owing to Lemma 1.3.4 each term of the equation (3.2.24) (together with the omitted smoothing remainder) is a parity and reality preserving and reversible vector field.

The term $\Phi_U^{\star}(\mathrm{i}ER_1^{(2)}(U))(\Phi_U^{\star})^{-1}V_3 + \Phi_U^{\star}(\mathrm{i}ER_2^{(2)}(U)U)$ in (3.2.22) is analyzed as follows. First of all both the operators $\Phi_U^{\star}(\mathrm{i}ER_1^{(2)}(U))(\Phi_U^{\star})^{-1}$ and $\Phi_U^{\star}(\mathrm{i}ER_2^{(2)}(U)$ are reversible, parity and reality preserving thanks to Lemma 1.3.3. Moreover we remark that the paracomposition operator may be written as

$$\Phi_U^{\star} = \Omega_{B(U)}(1) = U + \sum_{p=2}^{N-1} M_p(U,\ldots,U)U + M_N(U;t)U, \tag{3.2.28}$$

where $M_p \in \widetilde{\mathcal{M}}_p^m$, $M_N(U;t) \in \mathcal{M}^m_{K,\rho,N}[r]$ for some $m>0$ depending only on N. This is a consequence of Theorem 1.1.1. Therefore as a consequence of Prop. 1.1.3 the operators $\Phi_U^{\star}(\mathrm{i}ER_1^{(2)}(U))(\Phi_U^{\star})^{-1}$ and $\Phi_U^{\star}(\mathrm{i}ER_2^{(2)}(U)$ belong to the class $\Sigma\mathscr{R}^{-\rho+2m}_{K,\rho+3,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$.

We are left to study the term $(\partial_t\Phi_U^{\star})(\Phi_U^{\star})^{-1}V_3$ in (3.2.22). This is nothing but $-\Phi_U^{\star}(\partial_t\Phi_U^{\star})^{-1}V_3$, therefore by Prop. 1.1.4 it is equal to $\mathrm{Op}^{\mathscr{B}W}\big(e(U;t,x,\xi)\cdot\mathbb{1}\big) + R(U;t)\cdot\mathbb{1}$ with $e(U;t,x,\xi)$ and $R(U;t)$ given by Prop. 1.1.4. Since the map $\Phi_U^{\star}$ satisfies the reality condition (1.3.15) its derivative $\partial_t\Phi_U^{\star} = \partial_t\Omega_{B(U)}(1)\cdot\mathbb{1}$ is reality preserving, this follows by taking the derivative with respect to t to both side of the equation (1.1.85) and by using the fact that the fact that $\partial_t B(U;t,x,\tau,\xi)\cdot\mathbb{1}$ still satisfies the anti-reality condition (1.3.16). We deduce, by using Lemma 1.3.3, that the maps $-\Phi_U^{\star}(\partial_t\Phi_U^{\star})^{-1}$ is reality preserving. Since the map $\Phi_U^{\star}$ is reversibility preserving and parity preserving, one reasons in the same way as above to prove that its derivative with respect to t is parity preserving and reversible. Therefore thanks to Lemma 1.3.4 we can also assume that both the matrix of symbols $e(U;t,x,\xi)\cdot\mathbb{1}$ and the operator $R(U;t)$ above are reality and parity preserving and reversible. □

3.2.4 Reduction to constant coefficients: elimination of the term of order one

In this section we shall eliminate the matrix of order one by conjugating the system through a multiplication operator.

Proposition 3.2.4. *There exist $s_0>0$, $r_0>0$ such that for any $s\geq s_0$, $r\leq r_0$ and any $U\in B_s^K(I,r)$ solution of (3.1.1) the following holds. There exist two (R,R,P) maps*

$$\Phi_4(U),\ \Psi_4(U): C_{*\mathbb{R}}^{K-(\rho+4)}(I;\mathbf{H}^s(\mathbb{T}))\to C_{*\mathbb{R}}^{K-(\rho+4)}(I;\mathbf{H}^s(\mathbb{T}))$$

(i) there exists a constant C depending on s, r and K such that

$$\begin{aligned}\|\Phi_4(U)V\|_{K-(\rho+4),s}&\leq\|V\|_{K-(\rho+4),s}\big(1+C\|U\|_{K,s_0}\big)\\ \|\Psi_4(U)V\|_{K-(\rho+4),s}&\leq\|V\|_{K-(\rho+4),s}\big(1+C\|U\|_{K,s_0}\big)\end{aligned}\tag{3.2.29}$$

for any V in $C_{*\mathbb{R}}^{K-(\rho+4)}(I,\mathbf{H}^s)$;

(ii) $\Phi_4(U)-\mathbb{1}$, $\Psi_4(U)-\mathbb{1}$ belong to $\Sigma\mathcal{M}_{K,\rho+4,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, moreover $\Psi_4(U)\circ\Phi_4(U)=\mathbb{1}+R(U)U$ with $R(U)$ is in $\Sigma\mathcal{R}^{-\rho}_{K,\rho+4,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and it is reversible, parity and reality preserving.

(iii) The function $V_4=\Phi_4(U)V_3$ (where V_3 solves (3.2.20)) solves the problem

$$\partial_t V_4=\mathrm{i}E\big(\Lambda V_4+\mathrm{Op}^{\mathcal{B}W}(A^{(4)}(U;t,x,\xi))V_4+R_1^{(4)}(U)V_4+R_2^{(4)}(U)U\big),\tag{3.2.30}$$

where $A^{(4)}(U;t,x,\xi)$ is a (R,R,P) matrix of symbols in $\Sigma\Gamma^2_{K,\rho+4,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and it has the form

$$A^{(4)}(U;t,x,\xi)=A_2^{(3)}(U;t)(\mathrm{i}\xi)^2+\sum_{j=-(\rho-1)}^{0}A_j^{(4)}(U;t,x,\xi)\tag{3.2.31}$$

where the diagonal matrix $A_2^{(3)}(U;t)$ is x-independent and it is defined in (3.2.21), $A_j^{(4)}(U;t,x,\xi)$ are diagonal matrices belonging to $\Sigma\Gamma^{j}_{K,\rho+4,1}[r,N]$ $\otimes\mathcal{M}_2(\mathbb{C})$ for $j=-(\rho-1),\dots,0$. The operators $R_1^{(4)}(U)$ and $R_2^{(4)}(U)$ are (R,R,P) and belong to the class $\Sigma\mathcal{R}^{-\rho+m}_{K,\rho+4,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ for some $m\in\mathbb{N}$ depending on N.

Proof. Let $s(U;t,x)$ be a function in $\Sigma\mathcal{F}_{K,\rho+3,1}[r,N]$ to be chosen later, define the map

$$\Phi_4(U)[\cdot]:=\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} e^{s(U;t,x)} & 0\\ 0 & \overline{e^{s(U;t,x)}}\end{pmatrix}[\cdot]. \tag{3.2.32}$$

Suppose moreover that $\Phi_4(U)$ in (3.2.32) is a (R,R,P) map. Since $s(U;t,x)$ is in $\Sigma\mathcal{F}_{K,\rho+3,1}[r,N]$ then by Taylor expanding the exponential function one gets that the symbol $e^{s(U;t,x)}-1$ is in $\Sigma\mathcal{F}_{K,\rho+3,1}[r,N]$, in particular the map $\Phi_4(U)$ satisfies the condition (i) and (ii) in the statement. The matrix in (3.2.32) is invertible, therefore the map

$$\Psi_4(U)[\cdot]:=\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} e^{-s(U;t,x)} & 0\\ 0 & e^{-\overline{s(U;t,x)}}\end{pmatrix}[\cdot] \tag{3.2.33}$$

is an approximate inverse $\Psi_4(U)$ for the map $\Phi_4(U)$ (i.e. satisfying the conditions in the items (i) and (ii) of the statement). To prove this last claim one has to argue exactly as done in the proof of Prop. 3.2.1.
The function $V_4:=\Phi_4(U)V_3$ solves the equation

$$\begin{aligned}\partial_t V_4 &= (\partial_t\Phi_4(U))[\Psi_4(U)V_4]\\ &\quad+\Phi_4(U)\mathrm{i}E\big[\Lambda\Psi_4(U)V_4+\mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;t,x,\xi))\Psi_4(U)V_4\\ &\quad+R_1^{(3)}(U)\Psi_4(U)V_4+R_2^{(3)}(U)U\big]+\widetilde{R}(U)U,\end{aligned} \tag{3.2.34}$$

where $\widetilde{R}(U)U$ is equal to

$$\big(\partial_t\Phi_4(U)\big)R(U)U+\Phi_4(U)\mathrm{i}E\Big(\big[\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;t,x,\xi))\big]R(U)U+R_1^{(3)}(U)R(U)U\Big),$$

where $R(U)$ is the (R,R,P) smoothing operator in $\Sigma\mathcal{R}^{-\rho}_{K,\rho+4,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ such that $\Psi_4(U)\circ\Phi_4(U)V_4=V_4+R(U)U$ and the matrix

$$A^{(3)}(U;t,x,\xi)=\sum_{j=-(\rho-1)}^{2}A_j^{(3)}(U;t,x,\xi)$$

is defined in the statement of Prop. 3.2.3. Therefore $\widetilde{R}(U)$ is a (R,R,P) smoothing remainder in the class $\Sigma\mathscr{R}^{-\rho+m}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ thanks to Lemma 1.3.3, Prop. 1.1.3 and to the fact that $R(U)$ is a (R,R,P) smoothing remainder.
The term $(\partial_t\Phi_4(U))\Psi_4(U)$ is of order 0 thanks to Lemma 1.1.4, and Prop. 1.1.3; moreover it is reversible, parity and reality preserving thanks to Lemma 1.3.3 since $\Psi_4(U)$ is (R,R,P) and $(\partial_t\Phi_4(U))$ is reversible, parity and reality preserving.
The term $\Phi_4(U)\mathrm{i}ER_2^{(3)}(U)U$ is reversible, reality and parity preserving thanks to Lemma 1.3.3.
We now study the algebraic structure of the term

$$\Phi_4(U)\mathrm{i}E\big[\Lambda\Psi_4(U)V_4+\mathrm{Op}^{\mathscr{B}W}(A^{(3)}(U;t,x,\xi))\Psi_4(U)V_4\big], \tag{3.2.35}$$

coming from equation (3.2.34). First of all, recalling (0.4.7), (0.4.8), (0.4.6) and Remark 1.1.11 we have that $\Phi_4(U)\mathrm{i}E\mathfrak{P}\Psi_4(U)$ is equal to $\mathrm{i}E\mathfrak{P}$ up to a term of order 0 (actually by Remark 1.1.7 this is a symbol of negative order, but at this level it is enough to embed it in class of symbols of order 0). The conjugation of the term of order 2

$$\Phi_4(U)\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}\Big((\mathbb{1}+A_2^{(3)}(U;t,x)(\mathrm{i}\xi)^2)\Big)\Psi_4(U)$$

is equal to

$$\mathrm{i}E\begin{pmatrix} M & 0\\ 0 & \overline{M}\end{pmatrix},$$

$$M:=\mathrm{Op}^{\mathscr{B}W}\Big(e^{s(U;x)}\Big)\circ\mathrm{Op}^{\mathscr{B}W}\Big((1+a_2^{(3)}(U;x)\Big)(\mathrm{i}\xi)^2)\circ\mathrm{Op}^{\mathscr{B}W}\Big(e^{-s(U;x)}\Big).$$

Using symbolic calculus (Prop. 1.1.3) one can prove that up to contribution of order 0 the operator M is equal to

$$\mathrm{Op}^{\mathscr{B}W}\Big((1+a_2^{(3)}(U;x))(\mathrm{i}\xi)^2+2s_x(U;x)(1+a_2^{(3)}(U;x))(\mathrm{i}\xi)\Big).$$

Reasoning similarly one gets that the conjugation of the term of order one

$$\Phi_4(U)\mathrm{i}E\circ\mathrm{Op}^{\mathscr{B}W}(A_1^{(3)}(U;t,x,\xi))\circ\Psi_4(U)$$

in (3.2.35) is equal, up to contribution of order 0, to $\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(A_1^{(3)}(U;t,x,\xi))$, with $A_1^{(3)}(U;t,x,\xi)$ defined in (3.2.21).
Therefore the term of order one appearing in (3.2.34) is the following

$$\big(2s_x(U;t,x)(1+a_2^{(3)}(U;t))+a_1^{(3)}(U;t,x)\big)(\mathrm{i}\xi), \tag{3.2.36}$$

hence we have to choose the function $s(U;t,x)$ as

$$s(U;t,x) = -\partial_x^{-1}\left(\frac{a_1^{(3)}(U;t,x)}{2(1+a_2^{(3)}(U;t))}\right).$$

Note that the the function $s(U;t,x)$ is well defined since $a_1^{(3)}(U;t,x)$ is an odd function in x (therefore its mean is zero) and the denominator stays far away from zero since r_0 is small enough. With this choice the map $\Phi_4(U)$ defined in (3.2.32) is (R,R,P) and therefore the ansatz made at the beginning of the proof is correct. Furthermore the term (3.2.35) is equal to $\mathrm{i}E[\Lambda V_4 + \mathrm{Op}^{\mathscr{B}W}(\widetilde{A}(U;t,x,\xi)V_4 + Q(U)V_4)]$, where $Q(U)$ is a (R,R,P) smoothing remainder in $\Sigma\mathscr{R}^{-\rho}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and

$$\widetilde{A}(U;t,x,\xi) = A_2^{(3)}(U;t)(\mathrm{i}\xi)^2 + \sum_{j=-(\rho-1)}^{0}\widetilde{A}_j(U;t,x,\xi),$$

is a (R,R,P) diagonal matrix of symbols such that $\widetilde{A}_j(U;t,x,\xi)$ is in $\Sigma\Gamma^{j}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ thanks to Prop. 1.1.3 and Lemma 1.3.4. □

3.2.5 Reduction to constant coefficients: lower order terms

Here we reduce to constant coefficients all the symbols from the order 0 to the order $\rho-1$ of the matrix $A^{(4)}(U;t,x,\xi)$ in (3.2.31).

