Ist. di Fisica Matematica mod. A
First exercise session

Nicolò Piazzalunga (npiazza@sissa.it)

October 7th, 2015

Exercises are numbered as in the lecture notes of the course.

Exercise 2.2.4. *Let A and B be two linear differential operators of orders k and l with the principal symbols $a_k(x,\xi)$ and $b_l(x,\xi)$ respectively. Prove that the superposition $C = A \circ B$ is a linear differential operator of order $\le k + l$. Prove that the principal symbol of C is equal to*

$$c_{k+l}(x,\xi) = a_k(x,\xi) b_l(x,\xi) \tag{1}$$

in the case $\operatorname{ord} C = \operatorname{ord} A + \operatorname{ord} B$. *In the case of strict inequality* $\operatorname{ord} C < \operatorname{ord} A + \operatorname{ord} B$ *prove that the product* (1) *of principal symbols is identically equal to zero.*

Solution. Write

$$A = \sum_{|\mathbf{p}|\le k} a_{\mathbf{p}}(x)\, D^{\mathbf{p}}, \quad B = \sum_{|\mathbf{q}|\le l} b_{\mathbf{q}}(x)\, D^{\mathbf{q}},$$

so that the symbols of A and B are respectively

$$a(x,\xi) = \sum_{|\mathbf{p}|\le k} a_{\mathbf{p}}(x)\, \xi^{\mathbf{p}}, \quad b(x,\xi) = \sum_{|\mathbf{q}|\le l} b_{\mathbf{q}}(x)\, \xi^{\mathbf{q}}.$$

In particular, their principal symbols are

$$a_k(x,\xi) = \sum_{|\mathbf{p}|= k} a_{\mathbf{p}}(x)\, \xi^{\mathbf{p}}, \quad b_l(x,\xi) = \sum_{|\mathbf{q}|= l} b_{\mathbf{q}}(x)\, \xi^{\mathbf{q}}.$$

We can now calculate, using Leibniz rule,

$$\begin{aligned} C = A \circ B &= \left(\sum_{|\mathbf{p}|\le k} a_{\mathbf{p}}(x)\, D^{\mathbf{p}} \right) \circ \left(\sum_{|\mathbf{q}|\le l} b_{\mathbf{q}}(x)\, D^{\mathbf{q}} \right) = \\ &= \sum_{\substack{|\mathbf{p}|=k,\\ |\mathbf{q}|=l}} a_{\mathbf{p}}(x)\, D^{\mathbf{p}} \left(b_{\mathbf{q}}(x)\, D^{\mathbf{q}} \right) + \text{ l.o.t. } = \\ &= \sum_{\substack{|\mathbf{p}|=k,\\ |\mathbf{q}|=l}} a_{\mathbf{p}}(x) b_{\mathbf{q}}(x)\, D^{\mathbf{p}} \circ D^{\mathbf{q}} + \text{ l.o.t.} \end{aligned}$$

where "l.o.t." stands for "lower order terms" in the derivative operators D. Letting $\mathbf{r} := (\mathbf{p}, \mathbf{q}) \in \mathbb{N}^{k+l}$ be a new multi-index, we thus obtain

$$C = \sum_{|\mathbf{r}|=k+l} c_{\mathbf{r}}(x)\, D^{\mathbf{r}} + \text{ l.o.t.}$$

where

$$c_{\mathbf{r}}(x) = a_{\mathbf{p}}(x) b_{\mathbf{q}}(x) \quad \text{if } \mathbf{r} = (\mathbf{p}, \mathbf{q}),\ |\mathbf{p}| = k,\ |\mathbf{q}| = l$$

and $c_{\mathbf{r}}(x) = 0$ otherwise. From the expression for C we obtained above we immediately deduce that it is a linear differential operator of order at most $k + l$.

