# Ist. di Fisica Matematica mod. A
# Third exercise session

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Exercises are numbered as in the lecture notes of the course.

**Exercise 4.5.1.** *Find a function $u(x, y)$ satisfying*

$$\triangle u = x^2 - y^2$$

*for $r < a$ and the boundary condition $u\big|_{r=a} = 0$.*

*Solution.* Given the symmetry of the problem, we use polar coordinates

$$\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \end{cases} \qquad r \in [0, +\infty), \quad \varphi \in [0, 2\pi),$$

and look for solutions to the equation

$$\triangle u = r^2(\cos^2\varphi - \sin^2\varphi) = r^2\cos 2\varphi \tag{1}$$

in the form

$$u(r, \varphi) = \alpha_0(r) + \sum_{n=1}^{\infty}(\alpha_n(r)\cos n\varphi + \beta_n(r)\sin n\varphi) \tag{2}$$

on which we will then impose the boundary condition $u(a, \varphi) = 0$ for all $\varphi$.

The Laplace operator in polar coordinates has the form

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}. \tag{3}$$

We compute

$$u_r(r, \varphi) = \alpha_0'(r) + \sum_{n=1}^{\infty}(\alpha_n'(r)\cos n\varphi + \beta_n'(r)\sin n\varphi),$$
$$u_{rr}(r, \varphi) = \alpha_0''(r) + \sum_{n=1}^{\infty}(\alpha_n''(r)\cos n\varphi + \beta_n''(r)\sin n\varphi),$$
$$u_{\varphi\varphi}(r, \varphi) = -\sum_{n=1}^{\infty}n^2(\alpha_n(r)\cos n\varphi + \beta_n(r)\sin n\varphi).$$

Substituting in (1) we obtain

$$\begin{aligned}
&\alpha_0''(r) + \frac{\alpha_0'(r)}{r} \\
&+\sum_{n=1}^{\infty}\Big(\alpha_n''(r) + \frac{1}{r}\alpha_n'(r) - \frac{n^2}{r^2}\alpha_n(r)\Big)\cos n\varphi \\
&+\sum_{n=1}^{\infty}\Big(\beta_n''(r) + \frac{1}{r}\beta_n'(r) - \frac{n^2}{r^2}\beta_n(r)\Big)\sin n\varphi = r^2\cos 2\varphi,
\end{aligned}$$

from which we deduce the following infinite number of systems of ODEs

$$\begin{cases} \alpha_0''(r) + \frac{1}{r}\alpha_0'(r) = 0, \\ \alpha_0(a) = 0, \end{cases} \tag{4}$$

$$\begin{cases} \alpha_n''(r) + \frac{1}{r}\alpha_n'(r) - \frac{n^2}{r^2}\alpha_n(r) = \begin{cases} r^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \\ \alpha_n(a) = 0, \end{cases} \tag{5}$$

$$\begin{cases} \beta_n''(r) + \frac{1}{r}\beta_n'(r) - \frac{n^2}{r^2}\beta_n(r) = 0, \\ \beta_n(a) = 0. \end{cases} \tag{6}$$

The general solution to (4) is

$$\alpha_0(r) = C\log\frac{r}{a}. \tag{7}$$

In order for this function to be defined on the whole disk we have to set $C = 0$: hence $\alpha_0(r) = 0$. Moreover, the solution to (5) with $n \neq 2$, as well as to (6), is $\alpha_n(r) = 0$, $n \neq 2$, and $\beta_n(r) = 0$.

We now look for the solution to

$$\begin{cases} \alpha_2''(r) + \frac{1}{r}\alpha_2'(r) - \frac{4}{r^2}\alpha_2(r) = r^2 \\ \alpha_2(a) = 0 \end{cases} \tag{8}$$