Proposition 3.2.5. *There exist $s_0>0$, $r_0>0$, such that for any $s\geq s_0$, any $0<r\leq r_0$ and any $U\in B_s^K(I,r)$ solution of* (3.1.1) *the following holds. There exist two (R,R,P)-maps*

$$\Phi_5(U)[\cdot],\ \Psi_5(U)[\cdot] : C^{K-2\rho-4}_{*\mathbb{R}}(I,\mathbf{H}^s)\to C^{K-2\rho-4}_{*\mathbb{R}}(I,\mathbf{H}^s),$$

satisfying the following

(i) there exists a constant C depending on s, r and K such that

$$\begin{aligned}\|\Phi_5(U)V\|_{K-2\rho-4,s} &\leq \|V\|_{K-2\rho-4,s}\left(1+C\|U\|_{K,s_0}\right)\\ \|\Psi_5(U)V\|_{K-2\rho-4,s} &\leq \|V\|_{K-2\rho-4,s}\left(1+C\|U\|_{K,s_0}\right)\end{aligned}\tag{3.2.37}$$

for any V in $C^{K-2\rho-4}_{*\mathbb{R}}(I,\mathbf{H}^s)$;

(ii) $\Phi_5(U)[\cdot]-\mathbb{1}$ *and* $\Psi_5(U)[\cdot]-\mathbb{1}$ *belong to the class* $\Sigma\mathscr{M}_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$*;* $\Psi_5(U)[\Phi_5(U)[\cdot]]-\mathbb{1}$ *is a smoothing operator in the class* $\Sigma\mathscr{R}^{-\rho}_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$*;*

(iii) the function $V_5=\Phi_5(U)V_4$ *(where* V_4 *is the solution of* (3.2.30)*) solves the system*

$$\partial_t V_5 = \mathrm{i}E\big(\Lambda V_5+\mathrm{Op}^{\mathscr{B}W}(A^{(5)}(U;t,\xi))V_5+R_1^{(5)}(U)V_5+R_2^{(5)}(U)U\big), \quad (3.2.38)$$

where Λ *is defined in* (0.4.6)*,* $A^{(5)}(U;t,\xi)$ *is a (R,R,P) diagonal and constant coefficient in x matrix in* $\Sigma\Gamma^2_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(C)$ *of the form*

$$A^{(5)}(U;t,\xi)=A_2^{(3)}(U;t)(\mathrm{i}\xi)^2+A_0^{(5)}(U;t,\xi),$$

with $A_2^{(3)}(U;t)$ *of Prop. 3.2.3 and* $A_0^{(5)}(U;t,\xi)$ *in* $\Sigma\Gamma^0_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$*; the operators* $R_1^{(5)}(U)$ *and* $R_2^{(5)}(U)$ *are (R,R,P) smoothing remainders in the class* $\Sigma\mathscr{R}^{-\rho+m}_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ *for some* m *in* $\mathbb{N}$ *depending on* N*.*

Proof. We first construct a map which conjugates to constant coefficient the term of order 0 in (3.2.30) and (3.2.31). Consider a symbol $n_0(U;t,x,\xi)$ in $\Sigma\Gamma^{-1}_{K,\rho+4,1}[r,N]$ to be determined later and define

$$\Phi_{5,0}(U):=\mathbb{1}+\mathrm{Op}^{\mathscr{B}W}(N_0(U;t,x,\xi)):=\mathbb{1}+\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix} n_0(U;t,x,\xi) & 0\\ 0 & \overline{n_0(U;t,x,-\xi)}\end{pmatrix}.$$

Suppose moreover that the map $\Phi_{5,0}(U)$ defined above is (R,R,P). It is possible to construct an approximate inverse of the map above (i.e. satisfying items (i) and (ii) of the statement) of the form

$$\begin{aligned}\Psi_{5,0}(U)=\mathbb{1}-\mathrm{Op}^{\mathscr{B}W}(N_0(U;t,x,\xi))+\\ \mathrm{Op}^{\mathscr{B}W}\big((N_0(U;t,x,\xi))^2\big)+\mathrm{Op}^{\mathscr{B}W}(\widetilde{N}_0(U;t,x,\xi))\end{aligned} \quad (3.2.39)$$

proceeding as done in the proof of Prop. 3.2.2 by choosing a suitable matrix of symbols $\widetilde{N}_0(U;t,x,\xi)$ in the class $\Sigma\Gamma^{-3}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. Let $R(U)$ in the class $\Sigma\mathscr{R}^{-\rho}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ such that one has $\Psi_{5,0}(U)[\Phi_{5,0}(U)[\cdot]]-\mathbb{1}=R(U)$. Then the function $V_{5,0}=\Phi_{5,0}(U)V_4$ solves the following problem

$$\begin{aligned}\partial_t V_{0,5} =&(\partial_t\Phi_{5,0}(U))\Psi_{5,0}V_{5,0}\\ &+\Phi_{5,0}(U)\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(4)}(U;t,x,\xi))\big)\Psi_{5,0}(U)V_{5,0}\\ &+\Phi_{5,0}(U)\mathrm{i}ER_1^{(4)}(U)\Psi_{5,0}(U)V_{5,0}\\ &+\Phi_{5,0}(U)\mathrm{i}ER_2^{(4)}(U)U+\widetilde{R}(U)U,\end{aligned} \quad (3.2.40)$$

where $A^{(4)}(U;t,x,\xi)$, $R_1^{(4)}(U)$ and $R_2^{(4)}(U)$ are the ones of equation (3.2.30), while $\widetilde{R}(U)$ is the operator

$$\Big[(\partial_t\Phi_{5,0}(U))+\Phi_{5,0}(U)\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(4)}(U;t,x,\xi))+\Phi_{5,0}(U)\mathrm{i}ER_4^{(1)}(U)\Big]R(U).$$

The operator $\widetilde{R}(U)$ belongs to the class $\Sigma\mathscr{R}^{-\rho+m'}_{K,\rho+5,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ for some $m'\in\mathbb{N}$ thanks to Prop. 1.1.3, moreover it is reversible, parity and reality preserving by Lemma 1.3.3. By using (3.2.39) one deduces that the first summand in the r.h.s. of (3.2.40) is equal to $\mathrm{Op}^{\mathscr{B}W}(\partial_t N_0(U;t,x,\xi))\circ\mathrm{Op}^{\mathscr{B}W}(\mathbb{1}-N_0(U;t,x,\xi)+N_0(U;t,x,\xi)^2+\widetilde{N}_0(U;t,x,\xi))$, therefore by Lemmata 1.1.4, 1.3.4 and Prop. 1.1.3 can be decomposed as the sum of a para-differential operator of order -1 and a smoothing remainder, both of them reversible, parity and reality preserving. The third and the fourth summands in (3.2.40) are (R,R,P) remainders in the class $\Sigma\mathscr{R}^{-\rho+m'}_{K,2\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ by Lemma 1.3.3 and Prop. 1.1.3.
The remaining term $\Phi_{5,0}(U)\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A^{(4)}(U;t,x,\xi))\big)\Psi_{5,0}(U)V_{5,0}$ is equal to

$$\begin{aligned}
&\mathrm{i}E\big(\Lambda+\mathrm{Op}^{\mathscr{B}W}(A_2^{(3)}(U;t)(\mathrm{i}\xi)^2)\big)V_{5,0}+\\
&\Big[\mathrm{Op}^{\mathscr{B}W}(N(U;t,x,\xi),\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}((\mathbb{1}+A_2(U;t))(\mathrm{i}\xi)^2))\Big]_-V_{0,5}\\
&+\mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(A_0^{(4)}(U;t,x,\xi))V_{0,5}\\
&-\mathrm{Op}^{\mathscr{B}W}(N_0(U;t,x,\xi))\circ\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}E(\mathbb{1}+A_2(U;t))(\mathrm{i}\xi)^2)\circ\\
&\quad\circ\mathrm{Op}^{\mathscr{B}W}(N_0(U;t,x,\xi))V_{0,5}\\
&+\mathrm{Op}^{\mathscr{B}W}(\mathrm{i}E(\mathbb{1}+A_2(U;t))(\mathrm{i}\xi)^2)\circ\mathrm{Op}^{\mathscr{B}W}(N_0(U;t,x,\xi)^2)V_{0,5}
\end{aligned}\tag{3.2.41}$$

up to operators of order -1. Every operator here can be assumed to be reversible, parity and reality preserving thanks to Lemmata 1.3.3 and 1.3.4. The last two summands in (3.2.41) cancel out up to a (R,R,P) operator of order -1 thanks to Prop. 1.1.3. In order to reduce to constant coefficient the term of order 0 in (3.2.41) we develop the commutator $[\cdot,\cdot]_-$ and we choose $n_0(U;t,x,\xi)$ in such a way that the following equation is satisfied

$$\begin{aligned}
&2(n_0(U;t,x,\xi))_x(1+a_2^{(3)}(U;t)(\mathrm{i}\xi))+a_0^{(4)}(U;t,x,\xi)=\\
&\frac{1}{2\pi}\int_{\mathbb{T}}a_0^{(4)}(U;t,x,\xi)dx;
\end{aligned}\tag{3.2.42}$$

proceeding as done in the proof of Prop. 3.2.2 we choose the symbol $n_0(U;t,x,\xi)$

as follows:

$$n_0(U;t,x,\xi) = \frac{\partial_x^{-1}\Big(\frac{1}{2\pi}\int_{\mathbb{T}} a_0^{(4)}(U;t,x,\xi)dx - a_0^{(4)}(U;t,x,\xi)\Big)}{2(1+a_2^{(3)}(U;t))}\gamma(\xi);$$
$$\gamma(\xi) := \begin{cases} \frac{1}{\mathrm{i}\xi} & |\xi| \geq 1/2, \\ \text{odd continuation of class } C^\infty & |\xi| \in [0,1/2). \end{cases}$$

The symbol above is well defined since the denominator stays far away from zero since the function U is small, the numerator is well defined too since it is the periodic primitive of a zero mean function; moreover it is parity preserving and reversibility preserving, therefore the ansatz made at the beginning of the proof is satisfied. Therefore we have reduced the system to the following

$$\partial_t V_{5,0} = \mathrm{i}E\Big(\Lambda + \mathrm{Op}^{\mathscr{B}W}(A^{(5,0)}(U;t,x,\xi))V_{5,0} + R_1^{(5,0)}(U)V_{5,0} + R_2^{(5,0)}(U)U\Big),$$

where $R_1^{(5,0)}(U)$ and $R_2^{(5,0)}(U)$ are (R,R,P) smoothing remainder in $\Sigma\mathscr{R}^{-\rho+m'}_{K,\rho+4,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$, the matrix of symbols $A^{(5,0)}(U;t,x,\xi)$ is in $\Sigma\Gamma^2_{K,\rho+5,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and it has the form

$$A^{(5,0)}(U;t,x,\xi) = A_2^{(3)}(U;t)(\mathrm{i}\xi)^2 + A_0^{(5,0)}(U;t,\xi) + A_{-1}^{(5,0)}(U;t,x,\xi),$$

with $A_2^{(3)}(U;t)$ coming from Prop. 3.2.3, $A_0^{(5,0)}(U;t,\xi)$ is equal to

$$(\frac{1}{2\pi}\int A_0^{(4)}(U;t,x,\xi)dx)\gamma(\xi)$$

and $A_{-1}^{(5,0)}(U;t,x,\xi)$ is a matrix of symbols in $\Sigma\Gamma^{-1}_{K,\rho+5,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$.
Suppose now that there exist $j+1$, $j\geq 0$, (R,R,P) maps $\Phi_{5,0}(U),\dots,\Phi_{5,j}(U)$ maps such that the function $V_{5,j} := \Phi_{5,0}(U)\circ\cdots\circ\Phi_{5,j}(U)V_4$ solves the problem

$$\partial_t V_{5,j} = \mathrm{i}E\big(\Lambda V_{5,j} + \mathrm{Op}^{\mathscr{B}W}(A^{(5,j)}(U;t,x,\xi))V_{5,j} + R_1^{(5,j)}(U)V_{5,j} + R_2^{(5,j)}(U)U\big) \tag{3.2.43}$$

where $R_1^{(5,j)}(U)$, $R_2^{(5,j)}(U)$ are (R,R,P) smoothing remainders in $\Sigma\mathscr{R}^{-\rho+m'}_{K,\rho+5+j,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and where $A^{(5,j)}(U;t,x,\xi)$ is a (R,R,P) diagonal matrix of symbols in $\Sigma\Gamma^2_{K,\rho+5+j,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ of the form

$$A^{(5,j)}(U;t,x,\xi) = A_2^{(3)}(U;t)(\mathrm{i}\xi)^2 + \sum_{\ell=-j}^{0} A_\ell^{(5,j)}(U;t,\xi) + A_{-j-1}^{(5,j)}(U;t,x,\xi),$$

with $A_{\ell}^{(5,j)}(U;t,\xi)$ in $\Sigma\Gamma^{\ell}_{K,\rho+5+j,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and constant in x for $\ell=-j,\dots,0$, while $A_{-j-1}^{(5,j)}(U;t,x,\xi)$ is in $\Sigma\Gamma^{-j-1}_{K,\rho+5+j,1}[r,N]$ and may depend on x. We explain how to construct a map $\Phi_{5,j+1}(U)$ which put to coefficient in x the term of order $-j-1$. Let $n_{j+1}(U;t,x,\xi)$ be a symbol in $\Sigma\Gamma^{-j-2}_{K,\rho+5+j,1}[r,N]$ and consider the map

$$\begin{aligned}&\Phi_{5,j+1}(U):=\mathbb{1}+\mathrm{Op}^{\mathscr{B}W}(N_{j+1}(U;t,x,\xi)):=\\&\mathbb{1}+\mathrm{Op}^{\mathscr{B}W}\begin{pmatrix}n_{j+1}(U;t,x,\xi) & 0\\ 0 & \overline{n_{j+1}(U;t,x,-\xi)}\end{pmatrix}.\end{aligned}$$

Arguing as done in the proof of Prop. 3.2.2 one obtains the approximate inverse of the map above $\Psi_{5,j}(U)=\mathbb{1}-\mathrm{Op}^{\mathscr{B}W}(N_{j+1}(U;t,x,\xi))$ modulo lower order terms. The same discussion made at the beginning of the proof, concerning the conjugation through the map $\Phi_{5,0}(U)$, shows that the function $V_{5,j+1}:=\Phi_{5,j+1}(U)V_{5,j}$ solves the problem

$$\begin{aligned}&\partial_t V_{5,j+1}=\\&\mathrm{i}E\big(\Lambda V_{5,j+1}+\mathrm{Op}^{\mathscr{B}W}(A^{(5,j+1)}(U;t,x,\xi))V_{5,j+1}\big)+\\&\mathrm{i}E\big(R_1^{(5,j+1)}(U)V_{5,j+1}+R_2^{(5,j+1)}(U)U\big),\end{aligned}\tag{3.2.44}$$

where $R_1^{(5,j+1)}(U)$ and $R_2^{(5,j+1)}(U)$ are smoothing remainders in $\Sigma\mathscr{R}^{-\rho+m'}_{K,\rho+5+j,2}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and where $A^{(5,j+1)}(U;t,x,\xi)$ has the form

$$\begin{aligned}&A_2^{(3)}(U;t)(\mathrm{i}\xi)^2+\sum_{\ell=-j}^{0}A_{\ell}^{(5,j)}(U;t,\xi)\\&+A_{-j-1}^{(5,j)}(U;t,x,\xi)+2\big(N_{j+1}(U;t,x,\xi),\big)_x(1+A_2^{(3)}(U;t)(\mathrm{i}\xi)).\end{aligned}$$

The equation we need to solve is

$$2(n_{j+1}(U;t,x,\xi))_x(1+a_2^{(3)}(U;t)(\mathrm{i}\xi))+a_{-j-1}^{(5,j)}(U;t,x,\xi)=\frac{1}{2\pi}\int_{\mathbb{T}}a_{-j-1}^{(5,j)}(U;t,x,\xi)\,dx,$$

which has the same structure of (3.2.42) and hence one can define the symbol $n_{j+1}(U;t,x,\xi)$ as done above.
To conclude the proof we define the maps $\Phi_5(U):=\Phi_{5,0}(U)\circ\cdots\circ\Phi_{5,\rho-1}(U)$ and $\Psi_5(U):=\Psi_{5,\rho-1}(U)\circ\cdots\circ\Psi_{5,0}(U)$. □

At this point we can prove Theorem 3.2.1.

proof of Theorem 3.2.1. It is enough to define $\Phi(U):=\Phi_5(U)\circ\Phi_4(U)\circ\Phi_U^{\star}\circ\Phi_2(U)\circ\Phi_1(U)$ and $\Psi(U):=\Psi_1(U)\circ\Psi_2(U)\circ(\Phi_U^{\star})^{-1}\circ\Psi_4(U)\circ\Psi_5(U)$. □

3.3 Proof of the main theorem

The aim of this section is to prove the following Theorem which, together with Theorem 3.1.1, implies Theorem 0.3.1.

Theorem 3.3.1. *Fix $N > 0$ and assume $M \geq N$ (see (0.3.4)), $K \in \mathbb{N}$, $\rho \in \mathbb{N}$ such that $K \gg \rho \gg N$ and consider system (3.1.1). There is a zero measure set $\mathcal{N} \subseteq \mathcal{O}$ such that for any $\vec{m}$ outside the set $\mathcal{N}$ and if $\rho > 0$ is large enough there is $s_0 > 0$ such that for any $s \geq s_0$ there are $r_0, c, C > 0$ such that for any $0 \leq r \leq r_0$ the following holds. For all $U_0 \in \mathbf{H}^s_e$ with $\|U_0\|_{\mathbf{H}^s} \leq r$, there is a unique solution $U(t,x)$ of (3.1.1) with*

$$U \in \bigcap_{k=0}^{K} C^k([-T_r, T_r]; \mathbf{H}^{s-2k}_e(\mathbb{T}; \mathbb{C}^2)), \tag{3.3.1}$$

with $T_r \geq cr^{-N}$. Moreover one has

$$\sup_{t\in[-T_r,T_r]} \|\partial_t^k U(t,\cdot)\|_{\mathbf{H}^{s-2k}} \leq Cr, \quad 0 \leq k \leq K. \tag{3.3.2}$$

The proof of the result above is divided into two main steps.
We need some further notation. For any $n \in \mathbb{N}$, we define

$$\Pi_n^+ := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Pi_n, \qquad \Pi_n^- := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Pi_n \tag{3.3.3}$$

the composition of the spectral projector Π_n, defined in (1.1.3) with the projector from $\mathbb{C}^2$ to $\mathbb{C} \times \{0\}$ (resp. $\{0\} \times \mathbb{C}$). For a function U satisfying (1.3.3), i.e. of the form $U = (u, \bar{u})^T$, the projectors $\Pi_n^{\pm}$ act as follows. Let $\varphi_n(x) = 1/\sqrt{\pi}\cos(nx)$ be the Hilbert basis of the space of even $L^2(\mathbb{T}; \mathbb{C})$ functions, then, if $\hat{u}(n) = \int_{\mathbb{T}} u(x)\varphi_n(x)dx$, one has

$$\begin{gathered} \Pi_n U = \begin{pmatrix} \hat{u}(n) \\ \overline{\hat{u}(n)} \end{pmatrix} \varphi_n(x), \quad \Pi_n^+ U = \hat{u}(n) e_+ \varphi_n(x), \quad \Pi_n^- U = \overline{\hat{u}(n)} e_- \varphi_n(x), \\ e_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ e_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{gathered} \tag{3.3.4}$$

We now give the following definition.

