Ist. di Fisica Matematica mod. A
Fifth Exercise Session

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Exercises are numbered as in the lecture notes for the course.

Esercizio 6.5.1. *Derive the following formula for the solution of the Cauchy problem*

$$\delta v(x,0) = \phi(x)$$

for the linearized Burgers equation (6.2.4):

$$\delta v(x,t) = \frac{1}{2\sqrt{\pi\nu t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y-ct)^2}{4\nu t}} \phi(y)\, dy.$$

Solution. We remind the reader that the linearized Burgers equation has the form

$$\delta v_t + c\delta v_x = \nu \delta v_{xx}.$$

We present two possible approaches to complete the assignment.

1. We denote by $\delta\hat{v}(k,t)$ the Fourier transform of $\delta v(x,t)$ with respect to the variable x; then Burgers equation can be rewritten, in *momentum space* as

$$\delta\hat{v}_t = -(ick + \nu k^2)\delta\hat{v}.$$

One immediately gets

$$\delta\hat{v}(k,t) = e^{-(ick+\nu k^2)t}\delta\hat{v}(k,0) = e^{-ikct}e^{-\nu t k^2}\hat{\phi}(k).$$

Now we can apply the formula for the inverse Fourier transform

$$\begin{aligned}
\delta v(x,t) &= \int_{-\infty}^{+\infty} e^{ikx}\, e^{-ikct} e^{-\nu t k^2} \hat{\phi}(k)\, dk = \\
&= \int_{-\infty}^{+\infty} e^{ik(x-ct)} e^{-\nu t k^2} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iky} \phi(y)\, dy \right) dk = \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} e^{ik(x-y-ct)} e^{-\nu t k^2}\, dk \right) \phi(y)\, dy.
\end{aligned}$$

The term in the parentesis is the Fourier tranform of the function $g(k) = e^{-\nu t k^2}$, evaluated in $x - y - ct$. Since

$$\mathcal{F}_{x\to p}\left(e^{-x^2/2}\right)(p) = \frac{e^{-p^2/2}}{\sqrt{2\pi}}, \quad \mathcal{F}_{x\to p}\left(f(a\,x)\right)(p) = a^{-1}\mathcal{F}_{x\to p}(f)(a^{-1}p),$$

one immediately gets

$$\delta v(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{\sqrt{2\pi}}{\sqrt{2\nu t}}e^{-\frac{1}{2}\left(\frac{x-y-ct}{\sqrt{2\nu t}}\right)^2}\right)\phi(y)\,dy = \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y-ct)^2}{4\nu t}}\phi(y)\,dy$$

which proves the assignment.

2. We can perform a change of variables and reduce the linearized Burgers equation to a heat equation. We define

$$u(x,t) := \delta v(x+ct,t) \tag{1}$$

Then the heat equation for u reads

$$0 = u_t - \nu u_{xx} = \delta v_t + c\delta v_x - \nu \delta v_{xx}. \tag{2}$$

The general solution to the heat equation with initial condition $\phi(x)$ is given by the Poisson formula:

$$u(x,t) = \frac{1}{2\sqrt{\pi\nu t}}\int_{-\infty}^{+\infty} e^{\frac{(x-y)^2}{4\nu}}\phi(y)dy. \tag{3}$$

By performing the inverse change of variable we get to the required formula for the solution.

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Esercizio 6.5.2. *Obtain the following representation for solutions to the linearized KdV equation (6.2.7) with the initial data $\delta v(x,0) = \phi(x)$ rapidly decreasing at $|x| \to \infty$:*

$$\delta v(x,t) = \int_{-\infty}^{+\infty} A(x-y-ct, \epsilon^2 t)\phi(y)\,dy$$

where

$$A(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(kx+k^3t)}\,dk. \tag{4}$$

The integral (4) *converges and can be expressed via the Airy function.*

Solution. We remind the reader that the linearized KdV equation reads

$$\delta v_t + c\delta v_x + \epsilon^2 \delta v_{xxx} = 0.$$

We denote by $\delta\hat{v}(k,t)$ the Fourier transform of $\delta v(x,t)$ w.r.t the variable x; then we can rewrite the linearized KdV equation as

$$\delta\hat{v}_t = -i(ck - \epsilon^2 k^3)\delta\hat{v}$$

which immediately leads us to

$$\delta\hat{v}(k,t) = e^{-i(ck-\epsilon^2k^3)t}\delta\hat{v}(k,0) = e^{-ikct}e^{ik^3\epsilon^2 t}\hat{\phi}(k).$$