The solution to the associated homogeneous equation has the form $Ar^2 + Br^{-2}$. Again, requiring that this function be regular at the origin yields to $B = 0$. We now look for a particular solution in the class of polynomial functions: we find

$$\overline{\alpha_2}(r) = \frac{r^4}{12}.$$

Hence the general solution to the equation in the system (8) will be

$$\alpha_2(r) = Ar^2 + \frac{1}{12}r^4.$$

Imposing the condition $\alpha_2(a) = 0$ we obtain $A = -a^2/12$, and thus

$$\alpha_2(r) = \frac{1}{12} r^2 (r^2 - a^2).$$

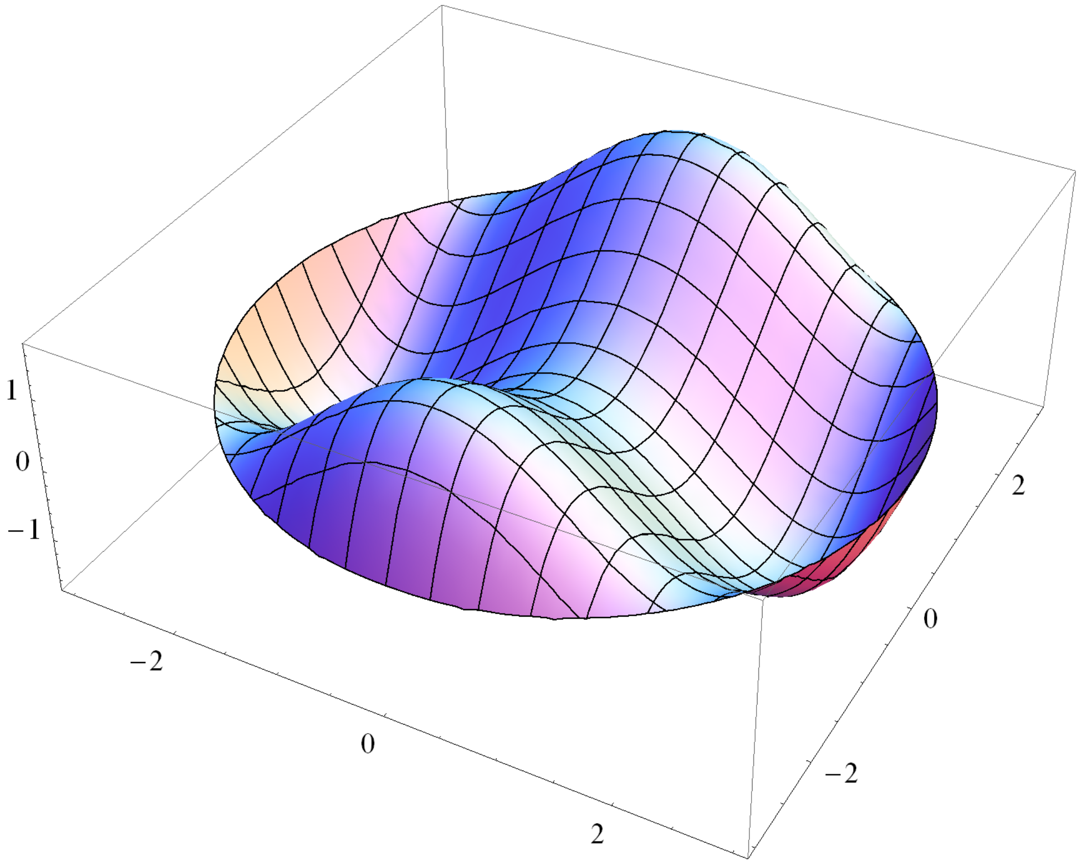
In conclusion, the solution $u$ to the Laplace problem under consideration is

$$u(r, \varphi) = \frac{1}{12} r^2 (r^2 - a^2) \cos 2\varphi,$$

namely, in Euclidean coordinates,

$$u(x, y) = \frac{1}{12} \left( x^4 - y^4 - a^2 (x^2 - y^2) \right).$$

The graph of this function, for $a = 3$, is depicted here.

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**Exercise 4.5.2.** *Find a harmonic function $u(r, \varphi)$ on the annular domain*

$$C_{a,b} = \{(r, \varphi) : a < r < b\}$$

*with the boundary conditions*

$$u(a, \varphi) = 1, \quad u_r(b, \varphi) = \cos^2 \varphi. \tag{9}$$

*Solution.* Given the symmetry of the domain, we look for solutions in the form (2) on which we will impose the boundary conditions (9).