Definition 3.3.1 (Kernel of the Adjoint Action). *Fix $p \in \mathbb{N}^*$, $\rho > 0$ and consider a symbol $a \in \widetilde{\Gamma}^2_p$.*

(i) *We denote by* $[\![a]\!](U;t,x,\xi)$ *the symbol in* $\widetilde{\Gamma}^2_p$ *defined, for* $n_1,\dots,n_p\in\mathbb{N}$, *as*

$$[\![a]\!]\Big(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_p}U;t,x,\xi\Big)=\\ a\Big(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_p}U;t,x,\xi\Big),$$

for p *even,* $\ell=p/2$ *and*

$$\{n_1,\dots,n_\ell\}=\{n_{\ell+1},\dots,n_p\}, \tag{3.3.5}$$

while for p *odd and* $0\le\ell\le p$, *or* p *even and* $0\le\ell\le p$ *with* $\ell\neq p/2$ *we set*

$$[\![a]\!]\Big(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_p}U;t,x,\xi\Big)=0.$$

(ii) *Let* $a\in\Sigma\Gamma^0_{K,K',1}[r,N]$ *of the form*

$$a(U;t,x,\xi)=\sum_{k=1}^{N-1}a_k(U;t,x,\xi)+a_N(U;t,x,\xi),\quad a_k\in\widetilde{\Gamma}^0_k,\ a_N\in\Sigma\Gamma^0_{K,K',N}[r,N],$$

we define the symbol $[\![a]\!](U;t,x,\xi)$ *as*

$$[\![a]\!](U;t,x,\xi):=\sum_{k=1}^{N-1}[\![a_k]\!](U;t,x,\xi)+a_N(U;t,x,\xi).$$

(iii) *For a diagonal matrix of symbols* $A\in\Sigma\Gamma^0_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *of the form*

$$A(U;t,x,\xi)=\begin{pmatrix}a(U;t,x,\xi) & 0\\ 0 & \overline{a}(U;t,x,-\xi)\end{pmatrix},$$

we define

$$[\![A]\!](U;t,x,\xi):=\begin{pmatrix}[\![a]\!](U;t,x,\xi) & 0\\ 0 & [\![\overline{a}]\!](U;t,x,-\xi)\end{pmatrix}. \tag{3.3.6}$$

In the following lemma we consider the problem

$$\begin{cases}\partial_t Z=\mathrm{i}E\Big(\Lambda Z+\mathrm{Op}^{\mathscr{B}W}(\mathtt{m}_2(U)(\mathrm{i}\xi)^2)Z+\mathrm{Op}^{\mathscr{B}W}\big([\![A]\!](U;t,\xi)\big)[Z]\Big),\\ Z(0,x)=Z_0\in\mathbf{H}^s_e\end{cases} \tag{3.3.7}$$

with $\mathtt{m}_2(U)$ in (3.2.3), A being a diagonal (R,R,P) matrix of bounded symbols independent of x in $\Sigma\Gamma^0_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and $U\in C^K_{*\mathbb{R}}(I,\mathbf{H}^s_e(\mathbb{T};\mathbb{C}^2))\cap B^K_{s_0}(I,r)$. We prove that the (R,R,P) structure of the matrix $A(U;t,\xi)$ (together with the fact that it is constant in x) guarantees a symmetry which produces a key cancellation in the energy estimates for the problem (3.3.7), more precisely we show that the multilinear part of the matrix $[\![A]\!](U;t,\xi)$ does not contribute to the energy estimates.

Lemma 3.3.1. *Let $N\in\mathbb{N}$, $r>0$, $K'\le K\in\mathbb{N}$. Using the notation above consider $Z=Z(t,x)$ the solution of the problem* (3.3.7)*. Then one has*

$$\frac{d}{dt}\|Z(t,\cdot)\|^2_{\mathbf{H}^s}\le C\|U(t,\cdot)\|^N_{K',s}\|Z(t,\cdot)\|^2_{\mathbf{H}^s}. \tag{3.3.8}$$

Proof. Consider the Fourier multipliers $\langle D\rangle^s:=\mathrm{Op}(\langle\xi\rangle^s)$. We have that

$$\begin{aligned}&\frac{d}{dt}\|Z\|^2_{\mathbf{H}^s}=\\&\Big(\langle D\rangle^s\Big[\mathrm{i}E(\Lambda Z+\mathrm{Op}^{\mathscr{B}W}(\mathtt{m}_2(U)(\mathrm{i}\xi)^2)+\mathrm{Op}^{\mathscr{B}W}([\![A]\!](U;t,\xi))[Z])\Big],\langle D\rangle^s Z\Big)_{\mathbf{H}^0}\\&+\Big(\langle D\rangle^s Z,\langle D\rangle^s\Big[\mathrm{i}E(\Lambda Z+\mathrm{Op}^{\mathscr{B}W}(\mathtt{m}_2(U)(\mathrm{i}\xi)^2)+\mathrm{Op}^{\mathscr{B}W}([\![A]\!](U;t,\xi))[Z])\Big]\Big)_{\mathbf{H}^0}\end{aligned} \tag{3.3.9}$$

where $(\cdot,\cdot)_{\mathbf{H}^0}$ is defined in (0.4.2). The contribution given by Λ and by the term $\mathrm{Op}^{\mathscr{B}W}(\mathtt{m}_0(U)(\mathrm{i}\xi)^2)[\cdot]$ is zero since they are independent of x (therefore they commute with $\langle D^s\rangle$) and their symbols are real valued (hence they are self-adjoint on $\mathbf{H}^0$ thanks to Remark 1.1.12). Let us consider the symbol $[\![A]\!](U;t,\xi)$. By definition we have that

$$[\![A]\!](U;t,\xi)=\sum_{p=1}^{N-1}[\![A_p]\!](U,\dots,U;t,\xi)+A_N(U;t,\xi)$$

with $A_p\in\widetilde{\Gamma}^0_p$, $p=1,\dots,N-1$, and $A_N\in\Sigma\Gamma^0_{K,K',N}[r,N]$. The contribution of $\mathrm{Op}^{\mathscr{B}W}(A_N(U;t,\xi))[Z]$ in the r.h.s. of (3.3.9) is bounded by the r.h.s. of (3.3.8). We show that $[\![A_p]\!]$ are real valued for $p=1,\dots,N-1$, this implies that, since they do not depend on x, they do not contribute to the sum in (3.3.9). By hypothesis the matrix of symbols $A(U;t,\xi)$ is reversibility preserving, i.e. satisfies (1.3.7), therefore, by Lemma 1.3.1, we may assume that A_p satisfy condition (1.3.13) for any $p=1,\dots,N-1$. Since A_p is reality preserving then we can write (see Remark 1.3.1)

$$A_p(U;t,\xi)=\begin{pmatrix}\tilde a_p(U;t,\xi) & 0\\ 0 & \overline{\tilde a_p(U;t,-\xi)}\end{pmatrix}, \tag{3.3.10}$$

for some symbol $\tilde a_p\in\widetilde{\Gamma}^m_p$ independent of x. Recalling Def. 3.3.1, since A_p is a symmetric function of its arguments, we have, for $\ell,p,n_1,\dots,n_p$ satisfying the conditions in (3.3.5), that

$$\begin{aligned}&[\![A_p]\!]\Big(\Pi^+_{n_1}SU,\dots,\Pi^+_{n_\ell}SU,\Pi^-_{n_1}SU,\dots,\Pi^-_{n_\ell}SU;t,\xi\Big)S\\&\quad=S[\![A_p]\!]\Big(\Pi^+_{n_1}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_1}U,\dots,\Pi^-_{n_\ell}U;t,\xi\Big).\end{aligned} \tag{3.3.11}$$

We recall that $\Pi_n^+ SU = \Pi_n^- U$ using (3.3.4). On the component $[\![\tilde a_p]\!]$ the condition (3.3.11) reads

$$\begin{aligned}&\overline{[\![\tilde a_p]\!]\Big(\Pi_{n_1}^- U,\dots,\Pi_{n_\ell}^- U,\Pi_{n_1}^+ U,\dots,\Pi_{n_\ell}^+ U;\xi\Big)}\\ &\qquad = [\![\tilde a_p]\!]\Big(\Pi_{n_1}^+ U,\dots,\Pi_{n_\ell}^+ U,\Pi_{n_1}^- U,\dots,\Pi_{n_\ell}^- U;-\xi\Big).\end{aligned} \tag{3.3.12}$$

The condition (1.3.8) (which holds since A_p is parity preserving) implies that $\tilde a_p(U,\dots,U;t,\xi)$ is even in ξ since it does not depend on x. Therefore by symmetry we deduce that

$$[\![\tilde a_p]\!]\Big(\Pi_{n_1}^+ U,\dots,\Pi_{n_\ell}^+ U,\Pi_{n_1}^- U,\dots,\Pi_{n_\ell}^- U;t,\xi\Big)$$

is real valued. This concludes the proof. □

The first important result is the following.

Theorem 3.3.2 (Normal form 1). *Let $N,\rho,K>0$ as in Theorem 3.2.1. There exists $\mathcal{N}\subset\mathcal{O}$ with zero measure such that for any $\vec m\in\mathcal{O}\setminus\mathcal{N}$ the following holds. There exist $K''>0$ such that $K':=2\rho+4<K''\ll K$, $s_0>0$, $r_0>0$ (possibly different from the ones given by Theorem 3.2.1) such that, for any $s\ge s_0$, $0<r\le r_0$ and any $U\in B_s^K(I,r)$ solution even in $x\in\mathbb{T}$ of* (3.1.1) *the following holds. There is an invertible (R,R,P)-map*

$$\Theta(U)[\cdot]\,:\,C_{*\mathbb{R}}^{K-K''}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2))\to C_{*\mathbb{R}}^{K-K''}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2)),$$

satisfying the following:

(i) there exists a constant C depending n s,r and K such that

$$\begin{aligned}\|\Theta(U)[V]\|_{K-K'',s}&\le\|V\|_{K-K'',s}(1+C\|U\|_{K,s_0}),\\ \|(\Theta(U))^{-1}[V]\|_{K-K'',s}&\le\|V\|_{K-K'',s}(1+C\|U\|_{K,s_0}),\end{aligned}\tag{3.3.13}$$

for any $V\in C_{*\mathbb{R}}^{K-K''}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2))$;

(ii) $\Theta(U)-\mathbb{1}$ and $(\Theta(U))^{-1}-\mathbb{1}$ belong to the class $\Sigma\mathcal{M}_{K,K'',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$;

(iii) the function $W=\Theta(U)[V]$, where V solves (3.2.2), *satisfies*

$$\begin{aligned}&\partial_t W=\\ &\mathrm{i}E\big(\Lambda W+\mathrm{Op}^{\mathscr{B}W}(L_1(U;t,\xi))[W]+Q_1^{(1)}(U;t)[W]+Q_2^{(1)}(U;t)[U]\big)\end{aligned}\tag{3.3.14}$$

where $Q_1^{(1)}, Q_2^{(1)} \in \Sigma\mathcal{R}^{-\rho+m_1}_{K,K'',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$, *for some* $m_1 > 0$ *depending on* N *(larger than* m *in Theorem 3.2.1), are (R,R,P)-operators and* $L_1(U;t,\xi)$ *is a (R,R,P)-matrix in* $\Sigma\Gamma^2_{K,K'',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *with constant coefficients in* $x \in \mathbb{T}$ *and which has the form (recalling Def. 3.3.1)*

$$L_1(U;t,\xi) := \begin{pmatrix} \mathtt{m}^{(1)}(U;t,\xi) & 0 \\ 0 & \overline{\mathtt{m}^{(1)}(U;-\xi)} \end{pmatrix}, \qquad \mathtt{m}^{(1)}(U;t,\xi) = \mathtt{m}_2(U)(\mathrm{i}\xi)^2 + [\![\mathtt{m}_0^{(1)}]\!](U;t,\xi), \tag{3.3.15}$$

where $\mathtt{m}_2(U)$ *is given in* (3.2.3) *and* $\mathtt{m}_0^{(1)}(U;t,\xi) \in \Sigma\Gamma^0_{K,K'',1}[r,N]$.

3.3.1 Non-resonance conditions

For $M \in \mathbb{N}$ and $\vec{m} = (m_1, \dots, m_M) \in \mathcal{O} := [-1/2, 1/2]^M$ we define, recalling (0.3.4), (0.4.4), for any $N \le M$ and $0 \le \ell \le N$ the function

$$\begin{aligned}\psi_N^\ell(\vec{m},\vec{n}) &= \lambda_{n_1} + \dots + \lambda_{n_\ell} - \lambda_{n_{\ell+1}} - \dots - \lambda_{n_N} \\ &= \sum_{j=1}^{\ell} (\mathrm{i}n_j)^2 - \sum_{j=\ell+1}^{N} (\mathrm{i}n_j)^2 + \sum_{k=1}^{M} m_k \left(\sum_{j=1}^{\ell} \frac{1}{\langle n_j \rangle^{2k+1}} - \sum_{j=\ell+1}^{N} \frac{1}{\langle n_j \rangle^{2k+1}} \right),\end{aligned} \tag{3.3.16}$$

where $\vec{n} = (n_1, \dots, n_N) \in \mathbb{N}^N$ with the convention that $\sum_{j=m}^{m'} a_j = 0$ when $m > m'$. We have the following lemma.