Moreover, the above calculation shows that if $\operatorname{ord} C = \operatorname{ord} A + \operatorname{ord} B$ then the pricipal symbol of C is

$$\begin{aligned} c_{k+l}(x,\xi) &= \sum_{|\mathbf{r}|=k+l} c_{\mathbf{r}}(x)\, \xi^{\mathbf{r}} = \sum_{\substack{|\mathbf{p}|=k,\\ |\mathbf{q}|=l}} a_{\mathbf{p}}(x) b_{\mathbf{q}}(x)\, \xi^{\mathbf{p}} \xi^{\mathbf{q}} = \\ &= \left(\sum_{|\mathbf{p}|=k} a_{\mathbf{p}}(x)\, \xi^{\mathbf{p}} \right) \left(\sum_{|\mathbf{q}|=l} b_{\mathbf{q}}(x)\, \xi^{\mathbf{q}} \right) = a_k(x,\xi) b_l(x,\xi) \end{aligned}$$

which is exactly Equation (1). If instead $\operatorname{ord} C < \operatorname{ord} A + \operatorname{ord} B$, then the principal symbol of C comes from the lower order terms, and hence this product must necessarily vanish.

[Additional question, solved in class: could you exhibit an example of two operators A and B such that, if C denotes their composition, then $\operatorname{ord} C < \operatorname{ord} A + \operatorname{ord} B$?] ♢

Exercise 2.2.5. *Let $a(x,\xi)$ and $b(x,\xi)$ be the symbols of two linear differential operators A and B with one spatial variable. Prove that the symbol of the superposition $A \circ B$ is equal to*

$$a \star b = \sum_{k \geq 0} \frac{(-i)^k}{k!}\, \partial_\xi^k a\, \partial_x^k b. \tag{2}$$

Solution. In one spatial dimension, "multi-indices" are just indices. We then write

$$A = \sum_{p\geq 0} a_p(x)\, D^p = \sum_{p \geq 0} (-i)^p a_p(x)\, \partial_x^p, \quad B = \sum_{q \geq 0} b_q(x)\, D^q$$

(the fact that $a_p(x) = 0$ for $p > \operatorname{ord} A$ and similarly for $b_q(x)$ is understood). Consequently

$$a(x,\xi) = \sum_{p\geq 0} a_p(x)\, \xi^p, \quad b(x,\xi) = \sum_{q \geq 0} b_q(x)\, \xi^q.$$

From these relations we immediately deduce

$$\partial_\xi^k a(x,\xi) = \sum_{p \geq 0} k! \binom{p}{k} a_p(x) \xi^{p-k}, \quad \partial_x^k b(x,\xi) = \sum_{q\geq 0} \left(\partial_x^k b_q\right)(x)\, \xi^q. \tag{3}$$

We compute

$$A \circ B = \left(\sum_{p \geq 0} (-i)^p a_p(x)\, \partial_x^p \right) \circ \left(\sum_{q \geq 0} b_q(x)\, D^q \right) =$$
$$= \sum_{p,q \geq 0} (-i)^p a_p(x)\, \partial_x^p \circ (b_q(x)\, D^q)\,.$$

Iterating Leibniz rule, we easily obtain

$$\partial_x^p \circ (b_q(x)\, D^q) = \sum_{k=0}^{p} \binom{p}{k} \left(\partial_x^k b_q \right)(x)\, \partial_x^{p-k} \circ D^q$$

so that

$$A \circ B = \sum_{p,q \geq 0} (-i)^p a_p(x) \sum_{k=0}^{p} \binom{p}{k} \left(\partial_x^k b_q \right)(x)\, \partial_x^{p-k} \circ D^q =$$
$$= \sum_{k \geq 0} (-i)^k \sum_{p,q \geq 0} \binom{p}{k} a_p(x) \left(\partial_x^k b_q \right)(x)\, D^{p-k} \circ D^q.$$