The first boundary condition implies

$$\alpha_0(a) = 1 \quad \text{and} \quad \alpha_n(a) = \beta_n(a) = 0 \quad \text{for } n = 1, 2, \ldots$$

Rewrite $\cos^2 \varphi = \frac{1}{2} + \frac{1}{2}\cos 2\varphi$. Using the expression for $u_r$ computed above and the second condition in (9) we obtain

$$\alpha_0'(b) = \frac{1}{2}, \quad \alpha_2'(b) = \frac{1}{2} \quad \text{and} \quad \alpha_n'(b) = 0 \quad \text{for } n \neq 0, 2.$$

Computing $u_{rr}$ and $u_{\varphi\varphi}$ and substituting in the Laplace equation, using the expression (3) for the Laplace operator in polar coordinates, we obtain the following systems of second order ODEs:

$$\begin{cases} \alpha_0''(r) + \frac{1}{r}\alpha_0'(r) = 0, \\ \alpha_0(a) = 1, \\ \alpha_0'(b) = \frac{1}{2}, \end{cases} \tag{10}$$

$$\begin{cases} \alpha_n''(r) + \frac{1}{r}\alpha_n'(r) - \frac{n^2}{r^2}\alpha_n(r) = 0, \\ \alpha_n(a) = 0, \\ \alpha_n'(b) = \begin{cases} \frac{1}{2} & \text{se } n = 2, \\ 0 & \text{altrimenti,} \end{cases} \end{cases} \tag{11}$$

$$\begin{cases} \beta_n''(r) + \frac{1}{r}\beta_n'(r) - \frac{n^2}{r^2}\beta_n(r) = 0, \\ \beta_n(a) = 0, \\ \beta_n'(b) = 0. \end{cases} \tag{12}$$

The solution to (10) is given by

$$1 + \frac{b}{2}\log\frac{r}{a}.$$

(Notice that this time this solution is admissible, since the annulus $C_{a,b}$ doesn't contain the origin.) The solution to (12) vanishes identically for all $n$, as well as the one to (11). The only case we have to treat more carefully is given by

$$\begin{cases} \alpha_2''(r) + \frac{1}{r}\alpha_2'(r) - \frac{4}{r^2}\alpha_2(r) = 0, \\ \alpha_2(a) = 0, \\ \alpha_2'(b) = \frac{1}{2}. \end{cases}$$

The general solution to this type of equation is of the form

$$\alpha_2(r) = Ar^2 + Br^{-2}.$$

Imposing the boundary conditions we find the following values for the constants $A$ and $B$:

$$A = \frac{b^3}{4\,(a^4 + b^4)}, \quad B = -\frac{a^4 b^3}{4\,(a^4 + b^4)}.$$

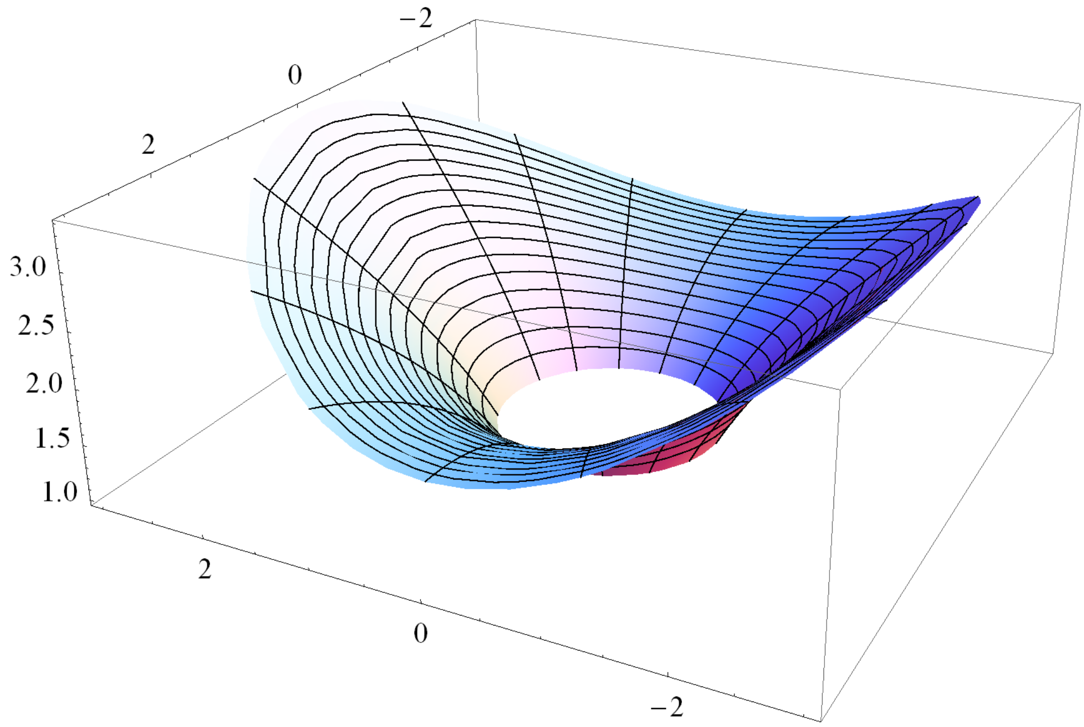
Hence the solution reads

$$\alpha_2(r) = \frac{b^3}{4\,(a^4 + b^4)} r^2 - \frac{a^4 b^3}{4\,(a^4 + b^4)} r^{-2}.$$

We only have to substitute the functions that we found in (2), which yields to the solution to the Laplace problem:

$$u(r, \varphi) = 1 + \frac{b}{2} \log \frac{r}{a} + \left( \frac{b^3}{4\,(a^4 + b^4)} r^2 - \frac{a^4 b^3}{4\,(a^4 + b^4)} r^{-2} \right) \cos 2\varphi. \tag{13}$$

The graph of this function is depicted here, for $a = 1$ and $b = 3$.

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**Exercise 4.5.3.** *Find the solution $u(x, y)$ to the Dirichlet b.v.p. in the rectangle*

$$R_{a,b} = \{(x, y) : 0 \le x \le a, \quad 0 \le y \le b\}$$

*satisfying the boundary conditions*

$$u(0,y) = Ay(b-y), \quad u(a,y) = 0, \quad u(x,0) = B\sin\frac{\pi x}{a}, \quad u(x,b) = 0. \tag{14}$$

*[Hint: use separation of variables in Euclidean coordinates.]*

*Solution.* It suffices to find two functions $u_1$ and $u_2$ satisfying the following Dirichlet problems:

$$\begin{cases} \triangle u_1 = 0 & \text{for } (x,y) \in R_{a,b}, \\ u_1(0,y) = Ay(b-y), & u_1(a,y) = 0, \\ u_1(x,0) = 0, & u_1(x,b) = 0, \end{cases}$$

and

$$\begin{cases} \triangle u_2 = 0 & \text{for } (x,y) \in R_{a,b}, \\ u_2(0,y) = 0, & u_2(a,y) = 0, \\ u_2(x,0) = B\sin\left(\frac{\pi}{a}x\right), & u_2(x,b) = 0. \end{cases}$$

Indeed, by the linearity of the Laplace problem the function $u := u_1 + u_2$ will solve the Dirichlet problem stated in the Exercise.

We look for the function $u_1$ in the form

$$u_1(x,y) = X_1(x)Y_1(y)$$

following the hint in the text. One has

$$\triangle u_1(x,y) = X_1''(x)Y_1(y) + X_1(x)Y_1''(y) = 0$$

which yields to

$$\frac{X_1''(x)}{X_1(x)} = -\frac{Y_1''(y)}{Y_1(y)} = \lambda$$

where $\lambda$ is a constant (indeed the first ratio depends only on $x$ while the second depends only on $y$). Imposing also the boundary conditions, we obtain that the function $Y_1$ solves

$$\begin{cases} Y_1''(y) = -\lambda Y_1(y) & \text{for } 0 \le y \le b, \\ Y_1(0) = 0 = Y_1(b), \end{cases}$$

hence we deduce that

$$\lambda = \lambda_n = \left(\frac{\pi}{b}n\right)^2 \quad \text{and} \quad Y_1(y) = C_n \sin\left(\frac{\pi}{b}ny\right), \quad n \in \mathbb{N}.$$

On the other hand, the solution to the equation

$$X_1''(x) = \lambda_n X_1(x) \quad \text{for } 0 \le x \le a$$

will be of the form

$$X_1(x) = D_n \exp\left(\frac{\pi}{b}nx\right) + D_n' \exp\left(-\frac{\pi}{b}nx\right).$$