Proposition 3.3.1 (**Non resonance condition).** *There exists* $\mathcal{N} \subset \mathcal{O}$ *with zero Lebesgue measure such that, for any* $\vec{m} \in \mathcal{O} \setminus \mathcal{N}$, *there exist* $\gamma, N_0 > 0$ *such that the inequality*

$$|\psi_N^\ell(\vec{m},\vec{n})| \ge \gamma \max(\langle n_1 \rangle, \dots, \langle n_N \rangle)^{-N_0}, \tag{3.3.17}$$

holds true for any $\vec{n} = (n_1, \dots, n_N) \in \mathbb{N}^N$ *if* N *is odd. In the case that* N *is even the* (3.3.17) *holds true if* $\ell \neq N/2$ *for any* $\vec{n} \in \mathbb{N}^N$*; if* $\ell = N/2$ *the condition* (3.3.17) *holds true for any* $\vec{n}$ *in* $\mathbb{N}^N$ *such that*

$$\{n_1, \dots, n_\ell\} \neq \{n_{\ell+1}, \dots, n_N\}. \tag{3.3.18}$$

Proof. First of all we show that, if $N, \ell, \vec{n}$ are as in the statement of the proposition, the function $\psi_N^\ell(\vec{m},\vec{n})$ is not identically zero as function of $\vec{m}$. We can write

$$\psi_N^\ell(\vec{m},\vec{n}) = a_0^{(\ell)}(\vec{n}) + \sum_{k=1}^{N} m_k a_k^{(\ell)}(\vec{n}) + \sum_{k=N+1}^{M} m_k a_k^{(\ell)}(\vec{n}),$$

where

$$a_0^{(\ell)}(\vec{n}) := \sum_{j=1}^{\ell} (\mathrm{i}n_j)^2 - \sum_{j=\ell+1}^{N} (\mathrm{i}n_j)^2;$$
$$a_k^{(\ell)}(\vec{n}) := \sum_{j=1}^{\ell} \frac{1}{\langle n_j\rangle^{2k+1}} - \sum_{j=\ell+1}^{N} \frac{1}{\langle n_j\rangle^{2k+1}},\ k \geq 1 \tag{3.3.19}$$

for $0 \leq \ell \leq N$. We show that there exists at least one non zero coefficient $a_k^{(\ell)}(\vec{n})$ for $1 \leq k \leq N$.
Let q in $\mathbb{N}^*$ such that there are $N_1, \ldots, N_q$ in $\mathbb{N}^*$ satisfying $N_1 + \ldots + N_q = N$ and

$$\{n_1, \ldots, n_N\} = \left\{n_{1,1}, \ldots, n_{1,N_1}, \ldots, n_{q,1} \ldots, n_{q,N_q}\right\}$$

where

$$\begin{cases} n_{j,i_1} = n_{j,i_2}\ \forall\ i_1, i_2 \in \{1, \ldots, N_j\}, \forall\ j \in \{1, \ldots, q\}, \\ n_{j,1} \neq n_{i,1}\ \forall\ i \neq j. \end{cases}$$

Note that, since $\langle x\rangle = \sqrt{1+x^2}$ for $x \in \mathbb{R}$, then $\langle n_{j,1}\rangle \neq \langle n_{i,1}\rangle$ for any $i \neq j$. According to this notation the element in (3.3.19) can be rewritten as

$$a_k^{(\ell)}(\vec{n}) = \sum_{j=1}^{q} \frac{1}{\langle n_{j,1}\rangle^{2k+1}} (N_j^+ - N_j^-), \tag{3.3.20}$$

where N_j^+, resp. N_j^-, is the number of times that the term $\langle n_{j,1}\rangle^{2k+1}$ appears in the sum in equation (3.3.19) with sign +, resp. with sign −, and hence $N_j^+ + N_j^- = N_j$. Note that if $N_j^+ - N_j^- = 0$ for any $j = 1, \ldots, q$, then the condition (3.3.18) is violated. Define the $(N \times q)$-matrix

$$A_q(\vec{n}) := \begin{pmatrix} \frac{1}{\langle n_{1,1}\rangle^3} & \cdots & \cdots & \frac{1}{\langle n_{q,1}\rangle^3} \\ \frac{1}{\langle n_{1,1}\rangle^5} & \cdots & \cdots & \frac{1}{\langle n_{q,1}\rangle^5} \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{\langle n_{1,1}\rangle^{2N+1}} & \cdots & \cdots & \frac{1}{\langle n_{q,1}\rangle^{2N+1}} \end{pmatrix}.$$

We have

$$\sum_{k=1}^{N} m_k a_k^{(\ell)}(\vec{n}) = \left(A_q(\vec{n})\vec{\sigma}_q^{(\ell)}\right) \cdot \vec{m}_N,$$
$$\vec{\sigma}_q^{(\ell)} := \left((N_1^+ - N_1^-), \ldots, (N_q^+ - N_q^-)\right)^T, \tag{3.3.21}$$

where $\vec{m}_N := (m_1, \ldots, m_N)$ and "$\cdot$" denotes the standard scalar product on $\mathbb{R}^N$. By the above reasoning the vector $\vec{\sigma}_q^{(\ell)}$ is different from $\vec{0}$. We claim that the vector $\vec{v} := A_q(\vec{n})\vec{\sigma}_q^{(\ell)}$ has at least one component different from zero.
Denote by $A_q^q(\vec{n})$ the $(q \times q)$-sub-matrix of $A_q(\vec{n})$ made of its firsts q rows. The matrix $A_q^q(\vec{n})$ is, up to rescaling the k-th column by the factor $\langle n_{1,k}\rangle^3$, a Vandermonde matrix, therefore

$$\det(A_q^q(\vec{n})) = \left(\prod_{j=1}^{q} \frac{1}{\langle n_{j,1}\rangle^3}\right) \prod_{1\le i<k\le q} \left(\frac{1}{\langle n_{i,1}\rangle^2} - \frac{1}{\langle n_{k,1}\rangle^2}\right), \tag{3.3.22}$$

which is different from zero since $n_{i,1} \neq n_{k,1}$ for any $1 \le i < k \le q$; this implies that $\mathrm{Rank}(A_q(\vec{n})) = q$, hence the claim follows since $\sigma_q^{(\ell)} \neq \vec{0}$.
Fix $\gamma > 0$, $N \le M \in \mathbb{N}$, $N_0 \in \mathbb{N}$ and $0 \le \ell \le N$; we introduce the following "bad" set

$$\mathscr{B}_{N,N_0}(\vec{n},\gamma,\ell) := \left\{\vec{m} \in \mathscr{O} \,:\, |\psi_N^\ell(\vec{m},\vec{n})| < \gamma \max\big(\langle n_1\rangle, \ldots, \langle n_N\rangle\big)^{-N_0}\right\}. \tag{3.3.23}$$

We give an estimate of the sub-levels of the function $\psi_N^\ell(\vec{m},\vec{n})$. By the discussion above there exists $1 \le k' \le q$ such that $a_{k'}^{(\ell)}(\vec{n}) \neq 0$, set $k_\infty \in \{1, \cdots, q\}$ the index such that $|a_{k_\infty}^{(\ell)}(\vec{n})| = \big(A_q^q(\vec{n})\vec{\sigma}^{(\ell)}\big)_{k_\infty} = \left\| A_q^q(\vec{n})\vec{\sigma}^\ell \right\|_\infty > 0$. We start by proving that there exist constants $\mathsf{c} \ll 1$ and $\mathsf{b} \gg 1$, both depending only on q (and hence only on N), such that

$$|\partial_{m_{k_\infty}} \psi_N^\ell(\vec{m},\vec{n})| \ge \frac{\mathsf{c}}{\max\big(\langle n_1\rangle, \ldots, \langle n_N\rangle\big)^{\mathsf{b}}}. \tag{3.3.24}$$

We have

$$|\partial_{m_{k_\infty}} \psi_N^\ell(\vec{m},\vec{n})| = |a_{k_\infty}^{(\ell)}(\vec{n})| \overset{(3.3.21)}{=} |(A_q^q(\vec{n})\vec{\sigma}_q^{(\ell)})_{k_\infty}| \ge K(\det A_q^q(\vec{n})), \tag{3.3.25}$$

where $1 \gg K = K(N) > 0$ depends only on N. The last inequality in (3.3.25) follows by the fact that $A_q^q(\vec{n})$ is invertible, hence

$$1 \le |(A_q^q(\vec{n}))^{-1} A_q^q(\vec{n})\vec{\sigma}_q^{(\ell)}| \le (\det A_q^q(\vec{n}))^{-1} N^2 C_N \left\| A_q^q(\vec{n})\vec{\sigma}_q^{(\ell)} \right\|_\infty,$$

with $C_N > 0$ and we have used $q \le N$. By formula (3.3.22) one can deduce that

$$|\det A_q^q(\vec{n})| \ge \frac{\widetilde{K}}{\max\big(\langle n_1\rangle, \ldots, \langle n_N\rangle\big)^{\mathsf{b}}}$$

where $\mathtt{b}$ and $\widetilde{K}$ depend only on N. The latter inequality, together with (3.3.25), implies the (3.3.24). Estimate (3.3.24) implies that

$$\text{meas}\Big(\mathscr{B}_{N,N_0}(\vec{n},\gamma,\ell)\Big) \leq \frac{\gamma}{\mathtt{c}\max\big(\langle n_1\rangle,\ldots,\langle n_N\rangle\big)^{N_0-\mathtt{b}}}.$$

Hence, for $N_0 \geq \mathtt{b}+2+N$, one obtains

$$\text{meas}\Big(\bigcap_{\gamma>0}\bigcup_{\vec{n}\in\mathbb{N}^N}\mathscr{B}_{N,N_0}(\vec{n},\gamma,\ell)\Big) \leq \lim_{\gamma\to 0}\frac{\gamma}{\mathtt{c}}\sum_{\vec{n}\in\mathbb{N}^N}\frac{1}{\max\big(\langle n_1\rangle,\ldots,\langle n_N\rangle\big)^{N_0-\mathtt{b}}} = 0.$$

By setting

$$\mathscr{N} := \bigcup_{0\leq N\leq M}\bigcap_{\gamma>0}\bigcup_{\vec{n}\in\mathbb{N}^N}\mathscr{B}_{N,N_0}(\vec{n},\gamma,\ell),$$

one gets the thesis. □

3.3.2 Normal forms

In this Section we prove Theorem 3.3.2. The proof will be based on an iterative use of the following lemma.

Lemma 3.3.2. *Fix $p,K,N\in\mathbb{N}$, $r,\rho>0$, $1\leq p\leq N-1$ and $K'\leq K$. For $U\in B^K_{s_0}(I,r)\cap C^K_{*\mathbb{R}}(I;\mathbf{H}^s_e)$ be a solution of* (3.1.1) *consider the system*

$$\partial_t V = \mathrm{i}E\big(\Lambda V + \mathrm{Op}^{\mathscr{B}W}(\tilde{L}^{(p)}(U;t,\xi))[V] + G_1^{(p)}(U;t)[V] + G_2^{(p)}(U;t)[U]\big), \quad (3.3.26)$$

where $G_1^{(p)}(U;t), G_2^{(p)}(U;t)\in\Sigma\mathscr{R}^{-\rho}_{K,K',1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ are (R,R,P)-operator and $\tilde{L}^{(p)}(U;t,\xi)$ is a diagonal and constant coefficients in x (R,R,P)-matrix of the form

$$\begin{aligned}\tilde{L}^{(p)}(U;t,\xi) &:= \begin{pmatrix}\mathtt{m}^{(p)}(U;t,\xi) & 0\\ 0 & \overline{\mathtt{m}^{(p)}(U;t,-\xi)}\end{pmatrix},\\ \mathtt{m}^{(p)}(U;t,\xi) &= \mathtt{m}_2(U;t)(\mathrm{i}\xi)^2 + \mathtt{m}_0^{(p)}(U;t,\xi),\end{aligned} \quad (3.3.27)$$

where $\mathtt{m}_2(U;t)$ is the real symbol in $\Sigma\mathscr{F}_{K,K',1}[r,N]$ given in (3.2.3)*, while $\mathtt{m}_0^{(p)}(U;t,\xi)\in$*

$\Sigma\Gamma^0_{K,K',1}[r,N]$ *is such that (recalling Def. 3.3.1)*

$$\begin{aligned}\mathtt{m}_0^{(1)}(U;t,\xi) &= \sum_{j=1}^{N-1} m_j^{(1)}(U,\dots,U;t,\xi) + m_N^{(1)}(U;t,\xi),\\ \mathtt{m}_0^{(p)}(U;t,\xi) &= \sum_{j=1}^{p-1} [\![m_j^{(p)}]\!](U,\dots,U;t,\xi)\\ &\quad + \sum_{j=p}^{N-1} m_j^{(p)}(U,\dots,U;t,\xi) + m_N^{(p)}(U;t,\xi),\end{aligned} \tag{3.3.28}$$

for $2 \le p \le N-1$ *and where*

$$m_j^{(p)} \in \widetilde{\Gamma}^0_j, \quad j = 1,\dots,N-1, \quad m_N^{(p)} \in \Sigma\Gamma^0_{K,K',N}[r,N]. \tag{3.3.29}$$

For r *small enough and* $\vec{m}$ *outside the subset* $\mathcal{N}$ *given by Proposition 3.3.1 the following holds. There is* $s_0 > 0$ *such that for* $s \ge s_0$*, there is an invertible (R,R,P)-map*

$$\Theta_p(U)[\cdot] : C_{*\mathbb{R}}^{K-K'}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2)) \to C_{*\mathbb{R}}^{K-K'}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2)), \tag{3.3.30}$$

satisfying the following:

(i) there exist C *depending on* s,r,K *such that*

$$\begin{aligned}\|\Theta_p(U)[V]\|_{K-K',s} &\le \|V\|_{K-K',s}(1 + C\|U\|_{K,s_0}),\\ \left\|\big(\Theta_p(U)\big)^{-1}[V]\right\|_{K-K',s} &\le \|V\|_{K-K',s}(1 + C\|U\|_{K,s_0}),\end{aligned} \tag{3.3.31}$$

for any $V \in C_{*\mathbb{R}}^{K-K'}(I,\mathbf{H}^s(\mathbb{T};\mathbb{C}^2))$*;*

(ii) $\Theta_p(U) - \mathbb{1}$ *and* $(\Theta_p(U))^{-1} - \mathbb{1}$ *belong to the class* $\Sigma\mathcal{M}_{K,K',1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$*;*

(iii) the function $W = \Theta_p(U)[V]$ *satisfies*

$$\begin{aligned}\partial_t W = &\mathrm{i}E\big(\Lambda W + \mathrm{Op}^{\mathscr{B}W}(\tilde{L}^{(p+1)}(U;t,\xi))[W]\big)\\ &+ \mathrm{i}E(G_1^{(p+1)}(U;t)[W] + G_2^{(p+1)}(U;t)[U]\big),\end{aligned} \tag{3.3.32}$$

where V *satisfies* (3.3.26)*. The operators* $G_1^{(p+1)}(U;t), G_2^{(p+1)}(U;t)$ *are (R,R,P)-operators in the class* $\Sigma\mathscr{R}^{-\rho+\tilde{m}}_{K,K'+1,1}[r,N] \otimes \mathcal{M}_2(\mathbb{C})$ *for some* $\tilde{m} > 0$*,* $\tilde{L}^{(p+1)}(U;t,\xi)$

is a (R,R,P)-matrix in $\Sigma\Gamma^{2}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ *with constant coefficients in* $x\in\mathbb{T}$ *and it has the form*

$$\tilde{L}^{(p+1)}(U;t,\xi) := \begin{pmatrix} \mathtt{m}^{(p+1)}(U;t,\xi) & 0 \\ 0 & \overline{\mathtt{m}^{(p+1)}(U;t,-\xi)} \end{pmatrix}, \qquad (3.3.33)$$
$$\mathtt{m}^{(p+1)}(U;t,\xi) = \mathtt{m}_2(U;t)(\mathrm{i}\xi)^2 + \mathtt{m}_0^{(p+1)}(U;t,\xi),$$

where $\mathtt{m}_2(U;t)$ *is given in* (3.2.3)*, the symbol* $\mathtt{m}_0^{(p+1)}(U;t,\xi)$ *is in* $\Sigma\Gamma^{0}_{K,K'+1,1}[r,N]$ *and it has the form*

$$\mathtt{m}_0^{(p+1)}(U;t,\xi) = \sum_{j=1}^{p}[\![m_j^{(p)}]\!](U,\dots,U;t,\xi) + \sum_{j=p+1}^{N-1} m_j^{(p+1)}(U,\dots,U;t,\xi) + m_N^{(p+1)}(U;t,\xi), \qquad (3.3.34)$$

where $m_j^{(p)}\in\widetilde{\Gamma}^0_j$*,* $j=1,\dots,p$ *are given in* (3.3.29) *and*

$$m_j^{(p+1)}\in\widetilde{\Gamma}^0_j, \quad j=p+1,\dots,N-1, \quad m_N^{(p+1)}\in\Gamma^0_{K,K'+1,N}[r]. \qquad (3.3.35)$$

Proof. Let $f(U;t,\xi)$ be a symbol in $\widetilde{\Gamma}^0_p$ which has constant coefficients in $x\in\mathbb{T}$. Consider the system

$$\partial_\tau W(\tau) = \mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U;t,\xi))[W(\tau)],$$
$$\widehat{F}^{(p)}(U;t,\xi) := \begin{pmatrix} f(U;t,\xi) & 0 \\ 0 & \overline{f(U;t,-\xi)} \end{pmatrix}. \qquad (3.3.36)$$

Suppose moreover that the matrix $\widehat{F}^{(p)}(U;t,\xi)$ is a (R,R,P)-matrix of symbols. By standard theory of ODEs on Banach spaces the flow $\Theta_p^\tau(U)[\cdot]$ of (3.3.36) is well defined for $\tau\in[0,1]$. We set $\Theta_p(U)[\cdot] := \Theta_p^\tau(U)[\cdot]_{|\tau=1}$. Estimates (3.3.31) hold by direct computation. Item (ii) follows by Taylor expanding $\Theta_p^\tau(U)[\cdot]$ in $\tau=0$ and by using Remark 1.1.14 and item (i) of Proposition 1.1.3. The same argument implies the following further properties of the map $\Theta_p(U)[\cdot]$:

$$\Theta_p(U)[\cdot] = \mathbb{1} + \mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U;t,\xi)) + \mathrm{Op}^{\mathscr{B}W}(C^+(U;t,\xi))[\cdot] + R^+(U;t)[\cdot], \qquad (3.3.37)$$

$$(\Theta_p(U))^{-1}[\cdot] = \mathbb{1} - \mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U;t,\xi)) + \mathrm{Op}^{\mathscr{B}W}(C^-(U;t,\xi))[\cdot] + R^-(U;t)[\cdot], \qquad (3.3.38)$$

for some (R,R,P)-matrices of symbols $C^{+}(U;t,\xi)$, $C^{-}(U;t,\xi)$ independent of x belonging to $\Sigma\Gamma^{0}_{K,K',p+1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ and some (R,R,P)-operators $R^{+}(U;t)[\cdot]$, $R^{-}(U;t)[\cdot]$ belonging to $\Sigma\mathcal{R}^{-\rho}_{K,K',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ (actually the homogeneity of these remainders is bigger than p but, at this level, we do not emphasize this property and we embed them in the remainders of homogeneity 1).