Now we can apply the formula for the inverse Fourier transform

$$\begin{aligned}\delta v(x,t) &= \int_{-\infty}^{+\infty} e^{ikx}\, e^{-ikct}e^{ik^3\epsilon^2 t}\hat{\phi}(k)\, dk = \\ &= \int_{-\infty}^{+\infty} e^{ik(x-ct)}e^{ik^3\epsilon^2 t}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iky}\phi(y)\, dy\right)\, dk = \\ &= \int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i[k(x-y-ct)+k^3\epsilon^2 t]}\, dk\right)\phi(y)\, dy.\end{aligned}$$

This leads to the required formula (note that, in order to apply Fubini's theorem and exchange the order of integration, it is necessary that ϕ is rapidly decreasing). $\diamondsuit$

Esercizio 6.5.3. *Derive the following* Stirling formula *for the asymptotic of the Gamma function*

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right), \quad x \to +\infty.$$

Hint*: after the substitution $t = x\, s$ the integral rewrites as follows:*

$$\Gamma(x+1) = x^{x+1}\int_0^\infty e^{-x(s-\log s)}\, ds.$$

Solution. We define $S(s) := s - \log s$ for $s > 0$. Then one has

$$S'(s) = 1 - \frac{1}{s} = 0 \iff s = 1, \qquad S''(s) = \frac{1}{s^2} > 0,$$

which tells us that $S(s)$ attains an absolute minimum in $s = 1$. We can apply Laplace's formula (Theorem 6.3.5), which yields that, for $\epsilon \to 0$

$$\int_0^\infty e^{-\frac{S(s)}{\epsilon}}\, ds = \sqrt{\frac{2\pi\epsilon}{S''(1)}}\cdot 1\cdot e^{-\frac{S(1)}{\epsilon}}\left(1+\mathcal{O}(\epsilon)\right) = \sqrt{2\pi\epsilon}\, e^{-\epsilon^{-1}}\left(1+\mathcal{O}(\epsilon)\right).$$

We can now set $\epsilon = 1/x$ and for $x \to +\infty$ we have

$$\begin{aligned}\Gamma(x+1) &= x^{x+1}\int_0^\infty e^{-x(s-\log s)}\, ds = x^{x+1}\sqrt{2\pi\frac{1}{x}}\, e^{-x}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right) = \\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right).\end{aligned}$$

$\diamondsuit$

Esercizio 3.8.5. *Prove that*

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2} \quad per \quad 0 < x < 2\pi. \tag{5}$$

Compute the sums of the following Fourier series for every other values of $x \in \mathbb{R}$.

Solution. We will compute the coefficients a_n, b_n of the Fourier series of the function $f(x) = (\pi - x)/2$. Using the change of variable $x \rightsquigarrow y = x - \pi$ we get

$$a_0 = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{y}{2}\, dy = 0$$

since it is an odd function. Moreover, using the trigonometric identity

$$\cos(ny + n\pi) = \cos(ny) \underbrace{\cos(n\pi)}_{=(-1)^n} - \sin(ny) \underbrace{\sin(n\pi)}_{=0} = (-1)^n \cos(ny) \tag{6}$$

we obtain for the same reason

$$a_n = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{y}{2} \cos(n(y+\pi))\, dy = -\frac{(-1)^n}{\pi} \int_{-\pi}^{\pi} \frac{y}{2} \cos(ny)\, dy = 0.$$

Now we compute the coefficients b_n: integrating by part we get

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{\pi - x}{2} \sin(nx)\, dx = \frac{1}{\pi} \left(\left[\frac{x - \pi}{2} \frac{\cos(nx)}{n} \right]_{x=0}^{x=2\pi} - \frac{1}{2n} \underbrace{\int_0^{2\pi} \cos(nx)\, dx}_{=\delta_{n,0}=0} \right) =$$
$$= \frac{1}{\pi} \left(\frac{\pi}{2} \frac{1}{n} - \frac{-\pi}{2} \frac{1}{n} \right) = \frac{1}{n}.$$

Therefore we proved (5).

The convergence (uniform and absolute) of the series for $x \in (0, 2\pi)$ is guarantee by the fact that in this interval f is of class $\mathcal{C}^1$. For the other values of $x \in \mathbb{R}$, we have that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \begin{cases} f(x') & \text{se } x = x' + 2k\pi,\ k \in \mathbb{Z},\ x' \in (0, 2\pi), \\ 0 & \text{se } x = 2k\pi,\ k \in \mathbb{Z}, \end{cases}$$

that is extending f by periodicity with period 2π (and putting it equal to zero in the multiples of 2π where the sine is zero). $\diamondsuit$