Ist. di Fisica Matematica mod. A
Fourth exercise session

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November 25th, 2015

Exercises are numbered as in the lecture notes of the course.

The following Exercises deal with the Fourier transform

$$\hat{f}(p) = \mathcal{F}_{x\to p}(f)(p) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ipx} f(x)\,dx. \tag{1}$$

Recall that the inversion formula

$$f(x) = \mathcal{F}_{p\to x}(\hat{f})(x) = \int_{-\infty}^{+\infty} e^{ipx} \hat{f}(p)\,dp \tag{2}$$

holds.

Exercise 5.7.2. *Let $\hat{f}(p)$ be the Fourier transform of the function $f(x)$. Prove that $e^{iap}\hat{f}(p)$ is the Fourier transform of the shifted function $f(x+a)$.*

Solutione. Denote by $(T_a f)(x) := f(x+a)$. Clearly $T_a f$ is absolutely integrable every time f is, since the Lebesgue measure dx is translation-invariant: hence the Fourier transform $(\widehat{T_a f})(p)$ is well-defined. One then has

$$\begin{aligned}
(\widehat{T_a f})(p) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ipx}(T_a f)(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ipx} f(x+a)\,dx = \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ip(y-a)} f(y)\,dy = \frac{e^{ipa}}{2\pi}\int_{-\infty}^{+\infty} e^{-ipy} f(y)\,dy = \\
&= e^{iap}\hat{f}(p)
\end{aligned}$$

where the third equality is realized by the change of variables $y = x + a$ (hence $dy = dx$). ♢

Exercise 5.7.3. *Find the Fourier transforms of the following functions.*

$$f(x) = \Pi_A(x) = \begin{cases} \dfrac{1}{2A} & \textit{if } |x| < A, \\ 0 & \textit{otherwise} \end{cases} \tag{5.7.2}$$

$$f(x) = \Pi_A(x)\,\cos(\omega x) \tag{5.7.3}$$

Solutione. As for (5.7.2), we compute

$$\begin{aligned} \hat{\Pi}_A(p) &= \frac{1}{2\pi}\int_{-A}^{A} e^{-ipx}\,\frac{1}{2A}\,dx = \frac{1}{4\pi A}\left[\frac{e^{-ipx}}{-ip}\right]_{x=-A}^{x=A} = \\ &= \frac{1}{2\pi}\frac{1}{Ap}\,\frac{e^{iAp}-e^{-iAp}}{2i} = \frac{1}{2\pi}\,\frac{\sin(Ap)}{Ap}. \end{aligned}$$

As for (5.7.3), first notice that, arguing as in the previous Exercise, it is easy to show that the Fourier transform of a function of the form $e^{\pm i\omega x}g(x)$ is given by the translated function $\hat{g}(p \mp \omega)$. Since

$$f(x) = \Pi_A(x)\,\cos(\omega x) = \frac{1}{2}\left(e^{i\omega x}\Pi_A(x) + e^{-i\omega x}\Pi_A(x)\right),$$

by the linearity of the Fourier transform we have that

$$\hat{f}(p) = \frac{1}{2}\left(\hat{\Pi}_A(p-\omega) + \hat{\Pi}_A(p+\omega)\right) = \frac{1}{4\pi}\left(\frac{\sin(A(p-\omega))}{A(p-\omega)} + \frac{\sin(A(p+\omega))}{A(p+\omega)}\right).$$

$\diamondsuit$

Exercise 5.7.4. *Find the function $f(x)$ if its Fourier transform is given by*

$$\hat{f}(p) = e^{-k|p|}, \quad k > 0. \tag{5.7.7}$$

Solutione. We use the inversion formula (2). We compute

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} e^{ipx}\,e^{-k|p|}\,dp = \\ &= \int_{-\infty}^{0} e^{ipx}\,e^{kp}\,dp + \int_{0}^{+\infty} e^{ipx}\,e^{-kp}\,dp = \\ &= 2\int_0^{\infty} \cos(px)\,e^{-kp}\,dp = \frac{2}{k}\int_0^{\infty} \cos\left(q\frac{x}{k}\right)e^{-q}\,dq \end{aligned}$$