Finally, since $\widehat{F}^{(p)}(U,\dots,U;t,\xi)$ is a (R,R,P)-matrix of symbols then $\Theta^{\tau}_{p}(U)[\cdot]$ is reversibility preserving. Indeed, by setting $G^{\tau}=S\Theta^{\tau}_{p}(U;-t)-\Theta^{\tau}_{p}(U_S;t)S$, one can note that

$$\partial_{\tau}G^{\tau}=\mathrm{Op}^{\mathscr{B}W}\Big(\widehat{F}^{(p)}(SU;t)\Big)G^{\tau},$$

with $G^{0}=0$, where we used that $S\widehat{F}^{(p)}(U;-t)=\widehat{F}^{(p)}(SU;t)$ (which is (1.3.18)). This implies that $G^{\tau}\equiv 0$ for $\tau=[0,1]$, which means that $\Theta_{p}(U;t)$ is reversibility preserving.
Since U solves (3.1.1), there is a (R,R,P)-map $M\in\Sigma\mathcal{M}^{\tilde{m}}_{K,0,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, for some $\tilde{m}>0$, such that

$$\partial_{t}U=\mathrm{i}E\Lambda U+\mathrm{i}EM(U;t)[U]. \tag{3.3.39}$$

Hence, by taking the derivative w.r.t. the variable t in (3.3.37), we have

$$\begin{aligned}\partial_{t}(\Theta_{p}(U))[\cdot]&=\sum_{j=1}^{p}\mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U,\dots,\underbrace{\partial_{t}U}_{j-th},U,\dots,U;t,\xi))\\&+\mathrm{Op}^{\mathscr{B}W}(\partial_{t}C^{+}(U;t,\xi))[\cdot]+(\partial_{t}R^{+}(U;t))[\cdot]\\&=\sum_{j=1}^{p}\mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U,\dots,\underbrace{\mathrm{i}E\Lambda U}_{j-th},U,\dots,U;t,\xi))+\\&+\mathrm{Op}^{\mathscr{B}W}(B(U;t,\xi))[\cdot]+\widetilde{R}^{+}(U;t)[\cdot],\end{aligned} \tag{3.3.40}$$

for some $B(U;t,\xi)\in\Sigma\Gamma^{0}_{K,K'+1,p+1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, and $\widetilde{R}^{+}(U;t)\in\Sigma\mathcal{R}^{-\rho+\tilde{m}}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$, where we used Proposition 1.1.3 (in particular items $(iv),(v)$) and $\tilde{m}$ is the loss given by M in (3.3.39). We fix $0\le\rho'=\rho-\tilde{m}$ which is possible since $\rho\gg 1$.

Now if $W = \Theta_p(U)[V]$ one has that

$$
\begin{aligned}
\partial_t W &= \Theta_p(U)\Big[\mathrm{i}E\big(\Lambda + \mathrm{Op}^{\mathscr{B}W}(\tilde{L}^{(p)}(U;t,\xi)) + G_1^{(p)}(U;t)\big)\Big](\Theta_p(U))^{-1}[W] \\
&\quad + \mathrm{i}E\Theta_p(U)G_2^{(p)}(U;t)U + (\partial_t\Theta_p(U))(\Theta_p(U))^{-1}[W] = \\
&= \mathrm{i}E\big(\Lambda W + \mathrm{Op}^{\mathscr{B}W}(\tilde{L}^{(p)}(U;t,\xi))[W]\big) \\
&\quad + \sum_{j=1}^{p} \mathrm{Op}^{\mathscr{B}W}(\widehat{F}^{(p)}(U,\ldots,\underbrace{\mathrm{i}E\Lambda U}_{j-th},U,\ldots,U;t,\xi))[W] + \\
&\quad + \mathrm{i}E\mathrm{Op}^{\mathscr{B}W}(C_1(U;t,\xi))[W] + \mathrm{i}EG_3(U;t)[W] + \mathrm{i}EG_4(U;t)[U],
\end{aligned}
\tag{3.3.41}
$$

for some (R,R,P)-matrix of symbols $C_1(U;t,\xi)$ independent of x belonging to $\Sigma\Gamma^0_{K,K'+1,p+1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$ and some (R,R,P)-operators $G_3(U;t)[\cdot], G_4(U;t)[\cdot]$ belonging to $\Sigma\mathscr{R}^{-\rho'}_{K,K'+1,1}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$. In the previous computation we used Proposition 1.1.3, the (3.3.37), (3.3.38) and (3.3.40). In particular we used also the fact that the matrix $\tilde{L}^{(p)}(U;t,\xi)$, together with the matrices of symbols appearing in (3.3.37) and (3.3.38), have constant coefficients in x. In the notation of item (iii) of the statement we look for $\widehat{F}^{(p)}(U,\ldots,U;t,\xi)$ such that

$$
\begin{aligned}
&\sum_{j=1}^{p} f(U,\ldots,\underbrace{\mathrm{i}E\Lambda U}_{j-th},U,\ldots,U;t,\xi) + \mathrm{i}m_p^{(p)}(U,\ldots,U;t,\xi) = \\
&\quad \mathrm{i}[\![m_p^{(p)}]\!](U;\ldots,U;t,\xi).
\end{aligned}
\tag{3.3.42}
$$

Recalling the definition of the operator Λ in (0.4.6) (see also (0.4.4), (0.3.4), and (3.3.16)), we have that, passing to Fourier series, the equation (3.3.42) is equivalent to

$$
\begin{aligned}
&\psi_p^{\ell}(\vec{m},\vec{n}) f\big(\Pi^+_{n_1}U,\ldots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\ldots,\Pi^-_{n_p}U;t,\xi\big) = \\
&- m_p^{(p)}\big(\Pi^+_{n_1}U,\ldots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\ldots,\Pi^-_{n_p}U;t,\xi\big)
\end{aligned}
$$

in the following cases:

- p is odd, $0\le\ell\le p$ and for any $\vec{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$;
- p is even , $0\le\ell\le p$ with $\ell\neq p/2$ and for any $\vec{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$;
- p is even, $\ell=p/2$ and for any $\vec{n}=(n_1,\ldots,n_p)\in\mathbb{N}^p$ such that

$$
\{n_1,\ldots,n_\ell\}\neq\{n_{\ell+1},\ldots,n_p\}.
$$

By estimate (3.3.17) on $\psi_p^\ell(\vec{m},\vec{n})$, we get that $f(U,\dots,U;t,\xi)$ is a symbol in $\widetilde{\Gamma}_p^0$ and does not depend on x since so does $m_p^{(p)}$. Furthermore, since $\widetilde{L}^{(p)}$ in (3.3.27) is a (R,R,P)-matrix of symbols, one has that $\widehat{F}^{(p)}(U,\dots,U;t,\xi)$ in (3.3.36) is even in ξ, and reality preserving (i.e. satisfies resp. (1.3.8) and (1.3.9)). Finally, since $m_p^{(p)}(U,\dots,U;t,\xi)$ satisfies (1.3.13) and the function $\psi_p^\ell(\vec{m},\vec{n})$ in (3.3.16) is real and even in each component of $\vec{n}$, one has that the symbol $\widehat{F}^{(p)}(U,\dots,U;t,\xi)$ satisfies (1.3.13). Thanks to the choice of f above the equation (3.3.41) has the form (3.3.32) for a suitable (R,R,P)-matrix of symbols $\tilde{L}^{(p+1)}$ of the form (3.3.33). □

Proof of Theorem 3.3.2. Let $\mathcal{N}$ be the set of parameters $\vec{m}\in[-1/2,1/2]^M$ given in Proposition 3.3.1. We apply Lemma 3.3.2 to the system (3.2.2) since it has the form (3.3.26) with $p=1$, $\tilde{L}^{(1)}\rightsquigarrow L$ in (3.2.3), $G_1^{(1)},G_2^{(1)}\rightsquigarrow Q_1,Q_2$, and $\rho\rightsquigarrow\rho-m$ (with m given by Theorem 3.2.1). The lemma guarantees the existence of a map $\Theta_1(U)[\cdot]$ (see (3.3.30)) such that the function $W_1=\Theta_1(U)[V]$ satisfies a system of the form (3.3.32) with $\tilde{L}^{(2)}$ given in (3.3.33) with $p=1$ and where $G_1^{(2)},G_2^{(2)}$ are some operators in $\Sigma\mathcal{R}^{-\rho^{(1)}}_{K,K'+1,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$. Here $\rho^{(1)}=\rho-m-\tilde{m}$ where $\tilde{m}$ is the loss of derivatives produced by the map $M(U;t)$ given in (3.3.39). This new system still satisfies the hypotheses of Lemma 3.3.2, hence we may apply it iteratively. We obtain a sequence of maps $\Theta_j(U)[\cdot]$ for $j=1,\dots,N-1$ such that $W_j:=\Theta_j(U)[W_{j-1}]$ satisfies a system of the form (3.3.32) for suitable matrices of symbols $\tilde{L}^{(j+1)}$ given in (3.3.33) with $p=j$ and where the remainders $G_1^{(j+1)},G_2^{(j+1)}$ belong to $\Sigma\mathcal{R}^{-\rho^{(j)}}_{K,K'+j,1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ where $\rho^{(j)}\sim\rho-m-j\tilde{m}$, which is positive since $\rho\gg N$ in Theorem 3.2.1. We set

$$\Theta(U)[\cdot]:=\Theta_{N-1}(U)\circ\dots\circ\Theta_1(U)[\cdot],$$

which satisfies items (i), (ii) because each map Θ_j, $j=1,\dots,N-1$ has similar properties by Lemma 3.3.2. With this choice, the constant coefficients in x matrix of symbols $L_1(U;t,\xi)$ in (3.3.14) is equal to $\tilde{L}^{(N-1)}(U;t,\xi)$ (given in (3.3.33), (3.2.3) and (3.3.34) with $p=N-1$), which satisfies (3.3.15). The smoothing remainders $Q_1^{(1)}(U;t),Q_2^{(2)}(U;t)$ belong to the class $\Sigma\mathcal{R}^{\rho-m_1}_{K,K'',1}[r,N]\otimes\mathcal{M}_2(\mathbb{C})$ where $K''\sim K'+N$ and $m_1=m+(N-1)\tilde{m}$. □

3.3.3 Modified energies

In this subsection we give the proof of Theorem 3.3.1. We introduce the classes of multilinear forms which will be used to construct modified energies for a system of the form (3.3.14).

Definition 3.3.2. *Let $\rho, s \in \mathbb{R}$ with $\rho, s \geq 0$ and $p \in \mathbb{N}$. One denotes by $\widetilde{\mathscr{L}}^{s,-\rho}_{p,\pm}$ the space of symmetric $(p+2)$-linear forms*

$$(U_0, \dots, U_{p+1}) \to L(U_0, \dots, U_{p+1})$$

defined on $C^\infty(\mathbb{T};\mathbb{C}^2)$ and satisfying for some $\mu \in \mathbb{R}_+$ and any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}$ and any $(U_0, \dots, U_{p+1}) \in (C^\infty(\mathbb{T};\mathbb{C}^2))^{p+2}$,

$$\begin{aligned}|L(\Pi_{n_0} U_0, \dots, \Pi_{n_{p+1}} U_{p+1})| \leq C \max(\langle n_0 \rangle, \dots, \langle n_{p+1} \rangle)^{2s-\rho} \\ \times \max_3(\langle n_0 \rangle, \dots, \langle n_{p+1} \rangle)^{\mu+\rho} \prod_{j=0}^{p+1} \|\Pi_{n_j} U_j\|_{L^2}\end{aligned} \tag{3.3.43}$$

where $\max_3(\langle n_0 \rangle, \dots, \langle n_{p+1} \rangle)$ is the third largest value among $\langle n_0 \rangle, \dots, \langle n_{p+1} \rangle$, and such that

$$L(\Pi_{n_0} U_0, \dots, \Pi_{n_{p+1}} U_{p+1}) \neq 0 \Rightarrow \sum_{j=0}^{p+1} \sigma_j n_0 = 0, \tag{3.3.44}$$

for some choice of the signs $\sigma_j \in \{+1, -1\}$ for $j = 0, \dots, p+1$, and for any $U_0, \dots, U_{p+1}$ satisfying (1.3.3),

$$L(SU_0, \dots, SU_{p+1}) = \pm L(U_0, \dots, U_{p+1}). \tag{3.3.45}$$

The following lemma collects some properties of the class $\widetilde{\mathscr{L}}^{s,-\rho}_{p,\pm}$.

Lemma 3.3.3. *The following facts hold true.*
(i) Fix $\rho \geq 0$, $p \in \mathbb{N}^*$ and consider $R \in \widetilde{R}^{-\rho}_p$ satisfying (1.3.24) (resp. (1.3.23)). One has that the $L(U_0, \dots, U_{p+1})$ defined as the symmetrization of

$$(U_0, \dots, U_{p+1}) \to \int_{\mathbb{T}} (\langle D \rangle^s SU_0)(\langle D \rangle^s R(U_1, \dots, U_p) U_{p+1}) dx \tag{3.3.46}$$

belongs to $\widetilde{\mathscr{L}}^{s,-\rho}_{p,+}$ (resp. $\widetilde{\mathscr{L}}^{s,-\rho}_{p,-}$).
(ii) Let $L \in \widetilde{\mathscr{L}}^{s,-\rho}_{p,\pm}$. Then for any $m \geq 0$ such that $\rho > m+1$ and any $s > \rho + \mu + \frac{1}{2}$, L extends as a continuous $(p+2)$-linear form on $H^s(\mathbb{T};\mathbb{C}^2) \times \dots \times H^s(\mathbb{T};\mathbb{C}^2) \times H^{s-m}(\mathbb{T};\mathbb{C}^2) \times H^s(\mathbb{T};\mathbb{C}^2) \times \dots \times H^s(\mathbb{T};\mathbb{C}^2)$.
(iii) Let $p = 2\ell$ with $\ell \in \mathbb{N}^*$ and $L \in \widetilde{\mathscr{L}}^{s,-\rho}_{p,-}$. For U even in x satisfying (1.3.3) one has, for $n_0, \dots, n_\ell \in \mathbb{N}^*$,

$$L(\Pi^+_{n_0} U, \dots, \Pi^+_{n_\ell} U, \Pi^-_{n_0} U, \dots, \Pi^-_{n_\ell} U) = 0. \tag{3.3.47}$$

(iv) Let $\vec{m}$ be outside the subset $\mathcal{N}$ and N_0 given by Proposition 3.3.1. Then for any $L \in \widetilde{\mathscr{L}}^{s,-\rho}_{p,-}$ there is $\tilde{L} \in \widetilde{\mathscr{L}}^{s,-\rho+N_0}_{p,+}$ such that

$$\sum_{j=0}^{p+1} \widetilde{L}(U,\dots,\underbrace{E\Lambda U}_{j-th},\dots,U) = \mathrm{i}L(U,\dots,U), \tag{3.3.48}$$

where E and Λ are defined in (0.4.5) *and* (0.4.6) *respectively.*
(v) Let $L \in \widetilde{\mathscr{L}}^{s,-\rho}_{p,\pm}$ and $M \in \Sigma\mathscr{M}^m_{K,K',q}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$ (see Def. 1.1.7) which is reality preserving and reversible (resp. reversibility preserving) according to Def. 1.3.3. Then

$$U \to L(U,\dots,U,M(U;t)U,U,\dots,U) \tag{3.3.49}$$

can be written as the sum $\sum_{q'=0}^{N-p-q-1} L_{q'}$, for some $L_{q'} \in \widetilde{\mathscr{L}}^{s,-\rho+m}_{p+q+q',\mp}$ (resp. $\widetilde{\mathscr{L}}^{s,-\rho+m}_{p+q+q',\pm}$), plus a term that, at any time t, is

$$O(\|U(t,\cdot)\|^{p+2}_{\mathbf{H}^s}\|U(t,\cdot)\|^{N-p}_{K',\sigma} + \|U(t,\cdot)\|^{p+1}_{\mathbf{H}^s}\|U(t,\cdot)\|^{N-p+1}_{K',\sigma}\|U(t,\cdot)\|_{K',s}) \tag{3.3.50}$$

if $s > \sigma \gg \rho$ and if $\|U(t,\cdot)\|_{K',\sigma}$ is bounded.

