Modular linear differential equations

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The title of my talk is vague, however, it is ambiguous.

The most naive definition of *modular linear differential equations* (MLDE) would be linear differential equations whose space of solutions are invariant under slash action of weight $k$ of $\Gamma = SL_2(\mathbb{Z})$. Then under an analytic condition for coefficients functions and Wronskians of the equations we have obvious expressions of MLDEs as

$$L(f) \;=\; \mathfrak{d}_k^n(f) + \sum_{i=2}^{n} P_{2i}\mathfrak{d}_k^{n-i}(f)\,,$$

where $P_{2i}$ is a modular form of of weight $2i$ on $SL_2(\mathbb{Z})$ and $\mathfrak{d}_k(f)$ is the *Serre derivative*. Of course, we could replace $\Gamma$ as a Fuchsian group of $SL_2(\mathbb{R})$ and modular forms $P_{2i}$ as being meromorphic.

MLDEs are often given in the form

$$\mathsf{L}(f) \;=\; D^n(f) + \sum_{i=1}^{n} Q_i D^i(f) \quad \text{where } D = q\frac{d}{dq}\,.$$

Then, first of all, it is not easy to know if the above equation is an MLDE. (It seems hopeless that we verify if $\mathsf{L}(f) = 0$ is a MLDE.)

Now, Y. Sakai (one of collaborators) found formulas $Q_i$ in terms of $P_{2i}$ and *inversion formulas*. One of remarkable facts he obtained is that the coefficients $Q_i$ are written by the *Rankin-Cohen brackets* $[f, g_{2n}]_*^{(*,*)}$ where $g_{2n}$ satisfy modular property of weight $2n$ and differential polynomials of $P_i$. More precisely, there is a recursion formula which determines $g_{2n}$ with an initial value as an $E_2$ like function $\phi$ which satisfies the same transformation low of $E_2$.

Finally, the most important point of my talk is that I will use a **black-board** instead of **slides**.