Imposing the boundary condition

$$X_1(a) = 0$$

yields to

$$0 = D_n \exp\left(\frac{\pi}{b}na\right)\left(1 + \frac{D_n'}{D_n}\exp\left(-\frac{2\pi}{b}na\right)\right) \quad \Longrightarrow \quad D_n' = -\exp\left(\frac{2\pi}{b}na\right) D_n$$

and hence

$$X_1(x) = D_n\left(\exp\left(\frac{\pi}{b}nx\right) - \exp\left(-\frac{\pi}{b}n(x-2a)\right)\right).$$

We have thus obtained a family of solutions, parametrized by $n \in \mathbb{N}$. By linearity, the sum of any two of these solutions is again a solution to the Laplace problem: as a consequence, the general form of the function $u_1$ will be

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n\left(\exp\left(\frac{\pi}{b}nx\right) - \exp\left(-\frac{\pi}{b}n(x-2a)\right)\right)\sin\left(\frac{\pi}{b}ny\right).$$

The coefficients $A_n = C_n D_n$ can now be computed by imposing the last boundary condition, namely

$$Ay(b-y) = u_1(0,y) = \sum_{n=1}^{\infty} A_n\left(1 - \exp\left(\frac{2\pi}{b}na\right)\right)\sin\left(\frac{\pi}{b}ny\right).$$

In order to do this, we compute the Fourier coefficients of the function $f(y) = Ay(b-y)$ extended by oddity on the interval $(-b,b)$; we want indeed to expand this function in a series of $2b$-periodic sines. One has

$$\begin{aligned}
&\frac{1}{b}\left[\int_0^b f(y)\,\sin\left(\frac{\pi}{b}ny\right)dy + \int_{-b}^0 (-f(-y))\,\sin\left(\frac{\pi}{b}ny\right)dy\right] = \\
&= \frac{2}{b}\int_0^b Ay(b-y)\,\sin\left(\frac{\pi}{b}ny\right)dy = \\
&= -\frac{2A}{\pi n}\left[y(b-y)\cos\left(\frac{\pi}{b}ny\right)\Big|_{y=0}^{y=b} - \int_0^b (b-2y)\,\cos\left(\frac{\pi}{b}ny\right)dy\right] = \\
&= \frac{2Ab}{\pi^2 n^2}\left[(b-2y)\sin\left(\frac{\pi}{b}ny\right)\Big|_{y=0}^{y=b} + 2\int_0^b \sin\left(\frac{\pi}{b}ny\right)dy\right] = \\
&= -\frac{4Ab^2}{\pi^3 n^3}\cos\left(\frac{\pi}{b}ny\right)\Big|_{y=0}^{y=b} = -\frac{4Ab^2}{\pi^3 n^3}((-1)^n - 1) = \\
&= \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{8Ab^2}{\pi^3 n^3} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

We conclude that

$$A_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \dfrac{8Ab^2}{\pi^3 n^3} \dfrac{1}{1 - \exp\left(\frac{2\pi}{b} na\right)} & \text{if } n \text{ is odd,} \end{cases}$$

and thus

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{8Ab^2}{\pi^3 (2n-1)^3} \frac{\exp\left(\frac{\pi}{b}(2n-1)x\right) - \exp\left(-\frac{\pi}{b}(2n-1)(x-2a)\right)}{1 - \exp\left(\frac{2\pi}{b}(2n-1)a\right)} \cdot \\ \cdot \sin\left(\frac{\pi}{b}(2n-1)y\right).$$

To find the function $u_2$, we proceed in the same way. We impose the form

$$u_2(x,y) = X_2(x) Y_2(y).$$

We find again

$$-\frac{X_2''(x)}{X_2(x)} = \frac{Y_2''(y)}{Y_2(y)} = \mu$$

with constant $\mu$. Imposing the boundary conditions, we obtain that $X_2$ solves

$$\begin{cases} X_2''(x) = -\mu X_2(x) & \text{for } 0 \le x \le a, \\ X_2(0) = 0 = X_2(a), \end{cases}$$

hence we deduce

$$\mu = \mu_n = \left(\frac{\pi}{a} n\right)^2 \quad \text{and} \quad X_2(x) = E_n \sin\left(\frac{\pi}{a} nx\right), \quad n \in \mathbb{N}.$$