Proof. We prove item (i). First of all we note the following

$$\begin{aligned} &|L(\Pi_{n_0}U_0,\dots,\Pi_{n_{p+1}}U_{p+1})| \le \\ &\quad \langle n_0\rangle^{2s}\left\|\Pi_{n_0}U_0\right\|_{L^2}\left\|\Pi_{n_0}R(\Pi_{n_1}U_1,\dots\Pi_{n_p}U_p)\Pi_{n_{p+1}}U_{p+1}\right\|_{L^2}, \end{aligned} \tag{3.3.51}$$

therefore the condition (3.3.44) is implied by (1.1.7), moreover by using (1.1.6) we can bound the r.h.s. of (3.3.51) by

$$\langle n_0\rangle^{2s}\frac{\max_2(\langle n_1\rangle,\dots,\langle n_{p+1}\rangle)^{\mu+\rho}}{\max(\langle n_1\rangle,\dots,\langle n_{p+1}\rangle)^{\rho}}\prod_{j=0}^{p+1}\left\|\Pi_{n_j}U_j\right\|_{L^2}. \tag{3.3.52}$$

We assume $n_1 \ge n_2 \ge \dots \ge n_{p+1}$, moreover thanks to (3.3.44) we have $|n_0 - n_1| \le Cn_2$ for some constant $C > 0$. If $n_0 \ge \frac{1}{2}n_1$ then $n_0 \sim n_1$, therefore

$$\max_2(n_1,\dots,n_{p+1}) \sim \max_3(n_0,\dots,n_{p+1}),$$

so the (3.3.43) follows. In the case that $n_0 \le \frac{1}{2}n_1$ we have $|n_0 - n_1| \le n_2$ so that $n_1 \sim n_2$. Hence we have $\langle n_0\rangle \le C\max_3(\langle n_0\rangle,\dots,\langle n_{p+1}\rangle)$, so that we can bound (3.3.52) by

$$\max(\langle n_0\rangle,\dots,\langle n_{p+1}\rangle)^{2s+\mu}\left(\frac{\max_3(\langle n_0\rangle,\dots,\langle n_{p+1}\rangle)}{\max(\langle n_0\rangle,\dots,\langle n_{p+1}\rangle)}\right)^{2s},$$

which, for $2s \geq \mu+\rho$, is controlled by the r.h.s. of (3.3.43). We still need to prove (3.3.45) for the form defined in (3.3.46), but this follows from the symmetry of the matrix S and from the property (1.3.18) (resp. (1.3.17)) of the operator R.
We prove item (ii). Fix an index ℓ_0 in $\{0,\dots,p+1\}$ and assume $0 \neq U_\ell \in H^s$ for $\ell = 0,\dots,p+1$ with $\ell \neq \ell_0$ and $0 \neq U_{\ell_0}$ in H^{s-m}. Let us assume $n_0 \geq n_1 \geq \dots \geq n_{p+1}$, then by using (3.3.43) we have

$$
\begin{aligned}
\sum_{n_0,\dots,n_{p+1}} &|L(\Pi_{n_0}U_0,\dots,\Pi_{n_{p+1}}U_{p+1})| \leq \\
&\leq \sum_{n_0,\dots,n_{p+1}} \langle n_0\rangle^{2s-\rho}\langle n_2\rangle^{\rho+\mu}\prod_{j=0}^{p+1}\left\|\Pi_{n_j}U_j\right\|_{L^2} \\
&= \sum_{n_0,\dots,n_{p+1}} \langle n_0\rangle^{2s-\rho}\langle n_2\rangle^{\rho+\mu}\Big(\prod_{\substack{j=0\\ j\neq\ell_0}}^{p+1} c^j_{n_j}\langle n_j\rangle^{-s}\Big)c^{\ell_0}_{n_{\ell_0}}\langle n_{\ell_0}\rangle^{-s+m} \\
&\quad\times\prod_{\substack{j=0\\ j\neq\ell_0}}^{p+1}\|U_j\|_{H^s}\|U_{\ell_0}\|_{H^{s-m}},
\end{aligned}
\tag{3.3.53}
$$

where we have denoted by $c^j_{n_j}$ the ℓ^2-sequence $\langle n_j\rangle^s\frac{\|\Pi_{n_j}U_j\|_{L^2}}{\|U\|_{H^s}}$ and by $c^{\ell_0}_{n_{\ell_0}}$ the ℓ^2-sequence $\langle n_{\ell_0}\rangle^{s-m}\frac{\|\Pi_{n_{\ell_0}}U_{\ell_0}\|_{L^2}}{\|U\|_{H^{s-m}}}$. The sum is restricted to the set of indices such that $\sum_{j=0}^{p+1}\varepsilon_j n_j = 0$ for a choice of signs $\varepsilon_j \in \{\pm1\}$, therefore we have $n_0 \sim n_1$. One needs to study the convergence of the series

$$
\sum_{n_0,n_1,n_2}\langle n_0\rangle^{s-\rho+m}c^0_{n_0}\langle n_1\rangle^{-s}c^1_{n_1}\langle n_2\rangle^{-s+\rho+\mu}c^2_{n_2},
$$

it is enough to write $\langle n_1\rangle^{-s} = \langle n_1\rangle^{-s+\frac12+\varepsilon}\langle n_1\rangle^{-\frac12-\varepsilon}$ for $\varepsilon > 0$, use the fact that $n_1 \sim n_2$ and Cauchy-Schwartz inequality.
We prove item (iii). By definition we have

$$
L(SU_0,\dots,SU_{p+1}) = -L(U_0,\dots,U_{p+1}). \tag{3.3.54}
$$

Write $U_j = [\hat u_{n_j},\bar{\hat u}_{n_j}]^T\varphi_{n_j} = \hat u_{n_j}\varphi_{n_j}e_+ + \bar{\hat u}_{n_j}\varphi_{n_j}e_-$, where the vectors e_+ and e_- are defined in (3.3.4). Note that $Se_\pm = e_\mp$, therefore from equation (3.3.54) we deduce

$$
\begin{aligned}
&L(\hat u_{n_0}\varphi_{n_0}e_- + \bar{\hat u}_{n_0}\varphi_{n_0}e_+,\dots,\hat u_{n_{p+1}}\varphi_{n_{p+1}}e_- + \bar{\hat u}_{n_{p+1}}\varphi_{n_{p+1}}e_+) = \\
&-L(\hat u_{n_0}\varphi_{n_0}e_+ + \bar{\hat u}_{n_0}\varphi_{n_0}e_-,\dots,\hat u_{n_{p+1}}\varphi_{n_{p+1}}e_+ + \bar{\hat u}_{n_{p+1}}\varphi_{n_{p+1}}e_-).
\end{aligned}
$$

By using the $\mathbb{C}$-linearity in each argument and identifying the terms of the form $\hat{u}_{n_0}\cdots\hat{u}_{n_\ell}\bar{\hat{u}}_{n_{\ell+1}}\cdots\bar{\hat{u}}_{n_{p+1}}$ on each side we get

$$\begin{gathered} L(\varphi_{n_0}e_-,\dots,\varphi_{n_\ell}e_-,\varphi_{n_{\ell+1}}e_+,\dots\varphi_{n_{\ell+1}}e_+) = \\ -L(\varphi_{n_0}e_+,\dots,\varphi_{n_\ell}e_+,\dots,\varphi_{n_{\ell+1}}e_-,\dots\varphi_{n_{p+1}}e_-). \end{gathered} \tag{3.3.55}$$

If p is even, $\ell=(p+2)/2$ and $n_0=n_{\ell+1},\dots,n_\ell=n_{p+1}$ then by symmetry we get the thesis.
We prove item (iv). Let us decompose

$$\begin{aligned} &L(U,\dots,U)= \\ &\sum_{\ell=-1}^{p+1}\binom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}}L(\Pi^+_{n_0},\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_{p+1}}U). \end{aligned} \tag{3.3.56}$$

Thanks to item (iii) this sum is restricted, in the case that p is even and $p=\ell/2$, to the set of indices such that $\{n_0,\dots,n_\ell\}\neq\{n_{\ell+1},\dots,n_{p+1}\}$. Let $\psi^\ell_{p+2}(\vec{m},\vec{n})$ be the function defined in (3.3.16). Then in order to solve equation (3.3.48) we define $\tilde{L}$ the $(p+2)$ symmetric linear form associated to the homogeneous map

$$U\mapsto \quad \mathrm{i}\sum_{\ell=-1}^{p+1}\binom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}}\psi^\ell_{p+2}(\vec{m},\vec{n})L(\Pi^+_{n_0}U,\dots,\Pi^+_{n_\ell}U,\Pi^-_{n_{\ell+1}}U,\dots,\Pi^-_{n_{p+1}}U). \tag{3.3.57}$$

By using (3.3.17) (and by using the fact that the sum excludes the indices such that p is even, $p=\ell/2$ and $\{n_1,\dots,n_\ell\}=\{n_{\ell+1},\dots,n_{p+1}\}$), we have that $\tilde{L}$ satisfies (3.3.43) with (ρ,μ) replaced by $(\rho-N_0,\mu+N_0)$. The form $\tilde{L}$ satisfies (3.3.44) since L satisfies the same property. It remains to prove that $\tilde{L}$ fulfils (3.3.45). Reasoning as above, and using that $S\Pi^+_{n_j}U=\Pi^-_{n_j}SU$ we have that $\tilde{L}(SU,\dots,SU)$ is equal to

$$\begin{aligned} &\mathrm{i}\sum_{\ell=-1}^{p+1}\binom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}}L(\Pi^-_{n_0}SU,\dots,\Pi^-_{n_\ell}SU,\Pi^+_{n_{\ell+1}}SU,\dots,\Pi^+_{n_{p+1}}SU) \\ &\quad\times(\sum_{j=0}^{\ell}\lambda_{n_j}-\sum_{j=\ell+1}^{p+1}\lambda_{n_j}), \end{aligned}$$

using the fact that the multilinear form L is symmetric and that satisfies (3.3.45) (with sign $-$) we obtain that the above quantity is equal to

$$\mathrm{i}\sum_{\ell=-1}^{p+1}\binom{p+2}{\ell+1}\sum_{n_0,\dots,n_{p+1}} L(\Pi^+_{n_{\ell+1}}U,\dots,\Pi^+_{n_{p+1}}U,\Pi^-_{n_0}U,\dots,\Pi^-_{n_\ell}U)$$
$$\times(-\sum_{j=0}^{\ell}\lambda_{n_j}+\sum_{j=\ell+1}^{p+1}\lambda_{n_j}),$$

which is the same quantity as in (3.3.57).
We prove item (v). By applying item (ii) of this lemma and (1.1.14) with $k=0$ we obtain the following estimate

$$\begin{aligned}&|L(U,\dots,M(U;t)U,\dots,U)|\le\\ &\quad\le C\|U(t,\cdot)\|_s^{p+1}\|M(U;t)U(t,\cdot)\|_{s-m}\\ &\quad\le C\|U(t,\cdot)\|^q_{\sigma,K'}\|U(t,\cdot)\|_s^{p+2}+\|U(t,\cdot)\|^N_{K',\sigma}\|U(t,\cdot)\|_{K',s}\|U(t,\cdot)\|_s^{p+1},\end{aligned}$$

therefore if $p+q\ge N$ the quantity above contribute to (3.3.50). Therefore we need to prove that if $\tilde M$ is in $\widetilde{M}^m_{\tilde q}\otimes\mathscr{M}_2(\mathbb{C})$ with $q\le\tilde q<N-p$ then the form

$$(U_0,\dots,U_{p+\tilde q+1})\mapsto L(U_0,\dots,U_p,\tilde M(U_{p+1},\dots,U_{p+\tilde q})U_{p+\tilde q+1})$$

is, up to symmetrization, a form $\tilde L$ in $\widetilde{\mathscr{L}}^{s,-\rho+m}_{p+\tilde q,\pm}$. By using (3.3.43) and (1.1.11) we have

$$\begin{aligned}&L(\Pi_{n_0}U_0,\dots,\Pi_{n_p}U_p,\tilde M(\Pi_{n_{p+1}}U_{p+1},\dots,\Pi_{n_{p+\tilde q}}U_{p+\tilde q})\Pi_{n_{p+\tilde q+1}}U_{p+\tilde q+1})\le\\ &\quad\sum_{n'\in\mathbb{N}}C\max\big(\langle n_0\rangle,\dots,\langle n_p\rangle,\langle n'\rangle\big)^{2s-\rho}\max_3\big(\langle n_0\rangle,\dots,\langle n_p\rangle,\langle n'\rangle\big)^{\rho+\mu}\\ &\qquad\times\big(\langle n'\rangle+\langle n_{p+1}\rangle+\dots+\langle n_{p+\tilde q+1}\rangle\big)^m\prod_{j=0}^{p+\tilde q+1}\Big\|\Pi_{n_j}U_j\Big\|_{L^2},\end{aligned}\tag{3.3.58}$$

where the sum is restricted to the set of indices such that $n'=\sum_{p+1}^{p+\tilde q+1}\varepsilon_\ell n_\ell$ for a choice of signs ε_ℓ in $\{\pm1\}$. The sum in (3.3.58) is bounded by the r.h.s. of (3.3.43) with (ρ,p) replaced by $(\rho-m,p+\tilde q)$. Moreover the form $\tilde L$ satisfies (3.3.44) since L fulfils the same condition and M satisfies (1.1.13). We need to check that condition (3.3.45) holds true with sign $\mp$ (resp. $\pm$) if M satisfies (1.3.16) and (1.3.17) (resp. (1.3.16) and (1.3.18)). Let us assume that M has the properties (1.3.16) and (1.3.17), the other case is similar. Let $\tilde M$ be a multilinear component of M and U

such that $SU = \bar{U}$. First of all since M satisfies the anti-reality condition (1.3.16) the homogeneous component $\tilde{M}$ satisfies

$$\overline{\tilde{M}(U,\dots,U)U} = S\tilde{M}(U,\dots,U)S\bar{U} = S\tilde{M}(U,\dots,U)U,$$

therefore $\tilde{M}(U,\dots,U)U$ has the same property of U. Since M has the property (1.3.17) by Lemma 1.3.2 each homogeneous component has the same property, so that

$$\tilde{M}(SU,\dots,SU)SU = -S(\tilde{M}(U,\dots,U)U),$$

and therefore thanks to (3.3.45) one has

$$\begin{aligned} L(SU,\dots,SU,\tilde{M}(SU,\dots,SU)SU) &= -L(SU,\dots,SU,S(\tilde{M}(U,\dots,U)U)) \\ &= \mp L(U,\dots,U,M(U,\dots,U)U), \end{aligned}$$

which concludes the proof. □

Lemma 3.3.4 (First energy inequality). *Let $U(t,x) \in B^K_s(I,r)$ be the solution of* (3.1.1) *with r small enough. If $\rho > 0$ is large enough there are constants $s \geq s_0 \gg K \geq \rho \gg \rho'' \gg N$ and multilinear forms $L_p \in \widetilde{\mathscr{L}}^{s,-\rho''}_{p,-}$, $p = 1,\dots,N-1$, such that for $s \geq s_0$ the following holds.*
Consider the functions $V = \Phi(U)[U]$, given by Theorem 3.2.1, and $W = \Theta(U)[V]$ given by Theorem 3.3.2. Then, for any $s \geq s_0$, one has

$$\frac{d}{dt}\int_{\mathbb{T}} |\langle D\rangle^s W(t,x)|^2 dx = \sum_{p=1}^{N-1} L_p(U,\dots,U) + O(\|U(t,\cdot)\|^{N+2}_{\mathbf{H}^s}) \tag{3.3.59}$$

for $t \in I$. Moreover

$$C_s^{-1}\|W\|_{\mathbf{H}^s} \leq \|U\|_{\mathbf{H}^s} \leq C_s\|W\|_{\mathbf{H}^s}, \tag{3.3.60}$$

for some constant $C_s > 0$.

