Tau functions in Integrable Systems: Theory and Applications

M. Bertola[†,‡][1]
[†] *Department of Mathematics and Statistics, Concordia University, Montréal, Canada*
[‡] *Area of Mathematics SISSA/ISAS, Trieste, Italy*

Integrable system theory in general terms can be thought of as the study of compatibility of overdetermined systems of differential equations, and often can be rephrased in terms of auxiliary linear problems, for example boundary value problems. The canonical example of the Korteveg-deVries nonlinear equation is instructive and also simple; in that case the solution of the equation is in-fact the logarithmic derivative of an infinite dimensional (possibly regularized) determinant of an associated linear system depending on parameters. The theme of associated linear systems and Lax representations is a common thread, if not the sole, to the many aspects of integrability; the (regularized) determinants associated to the solvability of these linear systems are generally referred to as tau functions and they end up playing a rather ubiquitous role in the area. The cases when a system is actually integrable in the classical Liouville sense are rather limited, though admittedly important: integrability is of course tied to symmetry. The surprise comes from the wide variety of areas in which integrability plays a role, varying from physically relevant PDEs, to random matrices, random processes, enumerative geometry and topological recursions, and combinatorics. The links are often not evident, and are sometimes only glimpsed through the recurrence of basic functions and symmetries; the theory of tau-functions, and now cluster algebras, appear in various guises, tying the theory to geometry.

In this course we shall review the main historical examples of integrable systems of Kadomtsev–Petviashvili and Korteveg-deVries type. These special functions are in a certain sense a generalization of Riemann Theta functions; they can be also thought of as (regularized) determinants. In certain cases they are genuine (finite or infinite-dimensional) determinants.

As such their vanishing determines the obstruction to the solvability of a certain linear problem.

The tau function was originally defined for isomonodromic systems (e.g. the associated linear systems for the Painlevé equations) but then the definition can be extended to more general Riemann–Hilbert problems; this extension does not require a linear ODE. Even in the case of the "isomonodromic" tau function, this extension allows to determine consistently the dependence on the (generalized) monodromy data. I will explain various applications to

1. integrable systems (KdV, Toda, KP)

2. Witten-Kontsevich special tau function and intersection numbers; generating functions.

[1] Marco.Bertola@{sissa.it, concordia.ca}

3. Gap probabilities (Tracy-Widom and generalizations thereof); Fredholm determinants.

4. (Multi)-Orthogonal polynomials and Hankel/Toeplitz determinants. Random Matrices.

The main references shall be course notes and [1], [9], [18], et al. The very tentative content is below.

1. Integrable dynamical systems and synopsis of Integrable systems [9];
   (a) Zakharov–Shabat construction;
   (b) Tau functions.
2. The KP hierarchy
3. The KdV hierarchy
4. Factorization method and Riemann–Hilbert problems;
   (a) What is a Riemann–Hilbert problem;
      i. Sovability and singular integral equations (after [7]);
      ii. Nontrivial example: Meromorphic matrix jumps on S^1
   (b) Deformation space of Riemann–Hilbert problems and the Malgrange one–form.
5. Malgrange forma and Tau function
   (a) KdV again;
   (b) Isomonodromic systems after the Japanese school[16]
   (c) Discrete Schlesinger transformations
6. Applications:
   (a) Determinantal Random Point Fields (DRPF) and Gap probabilities;
   (b) Fredholm determinants and the Its-Izergin-Korepin-Slavnov approach;
   (c) Intersection numbers on moduli spaces and tau functions after Kontsevich.

References

[1] Olivier Babelon, Denis Bernard, and Michel Talon. *Introduction to classical integrable systems*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2003.

[2] M. Bertola. The dependence on the monodromy data of the isomonodromic tau function. *Comm. Math. Phys.*, 294(2):539–579, 2010.

[3] M. Bertola and M. Cafasso. Fredholm determinants and pole-free solutions to the noncommutative Painlevé II equation. *Comm. Math. Phys.*, 309(3):793–833, 2012.

[4] M. Bertola and M. Cafasso. Darboux transformations and random point processes. *IMRN*, rnu122:56, 2014.

[5] Marco Bertola. Corrigendum: The dependence on the monodromy data of the isomonodromic tau function. 01 2016.

[6] Marco Bertola and Mattia Cafasso. The kontsevich matrix integral: convergence to the painlevé hierarchy and stokes' phenomenon. 03 2016.

[7] Kevin F. Clancey and Israel Gohberg. *Factorization of matrix functions and singular integral operators*, volume 3 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel-Boston, Mass., 1981.

[8] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.

[9] L. A. Dickey. *Soliton equations and Hamiltonian systems*, volume 26 of *Advanced Series in Mathematical Physics*. World Scientific Publishing Co. Inc., River Edge, NJ, second edition, 2003.

[10] Freeman J. Dyson. Fredholm determinants and inverse scattering problems. *Comm. Math. Phys.*, 47(2):171–183, 1976.

[11] F. D. Gakhov. *Boundary value problems*. Dover Publications, Inc., New York, 1990. Translated from the Russian, Reprint of the 1966 translation.

[12] J. Harnad and Alexander R. Its. Integrable Fredholm operators and dual isomonodromic deformations. *Comm. Math. Phys.*, 226(3):497–530, 2002.

[13] A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov. Differential equations for quantum correlation functions. In *Proceedings of the Conference on Yang-Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory*, volume 4, pages 1003–1037, 1990.

[14] Michio Jimbo and Tetsuji Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D*, 2(3):407–448, 1981.

[15] Michio Jimbo and Tetsuji Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III. *Phys. D*, 4(1):26–46, 1981/82.

[16] Michio Jimbo, Tetsuji Miwa, and Kimio Ueno. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and τ-function. *Phys. D*, 2(2):306–352, 1981.

[17] M Kontsevich. Intersection theory on the moduli space of curves and the matrix airy function. *Communications in Mathematical Physics*, Jan 1992.

[18] T. Miwa, M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*, volume 135 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000. Differential equations, symmetries and infinite-dimensional algebras, Translated from the 1993 Japanese original by Miles Reid.

[19] N. I. Muskhelishvili. *Singular integral equations*. Dover Publications, Inc., New York, 1992. Boundary problems of function theory and their application to mathematical physics, Translated from the second (1946) Russian edition and with a preface by J. R. M. Radok, Corrected reprint of the 1953 English translation.

[20] D. Zvonkine. An introduction to moduli spaces of curves and its intersection theory. 2011.