Zeroes of polynomial on definable hypersurfaces: pathologies exist, but they are rare

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Let p be a real homogeneous polynomial of degree d and denote by $Z(p) \subset \mathbb{R}\mathrm{P}^n$ its zero set. It is well known that if $\Gamma \subset \mathbb{R}\mathrm{P}^n$ is an *algebraic* hypersurface, the topology of $\Gamma \cap Z(p)$ is "polynomially bounded": for example the sum of its Betti numbers is at most $O(d^n)$ (the implied constant depends on the degree of Γ).

If we leave the real algebraic world, but we stay in the *definable* setting (for example considering a *subanalytic* Γ), the situation is much more interesting. Still there exists a function bounding the topology of $Z(p) \cap \Gamma$ in terms of the degree $d = \deg(p)$, but in general nothing can be said on the growth of this function.

In fact, given *any* sequence $\{Z_d\}_{d\in\mathbb{N}}$ of hypersurfaces in $\mathbb{R}^{n-1}$, we show that it is possible to construct a regular definable (e.g. subanalytic) $\Gamma \subset \mathbb{R}\mathrm{P}^n$ such that Z_d appears as one of the components of the zero set on Γ of some polynomial of degree d (up to extracting subsequences). We call this a "patological example".

However, pathologies are rare: if we endow the space of polynomials with a Gaussian invariant measure (e.g. the Fubini-Study measure), for a given definable Γ, the measure of the set of pathological polynomials become smaller and smaller as $d \to \infty$, and for "most" polynomials the complexity of $Z(p) \cap \Gamma$ still has at most polynomial growth.

(This is joint work with S. Basu and A. Natarajan)