Before giving the proof of Lemma 3.3.4 we need to prove the following.

Lemma 3.3.5. *Let $U(t,\cdot)$ be the solution of* (3.1.1) *defined on some interval $I \subset \mathbb{R}$ and belonging to $C^0(I;\mathbf{H}^s_e(\mathbb{T};\mathbb{C}^2))$. For any $0 \leq k \leq K$ there is a constant C_k such that, as long as $\|U(t,\cdot)\|_{\mathbf{H}^s} \leq 1$ with $s \gg K$, one has*

$$\|\partial_t^k U(t,\cdot)\|_{\mathbf{H}^{s-2k}} \leq C_k\|U(t,\cdot)\|_{\mathbf{H}^s}. \tag{3.3.61}$$

Proof. We argue by induction. Clearly (3.3.61) holds for $k=0$. Assume (3.3.61) holds for $k=0,\dots,k'\le K-1$. Since by assumption $\|U(t,\cdot)\|_{\mathbf{H}^s}\le 1$, then

$$\sum_{k=0}^{k'}\|\partial_t^k U(t,\cdot)\|_{\mathbf{H}^{s-2k}}\le \tilde{C}_{k'},$$

for some $\tilde{C}_{k'}$ uniformly for $t\in I$. In order to get (3.3.61) is it is sufficient to show $\|\partial_t^{k'+1}U(t,\cdot)\|_{\mathbf{H}^{s-2(k'+1)}}\le C\|U(t,\cdot)\|_{\mathbf{H}^s}$. Using (3.1.1) we have that

$$\begin{aligned}\partial_t^{k'+1}U &= \mathrm{i}E(\Lambda\partial_t^{k'}U+\partial_t^{k'}\big(\mathrm{Op}^{\mathscr{B}W}(A(U;t,x,\xi))[U]\big)+\partial_t^{k'}\big(R(U;t)[U]\big))\\ &= \mathrm{i}E\Lambda\partial_t^{k'}U+\mathrm{i}E\sum_{j_1+j_2=k'}C_{j_1}^{j_2}\mathrm{Op}^{\mathscr{B}W}(\partial_t^{j_1}A(U;t,x,\xi))[\partial_t^{j_2}U]\\ &+\mathrm{i}E\sum_{j_1+j_2=k'}(\partial_t^{j_1}R(U;t)))[\partial_t^{j_2}U],\end{aligned} \tag{3.3.62}$$

where $C_{j_1}^{j_2}$ are some binomial coefficients. By (1.1.44) in Proposition 1.1.1, (1.1.8) with $K'=0$ (recalling Remark 1.1.1), the inductive hypothesis and using that $\|U(t,\cdot)\|_{\mathbf{H}^s}\le 1$, we get

$$\|\partial_t^{k'+1}U(t,\cdot)\|_{\mathbf{H}^{s-2(k'+1)}}\le C\|U(t,\cdot)\|_{\mathbf{H}^s}. \tag{3.3.63}$$

This concludes the proof. □

Proof of Lemma 3.3.4. Since the maps Φ,Θ are (R,R,P)-maps, then the function $W=\Theta(U)[\Phi(U)[U]]$ is even in x and satisfies (1.3.3). In particular, by items (ii) of Theorems 3.2.1 and 3.3.2, we have that

$$W=U+\sum_{p=1}^{N-1}M_p(U,\dots,U)[U]+M_N(U;t)[U], \tag{3.3.64}$$

for some (R,R,P) maps $M_p\in\widetilde{\mathscr{M}}_p\otimes\mathscr{M}_2(\mathbb{C})$, $p=1,\dots,N-1$ and $M_N\in\Sigma\mathscr{M}_{K,K'',N}[r,N]\otimes\mathscr{M}_2(\mathbb{C})$.

We remark also that, by Lemma 3.3.5 and (1.1.1), we have

$$\|U(t,\cdot)\|_{K,s}\le C_{s,K}\|U(t,\cdot)\|_{\mathbf{H}^s}$$

for some $C_{s,K} > 0$. According to system (3.3.14), recalling (3.3.15), Remark 1.3.2 and Definition 3.3.1, we get

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}} |\langle D\rangle^s W(t,x)|^2 dx = \\
&\quad 2\mathrm{Re}\, \mathrm{i}\int_{\mathbb{T}} \overline{(\langle D\rangle^s W)}\big[\mathrm{Op}^{\mathscr{B}W}(\mathtt{l}(\xi) + \mathtt{m}_2(U;t)(\mathrm{i}\xi)^2)E\langle D\rangle^s W\big]dx \\
&\quad + 2\mathrm{Re}\, \mathrm{i}\int_{\mathbb{T}} \overline{(\langle D\rangle^s W)}\big[\mathrm{Op}^{\mathscr{B}W}([\![\mathtt{M}_0^{(1)}]\!](U;t,\xi)E\langle D\rangle^s W\big]dx \\
&\quad + 2\mathrm{Re}\, \mathrm{i}\int_{\mathbb{T}} \overline{(\langle D\rangle^s W)}(\langle D\rangle^s E Q_1^{(1)}(U;t)[W])dx \\
&\quad + 2\mathrm{Re}\, \mathrm{i}\int_{\mathbb{T}} \overline{(\langle D\rangle^s W)}(\langle D\rangle^s E Q_2^{(1)}(U;t)[U])dx,
\end{aligned}
\tag{3.3.65}
$$

where

$$
[\![\mathtt{M}_0^{(1)}]\!](U;t,\xi) = \begin{pmatrix} [\![\mathtt{m}_0^{(1)}]\!](U;t,\xi) & 0 \\ 0 & \overline{[\![\mathtt{m}_0^{(1)}]\!]}(U;t,-\xi) \end{pmatrix}
$$

The contribution of the first integral is zero since the symbol $\mathtt{l}(\xi) + \mathtt{m}_2(U;t)(\mathrm{i}\xi)^2$ is real. By using Lemmata 3.3.1 and 3.3.5 we have that the contribution of the second integral is bounded by

$$
O(\|U(t,\cdot)\|^N_{\mathbf{H}^s}\|W\|^2_{\mathbf{H}^s}).
$$

Let us consider the fourth integral term in (3.3.65). By definition we have that

$$
Q_2^{(1)}(U;t)[U] = \sum_{p=1}^{N-1} Q_{2,p}^{(1)}(U,\ldots,U)[U] + Q_{2,N}^{(1)}(U;t)[U]
$$

where $Q_{2,p}^{(1)} \in \widetilde{\mathscr{R}}_p^{-\rho'} \otimes \mathscr{M}_2(\mathbb{C})$, $p = 1,\ldots,N-1$ and $Q_{2,N}^{(1)} \in \mathscr{R}^{-\rho'}_{K,K'',N}[r] \otimes \mathscr{M}_2(\mathbb{C})$ are (R,R,P)-operators and with $\rho \gg \rho' \gg N$, $\rho' := \rho - m_1$ given in Theorem 3.3.2. The contribution given by the term $Q_{2,N}^{(1)}(U;t)$ is bounded by

$$
O(\|U(t,\cdot)\|^{N+2}_{\mathbf{H}^s}).
$$

Furthermore the operators $Q_{2,p}^{(1)}(U,\ldots,U)$ satisfy (1.3.24) by Lemma 1.3.2 ($\mathrm{i}EQ_{2,p}^{(1)}$ satisfies (1.3.23) by Remark 1.3.1). Hence the contribution to the fourth integral in (3.3.65) coming from the terms $Q_{2,p}^{(1)}(U,\ldots,U)$ can be written as in (3.3.46). By item (i) of Lemma 3.3.3 such contributions can be written as $\tilde{L}_p(U,\dots,U)$ for some multilinear form $\tilde{L}_p(U_0,\dots,U_{p+1})$ belonging to $\widetilde{\mathscr{L}}^{s,-\rho'}_{p,-}$. Consider now the operator $Q_1^{(1)}(U;t)$ in the third integral in (3.3.65). If $\rho' \gg \rho''$ is large enough, then, by (3.3.64) and item (iii) of Proposition 1.1.3, we get

$$Q_1^{(1)}(U;t)[W] = \sum_{p=1}^{N-1} \tilde{Q}_p(U,\dots,U)[U] + \tilde{Q}_N(U;t)[U],$$

for some $\tilde{Q}_p \in \widetilde{\mathscr{R}}^{-\rho''}_p \otimes \mathscr{M}_2(\mathbb{C})$, $p = 1,\dots,N-1$ and $\tilde{Q}_N \in \mathscr{R}^{-\rho''}_{K,K'',N}[r] \otimes \mathscr{M}_2(\mathbb{C})$ which are (R,R,P)-operators. Hence the contribution of the third integral can be studied as done for the term coming from $Q_2^{(1)}(U;t)$. This concludes the proof. □

Proof of Theorem 3.3.1. Let $s > 0$ large and $r > 0$ small enough. By Theorem 0.2.2 for any even function $u_0 \in H^s(\mathbb{T};\mathbb{C})$ with $\|u_0\|_{H^s} \le r$, there is a unique solution $u(t,x)$ of (0.2.1) with initial condition $u(0,x) = u_0(x)$ belonging to the space $C^1(I;H^{s-2}(\mathbb{T};\mathbb{C})) \cap C^0(I;H^s(\mathbb{T};\mathbb{C}))$ with $I = (-T_r,T_r)$, $T_r > 0$.
By Theorem 3.1.1 the function $U = (u,\bar{u})$ solves the problem (3.1.1) with initial condition $U_0 = (u_0,\bar{u}_0)$, furthermore by Lemma 3.3.5 such a solution belongs to the ball $B^K_s(I,r)$.
We now prove that $T_r \ge cr^{-N}$ for some $c > 0$ depending on s. By applying to the system (3.1.1) Theorems 3.2.1 and 3.3.2 we have that $U(t,x)$ solves (3.1.1) if and only if the function $W(t,x)$ given in Theorem 3.3.2 solves (3.3.14). By Lemma 3.3.4 we have that

$$\|U\|_{\mathbf{H}^s} \sim \|W\|_{\mathbf{H}^s}, \tag{3.3.66}$$

and that (3.3.59) holds.
We claim that there are multilinear forms $F_p \in \widetilde{\mathscr{L}}^{s,-\rho''+\tilde{\rho}}_{p,-}$ for $p = 1,\dots,N-1$, for some $\tilde{\rho} < \rho''$ (the constant ρ'' is given in Lemma 3.3.4), such that, by setting

$$\mathscr{G}(U,W) := \int_{\mathbb{T}} |\langle D\rangle^s W(t,x)|^2 dx + \sum_{p=1}^{N-1} F_p(U,\dots,U), \tag{3.3.67}$$

the following conditions hold:

$$\|U\|^2_{\mathbf{H}^s} \sim \|W\|^2_{\mathbf{H}^s} \sim \mathscr{G}(U,W), \tag{3.3.68}$$

$$\frac{d}{dt}\mathscr{G}(U,W) \le K_1 \|U(t,\cdot)\|^{N+2}_{\mathbf{H}^s}, \quad t \in [-T_r,T_r], \tag{3.3.69}$$

for some $K_1 > 0$ depending on s, N. To prove this fact we reason as follows. Note that system (3.1.1) can be written, by Remark 1.1.14, as

$$\partial_t U = \mathrm{i}E\Lambda U + M(U;t)[U], \tag{3.3.70}$$

for some $M \in \Sigma\mathscr{M}^m_{K,0,1}[r,N] \otimes \mathscr{M}_2(\mathbb{C})$, $m \geq 0$. We show that it is possible to find recursively multilinear forms

$$\begin{aligned} &\tilde{L}_p \in \widetilde{\mathscr{L}}^{s,-\rho''+(N_0+m)(p-1)+N_0}_{p,+}, \quad 1 \leq p \leq N-1, \\ &L^{(q)}_p \in \widetilde{\mathscr{L}}^{s,-\rho''+(N_0+m)q}_{p,-}, \quad q+1 \leq p \leq N-1, \end{aligned} \tag{3.3.71}$$

such that, for $q = 1, \dots, N-1$,

$$\begin{aligned} &\frac{d}{dt}\left[\int_{\mathbb{T}} |\langle D\rangle^s W(t,x)|^2 dx + \sum_{p=1}^{q} \tilde{L}_p(U(t,\cdot),\dots,U(t,\cdot))\right] \\ &\qquad = \sum_{p=q+1}^{N-1} L^{(q)}_p(U(t,\cdot),\dots,U(t,\cdot)) + O(\|U(t,\cdot)\|^{N+2}_{\mathbf{H}^s}). \end{aligned} \tag{3.3.72}$$

Here $m > 2$ is the loss coming form $M(U;t)$ in (3.3.70), the constant N_0 is in (3.3.17) of Proposition 3.3.1. This loss is compensated by the fact that $\rho > 0$ in Theorem 3.2.1 is arbitrary large, and hence also ρ'' can be taken large enough. We argue by induction on q. For $q = 0$ the (3.3.72) follows by (3.3.59). Assume inductively that (3.3.72) holds for $q-1$. Let us define $\tilde{L}_q \in \widetilde{\mathscr{L}}^{s,-\rho''+(N_0+m)(q-1)+N_0}$ as the multilinear form given by item (iv) of Lemma 3.3.3 applied to $L = L^{(q-1)}_q$. We get

$$\begin{aligned} \frac{d}{dt}\tilde{L}_q(U(t,\cdot),\dots,U(t,\cdot)) &= \mathrm{i}\sum_{j=0}^{q+1} \tilde{L}_q(\underbrace{U,\dots,U}_{j-times}, E\Lambda U, U,\dots,U) \\ &+ \mathrm{i}\sum_{j=0}^{q+1} \tilde{L}_j(\underbrace{U,\dots,U}_{j-times}, M(U;t)[U], U,\dots,U). \end{aligned} \tag{3.3.73}$$

Using items (iv) and (v) of Lemma 3.3.3 we have that

$$\begin{aligned} \frac{d}{dt}\tilde{L}_q(U(t,\cdot),\dots,U(t,\cdot)) &= -L^{(q-1)}_q(U,\dots,U) \\ &+ \sum_{j=0}^{N-q-3} L'_j(U,\dots,U) + O(\|U\|^{N+2}_{\mathbf{H}^s}), \end{aligned} \tag{3.3.74}$$

for some $L'_j \in \widetilde{\mathscr{L}}^{s,-\rho''+(N_0+m)(q-1)+m+N_0}_{p+2+j,-}$. Thus we get (3.3.72) at rank q. We conclude by setting F_p in (3.3.67) equals to $\tilde{L}_p$. Since r is small enough, then, thanks to item (ii) of Lemma 3.3.3, equation (3.3.66) and Lemma 3.3.5, we get

$$\mathscr{G}(U,W) \le C_s(\|U(t,\cdot)\|^2_{\mathbf{H}^s} + \|U\|^3_{\mathbf{H}^s}),$$

as long as $\|U(t,\cdot)\|_{\mathbf{H}^s} \le Cr$, therefore (3.3.68) holds. The (3.3.69) follows by (3.3.72) for $q = N-1$.
The thesis follows by using the following bootstrap argument. The integral form of (3.3.69) is

$$\mathscr{G}(U(t,\cdot),W(t,\cdot)) \le \mathscr{G}(U(0,\cdot),W(0,\cdot)) + K_1\int_0^t \|U(\tau,\cdot)\|^{N+2}_{\mathbf{H}^s}d\tau, \tag{3.3.75}$$

and by (3.3.68) we have that

$$\mathscr{G}(U(0,\cdot),W(0,\cdot)) \le c_0 r^2,$$

for some c_0 depending on s. Fix $K_2 = K_2(s,N) > 1$ and let $\bar{T}$ the supremum of those T such that

$$\sup_{t\in[-T,T]} \mathscr{G}(U(t,\cdot),W(t,\cdot)) \le K_2 r^2. \tag{3.3.76}$$

Assume, by contradiction, that $\bar{T} < \tilde{c} r^{-N}$. Then, if K_1 is the constant appearing in the r.h.s. of (3.3.75), we have

$$\begin{aligned}\mathscr{G}(U(t,\cdot),W(t,\cdot)) &\le c_0 r^2 + K_1\int_0^t K_2^{N+2} r^{N+2} d\tau\\ &\le c_0 r^2 + K_1 K_2^{N+2} r^{N+2}\bar{T}\\ &\le c_0 r^2 + K_1 K_2^{N+2} r^{N}\tilde{c} r^{-N} r^2\\ &\le r^2\big(c_0 + K_1 K_2^{N+2}\tilde{c}\big) \le K_2 r^2 \frac{3}{4},\end{aligned} \tag{3.3.77}$$

for $\tilde{c} > 0$ small enough and $K_2 \gg c_0$ large enough hence the contradiction. By (3.3.68) the reasoning above implies also that

$$\sup_{t\in[-T,T]} \|U(t,\cdot)\|_{\mathbf{H}^s} \le Cr, \quad T \ge \tilde{c} r^{-N},$$

for some fixed $C > 0$ depending on s, N. This is (3.3.2) for $k = 0$. Moreover by Lemma 3.3.5 we also obtain that $\partial_t^k U(t,\cdot)$ satisfies

$$\sup_{t\in[-T,T]} \|\partial_t^k U(t,\cdot)\|_{\mathbf{H}^{s-2k}} \le Cr, \quad T \ge \tilde{c} r^{-N},$$

if r is small, $s \gg K$ and where C is a large enough constant depending on K. □

Appendix A

Oscillatory integrals

In this short appendix we study the convergence of integrals of the form

$$\int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x) dx, \tag{A.0.1}$$

where $q(x)$ is a real valued function (usually called phase), and $a(x)$ has polynomial growth (usually called amplitude) and $d \geq 1$. Some non degeneracy on the phase $q(x)$ will be required in order to have "enough oscillations" to compensate the growth of the amplitude $a(x)$. These integrals are used in the Chapter 1 to define operations on the symbols. Therefore the aim of this appendix is not to write down a general theory, but only to establish some fundamental properties of these integrals in some particular cases that we shall use in Chapter 1. For a more complete and deep analysis of this subject we refer to the books by Alinhac-Gèrard [3] and by Saint Raymond [83].