upon substituting $q = kp$. Integrating by parts twice, we get

$$\begin{aligned} \int_0^{\infty} \cos\left(q\frac{x}{k}\right)e^{-q}\,dq &= \left[-e^{-q}\cos\left(q\frac{x}{k}\right)\right]_{q=0}^{q=\infty} - \frac{x}{k}\int_0^{\infty} \sin\left(q\frac{x}{k}\right)e^{-q}\,dq = \\ &= 1 - \frac{x}{k}\left(\left[-e^{-q}\sin\left(q\frac{x}{k}\right)\right]_{q=0}^{q=\infty} + \frac{x}{k}\int_0^{\infty} \cos\left(q\frac{x}{k}\right)e^{-q}\,dq\right) = \\ &= 1 - \frac{x^2}{k^2}\int_0^{\infty} \cos\left(q\frac{x}{k}\right)e^{-q}\,dq \end{aligned}$$

from which we deduce that

$$\int_0^\infty \cos\left(q\frac{x}{k}\right) e^{-q}\, dq = \frac{1}{1+\frac{x^2}{k^2}} = \frac{k^2}{x^2+k^2}.$$

In conclusion

$$f(x) = \frac{2}{k}\frac{k^2}{x^2+k^2} = \frac{2k}{x^2+k^2}. \tag{3}$$

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Exercise 5.7.5. *Let $u = u(x,y)$ be a solution to the Laplace equation on the half-plane $y \geq 0$ satisfying the conditions*

$$\begin{aligned} &\Delta u(x,y) = 0, \quad y > 0 \\ &u(x,0) = \phi(x) \\ &u(x,y) \to 0 \quad \textit{as} \quad y \to +\infty \quad \textit{for every} \quad x \in \mathbb{R}. \end{aligned} \tag{5.7.8}$$

1) Prove that the Fourier transform of u in the variable x

$$\hat{u}(p,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x,y)e^{-ipx}\, dx$$

has the form

$$\hat{u}(p,y) = \hat{\phi}(p)e^{-y|p|}.$$

Here $\hat{\phi}(p)$ is the Fourier transform of the boundary function $\phi(x)$.
2) Derive the following formula for the solution to the b.v.p. (5.7.8)*:*

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{(x-s)^2+y^2}\,\phi(s)\, ds.$$

Solutione. 1) Recall that (compare (5.3.12) in the lecture notes)

$$\mathcal{F}_{x\to p}\left(f'\right)(p) = ip\,\mathcal{F}_{x\to p}(f)(p). \tag{4}$$

If $u(x,y)$ is a solution to (5.7.8), using (4) one deduces that its Fourier tranform $\hat{u}(p,y)$ satisfies the following problem:

$$\begin{aligned} &(ip)^2\hat{u}(x,y) + \frac{\partial^2\hat{u}}{\partial y^2}(p,y) = 0, \quad y > 0 \\ &\hat{u}(p,0) = \hat{\phi}(p) \\ &\hat{u}(p,y) \to 0 \quad \text{as} \quad y \to +\infty \quad \text{for every} \quad p \in \mathbb{R}. \end{aligned}$$

As $p^2 \geq 0$, the differential equation

$$\frac{\partial^2\hat{u}}{\partial y^2}(p,y) = p^2\hat{u}(x,y)$$

admits as a general solution

$$\hat{u}(p,y) = c_1(p)\, e^{-y|p|} + c_2(p)\, e^{y|p|}.$$

Imposing that $\hat{u}(p,y) \to 0$ as $y \to +\infty$ yields that $c_2(p) \equiv 0$. Evaluating then at $y = 0$ one obtains $c_1(p) = \hat{\phi}(p)$. Consequently

$$\hat{u}(p,y) = \hat{\phi}(p) e^{-y|p|}.$$

2) Applying the inversion formula (2), we obtain

$$\begin{aligned} u(x,y) &= \int_{-\infty}^{+\infty} e^{ipx}\, \hat{u}(p,y)\, dp = \int_{-\infty}^{+\infty} e^{ipx}\, \hat{\phi}(p) e^{-y|p|}\, dp = \\ &= \int_{-\infty}^{+\infty} e^{ipx} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ips} \phi(s)\, ds \right) e^{-y|p|}\, dp = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{-ip(x-s)} e^{-y|p|}\, dp \right) \phi(s)\, ds = \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{F}_{p\to x}\left(e^{-y|p|} \right) (x-s)\, \phi(s)\, ds. \end{aligned}$$

[Exercise: justify the exchange of integration in ds and dp above.] Using the result (3) from the previous Exercise for $k = y$, we conclude that

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y}{(x-s)^2 + y^2}\, \phi(s)\, ds$$

as wanted. ♢