It is now clear, also in view of (3), that the symbol of $A \circ B$ equals

$$(a \star b)(x, \xi) = \sum_{k \geq 0} \frac{(-i)^k}{k!} \left(\sum_{p \geq 0} k! \binom{p}{k} a_p(x)\, \xi^{p-k} \right) \left(\sum_{q \geq 0} \left(\partial_x^k b_q \right)(x)\, \xi^q \right) =$$
$$= \sum_{k \geq 0} \frac{(-i)^k}{k!}\, \partial_\xi^k a(x, \xi)\, \partial_x^k b(x, \xi)$$

as wanted. ♢

Exercise 2.8.1. *Reduce to the canonical form the following equations:*

$$u_{xx} + 2u_{xy} - 2u_{xz} + 2u_{yy} + 6u_{zz} = 0, \tag{4a}$$
$$u_{xy} - u_{xz} + u_x + u_y - u_z = 0. \tag{4b}$$

Solution. As for equation (4a), the matrix A for the coefficients of the second order terms has the form

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 6 \end{pmatrix}.$$

To compute the signature of the quadratic form Q associated to A we must compute the sign of the eigenvalues of the latter matrix. In order to do so, we compute its characteristic polynomial:

$$P_A(\lambda) = -\lambda^3 + 9\lambda^2 - 18\lambda + 4.$$

We could explicitly compute the roots of this polynomial, but to find the signature it suffices to apply *Decartes' rule*: the number of positive roots of the polynomial equals the number of sign changes in its coefficients. Using this criterion we obtain in this case three positive roots: the signature will then be $(p = 3, q = 0)$. From this we obtain

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as the canonical form for Q.

The equation we started from does not contain terms of order less than 2. Consequently it is not necessary to compute the change of coordinates matrix

$$\xi_i = \sum_{k=1}^{3} c_{ki} \tilde{\xi}_k. \tag{5}$$

Indeed, it is immediate to verify [Additional exercise: do that!] that, if b_i denotes the (constant) coefficients of the first order terms of a linear differential operators, then after the coordinate change (5), bringing the equation to its canonical form, they change according to

$$\tilde{b}_k = \sum_{i=1}^{d} c_{ki} b_i \tag{6}$$

(notice the inverted order of indices with respect to (5)!). The canonical form of Equation (4a) thus reduces to

$$\varphi_{uu} + \varphi_{vv} + \varphi_{ww} = 0.$$

As for Equation (4b) the matrix A has the form

$$A = \begin{pmatrix} 0 & 1/2 & -1/2 \\ 1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{pmatrix}$$

Computing the characteristic polynomial, we obtain this time

$$P_A(\lambda) = -\lambda \left(\lambda^2 - \frac{1}{2} \right).$$

We deduce that the eigenvalues are $\lambda_0 = 0, \lambda_\pm = \pm 1/\sqrt{2}$, and that the canonical form is

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let's compute explicitly the eigenvectors (normalized to 1 for convenience) corresponding to these eigenvalues:

$$v_+ = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ 1 \\ -1 \end{pmatrix}, \quad v_- = \frac{1}{2} \begin{pmatrix} \sqrt{2} \\ -1 \\ 1 \end{pmatrix}, \quad v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

These vectors also give the columns for the matrix $T = (v_+|v_-|v_0)$ needed to change the basis and put A in the diagonal form, with its eigenvalues on the diagonal. Notice however that in the case under considerations we obtain

$$T^t A T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \tilde{A},$$

hence to compute the change of basis matrix $C = (c_{ki})_{1 \le k,i \le 3}$ bringing A to $\tilde{A}$ (as in (5)) we just need to rescale appropriately: $C = \sqrt[4]{2}\, T$. The transpose matrix (compare (6)) of the correct change of basis, needed to compute the new coefficients of the terms of order 1, is then

$$C^t = 2^{-3/4} \begin{pmatrix} \sqrt{2} & 1 & -1 \\ \sqrt{2} & -1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Applying this matrix to the "vector" of the linear terms in (4b), namely $b = (1, 1, -1)$, we obtain

$$\tilde{b} = 2^{-1/4} \left(1 + \sqrt{2}, 1 - \sqrt{2}, 0\right).$$

The canonical form of Equation (4b) is thus

$$\varphi_{uu} - \varphi_{vv} + \frac{1+\sqrt{2}}{\sqrt[4]{2}} \varphi_u + \frac{1-\sqrt{2}}{\sqrt[4]{2}} \varphi_v = 0.$$

◊