Arguing as before, we obtain also the solution to the problem

$$\begin{cases} Y_2''(y) = \mu_n Y_2(x) & \text{per } 0 \le y \le b, \\ Y_2(b) = 0 \end{cases}$$

in the form

$$Y_2(y) = F_n \left( \exp\left(\frac{\pi}{a} ny\right) - \exp\left(-\frac{\pi}{a} n(y-2b)\right)\right).$$

The general form of the function $u_2$ will thus be

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi}{a} nx\right) \left( \exp\left(\frac{\pi}{a} ny\right) - \exp\left(-\frac{\pi}{a} n(y-2b)\right)\right).$$

The coefficients $B_n = E_n F_n$ can be now computed by imposing the last boundary condition, namely

$$B \sin\left(\frac{\pi}{a} x\right) = u_2(x,0) = \sum_{n=1}^{\infty} B_n \left(1 - \exp\left(\frac{2\pi}{a} nb\right)\right) \sin\left(\frac{\pi}{a} nx\right).$$

We immediately obtain

$$B_n = \begin{cases} 0 & \text{if } n \neq 1, \\ \dfrac{B}{1 - \exp\left(\frac{2\pi}{a} b\right)} & \text{if } n = 1. \end{cases}$$

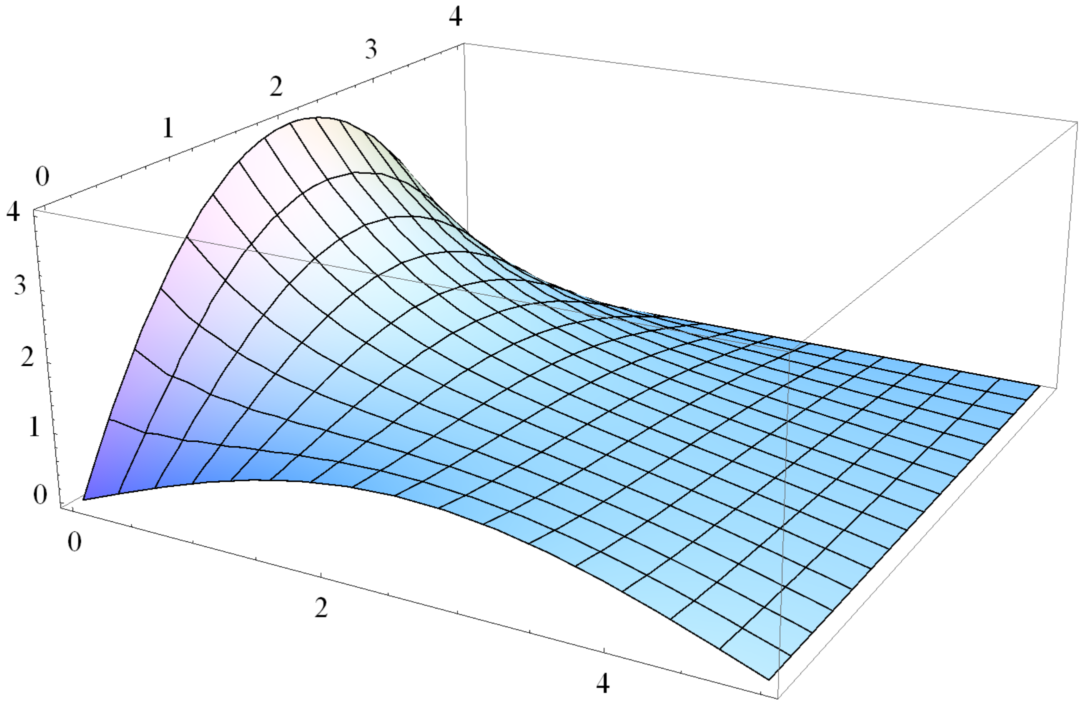
In conclusion

$$u_2(x, y) = B \sin\left(\frac{\pi}{a} x\right) \frac{\exp\left(\frac{\pi}{a} y\right) - \exp\left(-\frac{\pi}{a}(y - 2b)\right)}{1 - \exp\left(\frac{2\pi}{a} b\right)}.$$

The following figure illustrates the graph of the solution $u(x, y) = u_1(x, y) + u_2(x, y)$ for the following values of the parameters:

$$A = 1, \quad B = 10, \quad a = 5, \quad b = 4.$$

For "computational" reasons, only the first 5 terms in the series defining $u_1$ have been computed.

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