Definition A.0.1 (Amplitudes)**.** *For ρ in $(-\infty, 1]$, m in $\mathbb{R}$ we define $A^m_\rho(\mathbb{R}^d)$ the class of functions $a(x)$ in $C^\infty(\mathbb{R}^d;\mathbb{C})$ such that for any multi-index $\alpha \in \mathbb{N}^d$ there exists a constant $C_\alpha > 0$ such that*

$$\left|\partial_x^\alpha a(x)\right| \leq C_\alpha \langle x \rangle^{m-\rho|\alpha|}, \tag{A.0.2}$$

for any x in $\mathbb{R}^d$.

We have the following easy remarks.

Remark A.0.1. *1. The amplitudes "behave" like symbols in the following sense. The product of two amplitudes, and the derivative of an amplitude, are still amplitudes:*

- *if* $a \in A^{m_1}_\rho$ *and* $b \in A^{m_2}_\rho$ *then* $a \cdot b \in A^{m_1+m_2}_\rho$;
- $a \in A^m_\rho$ *and* $\beta \in \mathbb{N}$ *then* $\partial^\beta_x a \in A^{m-|\beta|}_\rho$;
- *if* $m_1 \leq m_2$ *then* $A^{m_1}_\rho \subset A^{m_2}_\rho$.

2. *The class* $A^m_\rho(\mathbb{R}^d)$ *turns out to be a Frechét space endowed with the family of semi-norms* $(\mathcal{N}^m_{\rho,k})_{k\in\mathbb{N}^*}$ *defined as*

$$\mathcal{N}^m_{\rho,k}(a) := \sup_{x\in\mathbb{R}^d} \max_{|\alpha|\leq k} \left|\partial^\alpha_x a(x)\right| \langle x\rangle^{-(m-\rho|\alpha|)}. \tag{A.0.3}$$

If $a(x)$ is an amplitude in $A^m_\rho(\mathbb{R}^d)$ we shall study the integral (A.0.1) in the case that the phase $q(x) : \mathbb{R}^d \to \mathbb{R}$ satisfies the following conditions:

- $q(x) \in \mathbb{C}^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$ is homogeneous of degree $\mu > 0$, i. e. $q(\lambda x) = \lambda^\mu q(x)$ for any λ in $\mathbb{R}$ and x in $\mathbb{R}^d$;
- $\nabla q(x)$ is different from 0 for any $x \in \mathbb{R}^d \setminus \{0\}$;
- $\mu > 1 - \rho$.

We have the following theorem.

Theorem A.0.5 (Definition of oscillatory integrals)**.** *Fix* $m \in \mathbb{R}$ *and* $\rho \in (-\infty, 1]$. *Let* $\psi(x)$ *be a rapidly decaying function in* $\mathcal{S}(\mathbb{R}^d; \mathbb{R})$ *such that* $\psi(0) = 1$. *If* $a(x)$ *is in* $A^m_\rho(\mathbb{R}^d)$, *and* $q(x) \in \mathbb{C}^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{R})$ *verifies the conditions above, then there exist finite the limit*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)\psi(\varepsilon x)\,dx; \tag{A.0.4}$$

moreover the limit does not depend on the choice of the function $\psi(x)$. *In the case that the amplitude* $a(x)$ *is in* $L^1(\mathbb{R}^d)$ *such a limit coincide with* $\int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)\,dx$.

Proof. If $a(x)$ is in $L^1(\mathbb{R}^d)$ then the thesis follows just by Lebesgue dominated convergence theorem. So let us prove the theorem in the case that $a \notin L^1$.
We define a dyadic partition of unity in the following way. Let $\phi(x)$ be a $C^\infty_0(\mathbb{R}^d)$ function such that $\phi(x) = 1$ for any x in the ball of center 0 and radius 1 and such that its support supp(ϕ) is contained in the ball of center 0 and radius 2. We define the function $\chi(x) := \phi(x/2) - \phi(x)$, so that we have $\mathrm{supp}(\chi) \subset \{1 \leq |x| \leq 2^2\}$. Analogously the function $\chi_k(x) := \chi(2^{-(k-1)}x)$ is supported in the set $\{2^{k-1} \leq |x| \leq 2^{k+1}\}$.

One has $1 = \phi(x) + \sum_{k=0}^{\infty} \chi_k(x)$. By using the Lebesgue dominated convergence theorem, it is enough to prove that there exists a constant $C > 0$ such that

$$\left| \int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)(1-\psi(\epsilon x))\chi_k(x)dx \right| \le C\epsilon 2^{-k}. \qquad \text{(A.0.5)}$$

Performing the change of variable $y = 2^{-(k-1)}x$, and using the μ-homogeneity of the function $q(x)$, we obtain that the l.h.s. of the inequality (A.0.5) is equal to

$$2^{(k-1)d} \left| \int_{\mathbb{R}^d} e^{\mathrm{i}2^{\mu(k-1)}q(y)} a(2^{k-1}y)(1-\psi(\epsilon 2^{k-1}y)\chi(y))dy \right|. \qquad \text{(A.0.6)}$$

Note that the vector field $L[\cdot] := \mathrm{i}2^{-\mu(k-1)} \frac{\langle \nabla q, \nabla \cdot \rangle}{|\nabla q|^2}$ and satisfies $L[e^{\mathrm{i}q(x)}] = e^{\mathrm{i}q(x)}$ and

$$L^*[v] = -\mathrm{i}2^{-\mu(k-1)} \sum_{j=1}^{d} \partial_{x_j} \frac{(\partial_{x_j} q)v}{|\nabla q|^2}.$$

One can easily prove that there exists a positive constant C such that

$$\left\| (L^*)^N v \right\|_\infty \le C \max_{|\alpha| \le N} 2^{-(k-1)N} \left\| \partial_x^\alpha v \right\|_\infty,$$

for any $v \in C_0^\infty(\mathbb{R}^n)$ and where the maximum is taken over all multi-indices α of length less than N. Thanks to this fact one can bound, remembering that the support of the function χ is contained in the ball of center 0 and radius 2, the integral in (A.0.6) by

$$C 2^{kd} 2^{-(k-1)\mu N} \sup_{|y| \le 2} \max_{|\alpha| \le N} \left| \partial_y^\alpha [a(2^{k-1}y)(1-\psi(\epsilon 2^{k-1}y))\chi(y)] \right|. \qquad \text{(A.0.7)}$$

Since $\psi(0) = 1$ one has $|\partial_y^\gamma (1-\psi(\epsilon 2^{k-1}y))| \le 2^{|\gamma| k-1} \epsilon 2^{k-1} y$, therefore, by using the Liebniz rule and (A.0.2), one can bound (A.0.7) by

$$C 2^{k\left(d-\mu N+|\alpha|(1-\rho)+m+1\right)} \epsilon \le C 2^{k\left(d+N(1-\rho-\mu)+m+1\right)},$$

where the constant C depends on everything but on k and ϵ. Since by hypothesis (remember the hypothesis on the phase q above the statement), we have $1-\rho-\mu < 0$ one gets the (A.0.5) by choosing $N > m+d+2$. □

Thanks to Theorem (A.0.5) we can define the oscillatory integral $\int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)dx$ by using formula (A.0.4).

We prove below a series of simple lemmas in which we show that the oscillatory integrals essentially “behave" as absolutely convergent integrals.

Lemma A.0.1 (Linear change of variable)**.** *Let $a(x)$ and $q(x)$ as in Theorem A.0.5. Consider $A \in \mathbb{R}^d \times \mathbb{R}^d$ an invertible and real matrix, then*

$$\int_{\mathbb{R}^d} e^{\mathrm{i}q(Ay)} a(Ay)\,|\det(A)|\,dy = \int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)\,dx.$$

Proof. Let $\psi(x)$ a rapidly decaying function in $\mathscr{S}(\mathbb{R}^d;\mathbb{R})$ and such that $\psi(0) = 1$. Then by definition we have

$$\begin{aligned}\int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)\,dx &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} e^{\mathrm{i}q(x)} a(x)\psi(\epsilon x)\,dx\\ &= \lim_{\epsilon\to 0}\int_{\mathbb{R}^d} e^{\mathrm{i}q(Ay)} a(Ay)\,|\det(A)|\,\psi(\epsilon Ay)\,dy\\ &= \int_{\mathbb{R}^d} e^{\mathrm{i}q(Ay)} a(Ay)\,|\det(A)|\,dy,\end{aligned}$$

where in the second passage we have used the “classic" change of variable $x = Ay$. □

Lemma A.0.2 (Integration by parts)**.** *Let $a \in A^m_\rho(\mathbb{R}^d \times \mathbb{R}^d)$ with $\rho > -1$. Then*

$$\int_{\mathbb{R}^{2d}} e^{-y\cdot\eta} y^\alpha a(y,\eta)\,dy\,d\eta = \int_{\mathbb{R}^{2d}} e^{-y\cdot\eta} D^\alpha_\eta(a(y,\eta))\,dy\,d\eta,$$

where we have denoted $D_\eta := \frac{1}{\mathrm{i}}\partial_\eta$ the Hörmdander derivative.

Proof. Without loss of generality it is enough to give the proof in the case that $\alpha = \vec{e}_j \in \mathbb{N}^d$. Let χ be a rapidly decaying function in $\mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R})$ such that $\chi(0,0) = 1$. By definition we have

$$\begin{aligned}\int_{\mathbb{R}^{2d}} e^{-\mathrm{i}y\cdot\eta} y_j a(y,\eta)\chi(\epsilon y,\epsilon\eta)\,dy\,d\eta &= \int_{\mathbb{R}^{2d}} (-D_{\eta_j})[e^{-\mathrm{i}y\cdot\eta}]\,a(y,\eta)\chi(\epsilon y,\epsilon\eta)\,dy\,d\eta\\ &= \int_{\mathbb{R}^{2d}} e^{-\mathrm{i}y\cdot\eta} D_{\eta_j}[a(y,\eta)\chi(\epsilon y,\epsilon\eta)]\,dy\,d\eta\end{aligned}$$

which when ϵ goes to 0 converges to $\int_{\mathbb{R}^{2d}} D_{\eta_j}[a(y,\eta)]\,dy\,d\eta$. □

Note that in the hypothesis of Lemma A.0.2 one proves the following very useful representation formula

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} e^{-\mathrm{i}y\cdot\eta} a(y,\eta)\,dy\,d\eta =\\ &\quad= \int_{\mathbb{R}^{2d}} e^{-\mathrm{i}y\cdot\eta} \langle y\rangle^{2\ell'} \langle\eta\rangle^{2\ell} \langle D_y\rangle^{2\ell} \langle D_\eta\rangle^{2\ell'} a(y,\eta)\,dy\,d\eta,\end{aligned} \tag{A.0.8}$$

where we used the standard Japanese brackets $\langle x\rangle := \sqrt{1+|x|^2}$ and the Hörmander derivatives $D_y := \frac{1}{\mathrm{i}}\partial_y$ and $D_\eta := \frac{1}{\mathrm{i}}\partial_\eta$. Note that the integral in the r.h.s. of (A.0.8) is absolutely convergent if $m-2\rho\ell < -d$ and $m-2\rho\ell' < -d$.

Lemma A.0.3 (Fubini's theorem for oscillatory integrals)**.** *Let $a(y,y',\eta,\eta')$ be an amplitude in $A^m_\rho(\mathbb{R}^{d+k}\times\mathbb{R}^{d+k})$. Then*

$$b(y,\eta) := \int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y'\cdot\eta'} a(y,y',\eta,\eta')\,dy'd\eta'$$

is an amplitude in $A^m_\rho(\mathbb{R}^d\times\mathbb{R}^d)$ and moreover

$$\partial^\alpha_y\partial^\beta_\eta b(y,\eta) = \int_{\mathbb{R}^{2k}} \partial^\alpha_y\partial^\beta_\eta a(y,y',\eta,\eta')\,dy'd\eta'.$$

Moreover one has

$$\int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y\cdot\eta}e^{-\mathrm{i}y'\cdot\eta'} a(y,y',\eta,\eta')\,dydy'd\eta d\eta' = \\ \int_{\mathbb{R}^{2d}} e^{-\mathrm{i}y\cdot\eta}\left(\int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y'\cdot\eta'} a(y,y',\eta,\eta')\,dy'd\eta'\right)dyd\eta.$$

Proof. Let us consider the quantity

$$\int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y'\cdot\eta'}\partial^\alpha_y\partial^\beta_\eta a(y,y'\eta,\eta')\,dy'd\eta', \tag{A.0.9}$$

by using the representation formula (A.0.8) it is equal to

$$\int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y'\cdot\eta'}\langle y'\rangle^{-2\ell'}\langle \eta'\rangle^{-2\ell}\langle D_{y'}\rangle^{2\ell}\langle D_{\eta'}\rangle^{2\ell'}\partial^\alpha_y\partial^\beta_\eta a(y,y',\eta,\eta')\,dy'd\eta',$$

if $2|\ell'| > (m+k)/\rho$ and $2|\ell| > (m+k)/\rho$ then the last integral is absolutely convergent and therefore one can use the fact that the amplitude a is in the class $A^m_\rho(\mathbb{R}^{d+k}\times\mathbb{R}^{d+k})$ to show that the integral (A.0.9) is an amplitude in $A^m_\rho(\mathbb{R}^d\times\mathbb{R}^d)$. For the same reason one takes the derivatives with respect to η and y outside of the integral and hence proves that the symbol $b(y,\eta)$ in the statement is an amplitude in $A^m_\rho(\mathbb{R}^d\times\mathbb{R}^d)$.
In order to prove the second part of the statement it is enough to note that

$$\langle y\rangle^{-2\ell'}\langle \eta\rangle^{-2\ell}\langle D_y\rangle^{2\ell'}\langle D_\eta\rangle^{2\ell} b(y,\eta) = \\ \int_{\mathbb{R}^{2k}} e^{-\mathrm{i}y'\cdot\eta'}(\langle y\rangle\langle y'\rangle)^{-2\ell'}(\langle D_\eta\rangle\langle D_{\eta'}\rangle)^{2\ell'}(\langle \eta\rangle\langle \eta'\rangle)^{-2\ell}\times \\ (\langle D_y\rangle\langle D_{y'}\rangle)^{-2\ell}\times a(y,y',\eta,\eta')\,dy'd\eta',$$

where the integrand in the r.h.s. is a function in $L^1(\mathbb{R}^{d+k} \times \mathbb{R}^{d+k})$. Hence the thesis follows by applying the Fubini's theorem. $\square$

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