Singularly perturbed gradient flows in infinite-dimensional spaces

We study the limit behavior, as ε goes to zero, of a family of solutions to the gradient flow

$$\varepsilon\dot{u}_\varepsilon(t) + \mathrm{D}_x\mathcal{E}(t, u_\varepsilon(t)) = 0,$$

where $\mathcal{E}$ is an energy functional defined on $[0, T] \times \mathsf{H}$ and H is a Hilbert space. To tackle the problem, we adopt a variational point of view and the assumptions we make on $\mathcal{E}$ are quite standard in this context.We also require the condition

$$\{x \in \mathsf{H} : \mathrm{D}_x\mathcal{E}(t, x) = 0\} \text{ discrete for every } t \in [0, T],$$

whose "genericity" can be rigorously proved. To recover the compactness of a family u_{ϵ_n} of solutions, we analyse in detail the integral quantities

$$\int_{t_1^n}^{t_2^n} \|\mathrm{D}_x\mathcal{E}(r, u_{\varepsilon_n}(r))\| \|\dot{u}_{\varepsilon_n}(r)\| dr,$$

for every $t \in [0, T]$ and all sequences $t_1^n \leq t_2^n$ converging to t. We show that these integrals are bounded below by a strictly positive *cost function* $c(t; x_1, x_2)$, whenever $u_{\varepsilon_n}(t_1^n)$ and $u_{\varepsilon_n}(t_2^n)$ converge to two different critical points x_1 and x_2 of $\mathcal{E}(t, \cdot)$. Some key properties of the cost function allow us to prove that u_{ε_n} convergence pointwise in $[0, T]$ to a limit solution u, which satisfies $\mathrm{D}_x\mathcal{E}(t, u(t)) = 0$ for a.e. $t \in (0, T)$. Moreover, we prove that u is continuous on $[0, T] \setminus J$, that the jump set J is a countable set, and that the left and the right limits $u_-(t)$, $u_+(t)$ always exist and satisfy $\mathcal{E}(t, u_-(t)) - \mathcal{E}(t, u_+(t)) = c(t; u_-(t), u_+(t))$.
This is a joint work with Riccarda Rossi and Giuseppe Savaré.